Numerical Analysis homework # 1

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I. The width of the interval in bisection

I-a

According to the bisection method, the interval decreases to half after each step. The initial width of the interval is 2(at the 0th step), so it will be 2^{1-n}at the nth step.

I-b

At the nth step, the width of the interval is 2^{1-n}, and the midpoint divides the interval into 2 equal parts, each part with a length of 2^{-n}. Therefore, no matter which part the root is located at, its distance to the midpoint would not greater than the width of the small interval, i.e. 2^{-n}. So 2^{-n} is an upper bound.

In addition, assume that the interval is $[a_0, b_0]$ with $b_0 - a_0 = 2^{1-n}$ at the nth step. Given any $\epsilon > 0$, the root can be located at $b_0 - \epsilon$ or $a_0 + \epsilon$, and both of them has the distance $2^{-n} - \epsilon$ to the midpoint $\frac{a_0+b_0}{2}$. It is satisfied for all $\epsilon > 0$, so 2^{-n} is the supremum.

II. Guarantee the relative error

Similar to Problem I, we can get that the supremum of absolute error at nth step with an initial interval $[a_0, b_0]$ is $(b_0-a_0)\cdot 2^{-(n+1)}$. And the relative error is $\frac{(b_0-a_0)\cdot 2^{-(n+1)}}{|r|}$. Due to $0 < a0 \leq r$, so if we can guarantee $\epsilon \leq \frac{(b_0-a_0)\cdot 2^{-(n+1)}}{a_0}$, $\epsilon \leq \frac{(b_0-a_0)\cdot 2^{-(n+1)}}{|r|}$ is satisfied naturally.

$$\epsilon \leq \frac{(b_0 - a_0)\cdot 2^{-(n+1)}}{a_0} \Leftrightarrow n \geq \frac{\log(b_0 - a_0) - \log\epsilon - \log a_0}{\log 2} - 1$$

III. 4 iterations of Newton's method

n	x_n	$p(x_n)$	$p'(x_n)$
0	-1	-3	16
1	-0.8125	-0.46582	11.1719
2	-0.770804	-0.0201379	10.2129
3	-0.768832	-4.37084×10^{-5}	10.1686
4	-0.768828	-2.07412×10^{-10}	10.1685

IV. Newton's method, but only use $f'(x_0)$

By Taylor's theorem and $f(\alpha) = 0$, we can get:

$$f(x_n) = f(\alpha) + f'(\alpha)(x_n - \alpha) + \frac{f''(\xi)}{2}(x_n - \alpha)^2 = f'(\alpha)e_n + \frac{f''(\xi)}{2}e_n^2$$

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where ξ is between α and x_n. Use the iteration formula in the problem:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_0)} = x_n - \frac{f'(\alpha)}{f'(x_0)}e_n - \frac{f''(\xi)}{2f'(x_0)}e_n^2$$

Substract α from both sides:

$$e_{n+1} = (1 - \frac{f'(\alpha)}{f'(x_0)})e_n - \frac{f''(\xi)}{2f'(x_0)}e_n^2 = (1 - \frac{f'(\alpha)}{f'(x_0)} - \frac{f''(\xi)}{2f'(x_0)}(x_n - \alpha))e_n$$

Therefore, $s = 1$, and $C = 1 - \frac{f'(\alpha)}{f'(x_0)} - \frac{f''(\xi)}{2f'(x_0)}(x_n - \alpha)$, where ξ is depended on x_n and α.

V. Converge or not ? $x_{n+1} = \tan^{-1} x_n$

Without generality, assmume that $x_0 > 0$ ($x_0 = 0$ is a trivial case, i.e. $x_1 = x_2 = \cdots = 0$). With $x_{n+1} = \arctan x_n < x_n$ and $x_n > 0$, $\{x_n\}$ has a limit α.

Therefore,

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \arctan x_n = \alpha$$

$$\lim_{n\to\infty} (\arctan x_n - x_n) = \arctan \alpha - \alpha = 0$$

then $\alpha = 0$, which means $\{x_n\}$ converge.

VI. Fixed point

According to the problem, we can construct $f(x) = \frac{1}{x+p}$, and getting the value of $x = \frac{1}{p+\frac{1}{p+\frac{1}{p+\cdots}}}$ is equivalent to getting the fixed point of $f(x)$.

Due to $p > 1$, $x_1 = \frac{1}{p} \in [0, 1]$. Then if $x \in [0, 1]$, $f(x) \in [0, 1]$.

In addition, since $p > 1$, $\exists \epsilon > 0$, s.t. $p > 1 + \epsilon$. Then $|f'(x)| = \frac{1}{(x+p)^2} < \frac{1}{(1+\epsilon)^2}$ for all $x \in [0, 1]$.

Let $\lambda = \frac{1}{(1+\epsilon)^2}$, then for $\forall x, y \in [0, 1]$, $|f(x) - f(y)| \leq \lambda |x - y|$. Therefore, f is a contractive mapping on $[0, 1]$.

For the fixed point α, it should satisfy:$\alpha = f(\alpha) = \frac{1}{\alpha+p}$.

Then we get $\alpha = \frac{-p+\sqrt{p^2+4}}{2}$(negative value is unreasonable).

VII. Is relative error always a good measure?

According to problem II, the relative error is $\frac{(b_0-a_0)\cdot 2^{-(n+1)}}{|r|}$. But in the case $a_0 < 0 < b_0, r \in [a_0, b_0]$, $|r|$ does not have a positive lower bound any more.

By instead $|r|$ with b_0 , we can get $\frac{(b_0-a_0)\cdot 2^{-(n+1)}}{|r|} \leq \frac{(b_0-a_0)\cdot 2^{-(n+1)}}{b_0}$.

Similar to problem II, by using $\epsilon \leq \frac{(b_0-a_0)\cdot 2^{-(n+1)}}{b_0}$, we have $n \geq \frac{\log(b_0-a_0)-\log\epsilon-\log b_0}{\log 2} - 1$.

However, it is just a necessary condition, not sufficient. Because $\epsilon \leq \frac{(b_0-a_0)\cdot 2^{-(n+1)}}{b_0}$ is not sufficient to indicate $\epsilon \leq \frac{(b_0-a_0)\cdot 2^{-(n+1)}}{|r|}$.

This way of approximation of relative is not reliable, and its accuracy would be worse as $|r|$ vanishes. In a extreme case, i.e. $r = 0$, relative error has no meaning at all.

VIII. Multiple zero of Newton's method

VIII-a

Assume that α is a p-th multiple zero of $f(x)$, then it can be written as $f(x) = (x - \alpha)^p g(x)$, where $g'(\alpha) \neq 0$.[1]

According to Newton's method, $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$. It can be seen as a fixed-point problem, with $h(x) = x - \frac{f(x)}{f'(x)}$.

Then iterate it into the first formula:

$$f'(x) = (x - \alpha)^p g'(x) + p(x - \alpha)^{p-1} g(x)$$

$$h(x) = x - \frac{(x - \alpha)g(x)}{pg(x) + (x - \alpha)g'(x)}$$

$$h'(x) = 1 - \frac{g(x)}{pg(x) - (x-\alpha)g'(x)} - (x-\alpha)\frac{d}{dx}(\frac{g(x)}{pg(x) + (x-\alpha)g'(x)})$$

Therefore, $h'(\alpha) = 1 - \frac{1}{p} \in (0, 1)$, if $p > 1$. And Newton's method converges linearly with a rate of $1 - \frac{1}{p}$.

Compared to the case that only has single zeros, which converges quadratically, multiple zero cases has a lower convergence rate, and this can be observed at the behavious of $(x_n, f(x_n))$.

VIII-b

According to the problem, we instead the formula of $h(x)$ with $\tilde{h}(x) = x - p\frac{(x-\alpha)g(x)}{pg(x)+(x-\alpha)g'(x)}$.

Then compared with $h'(x)$ in VIII-a, it is obvious to get $\tilde{h}'(\alpha) = 0$. Therefore, Newton's method after modification has a quadratical convergence.

References

[1] Kendall Atkinson. *An introduction to numerical analysis.* John wiley & sons, 1991.