Numerical Analysis homework # 2

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I

I-a

According to the linear interpolation, $p_1(f;x) = -\frac{1}{2}x + \frac{3}{2}$. $f(x) = \frac{1}{x}$ results in $f''(x) = \frac{2}{x^3}$. Therefore, the equation in the problem can be wriiten as:

$$\frac{1}{x} + \frac{1}{2}x - \frac{3}{2} = \frac{1}{\xi^3(x)}(x-1)(x-2)$$

In the interval $(1,2)$, we can get $\xi(x) = \sqrt[3]{2x}$.

I-b

According to the continuity, we can extend $\xi(x)$ to $[1,2]$ by define $\xi(1) = \lim\limits_{x\to 1^+} \xi(x) = \sqrt[3]{2}$ and $\xi(1) = \lim\limits_{x\to 2^-} \xi(x) = \sqrt[3]{4}$. Therefore, $\max\limits_{x\in[1,2]} \xi(x) = \xi(2) = \sqrt[3]{4}$, $\min\limits_{x\in[1,2]} \xi(x) = \xi(1) = \sqrt[3]{2}$.

In addition, $f''(\xi(x)) = \frac{1}{x^3}$, then $\max\limits_{x\in[1,2]} f''(\xi(x)) = f''(\xi(1)) = 1$.

II

Due to $f(x_i) \geq 0$ for each i, given $n+1$ distinct x_i and their function values $\sqrt{f_i}$, a unique $\tilde{p}(x) \in \mathbb{P}_n$ can be determined.

Then $p(x) = (\tilde{p}(x))^2 \in \mathbb{P}^+_{2n}$ and $p(x)$ satisfies that $p(x_i) = f_i$ with $f_i$for each $i = 0, 1, \cdots, n$.

III

III-a

In the case that $n = 1$, $f[t, t+1] = f(t+1) - f(t) = e^{t+1} - e^t = (e-1)e^t$, which satisfies the equation in the problem.

We assume that the equation holds when $n = m$, i.e. $f[t, t+1, \cdots, t+m] = \frac{(e-1)^m}{m!}e^t$, then $f[t, t+1, \cdots, t+m+1] = \frac{f[t+1,t+2,\cdots,t+m+1]-f[t,t+1,\cdots,t+m]}{m+1} = \frac{1}{m+1}(\frac{(e-1)^m}{m!}e^{t+1} - \frac{(e-1)^m}{m!}e^t) = \frac{(e-1)^{m+1}}{(m+1)!}e^t$.

It means that the equation also holds when $n = m+1$. By induction, the equation holds for each $n \in \mathbb{Z}^+$.

III-b

According to III-a, $f[0, 1, \cdots, n] = \frac{(e-1)^n}{n!}$. From Corollary 2.22 we know $\exists \xi \in (0, n)$, s.t. $f[0, 1, \cdots, n] = \frac{1}{n!}f^{(n)}(\xi)$. $f(x) = e^x$, so $f^{(n)}(x) = e^x$, which means $\frac{(e-1)^n}{n!} = \frac{e^\xi}{n!}$. Then $\xi = n\ln(e-1) > \frac{n}{2}$, ξ is located to the right of the midpoint $\frac{n}{2}$.

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IV

IV-a

According to the data given by the problem, we can construct the following table of divided difference:

$$\begin{array}{c|cccc} 0 & 5 & & & \\ 1 & 3 & -2 & & \\ 3 & 5 & 1 & 1 & \\ 4 & 12 & 7 & 2 & \frac{1}{4} \end{array}$$

Therefore, $p_3(f;x) = 5 - 2x + x(x-1) + \frac{1}{4}x(x-1)(x-3)$.

IV-b

According to the formula of $p_3(f;x)$ above, we can get $p_3'(f;x) = \frac{3}{4}x^2 - \frac{9}{4}$. Therefore, $p_3'(f;x)$ decreases monotonocally on $(1, \sqrt{3})$ and increases monotonocally on $(\sqrt{3}, 3)$, which means it has a minimum at $\sqrt{3}$.

V

V-a

According to the data given by the problem, we can construct the following table of divided difference:

$$\begin{array}{c|cccccc} 0 & 0 & & & & & \\ 1 & 1 & 1 & & & & \\ 1 & 1 & 7 & 6 & & & \\ 1 & 1 & 7 & 21 & 15 & & \\ 2 & 128 & 127 & 120 & 99 & 42 & \\ 2 & 128 & 448 & 321 & 201 & 102 & 30 \end{array}$$

Therefore, $f[0,1,1,1,2,2] = 30$.

V-b

According to Corollary 2.22, $f[0,1,1,1,2,2,] = \frac{1}{5!}f^{(5)}(\xi)$, where $\xi \in (0,2)$. $f(x) = x^7$, then $\frac{1}{5!}f^{(5)}(\xi) = 21\xi^2$. By V-a, this result is 30. Therefore, $\xi = \sqrt{\frac{10}{7}}$.

VI

VI-a

According to the data given by the problem, we can construct the following table of divided difference:

$$\begin{array}{c|ccccc} 0 & 1 & & & & \\ 1 & 2 & 1 & & & \\ 1 & 2 & -1 & -2 & & \\ 3 & 0 & -1 & 0 & \frac{2}{3} & \\ 3 & 0 & 0 & \frac{1}{2} & \frac{1}{4} & -\frac{5}{36} \end{array}$$

Therefore, $p_4(x) = 1 + x - 2x(x-1) + \frac{2}{3}x(x-1)^2 - \frac{5}{36}x(x-1)^2(x-3)$. Estimate $f(2)$ with $p_4(2) = \frac{11}{18}$

VI-b

According to Theorem 2.37, the error is $f(x) - p_4(x) = \frac{f^{(5)}(\xi)}{5!}x(x-1)^2(x-3)^2$.

Due to $x \in [0,3]$, $|f(x) - p_4(x)| \le \frac{M}{5!}x(x-1)^2(x-3)^2 \le \frac{MC}{5!}$, where $C = \max\limits_{x\in[0,3]} |x(x-1)^2(x-3)^2|$.(A continuous function has a maximum in a closed interval)

VII

In the case $k = 1$, $\Delta f(x) = f(x+h) - f(x) = hf[x, x+h]$.

We assume that $\Delta^k f(x) = k!h^k f[x_0, \cdots, x_k]$ holds for $k = m$. Then $\Delta^{m+1} f(x) = \Delta^m f(x+h) - \Delta^m f(x) = m!h^m(f[x_1, \cdots, x_{m+1}] - f[x_0, \cdots, x_m]) = m!h^m(m+1)hf[x_0, \cdots, x_{m+1}] = (m+1)!h^{m+1} f[x_0, \cdots, x_{m+1}]$, which implies that $\Delta^k f(x) = k!h^k f[x_0, \cdots, x_k]$ holds for $k = m+1$.By induction, the proof is done.

In the case of $\nabla f(x)$, it is similar and can be proved in the same way.

VIII

In the case $n = 1$, $\frac{\partial}{\partial x_0} f[x_0, x_1] = \frac{\partial}{\partial x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{-f'(x_0)(x_1 - x_0) + (f(x_1) - f(x_0))}{(x_1 - x_0)^2} = f[x_0, x_0, x_1]$.

Assume that it holds for $n = m$. Then :

$$f[x_0, x_0, \cdots, x_{m+1}] = \frac{f[x_0, \cdots, x_{m+1}] - f[x_0, x_0, x_1, \cdots, x_m]}{x_{m+1} - x_0} = \frac{f[x_0, \cdots, x_{m+1}] - \frac{\partial}{\partial x_0} f[x_0, x_1, \cdots, x_m]}{x_{m+1} - x_0}$$

$$\begin{aligned}
\frac{\partial}{\partial x_0} f[x_0, \cdots, x_{m+1}] &= \frac{-\frac{\partial}{\partial x} f[x_0, \cdots, x_m](x_{m+1} - x_0) + f[x_1, \cdots, x_{m+1}] - f[x_0, \cdots, x_m]}{(x_{m+1} - x_0)^2} \\
&= \frac{f[x_0, \cdots, x_{m+1}] - \frac{\partial}{\partial x_0} f[x_0, x_1, \cdots, x_m]}{x_{m+1} - x_0}
\end{aligned}$$

Hence $\frac{\partial}{\partial x_0} f[x_0, \cdots, x_n] = f[x_0, x_0, \cdots, x_n]$ holds for $n = m+1$. By induction, the proof is done.

IX

By Corollary 2.48, we have

$$\max_{x \in [-1,1]} |x^n + a_1 x^{n-1} + \cdots + a_n| \geq \frac{1}{2^{n-1}}$$

Let $\tilde{x} = \frac{b-a}{2} x + \frac{a+b}{2}$, then it follows that

$$\begin{aligned}
\max_{\tilde{x} \in [a,b]} |a_0 \tilde{x}^n + a_1 \tilde{x}^{n-1} + \cdots + a_n| &= \max_{x \in [-1,1]} |(\frac{b-a}{2})^n x^n + (\text{terms with degrees} < n)| \\
&\geq \frac{(b-a)^n}{2^{2n-1}}
\end{aligned}$$

The inequailty becomes equation when it is Chebyshev polynomial, therefore $\min \max x \in [a,b] |a_0 x^n + a_1 \tilde{x}^{n-1} + \cdots + a_n| = \frac{(b-a)^n}{2^{2n-1}}$

X

Suppose the equation in the problem does not hold, i.e. $\exists p \in \mathbb{P}_n^a$ s.t. $\|p\|_\infty < \|\hat{p}_n\|_\infty = \frac{1}{|T_n(a)|}$. Let $Q(x) = \hat{p}(x) - p(x)$, then $Q(x_k') = \frac{(-1)^k}{T_n(a)} - p(x_k')$ for $x_k' = \cos \frac{k}{n}\pi$, $k = 0, 1, \cdots, n$. $Q(x)$ has alternating signs at these $n+1$ points in [-1,1]. Hence $Q(x)$ must have n zeros in [-1,1].

On the other hand, $Q(x)$ is a polynomial with degree n so it has at most n real zeros. But $Q(a) = \hat{p}(a) - p(a) = 0$, which means $a > 1$ is also a zero not in n zeros above. Therefore it is a contradiction, so the equation in the problem holds.

XI

By definition,

$$\begin{aligned}
\frac{n-k}{n}b_{n,k}(t)+\frac{k+1}{n}b_{n,k+1}(t) &= \frac{n-k}{n}\binom{n}{k}t^k(1-t)^{n-k}+\frac{k+1}{n}t^{k+1}(1-t)^{n-k-1}\\
&= \binom{n-1}{k}t^k(1-t)^{n-k}+\binom{n-1}{k}t^{k+1}(1-t)^{n-k-1}\\
&= \binom{n-1}{k}t^k(1-t)^{n-k-1}(1-t+t)\\
&= \binom{n-1}{k}t^k(1-t)^{n-k-1}\\
&= b_{n-1,k}(t)
\end{aligned}$$

XII

By definition,

$$\begin{aligned}
\int_0^1 b_{n,k}(t)dt &= \int_0^1 \binom{n}{k}t^k(1-t)^{n-k}dt\\
&= \frac{1}{k+1}\binom{n}{k}\int_0^1 (1-t)^{n-k}dt^{k+1}\\
&= \frac{1}{k+1}\binom{n}{k}[((1-t)t^{k+1})|_0^1-\int_0^1 t^{k+1}d(1-t)^{n-k}]\\
&= \frac{n-k}{k+1}\binom{n}{k}\int_0^1 t^{k+1}(1-t)^{n-k-1}dt\\
&= \int_0^1 b_{n,k+1}(t)dt
\end{aligned}$$

Therefore, $\int_0^1 b_{n,k}(t)dt$ is independent of k. Hence,

$$\begin{aligned}
\int_0^1 b_{n,k}(t)dt &= \int_0^1 b_{n,n}(t)dt\\
&= \int_0^1 \binom{n}{n}t^n dt\\
&= \frac{1}{n+1}
\end{aligned}$$