Numerical Analysis homework # 3

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I

According to the problem, $s \in \mathcal{C}^k[0,2]$, i.e. $s(1^-) = s(1^+)$,$s'(1^-) = s'(1^+)$ and $s''(1^-) = s''(1^+)$. Assume that $p(x) = ax^3 + bx^2 + cx + d$, with $s(0) = p(0) = 0$, then:

$$\begin{aligned} d &= 0 \\ a + b + c + d &= 1 \\ 3a + 2b + c &= -3 \\ 6a + 2b &= 6 \end{aligned}$$

Solve the equations above and get $a = 7$, $b = -18$, $c = 12$, $d = 0$, then $p(x) = 7x^3 - 18x^2 + 12x$. $s''(0) = p''(0) = -36$ and $s(x)$ is not a natural cubic spline.

II

II-a

$x_1, x_2, \cdots, x_n$ divide [a,b] into $n-1$ parts, each part $[x_i, x_{i+1}]$ with a polynomial s_i, $n-1$ polynomials in total. Due to $s \in \mathbb{S}_2^1$, which means $s_i \in \mathbb{P}_2$ and their derivates should be continuous.

To determine a quadratic polynomial, we need 3 independent equations, so there would be $3(n-1)$ independent equations needed in total. Now consider what we have: Firstly for each interval $[x_i, x_{i+1}]$, function values of two endpoints are known, so we get $2(n-1)$ equations. In addition, to be 1-order continuous on $[a, b]$, derivates should be continuous at $x_2, \cdots, x_{n-1}$, therefore another $n-2$ equations.

In total we get $2(n-1) + n - 2 = 3n - 4$ equations, so an additional condition is still needed in order to determine s uniquely.

II-b

$p_i \in \mathbb{P}_2$, assume that $p_i = ax^2 + bx + c$. According to the problem, $p(x_i) = s(x_i) = f_i$, $p(x_{i+1}) = s(x_{i+1}) = f_{i+1}$, $p'(x_i) = s'(x_i) = m_i$, then:

$$\begin{aligned} ax_i^2 + bx_i + c &= f_i \\ ax_{i+1}^2 + bx_{i+1} + c &= f_{i+1} \\ 2ax_i + b &= m_i \end{aligned}$$

Solve the equations above and get:

$$\begin{aligned} a &= \frac{f_{i+1} - f_i}{(x_{i+1} - x_i)^2} - \frac{m_i}{x_{i+1} - x_i} \\ b &= \frac{x_{i+1} + x_i}{x_{i+1} - x_i} m_i - \frac{2x_i(f_{i+1} - f_i)}{(x_{i+1} - x_i)^2} \\ c &= f_i + \frac{x_i^2(f_{i+1} - f_i)}{(x_{i+1} - x_i)^2} - \frac{m_i x_i x_{i+1}}{x_{i+1} - x_i} \end{aligned}$$

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Therefore, $p_i(x) = (\frac{f_{i+1}-f_i}{(x_{i+1}-x_i)^2} - \frac{m_i}{x_{i+1}-x_i})x^2 + (\frac{x_{i+1}+x_i}{x_{i+1}-x_i}m_i - \frac{2x_i(f_{i+1}-f_i)}{(x_{i+1}-x_i)^2})x + f_i + \frac{x_i^2(f_{i+1}-f_i)}{(x_{i+1}-x_i)^2} - \frac{m_i x_i x_{i+1}}{x_{i+1}-x_i}$

II-c

According to II-b, we can get $p_1(x)$ with m_1, f_1 and f_2. Then we can calculate $m_2 = s'(x_2) = p_1^{'}(x_2)$. By using m_2, f_2 and f_3 we can get $p_2(x)$ then calculate m_3.

Repeat this process $n-2$ times and we get $m_2, m_3, \cdots, m_{n-1}$.

III

$s(x)$ is a natural cubic spline so $s(x) \in \mathbb{S}_3^2$, which means $s_1(0) = s_2(0)$, $s_1^{'}(0) = s_2^{'}(0)$ and $s_1^{''}(0) = s_2^{''}(0)$. Assume that $s_2(x) = a_3x^3 + a_2x^2 + a_1x + a_0$:

$$a_0 = 1 + c$$
$$a_1 = 3c$$
$$2a_2 = 6c$$

In addition, $s_2^{''}(1) = 0$ because $s(x)$ is a natural cubic spline, i.e. $6a_3 + 2a_2 = 0$.

Therefore, $s_2(x) = -cx^3 + 3cx^2 + 3cx + c + 1$.

If one wants $s(1) = 6c + 1 = -1$,then $c = -\frac{1}{3}$.

IV

IV-a

Denote the spline as $s(x)$:

$$s(x) = \begin{cases} s_1(x), x \in [-1, 0] \\ s_2(x), x \in [0, 1] \end{cases}$$

Assume that $s_1(x) = a_3x^3 + a_2x^2 + a_1x + a_0$ and $s_2(x) = b_3x^3 + b_2x^2 + b_1x + b_0$. $s(x) \in \mathbb{S}_3^2$, and $s(x)$ is a natural cubic spline, then:

$$s_1(-1) = f(-1)$$
$$s_1(0) = f(0)$$
$$s_1^{'}(0) = s_2^{'}(0)$$
$$s_1^{''}(0) = s_2^{''}(0)$$
$$s_2(0) = f(0)$$
$$s_2(1) = f(1)$$
$$s_1^{''}(-1) = 0$$
$$s_2^{''}(1) = 0$$

Solve the equations above and get $s_1(x) = -\frac{1}{2}x^3 - \frac{3}{2}x^2 + 1$, $s_2(x) = \frac{1}{2}x^3 - \frac{3}{2}x^2 + 1$.

IV-b

$f(x) = cos(\frac{\pi}{2}x)$, then $f(-1) = 0$, $f(0) = 1$, $f(1) = 0$. $g(x)$ is a quadratic polynomial and assume that $g(x) = ax^2 + bx + c$. With the function value at -1, 0, 1 we can get $g(x) = -x^2 + 1$. Now calculate bending energy of three functions:

$$\begin{aligned}
\int_{-1}^{1}[g''(x)]^2dx &= \int_{-1}^{1}(-2)^2dx \\
&= 8 \\
\int_{-1}^{1}[f''(x)]^2dx &= \int_{-1}^{1}[-\frac{\pi^2}{4}\cos(\frac{\pi}{2}x)]^2dx \\
&= \frac{\pi^4}{16} \\
\int_{-1}^{1}[s''(x)]^2dx &= \int_{-1}^{0}[s_1''(x)]^2dx + \int_{0}^{1}[s_2''(x)]^2dx \\
&= \int_{-1}^{0}[-3x-3]^2dx + \int_{0}^{1}[3x-3]^2dx \\
&= 6
\end{aligned}$$

Therefore, $s(x)$ has the minimum bending energy of three functions.

V

V-a

According to the book:

$$B_i^1(x) = \begin{cases} \frac{x-t_{i-1}}{t_i-t_{i-1}}, x \in (t_{i-1}, t_i] \\ \frac{t_{i+1}-x}{t_{i+1}-t_i}, x \in (t_i, t_{i+1}] \\ 0, otherwise \end{cases}$$

$B_i^2(x) = \frac{x-t_{i-1}}{t_{i+1}-t_{i-1}}B_i^1(x) + \frac{t_{i+2}-x}{t_{i+2}-t_i}B_{i+1}^1(x)$, then:

$$B_i^2(x) = \begin{cases} \frac{(x-t_{i-1})^2}{(t_i-t_{i-1})(t_{i+1}-t_{i-1})}, x \in (t_{i-1}, t_i] \\ \frac{(x-t_{i-1})(t_{i+1}-x)}{(t_{i+1}-t_i)(t_{i+1}-t_{i-1})} + \frac{(x-t_i)(t_{i+2}-x)}{(t_{i+1}-t_i)(t_{i+2}-t_i)}, x \in (t_i, t_{i+1}] \\ \frac{(t_{i+2}-x)^2}{(t_{i+2}-t_{i+1})(t_{i+2}-t_i)}, x \in (t_{i+1}, t_{i+2}] \\ 0, otherwise \end{cases}$$

V-b

$$\frac{d}{dx}B_i^2(x_i^-) = \frac{2(x-t_{i-1})}{(t_i-t_{i-1})(t_{i+1}-t_{i-1})}\bigg|_{x=t_i} = \frac{2}{t_{i+1}-t_{i-1}}$$

$$\frac{d}{dx}B_i^2(x_i^+) = [\frac{-2x+t_{i-1}+t_{i+1}}{(t_{i+1}-t_i)(t_{i+1}-t_{i-1})} + \frac{-2x+t_i+t_{i+2}}{(t_{i+1}-t_i)(t_{i+2}-t_i)}]\bigg|_{x=t_i} = \frac{2}{t_{i+1}-t_{i-1}}$$

$$\frac{d}{dx}B_i^2(x_{i+1}^-) = [\frac{-2x+t_{i-1}+t_{i+1}}{(t_{i+1}-t_i)(t_{i+1}-t_{i-1})} + \frac{-2x+t_i+t_{i+2}}{(t_{i+1}-t_i)(t_{i+2}-t_i)}]\bigg|_{x=t_{i+1}} = -\frac{2}{t_{i+2}-t_i}$$

$$\frac{d}{dx}B_i^2(x_{i+1}^+) = \frac{2(x-t_{i+2})}{(t_{i+2}-t_{i+1})(t_{i+2}-t_i)}\bigg|_{x=t_{i+1}} = -\frac{2}{t_{i+2}-t_i}$$

Therefore, $\frac{d}{dx}B_i^2(x)$ is continuous at t_i and t_{i+1}.

V-c

According to V-b, $\frac{d}{dx}B_i^2(x) > 0$ if $x \in (t_{i-1}, t_i]$, $\frac{d}{dx}B_i^2(x) < 0$ if $x \in (t_{i+1}, t_{i+2})$. In addition, $\frac{d}{dx}B_i^2(t_i) > 0$, $\frac{d}{dx}B_i^2(t_{i+1}) < 0$, $\frac{d}{dx}B_i^2(x)$ decreases monotonocally on (t_i, t_{i+1}) and it is continuous on (t_i, t_{i+1}), so there is only one $x^* \in (t_i, t_{i+1})$ which satisfies $\frac{d}{dx}B_i^2(x^*) = 0$.

Let $\frac{d}{dx}B_i^2(x^*) = 0$ and use the formula in V-b, we can get $x^* = \frac{t_{i+2}t_{i+1} - t_i t_{i-1}}{t_{i+2} + t_{i+1} - t_i - t_{i-1}}$.

V-d

According to V-c, $\frac{d}{dx}B_i^2(x)$ increases monotonocally on (t_{i-1}, x^*) and decreases monotonocally on (x^*, t_{i+2}).

$$B_i^2(t_{i-1}) = 0$$

$$B_i^2(t_{i+2}) = 0$$

$$B_i^2(x^*) = \frac{(t_{i+2} - t_{i-1})(t_{i+1} - t_{i-1})}{(t_{i+2} + t_{i+1} - t_i - t_{i-1})^2} + \frac{(t_{i+2} - t_i)(t_{i+2} - t_{i-1})}{(t_{i+2} + t_{i+1} - t_i - t_{i-1})^2} = \frac{t_{i+2} - t_{i-1}}{t_{i+2} + t_{i+1} - t_i - t_{i-1}} < 1$$

Therefore, $B_i^2(x) \in (0, 1]$

V-e

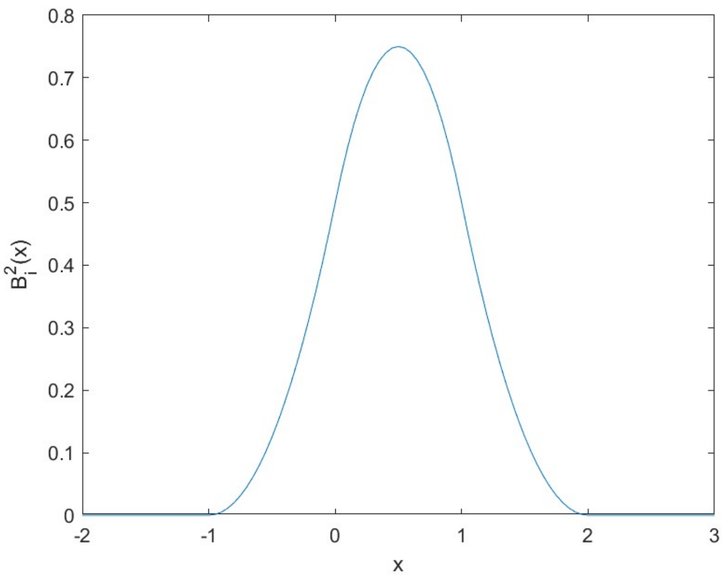

Figure 1: ProblemV-e

VI

$$\begin{aligned}
&(t_{i+2} - t_{i-1})[t_{i-1}, t_i, t_{i+1}, t_{i+2}](t - x)_+^2 \\
&= \frac{1}{t_{i+2} - t_i}\left(\frac{(t_{i+2} - x)_+^2 - (t_{i+1} - x)_+^2}{t_{i+2} - t_{i+1}} - \frac{(t_{i+1} - x)_+^2 - (t_i - x)_+^2}{t_{i+1} - t_i}\right) \\
&- \frac{1}{t_{i+1} - t_{i-1}}\left(\frac{(t_{i+1} - x)_+^2 - (t_i - x)_+^2}{t_{i+1} - t_i} - \frac{(t_i - x)_+^2 - (t_{i-1} - x)_+^2}{t_i - t_{i-1}}\right)
\end{aligned}$$

Let $\alpha(x) = \frac{1}{t_{i+2} - t_i}(\frac{(t_{i+2} - x)_+^2 - (t_{i+1} - x)_+^2}{t_{i+2} - t_{i+1}} - \frac{(t_{i+1} - x)_+^2 - (t_i - x)_+^2}{t_{i+1} - t_i})$ and $\beta(x) = \frac{1}{t_{i+1} - t_{i-1}}(\frac{(t_{i+1} - x)_+^2 - (t_i - x)_+^2}{t_{i+1} - t_i} - \frac{(t_i - x)_+^2 - (t_{i-1} - x)_+^2}{t_i - t_{i-1}})$.

$$\alpha(x)=\begin{cases}1, x\in(-\infty,t_i]\\ \dfrac{1}{t_{i+2}-t_i}(t_{i+2}+t_{i+1}-2x-\dfrac{(t_{i+1}-x)^2}{t_{i+1}-t_i}), x\in(t_i,t_{i+1}]\\ \dfrac{1}{t_{i+2}-t_i}\cdot\dfrac{(t_{i+2}-x)^2}{t_{i+2}-t_{i+1}}, x\in(t_{i+1},t_{i+2}]\\ 0, x\in(t_{i+2},+\infty)\end{cases}$$

$$\beta(x)=\begin{cases}1, x\in(-\infty,t_{i-1}]\\ \dfrac{1}{t_{i+1}-t_{i-1}}(t_{i+1}+t_i-2x-\dfrac{(t_i-x)^2}{t_i-t_{i-1}}), x\in(t_{i-1},t_i]\\ \dfrac{1}{t_{i+1}-t_{i-1}}\cdot\dfrac{(t_{i+1}-x)^2}{t_{i+1}-t_i}, x\in(t_i,t_{i+1}]\\ 0, x\in(t_{i+1},+\infty)\end{cases}$$

Therefore, it can be divided into several cases:

- $x\in(-\infty,t_{i-1}]$, $\alpha(x)-\beta(x)=1-1=0$.
- $x\in(t_{i-1},t_i]$, $\alpha(x)-\beta(x)=1-\dfrac{1}{t_{i+1}-t_{i-1}}(t_{i+1}+t_i-2x-\dfrac{(t_i-x)^2}{t_i-t_{i-1}})=\dfrac{(x-t_{i-1})^2}{(t_i-t_{i-1})(t_{i+1}-t_{i-1})}$.
- $x\in(t_i,t_{i+1}]$, $\alpha(x)-\beta(x)=\dfrac{1}{t_{i+2}-t_i}(t_{i+2}+t_{i+1}-2x-\dfrac{(t_{i+1}-x)^2}{t_{i+1}-t_i})-\dfrac{1}{t_{i+1}-t_{i-1}}\cdot\dfrac{(t_{i+1}-x)^2}{t_{i+1}-t_i}=\dfrac{(x-t_{i-1})(t_{i+1}-x)}{(t_{i+1}-t_i)(t_{i+1}-t_{i-1})}+\dfrac{(x-t_i)(t_{i+2}-x)}{(t_{i+1}-t_i)(t_{i+2}-t_i)}$
- $x\in(t_{i+1},t_{i+2}]$, $\alpha(x)-\beta(x)=\dfrac{(t_{i+2}-x)^2}{(t_{i+2}-t_{i+1})(t_{i+2}-t_i)}$
- $x\in(t_{i+2},+\infty)$, $\alpha(x)-\beta(x)=0$.

Therefore,

$$\alpha(x)-\beta(x)=\begin{cases}0, x\in(-\infty,t_{i-1}]\\ \dfrac{(x-t_{i-1})^2}{(t_i-t_{i-1})(t_{i+1}-t_{i-1})}, x\in(t_{i-1},t_i]\\ \dfrac{(x-t_{i-1})(t_{i+1}-x)}{(t_{i+1}-t_i)(t_{i+1}-t_{i-1})}+\dfrac{(x-t_i)(t_{i+2}-x)}{(t_{i+1}-t_i)(t_{i+2}-t_i)}, x\in(t_i,t_{i+1}]\\ \dfrac{(t_{i+2}-x)^2}{(t_{i+2}-t_{i+1})(t_{i+2}-t_i)}, x\in(t_{i+1},t_{i+2}]\\ 0, x\in(t_{i+2},+\infty)\end{cases}$$

Compared to the formula in V-a, and get the conclusion that $B_i^2(x)=(t_{i+2}-t_{i-1})[t_{i-1},t_i,t_{i+1},t_{i+2}](t-x)_+^2$.

VII

By Theorem 3.34 we know $\dfrac{d}{dx}B_i^n(x)=\dfrac{nB_i^{n-1}(x)}{t_{i+n-1}-t_{i-1}}-\dfrac{nB_{i+1}^{n-1}(x)}{t_{i+n}-t_i}$.

Take the integral of two sides from t_{i-1} to t_{i+n}:

$$\int_{t_{i-1}}^{t_{i+n}}\frac{d}{dx}B_i^n(x)dx=B_i^n(x)\bigg|_{t_{i-1}}^{t_{i+n}}=0$$

$$\int_{t_{i-1}}^{t_{i+n}}(\frac{nB_i^{n-1}(x)}{t_{i+n-1}-t_{i-1}}-\frac{nB_{i+1}^{n-1}(x)}{t_{i+n}-t_i})dx=\frac{n}{t_{i+n-1}-t_{i-1}}\int_{t_{i-1}}^{t_{i+n-1}}B_i^{n-1}(x)dx-\frac{n}{t_{i+n}-t_i}\int_{t_i}^{t_{i+n}}B_{i+1}^{n-1}(x)dx$$

Hence $\dfrac{n}{t_{i+n-1}-t_{i-1}}\displaystyle\int_{t_{i-1}}^{t_{i+n-1}}B_i^{n-1}(x)-\dfrac{n}{t_{i+n}-t_i}\int_{t_i}^{t_{i+n}}B_{i+1}^{n-1}(x)=0$

i.e.: $\dfrac{1}{t_{i+n-1}-t_{i-1}}\displaystyle\int_{t_{i-1}}^{t_{i+n-1}}B_i^{n-1}(x)=\dfrac{1}{t_{i+n}-t_i}\int_{t_i}^{t_{i+n}}B_{i+1}^{n-1}(x)$

Therefore, the scaled integral of $B_i^n(x)$ over its support is independent of its index.

VIII

VIII-a

Calculate $[x_i, x_{i+1}, x_{i+2}]x^4$ with the table of divided difference:

$$\begin{array}{c|ccc} x_i & x_i^4 & & \\ x_{i+1} & x_{i+1}^4 & x_{i+1}^3 + x_{i+1}^2 x_i + x_{i+1} x_i^2 + x_i^3 & \\ x_{i+2} & x_{i+2}^4 & x_{i+2}^3 + x_{i+2}^2 x_{i+1} + x_{i+2} x_{i+1}^2 + x_{i+1}^3 & x_i^2 + x_i x_{i+1} + x_i x_{i+2} + x_{i+1}^2 + x_{i+1} x_{i+2} + x_{i+2}^2 \end{array}$$

According to the definition of complete symmetric polynomials:

$$\begin{aligned} \tau_2(x_i, x_{i+1}, x_{i+2}) &= \sum_{i \le i_1 \le i_2 \le i+2} x_{i_1} x_{i_2} \\ &= x_i^2 + x_i x_{i+1} + x_i x_{i+2} + x_{i+1}^2 + x_{i+1} x_{i+2} + x_{i+2}^2 \end{aligned}$$

Compare two results and they are same.

VIII-b

Recursive relations of complete symmetric polynomials says that:

$$\tau_{k+1}(x_1, \cdots, x_n, x_{n+1}) = \tau_{k+1}(x_1, \cdots, x_n) + x_{n+1} \tau_k(x_1, \cdots, x_n, x_{n+1})$$

Then we have:

$$\begin{aligned} (x_{n+1} - x_1)\tau_k(x_1, \ldots, x_n, x_{n+1}) &= \tau_{k+1}(x_1, \ldots, x_n, x_{n+1}) - \tau_{k+1}(x_1, \ldots, x_n) \\ &\quad - x_1 \tau_k(x_1, \ldots, x_n, x_{n+1}) \\ &= \tau_{k+1}(x_2, \ldots, x_n, x_{n+1}) + x_1 \tau_k(x_1, \ldots, x_n, x_{n+1}) \\ &\quad - \tau_{k+1}(x_1, \ldots, x_n) - x_1 \tau_k(x_1, \ldots, x_n, x_{n+1}) \\ &= \tau_{k+1}(x_2, \ldots, x_n, x_{n+1}) - \tau_{k+1}(x_1, \ldots, x_n). \end{aligned}$$

For $n = 0$, $\tau_m(x_i) = x_i^m = [x_i]x^m$.

Suppose that $\forall m \in \mathbb{N}^+$, $\forall i \in \mathbb{N}$, $\forall n = 0, 1, \cdots, m$, $\tau_{m-n}(x_i, \cdots, x_{i+n}) = [x_i, \cdots, x_{i+n}]x^m$ holds for a non-negative integer $n < m$, then:

$$\begin{aligned} \tau_{m-n-1}(x_i, \ldots, x_{i+n+1}) &= \frac{\tau_{m-n}(x_{i+1}, \ldots, x_{i+n+1}) - \tau_{m-n}(x_i, \ldots, x_{i+n})}{x_{i+n+1} - x_i} \\ &= \frac{[x_{i+1}, \ldots, x_{i+n+1}]x^m - [x_i, \ldots, x_{i+n}]x^m}{x_{i+n+1} - x_i} \\ &= [x_i, \ldots, x_{i+n+1}]x^m. \end{aligned}$$

Therefore it holds for the case $n + 1$, by induction this proof is done.