(1) # Time Series

Date 24/08/2022

* Autocorrelation/ Partial autocorrelation.

Salmon → Avg Price S_t (this month) ←

↙ depend $S_{t-1}, S_{t-2}, S_{t-3}, ---$

$$S_t = f(S_{t-1}, S_{t-2}, S_{t-3}, ---)$$

$$\begin{tikzcd} S_{t-2} \arrow[r] \arrow[rr, bend left] & S_{t-1} \arrow[r] & S_t \end{tikzcd}$$

Price of salmon at t-2 affect price of salmon at t through its affect of price of salmon at t-1 but also it affects the price of salmon directly.

Autocorrelation Function

$ACF(S_{t-2}, S_t)$

↓

mathematically we can make combination of prices at a two month intervals and then can obtain correlation b/w them.

X	Y
t-2	t
t-4	t-2
t-6	t-4
t-8	t-6

→ Pearson Correlation

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ACF - theoretically considers both indirect and direct effect of t-2 on t. i.e two things

1. t-2 $\longrightarrow$ t
2. t-2 $\longrightarrow$ t-1 $\longrightarrow$ t

PACF : Partially Auto correlation factor
↳ It only cares about the direct relation i.e

t-2 $\longrightarrow$ t

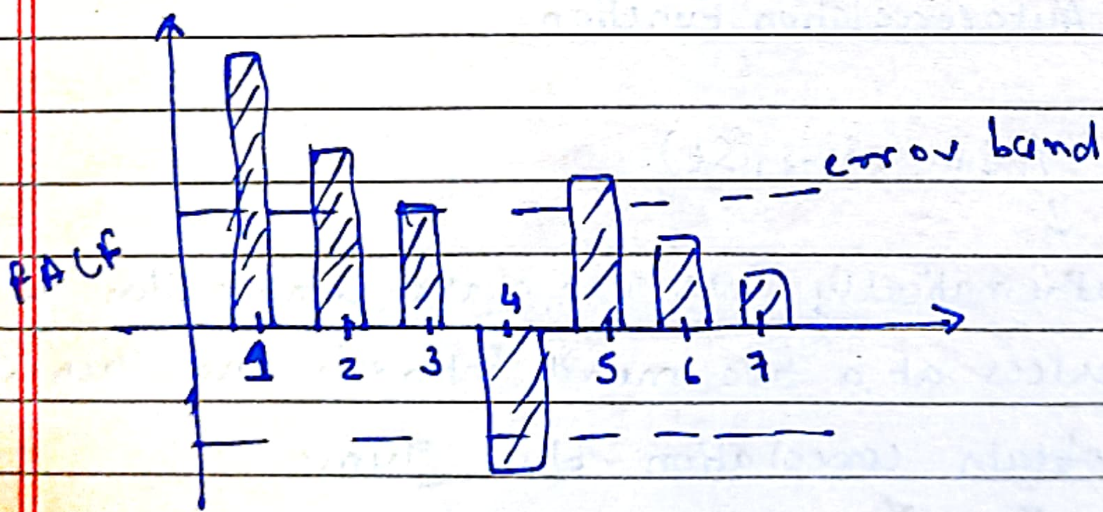
PACF for K=2

$$S_t = \phi_{21} S_{t-1} + \boxed{\phi_{22}} S_{t-2} + \varepsilon_t$$

↳ direct effect of t-2

ACF includes both direct and indirect effect
PACF includes only direct effect.

Regression



$$\text{Price of salmon today} = \beta_0 + \beta_1 S_{t-1} + \beta_2 S_{t-2} + \cdots + \varepsilon_t$$

↳ Auto Regressive Model.

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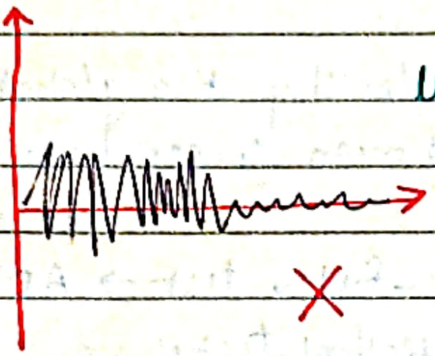
Stationarity.

Q. What make time series stationary?

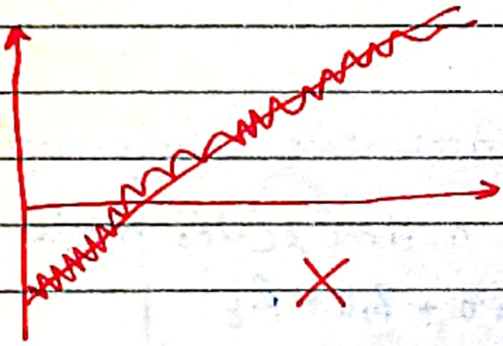
Stationority
- → μ (mean of time series) is constant
- → σ (std dev) is constant
- → no seasonality.

1



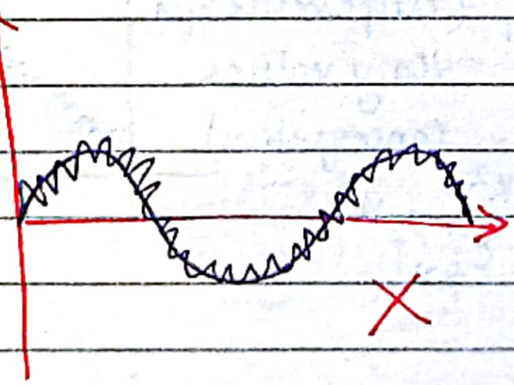
μ is constant, no seasonality
but σ is not const.
↳ hence not a time series.

2



μ is shifting
σ - const
no seasonality

3



μ → constant
σ → constant
but there is seasonality

All above three examples are not a stationary time series.

Issue of white noise?

White noise is stationary but not other way around.

How to check for stationarity?

1. Visually looking at the criteria's we just discussed
2. Global vs local test: looking at a global mean and then at a local mean to see how much difference is there.

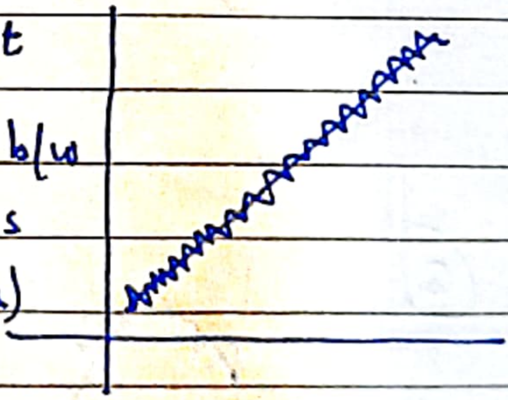
* 3. Augmented Dickey-Fuller test → ADF

↳ more concrete statistical test.

How to make a time series stationary?

Suppose we have a time series model

$$y = \beta_0 + \beta_1 t + \varepsilon_t$$



Let $z_t = y_t - y_{t-1}$ (difference b/w the y values consequetive)

$$z_t = \beta_1 + (\overset{K^2}{\varepsilon_t} - \overset{K^2}{\varepsilon_{t-1}})$$

$E(z_t)$ → mean/expectation of z_t $= \beta_1$

$Var(z_t) \to 2K^2 \to$ const

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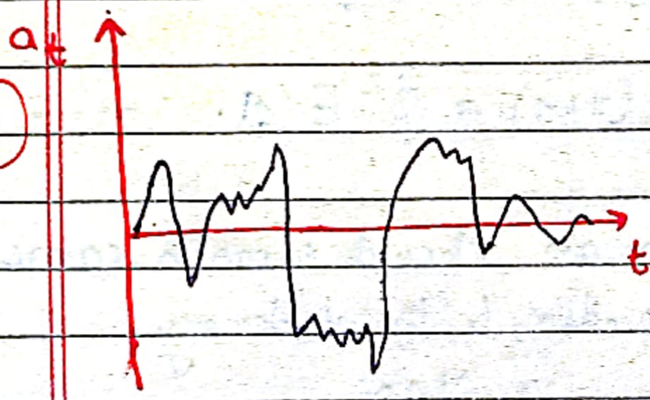
For a stationary series we can use auto regressive and moving average type models. ARMA/AR

Unit Roots

Unit roots come in our way of modelling a stationary time series. If a time series contain a unit root its not stationary and hence we cannot apply AR/ARMA models to that time series. At least we should know that there are unit roots present as we can seek another method or can also look for transforming the time series.

1



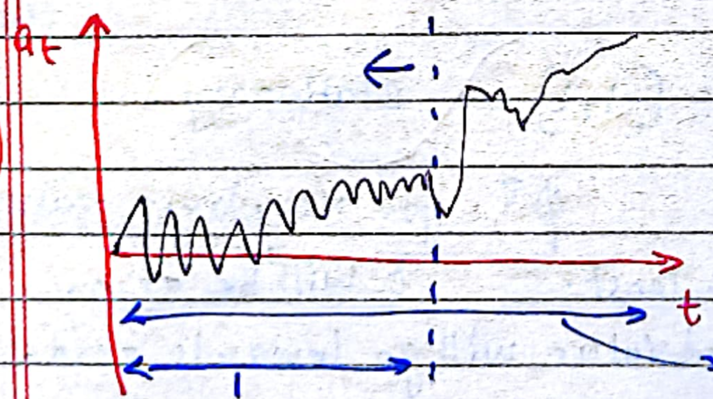
$|\phi|<1$

$E(a_t) \rightarrow$

$Var(a_t) \rightarrow \dfrac{\sigma^2}{1-\phi^2}$

visually appears stationary

2



$|\phi| = 1$

$E(a_t) =$

$var(a_t) = t\sigma^2$

UNIT ROOT

→ if one sees in full length - not stationary

If one see within this area it will appear stationary

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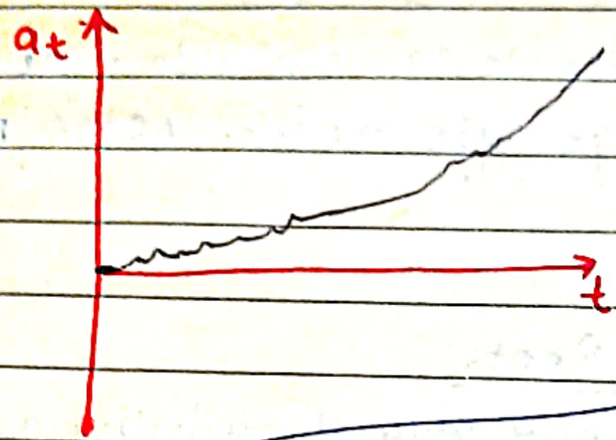
$|\phi| > 1$

3



Not stationary.

exploding μ

& exploding var

$$a_t = \phi a_{t-1} + \varepsilon_t$$

$$= \phi^t a_0 + \sum_{k=0}^{t-1} \phi^k \varepsilon_{t-k}$$

$$Var(a_t) = \sigma^2 \left[\phi^0 + \phi^2 + \phi^4 + \cdots + \phi^{2(t-1)} \right]$$

$$E(a_t) = \phi E(a_{t-1}) = \phi^2 E(a_{t-2}) = \cdots$$

Expected value/mean should remain same throughout in order to be stationary.

mean of ε fn $= 0$ ~~mean~~ var of ε fn $= \sigma^2$ → const.

1

$|\phi| < 1 \rightarrow \phi \in (-1, 1) \rightarrow$ stationary.

if $\phi = 0.5$ ϕ^t for very large values of t will be zero.

→ both will get const.

hence expected value will go towards zero - μ const

$$Var(a_t) = \frac{\sigma^2}{1-\phi^2}$$

7 $|\phi| = 1$ for AR-1. A time series has a unit root if $|\phi| = 1$. i.e $\phi = -1$ or $\phi = 1$

unit root becomes more difficult to define for higher order time series like AR-2, AR-3, ... we need to have some thing like characteristic eqn.

$E(a_t) = a_0 \rightarrow \mu$ is const.
$var(a_t) = t\sigma^2 \rightarrow$ as time gets bigger var will get bigger and bigger.

To make this stationary for AR1 model we can use something called as first difference.

$$\left.\begin{aligned} \text{let } d_t &= a_t - a_{t-1} \\ d_t &= \varepsilon_t \\ E(d_t) &= 0 \\ Var(d_t) &= \sigma^2 \end{aligned}\right] \rightarrow \text{stationary series.}$$

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Dickey-Fuller Test

dickey fuller test assumes that the time series in question for us is AR-1.

$$y_t = \mu + \phi_1 y_{t-1} + \varepsilon_t$$

$$\begin{cases} H_0 & \phi_1 = 1 \quad \text{- null hypothesis. unit root.} \\ H_1 & \phi_1 < 1 \quad \text{- no unit root} \end{cases}$$

∴) First difference.

$$y_t - y_{t-1} = \mu + (\phi_1 - 1) y_{t-1} + \varepsilon_t$$

$$\Delta y_t = \mu + \delta y_{t-1} + \varepsilon_t \quad \begin{cases} H_0 & \delta = 0 \\ H_1 & \delta < 0 \end{cases}$$

Assuming the null hypothesis as true our time series of Δy_t in question is stationary as this is our first difference.

calculate t-statistic for δ

$$t_{\hat{\delta}} = \frac{\hat{\delta}}{se(\hat{\delta})}$$

se, standard error

This is a simple t statistic but no instead of it being compared to t distribution we will compare this with dickey-fuller distribution.

(9)

$t_{\hat{\delta}} < DF_{critical} \rightarrow$ Reject Ho

$t_{\hat{\delta}} > DF_{critical} \rightarrow$ Don't reject Ho

Reject Ho → our series is stationary
Dont Reject Ho → our series is non-stationary

Now the time series model will be more complicated than AR 1, so now how we will extend this to them.

Augment Dicky Fuller Test

$$y_t = \mu + \sum_{i=1}^{p} \phi_i y_{t-i} + \varepsilon_t$$ for series other than AR-1

$$\Delta y_t = \mu + \delta y_{t-1} + \sum_{i=1}^{p} \beta_i \Delta y_{t-i} + \varepsilon_t$$

$H_0 : \delta = 0$

$H_1 : \delta < 0$

$$t_{\hat{\beta}} = \frac{\hat{B}_i}{se(\hat{\beta}_i)}$$ → normal t-statistic comparision

$$t_{\hat{\delta}} = \frac{\hat{\delta}}{se(\hat{\delta})}$$ → Dicky-fuller statistic comparision

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(10) White Noise

It is a just one type of time series. In this time series where mean and standard deviations are constant and there is no correlation b/w lags version and the time series.

when you should stop fitting the model. white noise is not predictable

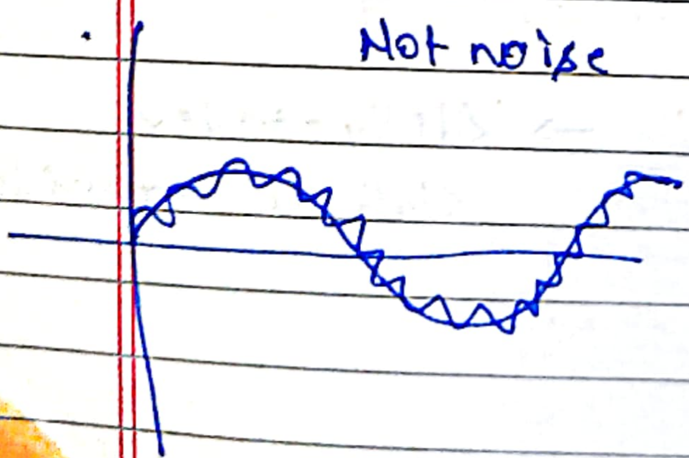
y_t = Signal + noise
↳ white noise
↳ best model.

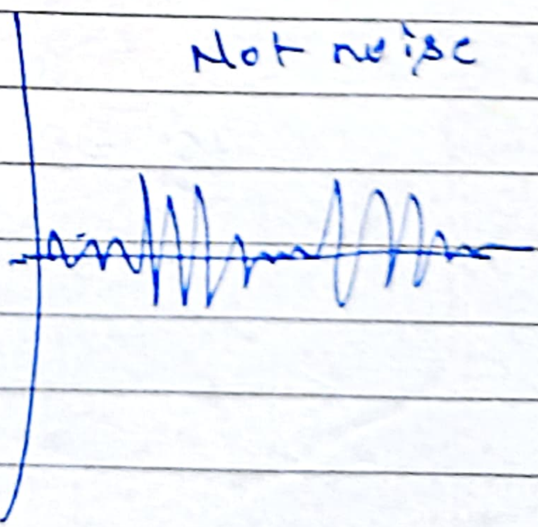
Testing for white noise ?

(1) Visual Tests

(2) Global vs local checks.
rolling window of 7 days or other time quantity over the entire series.

(3) Check ACF





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Backshift Operator

ARMA(3,3)

$$Y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \phi_3 y_{t-3} + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2} + \theta_3 \varepsilon_{t-3} + \varepsilon_t$$

Auto regressive

$$y_t - [\phi_1 y_{t-1} + \phi_2 y_{t-2} + \phi_3 y_{t-3}] = \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2} + \underbrace{\theta_3 \varepsilon_{t-3} + \varepsilon_t}_{\text{moving avg}}$$

L - Lag.

$$L y_t = y_{t-1}$$

$$L^2 y_t = y_{t-2}$$

$$\underbrace{(1 - \phi_1 L - \phi_2 L^2 - \phi_3 L^3)}_{\text{lag polynomials}} y_t = (1 + \theta_1 L + \theta_2 L^2 + \theta_3 L^3) \varepsilon_t$$

$$\Phi y_t = \Theta \varepsilon_t$$

$$\boxed{\Phi(L) y_t = \Theta(L) \varepsilon_t}$$

Lag operator saves us a lot of space.

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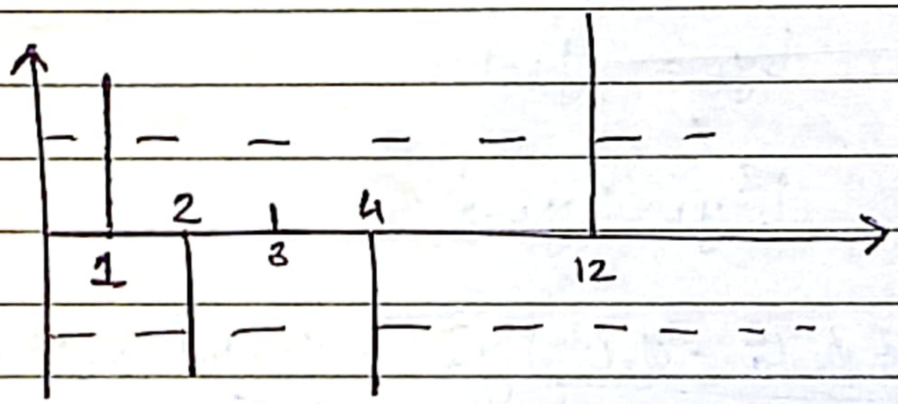
(12)

AR models

$M_t \quad M_{t-1} \quad -- \quad M_{t-12}$

If we provide every lag within our model, it might not be a very good idea as our models will be overfitting and will cause an statistical issue. We may have a look at ACF and PACF curve and this can provide us an understanding into our AR model.

PACF



$$\boxed{m_t = \beta_0 + \beta_1 m_{t-1} + \beta_2 m_{t-2} + \beta_4 m_{t-4} + \beta_{12} m_{t-12} + \varepsilon_t}$$

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Moving Average

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$\hat{f}_t = \mu + \phi_1 \varepsilon_{t-1}$ $\qquad \varepsilon_t \rightarrow N(\mu_{\varepsilon}, \sigma_{\varepsilon})$ $\quad$ 0 $\quad$ 1

$\mu = 10$, $\phi_1 = 0.5$

t	$\hat{f}t$	εt	$f t$
1	10	-2	8
2	9	1	10
3	10.5	0	10.5
4	10	2	12
5	11	1	12

$\mu \Rightarrow$ average number of cupcake that I will bring

$\phi_1 \varepsilon_{t-1} \Rightarrow$ Adjusting this to the previous error value that my professor said

but our value stays centered around our mean 10. Our value moves but in totality it stays centered around the mean value. this is called moving Average.

$\hat{f}t = \mu + \phi_1 \varepsilon_{t-1}$ $\qquad MA_1$ model

$\hat{f}t = \mu + \phi_1 \varepsilon_{t-1} + \phi_2 \varepsilon_{t-2}$ $\qquad MA(2)$ model

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$MA(q) - X_t = \mu + \phi_1 \varepsilon_{t-1} + \phi_2 \varepsilon_{t-2} + \ldots + \varepsilon_t$

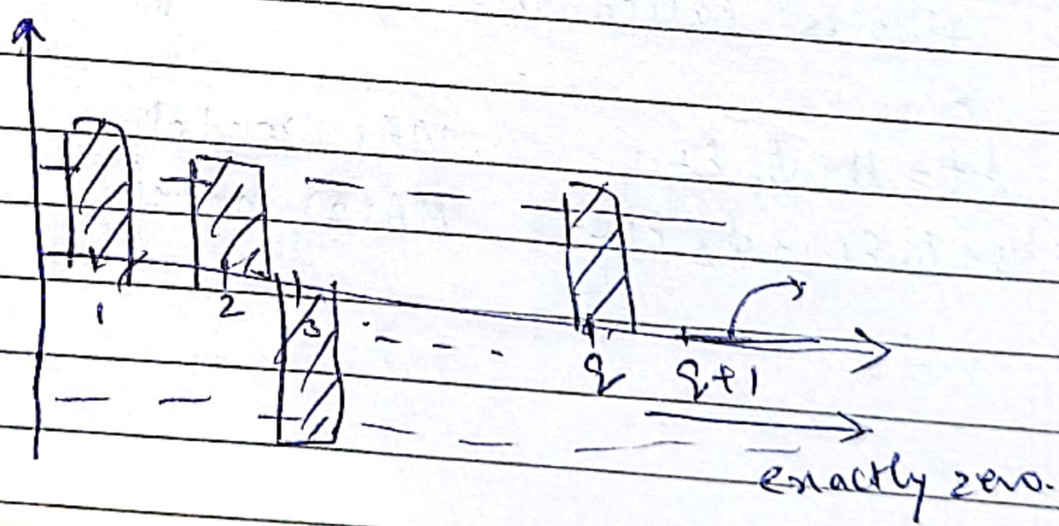
Let's say the data generating process for our time series X_t is indeed $MA(q)$.

ACF - To figure out the order of MA model we use ACF whereas to figure out the order for an AR model we use PACF.

$$\underbrace{corr(X_t, X_{t-k})}_{\text{ACF with lag } k} = \frac{E(X_t \cdot X_{t-k}) - E(X_t) * E(X_{t-k})}{\sqrt{Var(X_t)\, Var(X_{t-k})}}$$

* Auto correlation of our time series b/w t and t-k will not be equal to zero if our lag $k \leq q$ i.e the order for our time series generation

if $k > q$ than ACF = 0



For finding out whether a time series is a good fit for a MA(q) model we should look at its ACF and see whether we have zero ACF values after a certain point q which will be the model order for our MA time series.

* Invertibility

There is a connection b/w AR and MA models.

$$MA(1) \iff AR(\infty)$$

$$\left.C_t = -\phi \varepsilon_{t-1} + \varepsilon_t\right] \rightarrow \text{MA(1) series.}$$

$$C_t = (1 - \phi L^1)\varepsilon_t$$

$$\frac{C_t}{(1-\phi L^1)} = \varepsilon_t$$

$$\frac{1}{1-\phi L} \rightarrow \text{infinite geometric sum}$$

$$\underline{\phi L < 1} \quad |\phi| < 1$$

$$(1 + \phi L + \phi^2 L^2 + \text{---}) C_t = \varepsilon_t$$

$$C_t + C_t \phi L + C_t \phi^2 L^2 + \text{---} = \varepsilon_t$$

$$C_t = \underbrace{-C_t \phi L - C_t \phi^2 L^2 + \text{--} + \varepsilon_t}_{\text{AR infinite series.}}$$

$$C_t = -\phi C_{t-1} - \phi^2 C_{t-2} + \dots + \varepsilon_t$$

$$\begin{array}{ccc} C_t & \leftarrow & \varepsilon_t \\ \uparrow & & \\ \varepsilon_{t-1} & \leftarrow & C_{t-1} \\ \uparrow & & \\ \varepsilon_{t-2} & \leftarrow & C_{t-2} \\ \vdots & & \end{array}$$

$$\begin{array}{ccc} C_t & \leftarrow & \varepsilon_t \\ \uparrow & & \\ C_{t-1} & \leftarrow & \varepsilon_{t-1} \\ \uparrow & & \\ C_{t-2} & \leftarrow & \varepsilon_{t-2} \\ \uparrow & & \\ C_{t-3} & \leftarrow & \varepsilon_{t-3} \\ \vdots & \leftarrow & \\ \uparrow & & \end{array}$$

$$AR(1) \Leftrightarrow MA(\infty)$$

$$C_t = \phi C_{t-1} + \varepsilon_t \quad AR(1)$$

$$(1-\phi L) C_t = \varepsilon_t$$

$$\frac{\varepsilon_t}{(1-\phi L)} = C_t \qquad |\phi| < 1$$

$$(1 + \phi L + \phi^2 L^2 + \dots)\, \varepsilon_t = C_t$$

MA(∞)

$$\boxed{\phi \varepsilon_{t-1} + \phi^2 \varepsilon_{t-2} + \dots + \varepsilon_t = C_t}$$

(18) ARMA → Auto Regressive Moving Average

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$$l_t = \underline{\beta_0 + \beta_1 l_{t-1}} + \phi_1 \varepsilon_{t-1} + \varepsilon_t$$

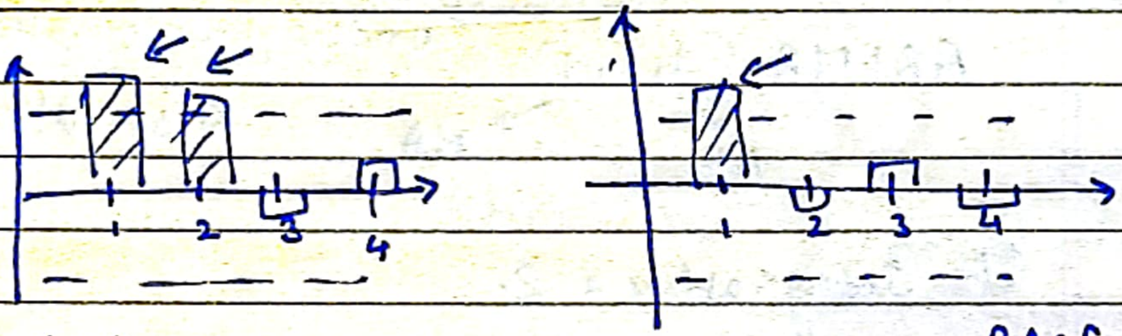
AR(1)

MA(1)

ARMA(1, 1) - basic ARMA model.

AR MA

$$\hat{l}_t = \beta_0 + \beta_1 l_{t-1} + \phi_1 \varepsilon_{t-1}$$

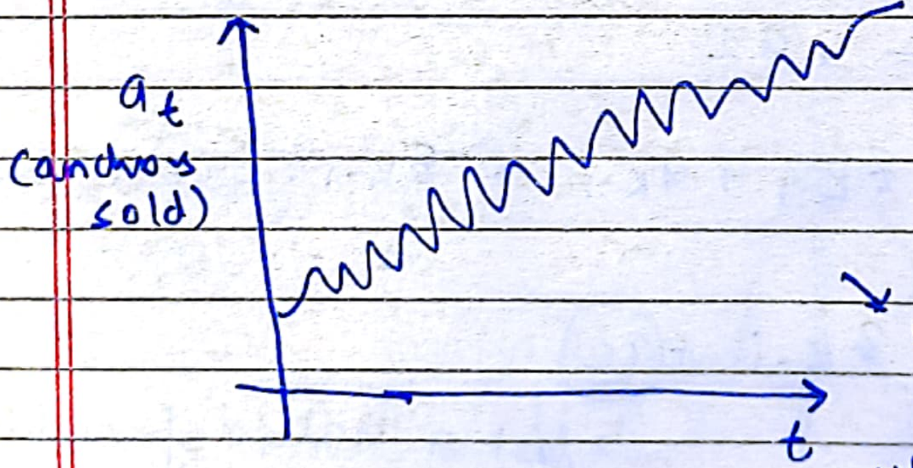


ACF → MA order

PACF → AR order

ARMA(1, 2)
- 1 → from PACF AR
- 2 → from ACF MA order.

ARIMA : Auto Regressive Integrative Moving Average



→ Not stationary time series.

→ We use ARIMA model in situation when the series is stationary except the $\mu \uparrow$.

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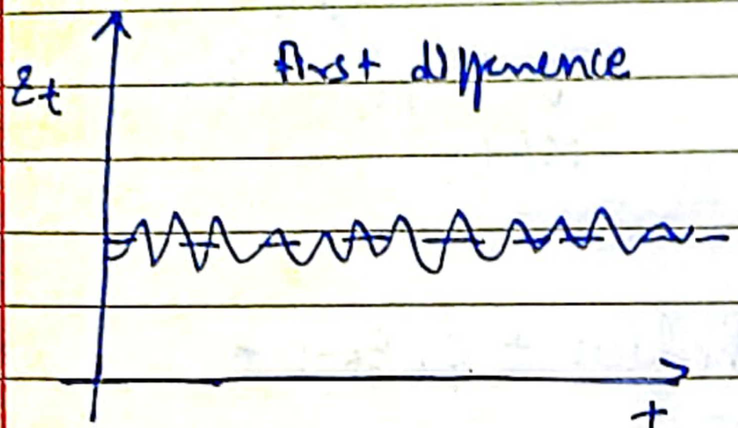
$z_t = a_{t+1} - a_t$



ARIMA (1, 1, 1) → order of Integrative part, first difference

AR, MA

if Integrative = 2.

$z_t = a_{t+1} - a_t$

$w_t = z_{t+1} - z_t$

use.

$z_t = \phi_1 z_{t-1} + \theta_1 \varepsilon_{t-1} + \varepsilon_t$

recover a_k?

$a_0, \ldots, a_l$

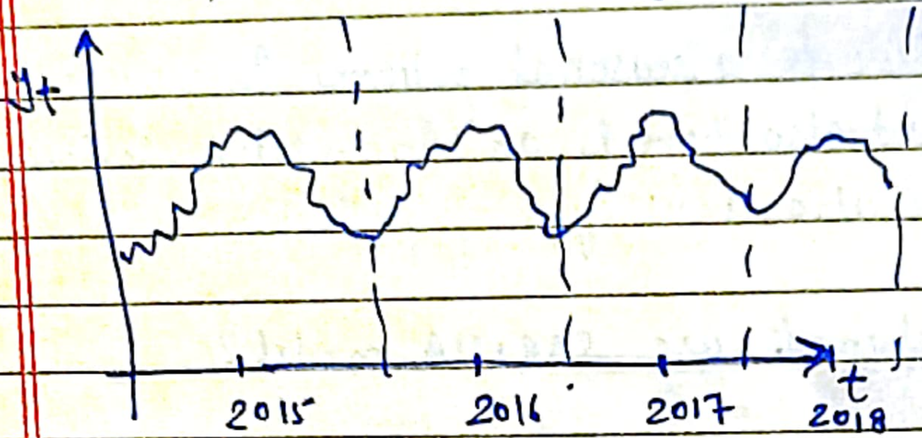
$a_k = z_{k-1} + a_{k-1} = z_{k-1} + z_{k-2} + a_{k-2}$

$= \ldots = \ldots$

$\sum_{i=1}^{k-l} z_{k-i} + a_l$

↳ last a value for which we have data for.

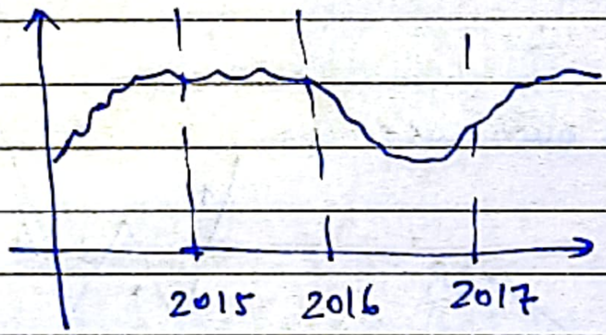
What is seasonality?



A repeating pattern <u>within a year</u> is called seasonality.
↳ Imp.

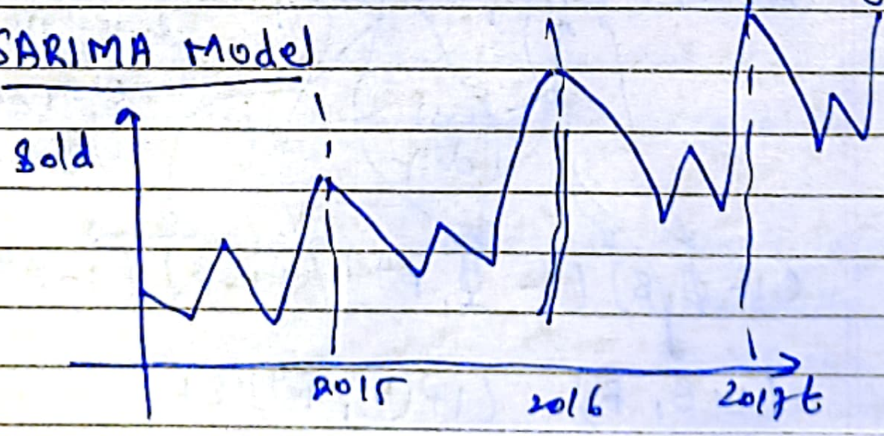
Removing Seasonality: $z_t = y_{t+365} - y_t$

<u>Cycles</u>



Cycles are not within a year. It is a trend over a course of several years.

<u>SARIMA Model</u>



There is a seasonal pattern in our series and also there is an increasing μ. There is also a W type of pattern

When to use SARIMA model?

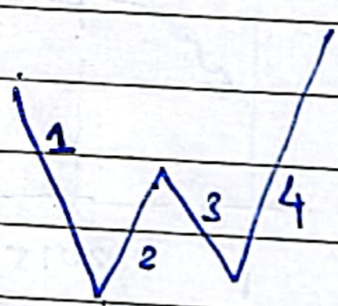
1. Seasonality ✓
2. AR | MA ✓
3. I - clear upward trend ✓

ARIMA $(p, d, q)\ (P, D, Q)_m$

AR I MA

m → seasonal factor

$m = 4$ in our case



AR

ARIM $(1,1,1)\ (1,1,1)_4$

first diff

backshift my time series by 4 time periods

$$(1-\phi_1 B)(1-\Phi_1 B^4)(1-B)(1-B^4)y_t =$$

$$(1+\theta_1 B)(1+\Theta_1 B^4)\varepsilon_t$$

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ARIMA $(1,0,0)(0,1,1)_4$

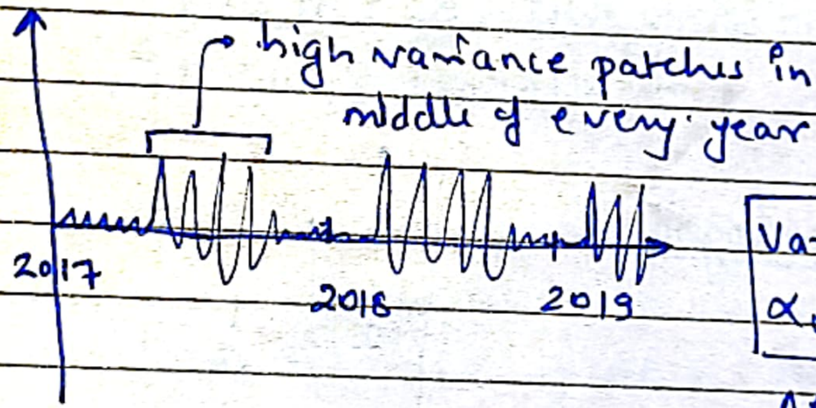
$$(1-\phi_1 B)(1-B^4)\,y_t = (1+\Theta_1 B^4)\,\varepsilon_t$$

$$z_t = y_t - y_{t-4} = \phi_1 y_{t-1} - \phi_1 y_{t-5} + \Theta_1 \varepsilon_{t-4} + \varepsilon_t$$

$$\boxed{z_t = \phi_1 z_{t-1} + \Theta_1 \varepsilon_{t-4} + \varepsilon_t}$$

ARCH Model

(1) Fit the best possible models to the data. (AR)
(2) Consider the residuals, ε_t



$$\boxed{\begin{aligned} Var(\varepsilon_t) &= \sigma_t^2 \\ &\alpha_0 + \alpha_1 \sigma_{t-1}^2 \end{aligned}}$$

ARCH(1) model

Our model does a very bad job in the middle of the year every year as volatility is high. As we approach the end of the year or beginning of the year it does a fine job.

Auto Regressive Conditional Heteroskedasticity
volatility,
variance
deviation

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Conditional means that there is a pattern or condition to your volatility (heteroskedasticity)

$$\varepsilon_t = w_t \sqrt{\alpha_0 + \alpha_1 \varepsilon_{t-1}^2}$$

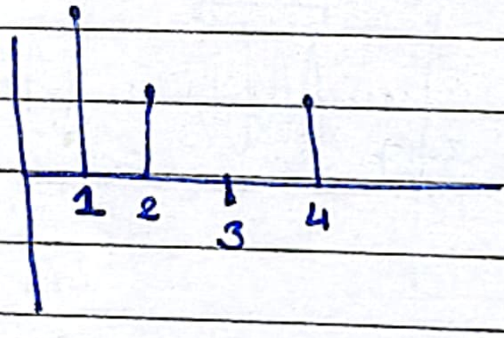
w_t — white noise

How do we test a series for ARCH?

- High errors today will lead to high errors tomorrow and low errors today will lead to low errors tomorrow.

$$\varepsilon_t^2 = w_t^2 (\alpha_0 + \alpha_1 \varepsilon_{t-1}^2)$$
$$= w_t^2 \alpha_0 + \alpha_1 w_t^2 \varepsilon_{t-1}^2$$

Correlogram



GARCH model

AR(1) $\quad a_t = \phi a_{t-1} + \varepsilon_t$

$a_t \leftarrow a_{t-1}$, $a_t \leftarrow \varepsilon_t$

ARMA(1,1)

$$a_t = \beta a_{t-1} + \phi \varepsilon_{t-1} + \varepsilon_t$$

$a_t \leftarrow a_{t-1}$, $a_t \leftarrow \varepsilon_{t-1}$, $a_t \leftarrow \varepsilon_t$

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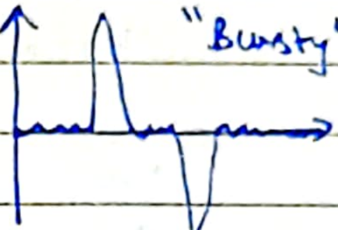
ARCH(1)

$$a_t = \underbrace{\varepsilon_t}_{\text{white noise}} \sqrt{\alpha + \alpha_1 a_{t-1}^2} = \varepsilon_t \sigma_t$$

$$\begin{array}{ccccc} & \varepsilon_t & & & \\ & \swarrow & & & \\ a_t & \longleftarrow & \sigma_t & \longleftarrow & a_{t-1} \end{array}$$



ARCH is not good at modelling these bursts. Hence we need a GARCH.

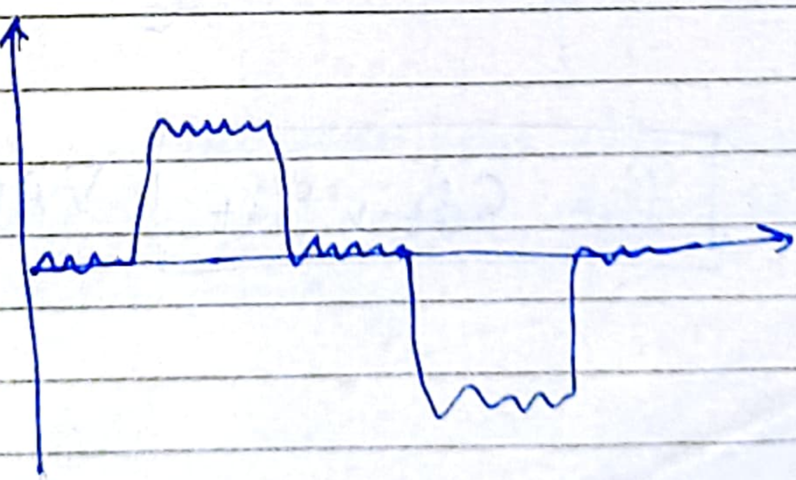
GARCH(1,1)

$$a_t = \varepsilon_t \underbrace{\sqrt{\alpha + \alpha_1 \underbrace{a_{t-1}^2}_{1} + \beta_1 \underbrace{\sigma_{t-1}^2}_{1}}}_{\sigma_t} = \varepsilon_t \sigma_t$$

$$\begin{array}{ccccc} & \varepsilon_t & & a_{t-1} & \\ & \swarrow & & \swarrow & \\ a_t & \longleftarrow & \sigma_t & & \\ & & & \nwarrow & \\ & & & & \sigma_{t-1} \end{array}$$

GARCH(1,1)

Volatility today is ~~so~~ not only affected by the value yesterday but also by the volatility yesterday. In case of ARCH the volatility is only impacted by value of yesterday.



G - generalised ARCH

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Vector Auto Regression.

$$a_t = C_{11} a_{t-1} + C_{12} b_{t-1} + \varepsilon_{t,a}$$

$$b_t = C_{21} a_{t-1} + C_{22} b_{t-1} + \varepsilon_{t,b}$$

Let's say a fruit shop want to know about how many banana it will sell the next month and how many apple it will sell the next month. Now we have two different variables that we need to know about based on the past data rather than just one variable. Now sell of apple will impact sell of banana and sell of banana will impact sell of apple.

VAR(1)

$$\underset{\uparrow \text{ vector}}{f_t = \begin{bmatrix} a_t \\ b_t \end{bmatrix}} = \underbrace{\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} a_{t-1} \\ b_{t-1} \end{bmatrix}}_{f_{t-1}} + \underbrace{\begin{bmatrix} \varepsilon_{at} \\ \varepsilon_{bt} \end{bmatrix}}_{\varepsilon_t}$$

$$\boxed{f_t = C f_{t-1} + \varepsilon_t} \quad \text{VAR(1)}$$

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Date...........................

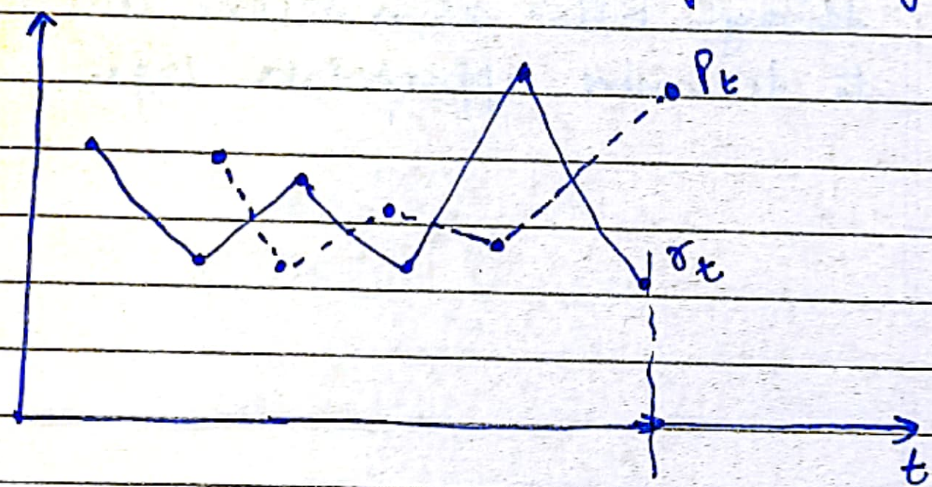
Granger Causality

How two time series are interacting with each other? whether one time series is cause of another time series.

If house prices go up in your neighbourhood then it may be a case that house prices will also go up in nearby neighbourhood.

Two cities / Rich city r_t : no. of goods exported
\ poor city P_t : no. of goods exported.

poor city goes for guidance to rich city and try to follow the pattern of rich city.



Now one can say that poor city graph is shifted version of rich city. Now look at it mathematically.

One time series leads to Granger Causality to another time series.

(27) Steps

(1) Best AR model for P_t

$$P_t = \phi_1 P_{t-1} + \phi_3 P_{t-3} + \varepsilon_t$$

(2) Add r_t terms

$$P_t = \phi_1 P_{t-1} + \phi_3 P_{t-3} + \psi_3 r_{t-3} + \psi_5 r_{t-5} + \varepsilon_t$$

How we get r_t data - we tried bunch of different lags, we do t test to determine which lag relates best.

~~They~~ Then we run all the selected lags through F test as a second line of check to determine appropriate lags.