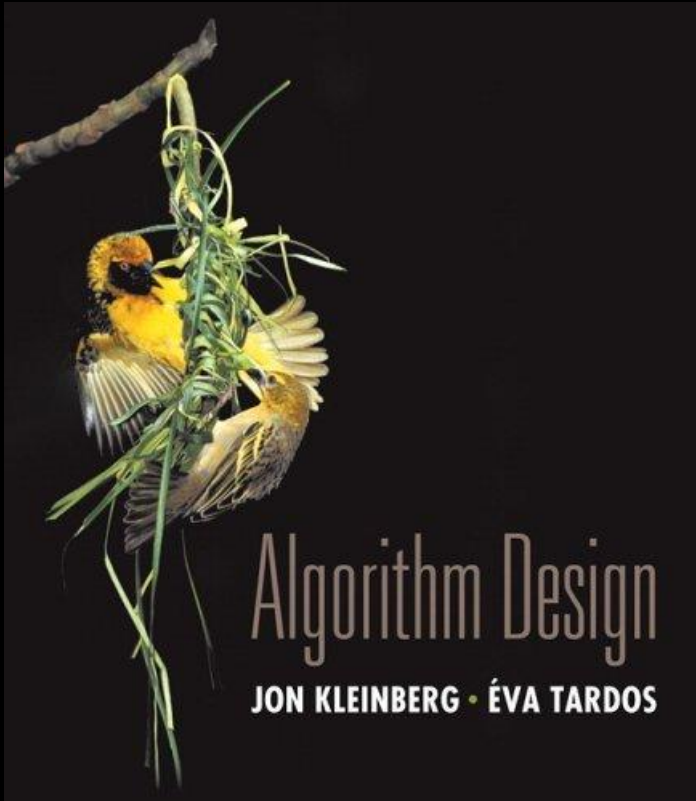


Chapter 7

Network Flow



Slides by Kevin Wayne.
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7.3 Choosing Good Augmenting Paths

Choosing good augmenting paths

Use care when selecting augmenting paths

- Some choices lead to exponential algorithms
- Clever choice lead to polynomial algorithms

Pathology. When edge capacities can be irrational, no guarantee that Ford-Fulkerson terminates (or converges to a maximum flow)!

Goal. Choose augmenting paths so that:

- Can find augmenting paths efficiently.
- Few iterations

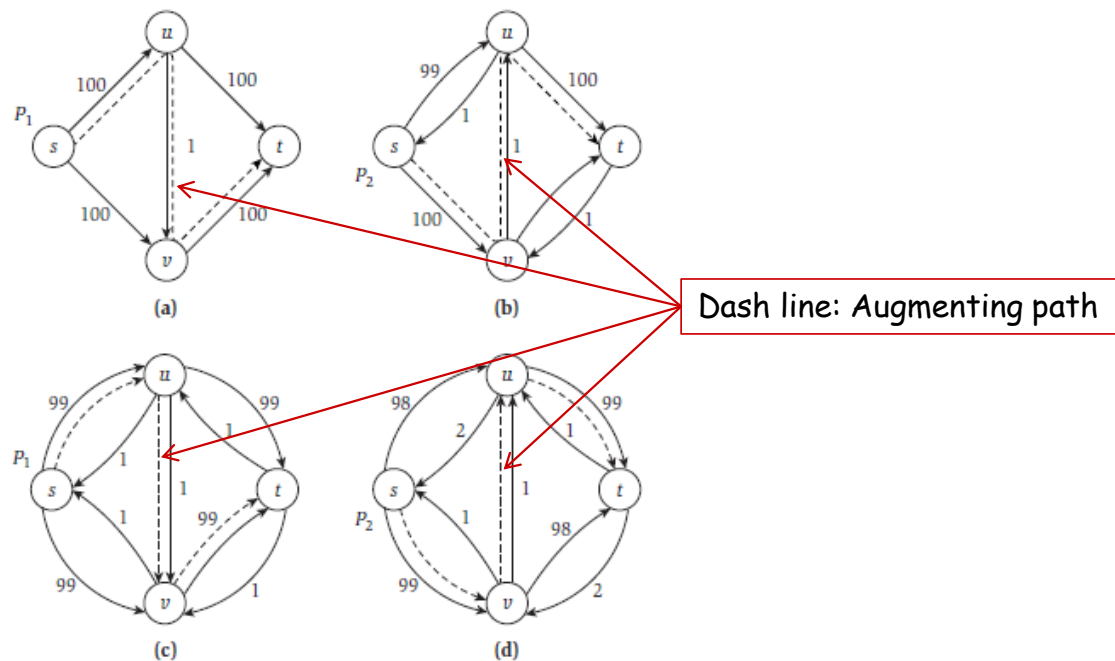


Figure Parts (a) through (d) depict four iterations of the Ford-Fulkerson Algorithm using a bad choice of augmenting paths: The augmentations alternate between the path P_1 through the nodes s, u, v, t in order and the path P_2 through the nodes s, v, u, t in order.

Choosing good augmenting paths

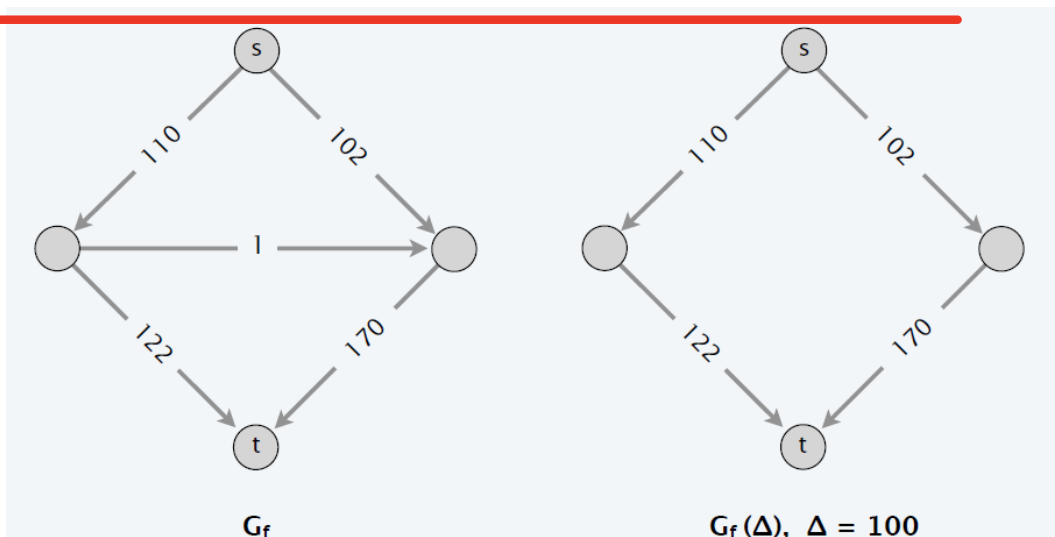
Choose augmenting paths with:

- Max bottleneck capacity ("fattest"). ← how to find?
- Fewest edges. ← ahead

Capacity-scaling algorithm

Overview. Choosing augmenting paths with “large” bottleneck capacity.

- Maintain scaling parameter Δ .
- Let $G_f(\Delta)$ be the part of the residual graph containing only those edges with capacity $\geq \Delta$.
- Any augmenting path in $G_f(\Delta)$ has bottleneck capacity $\geq \Delta$.



Scaling Max-Flow

Initially $f(e)=0$ for all e in G

Initially set Δ to be the largest power of 2 that is no larger than the maximum capacity out of s : $\Delta \leq \max_{e \text{ out of } s} c_e$

While $\Delta \geq 1$

While there is an s - t path in the graph $G_f(\Delta)$

Let P be a simple s - t path in $G_f(\Delta)$

$f' = \text{augment}(f, P)$

Update f to be f' and update $G_f(\Delta)$

Endwhile

$\Delta = \Delta/2$

Endwhile

Return f

Capacity-scaling algorithm: proof of correctness

Assumption: All edge capacities are integers between 1 and C .

Invariant. The scaling parameter Δ is a power of 2.

Pf. Initially a power of 2 (largest power with $2^\Delta \leq C$); each phase divides Δ by exactly 2.

Integrality invariant. Throughout the algorithm, every edge flow $f(e)$ and residual capacity $c_f(e)$ is an integer.

Pf. Same as for generic Ford-Fulkerson.

Theorem. If capacity-scaling algorithm terminates, then f is a max flow.

Pf.

- By integrality invariant, when $\Delta = 1 \rightarrow G_f(\Delta) = G_f$.
- Upon termination of $\Delta = 1$ phase, there are no augmenting paths.
- Result follows augmenting path theorem.

Capacity-scaling algorithm: analysis of running time

Lemma 1. There are $1 + \lfloor \log_2 C \rfloor$ scaling phases.

Pf. Initial $C/2 < \Delta \leq C$; Δ decreases by a factor of 2 in each iteration.

Lemma 2. Let f be the flow at the end of a Δ -scaling phase, then the max-flow value $\leq v(f) + m\Delta$.

Pf. Next slide.

Lemma 3. There are $\leq 2m$ augmentations per scaling phase.

Pf.

- Let f be the flow at the beginning of a Δ -scaling phase.
- Lemma 2 \rightarrow max-flow value $\leq v(f) + m(2\Delta)$.
- Each augmentation in a Δ -scaling phase increases $v(f)$ by at least Δ .

Theorem. The capacity-scaling algorithm takes $O(m^2 \log C)$ time.

Pf.

- Lemma 1 + Lemma 3 $\rightarrow O(m \log C)$ augmentations.
- Finding an augmenting path takes $O(m)$ time.

or equivalently,
at the end
of a 2Δ -scaling phase

$$m \geq 1 \quad v(f) + m(2\Delta) - (v(f') + m(\Delta)) \geq \Delta$$

Capacity-scaling algorithm: analysis of running time

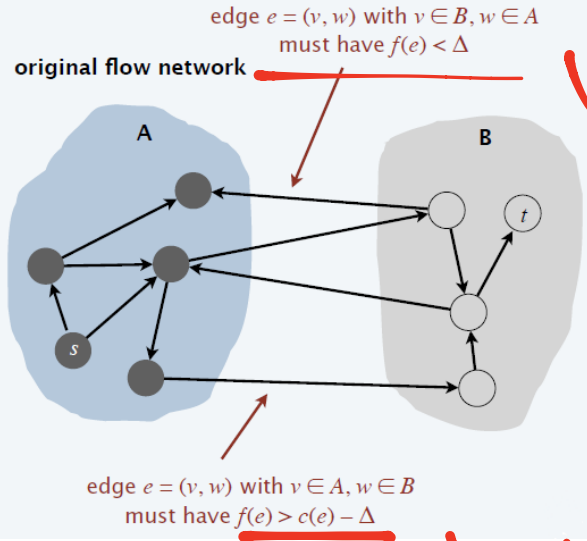
Lemma 2. Let f be the flow at the end of a Δ -scaling phase, then the max-flow value $\leq v(f) + m\Delta$.

Pf.

- We show there exists a cut (A, B) such that $\text{cap}(A, B) \leq v(f) + m\Delta$.
- Choose A to be the set of nodes reachable from s in $G_f(\Delta)$.
- By definition of flow f : $t \notin A$.

flow value lemma

$$\begin{aligned} \text{val}(f) &= \sum_{e \text{ out of } A} f(e) - \sum_{e \text{ in to } A} f(e) \\ &\geq \sum_{e \text{ out of } A} (c(e) - \Delta) - \sum_{e \text{ in to } A} \Delta \\ &\geq \sum_{e \text{ out of } A} c(e) - \sum_{e \text{ out of } A} \Delta - \sum_{e \text{ in to } A} \Delta \\ &\geq \text{cap}(A, B) - m\Delta \quad \blacksquare \end{aligned}$$



why?

why?

不相容
cut会改变

Shortest augmenting path

Q. How to choose next augmenting path in Ford-Fulkerson?

A. Pick one that uses the fewest edges.

can find via BFS

SHORTEST-AUGMENTING-PATH(G)

FOREACH $e \in E$: $f(e) \leftarrow 0$.

$G_f \leftarrow$ residual network of G with respect to flow f .

WHILE (there exists an $s \rightarrow t$ path in G_f)

$P \leftarrow \text{BREADTH-FIRST-SEARCH}(G_f)$.

$f \leftarrow \text{AUGMENT}(f, c, P)$.

Update G_f .

RETURN f .
