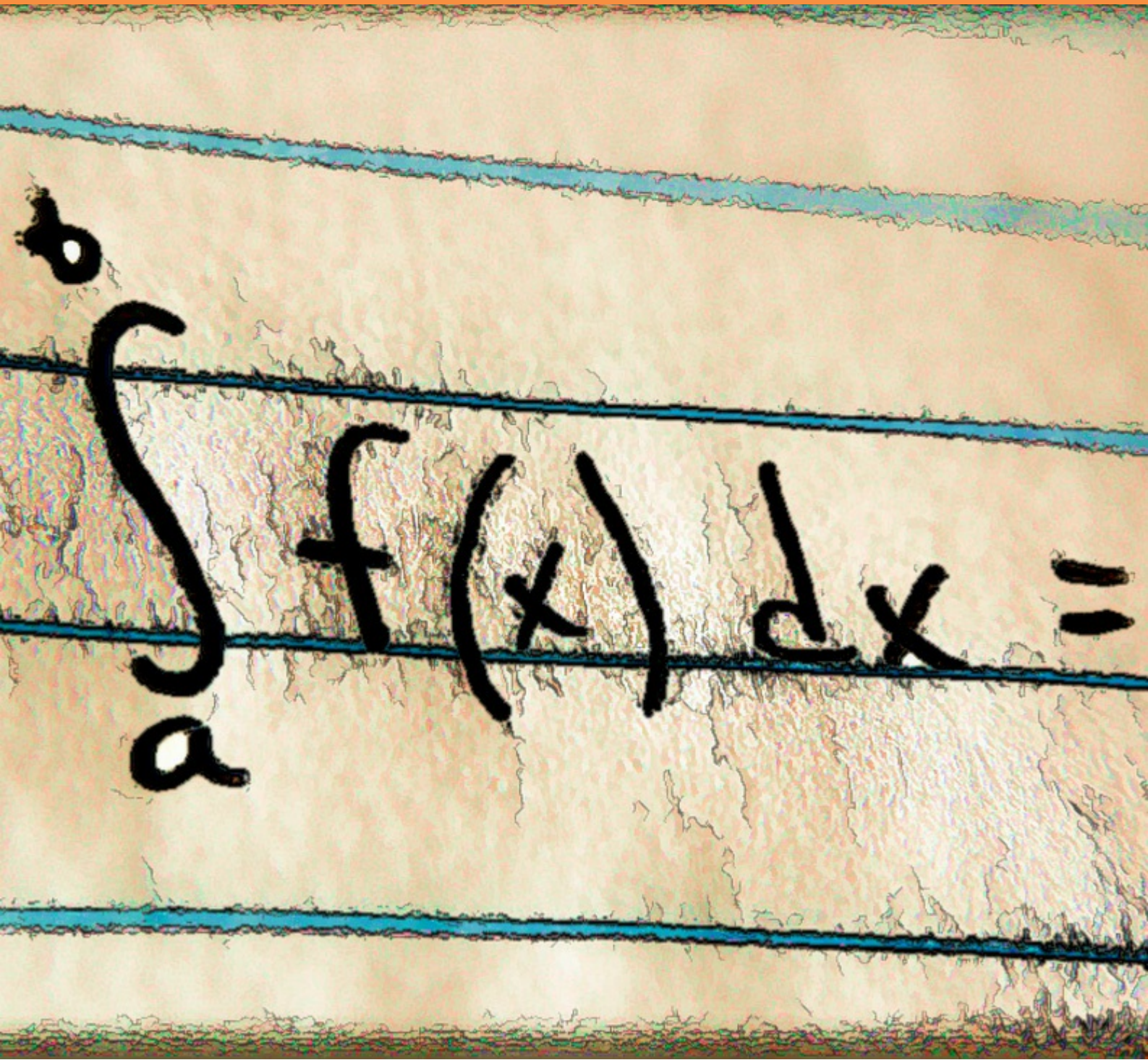


# Engineering Mathematics: YouTube Workbook

Christopher C. Tisdell

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Christopher C. Tisdell

# Engineering Mathematics: YouTube Workbook



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# Contents

<b>How to use this workbook</b>	<b>8</b>
<b>About the author</b>	<b>9</b>
<b>Acknowledgments</b>	<b>10</b>
<b>1 Partial derivatives &amp; applications</b>	<b>11</b>
1.1 Partial derivatives & partial differential equations	11
1.2 Partial derivatives & chain rule	12
1.3 Taylor polynomial approximations: two variables	13
1.4 Error estimation	14
1.5 Differentiate under integral signs: Leibniz rule	15
<b>2 Some max/min problems for multivariable functions</b>	<b>17</b>
2.1 How to determine & classify critical points	17
2.2 More on determining & classifying critical points	18
2.3 The method of Lagrange multipliers	20



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2.4	Another example on Lagrange multipliers	21
2.5	More on Lagrange multipliers: 2 constraints	22
<b>3</b>	<b>A glimpse at vector calculus</b>	<b>24</b>
3.1	Vector functions of one variable	24
3.2	The gradient field of a function	26
3.3	The divergence of a vector field	27
3.4	The curl of a vector field	28
3.5	Introduction to line integrals	29
3.6	More on line integrals	31
3.7	Fundamental theorem of line integrals	32
3.8	Flux in the plane + line integrals	34
<b>4</b>	<b>Double integrals and applications</b>	<b>36</b>
4.1	How to integrate over rectangles	36
4.2	Double integrals over general regions	37
4.3	How to reverse the order of integration	38
4.4	How to determine area of 2D shapes	40
4.5	Double integrals in polar co-ordinates	41
4.6	More on integration & polar co-ordinates	42



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<b>5</b>	<b>Ordinary differential equations</b>	<b>46</b>
5.1	Separable differential equations	46
5.2	Linear, first-order differential equations	47
5.3	Homogeneous, first-order ODEs	49
5.4	2nd-order linear ordinary differential equations	50
5.5	Nonhomogeneous differential equations	51
5.6	Variation of constants / parameters	53
<b>6</b>	<b>Laplace transforms and applications</b>	<b>54</b>
6.1	Introduction to the Laplace transform	54
6.2	Laplace transforms + the first shifting theorem	55
6.3	Laplace transforms + the 2nd shifting theorem	57
6.4	Laplace transforms + differential equations	58

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<b>7</b>	<b>Fourier series</b>	<b>61</b>
7.1	Introduction to Fourier series	61
7.2	Odd + even functions + Fourier series	63
7.3	More on Fourier series	64
7.4	Applications of Fourier series to ODEs	66
<b>8</b>	<b>PDEs &amp; separation of variables</b>	<b>68</b>
8.1	Deriving the heat equation	68
8.2	Heat equation & separation of variables	69
8.3	Heat equation & Fourier series	70
8.4	Wave equation and Fourier series	71
<b>9</b>	<b>Bibliography</b>	<b>73</b>

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# How to use this workbook

This workbook is designed to be used in conjunction with the author's free online video tutorials. Inside this workbook each chapter is divided into learning modules (subsections), each having its own dedicated video tutorial.

View the online video via the hyperlink located at the top of the page of each learning module, with workbook and paper / tablet at the ready. Or click on the Engineering Mathematics YouTube Workbook playlist where all the videos for the workbook are located in chronological order:

Engineering Mathematics YouTube Workbook Playlist

<http://www.youtube.com/playlist?list=PL13760D87FA88691D> [YTPL].

While watching each video, fill in the spaces provided after each example in the workbook and annotate to the associated text. You can also access the above via the author's YouTube channel

Dr Chris Tisdell's YouTube Channel

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The delivery method for each learning module in the workbook is as follows:

- Briefly motivate the topic under consideration;
- Carefully discuss a concrete example;
- Mention how the ideas generalize;
- Provide a few exercises (with answers) for the reader to try.

Incorporating YouTube as an educational tool means enhanced eLearning benefits, for example, the student can easily control the delivery of learning by pausing, rewinding (or fast-forwarding) the video as needed.

The subject material is based on the author's lectures to engineering students at UNSW, Sydney. The style is informal. It is anticipated that most readers will use this workbook as a revision tool and have their own set of problems to solve – this is one reason why the number of exercises herein are limited.

Two semesters of calculus is an essential prerequisite for anyone using this workbook.



# About the author

“With more than a million YouTube hits, Dr Chris Tisdell is the equivalent of a best-selling author or chart-topping musician. And the unlikely subject of this mass popularity? University mathematics.” [Sydney Morning Herald, 14/6/2012 <http://ow.ly/o7gti>].

Chris Tisdell has been inspiring, motivating and engaging large mathematics classes at UNSW, Sydney for over a decade. His lectures are performance-like, with emphasis on contextualisation, clarity in presentation and a strong connection between student and teacher.

He blends the live experience with out-of-class learning, underpinned by flexibility, sharing and openness. Enabling this has been his creation, freely sharing and management of future-oriented online learning resources, known as Open Educational Resources (OER). They are designed to empower learners by granting them unlimited access to knowledge at a time, location and pace that suits their needs. This includes: hundreds of YouTube educational videos of his lectures and tutorials; an etextbook with each section strategically linked with his online videos; and live interactive classes streamed over the internet.

His approach has changed the way students learn mathematics, moving from a traditional closed classroom environment to an open, flexible and forward-looking learning model.

Indicators of esteem include: a prestigious educational partnership with Google; an etextbook with over 500,000 unique downloads; mathematics videos enjoying millions of hits from over 200 countries; a UNSW Vice-Chancellor’s Award for Teaching Excellence; and 100% student satisfaction rating in teaching surveys across 15 different courses at UNSW over eight years.

Chris has been an educational consultant to The Australian Broadcasting Corporation and has advised the Chief Scientist of Australia on educational policy.

Subscribe to his YouTube channel for more <http://www.youtube.com/DrChrisTisdell>.

# Acknowledgments

THANK YOU for 500,000+ downloads of Engineering Mathematics: YouTube Workbook!!!

I'm so happy with the popularity and educational impact of my ebook. Its success is due to people like you for supporting online education! THANK YOU!

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The author thanks those who have read earlier drafts of this ebook and suggested improvements. This includes: Dr Bill Ellis; Mr Nicholas Fewster; and Mr Pingshun Huang – all at UNSW, Sydney.

Gratitude is also expressed to Ms Sophie Tergeist from bookboon.com Ltd. for inviting the author to undertake this work.

# 1 Partial derivatives & applications

## 1.1 Partial derivatives & partial differential equations

View this lesson at <http://www.youtube.com/watch?v=SV56UC31rbk>, [PDE].

### Motivation

Partial differential equations (PDEs) are very important in modelling as their solutions unlock the secrets to a range of important phenomena in engineering and physics. The PDE known as the wave equation models sound waves, light waves and water waves. It arises in fields such as acoustics, electromagnetics and fluid dynamics.

### Example

Consider the wave equation

$$\frac{\partial^2 w}{\partial t^2} = c^2 \frac{\partial^2 w}{\partial x^2}, \quad c \text{ is a constant.}$$

Show that a solution is given by

$$w(t, x) := \cos(2x + 2ct).$$

### The bigger picture

- When verifying solutions to PDEs, compute the necessary derivatives and show that the PDE holds.
- In certain circumstances, the chain rule will be required to compute the partial derivatives of interest.

**Exercises**

Consider the wave equation

$$\frac{\partial^2 w}{\partial t^2} = c^2 \frac{\partial^2 w}{\partial x^2}, \quad c \text{ is a constant.}$$

Show that another solution is given by

$$w(t, x) = \ln(3x + 3ct).$$

Show that the wave equation has solutions of a more general form

$$w(t, x) = f(x + ct)$$

where  $f$  is a sufficiently smooth function. (Hint: use the chain rule.)

## 1.2 Partial derivatives & chain rule

View this lesson at <http://www.youtube.com/watch?v=HOzMR22HsiA>, [Chain].

**Motivation**

The chain rule is an important technique for computing derivatives and better-understanding rates of change of functions. For functions of two (or more) variables, the chain rule takes various forms.

**Example**

Let the function  $w := f(x, y)$  have continuous partial derivatives. If we make a change of variables:  $x = r \cos \theta$ ;  $y = r \sin \theta$  then show that

$$\frac{\partial w}{\partial r} = \frac{\partial f}{\partial x} \cos \theta + \frac{\partial f}{\partial y} \sin \theta.$$

**The bigger picture**

- It is generally helpful to draw a simple diagram to understand which form of the chain rule to apply.
- For functions of two (or more) variables, the chain rule takes a more profound form than the one-variable case.



**Exercises**

Let the function  $w := f(x, y)$  have continuous partial derivatives. If we make a change of variables:  $x = r \cos \theta$ ;  $y = r \sin \theta$  then show that

$$\frac{\partial w}{\partial \theta} = -\frac{\partial f}{\partial x} r \sin \theta + \frac{\partial f}{\partial y} r \cos \theta.$$

**1.3 Taylor polynomial approximations: two variables**

View this lesson at [http://www.youtube.com/watch?v=ez\\_HZZ9H2ao](http://www.youtube.com/watch?v=ez_HZZ9H2ao), [Tay].

**Motivation**

Taylor polynomials are a very simple and useful way of approximating complicated functions. Taylor polynomials are desirable types of approximations as their polynomial structure make them easy to work with.

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**Example**

Using an appropriate Taylor polynomial, compute an approximation to

$$\sqrt{(1.02)^3 + (1.97)^3}.$$

**The bigger picture**

- The Taylor polynomial up to and including linear terms is usually a good starting approximation (unless a greater degree of accuracy is required)

$$T(x, y) := f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b).$$

- The use of a linear Taylor polynomial approximation geometrically equates to approximating surfaces by a suitable tangent plane.

**Exercises**

Using an appropriate Taylor polynomial, compute an approximation to

$$\sqrt{(0.98)^3 + (2.03)^3}.$$

[Ans: 2.9].

## 1.4 Error estimation

View this lesson at <http://www.youtube.com/watch?v=wLrHCkesOA8>, [Est].

**Motivation**

When taking measurements (say, some physical dimensions), errors in the recorded measurements are a fact of life. In many cases we require measurements to be within some prescribed degree of accuracy. We now look at the effects of small variations in measurements and estimate the errors involved.

**Example**

A cylindrical can has height  $h$  and radius  $r$ . We measure the height and radius and obtain 12cm and 5cm respectively, with errors in our measurements being no more than 0.5mm. As a result of the errors in our measurements for  $h$  and  $r$ , obtain an estimate on the percentage error in calculating the volume of the can.

**The bigger picture**

- The main inequality used can be derived from the linear Taylor polynomial approximation for the function involved.
- Don't forget to unify the units involved!

**Exercises**

- Re-examine the previous example by switching the recorded measurements for height and radius and then estimate the percentage error in  $V$ . Compare your estimate with the estimate of the previous example. [Ans: 11/6%]
- We measure the length  $l$  and width  $w$  of a plot of land to be 40m and 10m respectively, with the errors in these measurements being no more than 0.5cm. As a result of the errors in  $l$  and  $w$ , obtain an estimate on the percentage error in calculating the area of the plot. [Ans: 1/16%]

**1.5 Differentiate under integral signs: Leibniz rule**

View this lesson at <http://www.youtube.com/watch?v=cipIYpDu9YY>, [Leib].

**Motivation**

Leibniz rule:

$$I(x) := \int_a^b f(t, x) dt$$

has derivative

$$I'(x) = \int_a^b \frac{\partial f}{\partial x}(t, x) dt.$$

There are two important motivational points concerning Leibniz' rule:

- If a function  $I$  defined by an integral is useful for modelling purposes, then it would seem to make sense that the derivative  $I'$  would also give us insight into the problem under consideration;
- Leibniz' rule can be applied to evaluate very challenging integrals.

**Example**

If

$$F(x) := \int_1^2 \frac{\cos tx}{t} dt$$

then calculate  $F'(x)$ .**The bigger picture**

Let  $a = u(x)$  and  $b = v(x)$  be differentiable functions. If  $f$  and  $\partial f / \partial x$  are continuous on the region

$$S := \{(t, x) : u(x) \leq t \leq v(x), c \leq x \leq d\}$$

then

$$I(x) := \int_{u(x)}^{v(x)} f(t, x) dt$$

has derivative

$$I'(x) = \int_{u(x)}^{v(x)} \frac{\partial f}{\partial x}(t, x) dt + f(v(x), x)v'(x) - f(u(x), x)u'(x).$$

**Exercises**

If

$$F(x) := \int_2^3 \frac{\sin tx}{t} dt$$

then calculate  $F'(x)$ .[Ans:  $(\sin 3x - \sin 2x)/x$ .]



## 2 Some max/min problems for multivariable functions

### 2.1 How to determine & classify critical points

View this lesson at <http://www.youtube.com/watch?v=LGfILcLnSfE>, [CP1].

#### Motivation

Functions of two (or more) variables enable us to model complicated phenomena in a more accurate way than, for example, by using functions of one variable. The determination and classification of critical points of functions is very important in engineering and the applied sciences as this information is sought in a wide range of problems involving modelling.

#### Example

Determine and classify all the critical points of the function

$$f(x, y) := x^3 + y^3 - 3xy + 15.$$

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**The bigger picture**

Assume that  $f$  is continuous and has continuous partial derivatives up to and including all those of the second-order.

- Determine the critical points of  $f$  by solving the simultaneous equations:  $f_x = 0$  and  $f_y = 0$ .
- If  $(a, b)$  is a critical point determined from above and defining

$$D(a, b) := f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2$$

then the nature of the critical points can be then determined via the 2nd-derivative test:

- 1) If  $D(a, b) > 0$  and  $f_{xx}(a, b) < 0$  then  $f$  has a local maximum at  $(a, b)$ ;
- 2) If  $D(a, b) > 0$  and  $f_{xx}(a, b) > 0$  then  $f$  has a local minimum at  $(a, b)$ ;
- 3) If  $D(a, b) < 0$  then  $f$  has a saddle point at  $(a, b)$ ;
- 4) If  $D(a, b) = 0$  then the 2nd-derivative test cannot be used and some other method or technique must be sought to classify our critical points.

**Exercises**

Determine and classify all the critical points of the function

$$f(x, y) := x^2 + xy + 3x + 2y + 5.$$

[Ans: This  $f$  has one critical point at  $(-2, 1)$  with our  $f$  having a saddle point at  $(-2, 1)$ .]

**2.2 More on determining & classifying critical points**

View this lesson at <http://www.youtube.com/watch?v=lq2kdnRY5h8>, [CP2].

**Motivation**

The determination and classification of critical points of functions is very important in engineering and the applied sciences as this information is sought in a wide range of problems involving modelling. We now look at a more involved example.

**Example**

Let  $A > 0$  be a constant. Determine and classify all the critical points of the function

$$f(x, y) := x^2 - Axy + y^2 + 7.$$

Hint – Discuss the cases:  $A = 2$ ;  $0 < A < 2$ ;  $A > 2$ .

**The bigger picture**

Assume that  $f$  is continuous and has continuous partial derivatives up to and including all those of the second-order.

- Determine the critical points of  $f$  by solving the simultaneous equations:  $f_x = 0$  and  $f_y = 0$ .
- If  $(a, b)$  is a critical point determined from above and defining

$$D(a, b) := f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2$$

then the nature of the critical points can be then determined via the 2nd-derivative test:

- 1) If  $D(a, b) > 0$  and  $f_{xx}(a, b) < 0$  then  $f$  has a local maximum at  $(a, b)$ ;
- 2) If  $D(a, b) > 0$  and  $f_{xx}(a, b) > 0$  then  $f$  has a local minimum at  $(a, b)$ ;
- 3) If  $D(a, b) < 0$  then  $f$  has a saddle point at  $(a, b)$ ;
- 4) If  $D(a, b) = 0$  then the 2nd-derivative test cannot be used and some other method or technique must be sought to classify our critical points.

**Exercises**

Determine and classify all the critical points of the functions:

- (a)  $f(x, y) := x^2 + xy + y^2 + 3y + 3$ ;
- (b)  $f(x, y) := -x^2 - 1 - y^2$ .

[Ans: (a) This  $f$  has one critical point at  $(1, -2)$  with  $f$  having a local minimum at  $(1, -2)$ .

(b) This  $f$  has one critical point at  $(0, 0)$  with  $f$  having a local maximum at  $(0, 0)$ .]

## 2.3 The method of Lagrange multipliers

View this lesson at <http://www.youtube.com/watch?v=4E-uLaRhrcA>, [Lag1].

### Motivation

The method of Lagrange multipliers is a very powerful technique enabling us to maximize or minimize a function that is subject to a constraint. Such kinds of problems frequently arise in engineering and applied mathematics, eg, designing a cylindrical silo to maximize its volume subject to a certain fixed amount of building material.

### Example

Consider a thin, metal plate that occupies the region in the  $XY$ -plane

$$\Omega := \{(x, y) : x^2 + y^2 \leq 25\}.$$

If  $f(x, y) := 4x^2 - 4xy + y^2$  denotes the temperature (in degrees C) at any point  $(x, y)$  in  $\Omega$  then determine the highest and lowest temperatures on the edge of the plate.



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**The bigger picture**

Suppose  $f$  and  $g$  have continuous partial derivatives and:  $\partial g/\partial x \neq 0$ ;  $\partial g/\partial y \neq 0$ ; when  $g = 0$ .

- If  $f$  is our function to maximize/minimize and  $g = 0$  is our constraint then the method of Lagrange multipliers involves calculating the critical points of the Lagrangian function  $L := f - \lambda g$

where  $\lambda$  is the Lagrange multiplier.

- To solve the resulting simultaneous equations, some creativity is often required.
- If  $L$  has more than one critical point then compare values of  $f$  at each critical point to determine which gives max/min values of  $f$ .

**Exercises**

Determine the maximum value of  $f(x, y) := xy$  subject to the constraint  $g(x, y) := x + y - 16 = 0$ .

[Ans: At (8, 8),  $f$  has a maximum value of 64.]

## 2.4 Another example on Lagrange multipliers

View this lesson at <http://www.youtube.com/watch?v=5w-b1yU9hy4>, [Lag2].

**Motivation**

The method of Lagrange multipliers is a very powerful technique enabling us to maximize or minimize a function that is subject to a constraint. Such kinds of problems frequently arise in engineering and applied mathematics.

**Example**

The temperature at a point  $(x, y)$  on a metal plate in the  $XY$ -plane is given by

$$T(x, y) := 6xy.$$

Use the method of Lagrange to determine the maximum temperature on the circle  $x^2 + y^2 = 8$ .

**The bigger picture**

Suppose  $f$  and  $g$  have continuous partial derivatives and:  $\partial g/\partial x \neq 0$ ;  $\partial g/\partial y \neq 0$ ; when  $g = 0$ .

- If  $f$  is our function to maximize/minimize and  $g = 0$  is our constraint then the method of Lagrange multipliers involves calculating the critical points of the Lagrangian function  $L := f - \lambda g$

where  $\lambda$  is the Lagrange multiplier.

- To solve the resulting simultaneous equations, some creativity is often required.
- If  $L$  has more than one critical point then compare values of  $f$  at each critical point to determine which gives max/min values of  $f$ .

**Exercises**

Minimize  $f(x, y, z) := x^2 + y^2 + z^2$  subject to the constraint  $g(x, y) := 2x - y + z - 1 = 0$ .

[Ans: At  $(1/3, 1/6, -1/6)$ ,  $f$  has a maximum value of  $2/9$ .]

## 2.5 More on Lagrange multipliers: 2 constraints

View this lesson at <http://www.youtube.com/watch?v=7fRYd8PKeGY>, [Lag3].

**Motivation**

The method of Lagrange multipliers is a very powerful technique enabling us to maximize or minimize a function that is subject to a constraint. Sometimes we have two (or more) constraints. Such kinds of problems frequently arise in engineering and applied mathematics.

**Example**

Maximize the function  $f(x, y, z) := x^2 + 2y - z^2$  subject to the constraints:

$$g_1(x, y, z) := 2x - y = 0; \quad g_2(x, y, z) := y + z = 0.$$

**The bigger picture**

- If  $f$  is our function to maximize/minimize and  $g_1 = 0$ ,  $g_2 = 0$  are our constraints then the method of Lagrange multipliers involves calculating and comparing the critical points of the Lagrangian function

$$L := f - [\lambda_1 g_1 + \lambda_2 g_2]$$

where  $\lambda_1, \lambda_2$  are the Lagrange multipliers.

- To solve the resulting simultaneous equations, some creativity is often required.
- If  $L$  has more than one critical point then compare values of  $f$  at each critical point to determine which gives max/min values of  $f$ .

**Exercises**

Determine the minimum value of  $f(x, y, z) := x^2 + y^2 + z^2$  subject to the constraints:

$$g_1(x, y, z) := x + 2y + 3z - 6 = 0; \quad g_2(x, y, z) := x + 3y + 9z - 9 = 0.$$

[Ans: Min value occurs at  $(1 + 22/59, 2 + 5/59, 9/59)$ .]

## 3 A glimpse at vector calculus

### 3.1 Vector functions of one variable

View this lesson at <http://www.youtube.com/watch?v=hmNlDuk08Yk>, [VecFun].

#### Motivation

Vector-valued functions allow us more flexibility in the modelling of phenomena in two and three dimensions, such as the orbits of planets. The most basic of vector-valued functions are those involving one variable which can be used to describe and analyze curves in space.

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**Example**

Consider the function  $\mathbf{r}(t) := (\cos t, \sin t, t)$  for  $t \geq 0$ .

1. Sketch and describe the curve associated with  $\mathbf{r}$ .
2. If a particle travels along this curve (with  $t$  representing time) then calculate  $\mathbf{r}'(t)$  and show that its speed is constant.
3. Calculate

$$\int_0^\pi \mathbf{r}(t) dt.$$

**The bigger picture**

- Differentiation, integration and limits of vector-valued functions of one variable involve differentiation, integration and limits of each component function.
- For smooth curves,  $\mathbf{r}'(t)$  is a tangent vector to the curve and points in the direction of motion  $\mathbf{r}'(t)/\|\mathbf{r}'(t)\|$ .

**Exercises**

Consider the function  $\mathbf{r}(t) := (2 \cos t, 3 \sin t)$  for  $t \geq 0$ .

1. Sketch and describe the curve associated with  $\mathbf{r}$ .
2. If a particle travels along this curve (with  $t$  representing time) then calculate  $\mathbf{r}'(t)$ . Is the speed constant?
3. Calculate

$$\int_{-\pi}^\pi \mathbf{r}(t) dt.$$

[Ans: 1. The curve is the ellipse  $x^2/4 + y^2/9 = 1$ ; 2.  $\mathbf{r}'(t) = (-2 \sin t, 3 \cos t)$ . No.; 3.  $(0, 0)$ .]

### 3.2 The gradient field of a function

View this lesson at <http://www.youtube.com/watch?v=Xw5wsWDt1Cg>, [Grad].

#### Motivation

The gradient field of a function is one of the basic concepts of vector calculus. It can be used to construct normal vectors to curves and surfaces, it can be applied to calculate slopes of tangent lines to surfaces in any direction, and can be very useful when integrating over curves (line integrals) to compute work done.

#### Example

Consider the function  $f(x, y) := x^2 + 4y^2$ .

1. Compute  $\nabla f$ .
2. Show that  $\nabla f$  is normal to the level curve  $f(x, y) = 16$ .
3. Calculate the rate of change (directional derivative) of  $f$  at  $(1, 1)$  in the direction  $\mathbf{u} = (1, 1)$ .

#### The bigger picture

- The gradient readily generalizes to equations involving three or more variables,  $\nabla f = (f_x, f_y, f_z)$ .
- For differentiable functions  $f$  the vector  $\nabla f$  will always be normal to level curves / level surfaces of  $f$ .
- For differentiable functions of two variables the directional derivative of  $f$  at  $P(x_0, y_0)$  in the direction of  $\mathbf{u}$  is just the slope of the tangent line to the surface of  $f$  with the tangent line lying in the vertical plane that contains  $(x_0, y_0, f(x_0, y_0))$  and  $\mathbf{u}$ .
- Directional derivatives are generalizations of partial derivatives.

#### Exercises

Consider the function  $f(x, y, z) := x^2 + y^2 - z^2$ .

1. Compute  $\nabla f$ .
2. Compute a normal vector to the surface  $f(x, y, z) = -7$  at  $(1, 1, 3)$ .
3. Calculate the rate of change (directional derivative) of  $f$  at  $(1, 1, 1)$  in the direction  $\mathbf{u} = (1, 1, 1)$ .

[Ans: 1.  $(2x, 2y, -2z)$ ; 2.  $(2, 2, -6)$ ; 3.  $2/\sqrt{3}$ .]

### 3.3 The divergence of a vector field

View this lesson at <http://www.youtube.com/watch?v=quZlfp59iJg>, [Div].

#### Motivation

The divergence of a vector field is one of the basic concepts of vector calculus. It measures expansion and compression of a vector field and can be very useful when integrating over curves (line integrals) and surfaces to compute flux.

#### Example

- a) If  $\mathbf{F}(x, y, z) := (x^2 - y, y + z, z^2 - x)$  then compute  $\nabla \cdot \mathbf{F}$  at  $(1, 2, 3)$ .
- b) If  $\mathbf{G}(x, y) := (0, x)$  then compute  $\nabla \cdot \mathbf{G}$ . Sketch  $\mathbf{G}$  and show that there is a zero net outflow over each rectangle.

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**The bigger picture**

- The divergence measures net outflow of a vector field.
- If the divergence is positive everywhere, then there is a net outflow over every closed curve / surface.
- If the divergence is negative everywhere, then there is a net inflow over every closed curve / surface.
- A vector field with zero divergence everywhere is called “incompressible” with zero net outflow over every closed curve / surface.

**Exercises**

Consider  $\mathbf{F}(x, y, z) := (x \cos y, y \sin x, z^2 - 1)$  and  $\mathbf{G}(x, y) := (x, y)$ .

1. Compute  $\nabla \cdot \mathbf{F}$  and  $\nabla \cdot \mathbf{G}$ .
2. Sketch  $\mathbf{G}$  and show that there is a net outflow over each circle in the plane.

[Ans: 1.  $\cos y + \sin x + 2z$ , 2; 2. Sketch  $\mathbf{G}$  and graphically show the net outflow is positive.]

### 3.4 The curl of a vector field

View this lesson at <http://www.youtube.com/watch?v=qeOT0YufpSY>, [Curl].

**Motivation**

The curl of a vector field is one of the basic concepts of vector calculus. It measures rotation in a vector field and can be very useful when integrating over curves (line integrals) and surfaces.

**Example**

- a) If  $\mathbf{F}(x, y, z) := (x^2 - y, y + z, z^2 - x)$  then compute  $\nabla \times \mathbf{F}$  at  $(1, 2, 3)$ .
- b) If  $\mathbf{G}(x, y) := (y, 0)$  then compute the scalar curl of  $\mathbf{G}$ . Sketch  $\mathbf{G}$  and discuss the rotation in  $\mathbf{G}$ .

**The bigger picture**

- The curl measures rotation in a vector field.
- If the scalar curl is positive everywhere in the plane, then there is an anticlockwise rotation of a paddlewheel in the plane about its axis.
- If the scalar curl is negative everywhere in the plane, then there is clockwise rotation of a paddlewheel in the plane about its axis.
- A vector field with zero curl everywhere is called “irrotational”. In the plane this means a paddlewheel in the vector field not spinning about its axis.

**Exercises**

Consider  $\mathbf{F}(x, y, z) := (z \cos y, z \sin x, z^2 - y^3)$  and  $\mathbf{G}(x, y) := (x, y)$ .

1. Compute  $\nabla \times \mathbf{F}$  and the scalar curl of  $\mathbf{G}$ .
2. Sketch  $\mathbf{G}$  and graphically show that  $\mathbf{G}$  is irrotational.

[Ans: 1.  $(-3y^2 - \sin x, \cos y, z \cos x + z \sin y)$ , 0;  
2. Sketch  $\mathbf{G}$  and graphically show a paddlewheel cannot rotate about its axis.]

**3.5 Introduction to line integrals**

View this lesson at <http://www.youtube.com/watch?v=6YuyBnXBFXg>, [line1].

**Motivation**

A line integral involves the integration of functions over curves. Applications include: calculating work done; determining the total mass and center of mass of thin wires; and also finding flux over curves.

**Example**

- a) If  $f(x, y) := x + y$  and  $\mathcal{C}$  is the unit circle with center at  $(0, 0)$  then compute the line integral

$$\int_{\mathcal{C}} f \, ds.$$

- b) If  $\mathbf{F}(x, y) := (-y + 1)\mathbf{i} + x\mathbf{j}$  and  $\mathcal{C}$  is oriented via an anticlockwise rotation then compute the line integral

$$\int_{\mathcal{C}} \mathbf{F} \bullet d\mathbf{r}.$$

### The bigger picture

- For a smooth curve  $\mathcal{C}$  with parametrization  $\mathbf{r} = \mathbf{r}(t)$ ,  $a \leq t \leq b$

$$\int_{\mathcal{C}} f \, ds = \int_a^b f(\mathbf{r}(t)) \|\mathbf{r}'(t)\| dt.$$

- For an oriented smooth curve  $\mathcal{C}$  with parametrization  $\mathbf{r} = \mathbf{r}(t)$ ,  $a \leq t \leq b$

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = \int_{\mathcal{C}} \mathbf{F} \cdot \hat{\mathbf{T}} \, ds$$

where  $\hat{\mathbf{T}}$  is the unit tangent vector in the direction of motion.



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**Exercises**

Let  $\mathcal{C}$  be part of a parabola  $y = x^2$  parametrized by  $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j}$ ,  $0 \leq t \leq 2$ .

3. If  $f(x, y) := x^2 - y + x$  then compute

$$\int_{\mathcal{C}} f \, ds.$$

4. If  $\mathcal{C}$  is oriented in the direction from  $(0, 0)$  to  $(1, 1)$  and if  $\mathbf{F}(x, y) := 2x\mathbf{i} + y\mathbf{j}$  then compute

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}.$$

What answer would we obtain if we reversed the orientation of  $\mathcal{C}$ ?

[Ans: 1.  $27/12$ ; 2.  $3/2$ , while reversing the orientation would give  $-3/2$ .]

### 3.6 More on line integrals

View this lesson at <http://www.youtube.com/watch?v=S2rT2zK2bdo>, [line2].

**Motivation**

A line integral involves the integration of functions over curves. Applications include: calculating work done; determining the total mass and center of mass of thin wires; and also finding flux over curves. An important part of the integration process is to appropriately describe the curve of integration by “parametrization”.

**Example**

For the vector field

$$\mathbf{F} = 8e^{-x}\mathbf{i} + \cosh z\mathbf{j} - y^2\mathbf{k}$$

calculate:

- 1)  $\nabla \cdot \mathbf{F}$  (i.e.  $\text{div } \mathbf{F}$ ) and  $\nabla \times \mathbf{F}$  (i.e.  $\text{curl } \mathbf{F}$ );
- 2) The line integral of  $\mathbf{F}$  along the straight line from  $A(0, 1, 0)$  to  $B(\ln(2), 1, 2)$ .

**The bigger picture**

- For a smooth curve  $\mathcal{C}$  with parametrization  $\mathbf{r} = \mathbf{r}(t)$ ,  $a \leq t \leq b$

$$\int_{\mathcal{C}} f \, ds = \int_a^b f(\mathbf{r}(t)) \|\mathbf{r}'(t)\| dt.$$

- For an oriented smooth curve  $\mathcal{C}$  with parametrization  $\mathbf{r} = \mathbf{r}(t)$ ,  $a \leq t \leq b$

$$\int_{\mathcal{C}} \mathbf{F} \bullet d\mathbf{r} = \int_{\mathcal{C}} \mathbf{F} \bullet \hat{\mathbf{T}} \, ds$$

where  $\hat{\mathbf{T}}$  is the unit tangent vector in the direction of motion.

**Exercises**

Let  $\mathcal{C}$  be the curve parametrized by  $\mathbf{r}(t) = t^2\mathbf{i} + \sqrt{t}\mathbf{j} + \sqrt{t}\mathbf{k}$ , from  $t = 2$  to  $t = 4$ . If  $\mathbf{F}(x, y, z) := x^2\mathbf{i} - z\mathbf{j} + 2xy\mathbf{k}$  then compute

$$\int_{\mathcal{C}} \mathbf{F} \bullet d\mathbf{r}.$$

[Ans: 49013/3.]

### 3.7 Fundamental theorem of line integrals

View this lesson at <http://www.youtube.com/watch?v=nH5KCrEuvyA>, [FTL1].

**Motivation**

A line integral involves the integration of functions over curves. In certain cases the value of the line integral is independent of the curve between two points. In such a case we may apply a “fundamental theorem of line integrals”.

**Example**

Consider the vector field

$$\mathbf{F} := (2xyz + \cos x)\mathbf{i} + x^2z\mathbf{j} + (x^2y + e^z)\mathbf{k}.$$

- 1) Show that  $\nabla \times \mathbf{F} = \text{curl } \mathbf{F} = 0$ .
- 2) Find a scalar field  $\phi$  such that  $\nabla\phi = \text{grad } \phi = \mathbf{F}$ .
- 3) Hence or otherwise evaluate  $\int_C \mathbf{F} \bullet d\mathbf{r}$  where the path  $\mathcal{C}$  is parametrized by

$$\mathbf{r}(t) := (\sin t, \cos t, t), \quad \text{from } t = 0 \text{ to } t = 2\pi.$$

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**The bigger picture**

- If there is a scalar-valued function  $\phi$  such that  $\nabla\phi = \mathbf{F}$  then the line integral

$$\int_C \mathbf{F} \bullet d\mathbf{r}$$

is independent of the path  $C$  from starting point  $A$  to ending point  $B$ . Furthermore,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \phi(B) - \phi(A).$$

- If  $\text{curl } \mathbf{F} = 0$  then a  $\phi$  will exist such that  $\nabla\phi = \mathbf{F}$ .

**Exercises**

Consider the vector field

$$\mathbf{F} := yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}.$$

Evaluate  $\int_C \mathbf{F} \bullet d\mathbf{r}$  where  $C$  is any path from  $(0, 0, 0)$  to  $(2, 1, 3)$ .

[Ans: 3.]

**3.8 Flux in the plane + line integrals**

View this lesson at <http://www.youtube.com/watch?v=YHiTuLNu3Sc>, [flux].

**Motivation**

We show how line integrals can be used to calculate the outward flux (flow rate) of a vector field over a closed curve in the plane. Such ideas have important applications to fluid flow.

**Example**

Let  $C$  be the ellipse  $x^2/4 + y^2 = 1$ .

- Compute an outward-pointing normal vector  $\mathbf{n}$  to  $C$ .
- If  $\mathbf{F}(x, y) := 4x\mathbf{i} + y\mathbf{j}$  then compute the outward flux over  $C$ .

**The bigger picture**

- For a closed, smooth curve  $\mathcal{C}$  with parametrization  $\mathbf{r} = \mathbf{r}(t)$ ,  $a \leq t \leq b$  a normal vector to  $\mathcal{C}$  may be produced by calculating  $\mathbf{r}' \times \mathbf{k}$  (or  $\mathbf{k} \times \mathbf{r}'$ ).
- For a smooth curve  $\mathcal{C}$  with parametrization  $\mathbf{r} = \mathbf{r}(t)$ ,  $a \leq t \leq b$  and with outward pointing unit normal vector  $\hat{\mathbf{n}}$  we can compute the outward flux over  $\mathcal{C}$  via

$$\oint_{\mathcal{C}} \mathbf{F} \cdot \hat{\mathbf{n}} \, ds.$$

**Exercises**

Let  $\mathcal{C}$  be the circle with centre  $(0, 0)$  and radius 2.

1. Compute an outward-pointing normal vector  $\mathbf{n}$  to  $\mathcal{C}$ .
2. If  $\mathbf{F}(x, y) := (x + 1)\mathbf{i} + y\mathbf{j}$  the compute the outward flux over  $\mathcal{C}$ .
3. Compute

$$\iint_{\Omega} \nabla \cdot \mathbf{F} \, dA$$

where  $\Omega$  is the disc (bounded by  $\mathcal{C}$ ). Compare your answer with (b).

[Ans: 1. If  $\mathbf{r}(t) = (2 \cos t, 2 \sin t)$ ,  $0 \leq t \leq 2\pi$  then  $\hat{\mathbf{n}} = (\cos t, \sin t)$ ; 2.  $8\pi$ , 3.  $8\pi$ .]

# 4 Double integrals and applications

## 4.1 How to integrate over rectangles

View this lesson at <http://www.youtube.com/watch?v=My6sdEekbHM>, [DblInt1].

### Motivation

Double integrals are a generalisation of the basic single integral seen in high-school. Double integrals enable us to work with more complicated problems in higher dimensions and find many engineering applications, for example, in calculating centre of mass and moments of inertia of thin plates.

### Example

Evaluate

$$I := \int_1^3 \int_2^3 (x^2 - 2xy + 2y^3) dy dx.$$

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**The bigger picture**

- When evaluating a given double integral, perform the “inside” integral first and then move to the outside integral.
- Fubini’s Theorem (simple version): If  $f = f(x, y)$  is continuous on the rectangle  
 $R := \{(x, y) : a \leq x \leq b, c \leq y \leq d\}$

then

$$\iint_R f(x, y) \, dA = \int_c^d \int_a^b f(x, y) \, dx \, dy = \int_a^b \int_c^d f(x, y) \, dy \, dx.$$

- It is generally a good idea to sketch the region of integration so as to better-understand the geometry of the problem.

**Exercises**

Evaluate:

- (a)  $\int_0^1 \int_1^2 (x^2 - 2x + y) \, dy \, dx;$
- (b)  $\int_0^2 \int_0^1 (x^2 + y^2) \, dx \, dy.$

[Ans: (a) 5/6; (b) 10/3.]

## 4.2 Double integrals over general regions

View this lesson at <http://www.youtube.com/watch?v=9cAVY9niDnI>, [DblIntG].

**Motivation**

We now learn how to integrate over more general two-dimensional regions than just rectangles.

**Example**

Evaluate

$$I := \iint_{\Omega} (x^2 y + y^3) \, dy \, dx$$

where

$$\Omega := \{(x, y) : x^2 + y^2 \leq 1, x \geq 0, y \geq 0\}.$$

**The bigger picture**

- To form a integral over more general two-dimensional regions we appropriately describe its boundary (or edges).
- When evaluating a given double integral, perform the “inside” integral first and then move to the outside integral.
- It is generally a good idea to sketch the region of integration so as to better-understand the geometry of the problem.

**Exercises**

Evaluate:

(a)  $\int_0^1 \int_0^x (x^2 - 2x + y) \, dy \, dx;$

(b)  $\int_0^1 \int_0^y (x^2 + y^2) \, dx \, dy.$

[Ans: (a) 3/4; (b) 1/3.]

### 4.3 How to reverse the order of integration

View this lesson at <http://www.youtube.com/watch?v=zxyx73HAtfQ>, [DblInt2].

**Motivation**

The order of integration in double integrals can sometimes be reversed and can lead to a greatly simplified (but equivalent) double integral that is easier to evaluate than the original one. This “order reversion” technique can be used when performing calculations involving the applications associated with double integrals.

**Example**

Evaluate the following integral by reversing the order of integration

$$I := \int_0^1 \int_{\sqrt{x}}^1 \sqrt{1+y^3} \, dy \, dx.$$

### The bigger picture


When reversing the order of integration, the region of integration needs to be suitably redescribed. Appropriate diagrams will help with understanding the geometry of the problem and hence will assist with the redescription of the region of integration.

### Exercises

Evaluate the following integral by reversing the order of integration

$$I := \int_0^4 \int_{3x}^{12} \sin(y^2) \, dy \, dx.$$

[Ans:  $(1 - \cos 144)/6$ .]



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## 4.4 How to determine area of 2D shapes

View this lesson at <http://www.youtube.com/watch?v=wiFUGkHTmO0>, [Area].

### Motivation

Double integrals are a generalisation of the basic single integral seen in high-school. They can be used to determine the area of two-dimensional shapes, which can be an important part of, for example, determining the centroid.

### Example

Using double integrals, calculate the area of the region bounded by the curves:

$$y = \sqrt{x}; \quad y = x; \quad y = x/2.$$

### The bigger picture

- Assume that  $f$  is continuous on a two-dimensional region  $\Omega$ . We can calculate the area of  $\Omega$  by evaluating

$$\iint_{\Omega} dA.$$

- If  $\Omega$  is not a simple region then it may be possible to deconstruct  $\Omega$  into simple parts in order to calculate the integral(s) involved.
- It is generally a good idea to sketch the region of integration so as to better-understand the geometry of the problem.

### Exercises

- a) Use double integrals to determine the area of the region bounded by the lines:

$$y = x^3; \quad y = x^2.$$

- b) If you are familiar with double integrals in polar coordinates, then prove that the area of a circle with radius  $R > 0$  is  $\pi R^2$  using double integration.

[Ans: (a)  $1/12 \, u^2$  where  $u^2$  means “square units”.]

## 4.5 Double integrals in polar co-ordinates

View this lesson at <http://www.youtube.com/watch?v=zknGq9CvZTQ>, [Polar1].

### Motivation

We know from basic calculus that sometimes a change of variables can greatly simplify very complicated integrals. We now explore this idea in the more general setting of double integrals under a specific change of variables known as polar co-ordinates.

### Example

Consider the double integral

$$\int_0^{1/\sqrt{2}} \int_y^{\sqrt{1-y^2}} 3y \, dx \, dy.$$

- 1) Sketch of the region of integration.
- 2) Evaluate the double integral by applying polar coordinates.

### The bigger picture

When transferring double integrals from Cartesian co-ordinates to polar co-ordinates, the three essential substitutions are:

$$x = r \cos \theta; \quad y = r \sin \theta; \quad dA = r \, dr \, d\theta.$$

It is generally a good idea to sketch the region of integration so as to better-understand the geometry of the problem.

### Exercises

Describe, in polar co-ordinates, the set of points of the unit disc  $\Omega$  that has centre  $(0, 0)$  and that lie in the first quadrant ( $x \geq 0, y \geq 0$ ). Hence, evaluate

$$I := \int \int_{\Omega} \sqrt{x^2 + y^2} \, dA.$$

[Ans:  $\pi/6$ ].

## 4.6 More on integration & polar co-ordinates

View this lesson at <http://www.youtube.com/watch?v=TOa3hgRT7iA>, [Polar2].

**Motivation**

We know from basic calculus that sometimes a change of variables can greatly simplify very complicated integrals. We now explore this idea in the more general setting of double integrals under a specific change of variables known as polar co-ordinates.

**Example**

Evaluate by using polar co-ordinates

$$I := \int_0^2 \int_{-\sqrt{2y-y^2}}^{\sqrt{2y-y^2}} \sqrt{x^2 + y^2} \, dx \, dy.$$

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**The bigger picture**

- When transferring double integrals from Cartesian co-ordinates to polar co-ordinates, the three essential substitutions are:

$$x = r \cos \theta; \quad y = r \sin \theta; \quad dA = r \, dr \, d\theta.$$

- It is generally a good idea to sketch the region of integration so as to better-understand the geometry of the problem.

**Exercises**

Evaluate by using polar co-ordinates

$$I := \int_0^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} x^2 y^2 \, dy \, dx.$$

[Ans:  $4\pi/3$ ].

**4.7 Calculation of the centroid**

View this lesson at <http://www.youtube.com/watch?v=rIVB-OsNJvY>, [Centroid].

**Motivation**

In engineering, the word centroid means the geometric center of an object's shape. If a physical object (for example, a thin plate) has uniform density, then its center of mass is the same as the centroid of its shape. Double integrals can be used to calculate the centroid of simple two-dimensional regions.

**Example**

Calculate the centroid of the region that lies in the first quadrant of the  $XY$ -plane, bounded by the circle

$$x^2 + y^2 = a^2, \quad a > 0.$$



**The bigger picture**

- The co-ordinates  $(\bar{x}, \bar{y})$  of the centroid of a two-dimensional region  $\Omega$  can be calculated by the use of double integrals via:

$$\bar{x} = \frac{\iint_{\Omega} x \, dA}{\iint_{\Omega} dA}, \quad \bar{y} = \frac{\iint_{\Omega} y \, dA}{\iint_{\Omega} dA}$$

- It is generally a good idea to sketch the region of integration so as to better-understand the geometry of the problem. Look for symmetry.

**Exercises**

Calculate the centroid of the region lying in the first quadrant which is bounded by the curves:

$$y = 0; \quad y^2 = 2x; \quad x + y = 4.$$

[Ans:  $(\bar{x}, \bar{y}) = (64/35, 5/7)$ .]

## 4.8 How to calculate the mass of thin plates

View this lesson at [http://www.youtube.com/watch?v=oty3f7wHY\\_M](http://www.youtube.com/watch?v=oty3f7wHY_M), [Mass].

**Motivation**

A common and important challenge in engineering is that of calculating the mass of a given object. Double integrals can be used to calculate the mass of simple two-dimensional objects, such as thin plates or sheets, if the object has a known density function.

**Example**

A thin plate occupies the region bounded by the curves:

$$y = \sqrt{a^2 - x^2}; \quad x = \pm a; \quad y = -a.$$

If we associate the density function

$$\rho(x, y) := y + a$$

with each point in the plate, then calculate the total mass of the plate.

**The bigger picture**

- If a thin plate occupies a simple region in the  $XY$ -plane and has a known density function  $\rho = \rho(x, y)$  then the mass of the plate may be calculated by using double integrals via

$$\text{mass} = \iint_{\Omega} \rho(x, y) \, dA.$$

- It is generally a good idea to sketch the region of integration  $\Omega$  so as to better-understand the geometry of the problem.

**Exercises**

If we change the density function in the previous example to

$$\rho(x, y) = x + a$$

then recalculate the total mass of the thin plate.

[Ans:  $a^3(2 + \pi/2)$ ].



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# 5 Ordinary differential equations

## 5.1 Separable differential equations

View this lesson at <http://www.youtube.com/watch?v=uQuBVa5PPy4>, [SepODE].

### Motivation

Ordinary differential equations (ODEs) are used to mathematically model a range of phenomena from science and engineering. ODEs naturally arise when making simple physical assumptions about the model under consideration. By computing a solution to an ODE we can make precise predictions about future and past states of the model of interest.

Separable ODEs are a special type of ODE and have the form

$$\frac{dy}{dx} = f(x)g(y).$$

### Example

(a) Solve the separable ODE

$$\frac{dy}{dx} = -\frac{x}{2y}.$$

(b) If we have the extra information  $y(0) = 1$  then determine the constant in the solution in (a).

### The bigger picture

- To solve a separable ODE simply: separate the variables (via division and / or multiplication) and then integrate both sides.
- Sometimes some algebraic manipulation is required to identify a separable ODE.
- If an initial condition is known, then the arbitrary constant from the general solution can be determined.

**Exercises**

Solve the following separable ODEs:

$$(a) \frac{dy}{dx} = \frac{2x}{3(y+1)^2} \qquad (b) \frac{dy}{dx} = 2xy, \quad y(0) = 1.$$

[Ans: (a)  $y(x) = -1 + (x^2 + c)^{1/3}$ ; (b)  $y(x) = Ae^{x^2}$  is the general solution (where  $A$  is a constant); and  $y(x) = e^{x^2}$  is the particular solution yielded from the initial condition.]

## 5.2 Linear, first-order differential equations

View this lesson at <http://www.youtube.com/watch?v=hIsLGmrpoPo>, [LinODE].

**Motivation**

Ordinary differential equations (ODEs) are used to mathematically model a range of phenomena from science and engineering. ODEs naturally arise when making simple physical assumptions about the model under consideration. By computing a solution to an ODE we can make precise predictions about future and past states of the model of interest.

Linear ODEs are a special type of ODE and have the form

$$\frac{dy}{dx} + p(x)y = q(x).$$

**Example**

a) Solve the linear ODE

$$\frac{dy}{dx} + \frac{1}{x}y = x^2$$

for the solution  $y(x)$  for  $x > 0$ .

b) If we have the extra information  $y(1) = 0$  then determine the constant in the solution in (a).

**The bigger picture**

- Construct the integrating factor  $u(x) := e^{\int p(x) dx}$
- Multiply both sides of the ODE by  $u$  and collapse the left-hand-side to the derivative of  $y \cdot u(x)$ , ie  $\frac{d}{dx}(y \cdot u(x))$ .
- Integrate both sides and solve for  $y$ .
- If an initial condition is known, then the constant from the general solution can be determined.

**Exercises**

Solve the following separable ODEs:

(a)  $\frac{dy}{dx} - 2xy = e^{x^2}, \quad y(0) = 1.$

(b)  $\frac{dy}{dx} + 2xy = 2x.$

[Ans: (a)  $y(x) = e^{x^2} (x + c)$  is the general solution and the initial condition yields  $y(x) = e^{x^2} (x + 1)$ ; (b)  $y(x) = 1 + ce^{-x^2}$ .]

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### 5.3 Homogeneous, first-order ODEs

View this lesson at <http://www.youtube.com/watch?v=UpLQUGBznE4>, [FOHomogODE].

#### Motivation

Ordinary differential equations (ODEs) are used to mathematically model a range of phenomena from science and engineering. By computing a solution to an ODE we can make precise predictions about future and past states of the model of interest.

A homogenous, first order ODE is of the form

$$y' = f\left(\frac{y}{x}\right).$$

#### Example

Solve the homogeneous, first-order ODE

$$y' = \frac{y - x}{y + x}$$

for  $y = y(x)$  with  $x > 0$ .

#### The bigger picture

To solve a first-order, homogeneous ODE:

- Make the substitution

$$u = \frac{y}{x}$$

- compute the derivative  $y'$  in terms of  $u'$
- obtain a separable ODE in  $u$
- solve for  $u$  and then backsubstitute for to obtain  $y$ .

#### Exercises

Solve

$$y' = (2x + y)/x, \quad x > 0$$

[Ans:  $y = 2x \ln x + Cx$ ]

## 5.4 2nd-order linear ordinary differential equations

View this lesson at <http://www.youtube.com/watch?v=2symahFSBd8>, [2ndHomog].

### Motivation

Ordinary differential equations (ODEs) are used to mathematically model a range of phenomena from science and engineering. ODEs naturally arise when making simple physical assumptions about the model under consideration. By computing a solution to an ODE we can make precise predictions about future and past states of the model of interest.

We shall discuss “the method of undetermined coefficients” for the case

$$ay'' + by' + cy = 0$$

where  $a$ ,  $b$  and  $c$  are given constants. These kinds of problems arise in the study of vibrating systems.

### Example

Solve the ODEs:

(a)  $y'' + 4y' + 3y = 0$ ; (b)  $y'' + 4y' + 4y = 0$ ; (c)  $y'' + 2y' + 5y = 0$ .

### The bigger picture

To solve the ODE  $ay'' + by' + cy = 0$

- Solve the corresponding quadratic (characteristic) equation  $a\lambda^2 + b\lambda + c = 0$  for  $\lambda$ .
- If the solutions  $\lambda_1, \lambda_2$  are real and unequal then  $y(x) = Ae^{\lambda_1 x} + Be^{\lambda_2 x}$ .
- If the solutions are equal, ie  $\lambda_1 = \lambda_2 = \lambda$  then  $y(x) = Ae^{\lambda x} + Bxe^{\lambda x}$ .
- If the solutions  $\lambda_1, \lambda_2$  are complex conjugate pairs, ie  $\lambda_1 = a + \beta i, \lambda_2 = a - \beta i$  then  $y(x) = e^{\alpha x}(A \cos \beta x + B \sin \beta x)$ .



**Exercises**

Solve the following ODEs:

(a)  $y'' + 3y' + 2y = 0$ ; (b)  $y'' + 2y' + y = 0$ ; (c)  $y'' + 4y' + 5y = 0$ .

[Ans: (a)  $y(x) = Ae^{-x} + Be^{-2x}$ ; (b)  $y(x) = Ae^{-x} + Bxe^{-x}$ ;  
(c)  $y(x) = e^{-2x}(A \cos x + B \sin x)$ .]

## 5.5 Nonhomogeneous differential equations

View this lesson at [http://www.youtube.com/watch?atch?v=haAV-\\_6R5VU](http://www.youtube.com/watch?atch?v=haAV-_6R5VU), [2ndInHomog].

**Motivation**

Ordinary differential equations (ODEs) are used to mathematically model a range of phenomena from science and engineering. ODEs naturally arise when making simple physical assumptions about the model under consideration. By computing a solution to an ODE we can make precise predictions about future and past states of the model of interest.

We shall discuss the case

$$ay'' + by' + cy = f(x)$$

where  $a$ ,  $b$  and  $c$  are given constants and  $f$  is a reasonably simple type of function. These kinds of problems arise in the study of vibrating systems with  $F$  representing an external forcing function.

**Example**

Solve the ODEs:

(a)  $y'' - 5y' + 6y = 2x + 3$ ; (b)  $y'' - 5y' + 6y = 12e^{5x}$ ; (c)  $y'' - 5y' + 6y = 10e^{2x}$ .

**The bigger picture**

To solve the ODE  $ay'' + by' + cy = f(x)$

- Solve the corresponding homogeneous problem  $ay'' + by' + cy = 0$  for  $y_H$ .
- Construct a particular solution  $y_p$  to the nonhomogeneous ODE, based on the form of  $f(x)$
- The general solution is then  $y = y_p + y_H$ .

**Exercises**

Solve the following ODEs:

(a)  $y'' + 3y' + 2y = e^x$ ; (b)  $y'' + 2y' + y = 2x$  (c)  $y'' + 4y' + 5y = 3$ .

[Ans: (a)  $y(x) = Ae^{-x} + Be^{-2x} + e^{-x}/6$ ; (b)  $y(x) = Ae^{-x} + Bxe^{-x} + 2e^{-x} - 4$ ;  
(c)  $y(x) = e^{-2x} (A \cos x + B \sin x) + 3/5$ .]

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## 5.6 Variation of constants / parameters

View this lesson at <http://www.youtube.com/watch?v=rhKoNLjOhJg>, [VarParam].

### Motivation

The method of variation of constants is a very powerful technique for solving 2nd-order, linear ODEs. It is useful when the method of undetermined coefficients does not apply.

We shall discuss the case

$$ay'' + by' + cy = f(x)$$

where  $a$ ,  $b$  and  $c$  are given constants and  $f$  is a given function. These kinds of problems arise in the study of vibrating systems.

### Example

Solve the ODE

$$y'' + 4y = \operatorname{cosec} 2x.$$

### The bigger picture

To solve the ODE  $ay'' + by' + cy = f(x)$

- Formulate the general solution  $y_H$  to the homogeneous ODE  $ay'' + by' + cy = 0$ , that is,  $y_H = Ay_1(x) + By_2(x)$  with  $y_1$  and  $y_2$  being linearly independent.
- Construct a particular solution  $y_p$  to the inhomogeneous ODE  $ay'' + by' + cy = f(x)$  by using the  $y_1$  and  $y_2$  in

$$y_p = -y_1(x) \int \frac{y_2(x)f(x)}{aW(x)} dx + y_2(x) \int \frac{y_1(x)f(x)}{aW(x)} dx$$

where  $W(x) := y_1(x)y_2'(x) - y_2(x)y_1'(x)$ .

- The final (general) solution will have the form  $y = y_H + y_p$ .

### Exercises

Solve:

$$y'' + y = \sec x.$$

[Ans:  $y(x) = A \cos x + B \sin x + (\cos x) \ln |\cos x| + x \sin x$ .]

# 6 Laplace transforms and applications

## 6.1 Introduction to the Laplace transform

View this lesson at <http://www.youtube.com/watch?v=a7BzFOlJtKw>, [LT1].

### Motivation

Laplace transforms are used in solving initial value problems that involve linear, ordinary differential equations with constant coefficients. These types of problems usually arise in modelling of phenomena. Laplace transforms offer an advantage over other solution methods to initial value problems as they streamline the process and can easily deal with discontinuous forcing functions.

### Example

Define and denote the Laplace transform of a function  $f = f(t)$  by

$$\mathcal{L}\{f(t)\} := \int_0^{\infty} e^{-st} f(t) dt = F(s). \quad (7.1.1)$$

1) From the above definition, calculate

$$\mathcal{L}\{e^{-2t}\}, \quad \text{for all } s > -2.$$

2) If  $a$  is a constant then use (i) to make a conjecture concerning

$$\mathcal{L}\{e^{-at}\}, \quad \text{for all } s > -a.$$

### The bigger picture

- While it is possible to directly use the definition of the Laplace transform to perform calculations, it is standard practice to use a table of Laplace transforms. This can save a lot of time!
- The inverse of the Laplace transform is denoted by  $\mathcal{L}^{-1}$ . Roughly speaking, it “undoes” the Laplace transform and vice-versa. From (ii) we deduce

$$\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\} = e^{-3t}.$$

## Exercises

- a) From the above definition (7.1.1), calculate

$$\mathcal{L}\{5t + 3\}, \quad \text{for all } s > 0.$$

- b) If  $c$  and  $d$  are constants, then use (a) to make a conjecture about what

$$\mathcal{L}\{ct + d\}, \quad \text{for all } s > 0.$$

should be.

- c) Use (b) to calculate

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}.$$

[Ans: (a)  $5/s^2 + 3/s$ ; (b)  $c/s^2 + d/s$ ; (c)  $t$ .]

## 6.2 Laplace transforms + the first shifting theorem

View this lesson at <http://www.youtube.com/watch?v=WpUbbsULId8>, [LT2].



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**Motivation**

The first shifting theorem provides a convenient way of calculating the Laplace transform of functions that are of the form

$$f(t) := e^{-at}g(t)$$

where  $a$  is a constant and  $g$  is a given function.

**Example**

Consider the first shifting theorem

$$\mathcal{L}\{e^{-at}g(t)\} = G(s + a) \quad \text{where} \quad G(s) = \mathcal{L}\{g(t)\}.$$

Apply the result to calculate:

a)  $\mathcal{L}\{e^{2t}t^3\};$

b)  $\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^3}\right\}.$

**The bigger picture**

- For the Laplace transform via the first shifting theorem, calculate  $G(s)$  from  $g(t)$  and then shift it by  $a$  units in an appropriate manner.
- For the inverse Laplace transform via the first shifting theorem, identify  $a$  and  $G(s)$  from  $G(s + a)$ . Then calculate  $g(t)$  from  $G(s)$  and multiply by  $e^{-at}$ .

**Exercises**

Calculate

a)  $\mathcal{L}\{e^t \sin t\};$

b)  $\mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2}\right\}.$

[Ans: (a)  $1/[1 + (s - 1)^2]$ ; (b)  $te^{-2t}$ .]

### 6.3 Laplace transforms + the 2nd shifting theorem

View this lesson at <http://www.youtube.com/watch?v=UyoxbrO3eFU>, [LT3].

#### Motivation

Laplace transforms offer an advantage over other solution methods to initial value problems as they streamline the process and can easily deal with discontinuous forcing functions.

The second shifting theorem gives us a way of computing the Laplace transform of certain types of discontinuous functions.

#### Example

Consider the second shifting theorem

$$\mathcal{L}\{u(t-c)g(t-c)\} = e^{-cs}G(s) \quad \text{where } G(s) = \mathcal{L}\{g(t)\} \\ \text{and } u(t-c) \text{ is the Heaviside step function.}$$

Apply the result to calculate:

a)  $\mathcal{L}\{u(t-1)(t-1)^2\};$

b)  $\mathcal{L}^{-1}\left\{\frac{2e^{-4s}}{s^3}\right\}.$

#### The bigger picture

- For the Laplace transform via the second shifting theorem, identify  $c$  and  $g(t)$  and then calculate  $G(s)$ .
- For the inverse Laplace transform via the second shifting theorem, identify  $c$  and  $G(s)$ . Then calculate  $g(t)$  from  $G(s)$  and shift it to form  $g(t-c)$ .

#### Exercises

Apply the second shifting theorem to calculate:

a)  $\mathcal{L}\{u(t-2)(t-2)^3\};$

b)  $\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^5}\right\}.$

[Ans: (a)  $6e^{-2s} / s^4$ ; (b)  $u(t-2)(t-2)^4/24$ ].



## 6.4 Laplace transforms + differential equations

View this lesson at [http://www.youtube.com/watch?v=SNVfyQ4\\_avY](http://www.youtube.com/watch?v=SNVfyQ4_avY), [LT4].

### Motivation

Laplace transforms offer an advantage over other solution methods to initial value problems as they streamline the process and can easily deal with discontinuous forcing functions.

### Example

Consider the transform of derivatives identities

$$\mathcal{L}\{y'(t)\} = sY(s) - y(0), \quad \text{where } Y(s) = \mathcal{L}\{y(t)\}$$

$$\mathcal{L}\{y''(t)\} = s^2Y(s) - sy(0) - y'(0),$$

Apply the result to solve the initial value problem:

$$y'' + y = u(t - 1), \quad y(0) = 1, \quad y'(0) = 0.$$



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**The bigger picture**

When applying Laplace transforms to solve initial value problems:

1. take the Laplace transform of both sides of the differential equation
2. apply the transform of derivatives identities
3. incorporate the initial conditions
4. solve for  $Y(s)$
5. use inverse Laplace transforms to produce the solution  $y(t)$ .

**Exercises**

Apply the transform of derivatives identities to solve:

$$y'' + y = 2u(t - 1), \quad y(0) = 0, \quad y'(0) = 1.$$

[Ans:  $y(t) = \sin t + 2u(t - 1)[1 - \cos(t - 1)]$ .]

## LAPLACE TRANSFORMS

$$\mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t) dt = F(s)$$

$f(t)$	$F(s)$
1	$1/s$
$t$	$1/s^2$
$t^m,$	$m!/s^{m+1}$
$t^\nu, (\nu > -1)$	$\Gamma(\nu + 1)/s^{\nu+1}$
$e^{-at}$	$1/(s + a)$
$\sin bt$	$b/(s^2 + b^2)$
$\cos bt$	$s/(s^2 + b^2)$
$\sinh bt$	$b/(s^2 - b^2)$
$\cosh bt$	$s/(s^2 - b^2)$
$\sin bt - bt \cos bt$	$2b^3/(s^2 + b^2)^2$
$\sin bt + bt \cos bt$	$2bs^2/(s^2 + b^2)^2$
$t \sin bt$	$2bs/(s^2 + b^2)^2$
$te^{-at}$	$1/(s + a)^2$
$u(t - c)$	$e^{-cs}/s$
$e^{-at}g(t)$	$G(s + a)$
$tg(t)$	$-G'(s)$
$u(t - c)g(t - c)$	$e^{-cs}G(s)$
$g'(t)$	$sG(s) - g(0)$
$g''(t)$	$s^2G(s) - sg(0) - g'(0)$
$g'''(t)$	$s^3G(s) - s^2g(0) - sg'(0) - g''(0)$
$\int_0^t g(\tau) d\tau$	$G(s)/s.$

In this table,  $a$  and  $b$  are any real numbers,  $c$  is any non-negative real number,  $m$  is any non-negative integer,  $\nu$  is any real number larger than  $-1$ ,  $\Gamma(\nu + 1) = \int_0^{\infty} e^{-x} x^\nu dx$  and  $u(t)$  is defined by  $u(t) = 0$ , for  $t < 0$ ,  $u(0) = 1/2$ ;  $u(t) = 1$  for  $t > 0$ .

# 7 Fourier series

The **Fourier series** of a function  $f = f(t)$  with period  $2L$  is defined by

$$Sf(t) := a_0 + \sum_{n=1}^{\infty} \left[ a_n \cos \frac{n\pi t}{L} + b_n \sin \frac{n\pi t}{L} \right]$$

where

$$a_0 := \frac{1}{2L} \int_{-L}^L f(t) dt$$

$$a_n := \frac{1}{L} \int_{-L}^L f(t) \cos \frac{n\pi t}{L} dt$$

$$b_n := \frac{1}{L} \int_{-L}^L f(t) \sin \frac{n\pi t}{L} dt$$

and  $f$  and  $f'$  are piecewise continuous on  $(-L, L)$ .

## 7.1 Introduction to Fourier series

View this lesson at <http://www.youtube.com/watch?v=3Ai3dKVPLcQ>, [FS1].

### Motivation

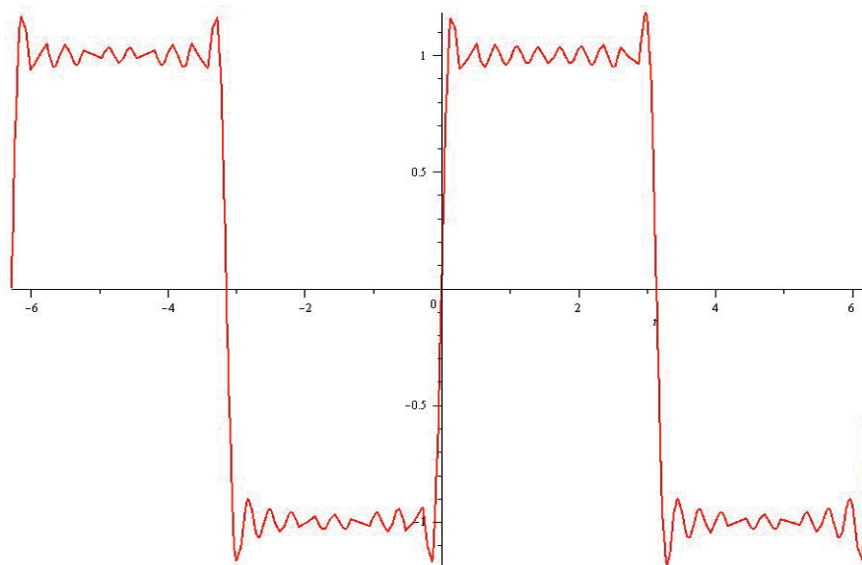
Fourier series are used in solving differential equations that arise in the study of heat flow and vibrations. Fourier series provide a means of approximating discontinuous periodic functions over intervals (rather than just near certain points).

### Example

Calculate the Fourier series of

$$f(t) := \begin{cases} 1, & \text{for } 0 < t < \pi; \\ 0, & \text{for } t = 0, \pm\pi; \\ -1, & \text{for } -\pi < t < 0; \end{cases} \quad (8.1.1)$$

with  $f(t) = f(t + 2\pi)$  for all  $t$ .



Maple has been used to plot the 10th partial sum of Fourier series of  $f$ .



**The bigger picture**

- Integration is required to compute the Fourier coefficients  $a_0$ ,  $a_n$  and  $b_n$  and the calculations will simplify when  $f$  is odd or even.
- If  $f$  is continuous at a point  $c$  then  $Sf(c) = f(c)$ .
- If  $f$  has a jump discontinuity at  $c$  then  $Sf(c) = [f(c^+) + f(c^-)]/2$ .

**Exercises**

Calculate the Fourier series of

$$f(t) := \begin{cases} 1, & \text{for } 0 < t < 1; \\ 0, & \text{for } -1 < t < 0; \end{cases} \quad (8.1.2)$$

with  $f(t) = f(t+2)$  for all  $t$ . [Ans:  $Sf(t) = \frac{1}{2} + \sum_{k=1}^{\infty} \frac{2}{(2k-1)\pi} \sin(2k-1)\pi t$ . Note that  $f$  is neither even nor odd, but can you see how to write it in terms of an odd function plus a constant? This can help to simplify the calculations!]

**7.2 Odd + even functions + Fourier series**

View this lesson at <http://www.youtube.com/watch?v=1BjOvv6G3dA>, [FSOE].

**Motivation**

Simplifications occur in the construction of Fourier series when dealing with odd or even functions.

**Example**

Let

$$f(x) := x, \quad 0 < x \leq 1.$$

Extend  $f$  as an odd (even) function  $f_o$  ( $f_e$ ) with period 2 and sketch their graphs on  $[-3, 3]$ . Find  $Sf_e$  and  $Sf_o$ . Show

$$\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = \frac{\pi^2}{8}.$$

**The bigger picture**

- Integration is required to compute the Fourier coefficients  $a_0$ ,  $a_n$  and  $b_n$  and the calculations will simplify when  $f$  is odd or even.
- Simplifications occur when dealing with odd or even functions.
- If  $f$  is continuous at a point  $c$  then  $Sf(c) = f(c)$ .
- If  $f$  has a jump discontinuity at  $c$  then  $Sf(c) = [f(c^+) + f(c^-)]/2$ .

**Exercises**

Calculate the Fourier series of

$$f(x) := \begin{cases} 3, & \text{for } 0 \leq x < 1, \\ -3, & \text{for } -1 < x < 0, \end{cases}$$

where  $f(x) = f(x + 2)$ . What is the value of  $Sf(0)$  and  $Sf(1/2)$ ?

$$[\text{Ans: } Sf(x) = \sum_{k=1}^{\infty} \frac{12}{(2k-1)\pi} \sin(2k-1)\pi x; Sf(0) = 0; Sf(1/2) = 3.]$$

**7.3 More on Fourier series**

View this lesson at <http://www.youtube.com/watch?v=toBJWQGPRB4>, [FSR].

**Motivation**

Fourier series are used in solving differential equations that arise in the study of heat flow and vibrations. Fourier series provide a means of approximating discontinuous periodic functions over intervals (rather than just near certain points).



**Example**


Let  $f(x) = |x|$  for  $x \in [-\pi, \pi]$  with  $f(x + 2\pi) = f(x)$ .

- 1) Make a sketch of  $f$  on the interval  $[-3\pi, 3\pi]$ .
- 2) Calculate the Fourier series of  $f$ .
- 3) Let

$$g(x) = \begin{cases} 1, & 0 < x < \pi \\ -1, & -\pi < x < 0, \end{cases}$$

with  $g(x + 2\pi) = g(x)$ . By differentiating the Fourier series in b) ii) (or otherwise) obtain the Fourier series of  $g$ .

- 4) What does the Fourier series of  $g$  converge to at  $x = \pi$ ?
- 5) Use the Fourier series for  $g$  to obtain a series representation for  $\pi/4$ .




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**The bigger picture**

- Integration is required to compute the Fourier coefficients  $a_0$ ,  $a_n$  and  $b_n$  and the calculations will simplify when  $f$  is odd or even.
- If  $f$  is continuous at a point  $c$  then  $Sf(c) = f(c)$ .
- If  $f$  has a jump discontinuity at  $c$  then  $Sf(c) = [f(c^+) + f(c^-)]/2$ .
- Simplifications occur when dealing with odd or even functions.

**Exercises**

Calculate the Fourier series of

$$f(x) := 3(x + 1), \quad -2 < x < 2$$

where  $f(x) = f(x + 4)$ .

$$[\text{Ans: } Sf(x) = 3 - 12 \sum_{n=1}^{\infty} \frac{(-1)^n}{n\pi} \sin \frac{n\pi x}{2}.]$$

## 7.4 Applications of Fourier series to ODEs

View this lesson at <http://www.youtube.com/watch?v=70KZhL7eVN8>, [FS2].

**Motivation**

Fourier series can be useful in solving differential equations that arise in the study of vibrations. They provide a means of approximating discontinuous periodic “forcing” functions over intervals (rather than just near certain points).

**Example**

Calculate the solution of the initial value problem

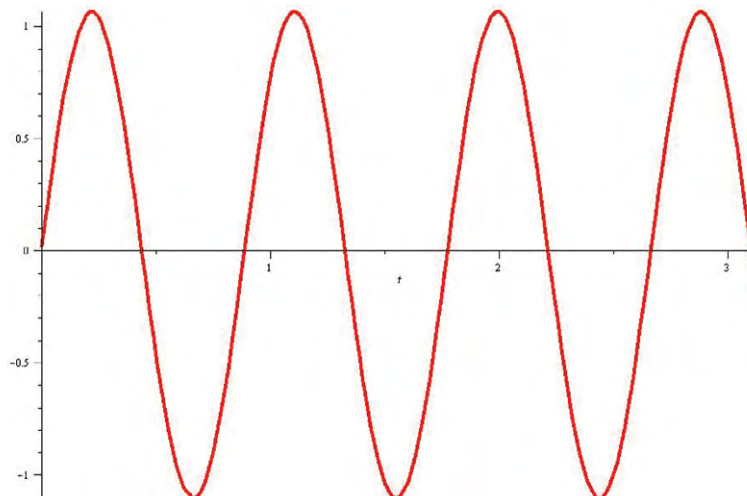
$$x'' + 50x = f(t), \quad x(0) = 0, \quad x'(0) = 0$$

where

$$f(t) := \begin{cases} 1, & \text{for } 0 < t < \pi; \\ 0, & \text{for } t = 0, \pi, 2\pi; \\ -1, & \text{for } \pi < t < 2\pi; \end{cases} \quad (8.4.3)$$

with  $f(t) = f(t + 2\pi)$  for all  $t$ . Hint: the Fourier sine series of  $f$  is

$$\sum_{k=1}^{\infty} \frac{4}{(2k-1)\pi} \sin(2k-1)t.$$



Maple has been used to plot the solution.

**The bigger picture**

When applying Fourier series to solve initial value problems:

- If it is not already given, then calculate the Fourier series of the forcing function.
- Apply the basic theory of differential equations to calculate  $y_H$  and  $y_p$ .

# 8 PDEs & separation of variables

## 8.1 Deriving the heat equation

View this lesson at <http://www.youtube.com/watch?v=K-CI61wV6JQ>, [HeartDerive].

### Motivation

In the early 1800s, J. Fourier began a mathematical study of heat. A deeper understanding of heat flow had significant applications in science and within industry. A basic version of Fourier's efforts is the problem

$$\alpha^2 u_{xx} = u_t; \quad u(0, t) = 0 = u(L, t); \quad u(x, 0) = f(x)$$

where:  $u(x, t)$  is the temperature at position  $x$  at time  $t$ ;  $\alpha^2$  is a constant; and  $f$  is a given function.



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**Example**

Derive the heat equation

$$u_t = \alpha^2 u_{xx} \quad (9.1.1)$$

where  $\alpha^2 := \kappa/\rho c$  called the thermal diffusivity (in (length)<sup>2</sup>/time) and  $\kappa$ ,  $\rho$  and  $c$  are positive constants that depend on the material of the bar.

**The bigger picture**

Our assumptions and method may be summarized as:

- The total amount of heat  $H = H(t)$  in  $D$  (say, in calories) is

$$H(t) = \int_{x_0}^{x_1} c\rho u(x, t) \, dx.$$

- Fourier's law of heat flow says that heat flows from hotter regions to colder regions and the flow rate is proportional to  $u_x$

$$\frac{dH}{dt} = -\kappa u_x(x_0, t) - (-\kappa u_x(x_1, t)).$$

- Differentiation then gives us two expressions for  $dH/dt$  which can be equated to form the heat equation.

## 8.2 Heat equation & separation of variables

View this lesson at <http://www.youtube.com/watch?v=Jfn6A1jZF5g>, [SepVar].

**Motivation**

The method of “separation of variables” is a powerful technique that enables us to solve linear partial differential equations that arise in the modelling of various physical processes. In some problems, Fourier series are an important element of the approach.

**Example**

Solve

$$4u_{xx} = u_t, \quad 0 < x < \pi, \quad t > 0; \quad (9.2.2)$$

$$u(0, t) = 0, \quad t > 0; \quad (9.2.3)$$

$$u(\pi, t) = 0, \quad t > 0; \quad (9.2.4)$$

$$u(x, 0) = f(x), \quad 0 \leq x \leq \pi. \quad (9.2.5)$$

with  $f(x) = \sin x - 2 \sin 2x$ .**The bigger picture**

Our method may be summarized as:

- Assume a solution is in the form  $u(x, t) = X(x)T(t)$ .
- Formulate and solve two ODEs for  $X$  and  $T$  determining which values of the separation constant  $\gamma$  are relevant.
- If required incorporate Fourier series.

**Exercises**

Solve (9.2.2)–(9.2.5) but with  $f(x) = -\sin 2x + 3 \sin 5x$  as the initial temperature. [Ans:  $(u(x, t) = -e^{-16t} \sin 2x + 3e^{-200t} \sin 5x)$ .]

**8.3 Heat equation & Fourier series**View this lesson at <http://www.youtube.com/watch?v=4jYgpHLP6xs>, [HeartFourier].**Motivation**

The method of “separation of variables” is a powerful technique that enables us to solve linear partial differential equations that arise in the modelling of various physical processes. In many problems Fourier series are an important element of the solution method.

**Example**

Solve

$$4u_{xx} = u_t, \quad 0 < x < \pi, \quad t > 0; \quad (9.6)$$

$$u(0, t) = 0, \quad t > 0; \quad (9.7)$$

$$u(\pi, t) = 0, \quad t > 0; \quad (9.8)$$

$$u(x, 0) = x, \quad 0 \leq x \leq \pi. \quad (9.9)$$

In previous lectures, we have applied the method of separation of variables to the above problem, obtaining

$$u(x, t) = \sum_{n=1}^{\infty} b_n e^{-4n^2 t} \sin nx.$$

where  $b_n$  are constants to be determined. In particular, (?) yields

$$u(x, 0) = x = \sum_{n=1}^{\infty} b_n \sin nx, \quad 0 \leq x \leq \pi.$$

**The bigger picture**

Our method may be summarized as:

- Assume a solution is in the form  $u(x, t) = X(x)T(t)$ .
- Formulate and solve two ODEs for  $X$  and  $T$  determining which values of the separation constant  $\gamma$  are relevant from the associated boundary conditions.
- If required incorporate Fourier series.

**8.4 Wave equation and Fourier series**

View this lesson at <http://www.youtube.com/watch?v=4UtGyd86FjI>, [Wave].



**Motivation**

The method of “separation of variables” is a powerful technique that enables us to solve linear partial differential equations that arise in the modelling of various physical processes. In some problems, Fourier series are an important element of the approach. We now apply these ideas to solve the wave equation.

**Example**

Solve

$$9u_{xx} = u_{tt}, \quad 0 < x < \pi, \quad t > 0; \quad (9.4.10)$$

$$u(0, t) = 0, \quad u(\pi, t) = 0, \quad t > 0; \quad (9.4.11)$$

$$u(x, 0) = f(x), \quad u_t(x, 0) = 0, \quad 0 \leq x \leq \pi. \quad (9.4.12)$$

where

$$f(x) := \begin{cases} x, & \text{for } 0 \leq x \leq \pi/2; \\ \pi - x, & \text{for } \pi/2 \leq x \leq \pi. \end{cases}$$

**The bigger picture**

Our method may be summarized as:

- Assume a solution is in the form  $u(x, t) = X(x)T(t)$ .
- Formulate and solve two ODEs for  $X$  and  $T$  determining which values of  $\gamma$  are relevant from the associated boundary conditions.
- If required incorporate Fourier series.

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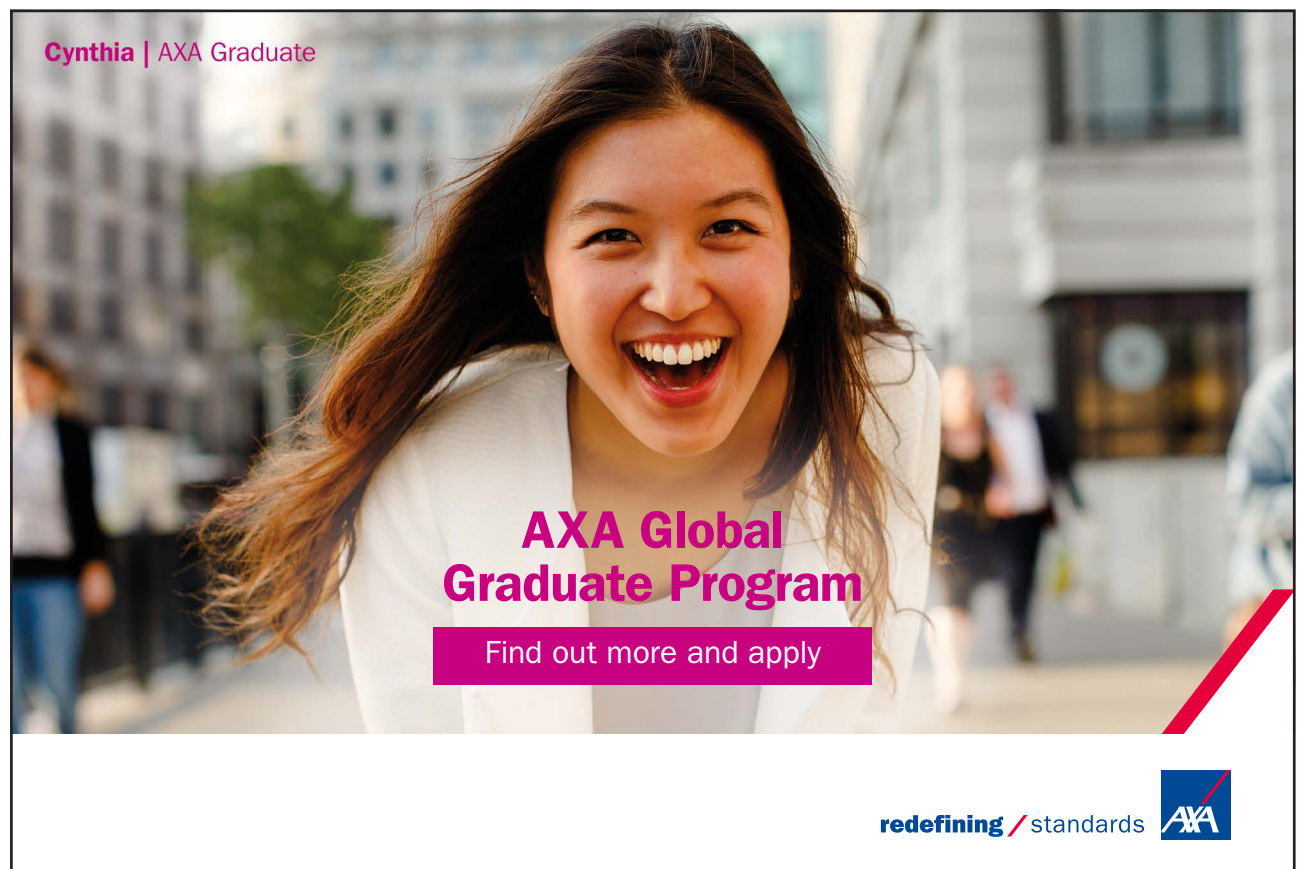
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