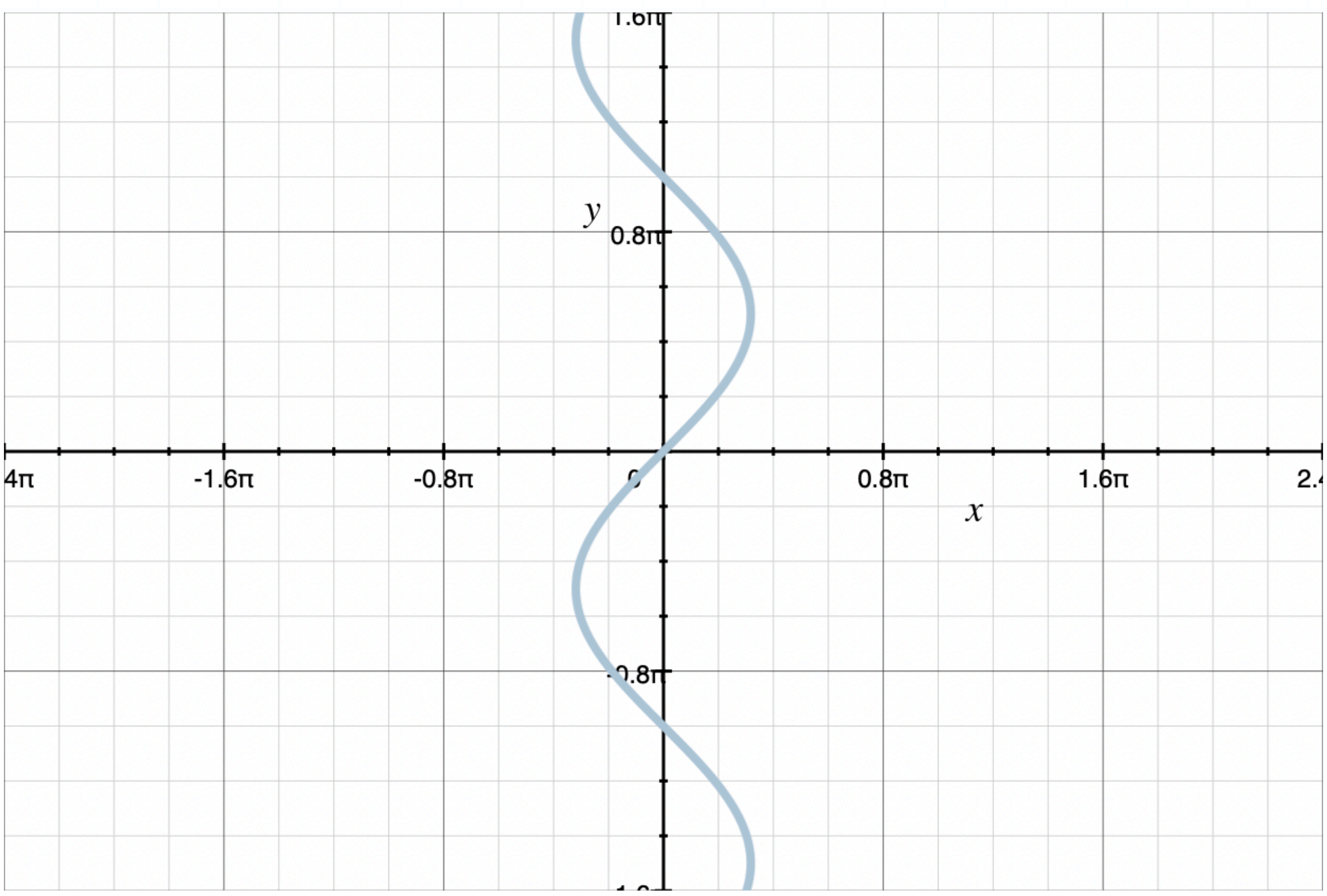
Inverse trig functions

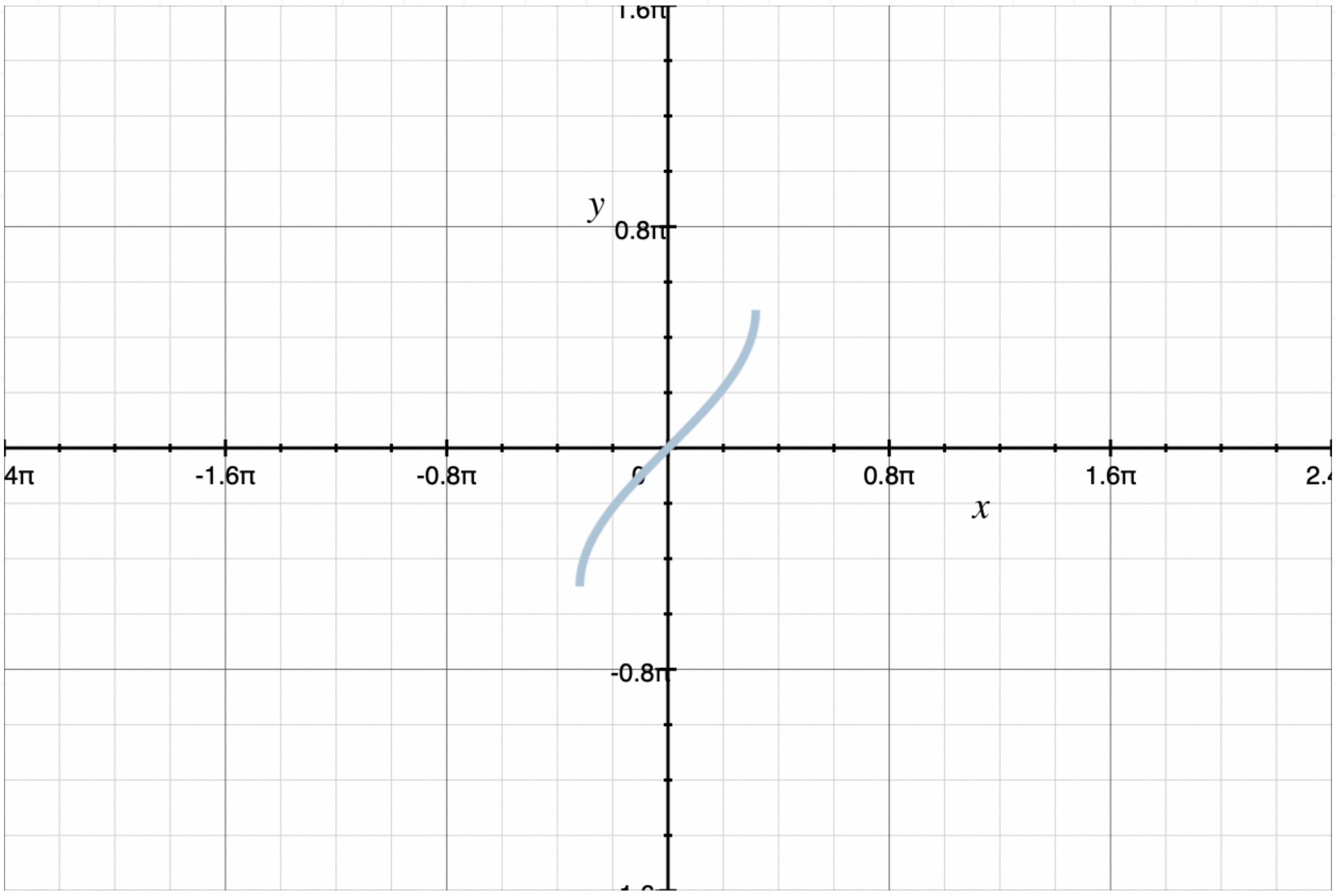
In the last lesson we looked at inverse trig relations, which were the equations we got when we flipped the variables in a standard trig function. In other words, starting with the standard trig function $y = \sin x$, we got its inverse trig relation by swapping the variables to get $x = \sin y$.

Remember though that we called $x = \sin y$ a "relation" because it wasn't a function, because it didn't pass the Vertical Line Test. The way that we turn the inverse trig relation into an inverse trig function is by limiting the range of the relation to just one period.

So in the last lesson we showed that the graph of $y = \sin^{-1} x$ was



But if we want to turn this relation into a function, then we limit the range to $-\pi/2 \le y \le \pi/2$. When we restrict the range that way, the graph gets cut down to just one portion:



Notice how this curve now represents a function, since it could pass the Vertical Line Test. Again, that means we could draw any perfectly vertical line anywhere along this curve, and it would only intersect the curve at one point. Therefore, if we name the curve this way,

$$y = \sin^{-1} x \text{ for } -\frac{\pi}{2} \le y \le \frac{\pi}{2}$$

then we can call it an **inverse trig function**, because it's now a function that will pass the Vertical Line Test. Sometimes we indicate an inverse trig function, as opposed to an inverse trig relation, with a capital letter. So if you see

$$y = \text{Sin}^{-1} x \text{ or } y = \text{Arcsin} x$$

it indicates an inverse trig function where the range is restricted. But if you see just

$$y = \sin^{-1} x \text{ or } y = \arcsin x$$

without any inequality restricting the range, it means we're dealing with the inverse relation. Of course, if we add an inequality to restrict the range, even if we use a lower case letter, then we're still defining the inverse trig function.

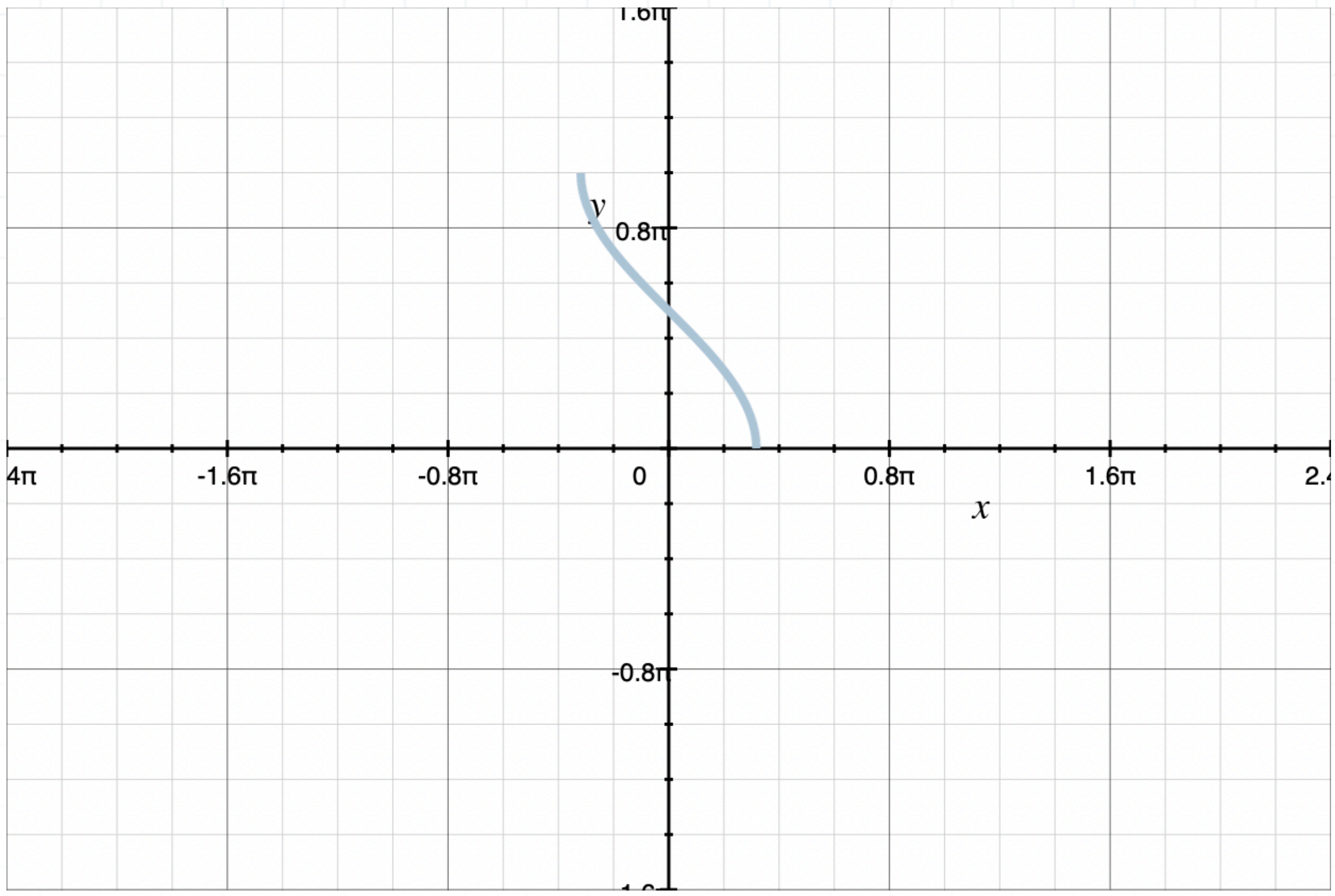
The other inverse trig functions

We've already defined the inverse sine function as

$$y = \sin^{-1} x \text{ for } -\frac{\pi}{2} \le y \le \frac{\pi}{2}$$

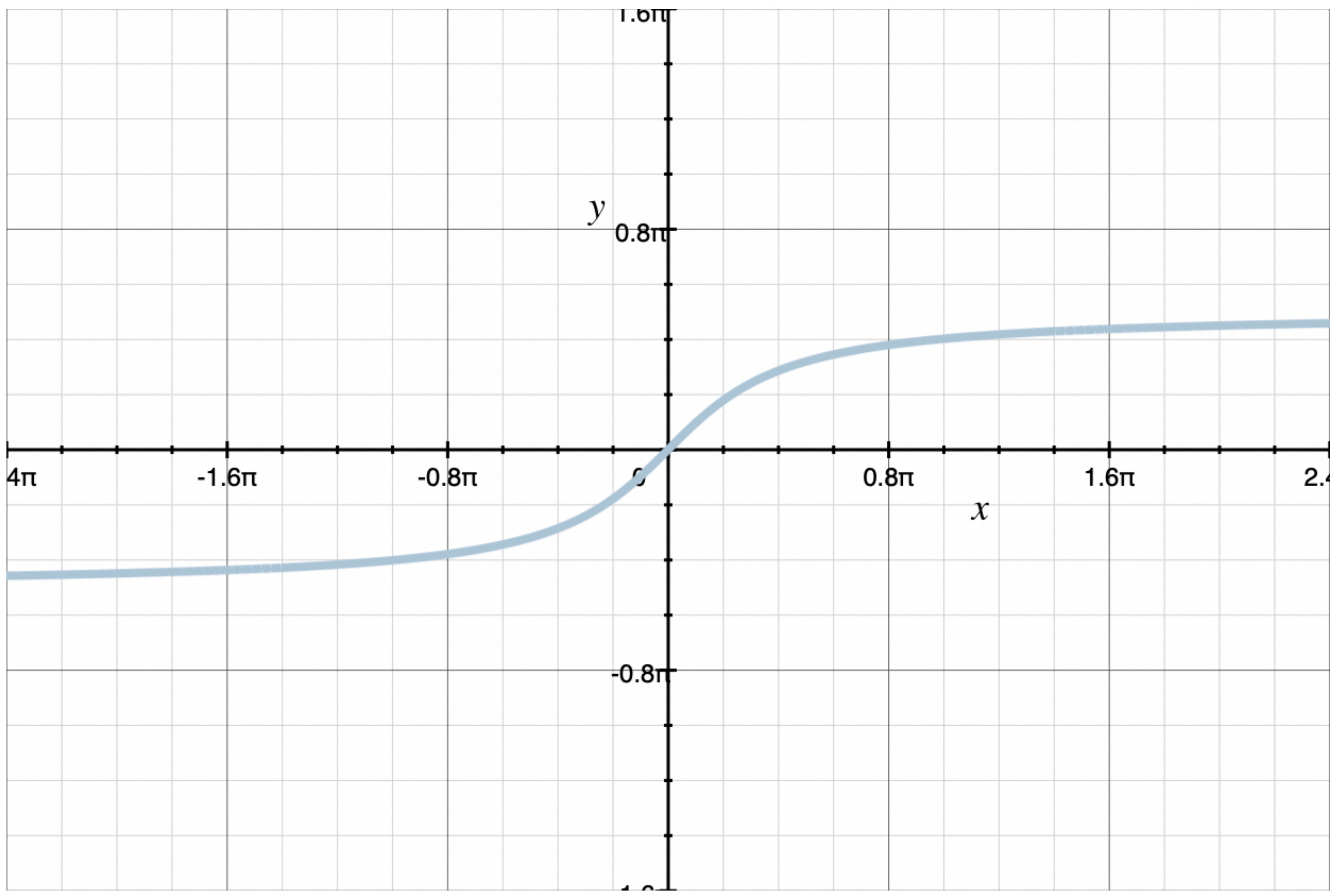
Now we want to define the inverse cosine and tangent functions. The inverse cosine function and its graph are

$$y = \cos^{-1} x \text{ for } 0 \le y \le \pi$$



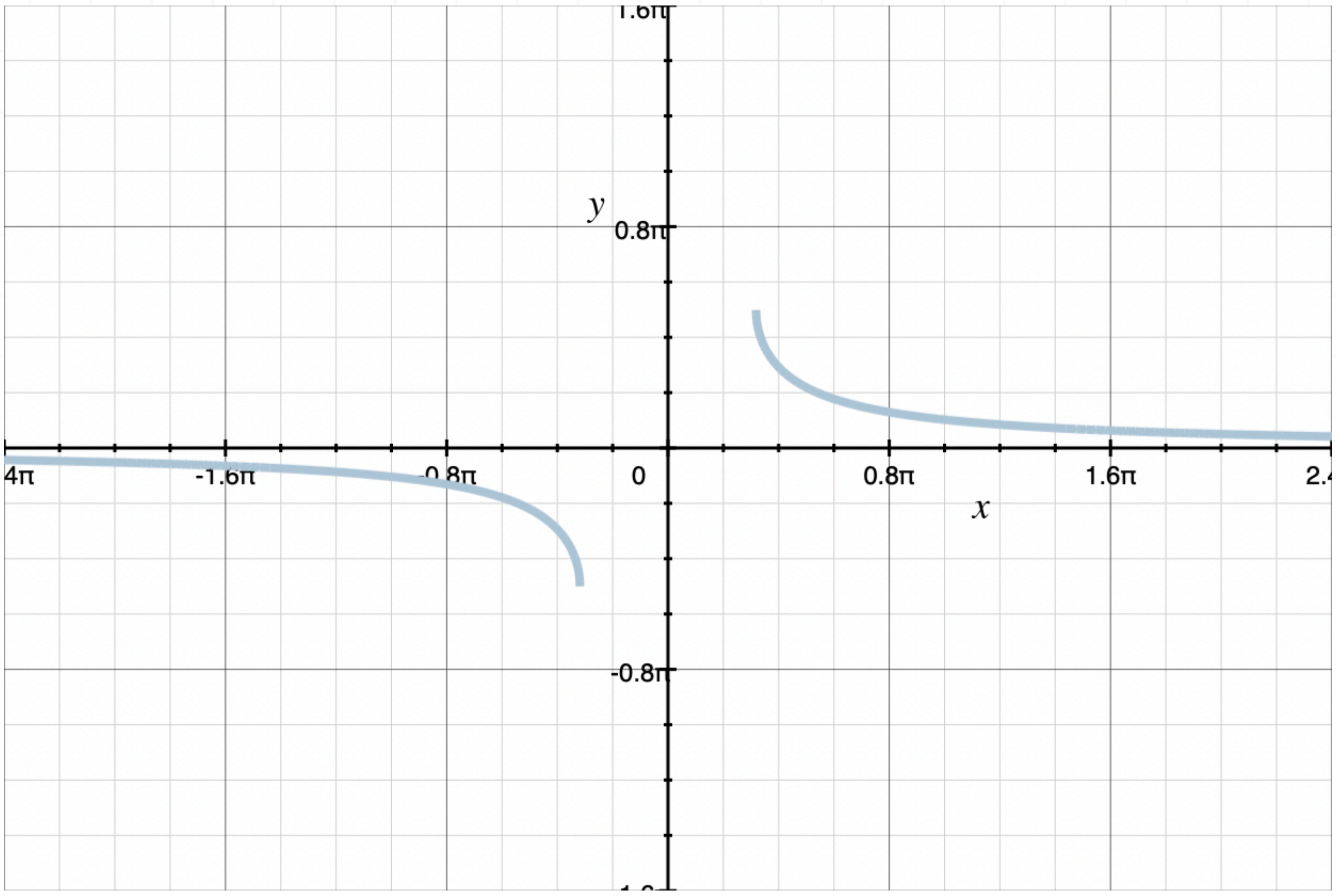
and the inverse tangent function and its graph are

$y = \tan^{-1} x$ for $-\dfrac{\pi}{2} < y < \dfrac{\pi}{2}$

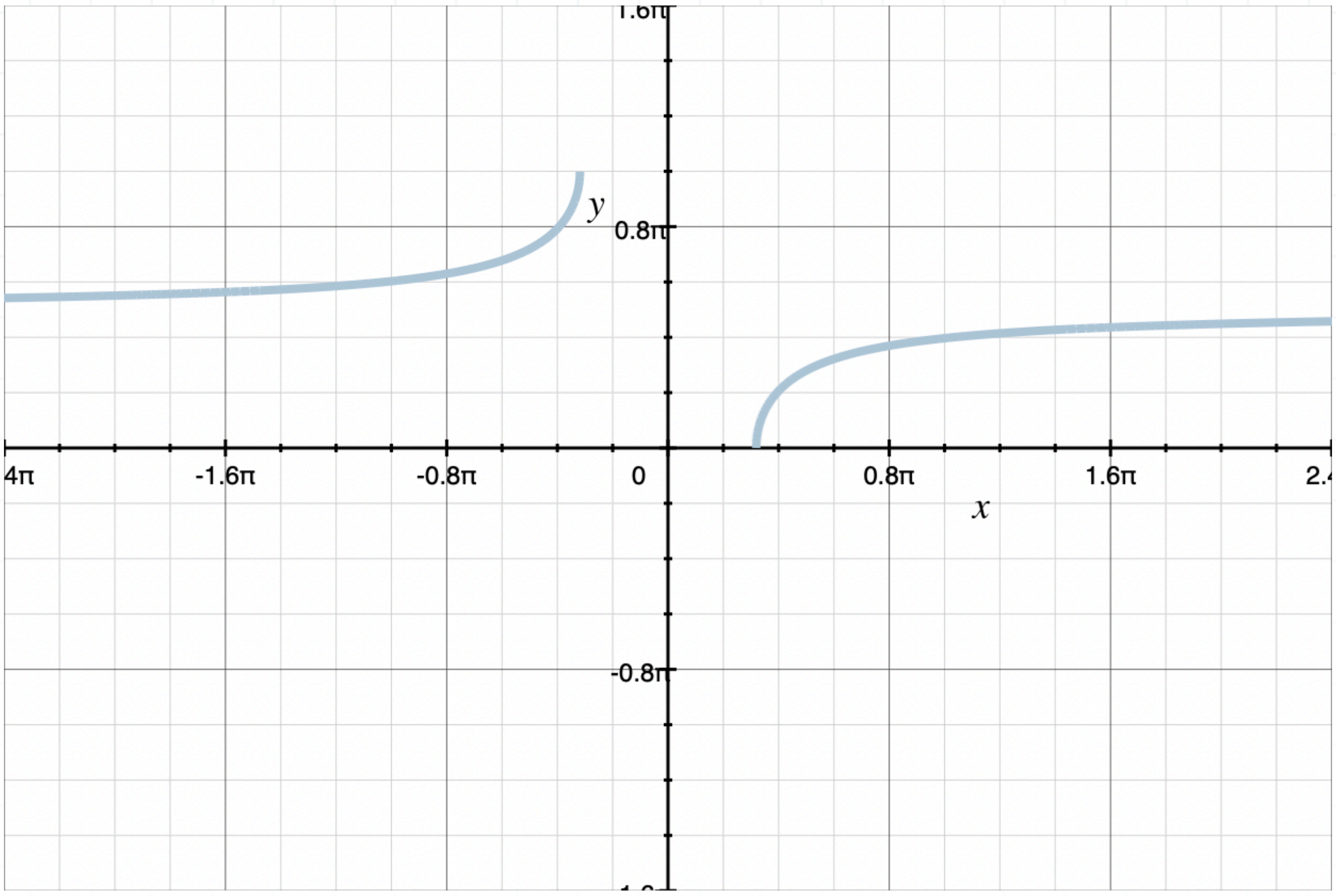


We can evaluate the other three trig functions as the reciprocals of the inverse sine, cosine, and tangent functions.

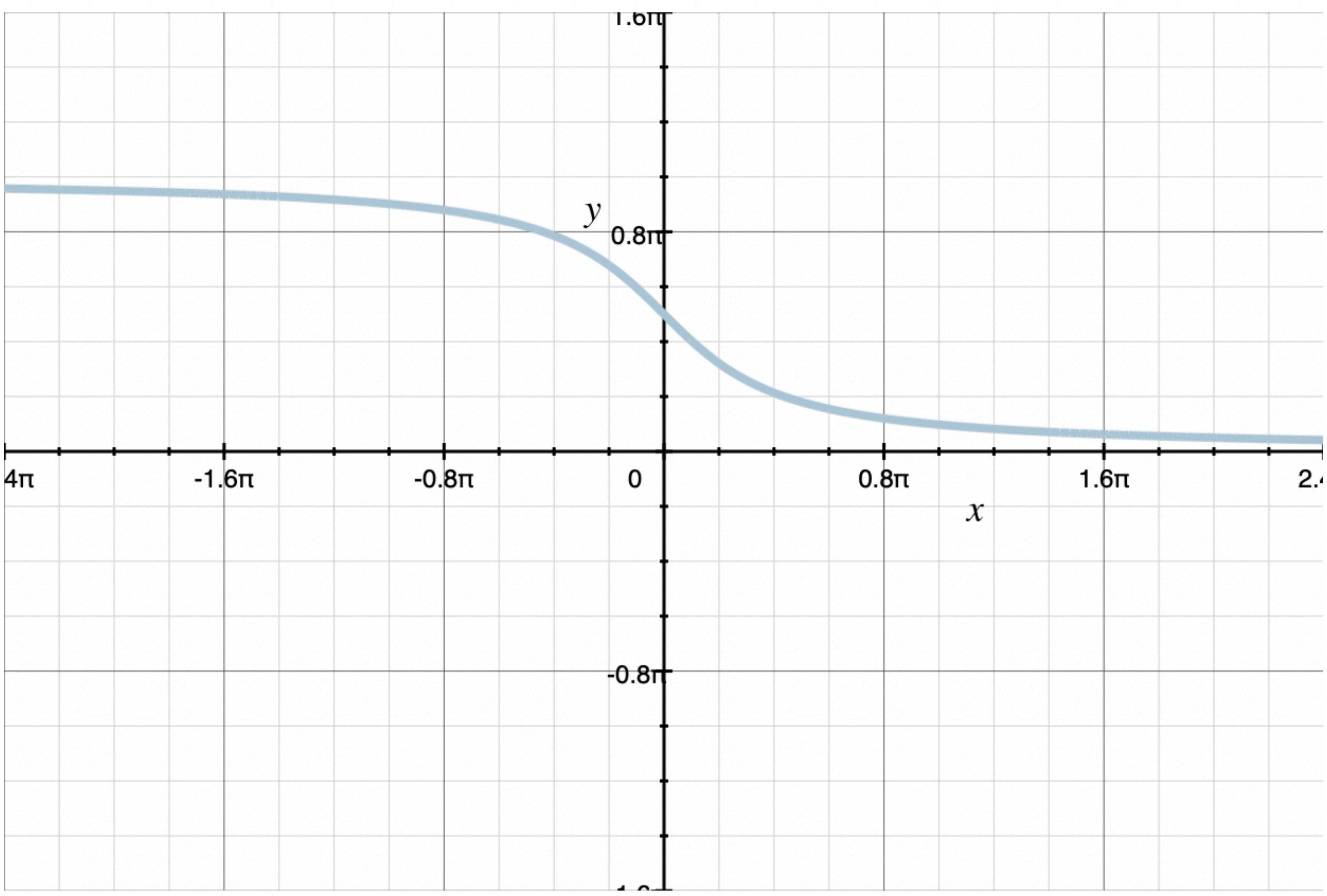
$$y = \csc^{-1} x = \frac{1}{\sin^{-1} x} \text{ for } -\frac{\pi}{2} \le y \le \frac{\pi}{2}, \text{ except at } y = 0$$



$$y = \sec^{-1} x = \frac{1}{\cos^{-1} x} \text{ for } 0 \le y \le \pi, \text{ except at } y = \frac{\pi}{2}$$



$$y = \cot^{-1} x = \frac{1}{\tan^{-1} x} \text{ for } 0 < y < \pi$$



Here's a summary of the domain and range of all six inverse trig functions.

Inverse function	**Domain**	**Range**
$y = \sin^{-1} x$	$x = [-1,1]$	$y = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$y = \cos^{-1} x$	$x = [-1,1]$	$y = [0,\pi]$
$y = \tan^{-1} x$	$x = (-\infty, \infty)$	$y = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
$y = \csc^{-1} x$	$x = (-\infty, -1] \cup [1,\infty)$	$y = \left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$
$y = \sec^{-1} x$	$x = (-\infty, -1] \cup [1,\infty)$	$y = \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$
$y = \cot^{-1} x$	$x = (-\infty, \infty)$	$y = (0,\pi)$

Remember that to evaluate inverse trig functions, we need to "think backwards," meaning that we'll start from the value of the trig function, and then figure out the angle to which it corresponds.

Let's do an example with inverse sine.

Example

Find the value of the inverse sine function.

$$\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$$

If we look at the unit circle, we can see that the sine function is $-\sqrt{2}/2$ when $\theta = 5\pi/4$ and when $\theta = 7\pi/4$. But because we're dealing with the inverse sine function, we only want an angle in the interval $[-\pi/2,\pi/2]$.

Both $\theta = 5\pi/4$ and $\theta = 7\pi/4$ fall outside the interval $[-\pi/2,\pi/2]$, which means we'll need to find an angle coterminal with either $\theta = 5\pi/4$ or $\theta = 7\pi/4$ that falls within $[-\pi/2,\pi/2]$.

Remember that the interval $[-\pi/2,\pi/2]$ defines the fourth and first quadrant. The angle $\theta = 5\pi/4$ is in the third quadrant, and the angle $\theta = 7\pi/4$ is in the fourth quadrant. Which means the angle we need is one that's coterminal with $\theta = 7\pi/4$, but in the interval $[-\pi/2,\pi/2]$. That angle that works is $\theta = -\pi/4$, so we'll say

$$\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right) = -\frac{\pi}{4}$$

Let's do another example, this time with the inverse cosine function.

Example

Find the value of the inverse cosine function.

$$\cos^{-1}\left(\frac{1}{2}\right)$$

If we look at the unit circle, we can see that the cosine function is $1/2$ when $\theta = \pi/3$ and when $\theta = 5\pi/3$. But because we're dealing with the inverse cosine function, we only want an angle in the interval $[0,\pi]$.

The angle $\theta = \pi/3$ is the only angle in $[0,\pi]$, so

$$\cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$$

To get the inverses for the reciprocal functions, you do the same thing, but we'll take the reciprocal of what's in the parentheses and then use the "normal" trig functions. For example, to get $\sec^{-1}(2)$, we have to look for $\cos^{-1}(1/2)$, which is $\pi/3$ and $5\pi/3$. But because we're dealing with the inverse cosine function, we only want an angle in the interval $[0,\pi]$. Therefore, $\sec^{-1}(2) = \cos^{-1}(1/2) = \pi/3$, or 60°.