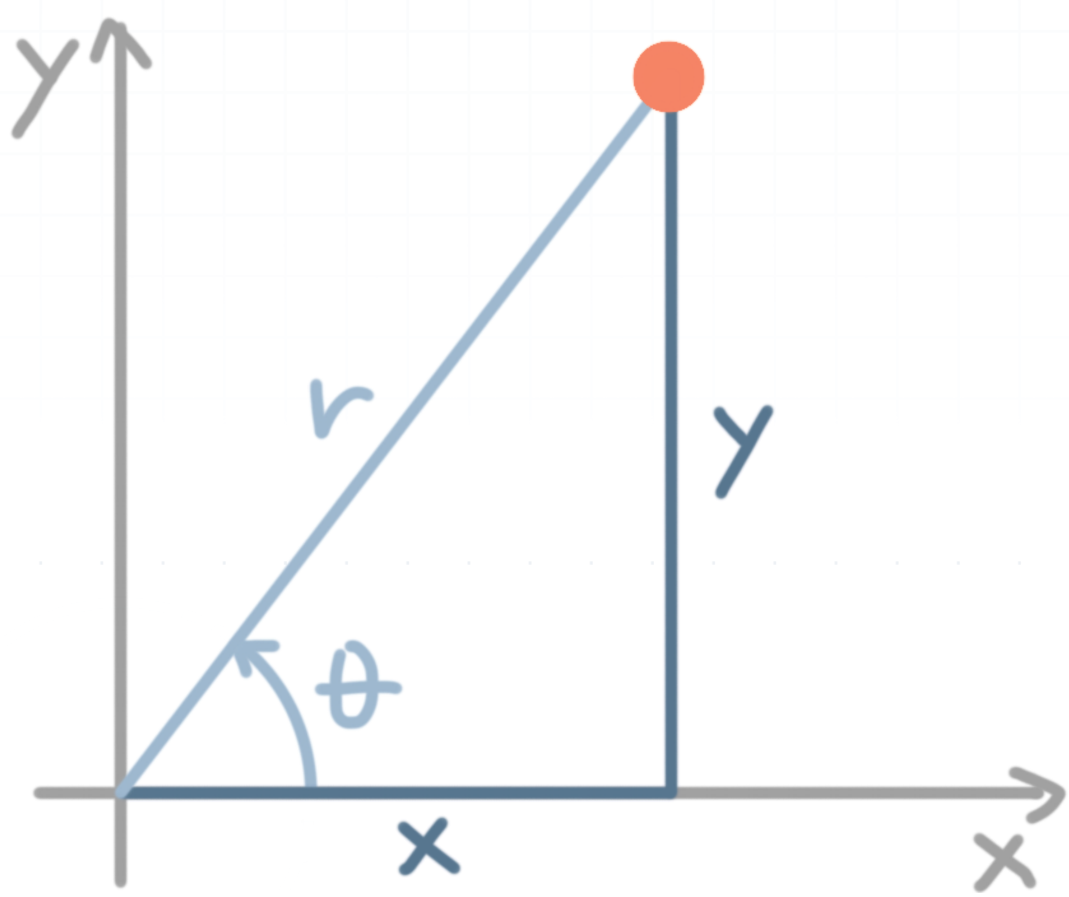
Converting polar points to rectangular

We already know how to get the rectangular coordinates (x, y) of a point in the plane, but now we want to change those (x, y) rectangular points into (r, θ) polar points.

Whereas (x, y) points give us the horizontal distance x from the origin and the vertical distance y from the origin, (r, θ) points give us distance r from the origin and angle θ between the point and the positive direction of the horizontal axis.



Notice how both the rectangular system and the polar system can get us to the same point. If we use the rectangular system, we move horizontally along the x-axis, then vertically parallel to the y-axis, until we arrive at our point. But if we use the polar system, we stand at the origin and turn an angle of θ in place until we're facing the point, and then walk a distance of r straight out from the origin until we arrive at our point.

The triangle we drew in the diagram is a right triangle, which means we can use right-triangle Trigonometry to describe relationships between x, y, r, and θ. Remembering SOH-CAH-TOA (sine equals opposite over hypotenuse, cosine equals adjacent over hypotenuse, and tangent equals opposite over adjacent), we can say

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{y}{r}$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{x}{r}$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{y}{x}$$

If we solve the sine equation for y by multiplying both sides by r, and solve the second equation for x by multiplying both sides by r, we get

$$y = r\sin\theta$$

$$x = r\cos\theta$$

These are the two most important equations we'll use in order to convert back and forth between rectangular and polar coordinates. Given a point or equation in rectangular coordinates (x, y), we'll just substitute $r\sin\theta$ for y and $r\cos\theta$ for x, and we'll have converted into polar coordinates.

Let's do an example so that we can see how to use these equations to convert a polar coordinate point into a rectangular coordinate point.

Example

Convert the polar point into rectangular coordinates (x, y).

$$\left(2, \frac{\pi}{3}\right)$$

The polar point tells us that $r = 2$ and $\theta = \pi/3$. To find the rectangular point that corresponds to these values, we'll just plug them into the conversion equations.

$$y = r \sin\theta$$

$$y = 2\sin\frac{\pi}{3} = 2\left(\frac{\sqrt{3}}{2}\right) = \sqrt{3}$$

and

$$x = r\cos\theta$$

$$x = 2\cos\frac{\pi}{3} = 2\left(\frac{1}{2}\right) = 1$$

So the polar point $(r, \theta) = (2, \pi/3)$ is equivalent to the rectangular point $(x, y) = (1, \sqrt{3})$.

In this last example, we converted a point in the first quadrant. Let's do another example where we convert a point in the second quadrant.

Example

Convert the polar point into rectangular coordinates (x, y).

$$\left(6,\frac{3\pi}{4}\right)$$

The polar point tells us that $r=6$ and $\theta=3\pi/4$. To find the rectangular point that corresponds to these values, we'll just plug them into the conversion equations.

$$y=r\sin\theta$$

$$y=6\sin\frac{3\pi}{4}=6\left(\frac{\sqrt{2}}{2}\right)=3\sqrt{2}$$

and

$$x=r\cos\theta$$

$$x=6\cos\frac{3\pi}{4}=6\left(-\frac{\sqrt{2}}{2}\right)=-3\sqrt{2}$$

So the polar point $(r,\theta)=(6,3\pi/4)$ is equivalent to the rectangular point $(x,y)=(-3\sqrt{2},3\sqrt{2})$.