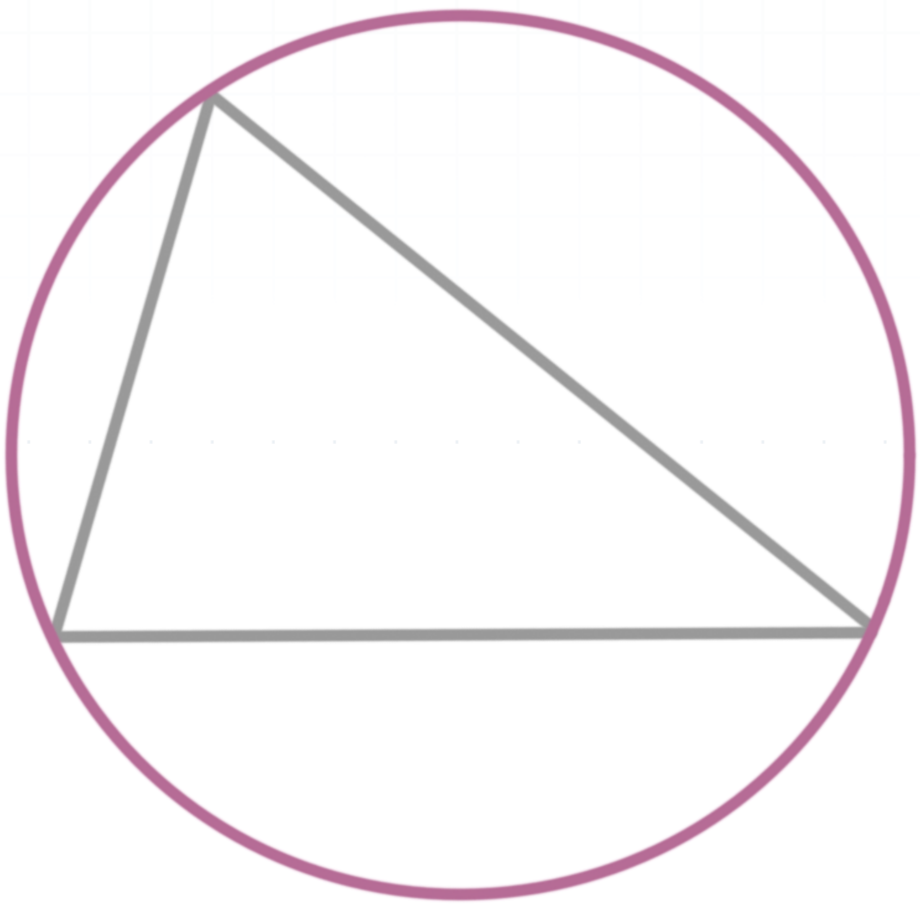
Circumscribed and inscribed circles of a triangle

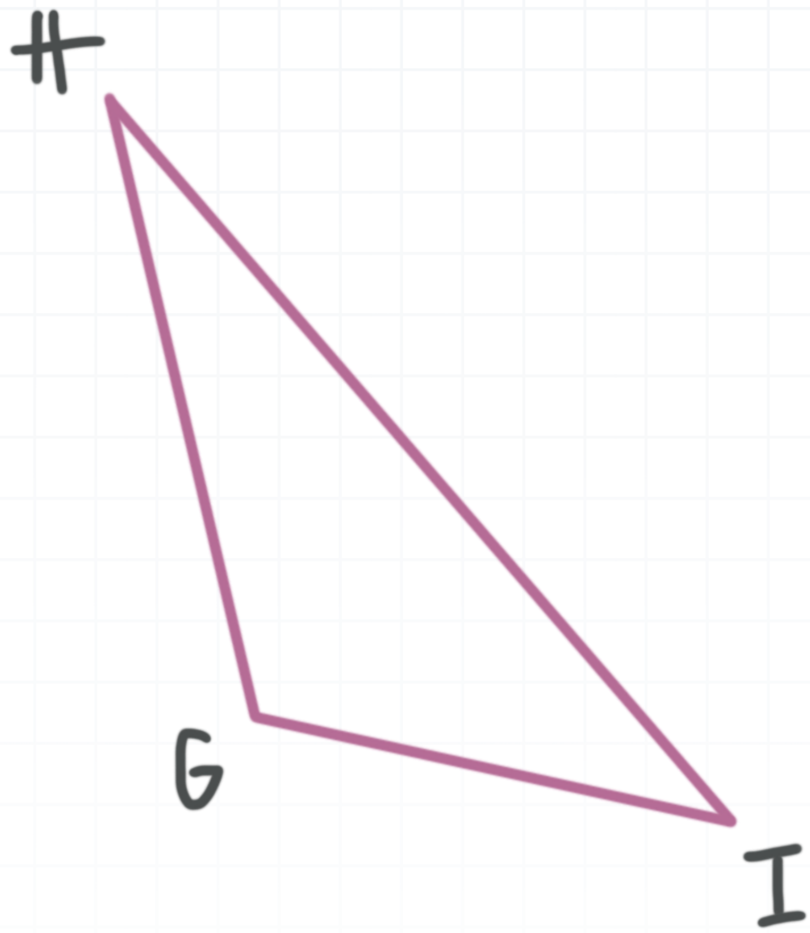
In this lesson we'll look at circumscribed and inscribed circles of a triangle and the special relationships between these circles and the corresponding triangles.

Circumscribed circles

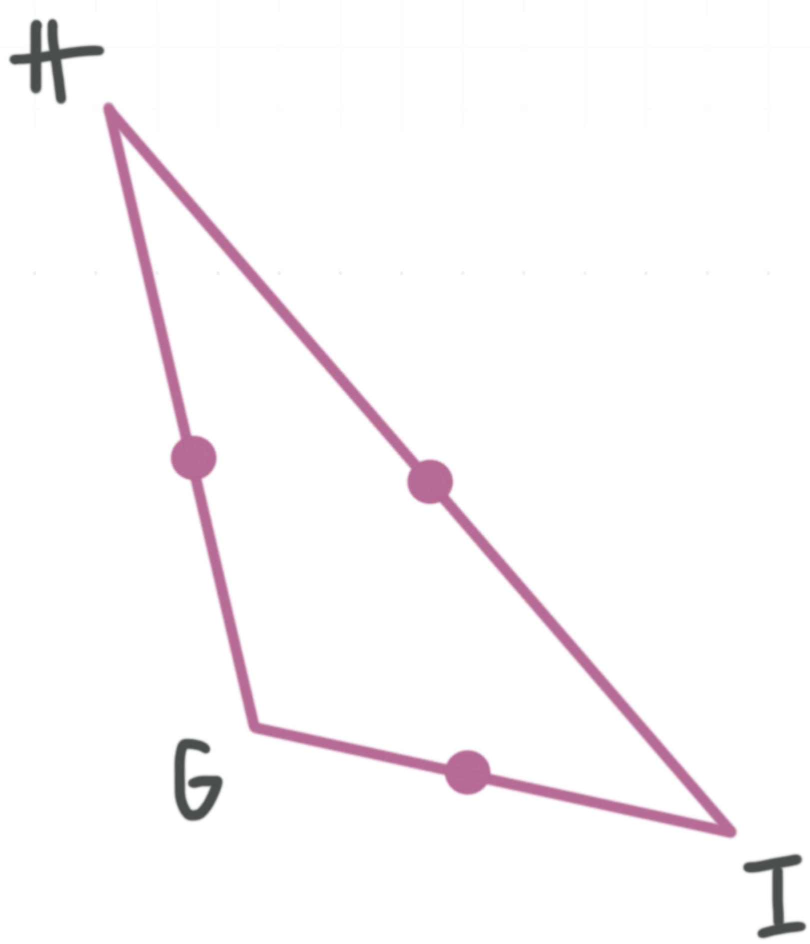
When a circle circumscribes a triangle, the interior of the triangle is inside the circle and all the vertices of the triangle lie on the circle.



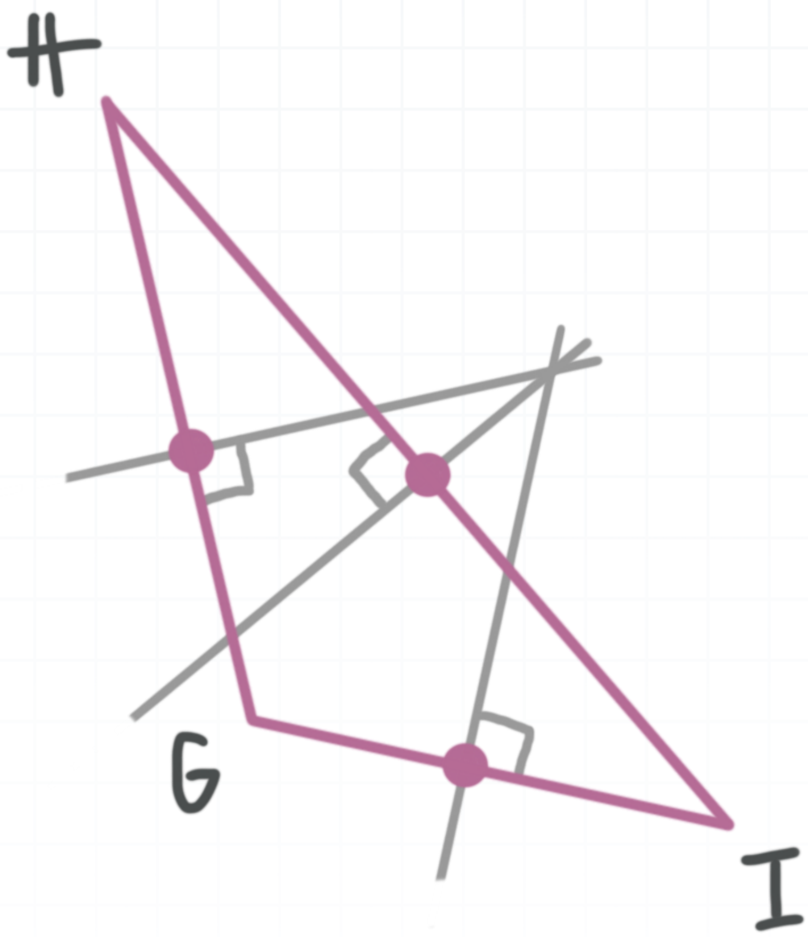
You use the perpendicular bisectors of the sides of the triangle to find the the center of the circle that will circumscribe the triangle. For example, given $\triangle GHI$,



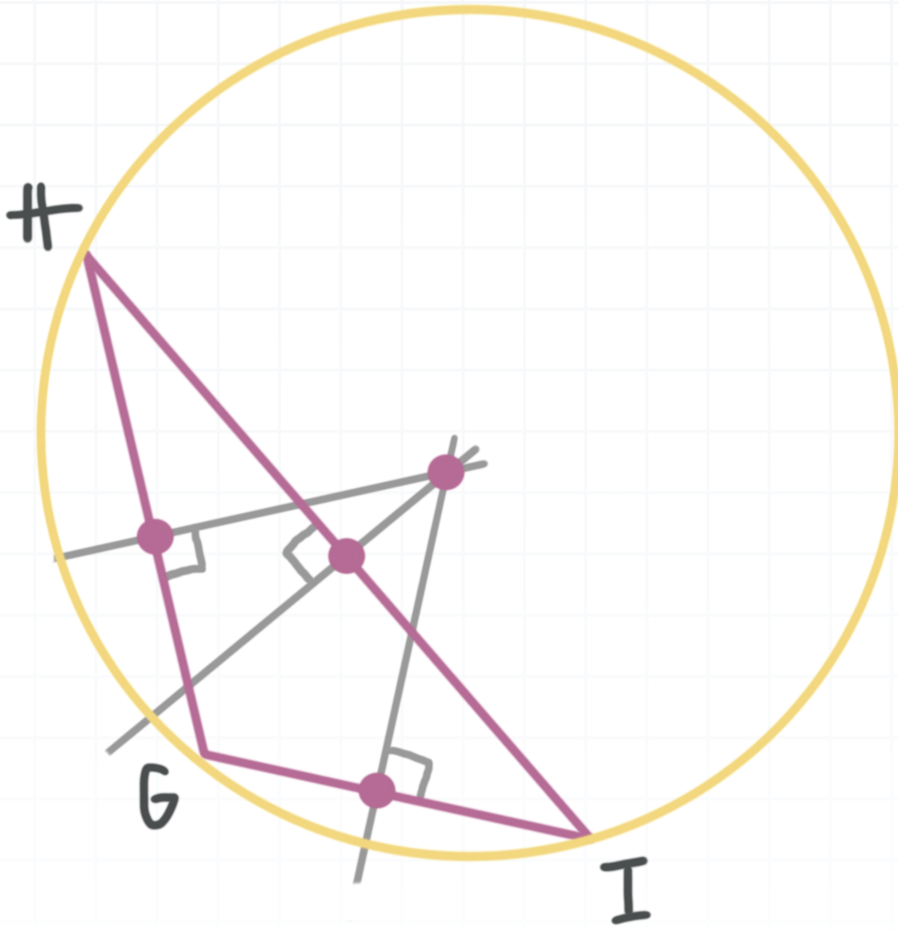
you first find the midpoint of each side.



After that, you draw the perpendicular bisector of each side of the triangle.



The point where the perpendicular bisectors intersect is the center of the circle.



The center of the circumscribed circle of a triangle is the **circumcenter** of the triangle.

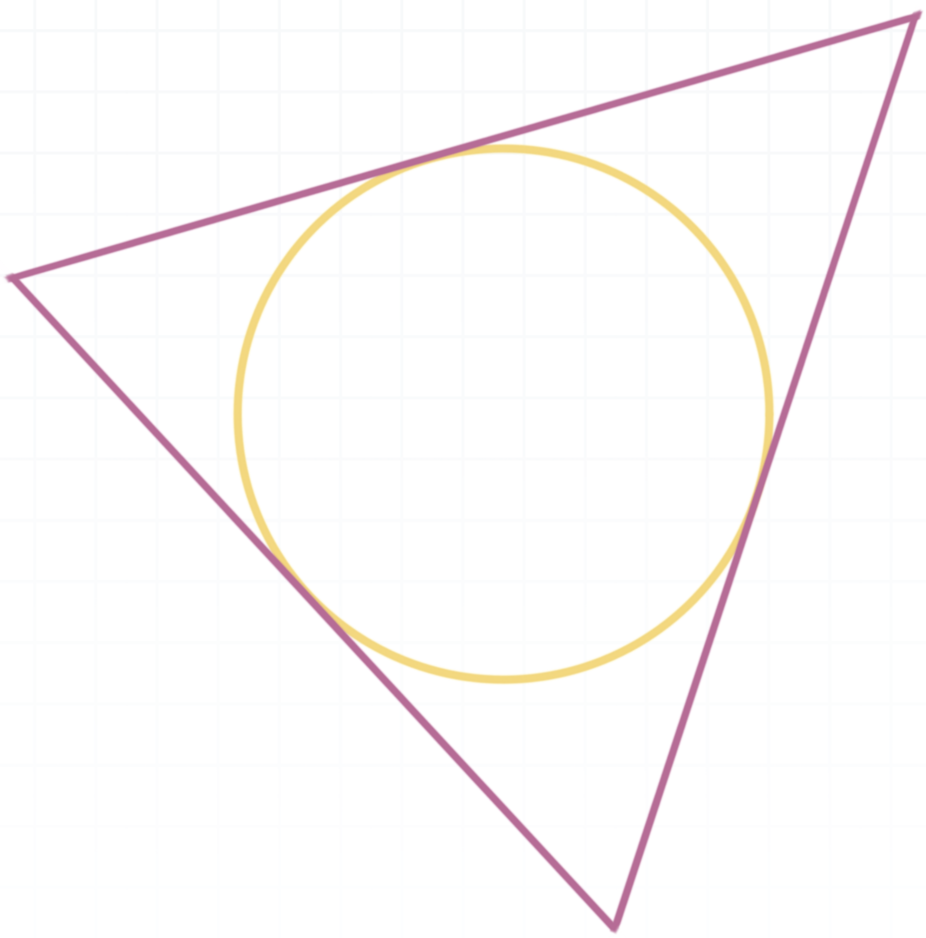
- For an acute triangle (one in which all the interior angles are acute, that is, they all have measure less than 90°), the circumcenter is inside the triangle.

- For a right triangle, the circumcenter is on the side opposite the right angle.

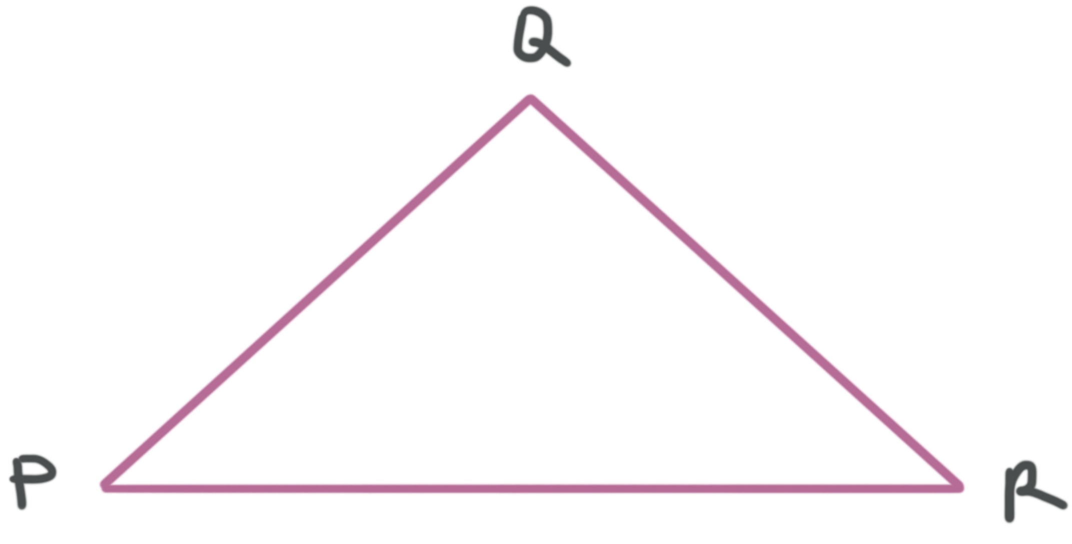
- For an obtuse triangle (one in which there’s an interior angle with measure greater than 90°), the circumcenter is outside the triangle.

Inscribed circles

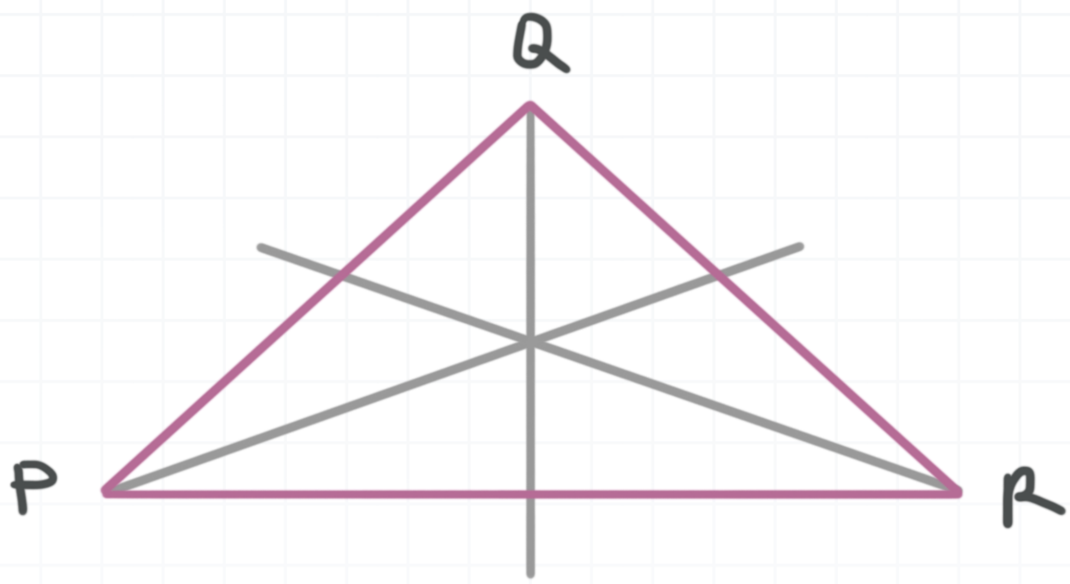
When a circle is inscribed in a triangle, the interior of the circle is inside the triangle, and exactly one point on each side of the triangle lies on the circle. Therefore, the sides of the triangle are tangent to the circle.



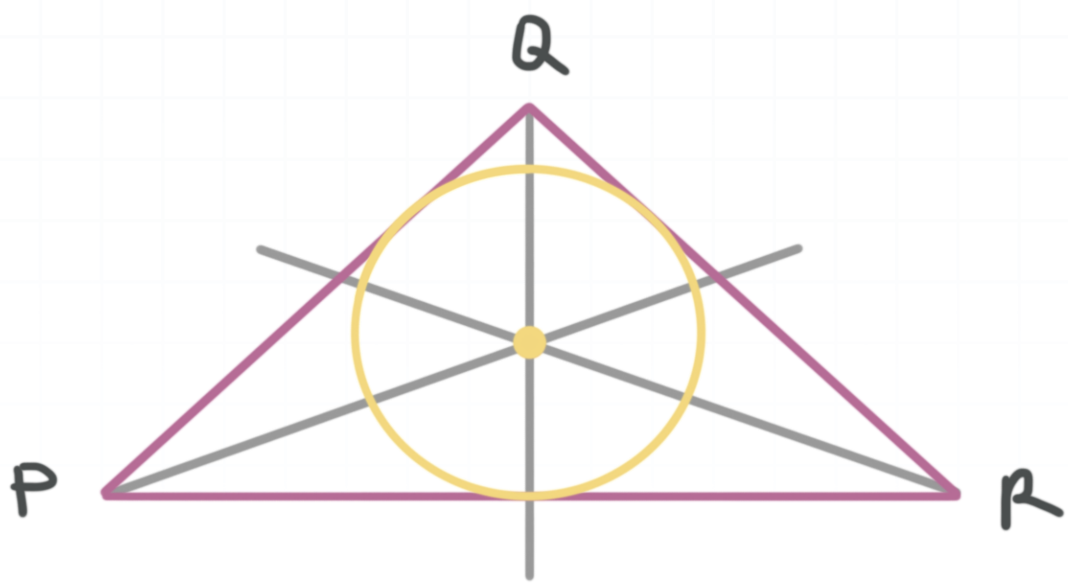
To find the center of the inscribed circle of a triangle, you draw the angle bisector of each interior angle of the triangle. For example, given $\triangle PQR$,



draw in the angle bisectors.



The intersection of the angle bisectors is the center of the inscribed circle.



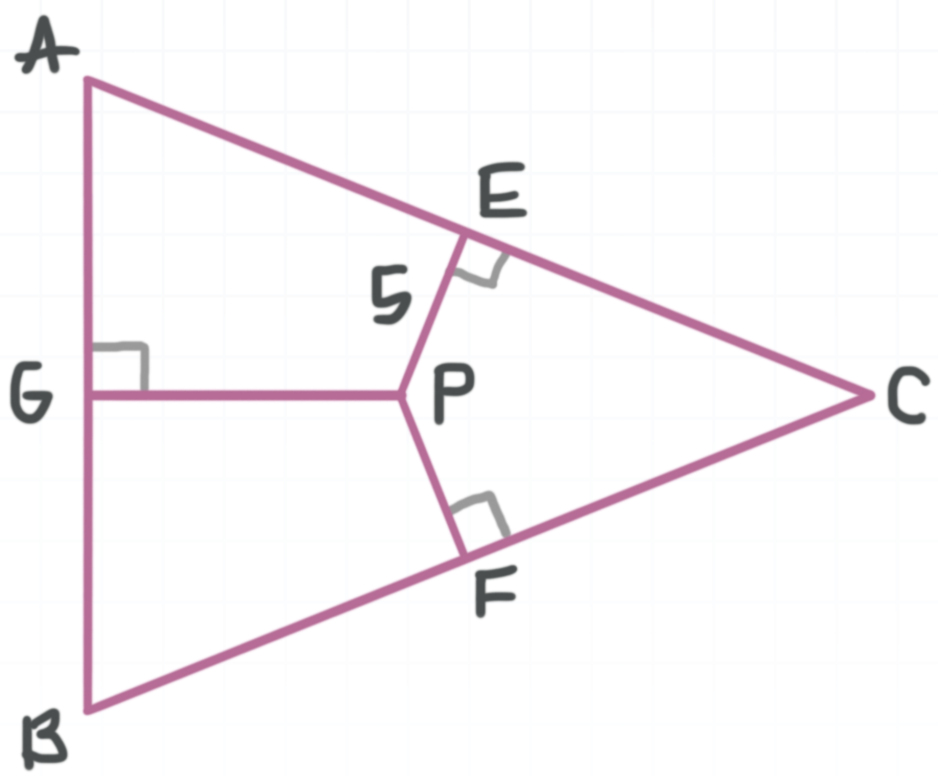
Remember that each side of the triangle is tangent to the circle, so if you draw a radius from the center of the circle to the point where the circle intersects a side of the triangle, that radius will form a right angle with that side of the triangle.

The center of the inscribed circle of a triangle is the **incenter** of the triangle. The incenter will always be inside the triangle.

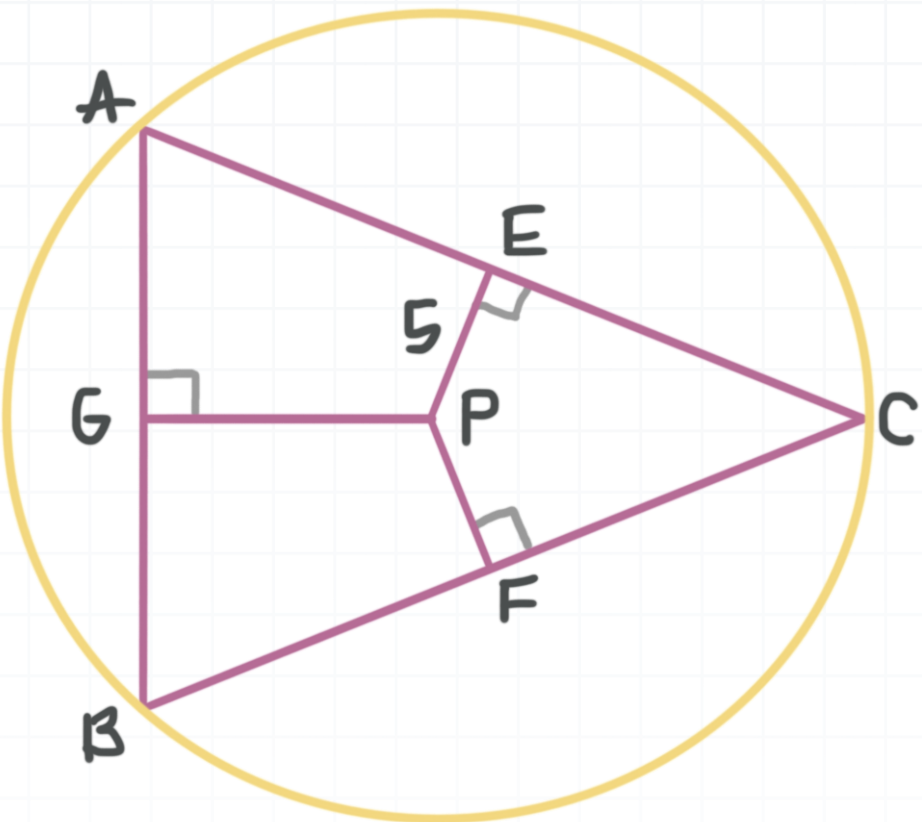
Let's use what we know about these constructions to solve a few problems.

Example

$\overline{GP}$, $\overline{EP}$, and $\overline{FP}$ are the perpendicular bisectors of sides $\overline{AB}$, $\overline{AC}$, and $\overline{BC}$, respectively, of $\triangle ABC$, and $\overline{AC} = 24$ units. What is the radius of the circumscribed circle of $\triangle ABC$?

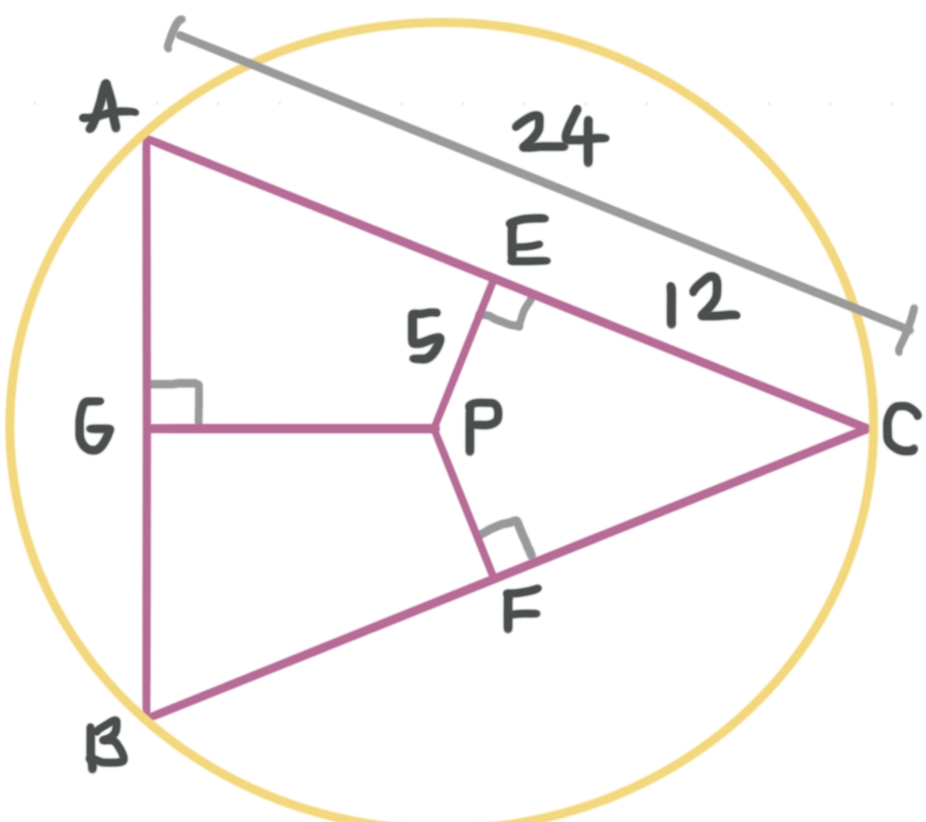


Point P is the circumcenter of $\triangle ABC$, because it's the point where the perpendicular bisectors of the sides of the triangle intersect. Since all the vertices of a triangle lie on its circumscribed circle, we can draw the circle.

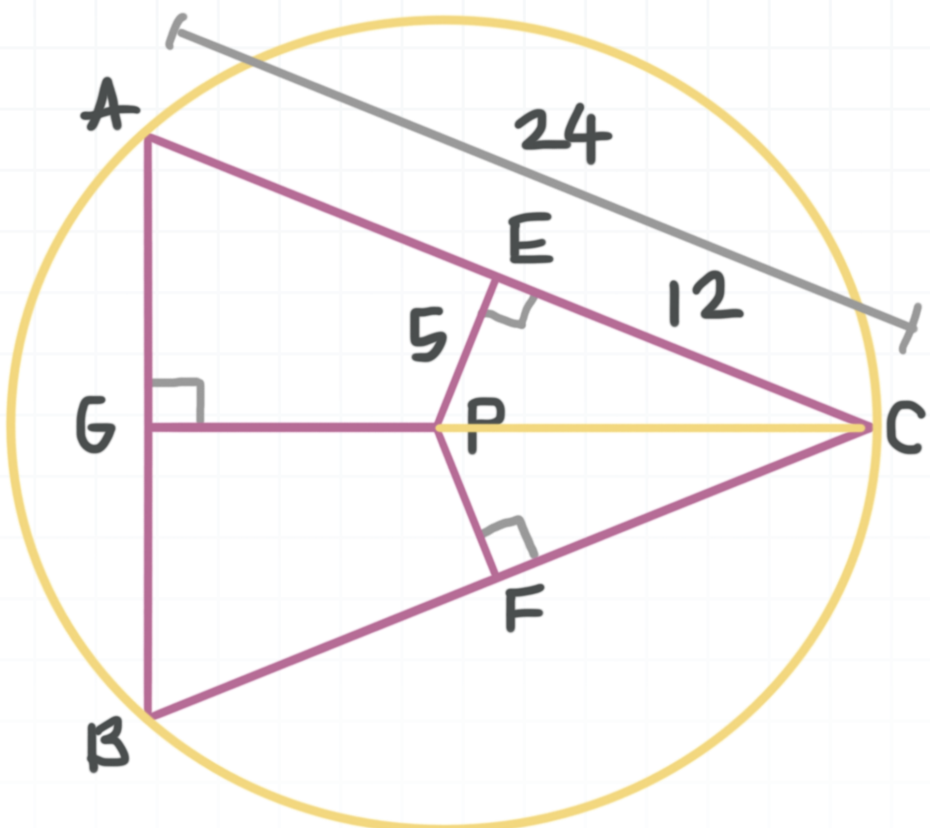


We also know that $\overline{AC} = 24$, and since $\overline{EP}$ is a perpendicular bisector of $\overline{AC}$, point E is the midpoint. Therefore,

$$\overline{EC} = \frac{1}{2}(\overline{AC}) = \frac{1}{2}(24) = 12$$



Now we can draw the radius from point P, the center of the circle, to point C, which is a vertex of the triangle, so it lies on the circle.



We can use right triangle PEC and the Pythagorean theorem to solve for the length of radius $\overline{PC}$.

$$5^2 + 12^2 = (\overline{PC})^2$$

$$25 + 144 = (\overline{PC})^2$$

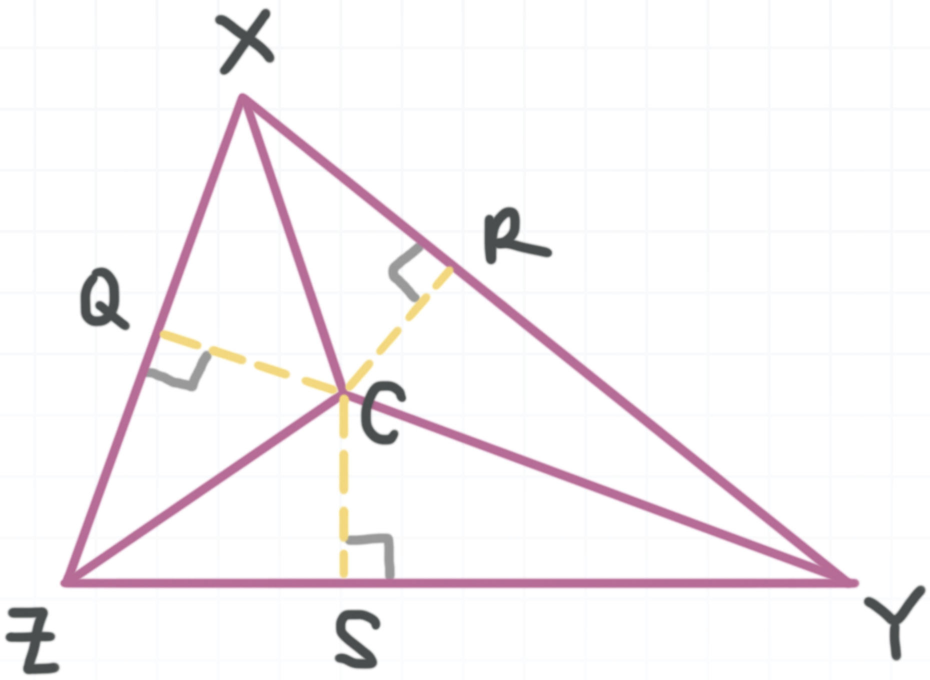
$$169 = (\overline{PC})^2$$

$$\overline{PC} = 13$$

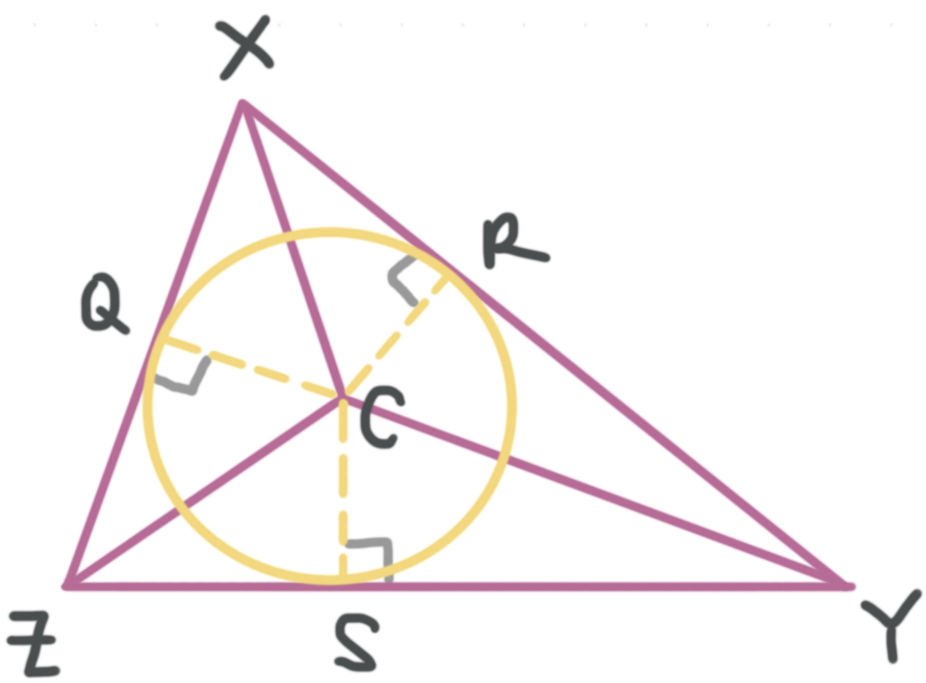
Let's try a different problem.

Example

If $\overline{CQ} = 2x - 7$ and $\overline{CR} = x + 5$, what is the length of $\overline{CS}$, given that $\overline{XC}$, $\overline{YC}$, and $\overline{ZC}$ are angle bisectors of the interior angles of $\triangle XYZ$?



Because $\overline{XC}$, $\overline{YC}$, and $\overline{ZC}$ are angle bisectors of the interior angles of $\triangle XYZ$, C is the incenter of the triangle. The circle with center C will be tangent to each side of the triangle at the point of intersection.



$\overline{CQ}$, $\overline{CR}$, and $\overline{CS}$ are the radii drawn from the incenter, C, to the points of intersection of the circle with the sides of $\triangle XYZ$. Since they're all radii of the same circle, they're all of equal length.

$$\overline{CQ} = \overline{CR} = \overline{CS}$$

We need to find the radius of the circle. We know that $\overline{CQ} = 2x - 7$ and $\overline{CR} = x + 5$, so

$$2x - 7 = x + 5$$

$$x = 12$$

Therefore,

$$\overline{CS} = \overline{CR} = x + 5$$

$$\overline{CS} = \overline{CR} = 12 + 5$$

$$\overline{CS} = \overline{CR} = 17$$