Geometry Notes

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MATH

Naming simple geometric figures

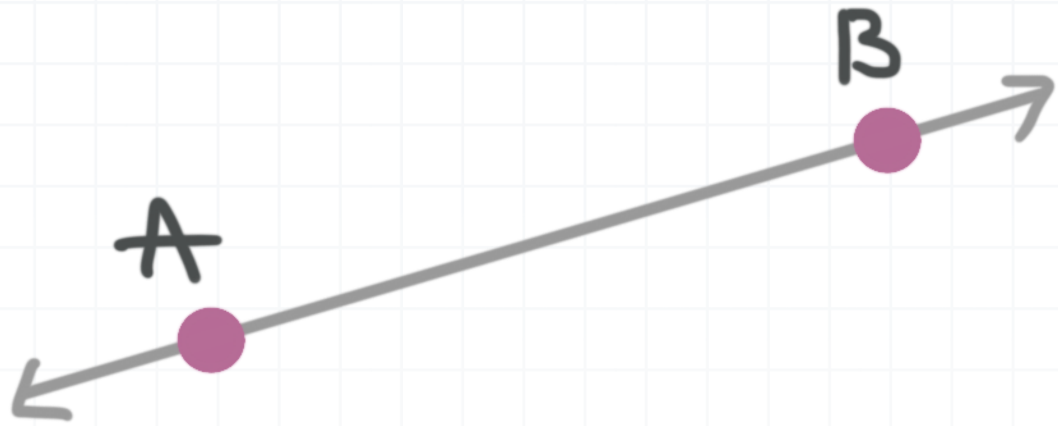
In this lesson we'll look at basic geometric figures like points, lines, line segments, rays, and angles, and we'll talk about how to name them.

Points

A **point** is a specific location in space. A point is usually named with a single letter, and it's represented by a dot. Point A might look like this:

Lines

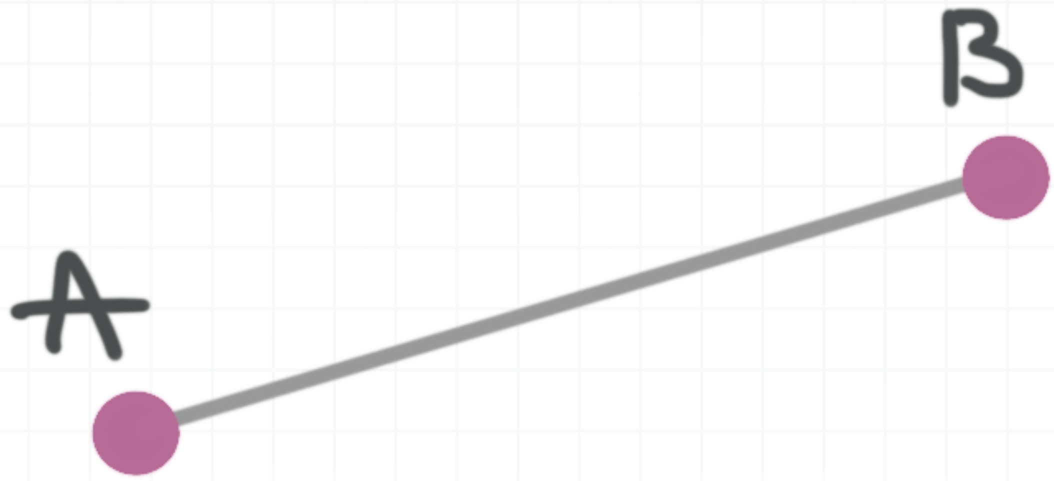
A **line** extends to infinity in two opposite directions, so it can be thought of as the straight path that connects two points (and extends beyond each of them, to infinity). A line is usually named with two letters that represent two points on the line, with an arrow drawn over them that points in both directions to indicate that the line extends forever in both directions. A drawing of a line needs to be straight and have arrows on both ends. Line $\overleftrightarrow{AB}$ (also called line $\overleftrightarrow{BA}$), might look like this:



A line can also be named with a lowercase letter. Line p might look like this:

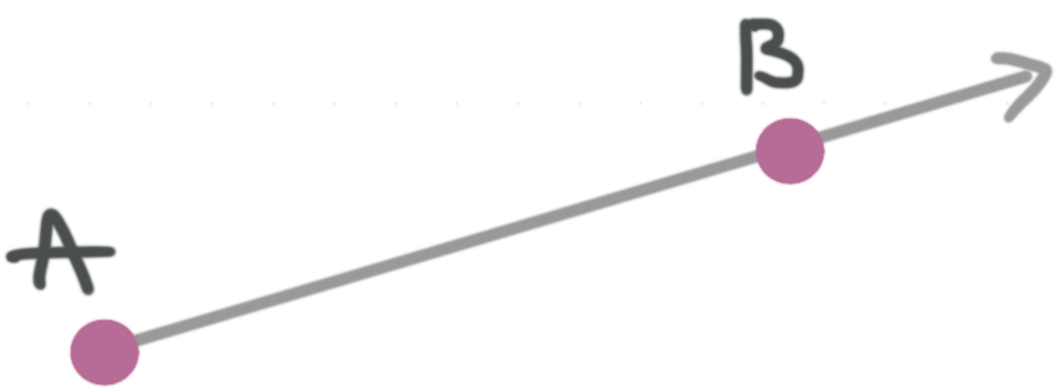
Line segments

A **line segment** is a finite piece of a line. In other words, line segments don't extend infinitely in both directions, unlike a line. The points at the ends of a line segment are called endpoints. You name a line segment by its endpoints, with a line (but no arrow) over them. Line segment $\overline{AB}$ might look like this:

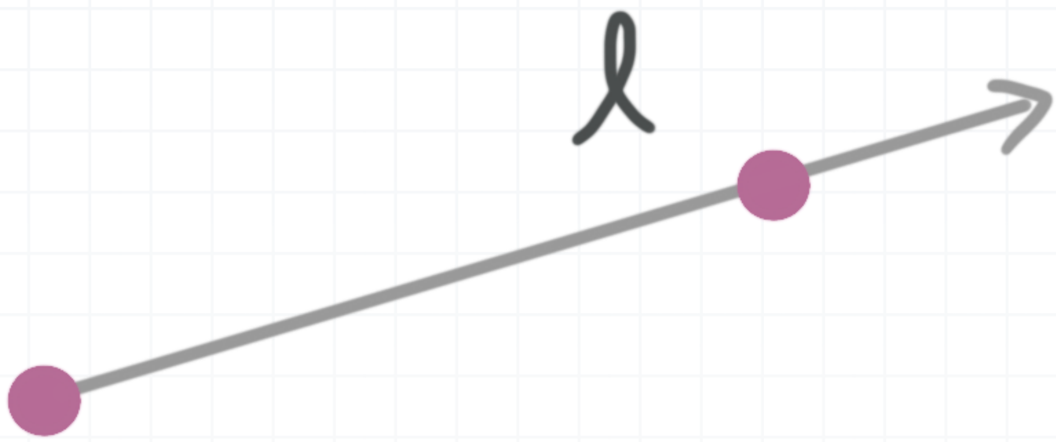


Rays

Another type of “partial line” a line is a **ray**. A ray is a part of a line that has one endpoint, and one side that goes on to infinity in the direction that doesn’t include the endpoint. We name a ray with its endpoint and any other point on the ray, with a one-sided arrow over the two letters that points in the direction away from the endpoint. Ray $\overrightarrow{AB}$ might look like this:



A ray can also be named with a lowercase letter. Ray l might look like this:



Angles

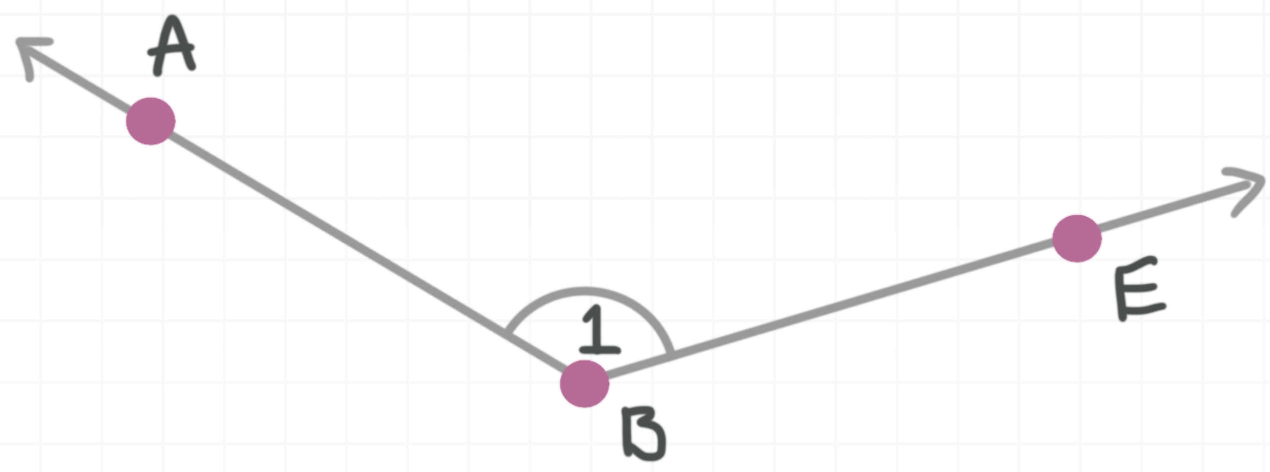
When two rays have a common endpoint, they form an angle. The common endpoint of the angle is called the vertex. These are ways to name angles:

1. Writing the **angle symbol** $\angle$ and then the name of the vertex.

2. Writing the angle symbol and then three letters: the name of some point on one ray, the name of the vertex, and the name of some point on the last ray.

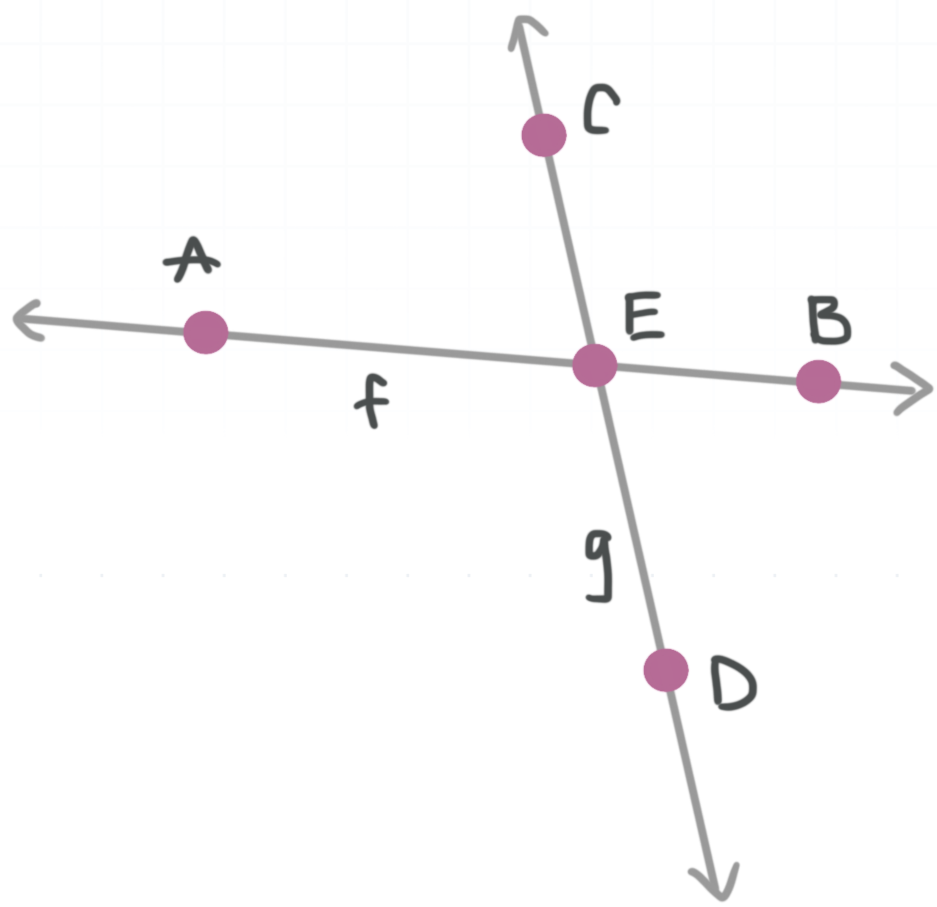
3. You can also name an angle with a lowercase letter or number. If the lowercase letter or number is written inside the angle, then you write the name of the angle inside the angle.

If you choose method 2, be sure to put the name of the vertex in the middle of the name! Here's an angle that can be named in any of four ways: $\angle B$ or $\angle ABE$ or $\angle EBA$ or $\angle 1$.



Example

What does the letter f represent in the diagram?

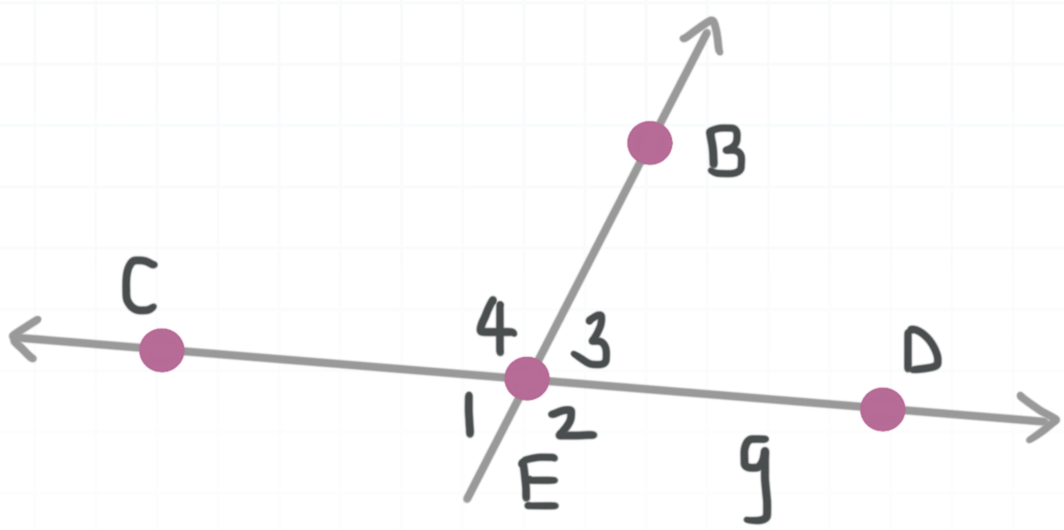


The letter f represents a line. That line could also be called line $\overleftrightarrow{AB}$ or line $\overleftrightarrow{BA}$.

Let's look at naming angles.

Example

What are three other names for $\angle 4$? Are any of the names a bad choice? Why or why not?



E is the vertex of $\angle 4$, so it can be named $\angle E$, $\angle CEB$, or $\angle BEC$. In this case $\angle E$ would be a bad choice for the name, because E is also the vertex of angles 1, 2, and 3.

Let's try an example where we identify geometric figures by the symbols used in naming them.

Example

Match the symbol on the left with the name of the figure on the right.

A An angle

$\overrightarrow{AB}$	A point
$\overleftrightarrow{EF}$	A line
$\angle E$	A line segment
$\overline{JI}$	A ray

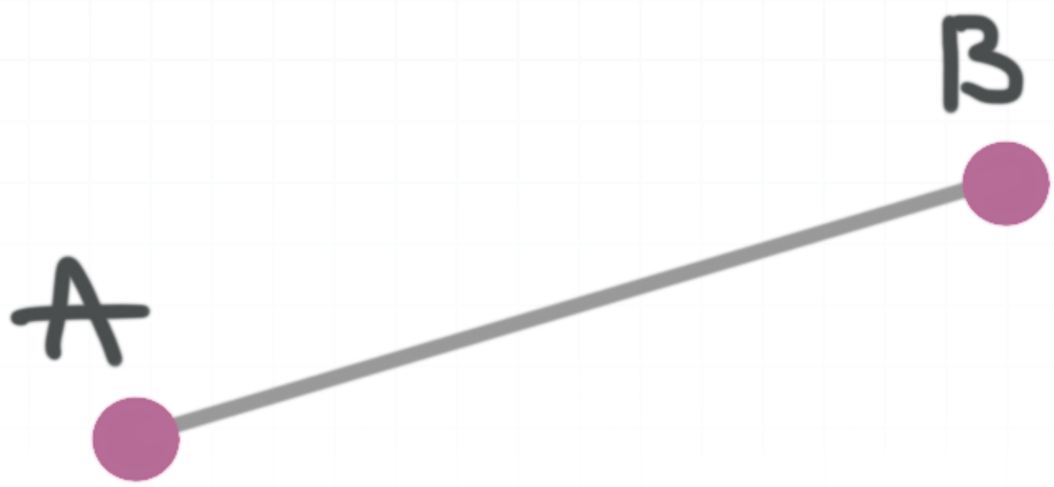
Remembering what type of geometric figure each symbol represents can go a long way toward helping us interpret and solve geometry problems. We'll rearrange the column on the right so that the name of each figure corresponds to the correct symbol in the column on the left.

A	A point
$\overrightarrow{AB}$	A ray
$\overleftrightarrow{EF}$	A line
$\angle E$	An angle
$\overline{JI}$	A line segment

Length of a line segment

In this lesson we'll look at how to find the length of a line segment algebraically when we're given information and measurements about parts of the line segment.

Remember that a line segment is a finite piece of a line, named by its endpoints. For instance, the line segment $\overline{AB}$ might look like this:



Line segments and distance

The distance between two points on a line segment is called the length of the segment. We usually use the same symbol for the length of the line segment that we use for the segment itself. So $\overline{AB}$ could be used to represent the segment itself, but also the length of the segment.

In this example, the distance between points A and B is

$$\overline{AB} = |3 - (-2)|$$

$$\overline{AB} = |3 + 2|$$

$$\overline{AB} = |5|$$

$$\overline{AB} = 5$$

In this example, you could also count from A to B and get a distance of 5. As you can see, sometimes it may be helpful to draw a number line in order to visualize the length of a line segment.

Example

Points S, T, U, V, and W lie, in order from left to right, on a number line. Point U is at -2. Where are the rest of the points located?

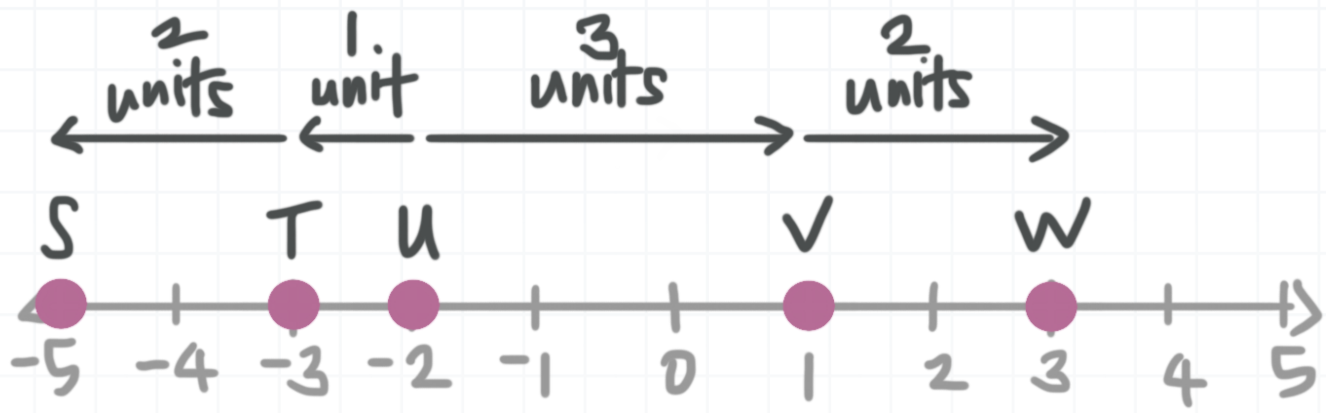
$$\overline{ST} = 2$$

$$\overline{TU} = 1$$

$$\overline{UV} = 3$$

$$\overline{VW} = 2$$

If we plot point U at -2, then S, T, U, V, and W must be plotted this way:

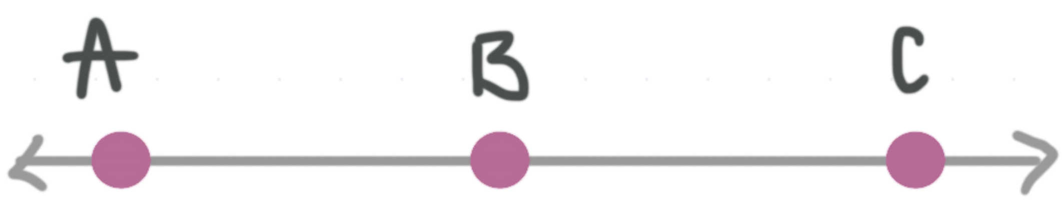


We know point U is at -2 and $\overline{TU} = 1$. This lets us locate point T at -3. Now we can use $\overline{ST} = 2$ to find that $S = -5$, and $\overline{UV} = 3$ to find that $V = 1$. Now we can locate point W by using $\overline{VW} = 2$, so $W = 3$.

Let's look at another example.

Example

Find $\overline{AB}$, if $\overline{AC} = 12$ and $\overline{BC} = 7$.



We know that $\overline{AC} = 12$ and $\overline{BC} = 7$. From the diagram, we also know that $\overline{AB}$ is part of $\overline{AC}$.

We can see that $\overline{AC} - \overline{BC} = \overline{AB}$, so we have $\overline{AB} = 12 - 7 = 5$.

Let's look at one last example.

Example

The locations of four points on a number line are $A = 2$, $B = 4$, $C = -3$, and $D = -6$. What is the value of $\overline{AB} + \overline{CD}$?

We can draw a number line to help solve the problem.

Now we can see that $\overline{AB} = |4 - 2| = 2$ and $\overline{CD} = |-3 - (-6)| = 3$. So $\overline{AB} + \overline{CD} = 2 + 3 = 5$.

Slope and midpoint of a line segment

In this lesson we'll look at how to find the slope and midpoint of a line segment in the Cartesian plane (the xy-plane). We'll start with how to find the slope.

Slope

The **slope** of a line segment is the rate at which the line segment is increasing or decreasing.

In other words, if we graph the line segment in the xy-plane, the slope will tell us how fast the segment is going "uphill" (a positive slope) or "downhill" (a negative slope) from left to right. You can find the slope of a line segment if you have two points that lie on it.

Let's say we have points with coordinates (x_1, y_1) and (x_2, y_2). Then the formula for the slope, m, is

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Let's start by working through an example.

Example

What is the slope of any line segment that passes through the points $(4, -5)$ and $(-3,6)$?

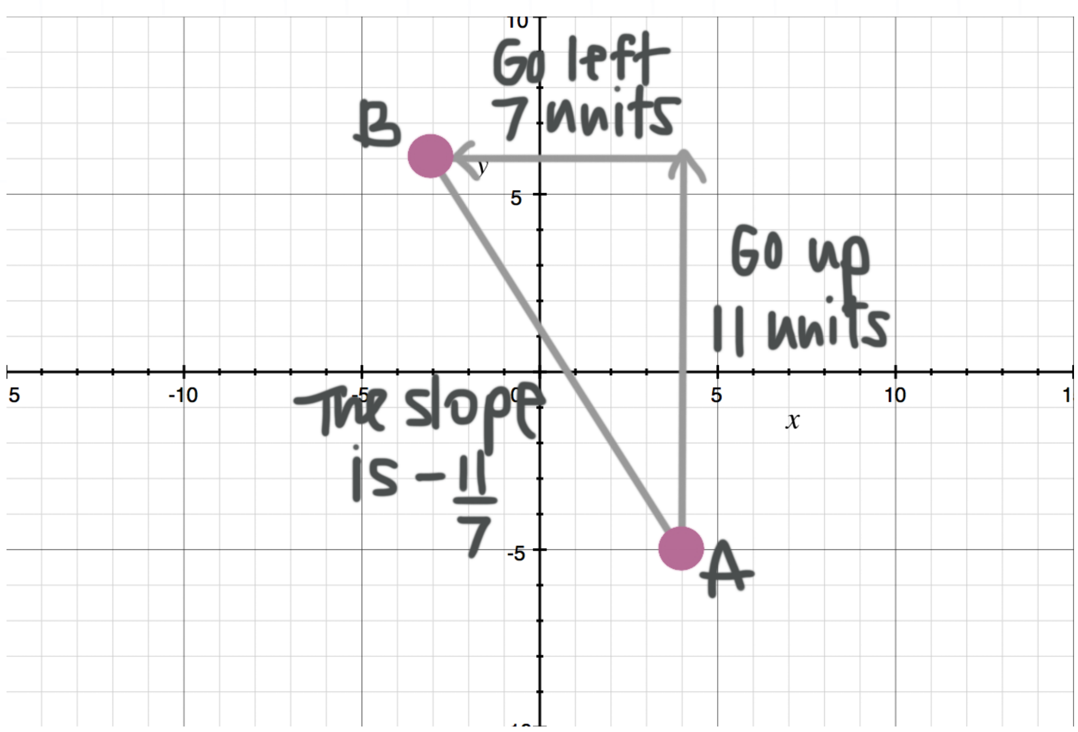
Let $(x_1, y_1) = (4, -5)$ and $(x_2, y_2) = (-3,6)$. We'll plug the coordinates of these points into the formula for the slope of a line segment.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{6 - (-5)}{-3 - 4} = \frac{6 + 5}{-3 - 4} = \frac{11}{-7} = -\frac{11}{7}$$

It doesn't matter which point we use first. If we switch the points and let $(x_1, y_1) = (-3,6)$ and $(x_2, y_2) = (4, -5)$, we get the same slope.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-5 - 6}{4 - (-3)} = \frac{-5 - 6}{4 + 3} = \frac{-11}{7} = -\frac{11}{7}$$

We can also find the slope by plotting the points:



Midpoint

The midpoint of a line segment is the point that's halfway between the endpoints, so it divides the line segment into two equal parts.

The formula for the coordinates of the midpoint, M, of a line segment with endpoints (x_1, y_1) and (x_2, y_2) is

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

Let's look at an example.

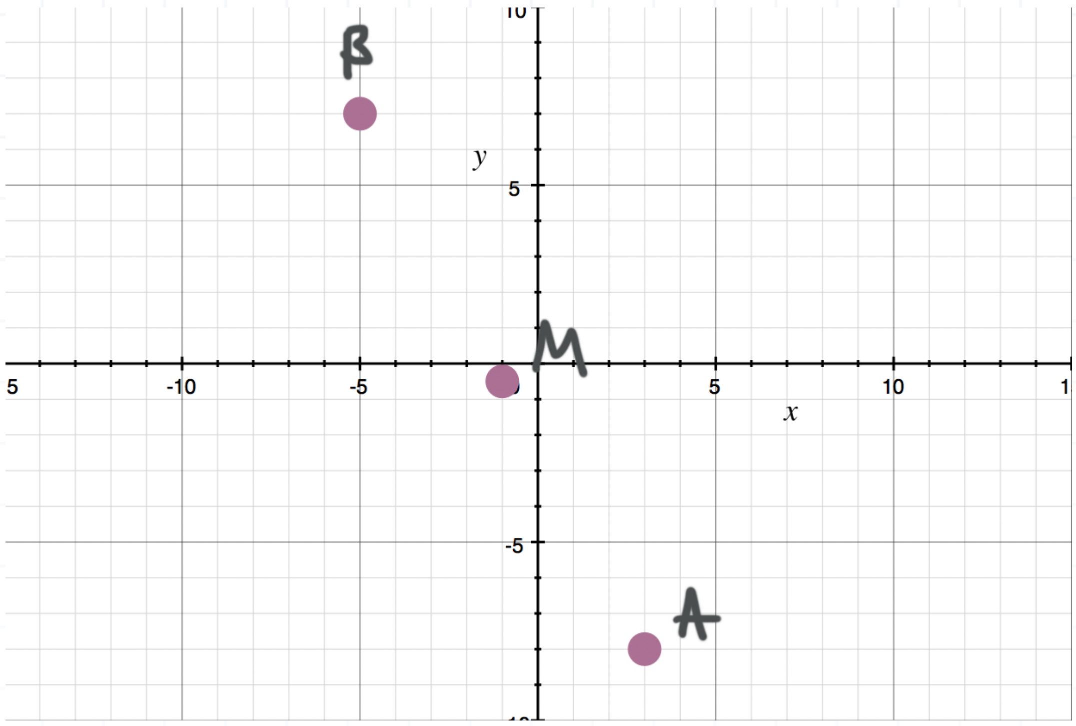
Example

What is the midpoint of the line segment with endpoints $(3, -8)$ and $(-5,7)$?

Let $(x_1, y_1) = (3, -8)$ and $(x_2, y_2) = (-5,7)$, then plug these coordinates into the formula for the midpoint.

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) = \left(\frac{3 + (-5)}{2}, \frac{-8 + 7}{2}\right) = \left(\frac{-2}{2}, \frac{-1}{2}\right) = \left(-1, -\frac{1}{2}\right)$$

We can also plot the endpoints, and the midpoint, on a graph.

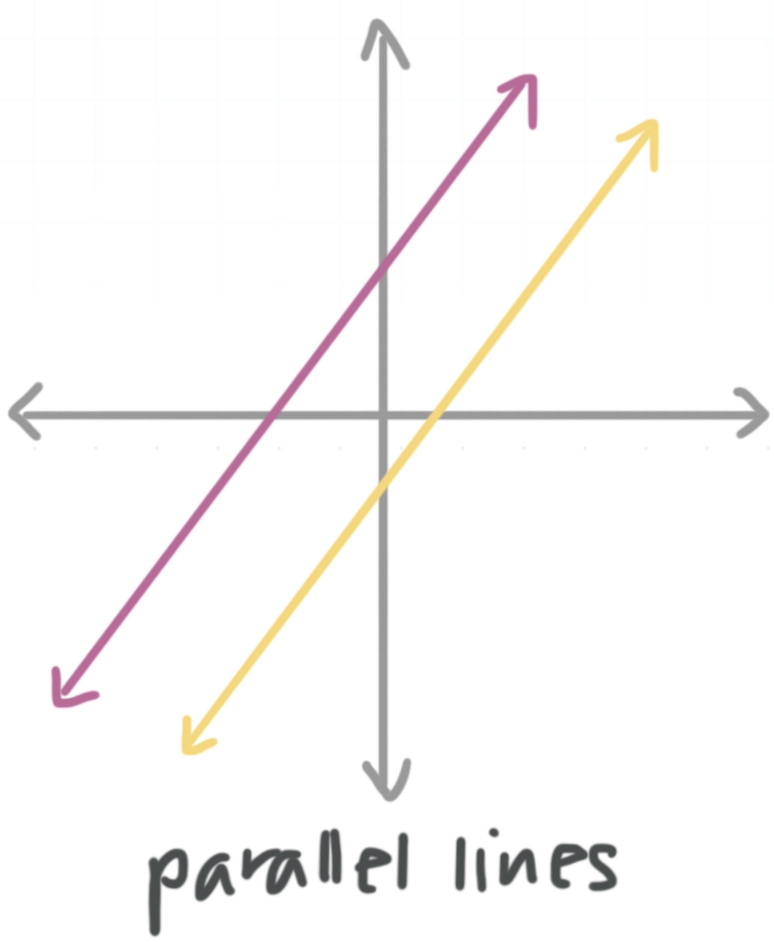

B
M
A
y
x
5
-10
-5
5
10
-5

Parallel, perpendicular, or neither

In this lesson we'll look at how to use the slopes of two lines in the Cartesian plane (the xy-plane) to see if the lines are perpendicular, parallel, or neither.

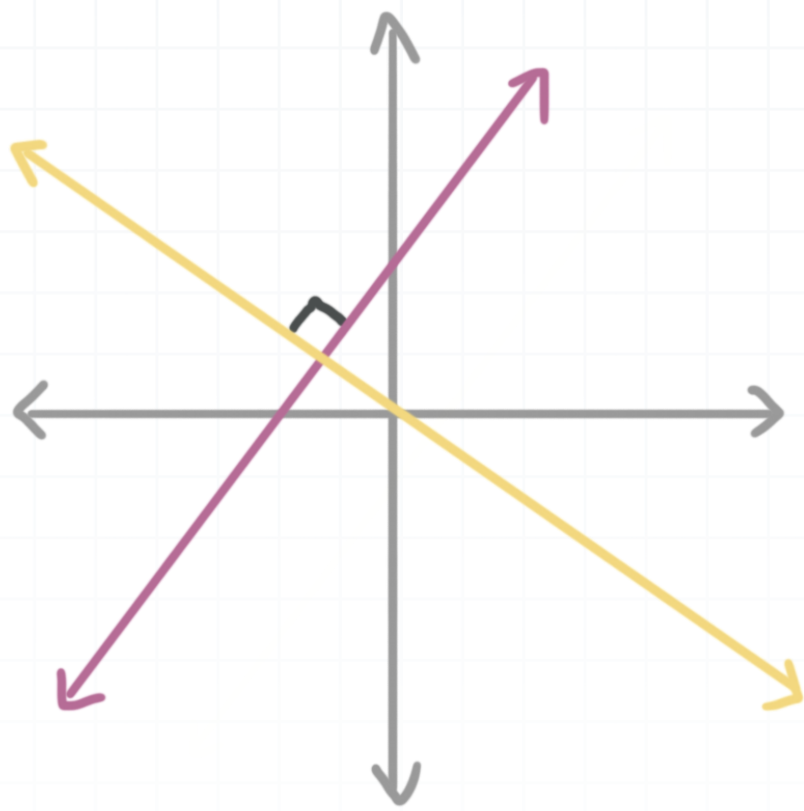
Parallel lines

Parallel lines are lines with equal slopes. Parallel lines will never intersect each other, because they'll always be the same distance apart.



Perpendicular lines

Perpendicular lines have slopes that are negative reciprocals of each other, and they intersect to form 90° angles.



Let's do a few examples.

Example

Each pair of points in the table below are points that lie on the given line. Which lines are parallel to each other, and which lines are perpendicular?

Line	Point 1	Point 2
Line AB	(3,1)	(-3,11)
Line CD	(0,2)	(5,5)
Line EF	(-5,-5)	(0,-2)

Use the slope formula for each line.

Slope of line $\overleftrightarrow{AB}$: $$m=\frac{y_2-y_1}{x_2-x_1}=\frac{11-1}{-3-3}=\frac{10}{-6}=-\frac{5}{3}$$

Slope of line $\overleftrightarrow{CD}$: $$m=\frac{y_2-y_1}{x_2-x_1}=\frac{5-2}{5-0}=\frac{3}{5}$$

Slope of line $\overleftrightarrow{EF}$: $$m=\frac{y_2-y_1}{x_2-x_1}=\frac{-2-(-5)}{0-(-5)}=\frac{-2+5}{0+5}=\frac{3}{5}$$

Lines $\overleftrightarrow{CD}$ and $\overleftrightarrow{EF}$ have the same slope, $3/5$, so these two lines are parallel, unless they're one and the same line. Lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ have slopes that are negative reciprocals of each other, so these two lines are perpendicular. Lines $\overleftrightarrow{EF}$ and $\overleftrightarrow{AB}$ are also perpendicular, for the same reason.

Let's see how we can find the slope of any line that's parallel to a given line.

Example

What is the slope of any line that's parallel to $\overleftrightarrow{CD}$, if $\overleftrightarrow{CD}$ passes through the points $(4,5)$ and $(-2,8)$?

Parallel lines have the same slope, so we need to find the slope of $\overleftrightarrow{CD}$. Using the slope formula, we get

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{8 - 5}{-2 - 4} = \frac{3}{-6} = -\frac{1}{2}$$

Any line that's parallel to $\overleftrightarrow{CD}$ will have a slope of $-1/2$.

Now let's look at how to find the slope of any line that's perpendicular to a given line.

Example

What is the slope of any line that's perpendicular to $\overleftrightarrow{WX}$, if $\overleftrightarrow{WX}$ passes through the points $(-3,5)$ and $(2,-6)$?

Perpendicular lines have slopes that are negative reciprocals of each other, so we first need to find the slope of $\overleftrightarrow{WX}$. Using the slope formula, we get

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-6 - 5}{2 - (-3)} = \frac{-11}{5} = -\frac{11}{5}$$

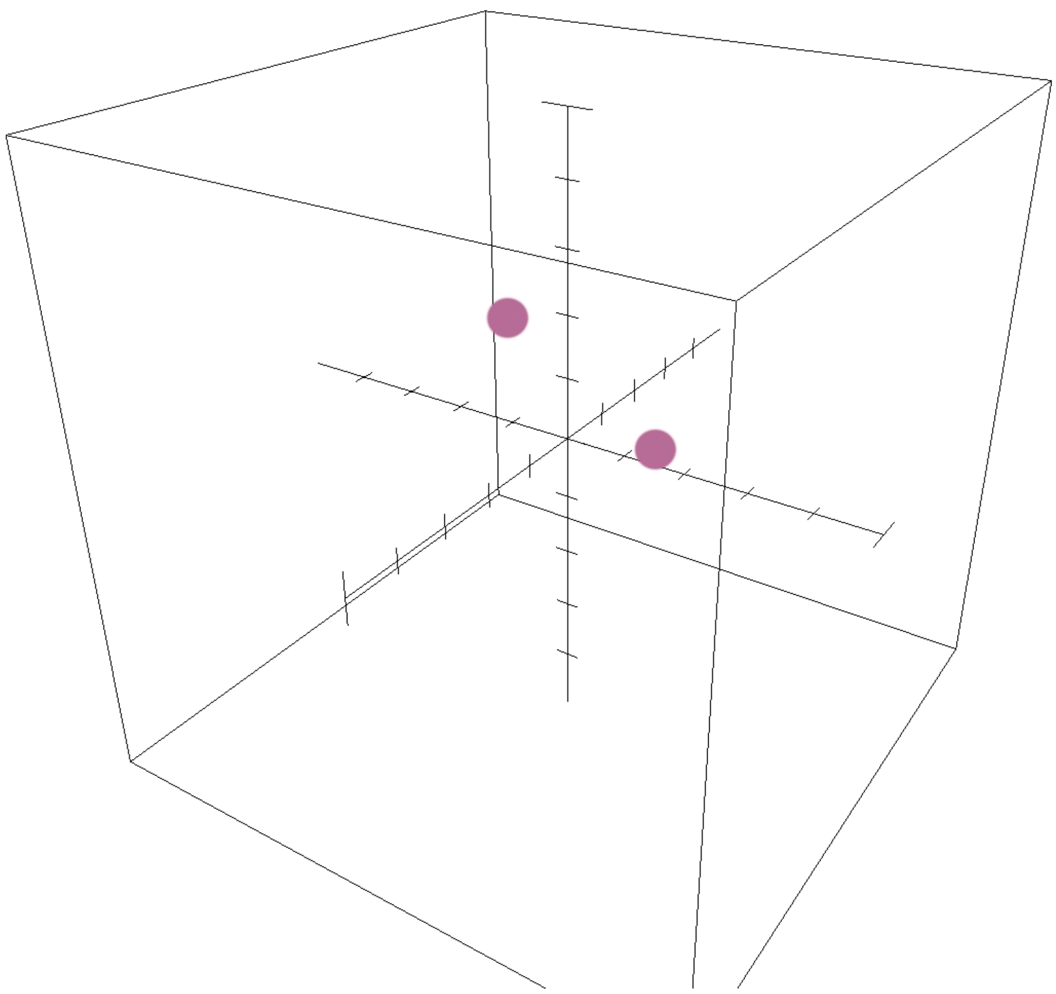
Now we find the negative reciprocal of $-11/5$ by turning it upside down and multiplying by -1, which gives $5/11$. The slope of any line that's perpendicular to $\overleftrightarrow{WX}$ is $5/11$.

Distance between two points in three dimensions

In this lesson we'll look at points that are plotted in three dimensions, and learn how to find the distance between them.

Points

When we plot points in three dimensions, we use a coordinate system with three axes: the x-axis, the y-axis, and the z-axis. So a point is represented by three coordinates, as (x, y, z). Here are two points plotted in three-dimensional space:



Distance formula

We can use the distance formula for three dimensions to find the length of the line segment that connects two points in three-dimensional space.

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$

where $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$.

Simplifying radicals

Since use of the distance formula involves taking a square root, we'll need to know how to simplify radicals.

Remember that, when the value under the radical isn't a perfect square, we can simplify it by factoring out the perfect squares, separating each factor into its own root, and then taking the square root of any perfect squares. For example,

$$\sqrt{18} = \sqrt{9 \cdot 2} = \sqrt{9} \cdot \sqrt{2} = 3\sqrt{2}$$

Now let's do an example of finding the distance between two points.

Example

Calculate the distance between Points B and C.

$$B = (4, -5, 8)$$

$$C = (1, -3,2)$$

We need to use the distance formula.

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$

$$d = \sqrt{(4 - 1)^2 + (-5 - (-3))^2 + (8 - 2)^2}$$

$$d = \sqrt{(3)^2 + (-2)^2 + (6)^2}$$

$$d = \sqrt{9 + 4 + 36}$$

$$d = \sqrt{49}$$

$$d = 7$$

Let's try another one.

Example

Calculate the distance between Points A and D.

$$A = (3,9, -2)$$

$$D = (0, -5,4)$$

We need to use the distance formula.

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$

$$d = \sqrt{(3 - 0)^2 + (9 - (-5))^2 + (-2 - 4)^2}$$

$$d = \sqrt{(3)^2 + (14)^2 + (-6)^2}$$

$$d = \sqrt{9 + 196 + 36}$$

$$d = \sqrt{241} \approx 15.5$$

Midpoint of a line segment in three dimensions

In this lesson we'll look at how to find the midpoint of a line segment in three dimensions when we're given the endpoints of the line segment as coordinates in three-dimensional space.

Midpoint formula

We can use the midpoint formula for three dimensions to find the middle of the line segment that has endpoints $P_1=(x_1,y_1,z_1)$ and $P_2=(x_2,y_2,z_2)$, which is

$$M=\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2},\frac{z_1+z_2}{2}\right)$$

Let's work through an example.

Example

Find the midpoint of the line segment with endpoints P_1 and P_2.

$$P_1=(4,-6,8)$$

$$P_2=(4,3,-5)$$

We'll use the formula for the midpoint M of a line segment in three dimensions. We'll plug in the coordinates of the given points, $P_1 = (4, -6, 8)$ and $P_2 = (4, 3, -5)$.

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$$

$$M = \left(\frac{4 + 4}{2}, \frac{-6 + 3}{2}, \frac{8 + -5}{2}\right)$$

$$M = \left(\frac{8}{2}, \frac{-3}{2}, \frac{3}{2}\right)$$

$$M = \left(4, \frac{-3}{2}, \frac{3}{2}\right)$$

Let's work through a different type of example.

Example

Find the coordinates of point A if M is the midpoint of $\overline{AB}$.

$$M = (4.5, -3.5, 3)$$

$$B = (2, -4, 8)$$

Let's use (x_1, y_1, z_1) for A and (x_2, y_2, z_2) for B, and then use the midpoint formula and plug in what we know.

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$$

$$(4.5, -3.5, 3) = \left(\frac{x_1 + 2}{2}, \frac{y_1 + (-4)}{2}, \frac{z_1 + 8}{2}\right)$$

Now we'll equate the numbers on the left-hand side to the corresponding expressions on the right-hand side.

$$4.5 = \frac{x_1 + 2}{2}$$

$$-3.5 = \frac{y_1 + (-4)}{2}$$

$$3 = \frac{z_1 + 8}{2}$$

Finally, we'll solve these three equations separately, and we'll get

$$4.5 = \frac{x_1 + 2}{2}$$

$$2(4.5) = x_1 + 2$$

$$9 = x_1 + 2$$

$$7 = x_1$$

and

$$-3.5 = \frac{y_1 + (-4)}{2}$$

$$2(-3.5) = y_1 - 4$$

$$-7 = y_1 - 4$$

$$-3 = y_1$$

and

$$3 = \frac{z_1 + 8}{2}$$

$$2(3) = z_1 + 8$$

$$6 = z_1 + 8$$

$$-2 = z_1$$

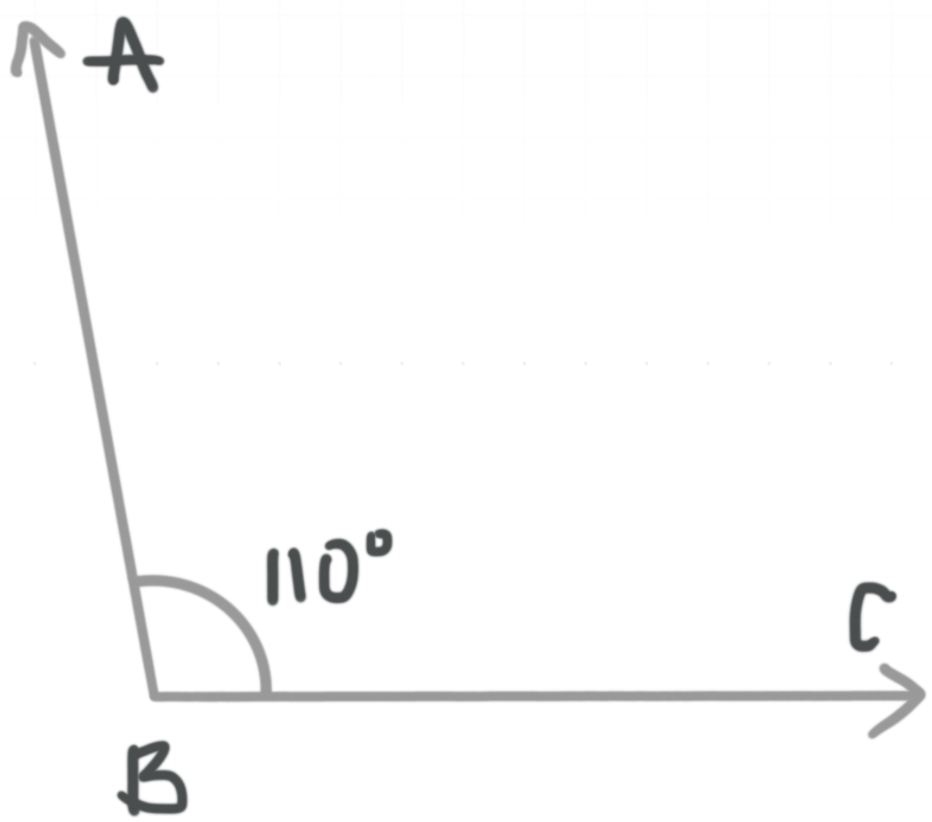
So the coordinates of point A are $(7, -3, -2)$.

Measures of angles

In this lesson we'll look at how to find the measures of angles, in degrees, algebraically.

The measure of angles

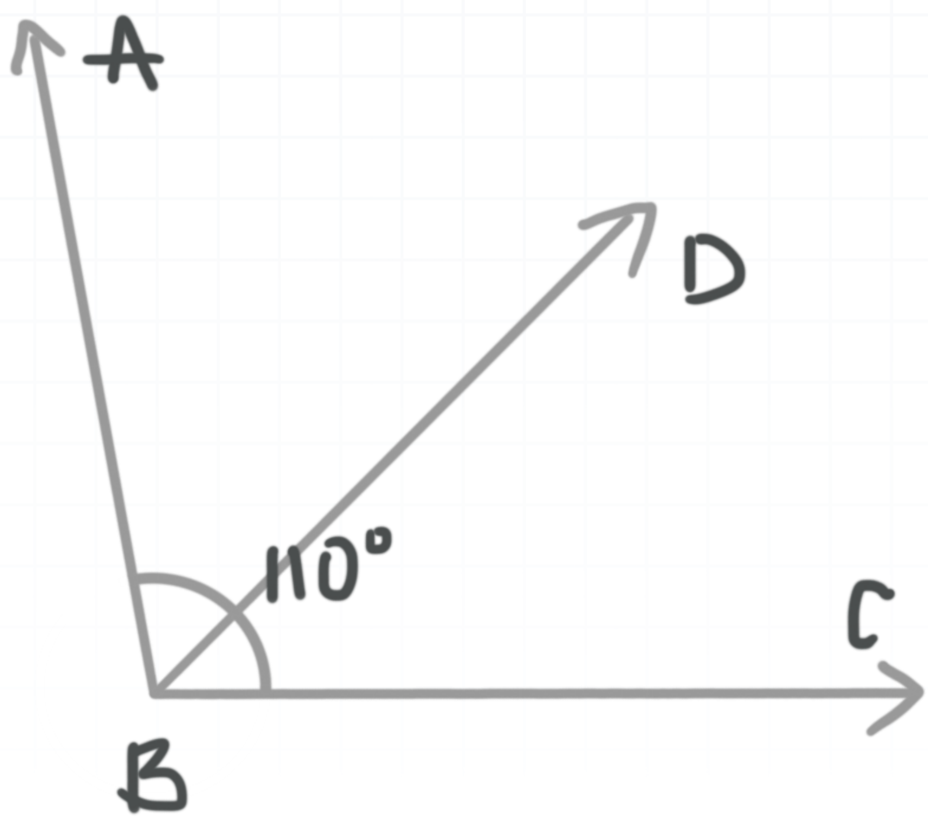
An angle is a fraction of a circle. The measure of an angle is the size of the "turn" (rotation) that's needed to get from the ray that forms one side of the angle to the ray that forms its other side. Angles are measured in degrees (or in radians, which you'll learn about if you study trigonometry).



The name of this angle is $\angle ABC$. When we talk about the measure of an angle, we use an m in front of the angle sign. For this angle, which has a measure of 110°, we write $m\angle ABC = 110^\circ$.

Angle addition

The individual parts of an angle add together to form the entire angle. For instance, in this figure,



the two smaller angles add together to equal the larger angle.

$$m\angle ABC = m\angle ABD + m\angle DBC$$

Which means that, if you know $m\angle DBC = 55^\circ$ and $m\angle ABC = 110^\circ$, you can find $m\angle ABD$ using angle addition.

$$m\angle ABC = m\angle ABD + m\angle DBC$$

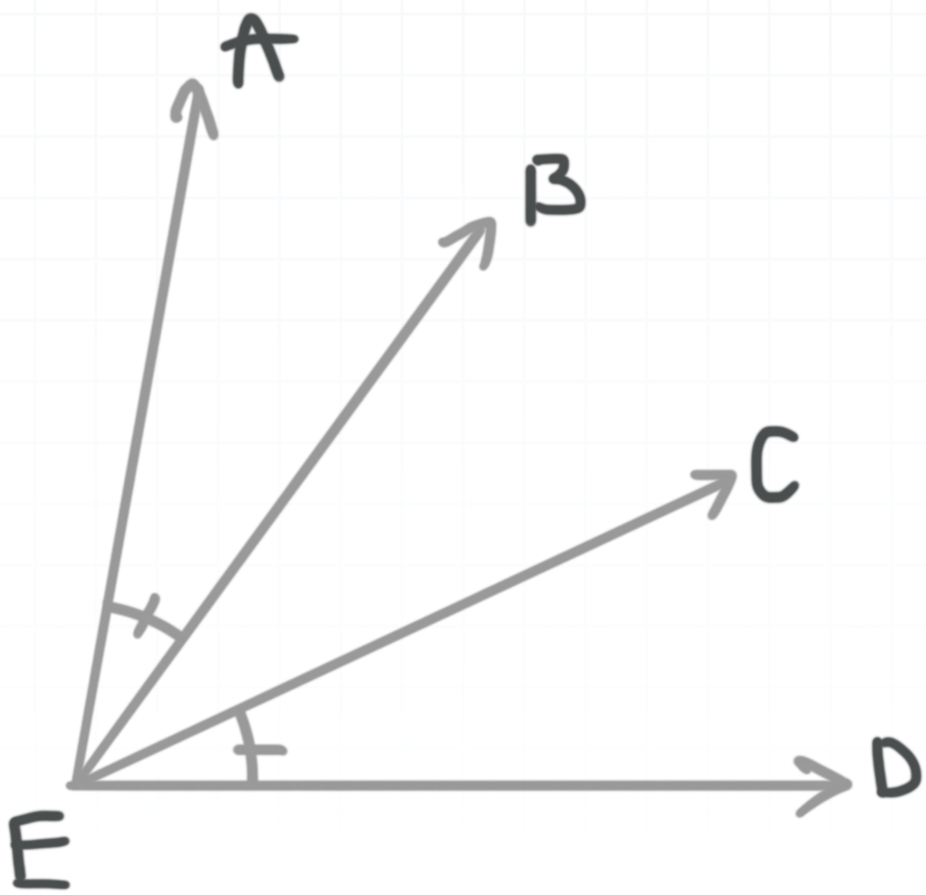
$$110^\circ = m\angle ABD + 55^\circ$$

$$55^\circ = m\angle ABD$$

Let's do a few examples.

Example

If $m\angle AED = 80^\circ$ and $m\angle AEB = 30^\circ$, what is $m\angle BEC$?



Let's organize what we know. In the diagram, the short line segments that cross the circular arcs inside angles AEB and CED mean that $\angle AEB$ is congruent to $\angle CED$, which is another way of saying that the measures of angles AEB and CED are equal. So we know that $m\angle CED = m\angle AEB = 30^\circ$.

We also know the measure of the entire angle, $m\angle AED = 80^\circ$, and that $m\angle AED = m\angle AEB + m\angle BEC + m\angle CED$. Let's let $x = m\angle BEC$. Then we get

$$m\angle AED = m\angle AEB + x + m\angle CED$$

Now we'll substitute the measures of the other three angles.

$$80^\circ = 30^\circ + x + 30^\circ$$

$$80^\circ = 60^\circ + x$$

$$x = 20^\circ$$

So $m\angle BEC = 20^\circ$.

Here's another type of problem you might see.

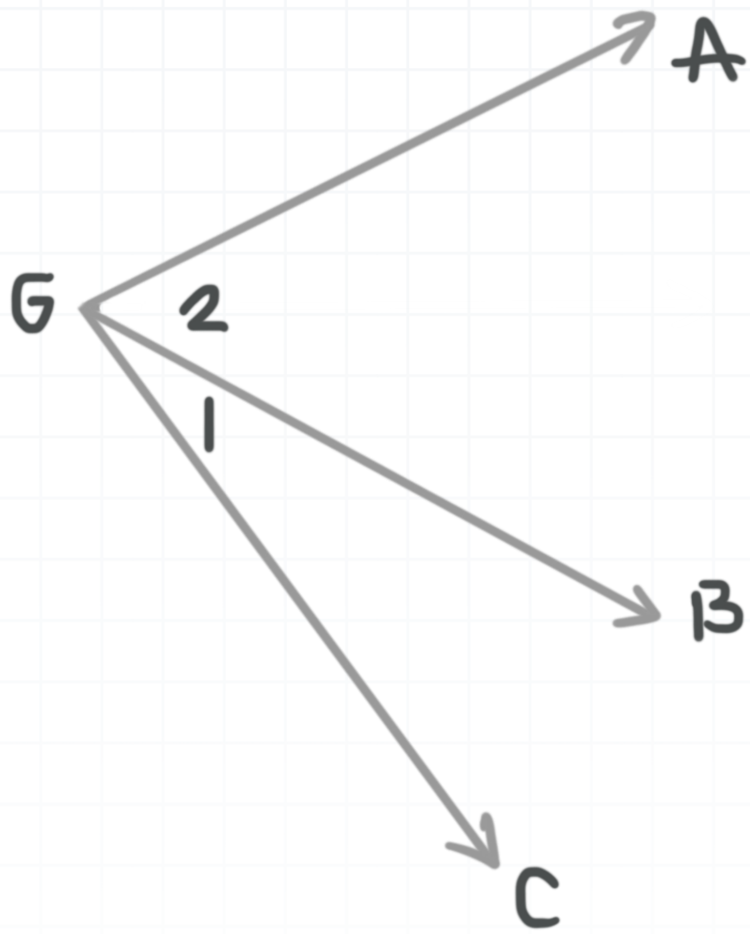
Example

Find the measure of angle 2 if x is in degrees.

$$m\angle 1 = 2x$$

$$m\angle 2 = 5x + 5^\circ$$

$$m\angle AGC = 105^\circ - 3x$$



We can set up an equation, solve for x, then substitute back in to find $m\angle 2$.

$$m\angle 1 + m\angle 2 = m\angle AGC$$

$$2x + 5x + 5^\circ = 105^\circ - 3x$$

$$7x + 5^\circ = 105^\circ - 3x$$

$$7x + 3x + 5^\circ = 105^\circ$$

$$10x + 5^\circ = 105^\circ$$

$$10x = 100^\circ$$

$$x = 10^\circ$$

Substituting 10° for x in the equation $m\angle 2 = 5x + 5^\circ$, we get

$m\angle 2 = 5(10^\circ) + 5^\circ = 55^\circ$.

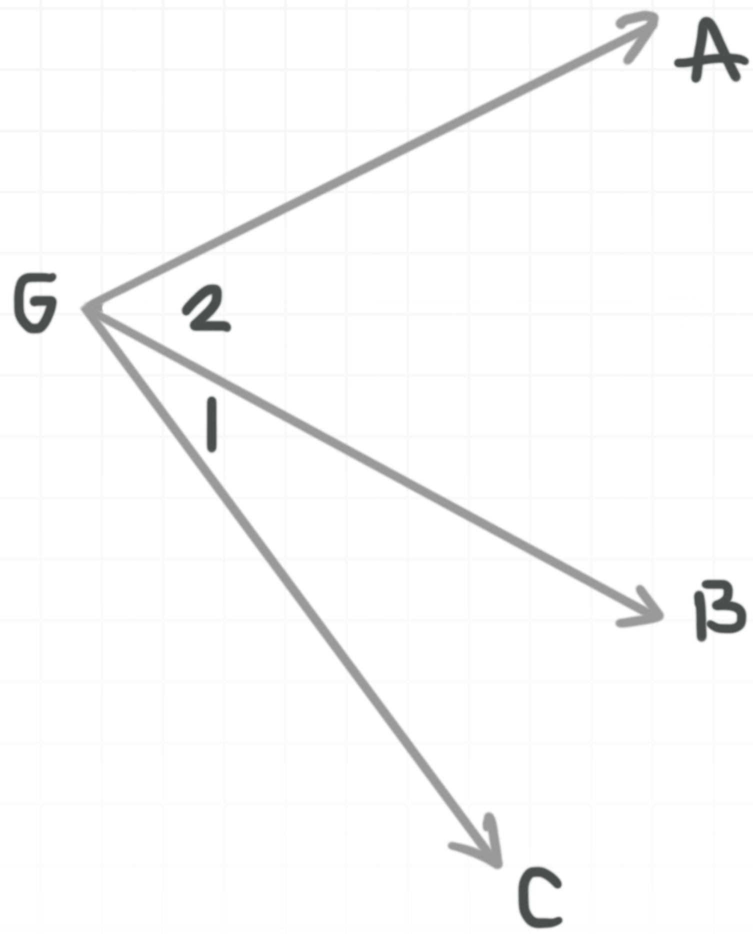
Adjacent and nonadjacent angles

In this lesson we'll look at how to identify adjacent angles in a diagram and how to interpret the name of an angle if it's given as a sequence of three letters.

Adjacent angles

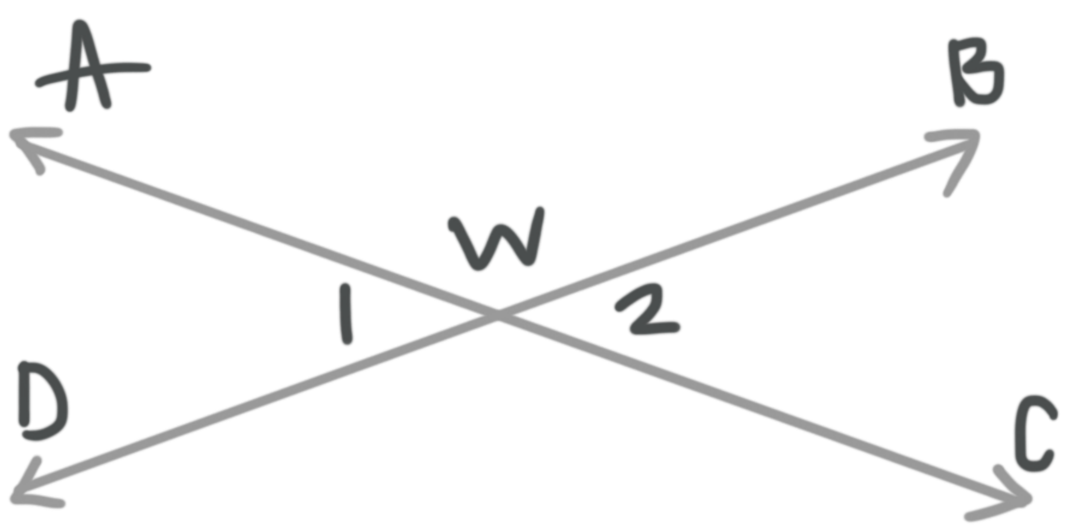
A pair of angles are **adjacent angles** if (a) they share a vertex and one ray (or side) and (b) their interiors don't overlap. The **interior of an angle** is made of all the points between the two rays that make the angle. It excludes the vertex and the rays themselves.

For instance, in this diagram of $\angle AGC$, angles 1 and 2 are adjacent because they share vertex G and ray $\overrightarrow{GB}$, and their interiors don't overlap.



Nonadjacent angles

Nonadjacent angles may or may not share a vertex, but either they don't share a ray or (even if they do share a ray) their interiors overlap.

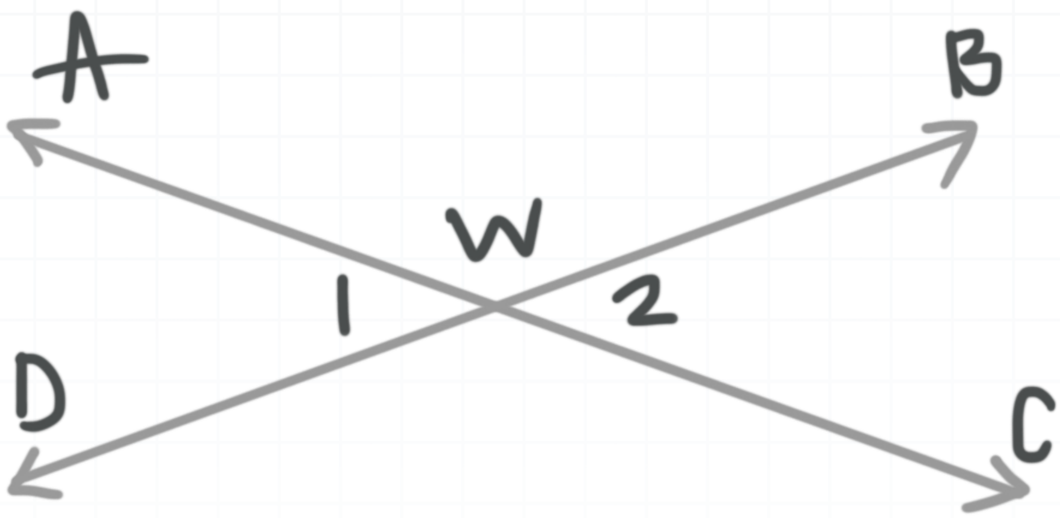


For instance, in this diagram, angles 1 and 2 are not adjacent, even though they share vertex W, because they do not share a ray.

Let's do a few example problems.

Example

List the pairs of adjacent angles in the diagram.



Adjacent angles share a vertex and one ray, and their interiors don't overlap.

- $\angle DWA$ is adjacent to $\angle AWB$; they share vertex W and ray $\overrightarrow{WA}$, and their interiors don't overlap.
- $\angle DWA$ is adjacent to $\angle DWC$: they share vertex W and ray $\overrightarrow{WD}$, and their interiors don't overlap.
- $\angle DWC$ is adjacent to $\angle CWB$; they share vertex W and ray $\overrightarrow{WC}$, and their interiors don't overlap.
- $\angle CWB$ is adjacent to $\angle AWB$; they share vertex W and ray $\overrightarrow{WB}$, and their interiors don't overlap.

Notice that you can tell from the way the names of the angles are written which ray they share. For example,

> $\angle CWB$ is adjacent to $\angle AWB$; they share vertex W (which is the "middle letter" in both of their names) and ray $\overrightarrow{WB}$, since $\angle AWB$ and $\angle CWB$ both have the letters W and B (from ray $\overrightarrow{WB}$) in their name.

Let's see how to identify possible adjacent angles without a diagram.

Example

All of the following pairs of angles are from the same diagram. For each pair, determine whether they're possibly adjacent or definitely nonadjacent.

A. $\angle ABC$ and $\angle CBD$

B. $\angle XYZ$ and $\angle ZYX$

C. $\angle LMN$ and $\angle MNP$

D. $\angle CED$ and $\angle CEP$

Remember that adjacent angles have the same vertex and share one ray. If the name of an angle is given as a sequence of three letters, the letter that corresponds to the vertex is always in the middle. You can then use the name of the angle to identify the rays that make the angle.

Let's look at each pair of angles.

A. $\angle ABC$ and $\angle CBD$. Both of these angles have vertex B. Now we need to look for one common ray. $\angle ABC$ has rays $\overrightarrow{BA}$ and $\overrightarrow{BC}$, and $\angle CBD$ has rays $\overrightarrow{BC}$ and $\overrightarrow{BD}$, which means that these angles share ray $\overrightarrow{BC}$. Therefore, $\angle ABC$ and $\angle CBD$ could be adjacent, but we can't tell for sure, because we don't know if their interiors overlap.

B. $\angle XYZ$ and $\angle ZYX$. Both of these angles have vertex Y. Now we need to look for one common ray. $\angle XYZ$ has rays $\overrightarrow{YX}$ and $\overrightarrow{YZ}$, and $\angle ZYX$ has rays $\overrightarrow{YZ}$ and $\overrightarrow{YX}$, which means that these angles share rays $\overrightarrow{YX}$ and $\overrightarrow{YZ}$. Therefore, $\angle XYZ$ and $\angle ZYX$ are just different ways to name the same angle, so they're nonadjacent.

C. $\angle LMN$ and $\angle MNP$. $\angle LMN$ has vertex M, and $\angle MNP$ has vertex N, so these angles do not share a vertex and therefore are nonadjacent.

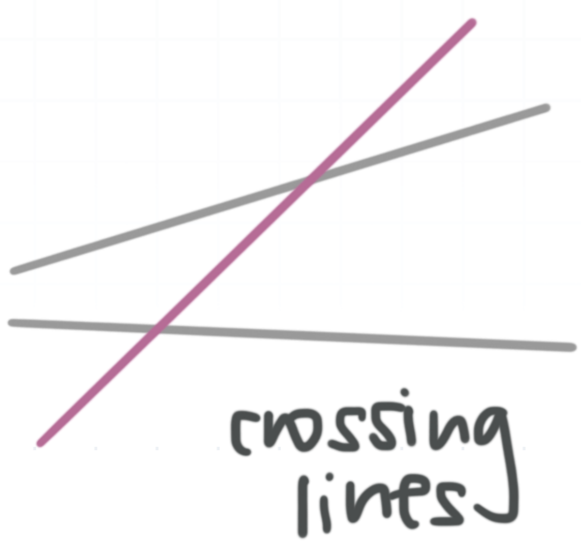
D. $\angle CED$ and $\angle CEP$. Both of these angles have vertex E. Now we need to look for one common ray. $\angle CED$ has rays $\overrightarrow{EC}$ and $\overrightarrow{ED}$, and $\angle CEP$ has rays $\overrightarrow{EC}$ and $\overrightarrow{EP}$, which means that these angles share ray $\overrightarrow{EC}$. Therefore, $\angle CED$ and $\angle CEP$ could be adjacent, but we can't tell for sure, because we don't know if their interiors overlap.

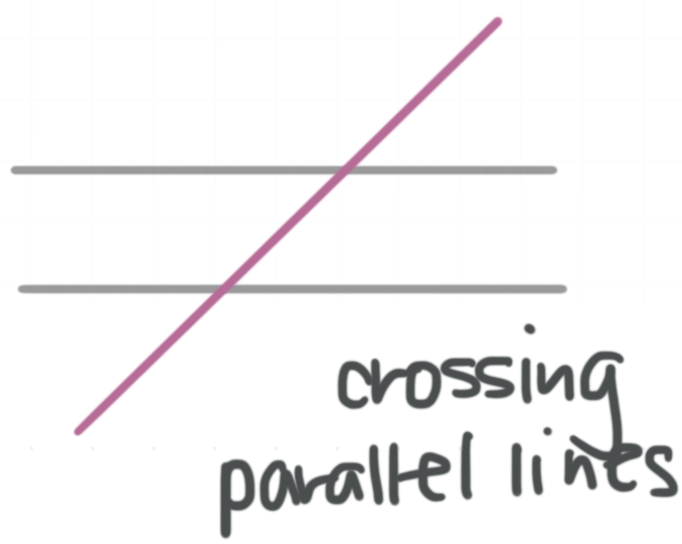
Angles and transversals

In this lesson we'll look at the angles formed when a pair of parallel lines is crossed by another line, called a "transversal."

Transversals

A **transversal** is a line that crosses at least two other lines. In the figure on the left, the transversal is crossing two non-parallel lines, and in the figure on the right, the transversal is crossing two parallel lines.





Special angle pairs

When a transversal crosses a pair of parallel lines, pairs of angles with special angle relationships are formed. In terms of angle measure, there are two types of angle pairs with these special relationships.

Congruent angles, which have the same measure.

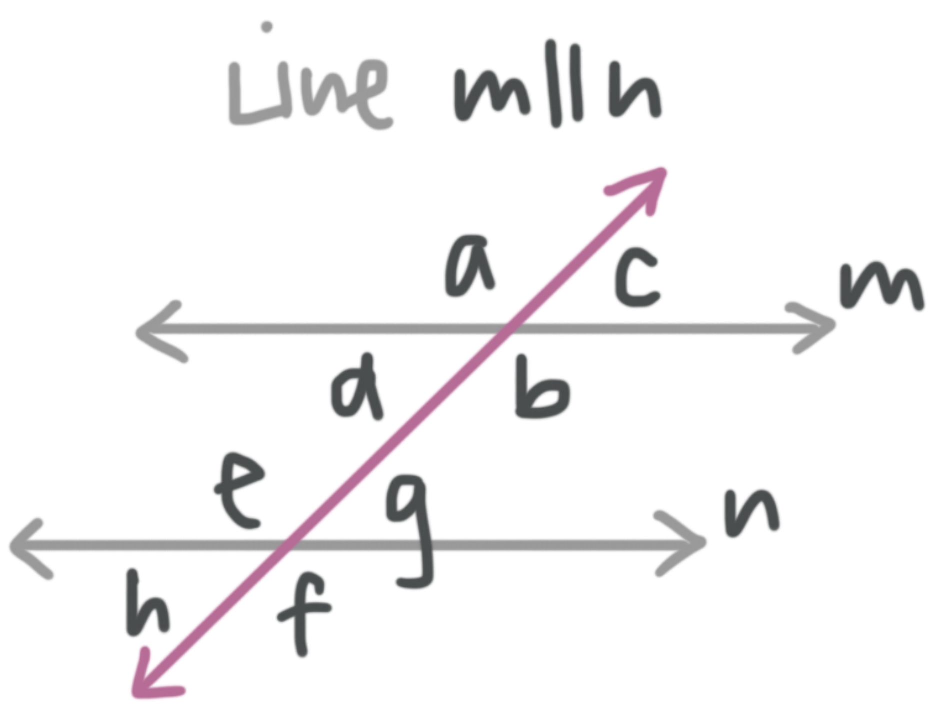
Supplementary angles, which have measures that add to 180°.

Actually, any pair of angles that have the same measure are congruent, not just pairs of angles that are formed when a transversal crosses a pair of parallel lines. And any pair of angles whose measures add up to 180° are supplementary.

Types of special angle pairs

Vertical angles share a vertex (which is the point of intersection of the transversal and one of the lines in the pair of parallel lines crossed by the transversal) but no ray, and they lie on opposite sides of the transversal. Vertical angle pairs are congruent.

There are four pairs of vertical angles in the figure below: (a, b), (c, d), (e, f), and (g, h). If a pair of lines are parallel (and we call them, say, m and n), we denote that by $m \parallel n$.



$$m\angle a = m\angle b$$

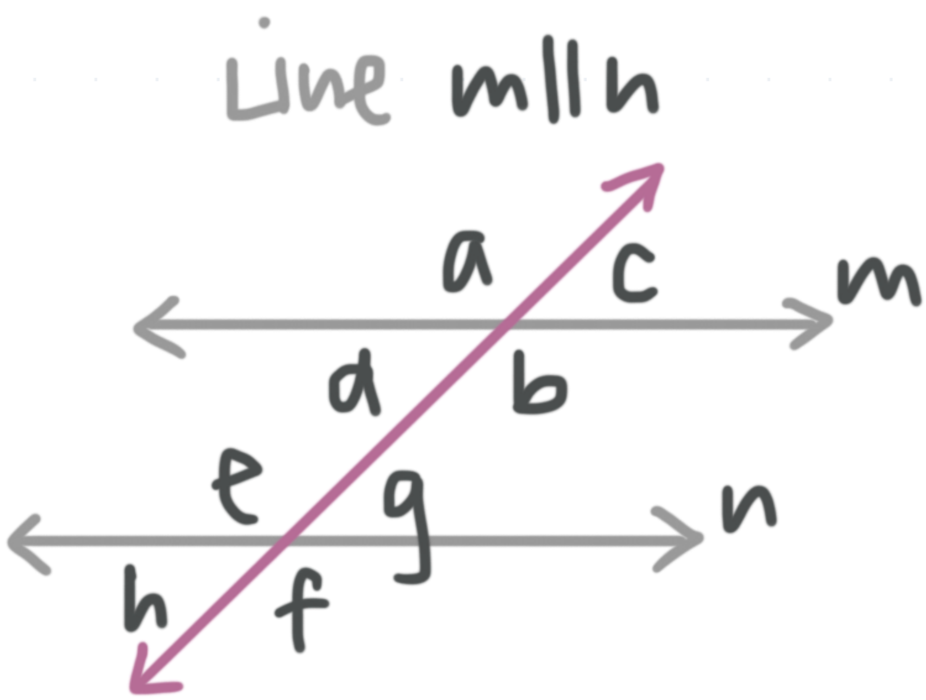
$$m\angle c = m\angle d$$

$$m\angle e = m\angle f$$

$$m\angle g = m\angle h$$

Corresponding angles are angles that lie on the same side of the transversal, their vertices are on opposite lines (in the pair of parallel lines crossed by the transversal), the interior of one of the angles in the pair lies partially inside the region between the parallel lines, and the interior of the other angle lies entirely outside the region between the parallel lines but entirely within the interior of the first angle. Corresponding angle pairs are congruent.

There are four pairs of corresponding angles in the figure below: (a, e), (b, f), (c, g), and (d, h).



$$m\angle a = m\angle e$$

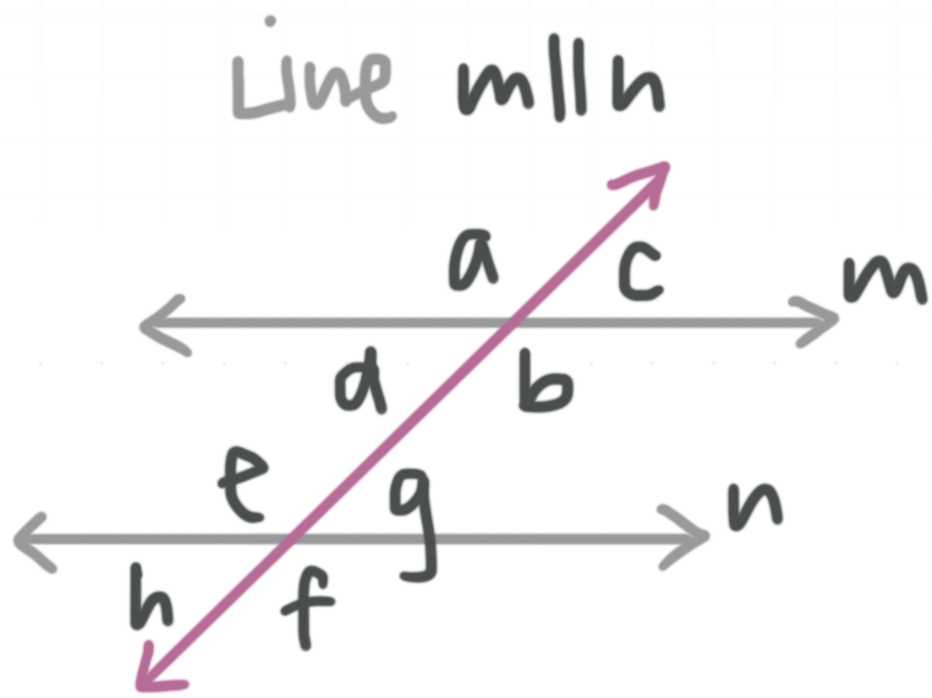
$$m\angle d = m\angle h$$

$$m\angle c = m\angle g$$

$$m\angle b = m\angle f$$

Alternate interior angles are angles that lie on opposite sides of the transversal, their vertices are on opposite lines (in the pair of parallel lines crossed by the transversal), and the interior of each angle in the pair lies partially inside the region between the parallel lines. Alternate interior angle pairs are congruent.

There are two pairs of alternate interior angles in the figure below: (d, g) and (b, e).

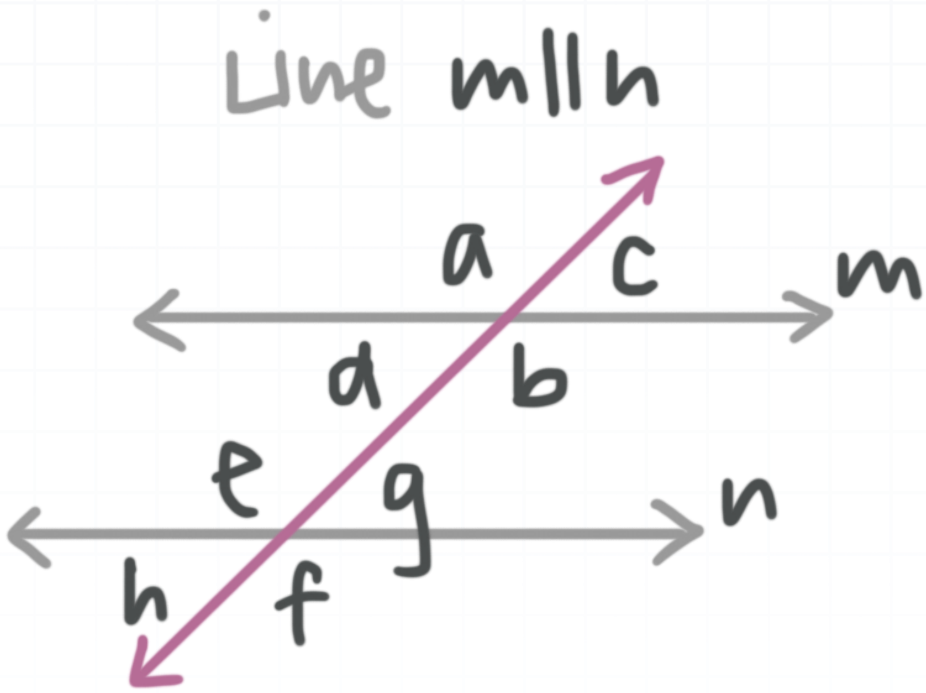


$$m\angle d = m\angle g$$

$$m\angle b = m\angle e$$

Alternate exterior angles are angles that lie on opposite sides of the transversal, their vertices are on opposite lines (in the pair of parallel lines crossed by the transversal), and the interior of each angle in the pair lies entirely outside the region between the parallel lines. Alternate exterior angle pairs are congruent.

There are two pairs of alternate exterior angles in the figure below: (a, f) and (c, h).

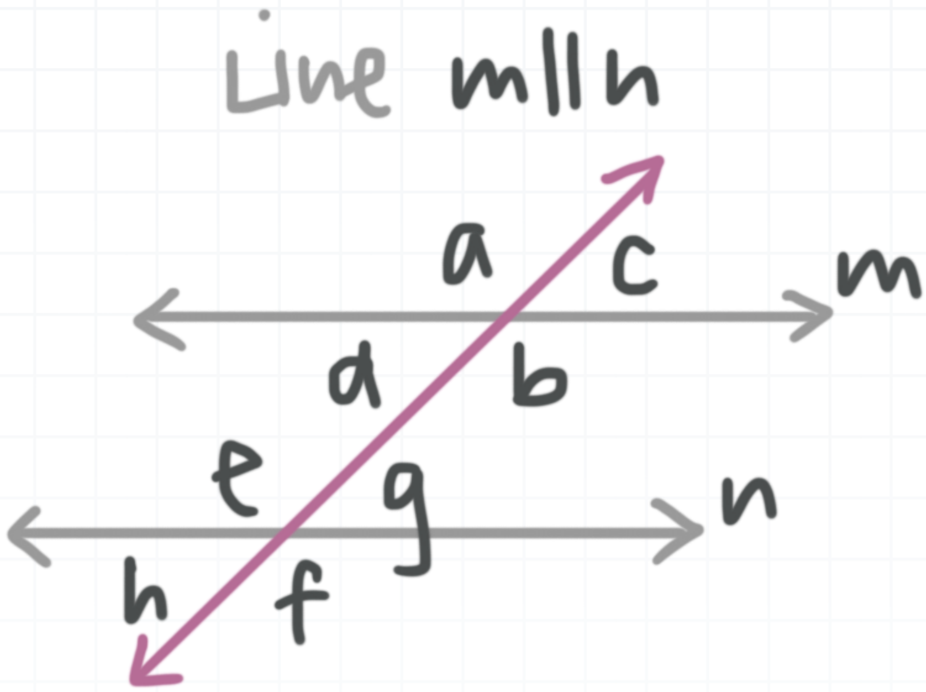


$$m\angle a = m\angle f$$

$$m\angle c = m\angle h$$

Consecutive interior angles are angles that lie on the same side of the transversal, their vertices are on opposite lines (in the pair of parallel lines crossed by the transversal), the interior of each angle in the pair lies partially inside the region between the parallel lines, and the interiors of the angles overlap in the region between the parallel lines. Consecutive interior angle pairs are supplementary.

There are two pairs of consecutive interior angles in the figure below: (b, g) and (d, e).

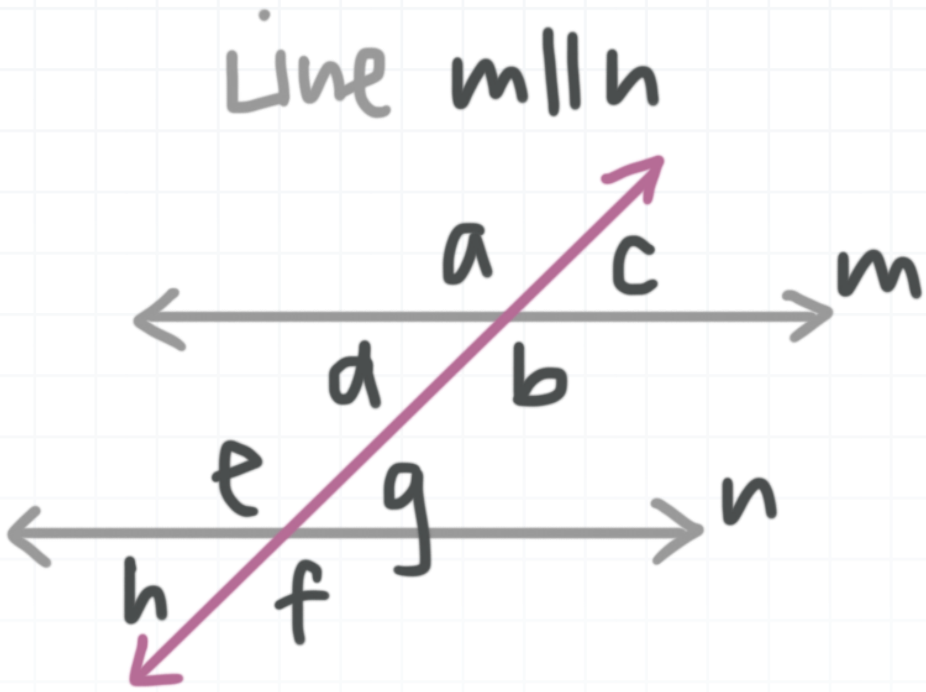


$$m\angle d + m\angle e = 180^\circ$$

$$m\angle b + m\angle g = 180^\circ$$

Adjacent angles are angles that share a vertex and one ray, and they can be on the same side of the transversal or on opposite sides of it, but (like all pairs of adjacent angles, not just those that are formed when a transversal crosses a pair of parallel lines) their interiors do not overlap. Adjacent angle pairs are supplementary.

There are eight pairs of adjacent angles in the figure below: four pairs that lie on the same side of the transversal ((a, d), (b, c), (e, h), and (f, g)) and four pairs that lie on opposite sides of the transversal ((a, c), (b, d), (e, g), and (f, h)).



$m\angle a + m\angle d = 180^\circ$

$m\angle a + m\angle c = 180^\circ$

$m\angle b + m\angle c = 180^\circ$

$m\angle d + m\angle b = 180^\circ$

$m\angle e + m\angle g = 180^\circ$

$m\angle e + m\angle h = 180^\circ$

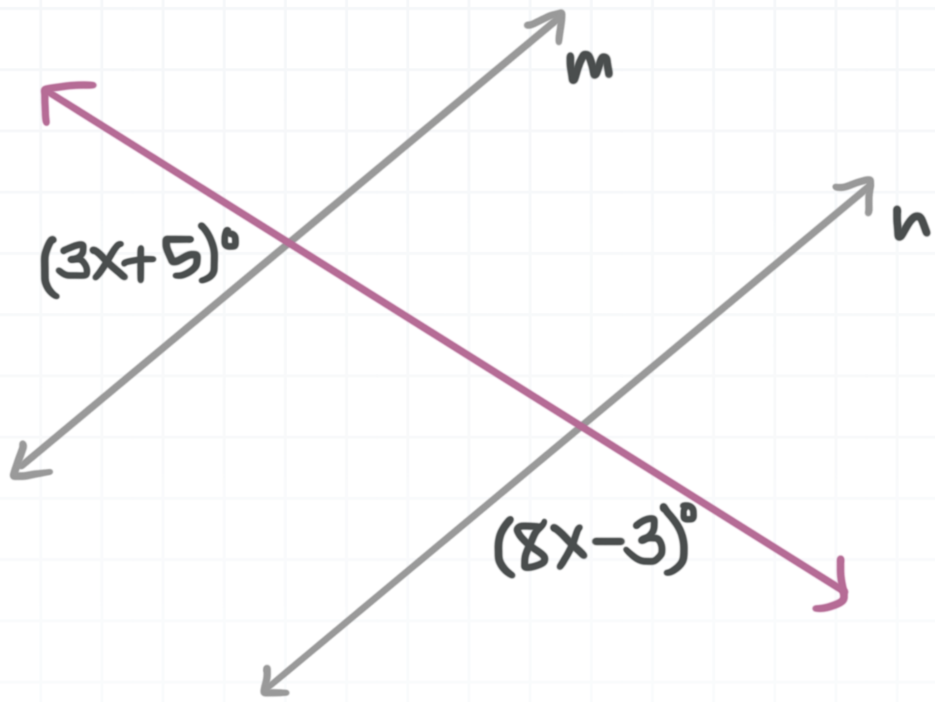
$m\angle g + m\angle f = 180^\circ$

$m\angle f + m\angle h = 180^\circ$

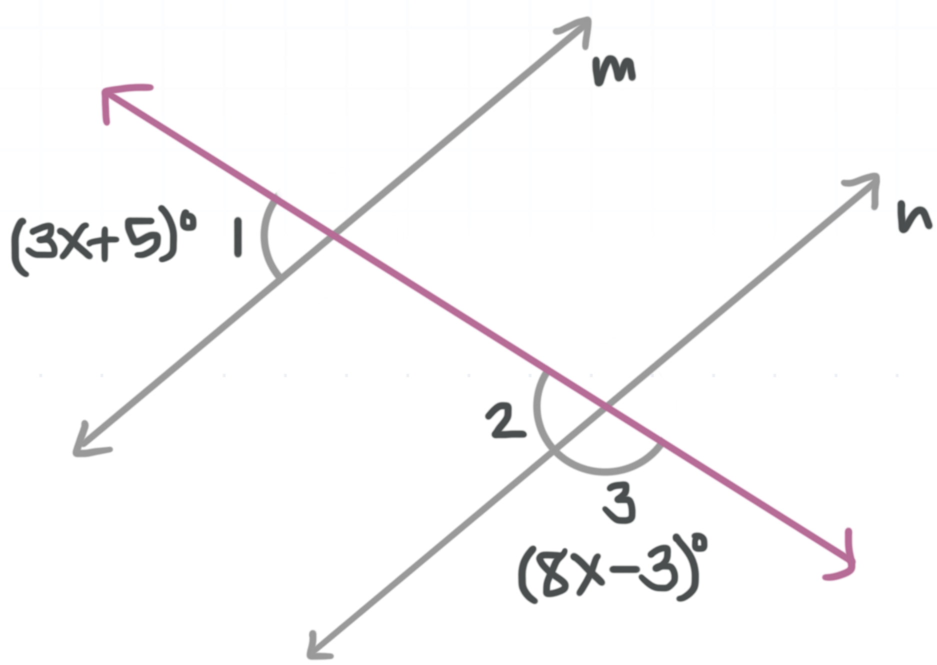
We often use these angle pair relationships to solve problems.

Example

Solve for the variable. Find the value of x to the nearest tenth, given that $m \parallel n$.



Let's think about which angle pair relationships to use.



Angles 1 and 2 are congruent because they are a pair of corresponding angles. Angles 2 and 3 are supplementary because they are adjacent angles. Combining these facts, we see that angles 1 and 3 are supplementary, so

$$(3x+5)^\circ + (8x-3)^\circ = 180^\circ$$

$$3x^\circ + 5^\circ + 8x^\circ - 3^\circ = 180^\circ$$

$$11x^\circ + 2^\circ = 180^\circ$$

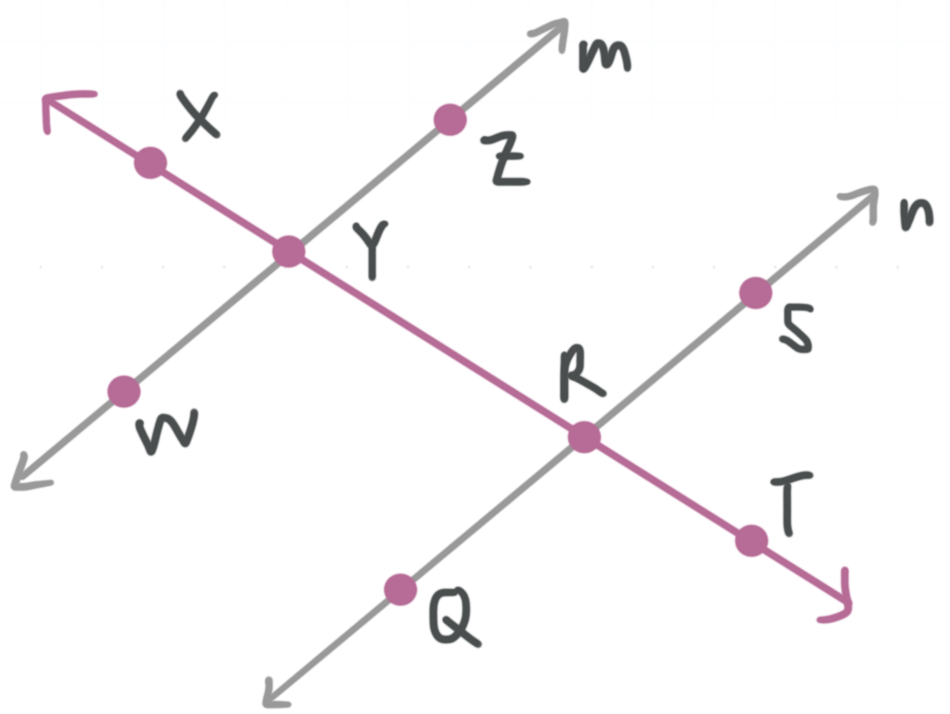
$$11x^\circ = 178^\circ$$

$$x \approx 16.2$$

Let's do one more.

Example

What is the measure of $\angle XYZ$, given that $m \parallel n$, $m\angle ZYR = (5x + 35)^\circ$, and $m\angle QRY = (15x - 5)^\circ$?



Looking at the diagram, we can see that $\angle XYZ$ and $\angle ZYR$ are adjacent angles, so they're supplementary. $\angle ZYR$ and $\angle QRY$ are alternate interior angles, so they're congruent.

Now we can use these facts to find $m\angle XYZ$. Let's begin by solving for x. $m\angle ZYR = m\angle QRY$ because they are congruent angles.

$$m\angle ZYR = m\angle QRY$$

$$(5x + 35)^\circ = (15x - 5)^\circ$$

$$35^\circ = 10x^\circ - 5^\circ$$

$$40^\circ = 10x^\circ$$

$$x = 4$$

Now we can find $m\angle ZYR$ by substituting 4 for x.

$$m\angle ZYR = (5x + 35)^\circ$$

$$m\angle ZYR = (5 \cdot 4 + 35)^\circ$$

$$m\angle ZYR = (20 + 35)^\circ$$

$$m\angle ZYR = 55^\circ$$

And because $\angle XYZ$ and $\angle ZYR$ are supplementary angles,

$$m\angle XYZ + m\angle ZYR = 180^\circ$$

$$m\angle XYZ + 55^\circ = 180^\circ$$

$$m\angle XYZ + 55^\circ - 55^\circ = 180^\circ - 55^\circ$$

$m\angle XYZ = 125^\circ$

Interior angles of polygons

In this lesson we'll look at how to find the measures of the interior angles of polygons.

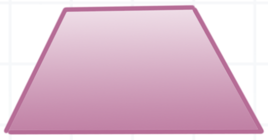
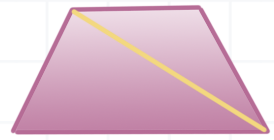
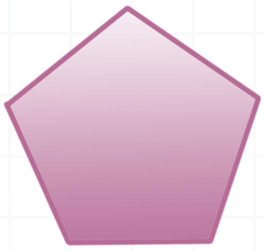
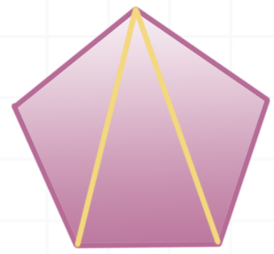
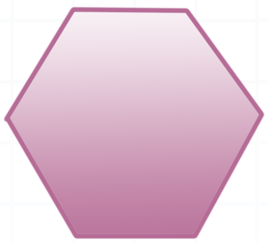
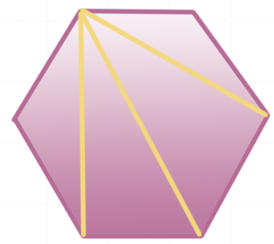
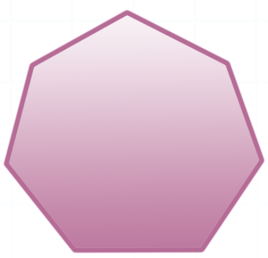
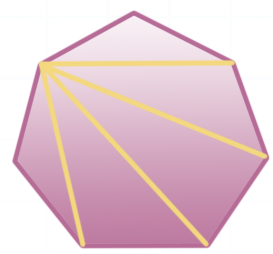
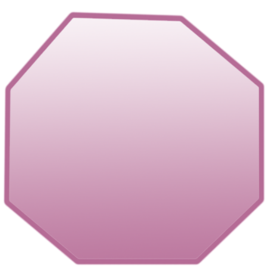
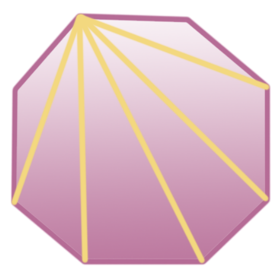
Triangles

Triangles are 3-sided polygons. The measures of the three interior angles of any triangle (the three angles inside the triangle) add up to 180°. For instance, in this figure, $m\angle 1+m\angle 2+m\angle 3=180^\circ$.

Polygons

The word "polygon" means "many-sided figure." A polygon has the same number of interior angles as it has sides, and a regular polygon has equal angles and equal sides.

Any polygon can be divided into triangles.

Picture	Name	Sides	Triangles	Degrees inside
	Quadrilateral	4	2: 	$2(180^\circ) = 360^\circ$
	Pentagon	5	3: 	$3(180^\circ) = 540^\circ$
	Hexagon	6	4: 	$4(180^\circ) = 720^\circ$
	Heptagon	7	5: 	$5(180^\circ) = 900^\circ$
	Octagon	8	6: 	$6(180^\circ) = 1{,}080^\circ$
	...	...	...	...
	n-gon	n	$n-2$	$(n-2)180^\circ$

Let's start by working through an example.

Example

What is the measure of each interior angle in a regular icosagon (a 20 -sided figure)?

The sum of the measures of the interior angles in a polygon is $(n-2)180^\circ$, where n is the number of sides in the polygon. For an icosagon, which is a 20-sided figure, that would be

$$(20-2)180^\circ = 3{,}240^\circ$$

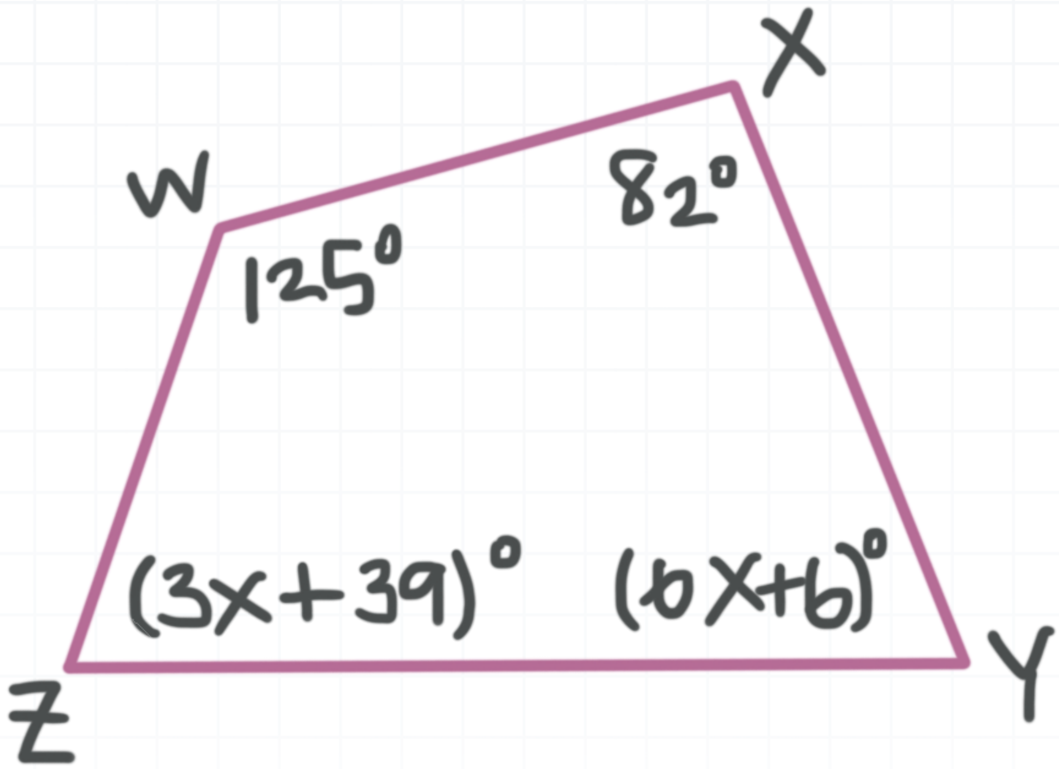
There are 20 congruent interior angles because the shape is regular, so each interior angle measures

$$3{,}240^\circ \div 20 = 162^\circ$$

If the shape is not regular, then we can't assume that all of the angles are congruent. Let's look at an example of a non-regular quadrilateral in which the angles aren't equal.

Example

What is the measure of $\angle Z$?



The sum of the measures of the interior angles in a polygon with n sides is $(n-2)180^\circ$. For a quadrilateral, that would be $(4-2)180^\circ = 360^\circ$. Set the sum of the four angles equal to 360° and then solve for x.

$$125^\circ + 82^\circ + (3x + 39)^\circ + (6x + 6)^\circ = 360^\circ$$

$$(125 + 82 + 39 + 6)^\circ + (3x + 6x)^\circ = 360^\circ$$

$$252^\circ + 9x^\circ = 360^\circ$$

$$9x^\circ = 108^\circ$$

$$x = 12$$

Substitute 12 for x in $(3x + 39)^\circ$ to find $m\angle Z$.

$$m\angle Z = (3 \cdot 12 + 39)^\circ$$

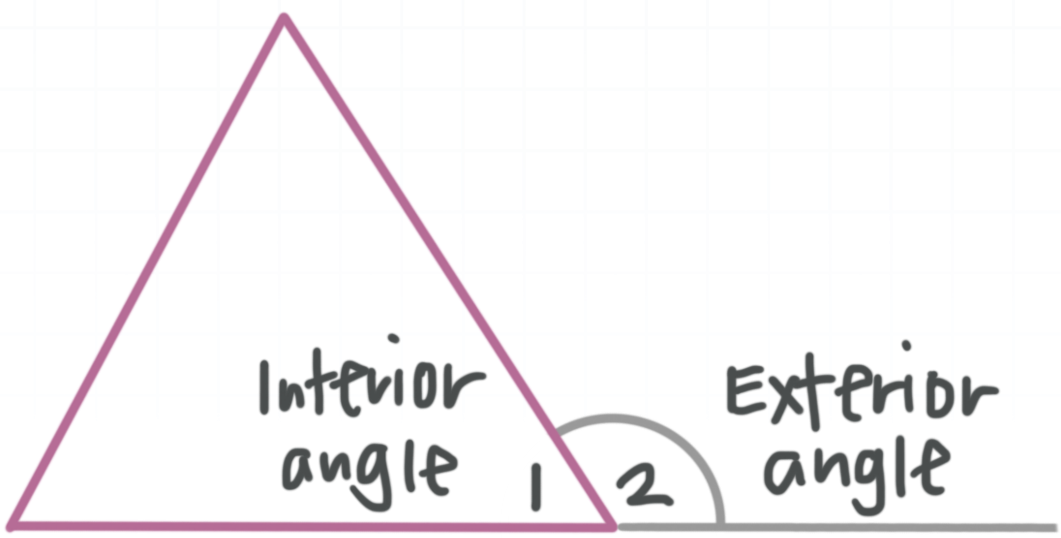
$$m\angle Z = (36 + 39)^\circ$$

$m\angle Z = 75^\circ$

Exterior angles of polygons

In this lesson we'll look at exterior angles of polygons and the relationship between those and their corresponding interior angles.

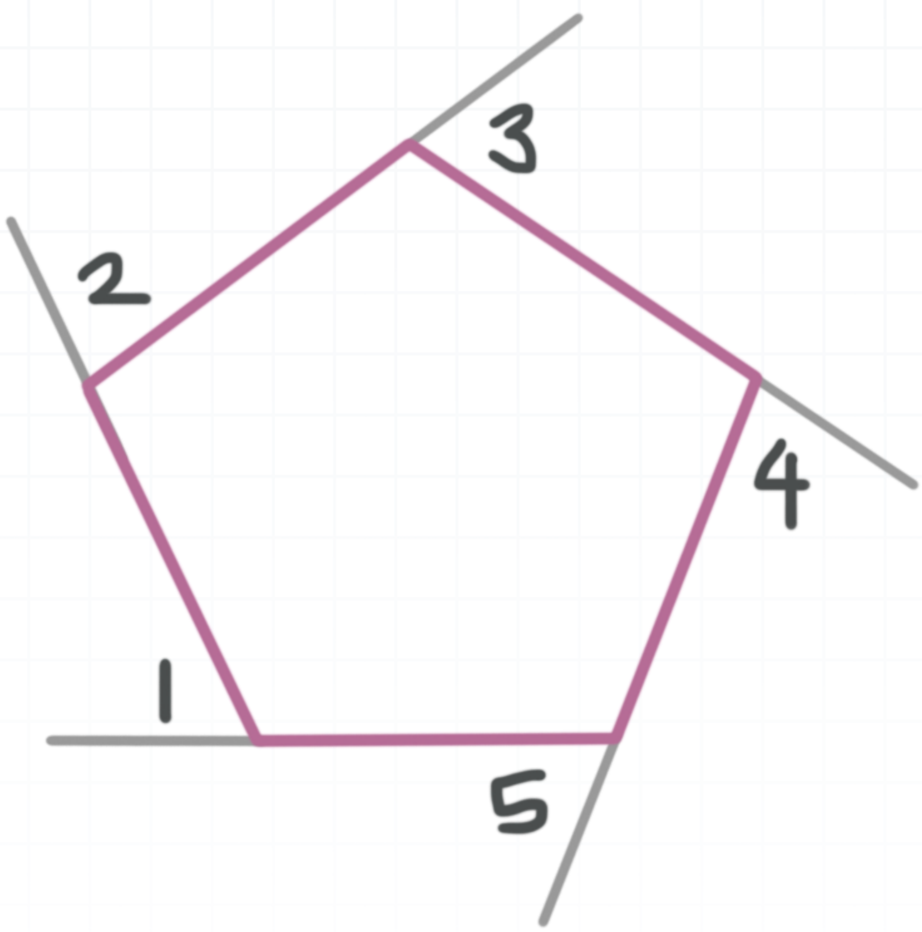
An exterior angle of a polygon is an angle that's supplementary to one of the interior angles of the polygon, has its vertex at the vertex of that interior angle, and is formed by extending one of the two sides of the polygon (at that vertex) in the direction opposite (180° away from) that side.



Realize that at each vertex of a polygon there are two exterior angles: one formed by extending one of the two sides at that vertex, and one formed by extending the other side at that vertex. Since both of the exterior angles at a given vertex are supplementary to the same interior angle, the two exterior angles always have equal measure.

This means that, in the figure above, $m\angle 1 + m\angle 2 = 180^\circ$.

The sum of the measures of the exterior angles in any polygon is 360° if we include only one of the two exterior angles at each vertex, which is what we'll mean going forward when we talk about a polygon's exterior angles. Here is an example of the exterior angles of a pentagon adding to 360°.

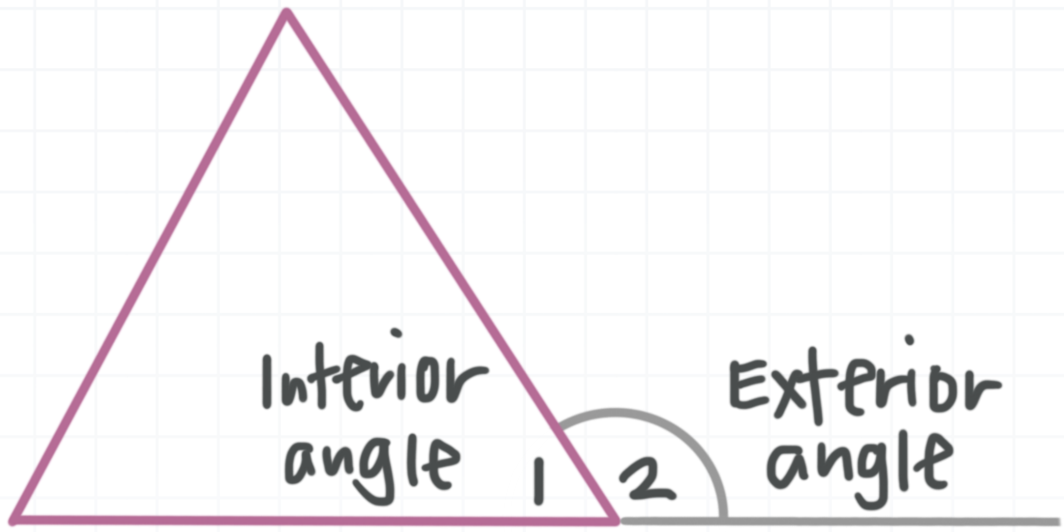


$$m\angle 1 + m\angle 2 + m\angle 3 + m\angle 4 + m\angle 5 = 360^\circ$$

Remember also that the sum of the measures of the interior angles of a polygon with n sides is $(n-2)180^\circ$.

Example

Given that the triangle in the diagram is equilateral, what is the measure of angle 2? ("Lateral" means "side," so an equilateral polygon is a polygon in which all sides have equal length.)



The measures of the interior angles in a triangle add to 180°. An equilateral triangle is also equiangular (which means that all of its interior angles have the same measure). That means that each interior angle measures $180^\circ \div 3 = 60^\circ$, so $m\angle 1 = 60^\circ$. $\angle 1$ and $\angle 2$ are supplementary, which means that

$$m\angle 1 + m\angle 2 = 180^\circ$$

$$60^\circ + m\angle 2 = 180^\circ$$

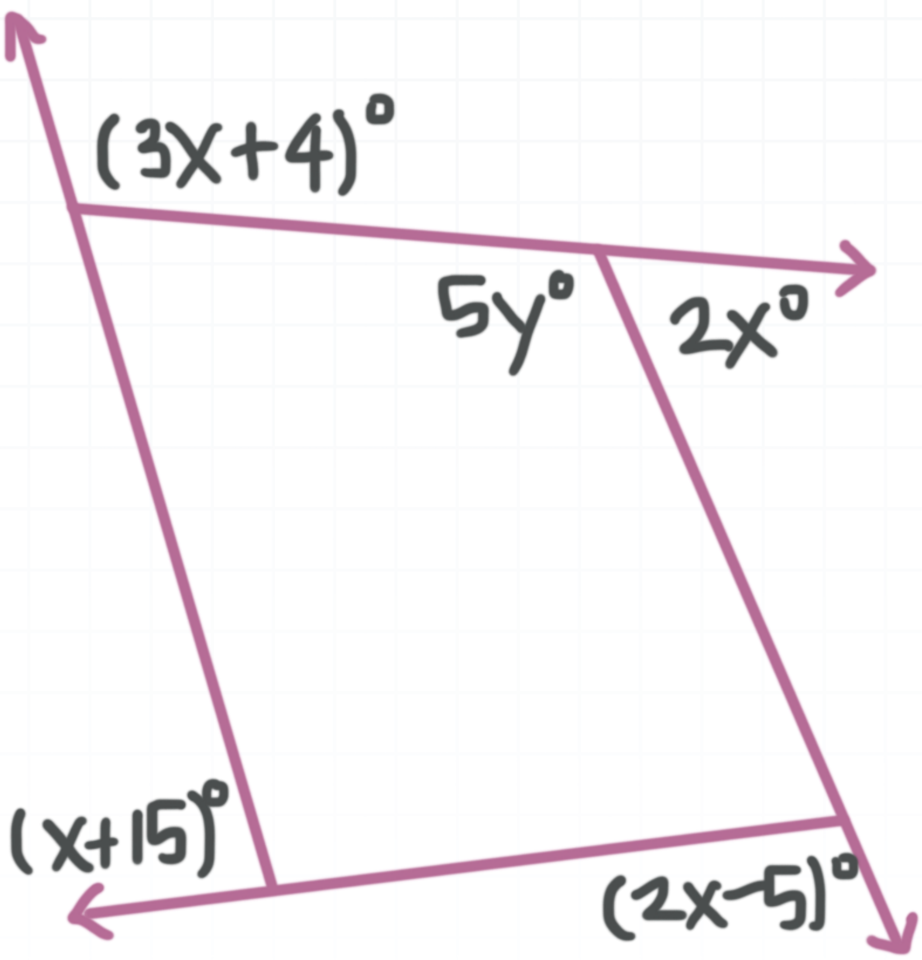
$$m\angle 2 = 120^\circ$$

We could also have solved this problem by using the fact that all of the exterior angles sum to 360°. A triangle has three interior angles, so it also has three exterior angles (if we include only one of the two exterior angles at each vertex). Since all of the interior angles are congruent, all of the exterior angles will also be congruent. This means that $m\angle 2 = 360^\circ \div 3 = 120^\circ$.

Let's look at another example.

Example

Find the value of y.



The sum of the measures of the quadrilateral's four exterior angles, shown in the figure, must be 360°. Therefore,

$$3x^\circ + 4^\circ + 2x^\circ + 2x^\circ - 5^\circ + x^\circ + 15^\circ = 360^\circ$$

$$3x^\circ + 2x^\circ + 2x^\circ + x^\circ + 4^\circ - 5^\circ + 15^\circ = 360^\circ$$

$$8x^\circ + 14^\circ = 360^\circ$$

$$8x^\circ = 346^\circ$$

$$x = 43.25^\circ$$

The interior angle of measure $5y^\circ$ and the exterior angle of measure $2x^\circ$ are supplementary.

$$5y^\circ + 2x^\circ = 180^\circ$$

Substitute 43.25 for x and solve for y.

$$5y^\circ + 2(43.25)^\circ = 180^\circ$$

$$5y^\circ + 86.5^\circ = 180^\circ$$

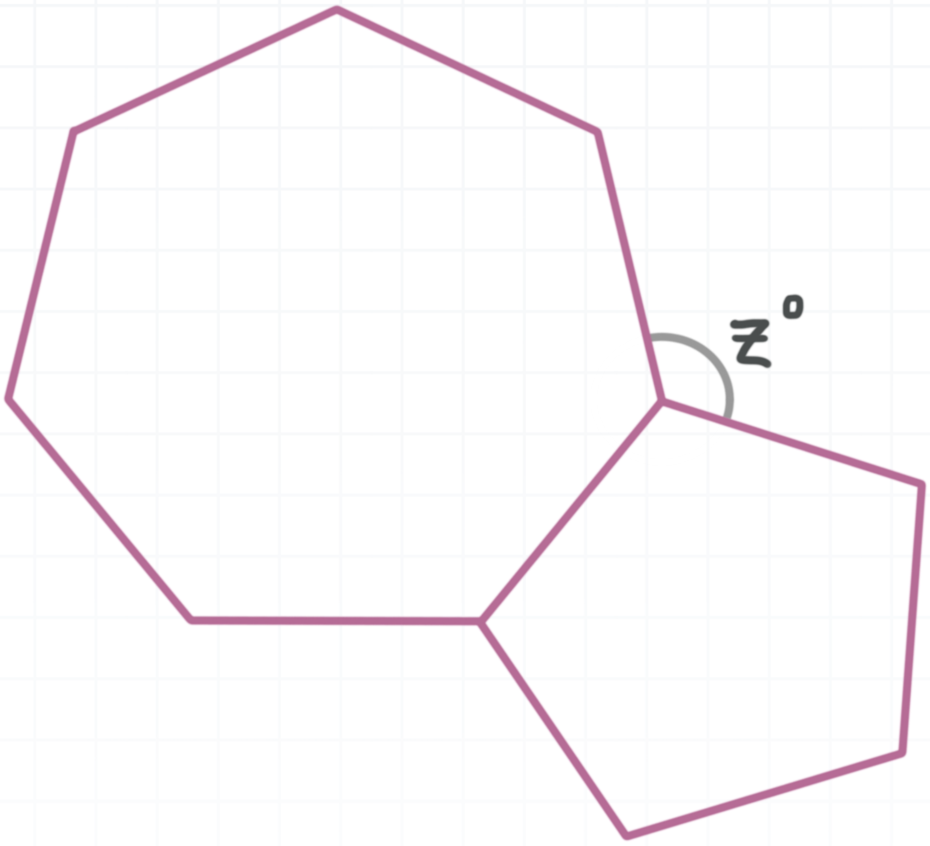
$$5y^\circ = 93.5^\circ$$

$$y = 18.7$$

Let's look at one more example.

Example

The figure shows a regular heptagon and a regular pentagon. Find the value of z to the nearest hundredth.



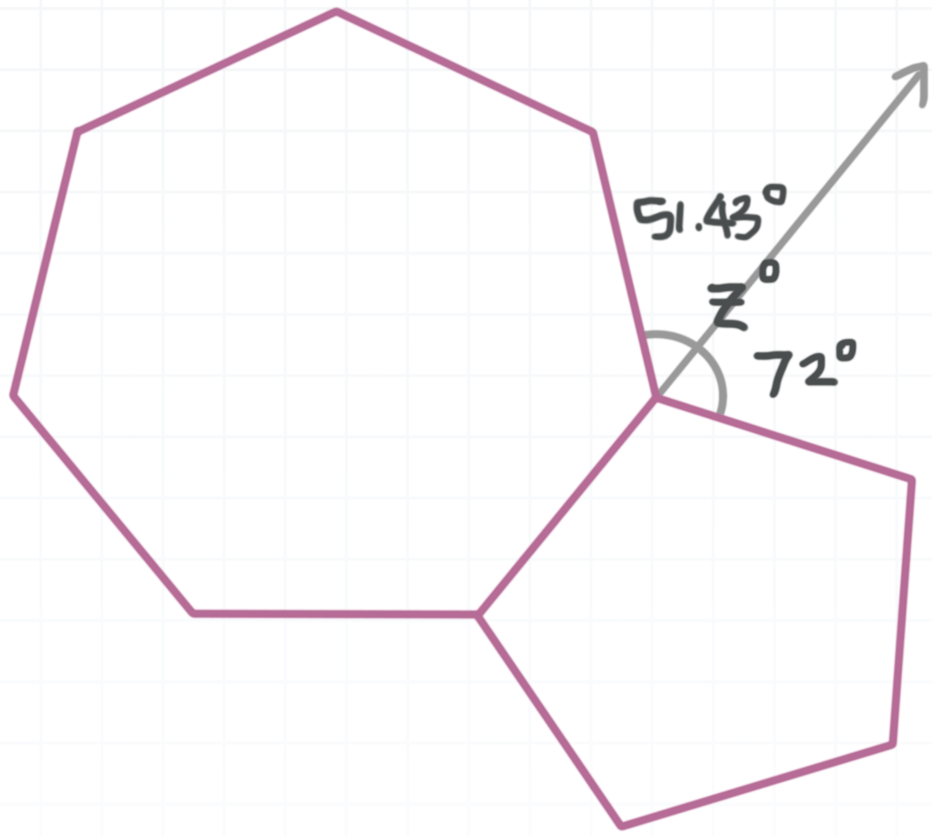
Notice that the angle of measure z° is formed from an exterior angle of the heptagon and an exterior angle of the pentagon, which are a pair of adjacent angles, and that the sum of their measures is z°.

An exterior angle of a regular heptagon has a measure of

$$360^\circ \div 7 \approx 51.43^\circ$$

An exterior angle of a regular pentagon has a measure of

$$360^\circ \div 5 = 72^\circ$$



Therefore,

$$z^\circ \approx 51.43^\circ + 72^\circ$$

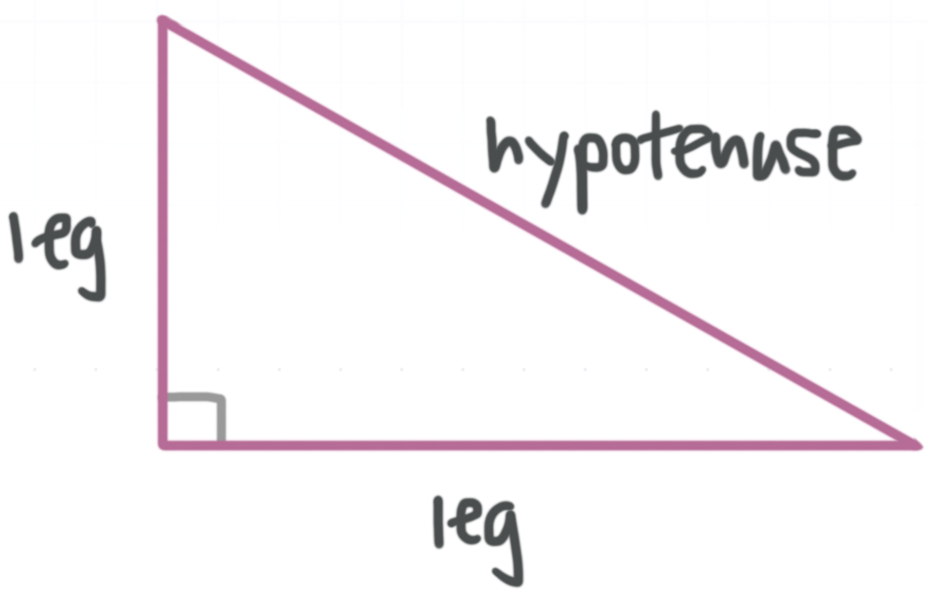
$$z^\circ \approx 123.43^\circ$$

Pythagorean theorem

In this lesson we'll look at how to use the Pythagorean theorem to find the length of one side of a **right triangle**, which is a triangle in which one of the angles is a right angle, that is, a 90° angle) if the lengths of the other two sides are known.

Right triangles

The sides of a right triangle that form the right angle are the **legs** of the triangle, and the side opposite the right angle is the **hypotenuse**.

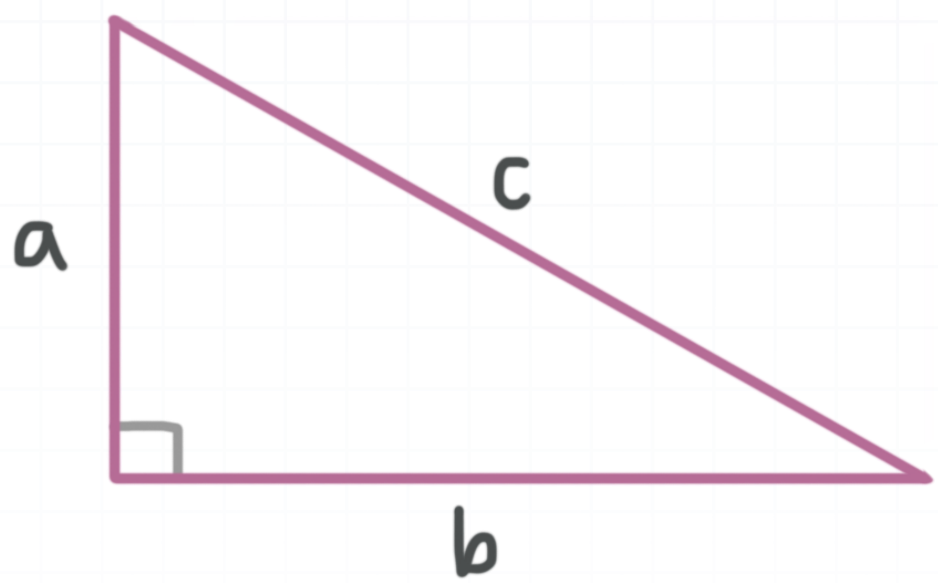


Pythagorean theorem

The Pythagorean theorem shows how the lengths of the sides of a right triangle are related to one another. It says that the sum of the squares of the lengths of the legs is equal to the square of the length of the hypotenuse. We'll usually see it written as

$$a^2 + b^2 = c^2$$

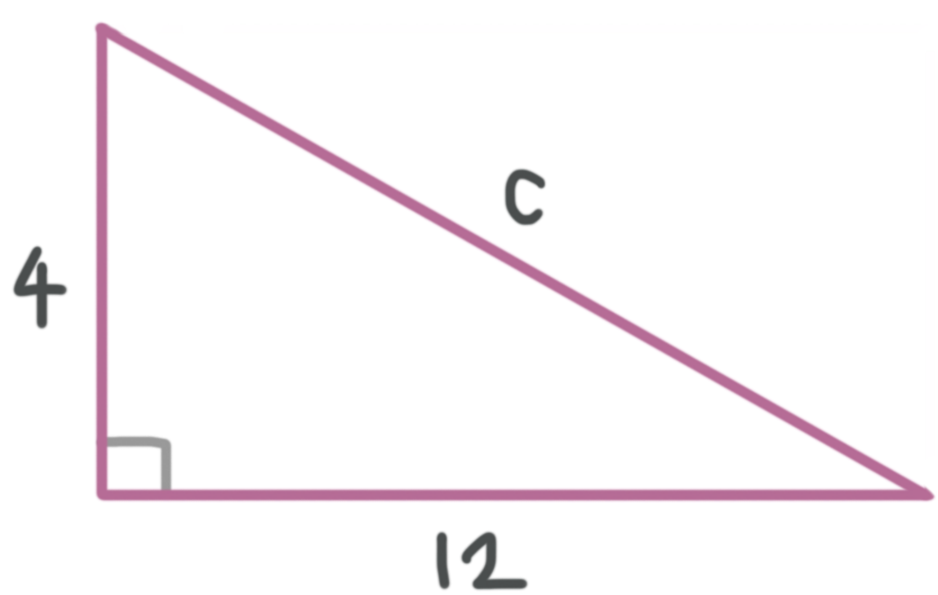
You can choose which leg to call leg a and which leg to call leg b, but the hypotenuse of the triangle is always labeled c.



Let's start by working through an example.

Example

Find the length of the hypotenuse.



Since this is a right triangle and we already know the lengths of two of its sides, we can use the Pythagorean theorem to find the length of the third side (in this case the hypotenuse). The Pythagorean theorem is

$$a^2 + b^2 = c^2$$

Plugging in the lengths we've been given (the lengths of the legs), we get

$$4^2 + 12^2 = c^2$$

$$16 + 144 = c^2$$

$$160 = c^2$$

$$c = \sqrt{160}$$

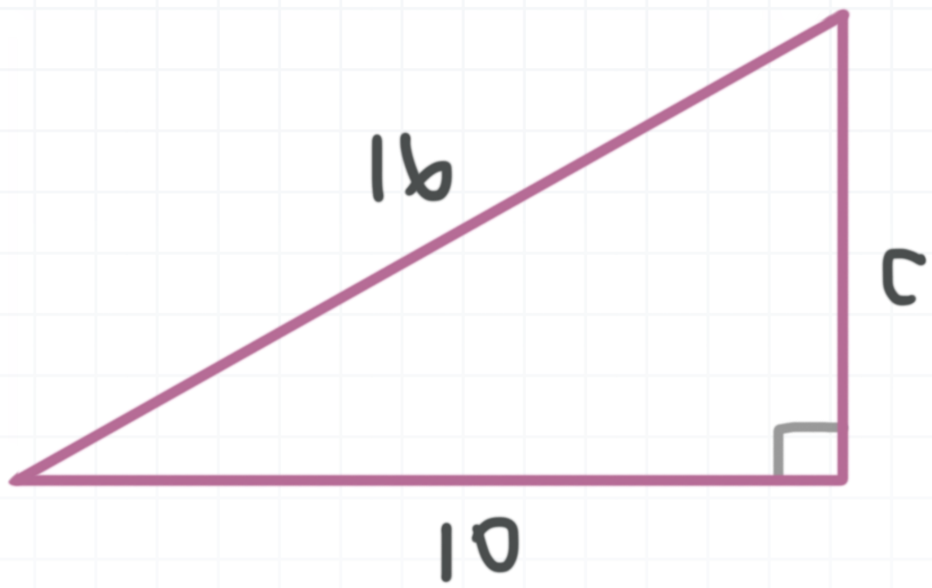
$$c = \sqrt{16 \cdot 10}$$

$$c = 4\sqrt{10}$$

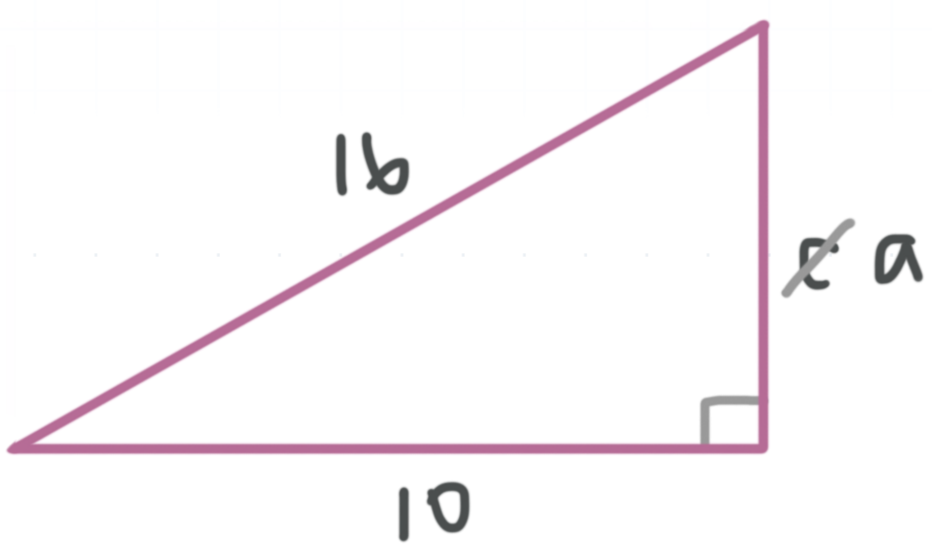
In some problems, the sides of a right triangle are deliberately labeled in a way that's intended to throw us off, so be careful.

Example

Find the length of side c.



Since this is a right triangle and we already know the lengths of two of its sides, we can use the Pythagorean theorem to find the length of the third side. The only thing we need to notice in this example is that the side labeled c is not the hypotenuse. Let's call it side a, since it's a leg.



The Pythagorean theorem is

$$a^2 + b^2 = c^2$$

Plugging in the lengths we've been given (in this case the length of the other leg and the length of the hypotenuse), we get

$$a^2 + 10^2 = 16^2$$

$$a^2 + 100 = 256$$

$$a^2 = 156$$

$$a = \sqrt{156}$$

$$a = \sqrt{4 \cdot 39}$$

$$a = 2\sqrt{39}$$

Remember that the side we're calling a was originally labeled c, so we need to state the answer as

$$c = 2\sqrt{39}$$

Pythagorean inequalities

In this lesson we'll look at different types of triangles and how to use Pythagorean inequalities to determine what kind of triangle we have based on their angle measures and side lengths.

Types of triangles by angle sizes

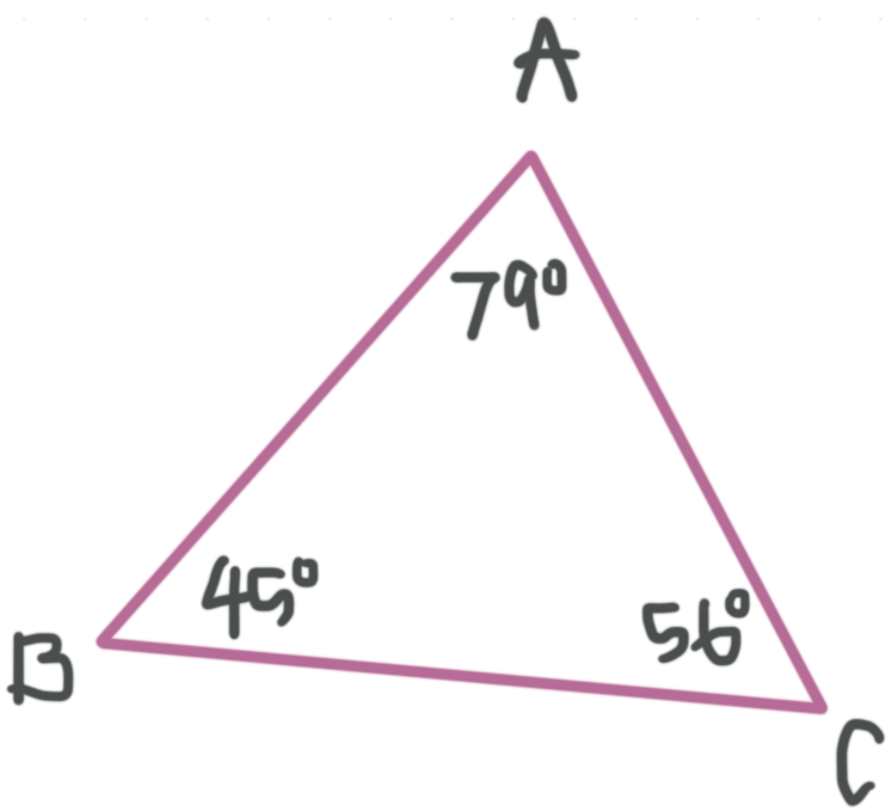
Acute triangle

All of the angles are smaller than 90°.

$$m\angle A = 79^\circ$$

$$m\angle B = 45^\circ$$

$$m\angle C = 56^\circ$$



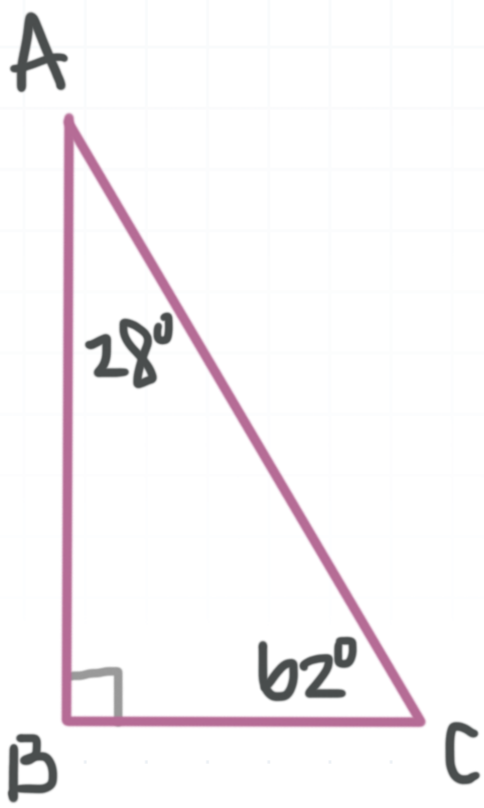
Right triangle

The triangle has a right angle.

$m\angle A = 28^\circ$

$m\angle B = 90^\circ$

$m\angle C = 62^\circ$



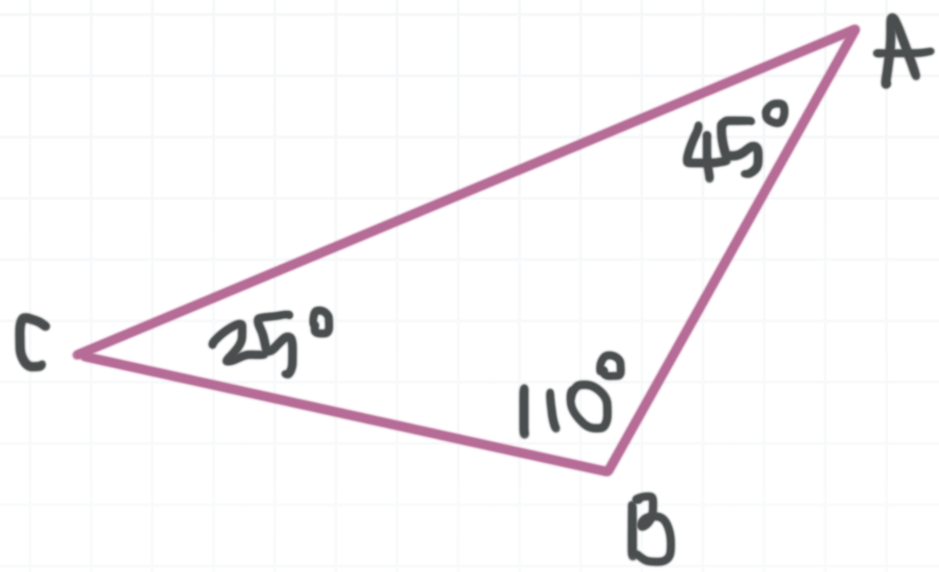
Obtuse triangle

The triangle has an angle greater than 90°.

$m\angle A = 45^\circ$

$m\angle B = 110^\circ$

$m\angle C = 25^\circ$

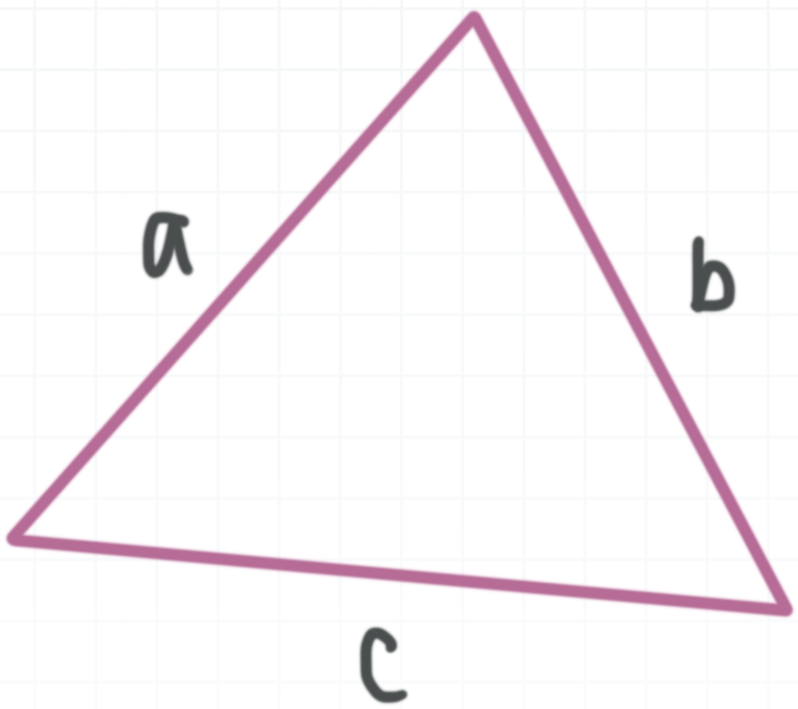


Pythagorean inequalities

There is a relationship between the lengths of the two shortest sides of a triangle and the length of its longest side. If the triangle is not a right triangle, then the relationship is an inequality. Just like in the Pythagorean theorem, we call the short sides a and b and the long side c.

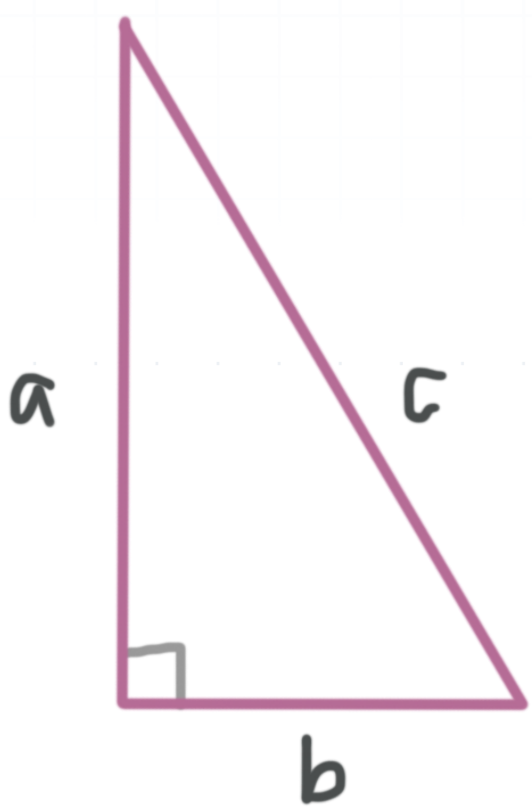
Acute triangle

$$a^2 + b^2 > c^2$$



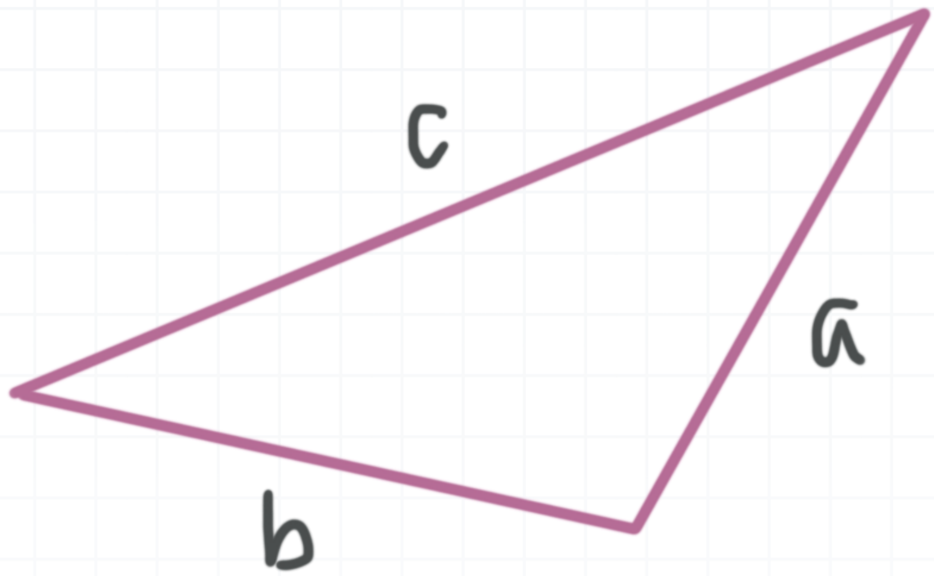
Right triangle

$$a^2 + b^2 = c^2$$



Obtuse triangle

$$a^2 + b^2 < c^2$$



Angles and sides in triangles

One thing to remember about triangles is that the smallest angle is always opposite the shortest side and the biggest angle is always opposite the longest side. Sometimes this can help you when you think through Pythagorean inequality problems.

Let's start by working through an example.

Example

Classify the triangle with sides of length 10, 5, and 9 as acute, obtuse, or right.

Use Pythagorean inequalities to classify the triangle. The two shortest sides are a and b and the longest side is c, so we can assign the letters a, b, and c as follows:

$a = 5$

$b = 9$

$c = 10$

Let's see how $a^2 + b^2$ compares with c^2.

$a^2 + b^2 \quad ? \quad c^2$

$5^2 + 9^2 \quad ? \quad 10^2$

$25 + 81 \quad ? \quad 100$

$106 \quad ? \quad 100$

$106 > 100$

Because $a^2 + b^2 > c^2$, this is an acute triangle.

Let's do one more like the first one.

Example

Classify the triangle with sides of length 9, 7, and 12 as acute, obtuse, or right.

Use Pythagorean inequalities to classify the triangle. The two shortest sides are a and b and the longest side is c, so we can assign the letters a, b, and c as follows:

$$a = 7$$

$$b = 9$$

$$c = 12$$

Let's see how $a^2 + b^2$ compares with c^2.

$$a^2 + b^2 \quad ? \quad c^2$$

$$7^2 + 9^2 \quad ? \quad 12^2$$

$$49 + 81 \quad ? \quad 144$$

$$130 \quad ? \quad 144$$

$$130 < 144$$

Because $a^2 + b^2 < c^2$, this is an obtuse triangle.

Let's try one with a bit more reasoning involved.

Example

The lengths of two sides of a certain triangle are 13 and 12. If the remaining side is the longest side, what is the smallest integer value its length can take that would make the triangle obtuse?

For a triangle to be obtuse, the length of its sides need to satisfy the inequality $a^2 + b^2 < c^2$. We want to find the smallest perfect square that's bigger than $a^2 + b^2$.

We can set $a = 13$ and $b = 12$, so we have

$$a^2 + b^2 = 13^2 + 12^2 = 169 + 144 = 313$$

We want 313 to be less than c^2, and we want c to be the smallest integer that satisfies $313 < c^2$, so let's take the square root of 313 and round up to the next integer.

$$\sqrt{313} \approx 17.692$$

This number rounds up to 18, so $c^2 = 18^2 = 324$, and the smallest integer value the length of the third side can take that makes the triangle obtuse is 18.

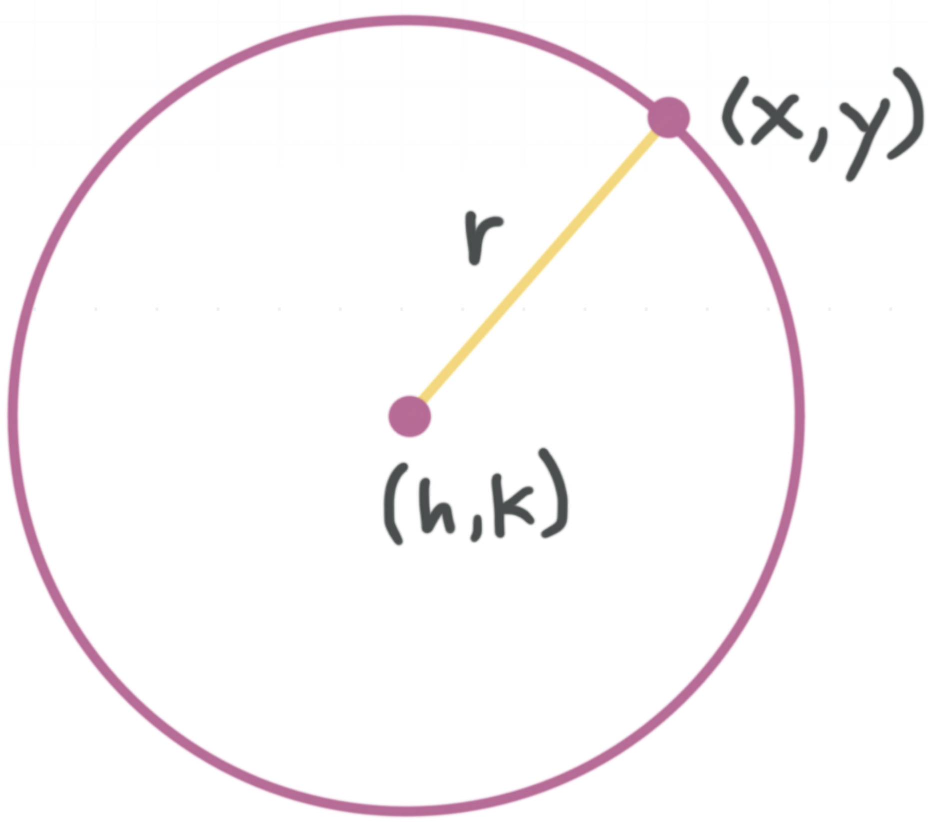
Equation of a circle

In this lesson we'll look at the equation of a circle, $(x-h)^2+(y-k)^2=r^2$, and how to use it to graph a circle, interpret points on a circle, and write the equation of a circle, given a graph or special features of the circle.

A circle can be defined by the point at its center and its radius. In the equation of a circle

$$(x-h)^2+(y-k)^2=r^2$$

the coordinates of the center are (h,k) and the radius is r. From the equation, you can see that the circle is the collection of all the points (x,y) that are a distance r from the center.

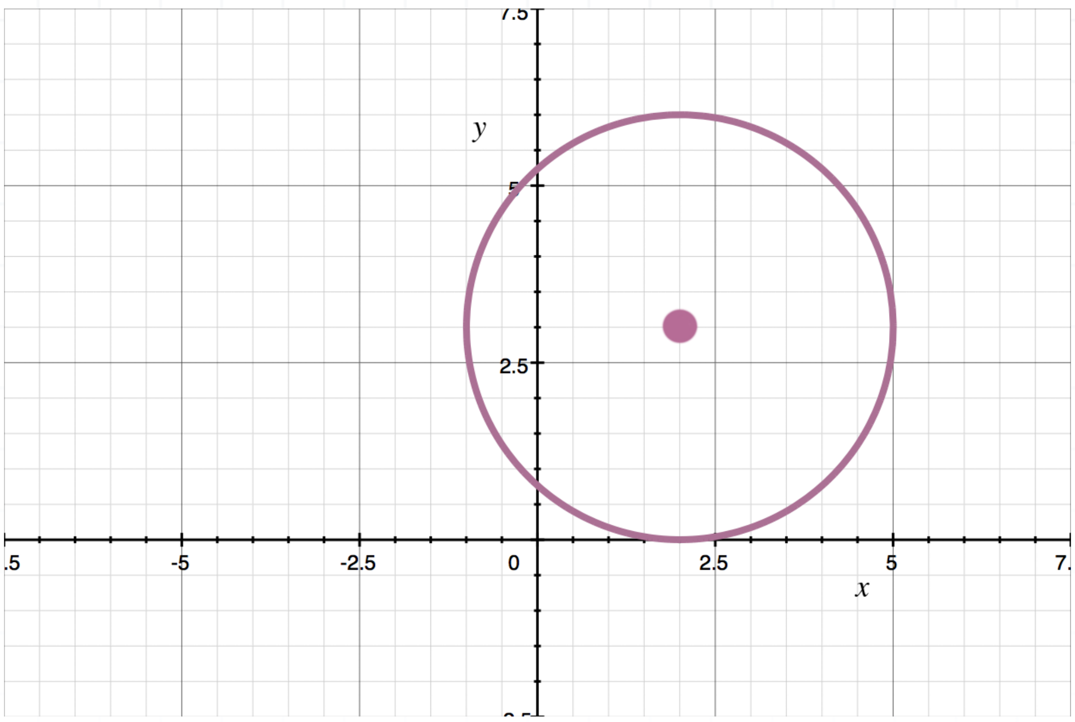


That collection of points includes only the points on the arc (curve). The center of the circle isn't actually a point of the circle.

Let's start by working through an example.

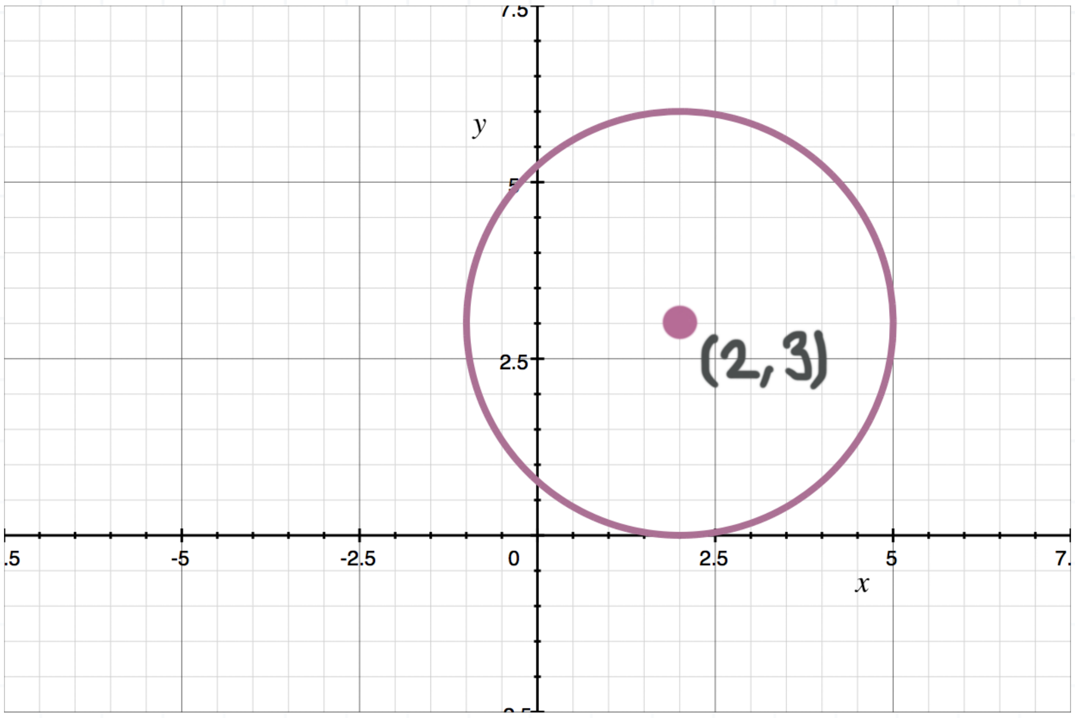
Example

What is the equation of the circle?

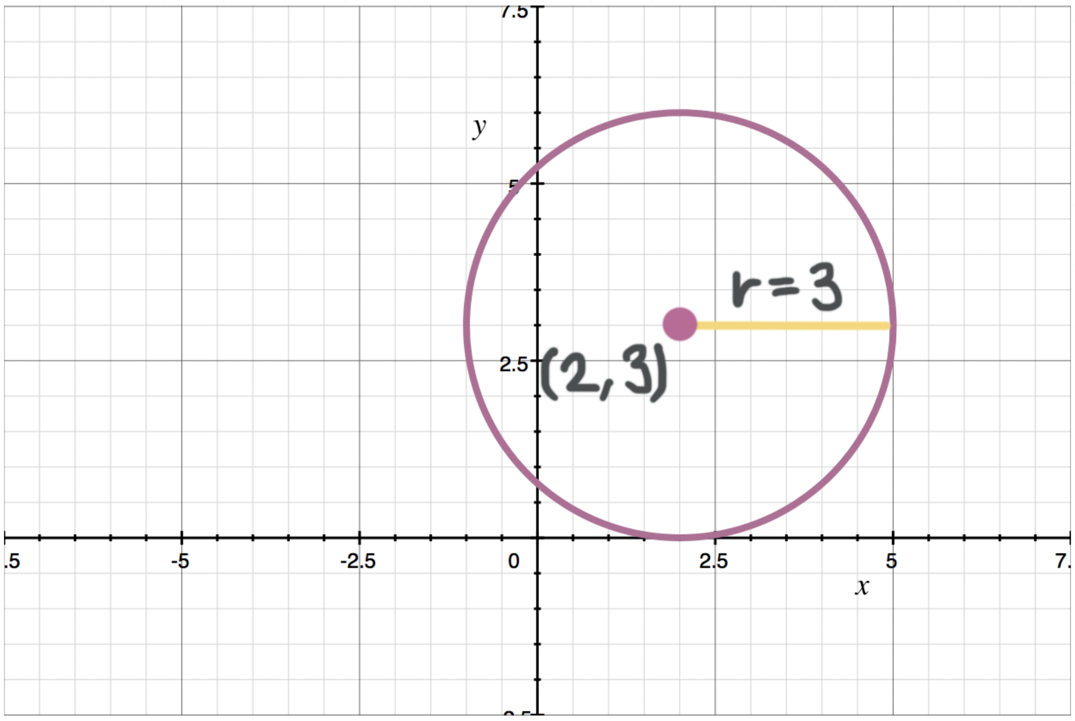


We need to find the equation of this circle in the form $(x-h)^2+(y-k)^2=r^2$, which means we need to find the coordinates of its center, (h,k), and its radius r.

The center is at $(2,3)$, so $h=2$ and $k=3$.



Now let's count from the center to a point on the circle to find the radius.



One point on the circle is $(5,3)$. The distance of that point from the center of the circle is 3, so $r=3$. Now let's plug everything into the standard form of the equation of a circle.

$$(x-2)^2+(y-3)^2=3^2$$

$$(x-2)^2+(y-3)^2=9$$

Sometimes we want to know the center and radius of a circle given the equation of the circle.

Example

What are the center and radius of the circle?

$$x^2 + (y - 3)^2 = 27$$

We can rewrite the equation as

$$(x - 0)^2 + (y - 3)^2 = 27$$

Which lets us identify h and k as 0 and 3, respectively, so the center is at $(0,3)$. And the radius is $\sqrt{27}$, so

$$r = \sqrt{27}$$

$$r = \sqrt{9 \cdot 3}$$

$$r = \sqrt{9} \cdot \sqrt{3}$$

$$r = 3\sqrt{3}$$

Sometimes we want to know the x-intercepts of a circle.

Example

What are the x-intercepts of the circle?

$$(x-2)^2+(y+1)^2=16$$

The x-intercepts are the points at which $y=0$, so set y to 0 and solve for x.

$$(x-2)^2+(y+1)^2=16$$

$$(x-2)^2+(0+1)^2=16$$

$$(x-2)^2+1^2=16$$

$$x^2-4x+4+1=16$$

$$x^2-4x+5=16$$

$$x^2-4x-11=0$$

If you can't factor the equation to solve for x, you can use the quadratic formula. In this case, $a=1$, $b=-4$, and $c=-11$.

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$x=\frac{4\pm\sqrt{(-4)^2-4(1)(-11)}}{2(1)}$$

$$x=\frac{4\pm\sqrt{60}}{2}=\frac{4\pm\sqrt{4\cdot 15}}{2}=\frac{4\pm 2\sqrt{15}}{2}=2\pm\sqrt{15}$$

The x-intercepts are $(2+\sqrt{15},0)$ and $(2-\sqrt{15},0)$.

Sometimes to find out information about a circle, you'll need to know how to complete the square.

Example

Find the center and radius of the circle.

$$x^2+y^2+24x+10y+160=0$$

In order to find the center and radius, we need to convert the equation of the circle into standard form, $(x-h)^2+(y-k)^2=r^2$. In order to get the equation into standard form, we have to complete the square with respect to both variables.

Grouping x's and y's together and moving the constant to the right side, we get

$$(x^2+24x)+(y^2+10y)=-160$$

Completing the square with respect to any variable requires us to take the coefficient of the first-degree term in that variable, divide it by 2, and then square the result before adding it to both sides of the equation.

The coefficient of the first-degree term in x is 24, so

$$\frac{24}{2}=12 \quad \to \quad 12^2=144$$

The coefficient of the first-degree term in y is 10, so

$$\frac{10}{2}=5 \quad \to \quad 5^2=25$$

So we add 144 inside the parentheses with the x terms, and 25 inside the parentheses with the y terms, and we also add 144 and 25 to the right side of the equation (to -160).

$$(x^2+24x+144)+(y^2+10y+25)=-160+144+25$$

$$(x+12)^2+(y+5)^2=9$$

Therefore, the center of the circle is at $(h,k)=(-12,-5)$ and its radius is $r=\sqrt{9}=3$.

Degree measure of an arc

In this lesson we'll look at arcs of circles and how to find their degree measure. Arcs also have length, but we won't consider that in this lesson, so we'll use "measure" to mean "degree measure."

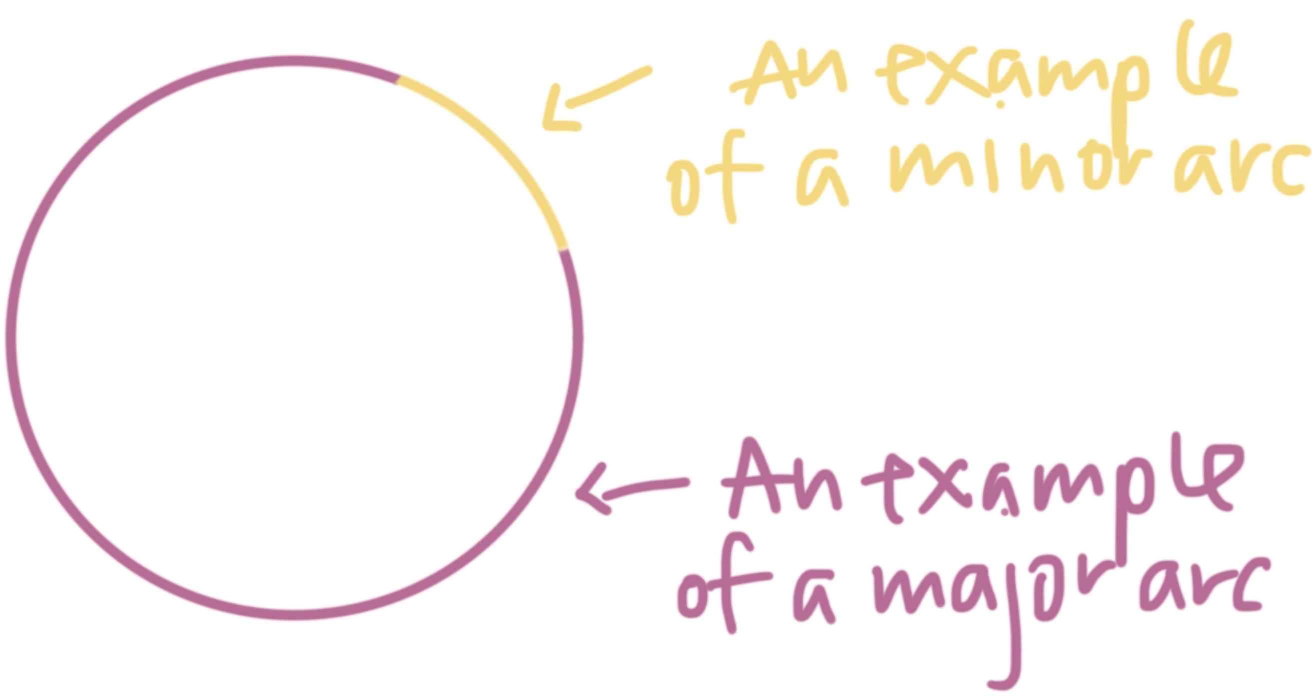
Arcs

An **arc** is a continuous part of a circle (a part that has no holes in it). There are three types of arcs.

1. **Minor arcs** have measure less than 180°.

2. **Major arcs** have measure more than 180°.

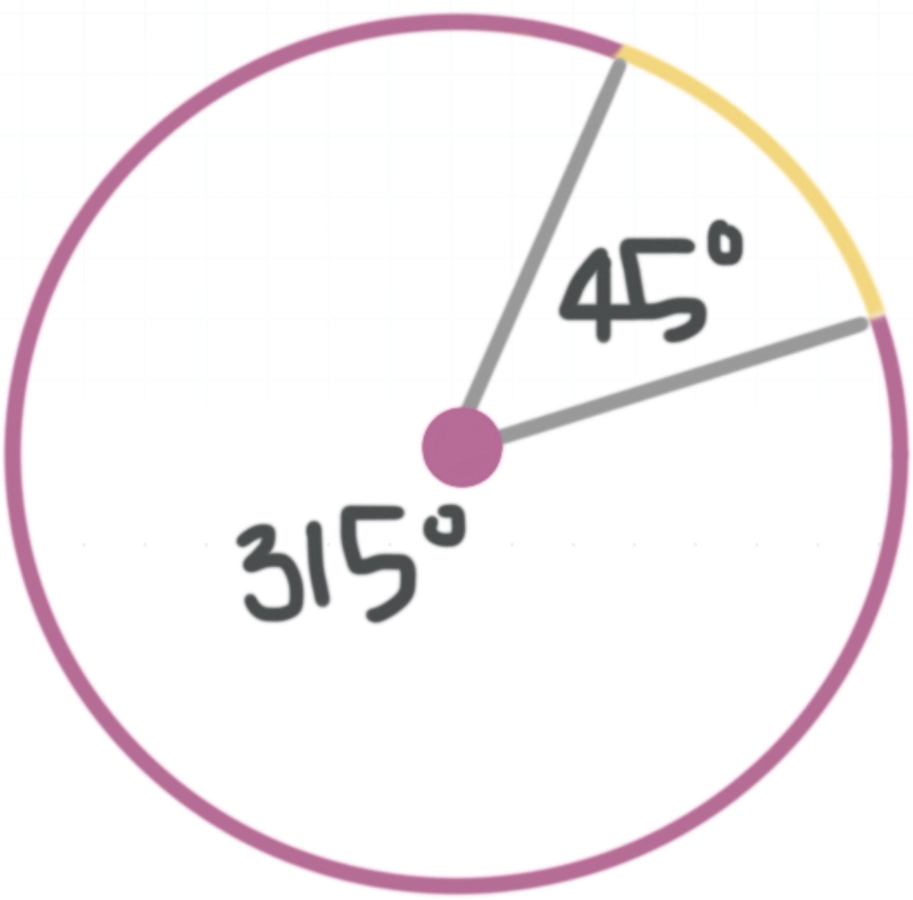
3. **Semicircular arcs** (also called **semicircles**) consist of half a circle and have measure of exactly 180°.



Measures of arcs and central angles

A **central angle** of a circle is an angle whose vertex is at the center of the circle. The central angle and the corresponding arc (the arc whose endpoints are the points of intersection of the sides of the central angle with the circle) have the same measure.

Here the measure of the central angle that corresponds to the minor arc is 45°, so the measure of the minor arc is 45°.



The measure of the central angle that corresponds to the major arc is 315°, so the measure of the major arc is 315°. Notice that the sum of the measures of a pair of central angles of a circle that correspond to a pair of arcs which (together) make up the entire circle (but intersect only at their endpoints) is 360°.

Naming arcs and their measures

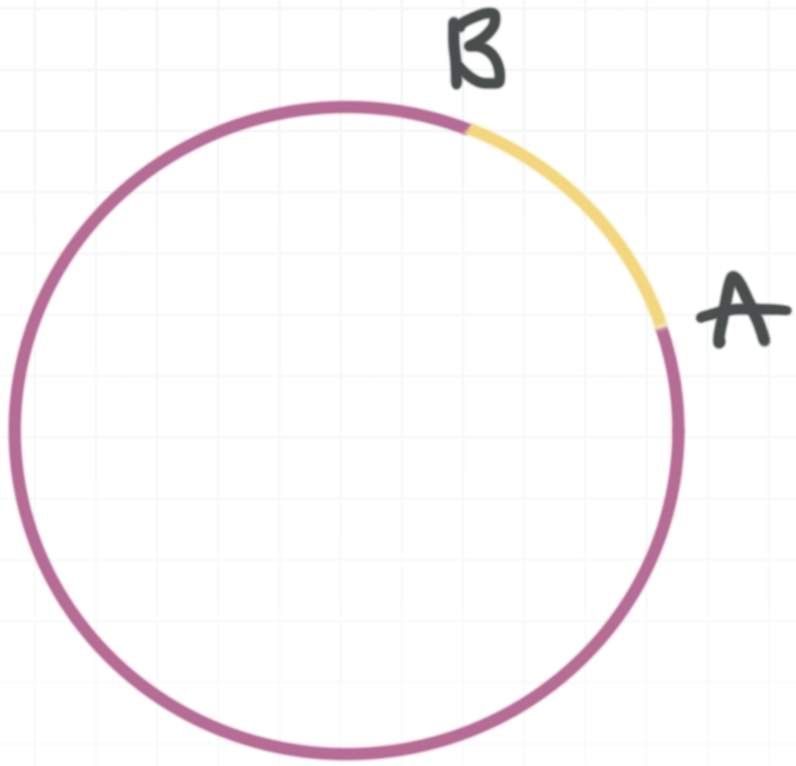
You can name an arc by the letters that name its endpoints.

Notice that you could take a pencil, place it at one of the endpoints of an arc, and "trace out" the arc until you get to the other endpoint. However, you could choose to go around the circle in either direction - clockwise or counterclockwise, when you do the tracing, which would give you two different arcs.

Therefore, the endpoints of the arc don't uniquely determine the arc. If you ever study trigonometry, you'll learn that an arc with a positive measure is traced out in the counterclockwise direction, while an arc with a negative measure is traced out in the clockwise direction.

In this course, we'll only deal with angles of positive measure, so we'll always name an arc in such a way that the first letter in its name is the endpoint you'd start from if you were to trace the arc by moving counterclockwise around the circle. We'll name the corresponding central angle the same way, where the first letter in the name of the arc is the starting point, and the last letter in the name of the arc is the endpoint point, after tracing out the arc in a counterclockwise direction.

For example, the yellow arc shown below is arc AB. Instead of writing the word "arc," we can use a curved symbol over the letters to indicate the arc: $\overset{\frown}{AB}$.



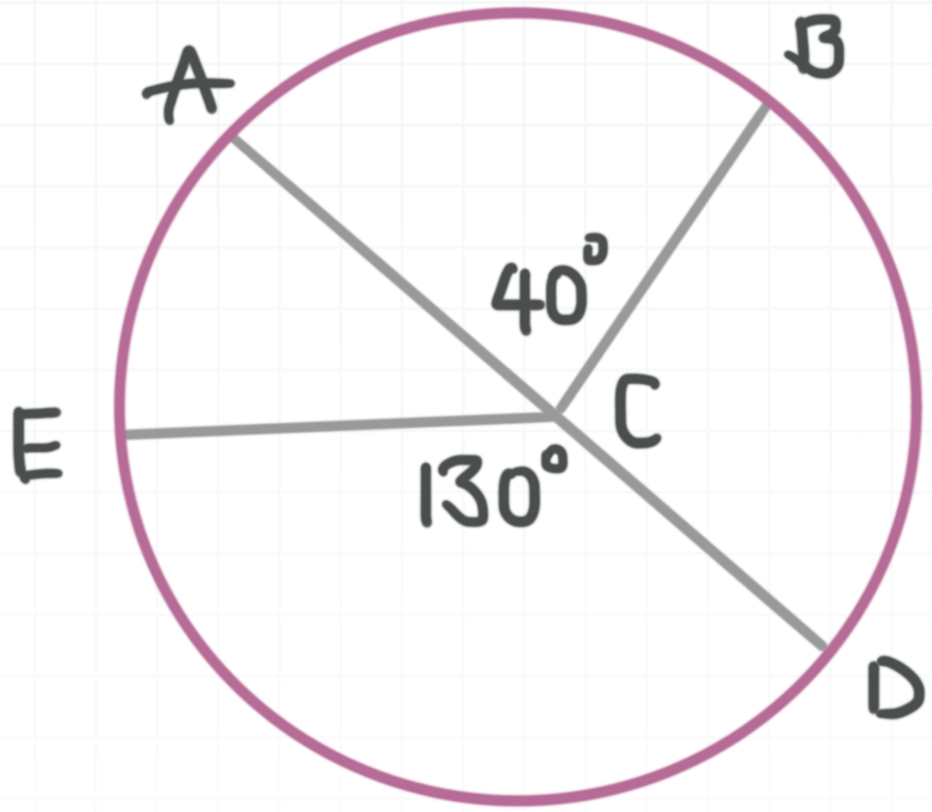
To indicate the measure of an arc, you can use an m or the word "measure." In this case we can say the measure of arc AB is 45°. If we used symbols instead of words, we could write $m\overset{\frown}{AB} = 45^\circ$.

Let's look at a few examples.

Example

If $\overline{AD}$ is a diameter of the circle (with center at C) in the figure, what is the difference between the measures of $\overset{\frown}{DB}$ and arc $\overset{\frown}{AE}$?

Note: A **diameter** is any line segment that passes through the center of a circle and has both of its endpoints on the circle. Notice that the diameter is equal to twice the radius, and that the diameter splits a circle into two semicircles. In this problem, $\overline{AD}$ is a diameter that splits the circle into semicircles AD and DA.



$\overline{AD}$ is a diameter of the circle, which means that the sum of the measures of arc DB and arc BA is 180°.

$$m\widehat{DB} + m\widehat{BA} = 180^\circ$$

$$m\widehat{DB} + 40^\circ = 180^\circ$$

$$m\widehat{DB} = 140^\circ$$

Likewise, the sum of the measures of $\widehat{AE}$ and $\widehat{ED}$ is 180°.

$$m\widehat{AE} + m\widehat{ED} = 180^\circ$$

$$m\widehat{AE} + 130^\circ = 180^\circ$$

$$m\widehat{AE} = 50^\circ$$

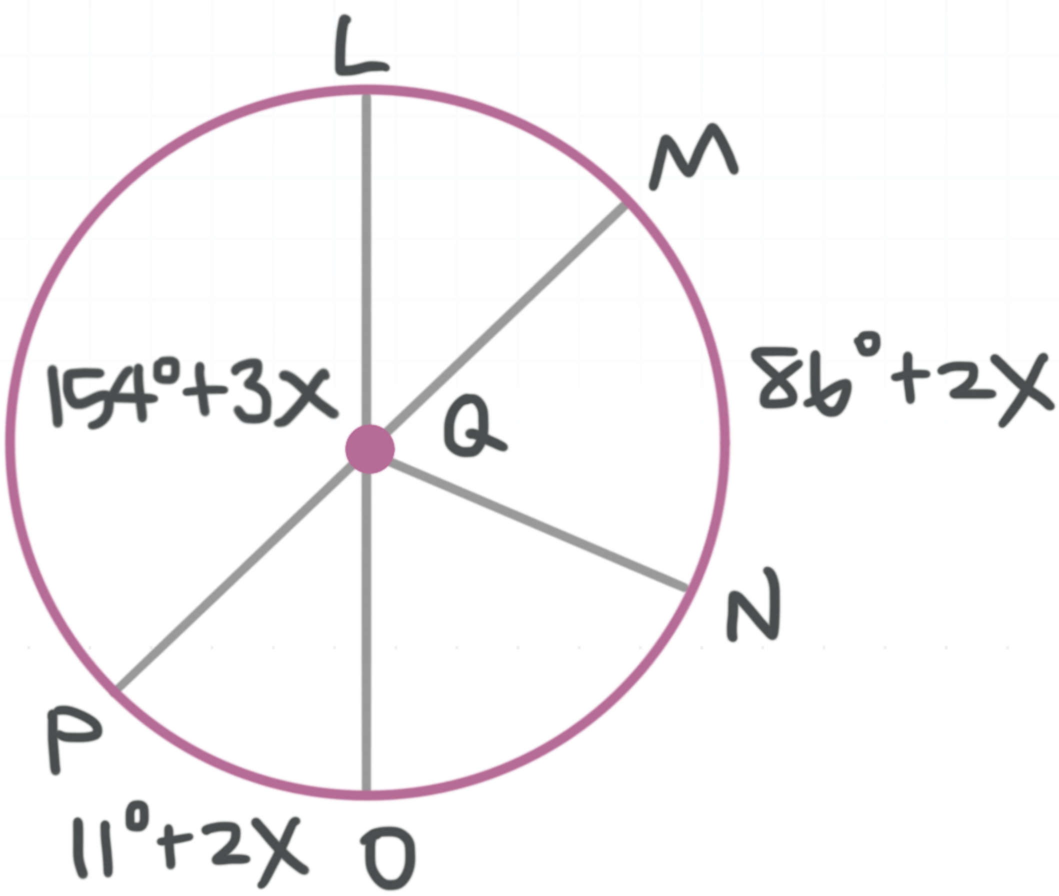
The difference between the measures of arcs $\widehat{DB}$ and $\widehat{AE}$ is

$$140^\circ - 50^\circ = 90^\circ$$

Let's look at one more example.

Example

What is the measure of $\overset{\frown}{NM}$, given that $\overline{LO}$ is a diameter of the circle (with center at Q) in the figure?



We need to solve for x and then plug it back into the expression for the measure of $\overset{\frown}{NM}$. We know that $\overline{LO}$ is a diameter of the circle, so the sum of the measures of arc LP and arc PO is 180°.

We can use this information to find the value of x.

$$m\widehat{LP} + m\widehat{PO} = 180^\circ$$

Central angle LQP has measure $154^\circ + 3x$, which means that $\widehat{LP}$ also has measure $154^\circ + 3x$. Therefore,

$$(154^\circ + 3x) + (11^\circ + 2x) = 180^\circ$$

$$165^\circ + 5x = 180^\circ$$

$$5x = 15^\circ$$

$$x = 3^\circ$$

Now we can find the measure of $\widehat{NM}$.

$$m\widehat{NM} = 86^\circ + 2x$$

$$m\widehat{NM} = 86^\circ + 2(3^\circ)$$

$$m\widehat{NM} = 92^\circ$$

Arc length

In this lesson we'll look at how to find the length of an arc when we're given the radius and the measure of the corresponding central angle in degrees.

The arc length formula

An arc can be measured in degrees, but it can also be measured in units of length.

The circumference of a circle is the total length of the circle (the "distance around the circle"). An arc is part of a circle. In fact, the ratio of the measure m of an arc of a circle (in degrees) to the measure of the entire circle (in degrees), that is, the ratio of (the numerical value of) m to 360, is equal to the ratio of the arc length L to the circumference (total length) C of the circle:

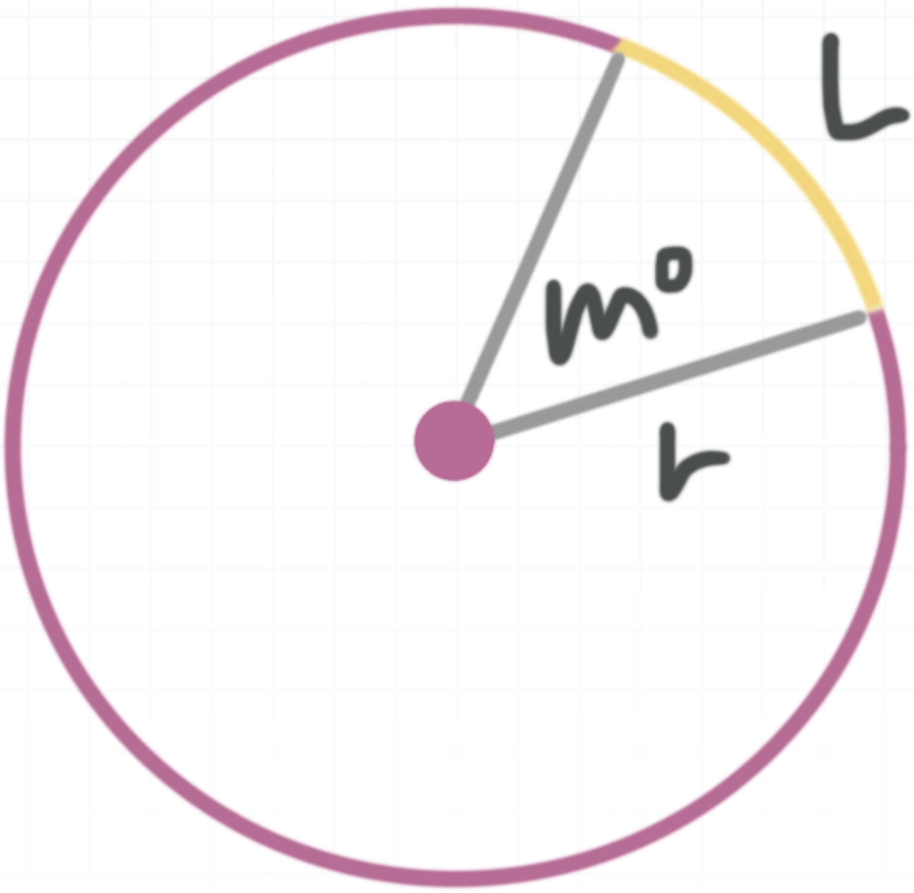
$$\frac{m}{360} = \frac{L}{C}$$

The formula for the circumference of a circle is $C = 2\pi r$, where r is the radius of the circle. Therefore,

$$\frac{m}{360} = \frac{L}{2\pi r}$$

Solving for L, we get the arc length formula:

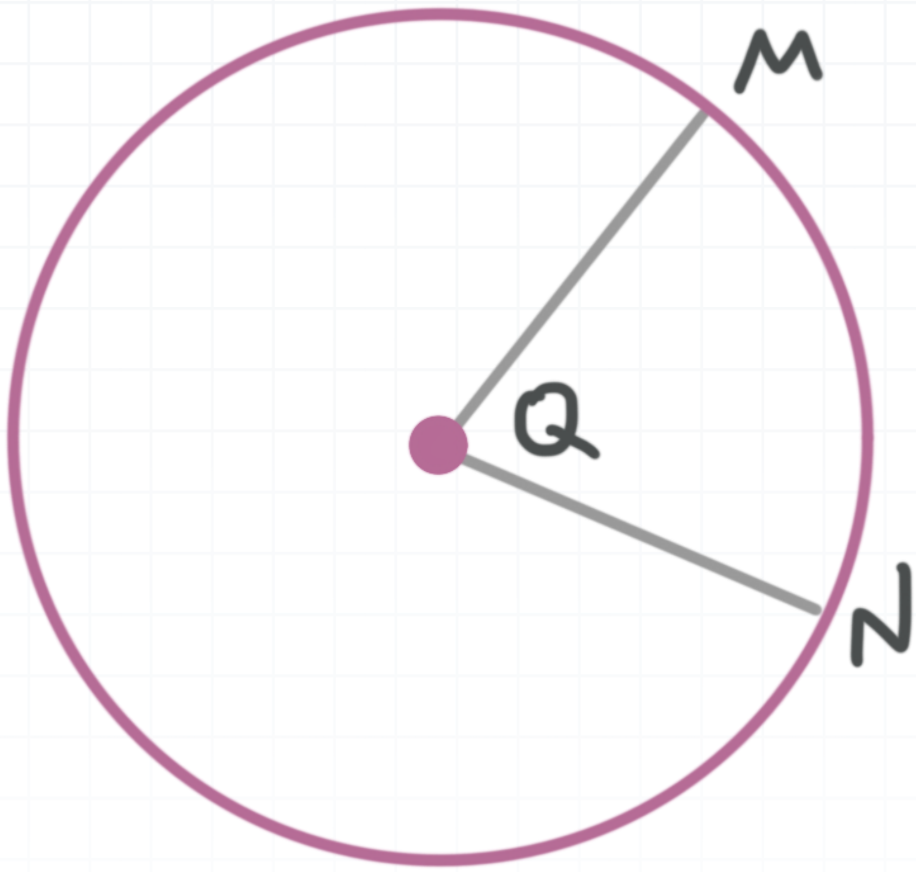
$$L = \frac{m}{360} \cdot 2\pi r$$



Many textbooks will refer to the arc length with the lowercase letter s. Let's start by working through an example.

Example

The radius of the circle (with center at Q) in the figure is 12 cm, and the measure of angle NQM is 86°. What is the length of $\overset{\frown}{NM}$?



We know that the radius of the circle is 12 cm and the central angle that corresponds to $\overset{\frown}{NM}$ measures 86°, so $m\overset{\frown}{NM}=86^\circ$. Now we can use the arc length formula.

$$L=\frac{m}{360}\cdot 2\pi r$$

$$L=\frac{86}{360}\cdot 2\pi(12)$$

$$L=\frac{43}{180}\cdot 24\pi$$

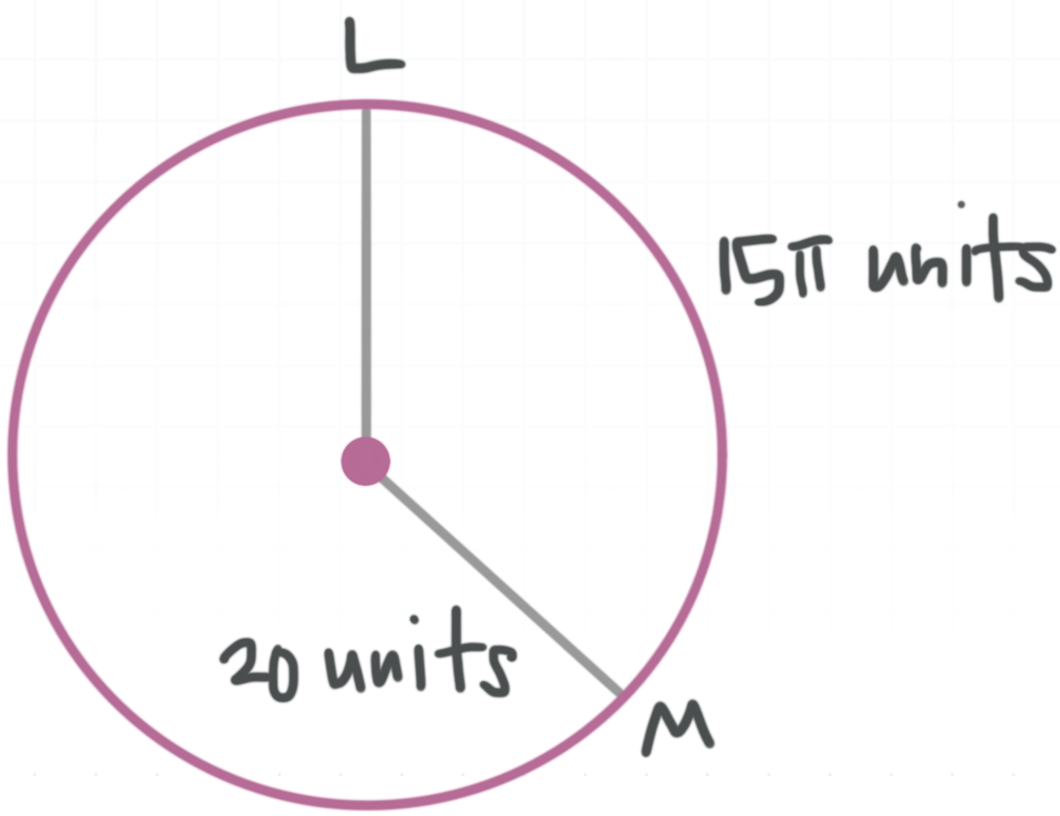
$$L=\frac{86}{15}\pi$$

$$L\approx 18.01$$

Sometimes you will be given the arc length and asked to find the degree measure of the arc or the measure of the corresponding central angle.

Example

Find the measure of the central angle (in degrees) that corresponds to $\widehat{ML}$.



Remember that the degree measure m of the arc is equal to the measure of the central angle that corresponds to the arc. We'll use the arc length formula

$$L = \frac{m}{360} \cdot 2\pi r$$

We know that $L = 15\pi$ and $r = 20$, so we'll solve for m.

$$L = \frac{m}{360} \cdot 2\pi r$$

$$15\pi = \frac{m}{360} \cdot 2\pi(20)$$

$$15\pi = \frac{m}{360} \cdot 40\pi$$

$$\frac{15\pi}{40\pi} = \frac{m}{360}$$

$$\frac{3}{8} = \frac{m}{360}$$

$$\frac{3}{8} \cdot 360 = m$$

$$m = 135$$

Therefore, the measure of the central angle that corresponds to $\overset{\frown}{ML}$ is 135°.

Inscribed angles of circles

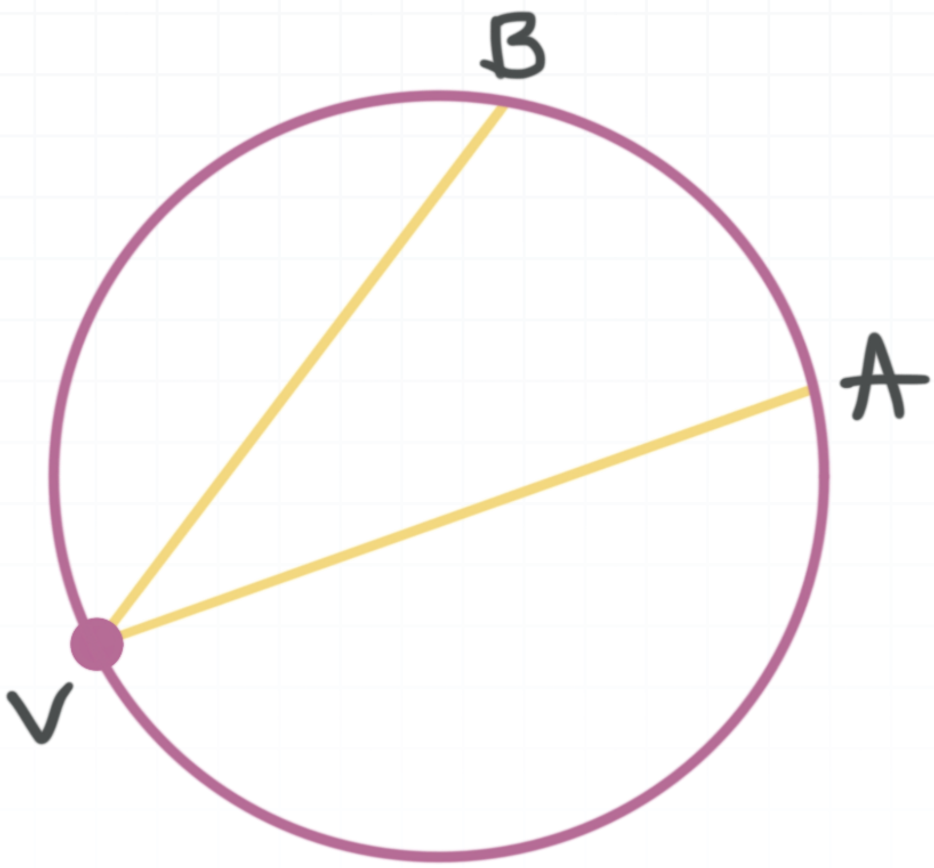
In this lesson we'll look at inscribed angles of circles and how they're related to arcs, called intercepted arcs.

Chord of a circle

The **chord** of a circle is a line segment that has both of its endpoints on the circle. A diameter of a circle is a special type of chord that passes through the circle's center. The yellow line is an example of a chord.

Inscribed angle

An inscribed angle is formed by two chords that have one endpoint in common, which is the vertex of the angle. The points at which the other endpoint of each chord intersects the circle are the endpoints of an arc which is called the **intercepted arc**.



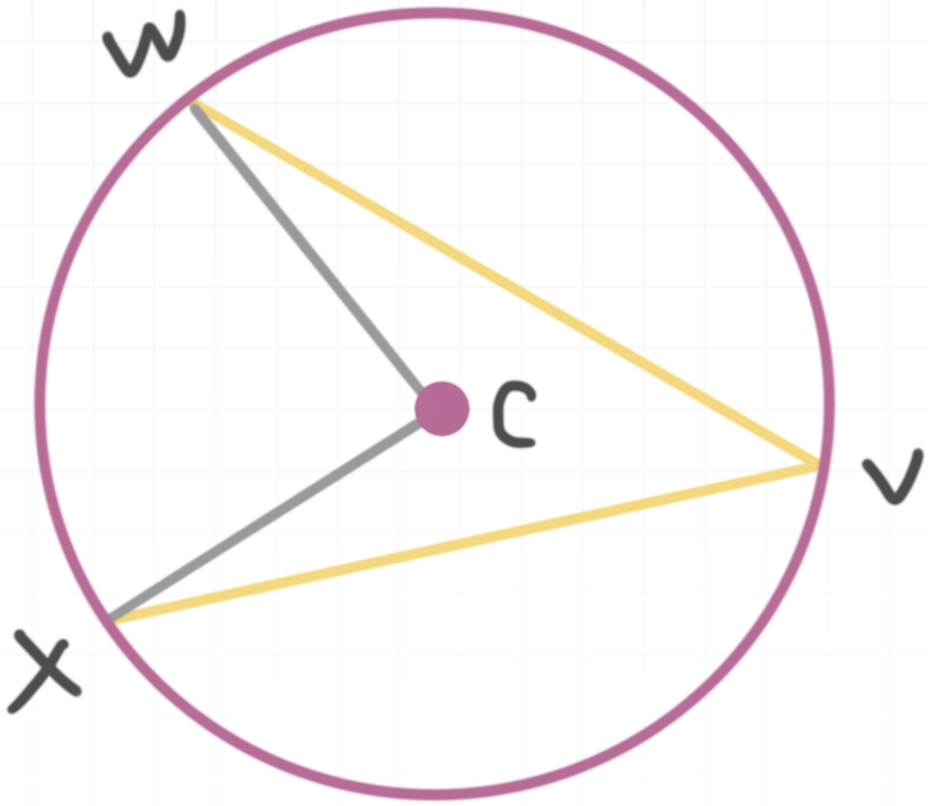
Angle AVB is an **inscribed angle**, and arc AB is the intercepted arc.

Inscribed and central angles, and intercepted arcs

The measure of an intercepted arc is equal to the measure of the central angle that corresponds to it.

The measure of an inscribed angle is equal to half the measure of the central angle that corresponds to the intercepted arc.

The measure of an inscribed angle is equal to half the measure of its intercepted arc.



$$m\widehat{WX} = m\angle WCX$$

$$\frac{1}{2}m\angle WCX = m\angle WVX$$

$$\frac{1}{2}m\widehat{WX} = m\angle WVX$$

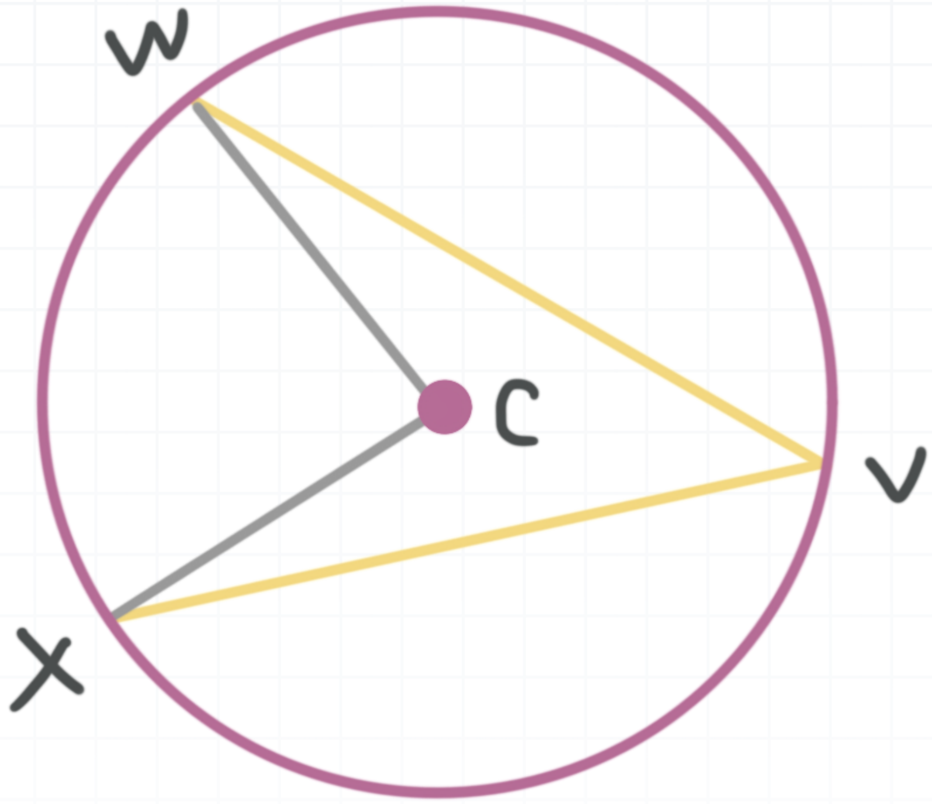
We can do some algebra to show that the following is also true.

$$m\widehat{WX} = m\angle WCX = 2m\angle WVX$$

Let's do a couple of example problems.

Example

Find the measure of the inscribed angle WVX if $m\angle WCX = 88^\circ$.



We can see from the figure that $\angle WCX$ is the central angle that corresponds to the intercepted arc $\overset{\frown}{WX}$. Which means that the measure of the inscribed angle ($\angle WVX$) is half that of $\angle WCX$. We know

$$\frac{1}{2}m\angle WCX = m\angle WVX$$

and

$$m\angle WCX = 88^\circ$$

So

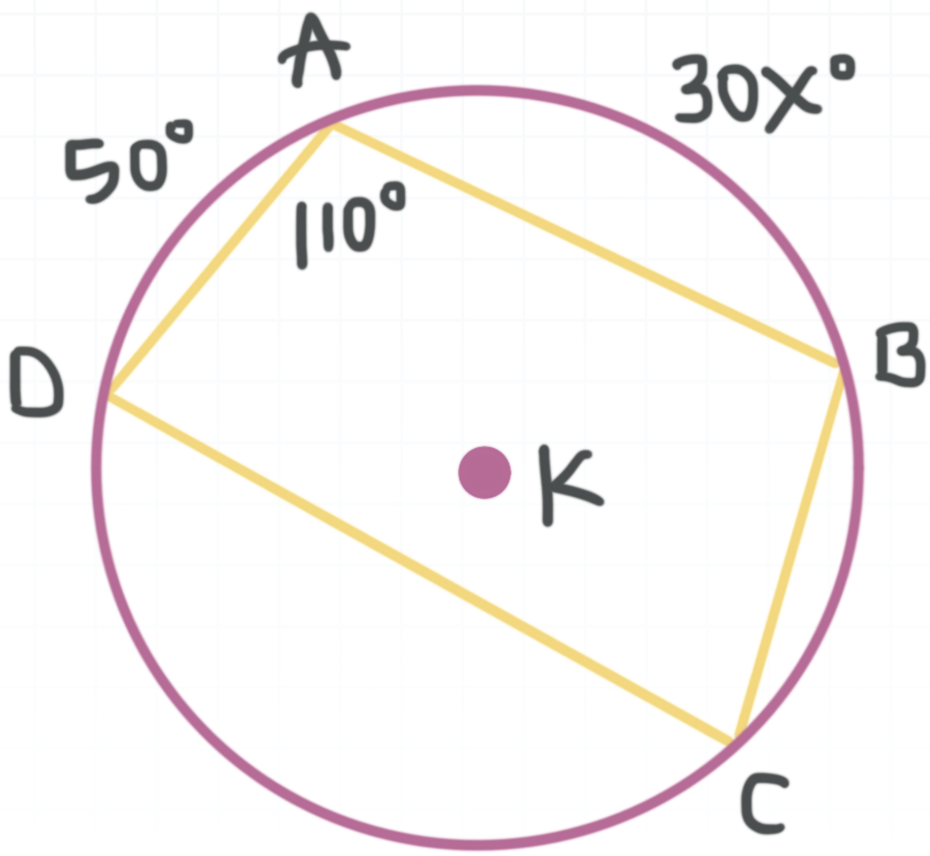
$$\frac{1}{2}(88^\circ) = m\angle WVX$$

$$44^\circ = m\angle WVX$$

Let's do a problem with a few more steps.

Example

Find the value of x.



The arc that consists of the complete circle has measure 360°. If we can find the measure of $\overset{\frown}{DB}$ (remember that this is the arc we'd trace out by starting at point D and going counterclockwise, through C, around the circle to point B), we can set up an equation to solve for x, because $m\overset{\frown}{DCB} + m\overset{\frown}{BA} + m\overset{\frown}{AD} = 360^\circ$, and we know that $m\overset{\frown}{BA} = 30x^\circ$ and $m\overset{\frown}{AD} = 50^\circ$.

From the diagram, we can see that $m\angle DAB = 110^\circ$. The arc intercepted by this inscribed angle is $\overset{\frown}{DB}$. The intercepted arc has measure twice that of the inscribed angle.

$$m\widehat{DB} = 2m\angle DAB = 2(110^\circ)$$

$$m\widehat{DB} = 220^\circ$$

Now we can use the equation we wrote earlier to solve for x.

$$m\widehat{DB} + m\widehat{BA} + m\widehat{AD} = 360^\circ$$

$$220^\circ + 30x^\circ + 50^\circ = 360^\circ$$

$$270^\circ + 30x^\circ = 360^\circ$$

$$30x^\circ = 90^\circ$$

$$x = 3$$

Vertex on, inside, and outside the circle

In this lesson we'll look at angles whose sides intersect a circle in certain ways and how the measures of such angles are related to the measures of certain arcs of that circle.

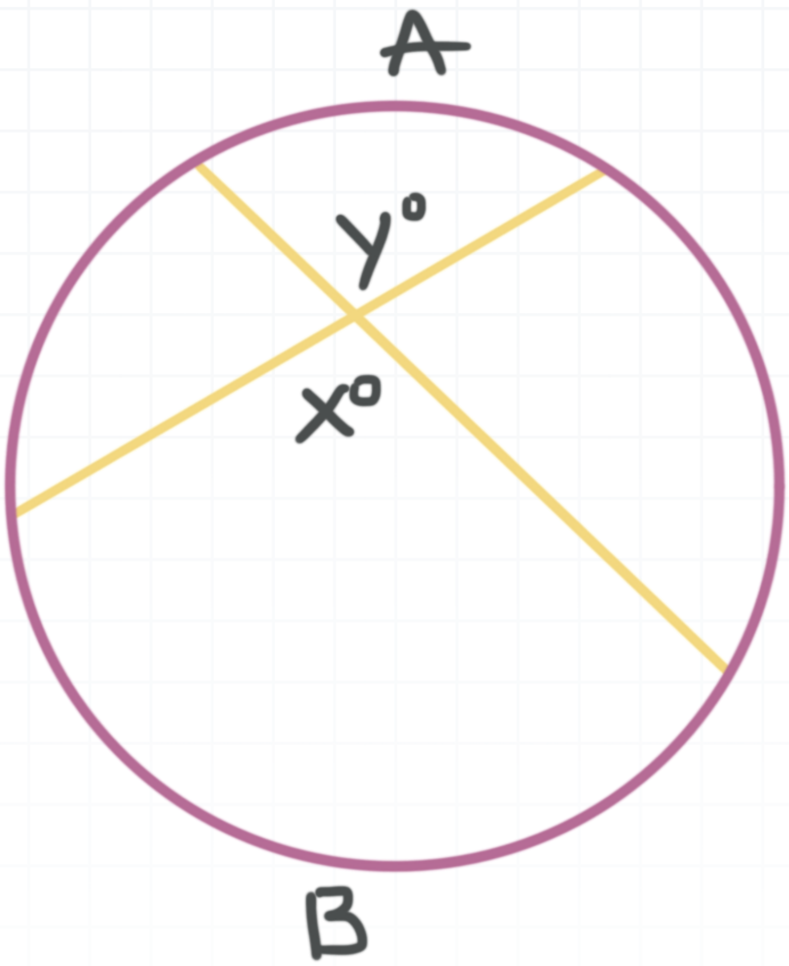
As we work through this lesson, remember that a **chord** of a circle is a line segment that has both of its endpoints on the circle. Besides that, well use the term **secant** for a line segment that has one endpoint outside the circle and intersects the circle at two points. Finally, we'll use the term **tangent** for a line that intersects the circle at just one point.

Here we look at the three possible locations for the vertex of an angle that intersects a circle, together with certain types of geometric figures (chords, secants, tangents) that intersect at the vertex of such an angle and their relationship to certain arcs of the circle.

Vertex inside the circle

When the vertex of the angle is inside the circle, two pairs of vertical angles are formed. We learned about vertical angles formed when a transversal crosses a pair of parallel lines, but actually any pair of angles formed by two lines crossing each other (like the angles of x° and y° in the figure below), are vertical angles.

So given vertical angles in the figure below,



we can actually say three things:

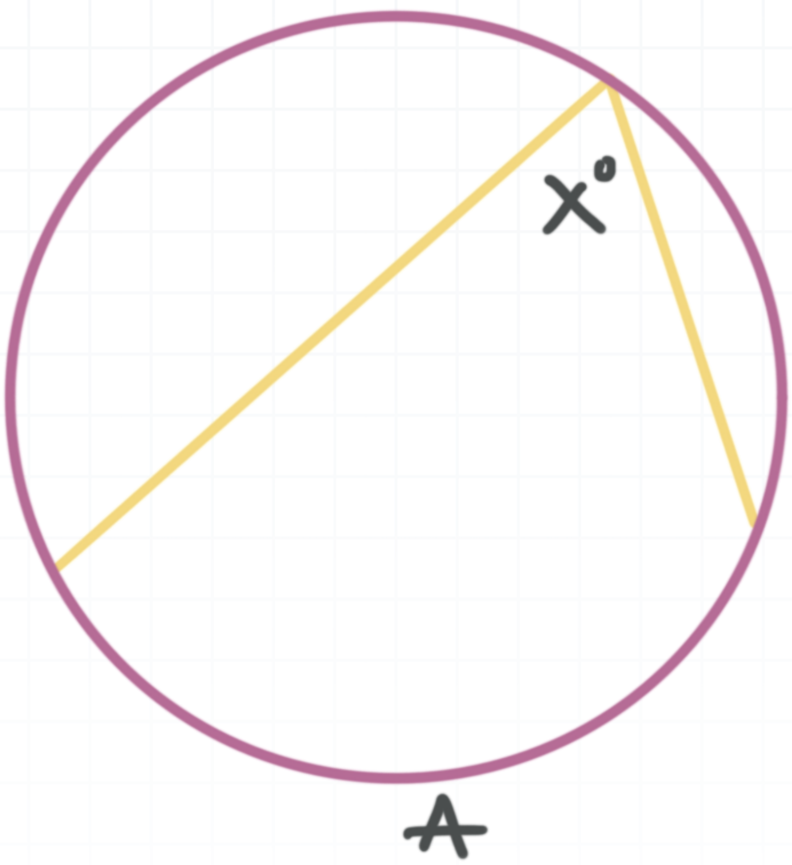
$$x^\circ = y^\circ$$

$$x^\circ = \frac{m\overset{\frown}{A} + m\overset{\frown}{B}}{2}$$

$$y^\circ = \frac{m\overset{\frown}{A} + m\overset{\frown}{B}}{2}$$

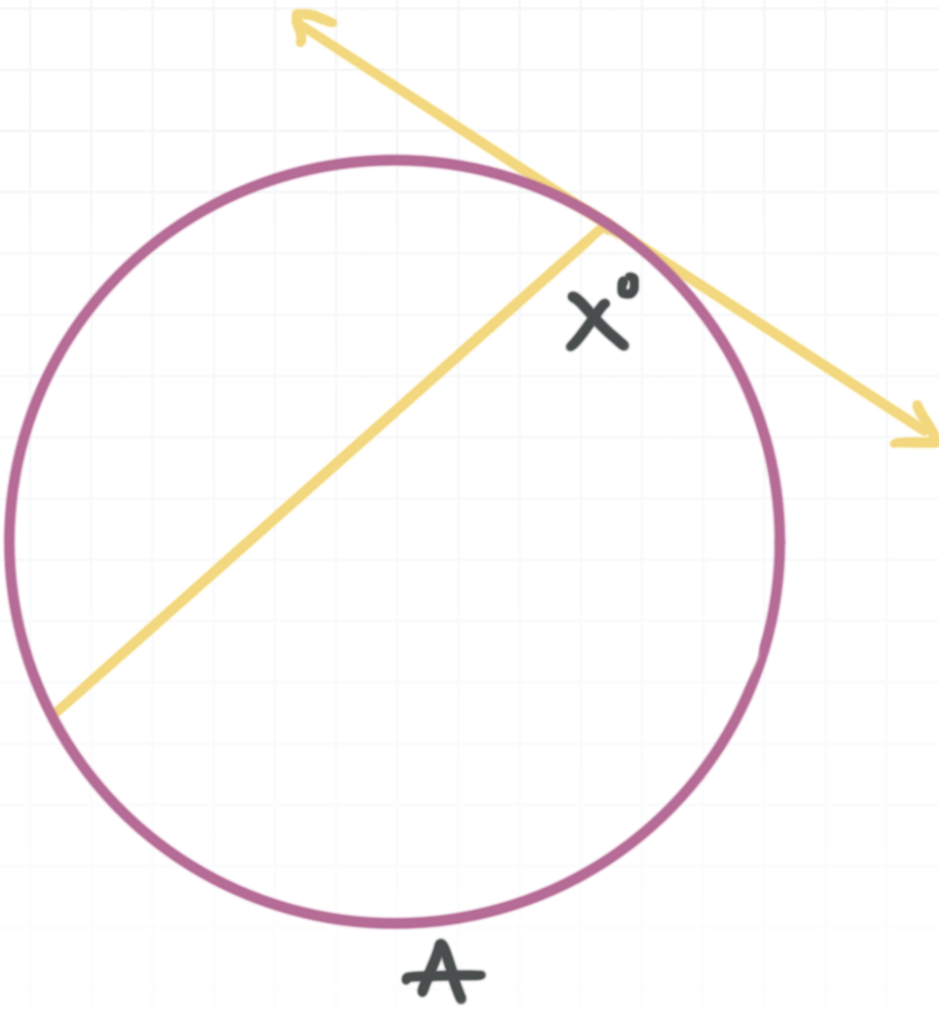
Vertex on the circle

When the vertex of the angle is on the circle, at the intersection of two chords, or of one chord and one tangent, the angle is called an **inscribed angle**. Remember from the last lesson that such an angle has only one intercepted arc, and that the measure of the angle is half the measure of its intercepted arc.



$$x^\circ = \frac{m\overset{\frown}{A}}{2}$$

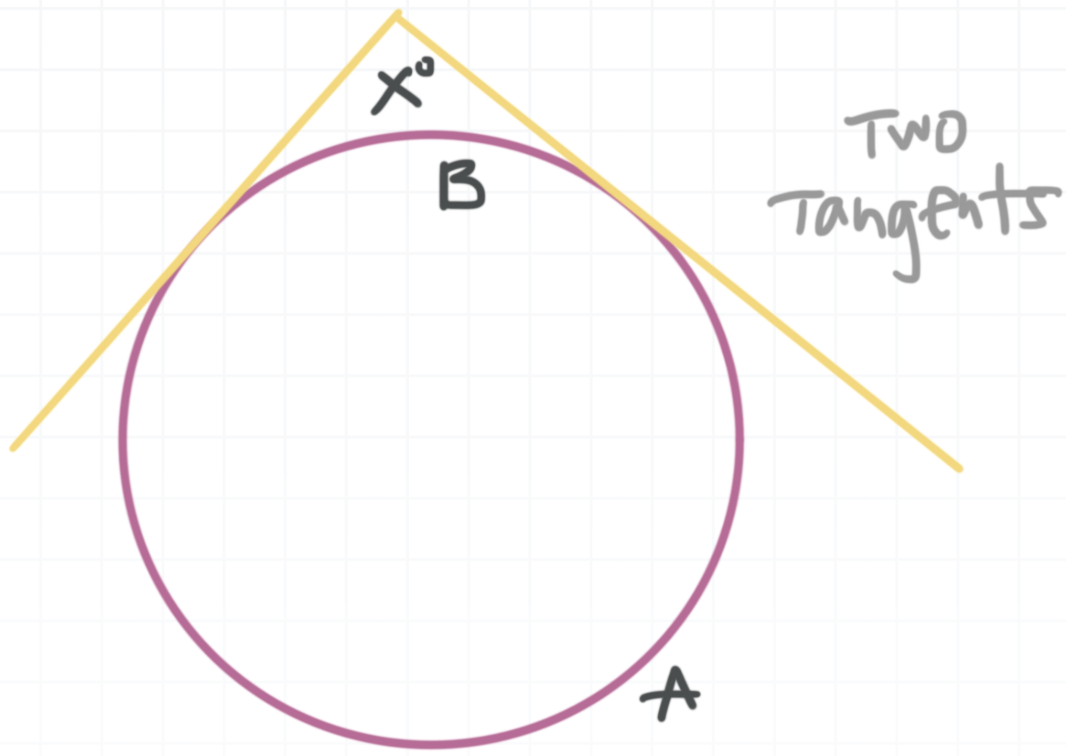
When the vertex of an angle is on the circle and it's the intersection of one chord and one tangent, it has only one intercepted arc. Such an angle is also considered to be an **inscribed angle**, and its measure is half the measure of its intercepted arc.

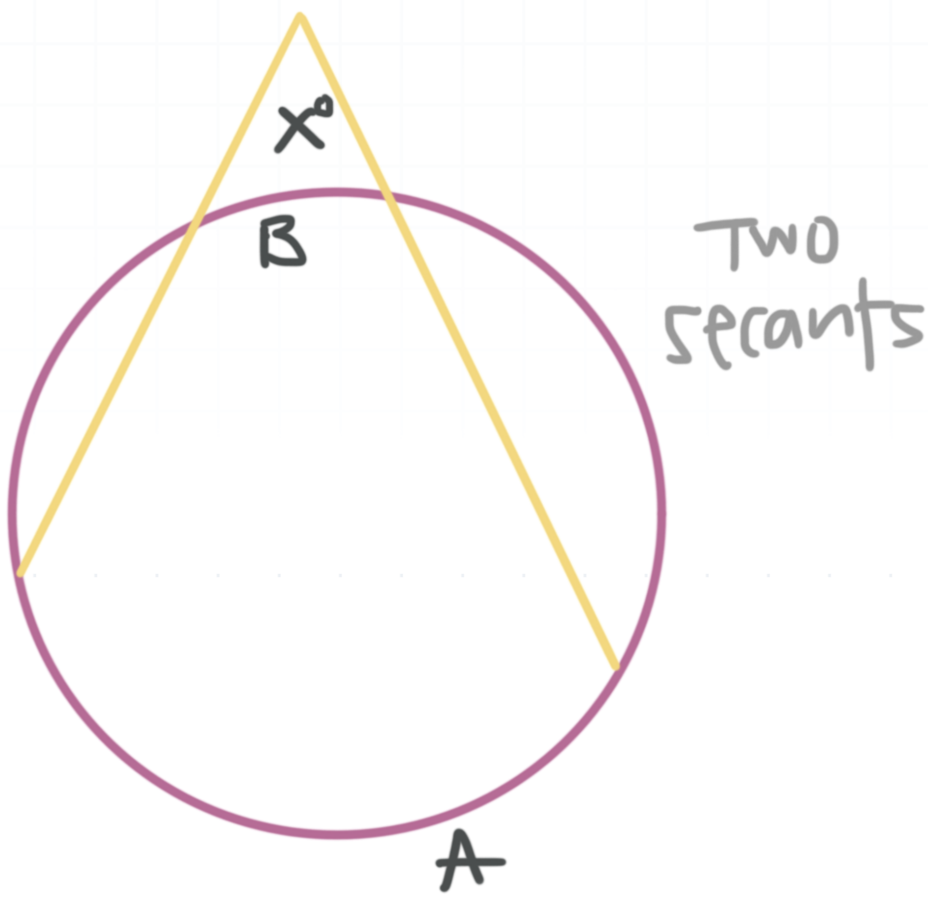


$$x^\circ = \frac{m\widehat{A}}{2}$$

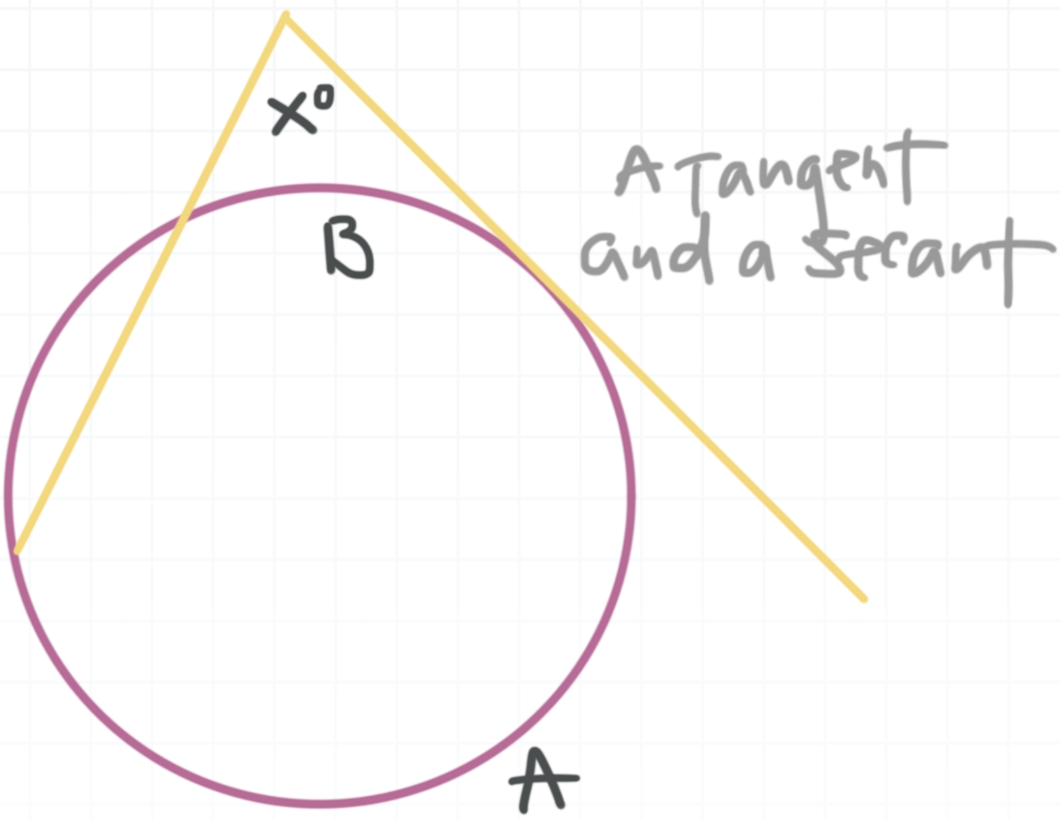
Vertex outside the circle

When the vertex of the angle is outside the circle, and at the intersection of two tangents, or of two secants, or of one tangent and one secant, it has two intercepted arcs, and the measure of the angle is half the difference between the measures of its intercepted arcs.





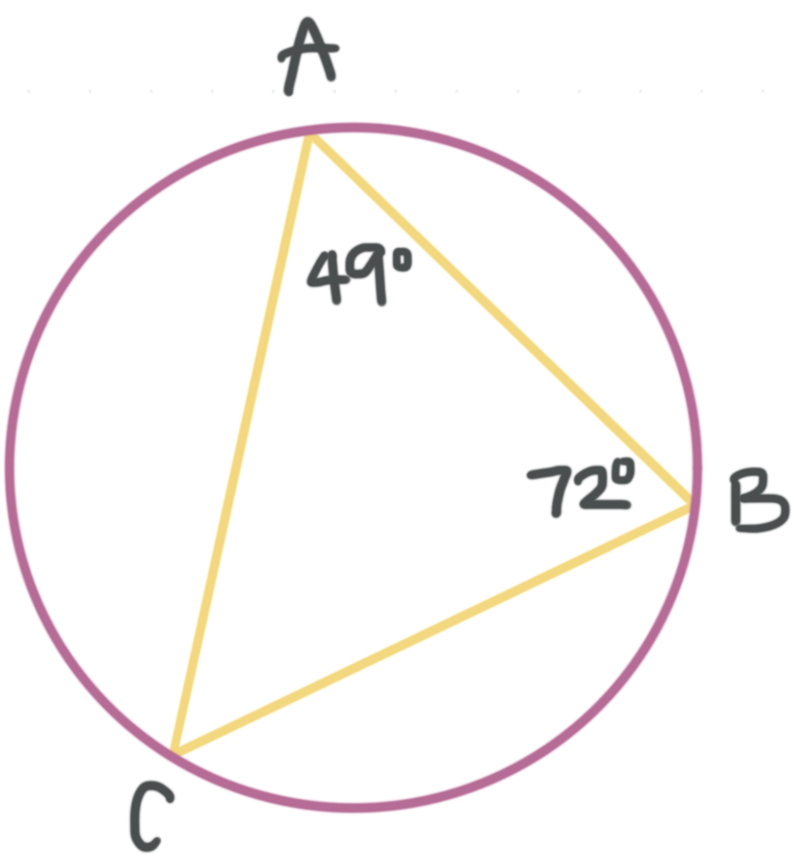
$$x^\circ = \frac{m\overset{\frown}{A} - m\overset{\frown}{B}}{2}$$



Let's start by working through an example.

Example

What is the measure of $\overset{\frown}{BA}$?



Arc $\overset{\frown}{BA}$ is the intercepted arc of $\angle BCA$, which is an inscribed angle, so the measure of $\angle BCA$ is half that $\overset{\frown}{BA}$. We know that the measures of the three interior angles of a triangle add to 180°, so we can find $m\angle BCA$ and use it to find the measure of $\overset{\frown}{BA}$.

$$m\angle BCA + 72^\circ + 49^\circ = 180^\circ$$

$$m\angle BCA + 121^\circ = 180^\circ$$

$$m\angle BCA = 59^\circ$$

And we know that

$$m\angle BCA = \frac{1}{2}m\overset{\frown}{BA}$$

So

$$2m\angle BCA = m\overset{\frown}{BA}$$

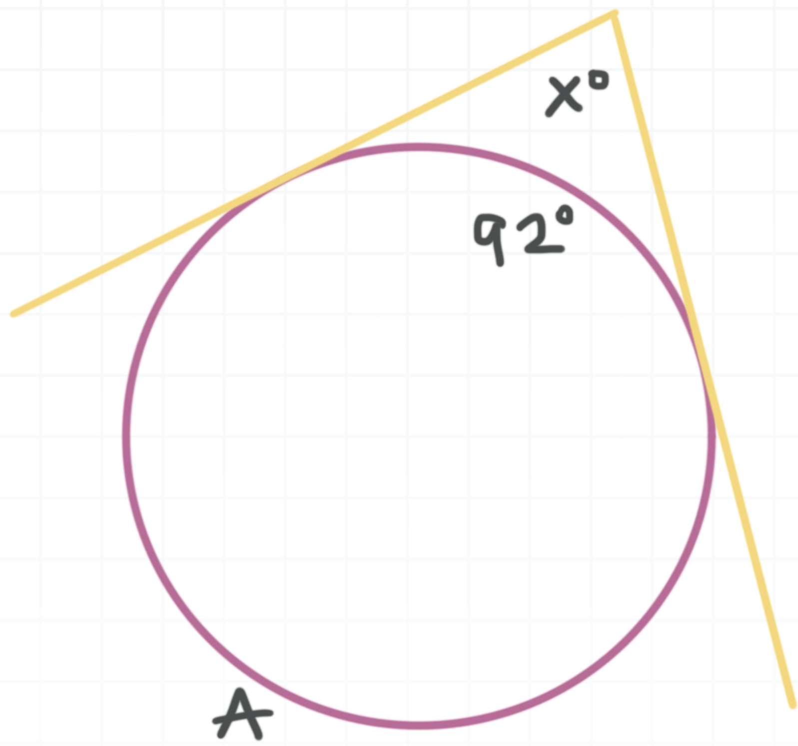
$$2(59^\circ) = m\overset{\frown}{BA}$$

$$118^\circ = m\overset{\frown}{BA}$$

Let's do two more examples.

Example

What is the value of x?



The angle measure x° is the measure of an angle whose vertex is outside the circle and at the intersection of two lines that are tangent to the circle. The measure of such an angle is half the difference between the measures of its intercepted arcs.

We know the measure of one of the intercepted arcs is 92°, so we need to find the measure of $\overset{\frown}{A}$. The measure of a complete circle is 360°, so we can find the measure of arc A by subtracting 92° from 360°.

$$m\overset{\frown}{A} = 360^\circ - 92^\circ$$

$$m\overset{\frown}{A} = 268^\circ$$

Now we can use the measures of the two intercepted arcs to find the value of x.

$$x^\circ = \frac{268^\circ - 92^\circ}{2}$$

$$x^\circ = \frac{176^\circ}{2}$$

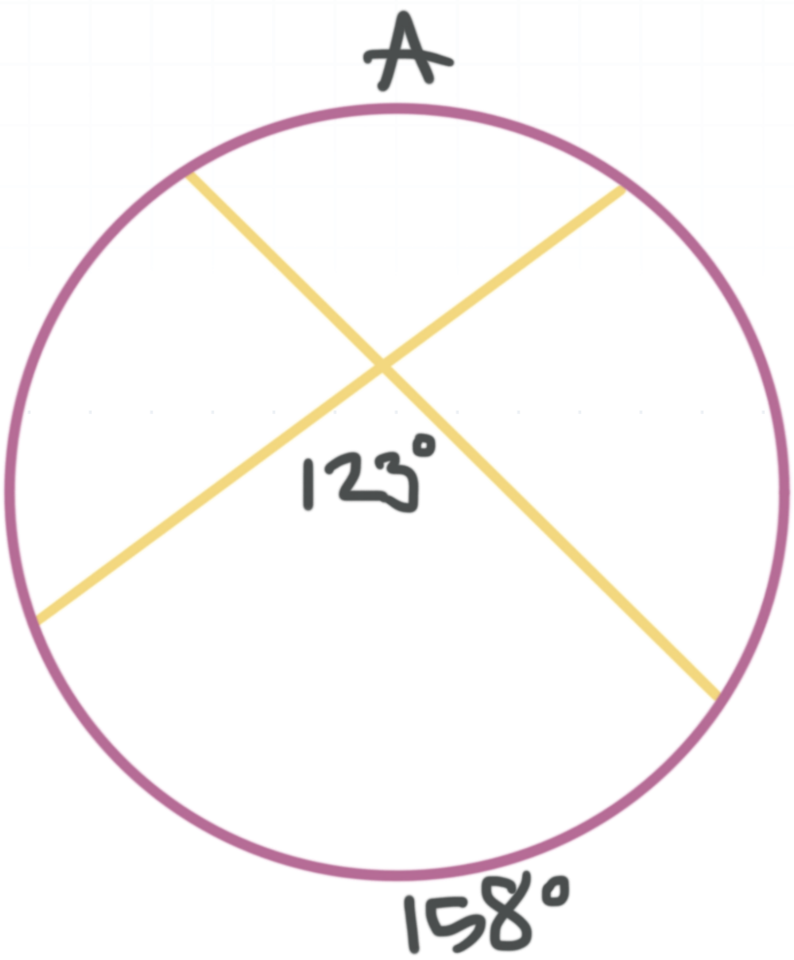
$$x^\circ = 88^\circ$$

$$x = 88$$

Let's try one with a vertex inside the circle.

Example

What is the measure of arc A?



We know that when the vertex of an angle is inside the circle, the measure of that angle is half the sum of the arc intercepted by that angle and the arc intercepted by the other angle in the pair.

Here, the arc intercepted by the 123° angle has measure 158°, and the arc intercepted by the other angle in the pair of vertical angles is $\widehat{A}$, so we can set up an equation and then solve for $m\widehat{A}$.

$$123^\circ = \frac{158^\circ + m\widehat{A}}{2}$$

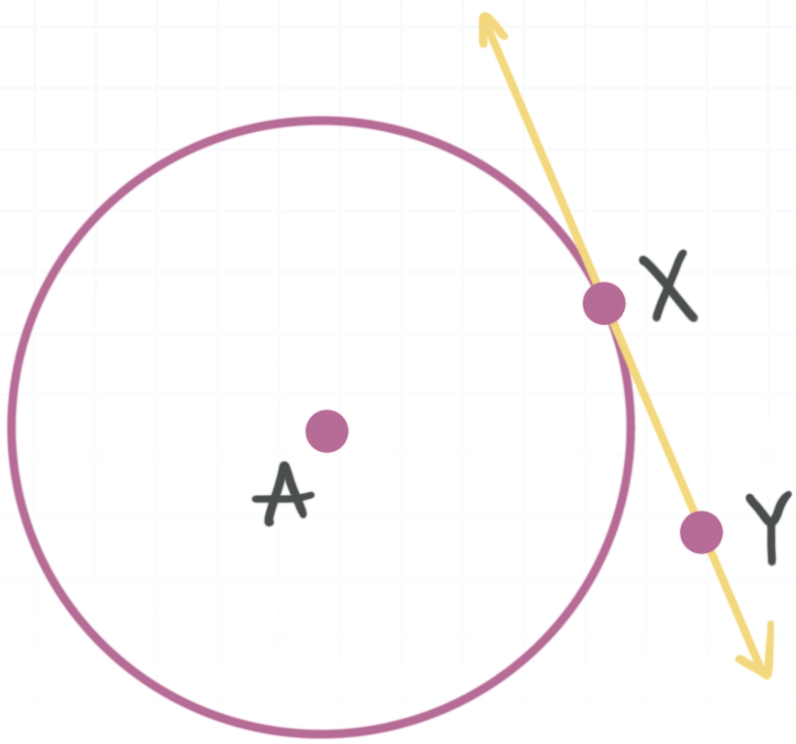
$$2(123^\circ) = 158^\circ + m\widehat{A}$$

$$246^\circ = 158^\circ + m\widehat{A}$$

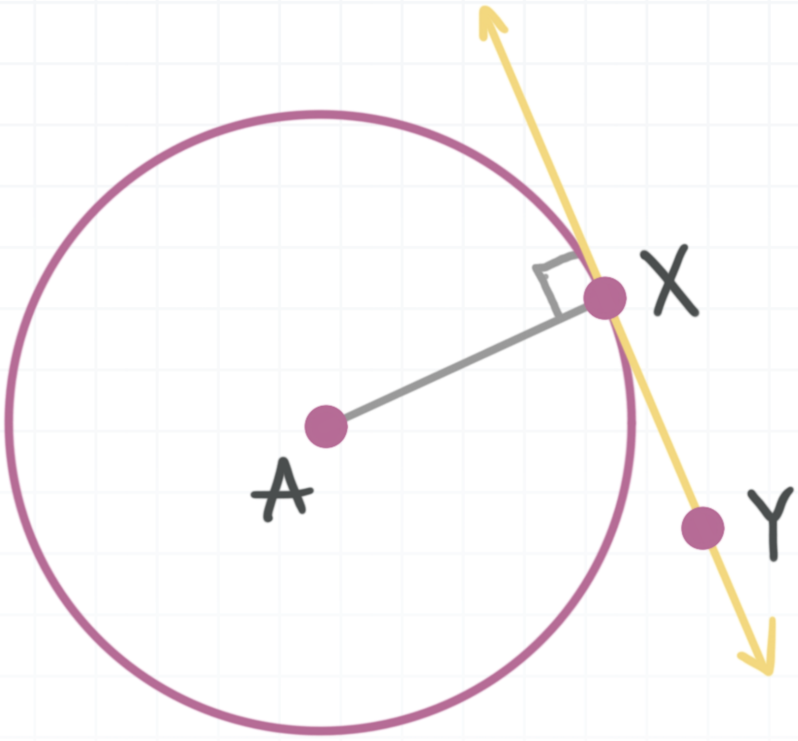
$$m\widehat{A} = 88^\circ$$

Tangent lines of circles

A **tangent line** of a circle is a line that intersects the circle at exactly one point. In the circle in the figure (with A at its center), line XY is tangent to the circle at point X. Point X is called the **point of tangency**.



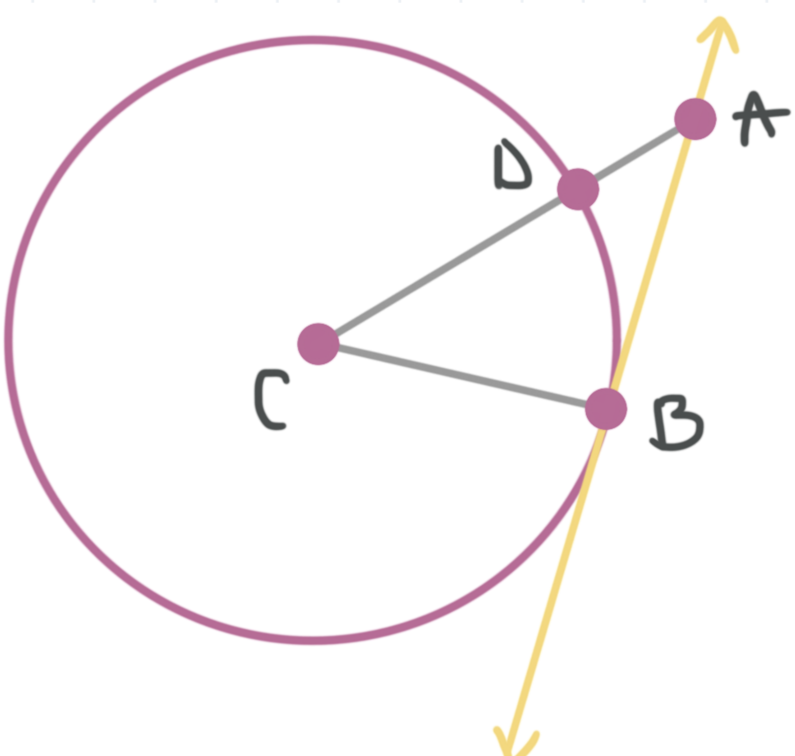
The radius drawn from the center of the circle to the point of tangency is always perpendicular to the tangent line. In the figure below, Radius $\overline{AX}$ is perpendicular to $\overleftrightarrow{XY}$.



Let's start by working through an example.

Example

The radius of the circle in the figure (with C at its center) is 5. Also, $\overline{AB}=6$, $\overline{DA}=3$, and line $\overleftrightarrow{AB}$ intersects the circle at B. Determine whether line $\overleftrightarrow{AB}$ is tangent to the circle.



If line $\overleftrightarrow{AB}$ is tangent to the circle, then radius $\overline{CB}$ will be perpendicular to line $\overleftrightarrow{AB}$, and $\angle ABC$ will be a right angle, so triangle ABC will be a right triangle. That will be true if and only if the Pythagorean theorem is satisfied for the triangle.

So we want to determine whether the following equation is true.

$$(\overline{CB})^2 + (\overline{AB})^2 = (\overline{CA})^2$$

Since $\overline{CB}$ is a radius, we know that $\overline{CB} = 5$. We also know that $\overline{CA} = \overline{CD} + \overline{DA}$, and that $\overline{CD}$ is a radius, so $\overline{CD} = 5$. Since $\overline{DA} = 3$, we see that $\overline{CA} = 5 + 3 = 8$. Now we can check the Pythagorean theorem.

$$(\overline{CB})^2 + (\overline{AB})^2 = (\overline{CA})^2$$

$$5^2 + 6^2 = 8^2$$

$$25 + 36 = 64$$

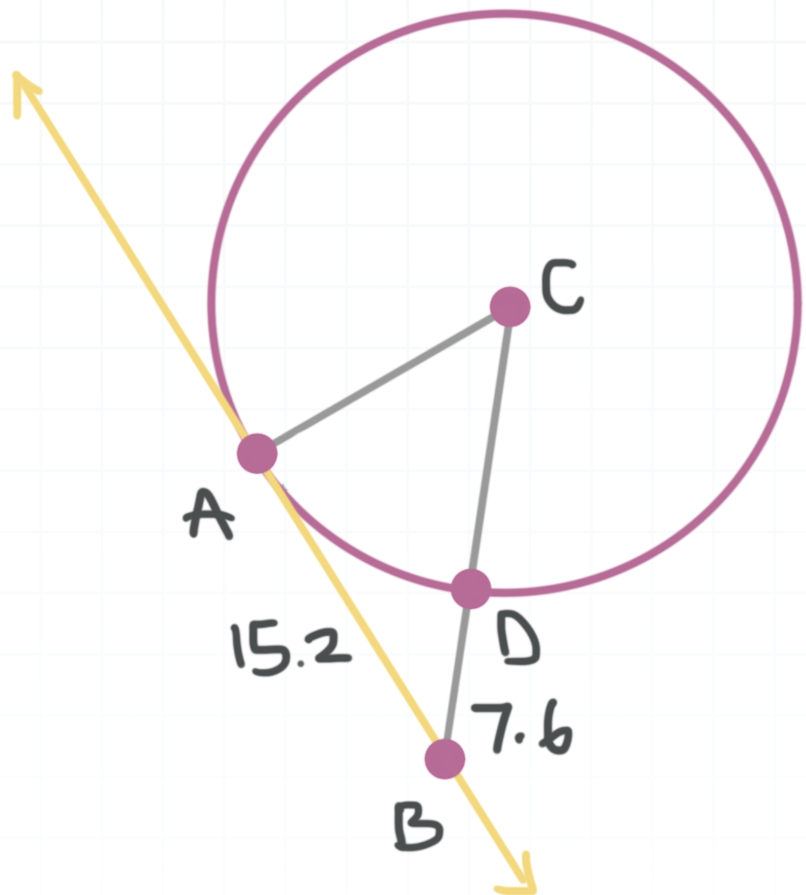
$$61 \neq 64$$

This means that $\angle ABC$ is not a right angle, and line AB is therefore not tangent to the circle.

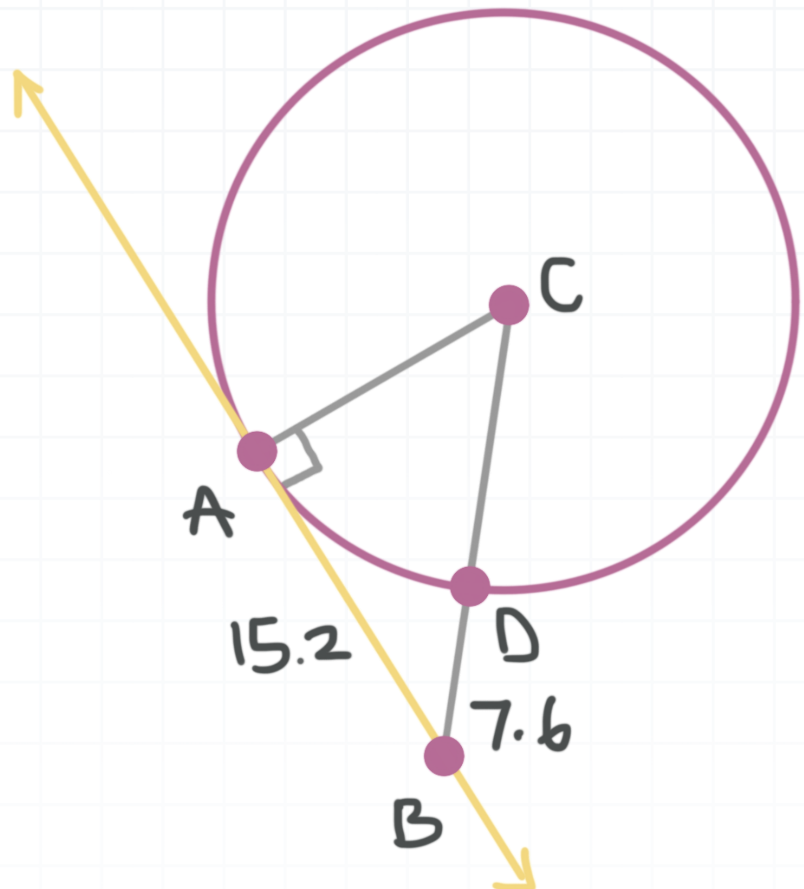
Let's do one more.

Example

Find the radius of the circle in the figure (with C at its center), given that $\overline{AB} = 15.2$, $\overline{DB} = 7.6$, and $\overleftrightarrow{AB}$ is tangent to the circle at A.



A radius drawn to point A will be perpendicular to $\overleftrightarrow{AB}$ and form right triangle BAC.



Let's call the radius x. Then $\overline{AC} = x$ and $\overline{CD} = x$. Now we can use the Pythagorean theorem to set up an equation and solve for x.

$$(\overline{AC})^2 + (\overline{AB})^2 = (\overline{CB})^2$$

$$x^2 + 15.2^2 = (x + 7.6)^2$$

$$x^2 + 231.04 = x^2 + 15.2x + 57.76$$

Subtract x^2 and 57.76 from both sides.

$$173.28 = 15.2x$$

$$x = 11.4$$

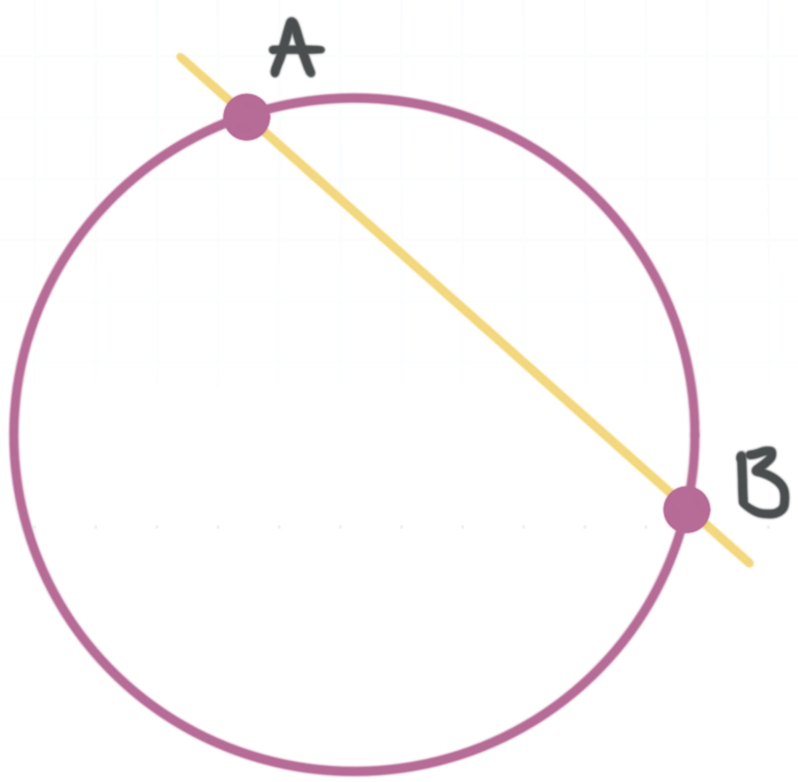
The radius of the circle is 11.4.

Intersecting tangents and secants

In this lesson we'll look at the relationships formed from intersecting tangents and secants in circles.

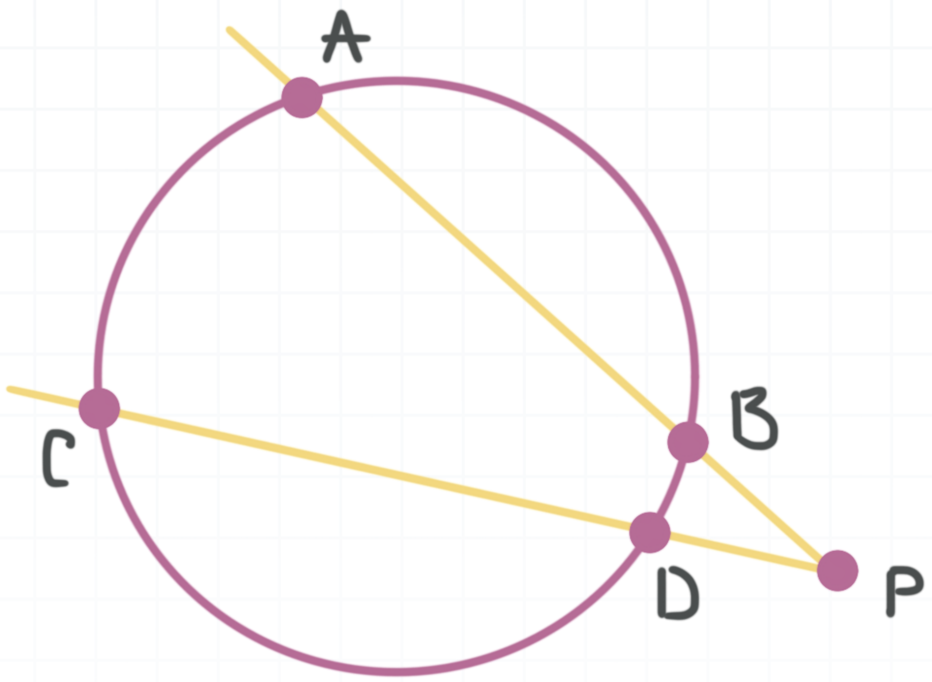
Secant

A **secant** of a circle is a line or line segment that intersects the circle at two points. $\overline{AB}$ is a secant of this circle.



Intersecting secants theorem

There's a special relationship between two secants that intersect at some point P outside a circle. The product of the lengths of the "outside" and "whole" segments of one of the secants is equal to the product of the lengths of the "outside" and "whole" segments of the other secant. In the following circle, secants $\overline{AP}$ and $\overline{CP}$ intersect at point P,



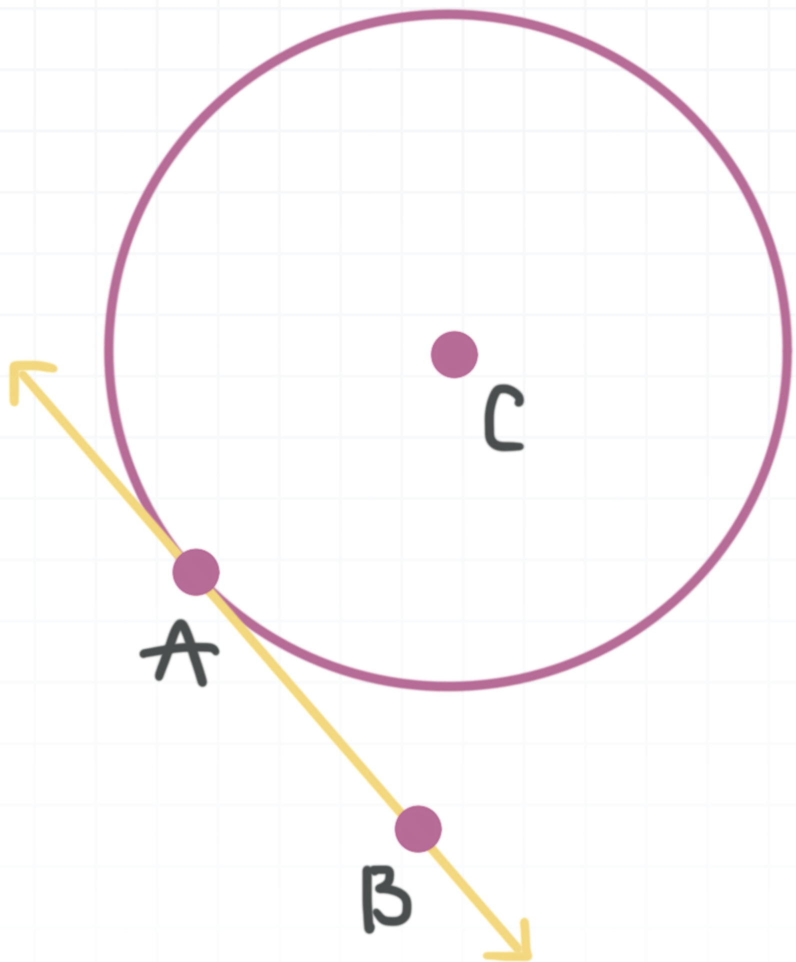
so

$$\text{outside} \cdot \text{whole} = \text{outside} \cdot \text{whole}$$

$$\overline{BP} \cdot \overline{AP} = \overline{DP} \cdot \overline{CP}$$

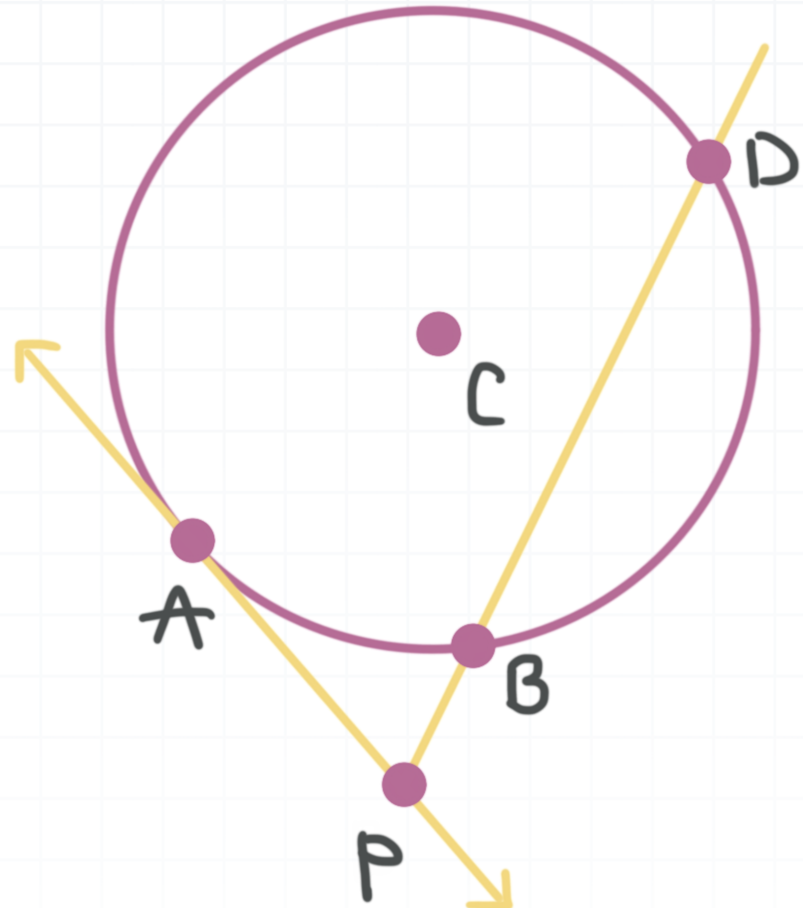
Tangents

A **tangent** of a circle is a line that intersects the circle at only one point. $\overleftrightarrow{AB}$ is a tangent of this circle.



Intersecting tangent-secant theorem

There is also a special relationship between a tangent and a secant that intersect at some point P outside a circle. The square of the length of (the segment of) the tangent from point P to the point of tangency is equal to the product of the lengths of the "outside" and "whole" segments of the secant. In the following circle, tangent $\overleftrightarrow{AP}$ and secant $\overline{DP}$ intersect at point P,



so

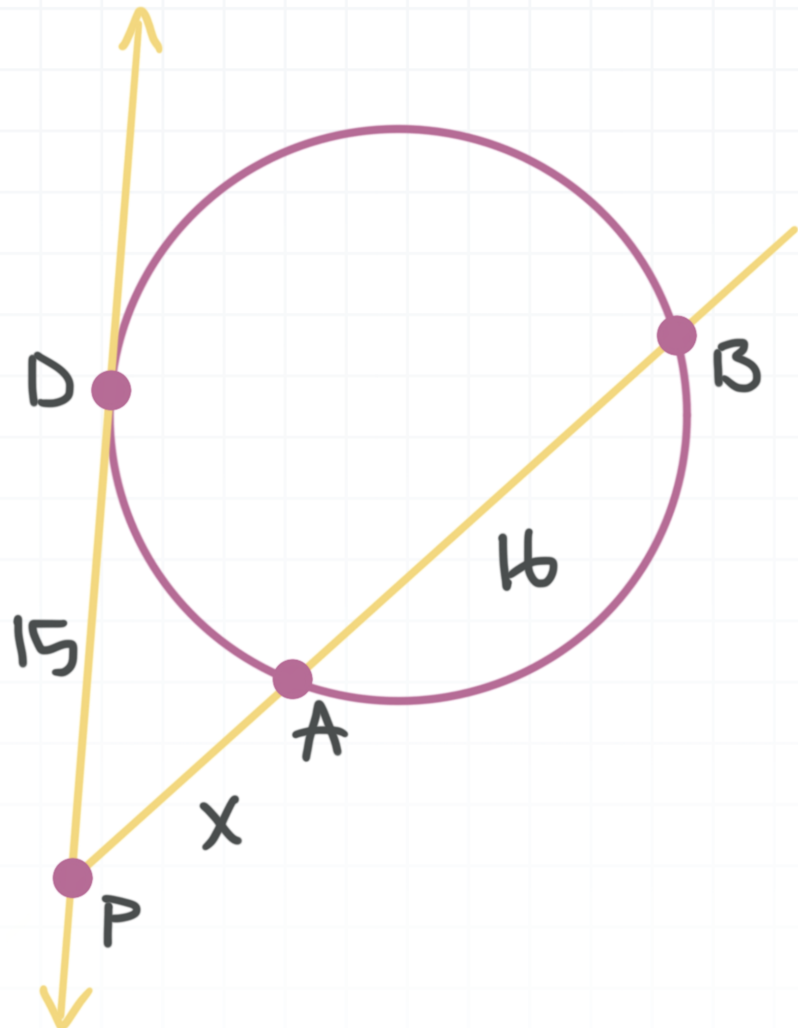
$$\text{tangent}^2 = \text{outside} \cdot \text{whole}$$

$$(\overline{AP})^2 = \overline{BP} \cdot \overline{DP}$$

Let's start by working through an example.

Example

Find the value of x in the figure, assuming that $\overleftrightarrow{DP}$ is a tangent.



Because there's a secant intersecting with a tangent, we can follow the formula and plug in the lengths shown in the figure.

$\text{tangent}^2 = \text{outside} \cdot \text{whole}$

$15^2 = x(x + 16)$

$225 = x^2 + 16x$

$0 = x^2 + 16x - 225$

$0 = (x + 25)(x - 9)$

$0 = x + 25$ or $0 = x - 9$

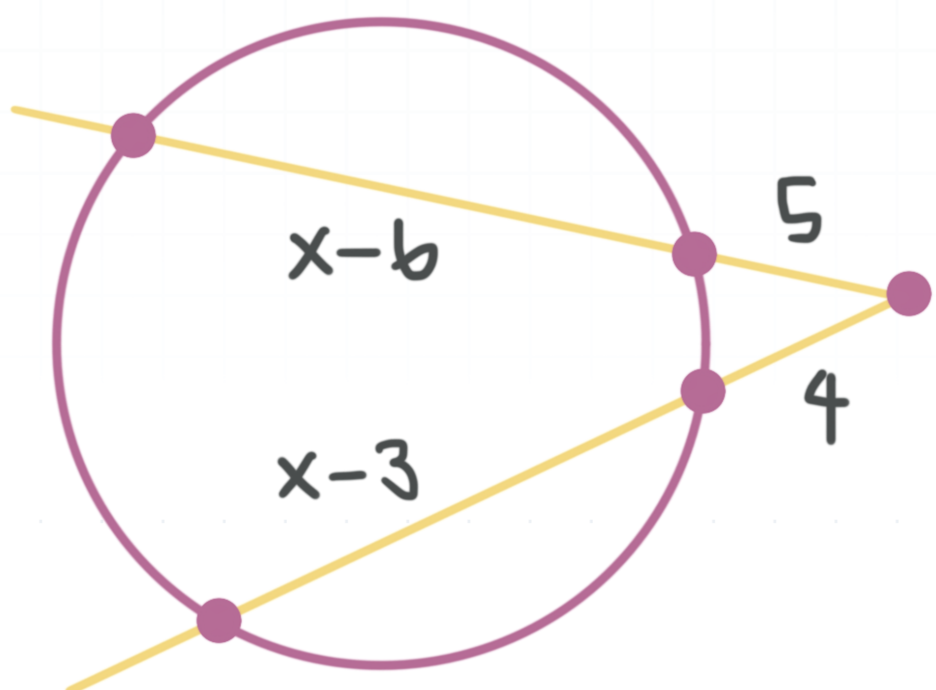
$$x = -25 \text{ or } x = 9$$

A line segment can't have a negative length, so rule out $x = -25$ and conclude that $x = 9$.

Let's do one more problem.

Example

Given the lengths in the figure, find the value of x.



Because there are two secants that intersect outside the circle, we can follow the formula and plug in the lengths shown in the figure.

$$\text{outside} \cdot \text{whole} = \text{outside} \cdot \text{whole}$$

$$5[(x-6)+5] = 4[(x-3)+4]$$

$$5(x-1)=4(x+1)$$

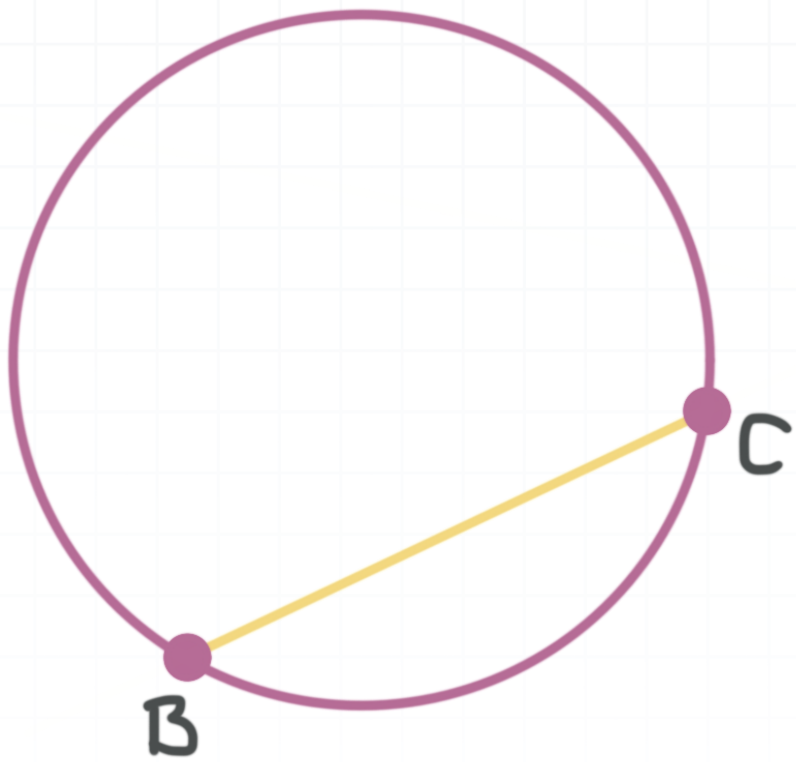
$$5x-5=4x+4$$

$$x-5=4$$

$$x=9$$

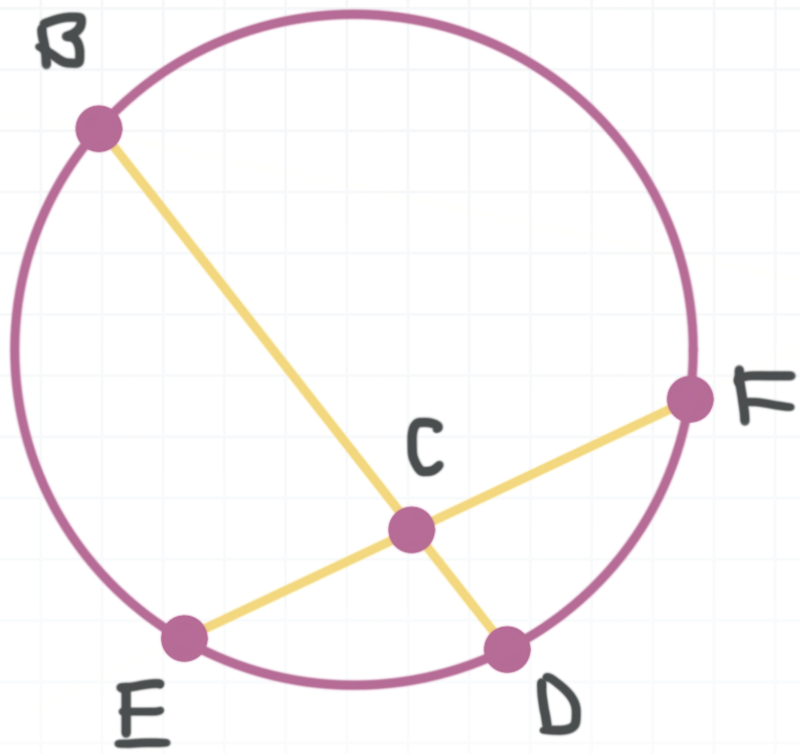
Intersecting chords

A **chord** of a circle is a line segment that has both of its endpoints on the circle. $\overline{BC}$ is an example of a chord.



Intersecting chord theorem

The intersecting chord theorem states that the products of chord segments are always equal. For instance, consider chords $\overline{BD}$ and $\overline{EF}$.



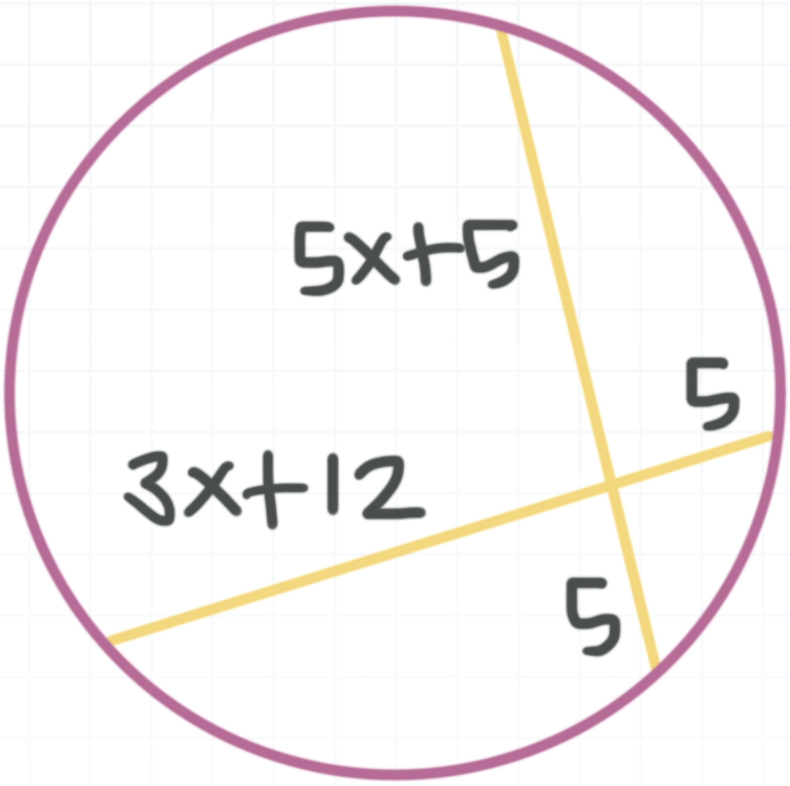
The intersecting chord theorem says that

$$\overline{BC} \cdot \overline{CD} = \overline{EC} \cdot \overline{CF}$$

Let's start by working through an example.

Example

Find the value of x in the figure.



The products of the chord segments are equal. So we can set up an equation.

$$5(5x+5)=5(3x+12)$$

$$25x+25=15x+60$$

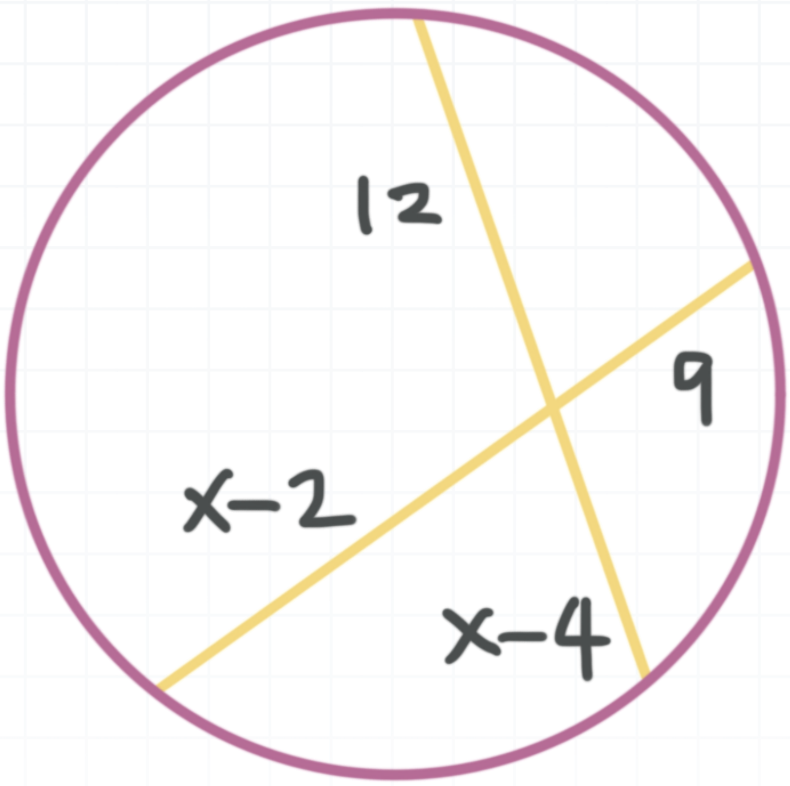
$$10x=35$$

$$x=3.5$$

Let's do one more.

Example

Find the length of each chord.



First we need to find the value of x, and then use that to find the lengths of the chords. The products of the chord segments are equal, so

$$12(x-4)=9(x-2)$$

$$12x-48=9x-18$$

$$3x=30$$

$$x=10$$

Now we can find the length of each chord. One chord has a length of

$$12+x-4$$

$$12+10-4$$

$$18$$

The other chord has a length of

$$x - 2 + 9$$

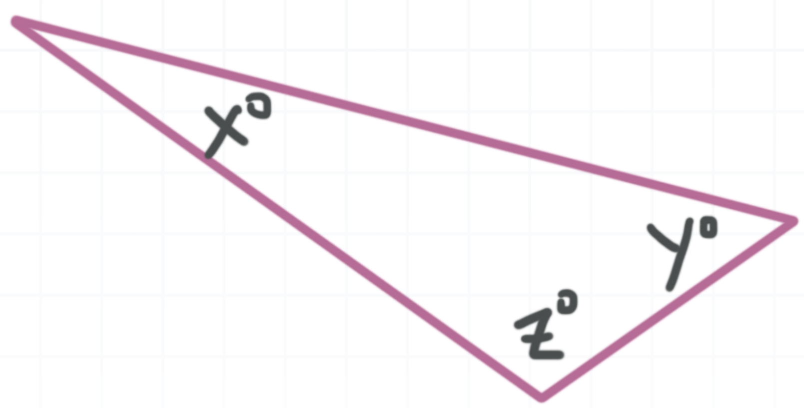
$$10 - 2 + 9$$

$$17$$

The chords have lengths of 17 and 18.

Interior angles of triangles

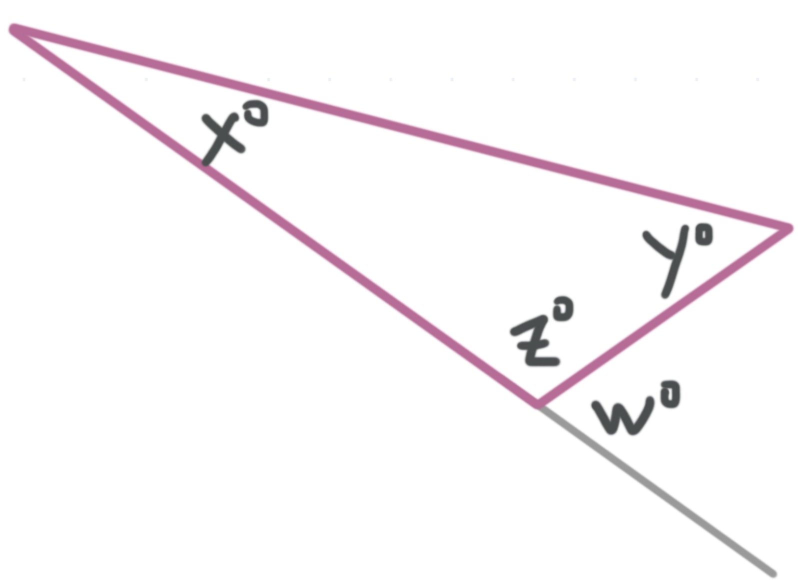
The interior angles of a triangle are the three angles on the inside of a triangle. The measures of these three angles always sum to 180°.



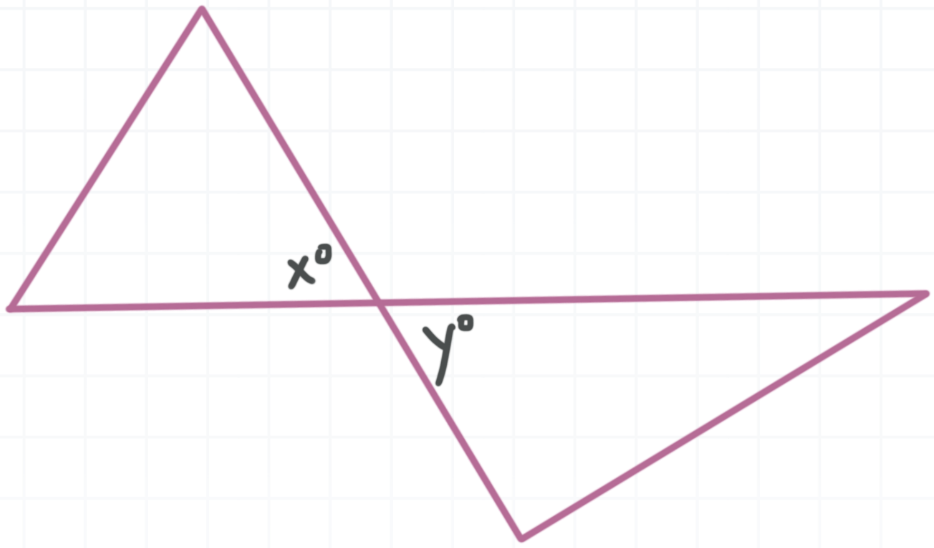
$$x^\circ + y^\circ + z^\circ = 180^\circ$$

There are a few other angle relationships we need to remember:

The measures of a pair of adjacent angles that (together) form a straight line add to 180°, so $z^\circ + w^\circ = 180^\circ$.



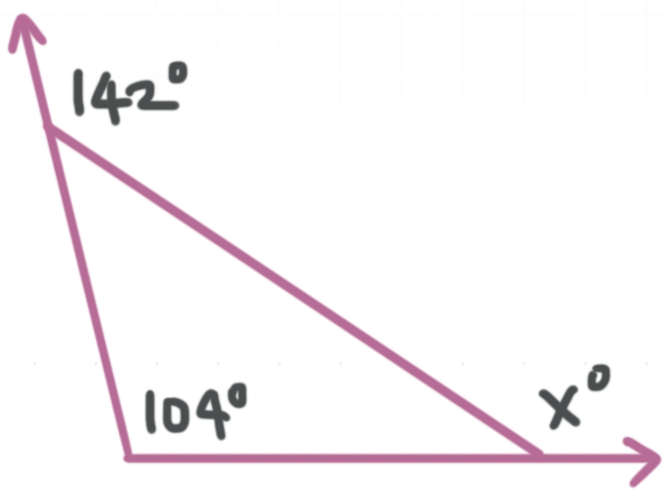
Vertical angles are congruent, so $x^\circ = y^\circ$.



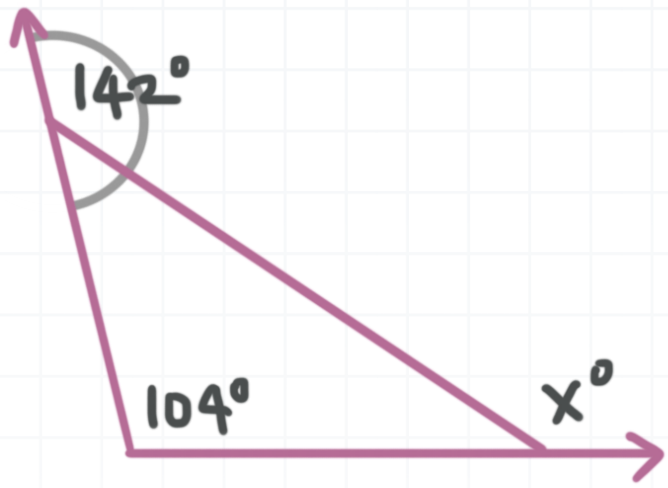
Let's start by working through an example.

Example

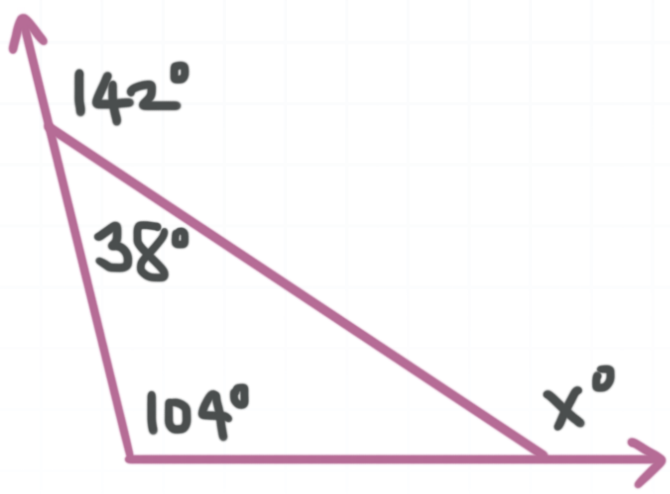
What is the value of x?



We know that the angle of measure 142° and the interior angle adjacent to it, together, form a straight line:

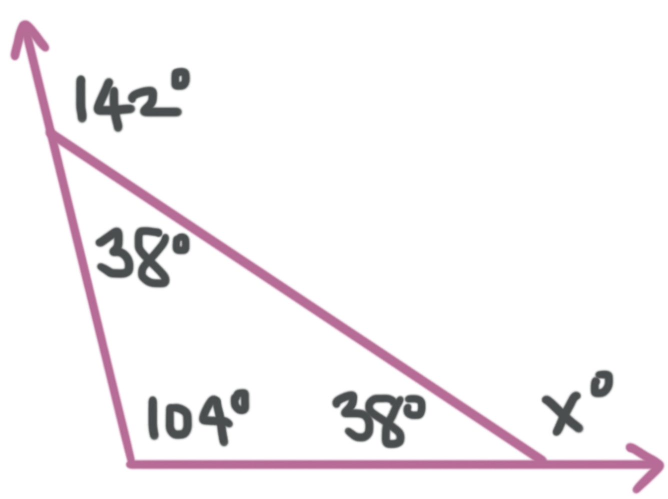


We can find the measure of that interior angle by subtraction: $180^\circ - 142^\circ = 38^\circ$.



The measures of the three angles inside a triangle sum to 180°, so the measure of the third interior angle is

$$180^\circ - 104^\circ - 38^\circ = 38^\circ$$



We can see that the adjacent angles of measure x° and 38°, together, form a straight line, so

$$x^\circ + 38^\circ = 180^\circ$$

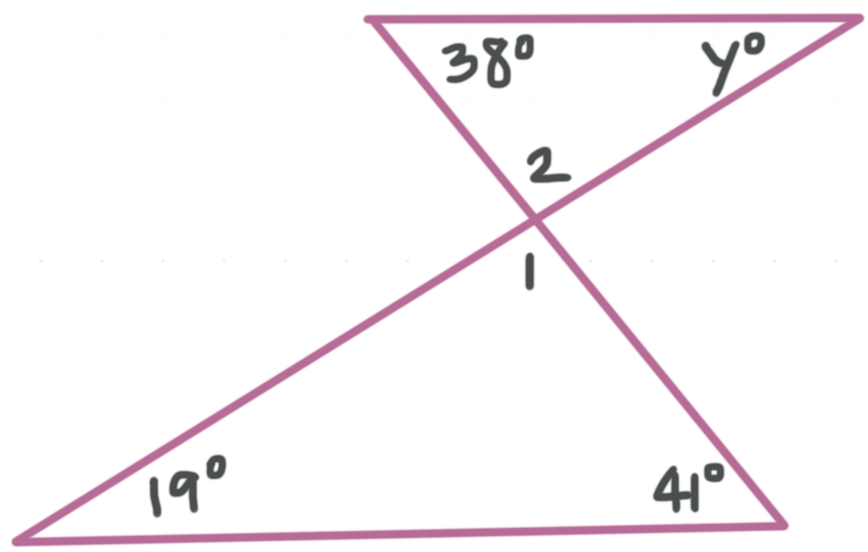
$$x^\circ = 142^\circ$$

$$x = 142$$

Let's try another one.

Example

What is the value of y?



The measures of the three interior angles of a triangle sum to 180°, so

$$m\angle 1 + 19^\circ + 41^\circ = 180^\circ$$

$$m\angle 1 = 120^\circ$$

Angle 1 and angle 2 are a pair of vertical angles, and vertical angles are congruent, so

$$m\angle 1 = m\angle 2 = 120^\circ$$

Again, the measures of the three interior angles of a triangle sum to 180°, so we see that

$$m\angle 2 + 38^\circ + y^\circ = 180^\circ$$

$$120^\circ + 38^\circ + y^\circ = 180^\circ$$

$$y^\circ = 22^\circ$$

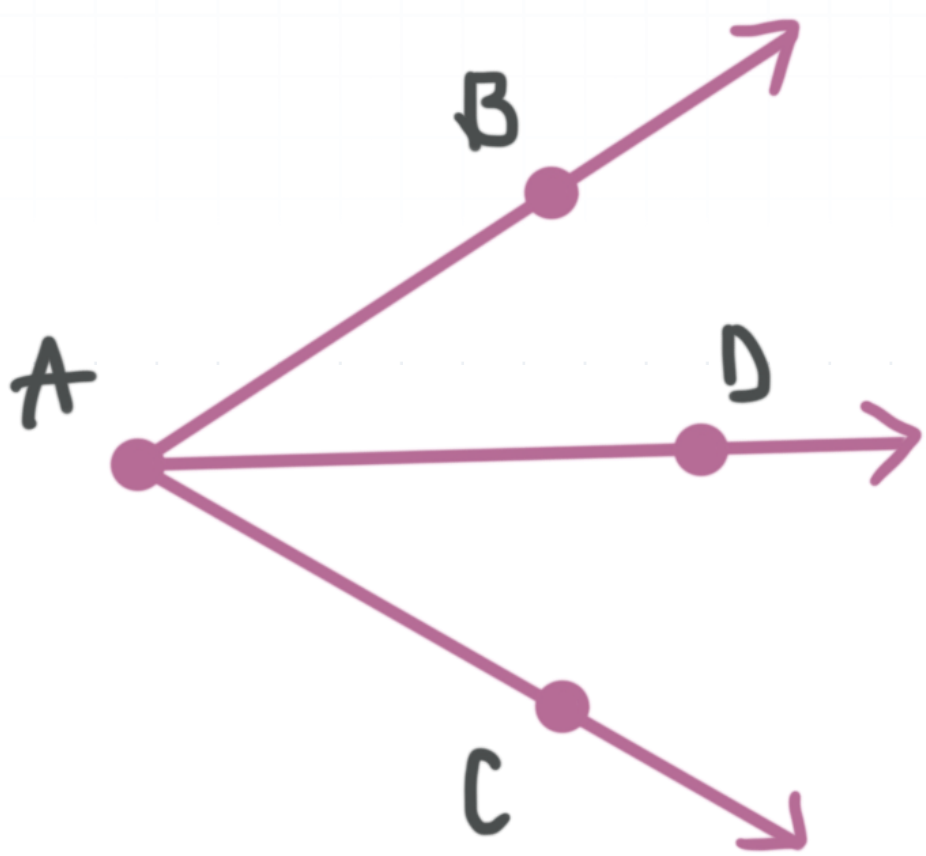
$$y = 22$$

Perpendicular and angle bisectors

In this lesson we'll look at how to use the properties of perpendicular and angle bisectors to get information about geometric figures.

Angle bisectors

An angle bisector is a line, line segment, or ray that goes through the vertex of an angle and divides the angle into two congruent angles that each have measure equal to half that of the original angle. If $\overrightarrow{AD}$ bisects $\angle CAB$,



then

$$m\angle DAB = m\angle CAD$$

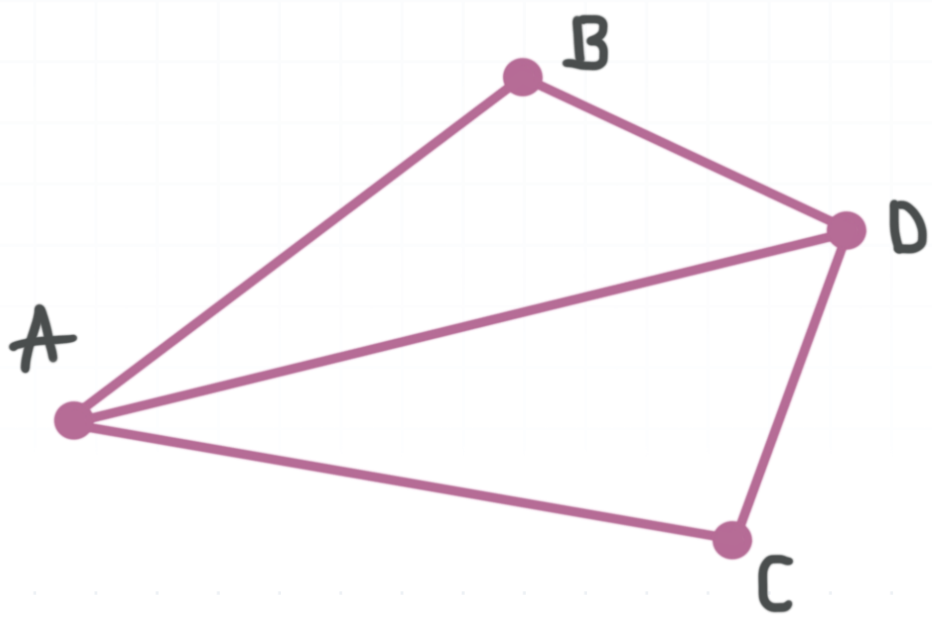
$$m\angle DAB = \frac{1}{2}m\angle CAB = m\angle CAD$$

$$2m\angle DAB = m\angle CAB = 2m\angle CAD$$

Let's look at an example.

Example

If $m\angle DAB = 31^\circ$ and $m\angle BDC = 66^\circ$, and $\overline{AD}$ is a bisector of both $\angle CAB$ and $\angle BDC$, what is $m\angle DCA$?



Using what we already know, we see that

$$m\angle CAD = m\angle DAB = 31^\circ$$

and

$$m\angle ADC = \frac{1}{2}m\angle BDC = \frac{1}{2}\cdot 66^\circ = 33^\circ$$

The measures of the three interior angles of any triangle add up to 180°. Applying that to $\triangle ACD$, we have

$$m\angle CAD + m\angle ADC + m\angle DCA = 180^\circ$$

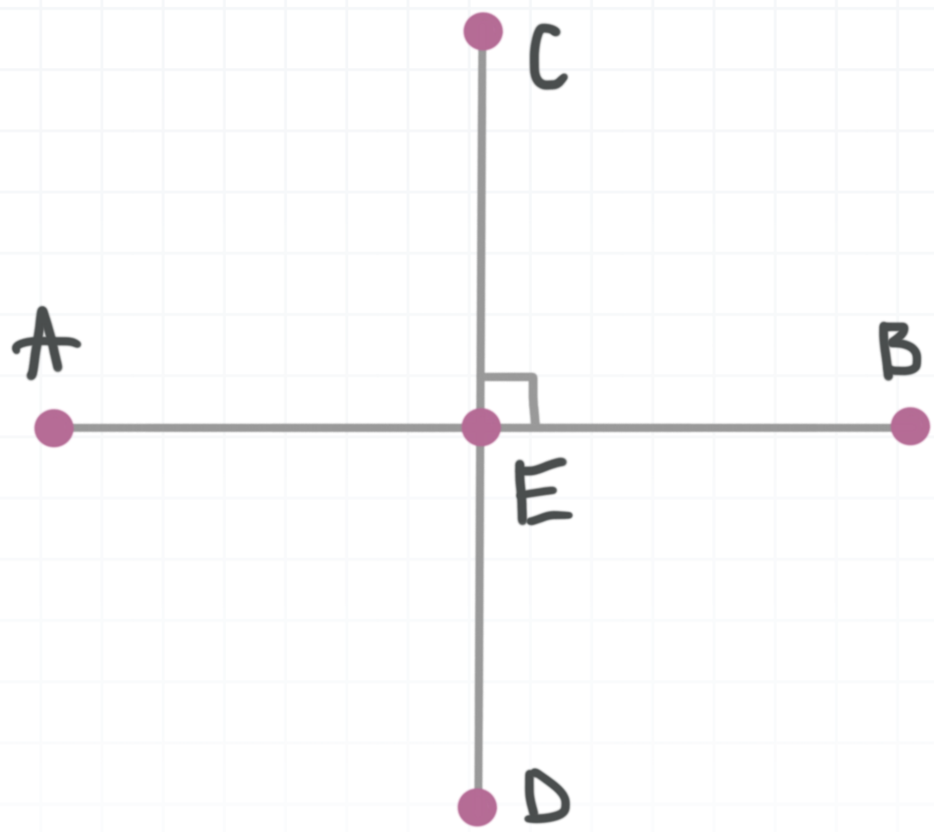
$$31^\circ + 33^\circ + m\angle DCA = 180^\circ$$

$$64^\circ + m\angle DCA = 180^\circ$$

$$m\angle DCA = 116^\circ$$

Perpendicular bisectors

A perpendicular bisector is a line, line segment, or ray that crosses a line segment at its midpoint and forms a right angle where it crosses. $\overline{CD}$ is a perpendicular bisector of $\overline{AB}$ at point E.



This tells us that

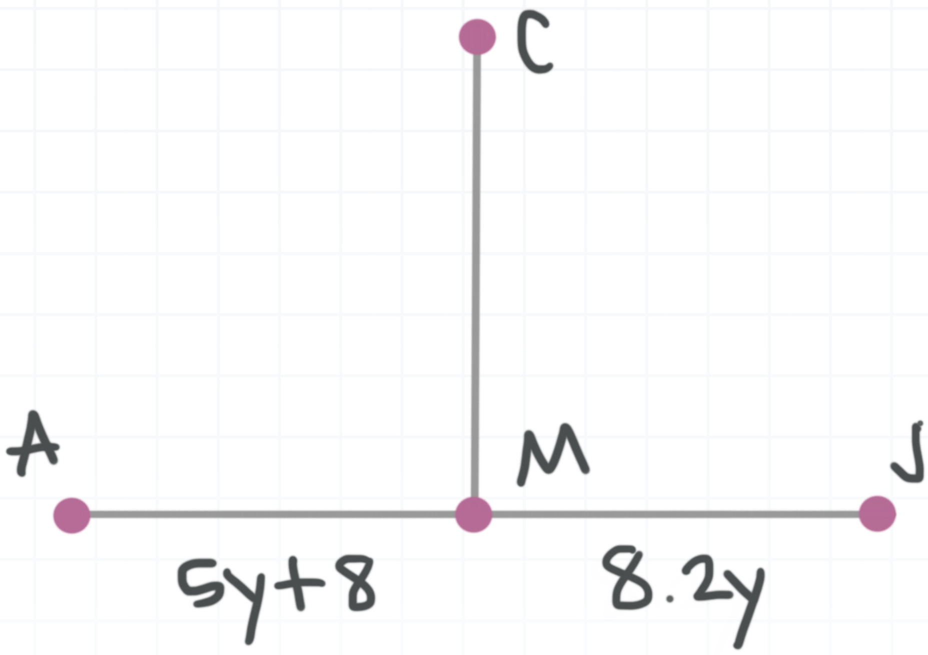
$$m\angle CEA = m\angle BEC = m\angle AED = m\angle DEB = 90^\circ$$

$$\overline{AE} = \overline{EB}$$

Let's look at a few more example problems.

Example

Find the value of y if $\overline{CM}$ is a perpendicular bisector of $\overline{AJ}$.



Because $\overline{CM}$ is a perpendicular bisector of $\overline{AJ}$, we know that $\overline{AM} = \overline{MJ}$, so

$$5y + 8 = 8.2y$$

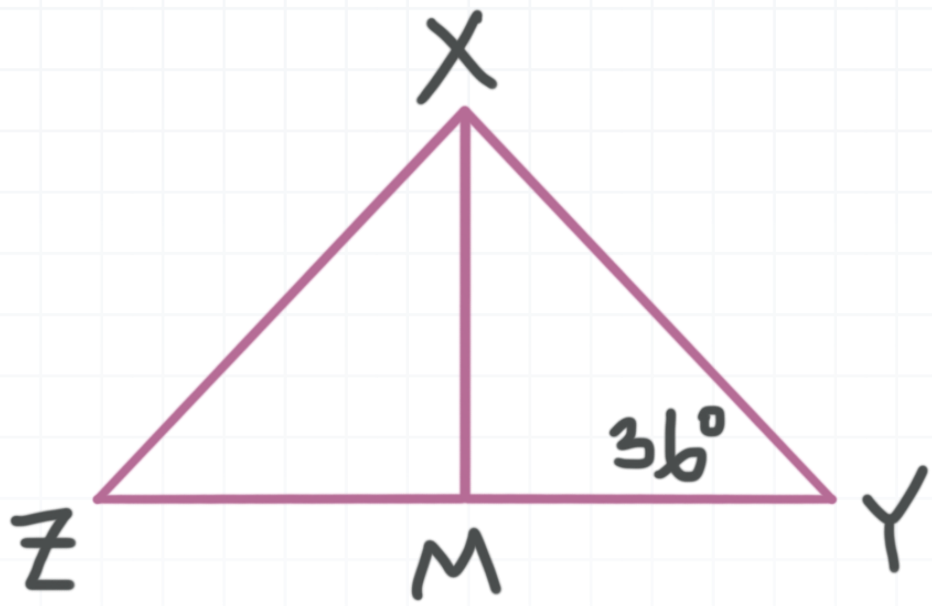
$$8 = 3.2y$$

$$y = 2.5$$

Let's look at one more problem.

Example

Find $m\angle MXY$ if $\overline{XM}$ is a perpendicular bisector of $\overline{ZY}$.



We know $\angle YMX$ is a right angle, so $m\angle YMX = 90^\circ$. The measures of the three interior angles of any triangle add up to 180°. Applying that to $\triangle XMY$, we have

$$m\angle YMX + m\angle XYM + m\angle MXY = 180^\circ$$

$$90^\circ + 36^\circ + m\angle MXY = 180^\circ$$

$$126^\circ + m\angle MXY = 180^\circ$$

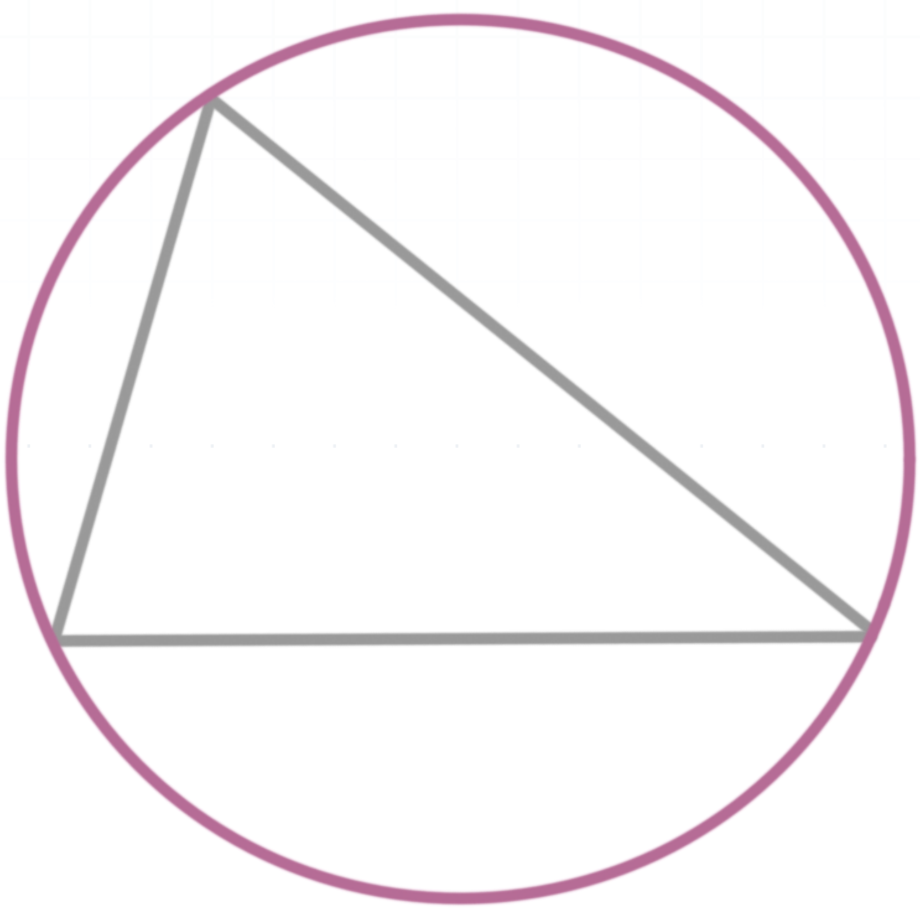
$$m\angle MXY = 54^\circ$$

Circumscribed and inscribed circles of a triangle

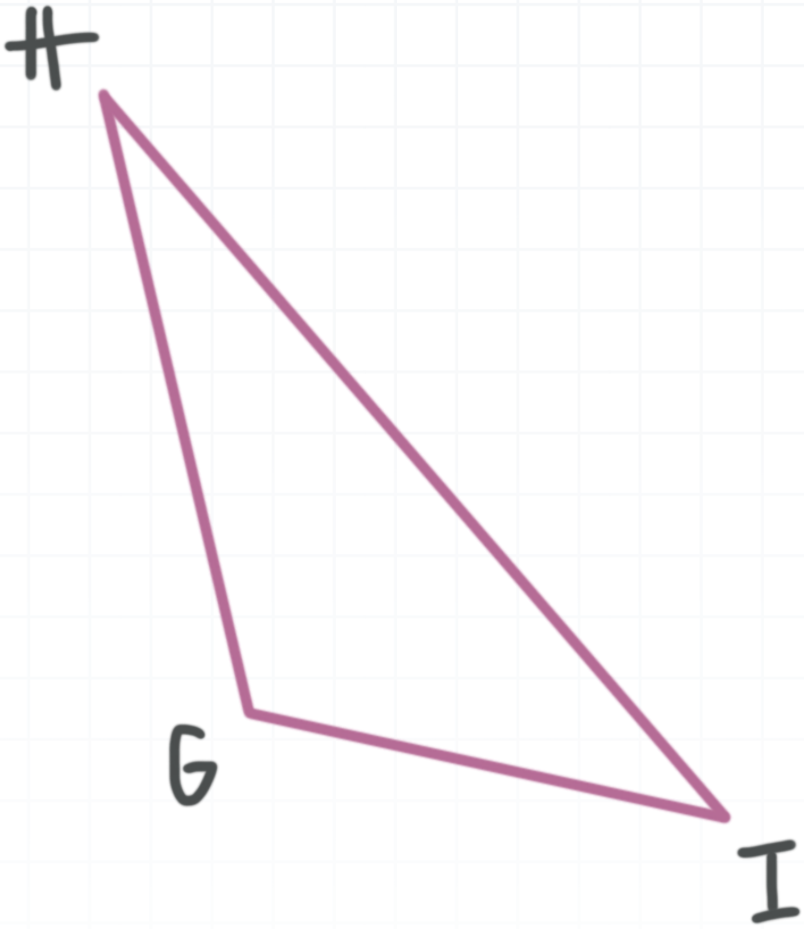
In this lesson we'll look at circumscribed and inscribed circles of a triangle and the special relationships between these circles and the corresponding triangles.

Circumscribed circles

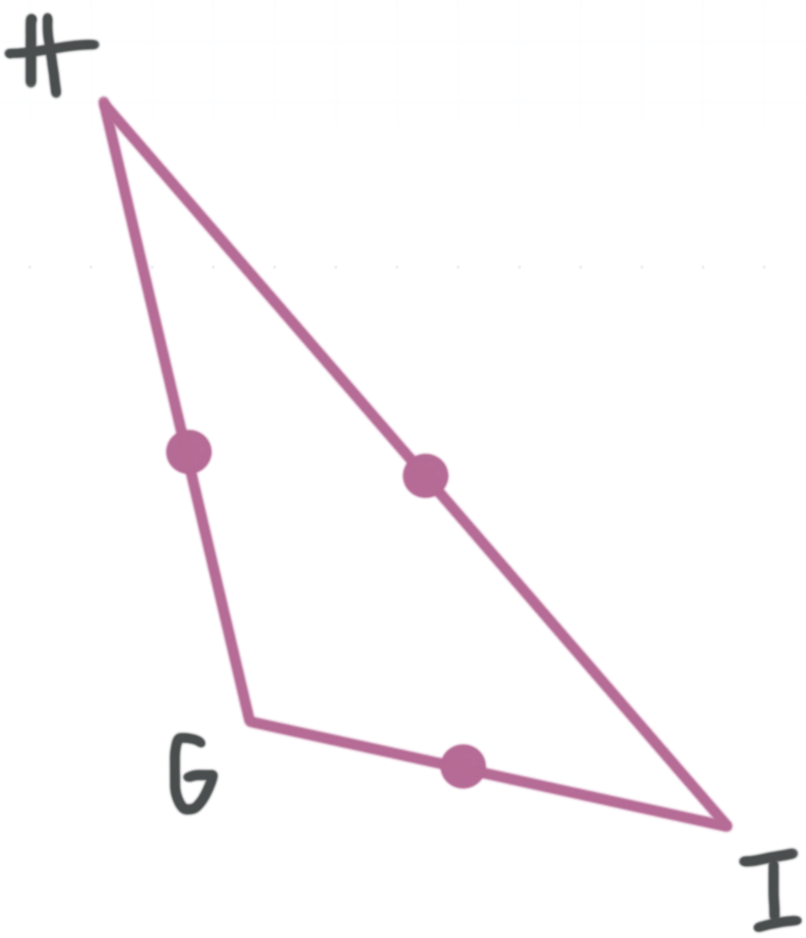
When a circle circumscribes a triangle, the interior of the triangle is inside the circle and all the vertices of the triangle lie on the circle.



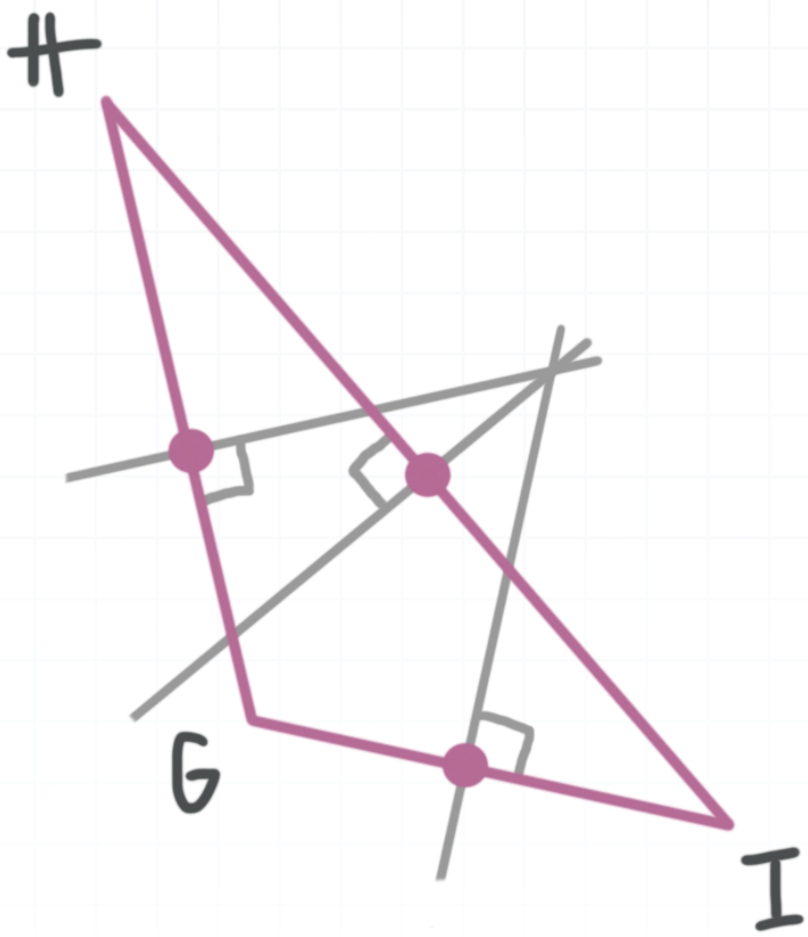
You use the perpendicular bisectors of the sides of the triangle to find the the center of the circle that will circumscribe the triangle. For example, given $\triangle GHI$,



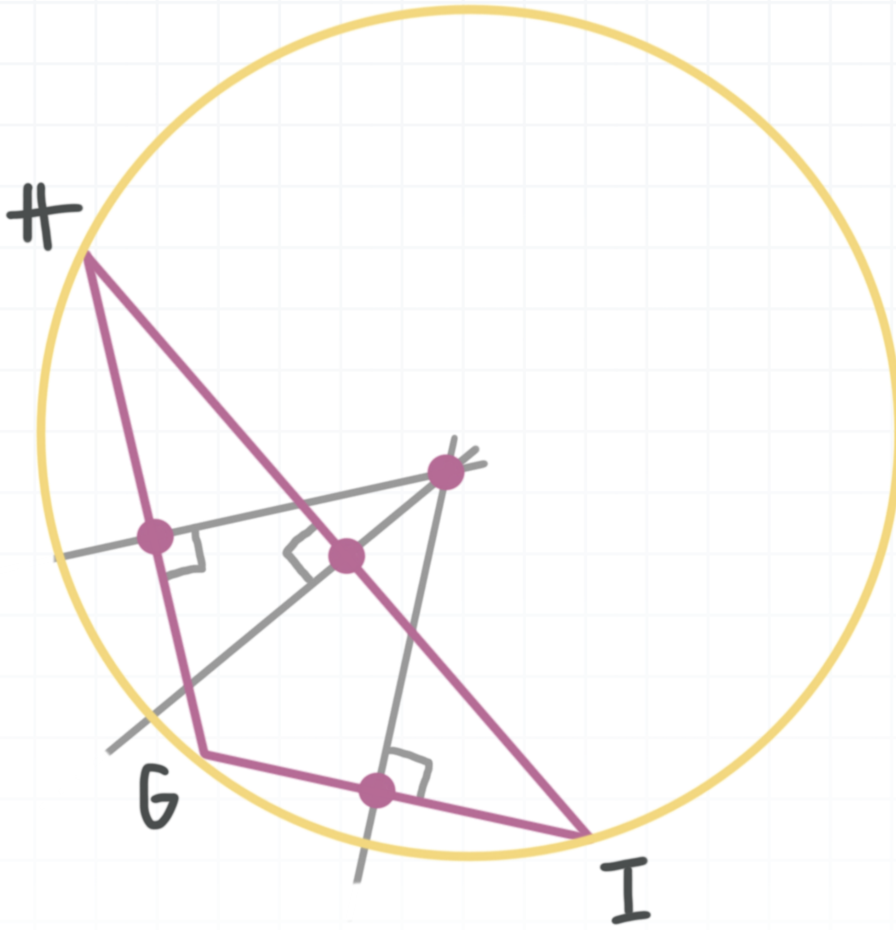
you first find the midpoint of each side.



After that, you draw the perpendicular bisector of each side of the triangle.



The point where the perpendicular bisectors intersect is the center of the circle.



The center of the circumscribed circle of a triangle is the **circumcenter** of the triangle.

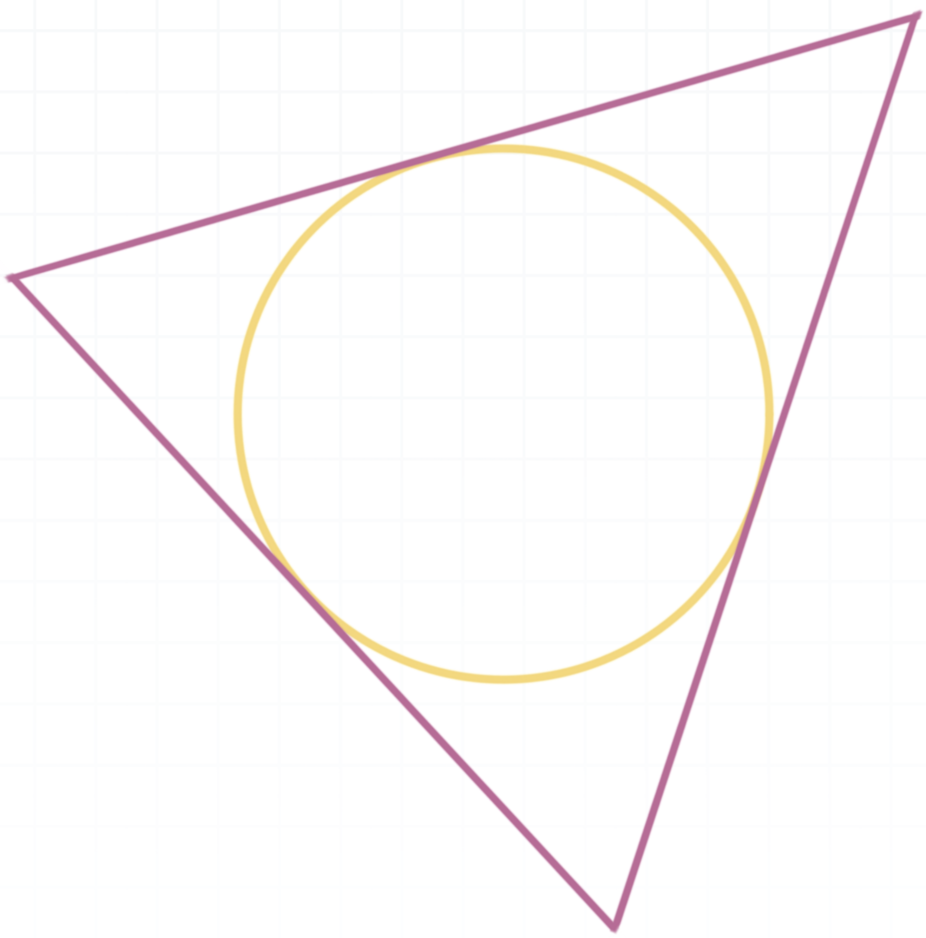
- For an acute triangle (one in which all the interior angles are acute, that is, they all have measure less than 90°), the circumcenter is inside the triangle.

- For a right triangle, the circumcenter is on the side opposite the right angle.

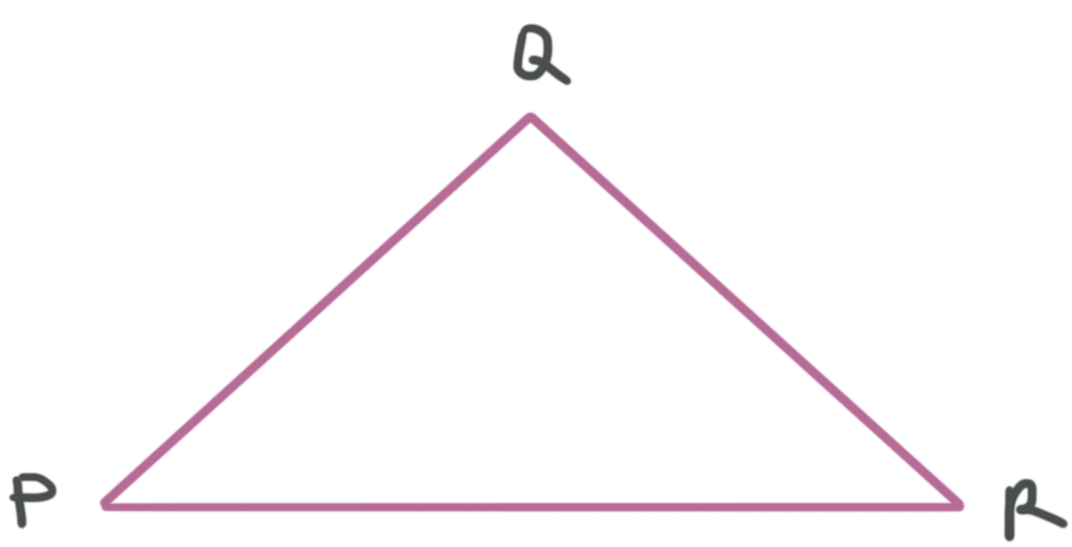
- For an obtuse triangle (one in which there's an interior angle with measure greater than 90°), the circumcenter is outside the triangle.

Inscribed circles

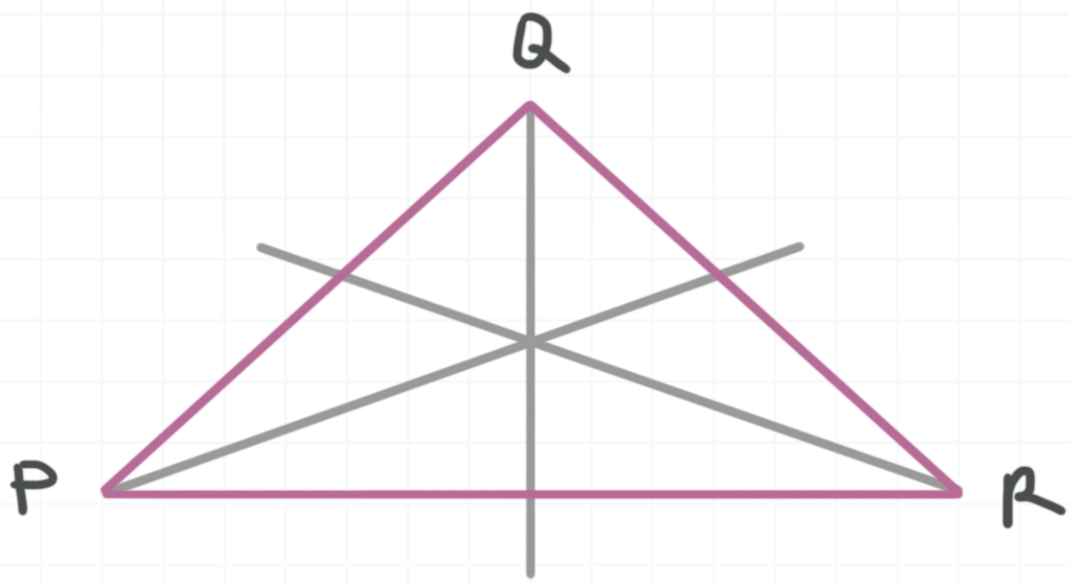
When a circle is inscribed in a triangle, the interior of the circle is inside the triangle, and exactly one point on each side of the triangle lies on the circle. Therefore, the sides of the triangle are tangent to the circle.



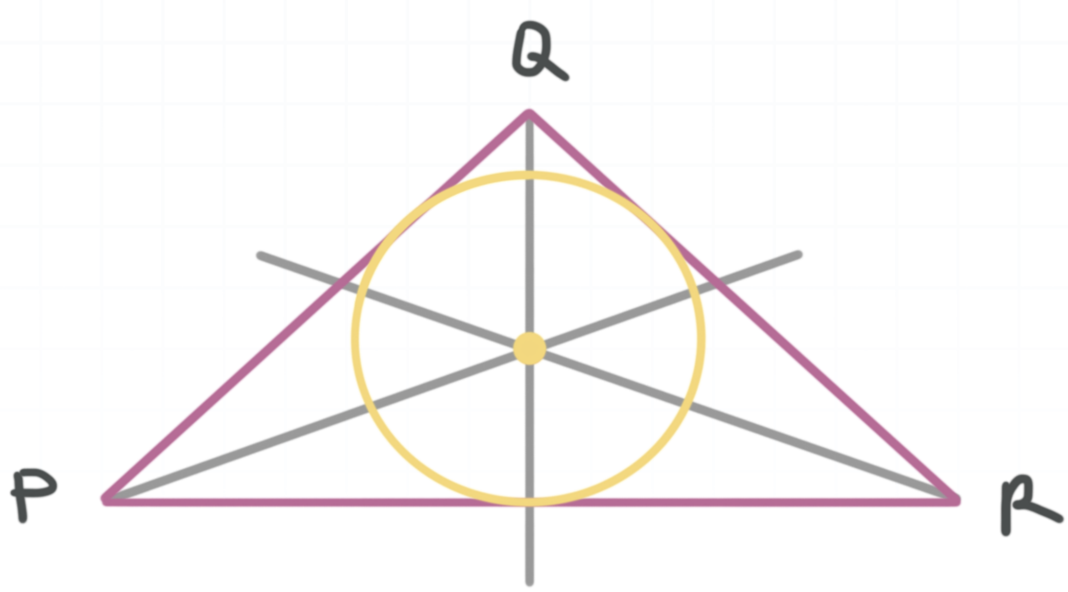
To find the center of the inscribed circle of a triangle, you draw the angle bisector of each interior angle of the triangle. For example, given $\triangle PQR$,



draw in the angle bisectors.



The intersection of the angle bisectors is the center of the inscribed circle.



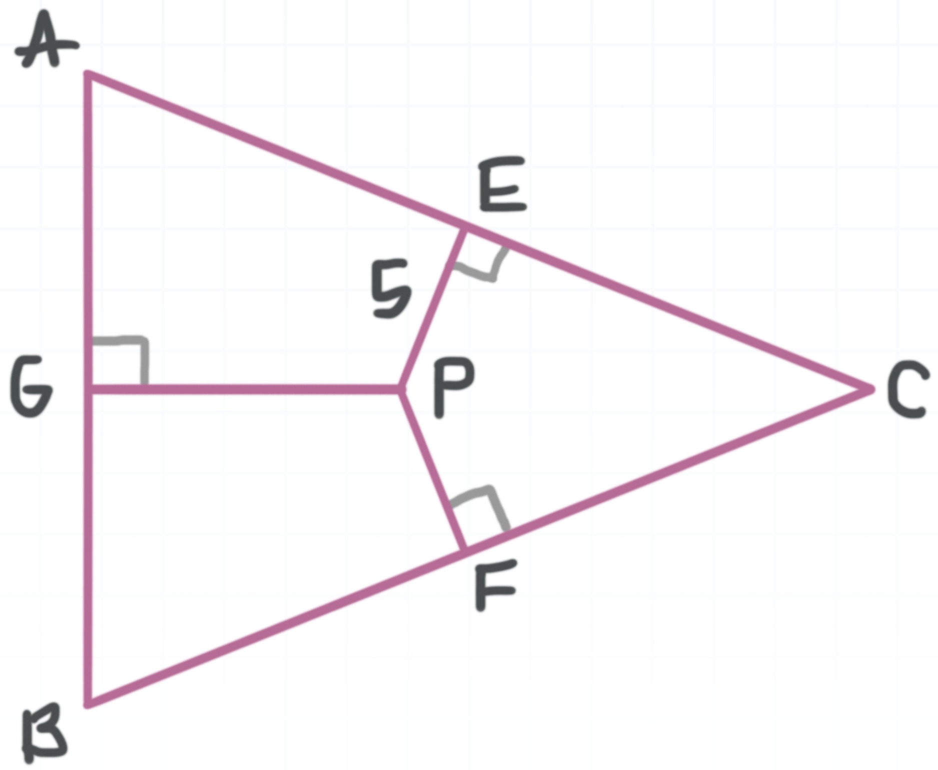
Remember that each side of the triangle is tangent to the circle, so if you draw a radius from the center of the circle to the point where the circle intersects a side of the triangle, that radius will form a right angle with that side of the triangle.

The center of the inscribed circle of a triangle is the **incenter** of the triangle. The incenter will always be inside the triangle.

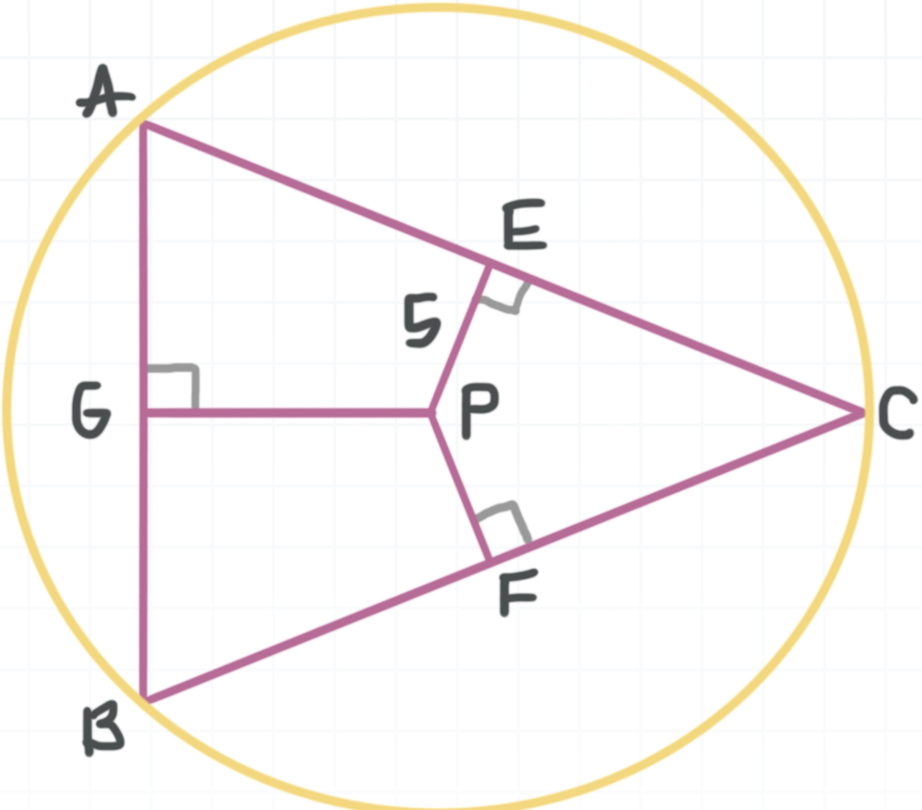
Let's use what we know about these constructions to solve a few problems.

Example

$\overline{GP}$, $\overline{EP}$, and $\overline{FP}$ are the perpendicular bisectors of sides $\overline{AB}$, $\overline{AC}$, and $\overline{BC}$, respectively, of $\triangle ABC$, and $\overline{AC} = 24$ units. What is the radius of the circumscribed circle of $\triangle ABC$?

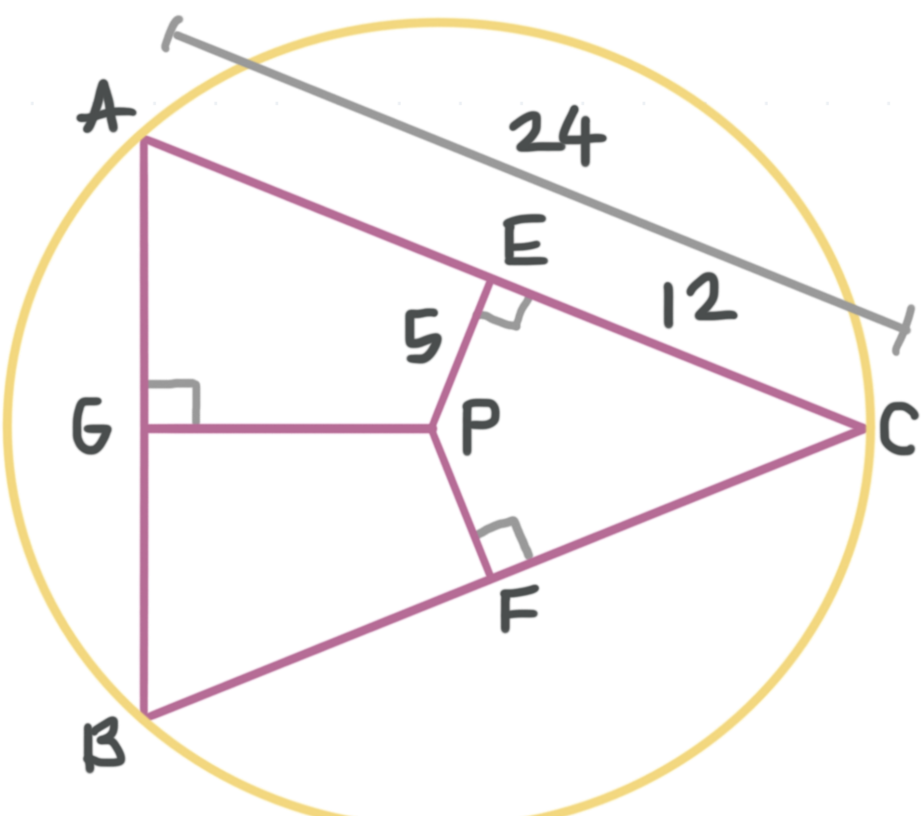


Point P is the circumcenter of $\triangle ABC$, because it's the point where the perpendicular bisectors of the sides of the triangle intersect. Since all the vertices of a triangle lie on its circumscribed circle, we can draw the circle.

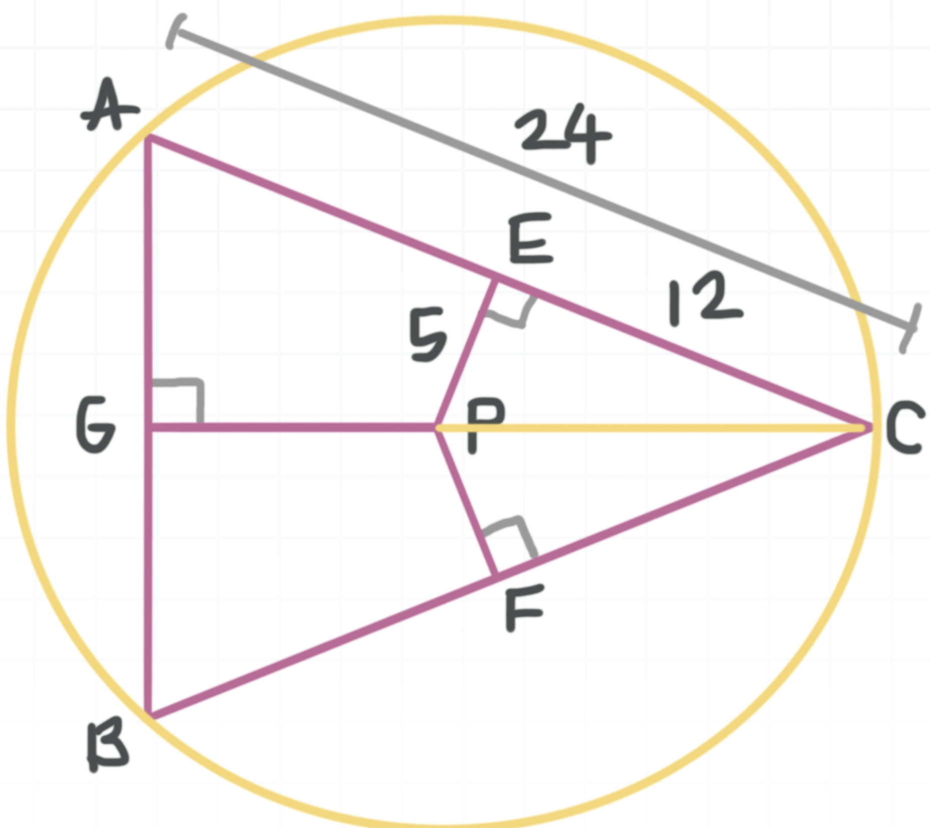


We also know that $\overline{AC} = 24$, and since $\overline{EP}$ is a perpendicular bisector of $\overline{AC}$, point E is the midpoint. Therefore,

$$\overline{EC} = \frac{1}{2}(\overline{AC}) = \frac{1}{2}(24) = 12$$



Now we can draw the radius from point P, the center of the circle, to point C, which is a vertex of the triangle, so it lies on the circle.



We can use right triangle PEC and the Pythagorean theorem to solve for the length of radius $\overline{PC}$.

$$5^2 + 12^2 = (\overline{PC})^2$$

$$25 + 144 = (\overline{PC})^2$$

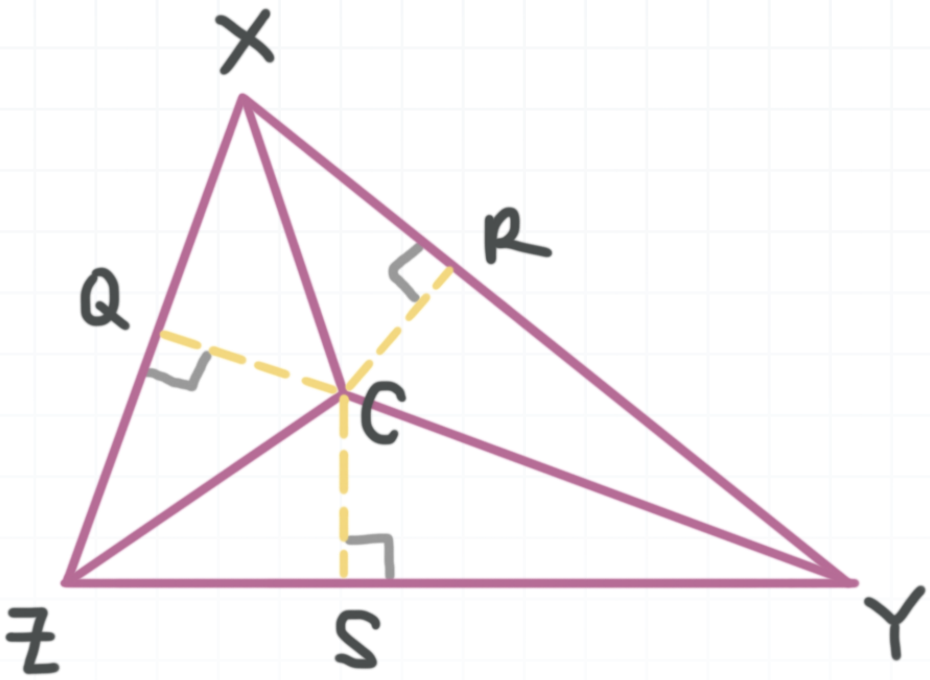
$$169 = (\overline{PC})^2$$

$$\overline{PC} = 13$$

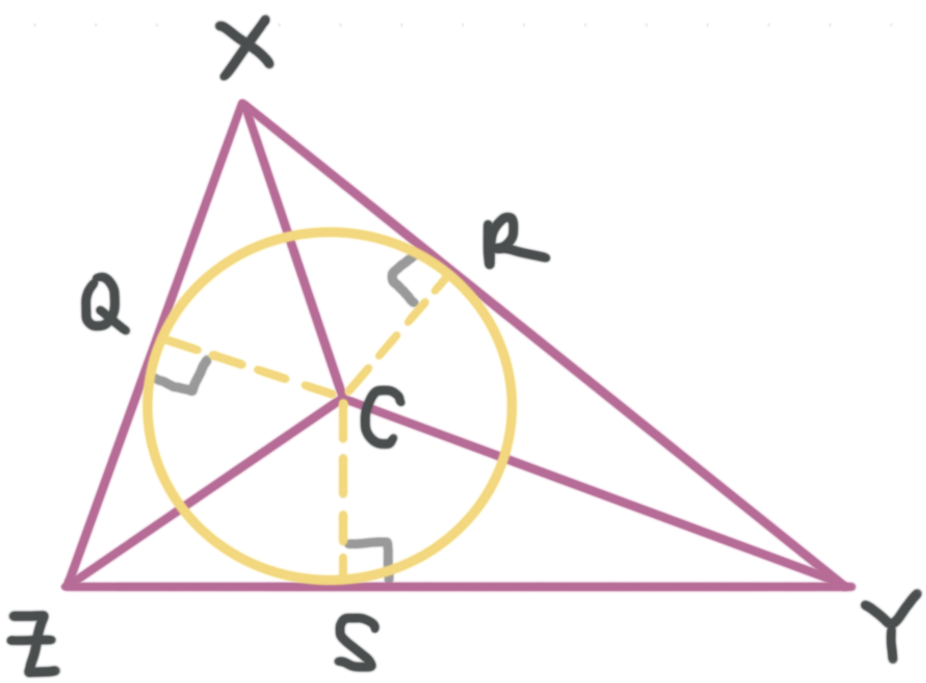
Let's try a different problem.

Example

If $\overline{CQ} = 2x - 7$ and $\overline{CR} = x + 5$, what is the length of $\overline{CS}$, given that $\overline{XC}$, $\overline{YC}$, and $\overline{ZC}$ are angle bisectors of the interior angles of $\triangle XYZ$?



Because $\overline{XC}$, $\overline{YC}$, and $\overline{ZC}$ are angle bisectors of the interior angles of $\triangle XYZ$, C is the incenter of the triangle. The circle with center C will be tangent to each side of the triangle at the point of intersection.



$\overline{CQ}$, $\overline{CR}$, and $\overline{CS}$ are the radii drawn from the incenter, C, to the points of intersection of the circle with the sides of $\triangle XYZ$. Since they're all radii of the same circle, they're all of equal length.

$$\overline{CQ} = \overline{CR} = \overline{CS}$$

We need to find the radius of the circle. We know that $\overline{CQ} = 2x - 7$ and $\overline{CR} = x + 5$, so

$$2x - 7 = x + 5$$

$$x = 12$$

Therefore,

$$\overline{CS} = \overline{CR} = x + 5$$

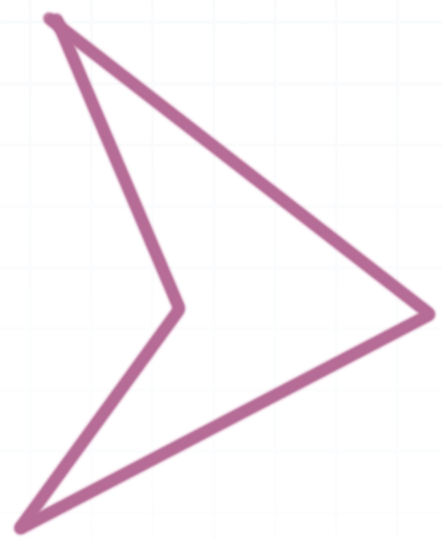
$$\overline{CS} = \overline{CR} = 12 + 5$$

$$\overline{CS} = \overline{CR} = 17$$

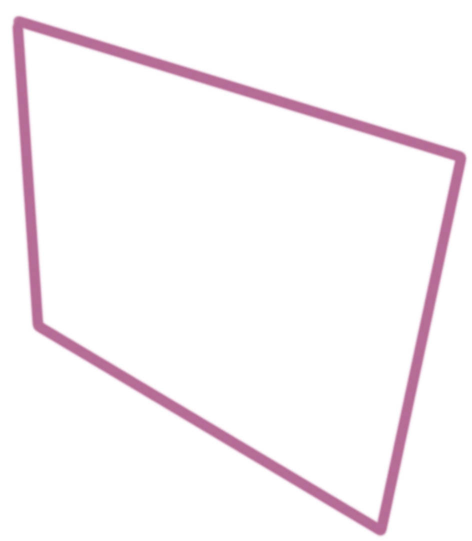
Measures of quadrilaterals

A quadrilateral is any closed four-sided figure. There are two types of quadrilaterals: concave and convex.

A concave quadrilateral has an interior angle with measure greater than 180°



A convex quadrilateral has interior angles that are all of measure less than 180°

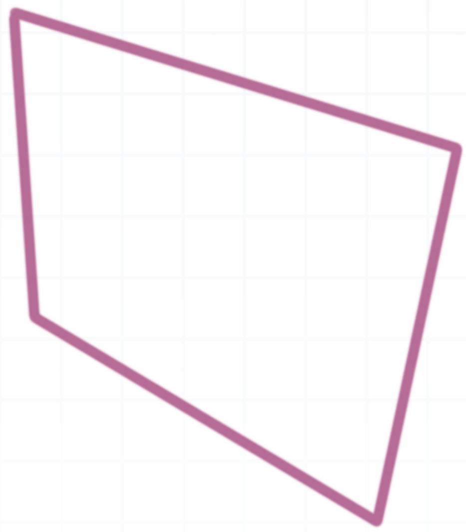


All quadrilaterals have four sides (edges), four corners (vertices), and four interior angles whose measures add up to $(4-2)180^\circ = 360^\circ$.

Here are some special types of convex quadrilaterals and their properties:

Trapezium

No pair of parallel sides and no pair of congruent sides (no pair of sides that have the same length)

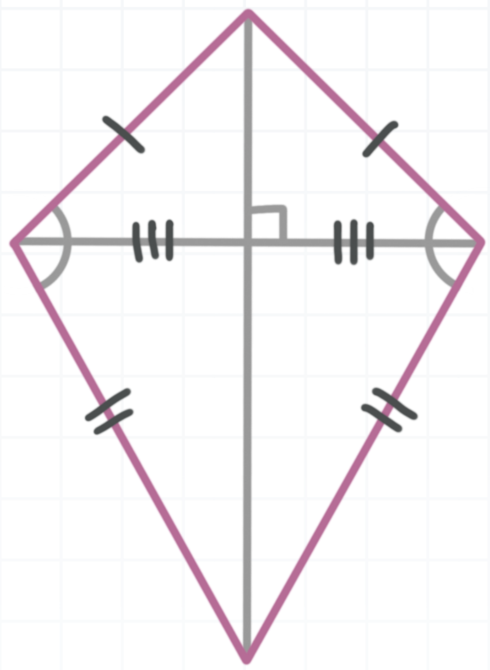


Kite

Has two pairs of adjacent congruent sides

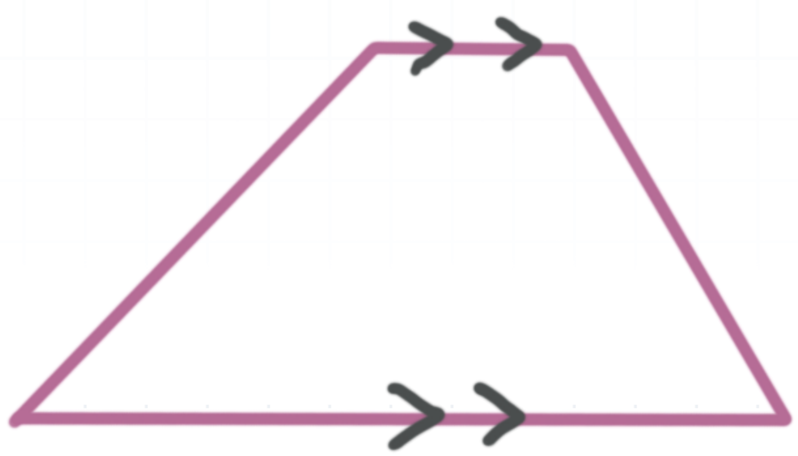
Has a pair of opposite congruent angles

Diagonals (line segments whose endpoints are a pair of opposite corners/vertices) cross to form right angles, and one of the diagonals bisects the other (cuts it in half)



Trapezoid

Has exactly one pair of opposite parallel sides



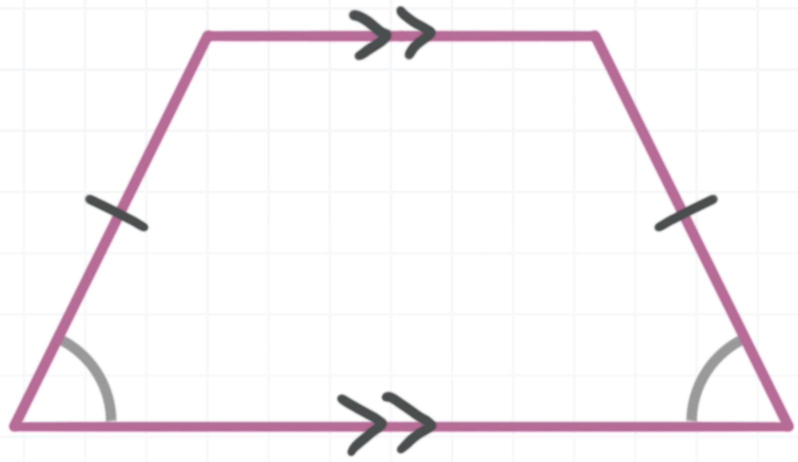
Isosceles trapezoid

Has exactly one pair of opposite parallel sides

Opposite non-parallel sides are congruent

Base angles are congruent

Diagonals are congruent



Parallelogram

Two pairs of opposite parallel sides

Opposite sides are congruent

Opposite angles are congruent

$m\angle 1 = m\angle 3$

$m\angle 2 = m\angle 4$

Consecutive angles are supplementary

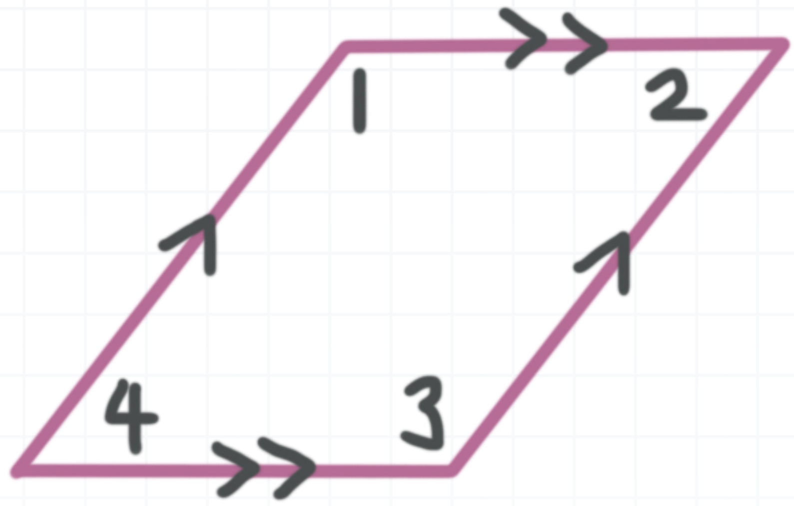
$m\angle 1 + m\angle 2 = 180^\circ$

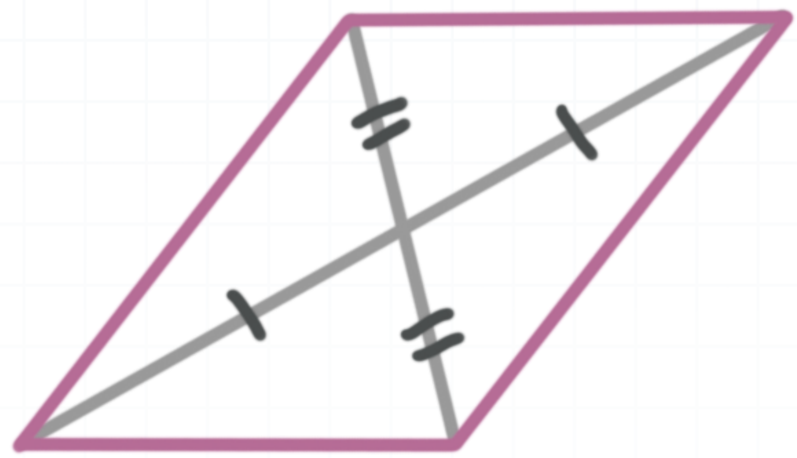
$m\angle 2 + m\angle 3 = 180^\circ$

$m\angle 3 + m\angle 4 = 180^\circ$

$m\angle 4 + m\angle 1 = 180^\circ$

Diagonals bisect each other (cut each other in half)





Rectangle

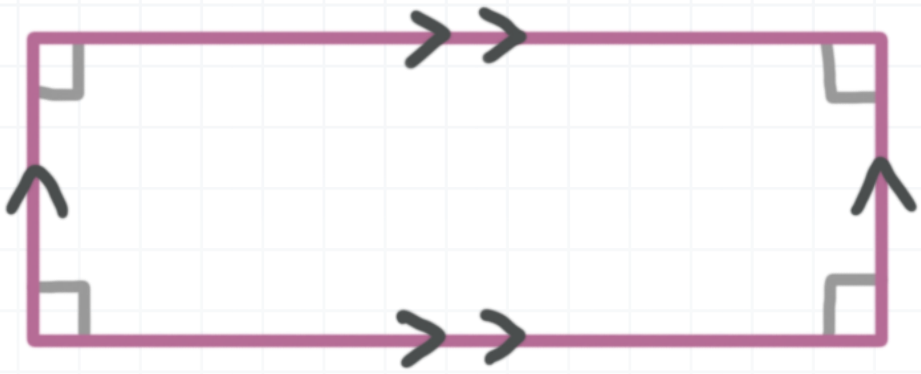
Two pairs of opposite parallel sides

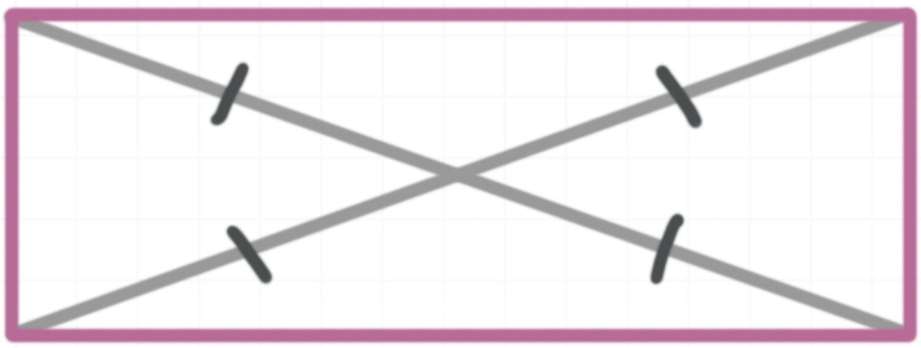
Opposite sides are congruent

All angles are right angles (90°)

Diagonals bisect each other (cut each other in half)

Diagonals are congruent





Rhombus

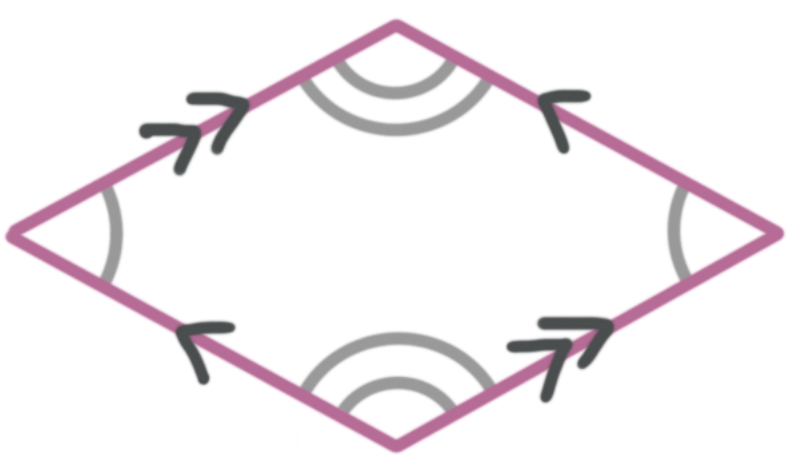
Two pairs of opposite parallel sides

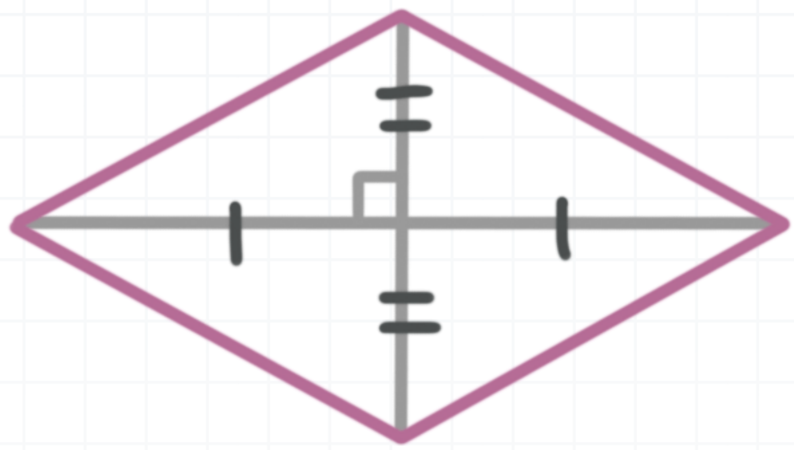
All sides are congruent

Opposite angles are congruent

Consecutive angles are supplementary

Diagonals are perpendicular bisectors of each other (cut each other in half and form right angles)





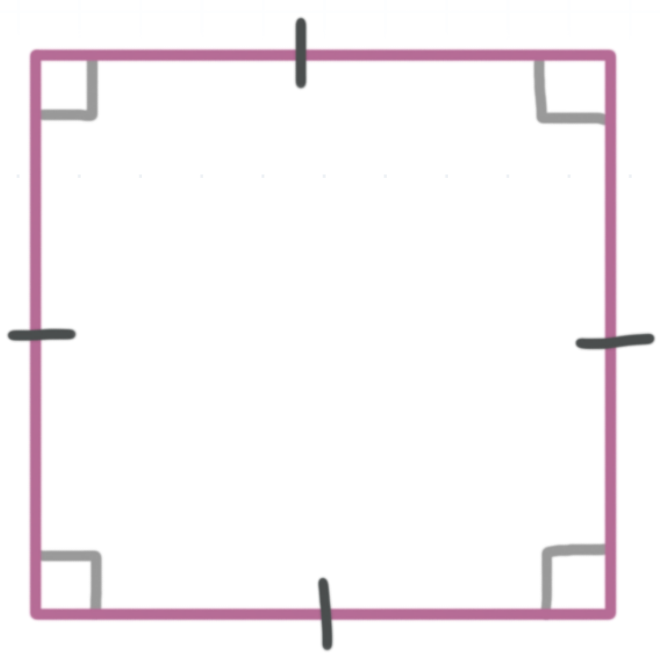
Square

Two pairs of opposite parallel sides

All angles are right angles

All sides are congruent

Diagonals are perpendicular bisectors of each other (cut each other in half and form right angles)



Notice the following:

Every rectangle is a parallelogram, but not every parallelogram is a rectangle.

Every square is a rectangle (and therefore also a parallelogram), but not every rectangle is a square.

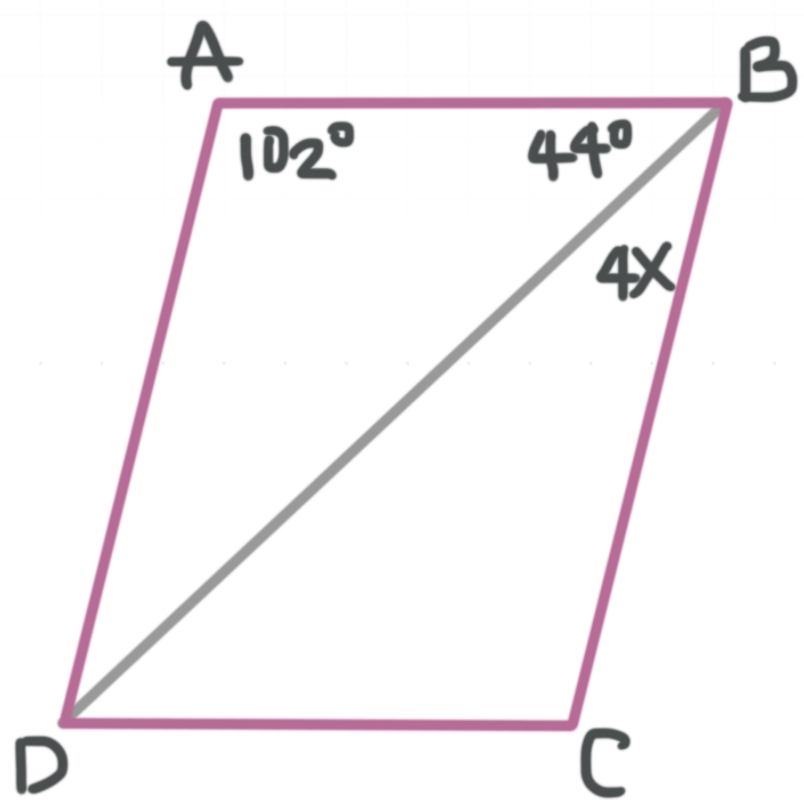
Every rhombus is a parallelogram, but not every parallelogram is a rhombus.

Every square is a rhombus, but not every rhombus is a square. A rhombus that isn't a square is in the shape of a diamond.

Let's start by working through an example.

Example

The figure below is a parallelogram. What is the value of x?



Angles DAB and ABC are consecutive angles in this parallelogram (they're next to each other, not across the figure from each other), so they're supplementary.

$$m\angle DAB + m\angle ABC = 180^\circ$$

$$102^\circ + (44^\circ + 4x) = 180^\circ$$

$$146^\circ + 4x = 180^\circ$$

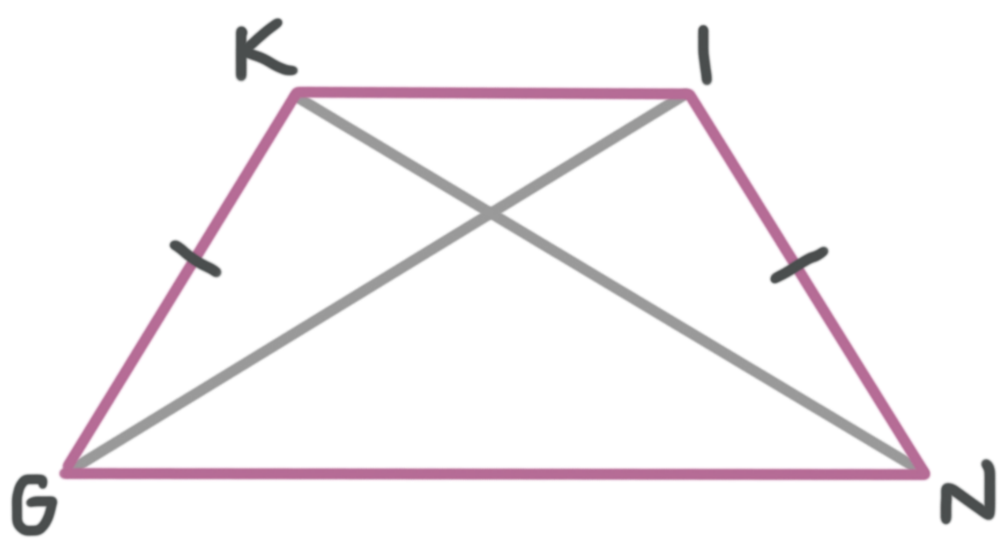
$$4x = 34^\circ$$

$$x = 8.5^\circ$$

Let's look at one more example.

Example

The figure below is a trapezoid. What is the length of $\overline{KN}$ if $\overline{KN} = 5x + 2$ and $\overline{IG} = 4x + 20$?



The side lengths of $\overline{KG}$ and $\overline{IN}$ are marked as being congruent, which means this is an isosceles trapezoid. The diagonals of an isosceles trapezoid are congruent, which means that $\overline{KN} = \overline{IG}$. Therefore,

$$5x + 2 = 4x + 20$$

$$5x = 4x + 18$$

$$x = 18$$

So we see that

$$\overline{KN} = 5x + 2$$

$$\overline{KN} = 5(18) + 2$$

$$\overline{KN} = 92$$

Measures of parallelograms

A **parallelogram** is a quadrilateral that has opposite sides that are parallel. The parallel sides let you know a lot about a parallelogram. Here are the special properties of parallelograms:

Parallelogram

Two pairs of opposite parallel sides

Opposite sides are congruent

Opposite angles are congruent

$$m\angle 1 = m\angle 3$$

$$m\angle 2 = m\angle 4$$

Consecutive angles are supplementary

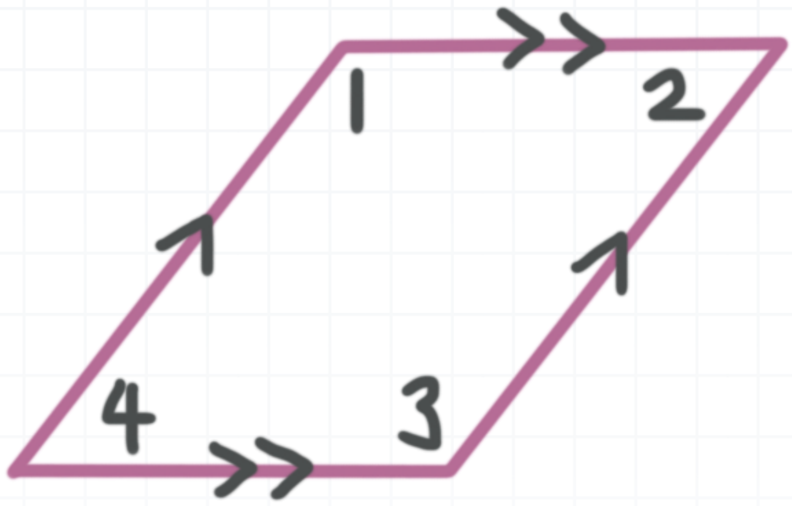
$$m\angle 1 + m\angle 2 = 180^\circ$$

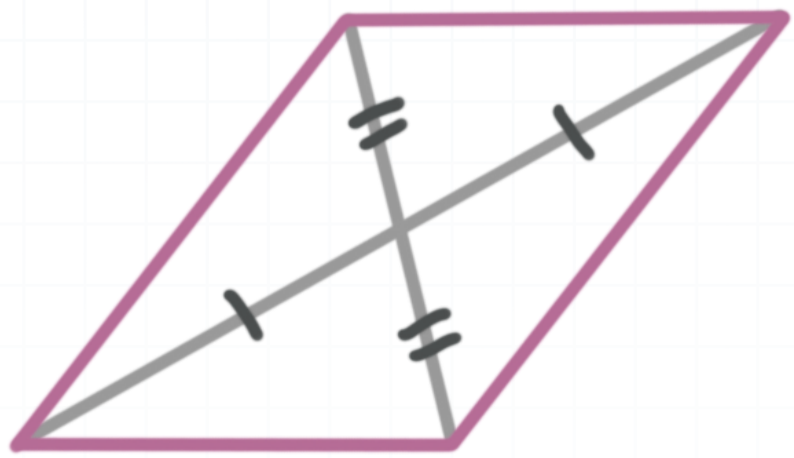
$$m\angle 2 + m\angle 3 = 180^\circ$$

$$m\angle 3 + m\angle 4 = 180^\circ$$

$$m\angle 4 + m\angle 1 = 180^\circ$$

Diagonals bisect each other (cut each other in half)

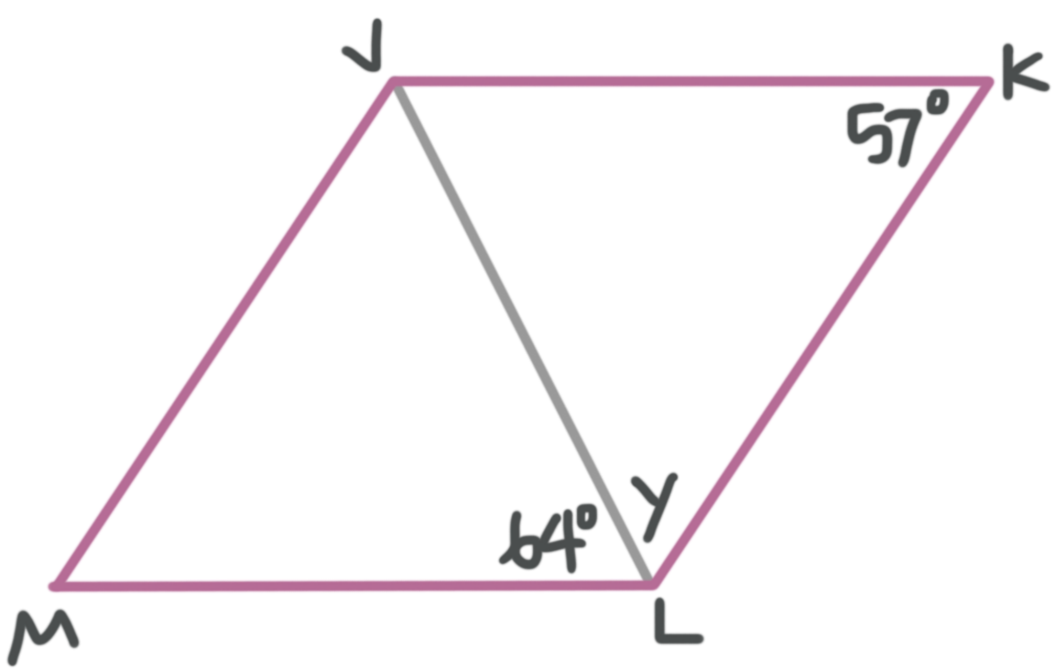




Let's look at a few examples.

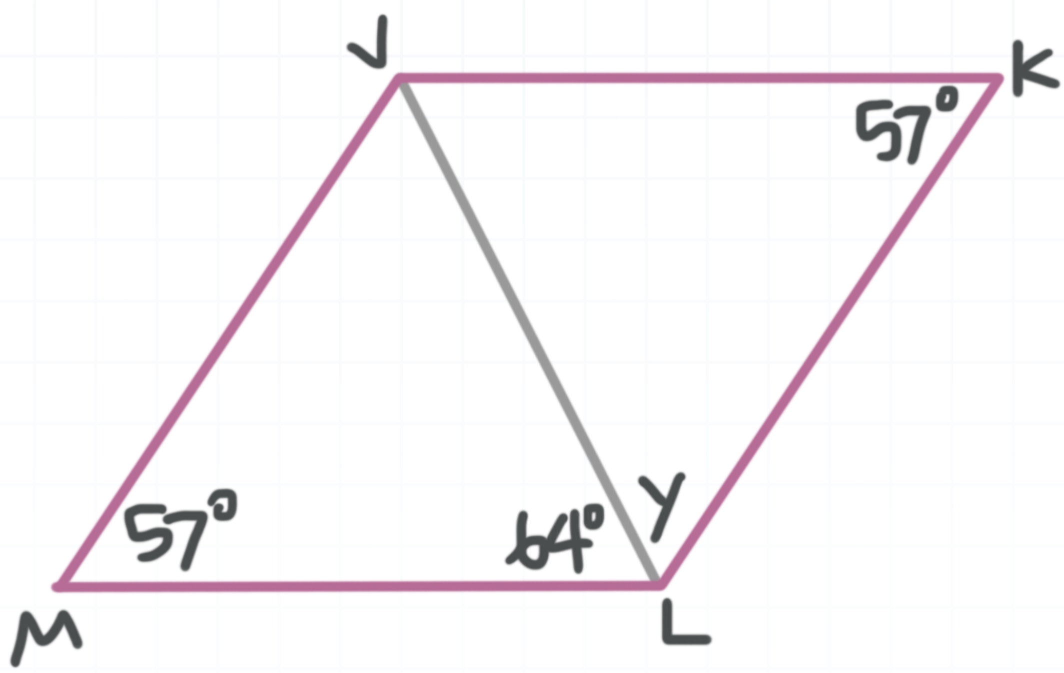
Example

The quadrilateral $JKLM$ is a parallelogram. Find the the value of y.

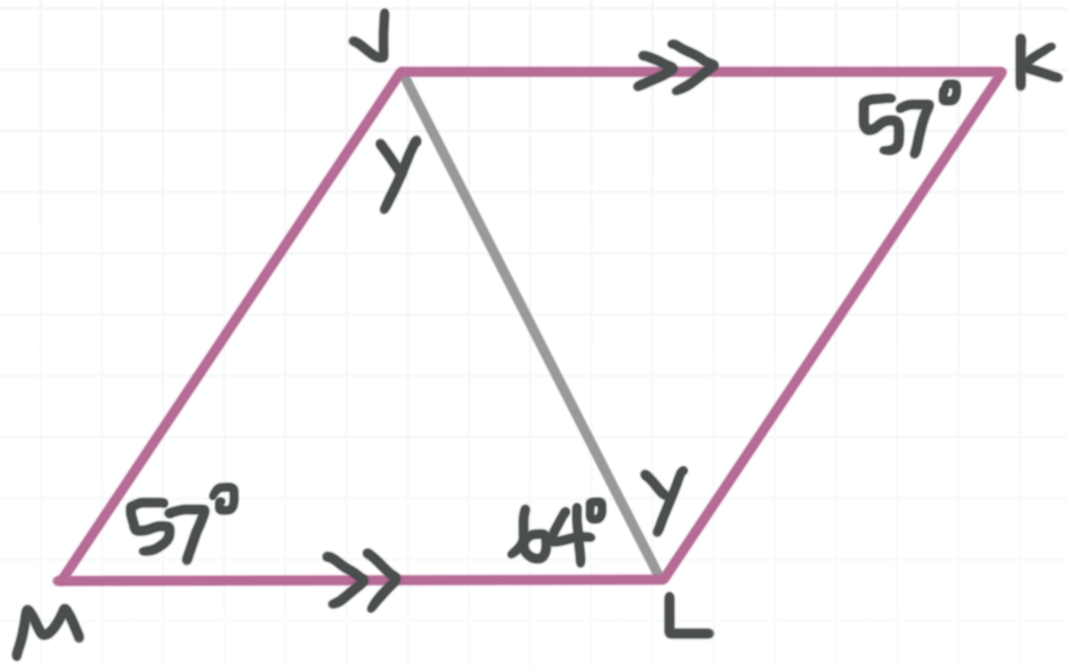


Opposite angles of a parallelogram are congruent, so

$$m\angle LMJ = m\angle JKL = 57^\circ$$



Now we can use the fact that opposite sides of a parallelogram are parallel to state that $\overline{JK} \parallel \overline{ML}$. This means that the diagonal $\overline{JL}$ is also a transversal that crosses a pair of parallel lines (the extensions of $\overline{JK}$ and $\overline{ML}$ to infinity in both direction). This means that $\angle KLJ$ and $\angle MJL$ are a pair of alternate interior angles. Alternate interior angles are congruent, so $m\angle MJL = m\angle KLJ = y$.



The measures of the three interior angles of a triangle add up to 180°, so we can set up an equation for the sum of the interior angles of $\triangle JML$ and solve for y.

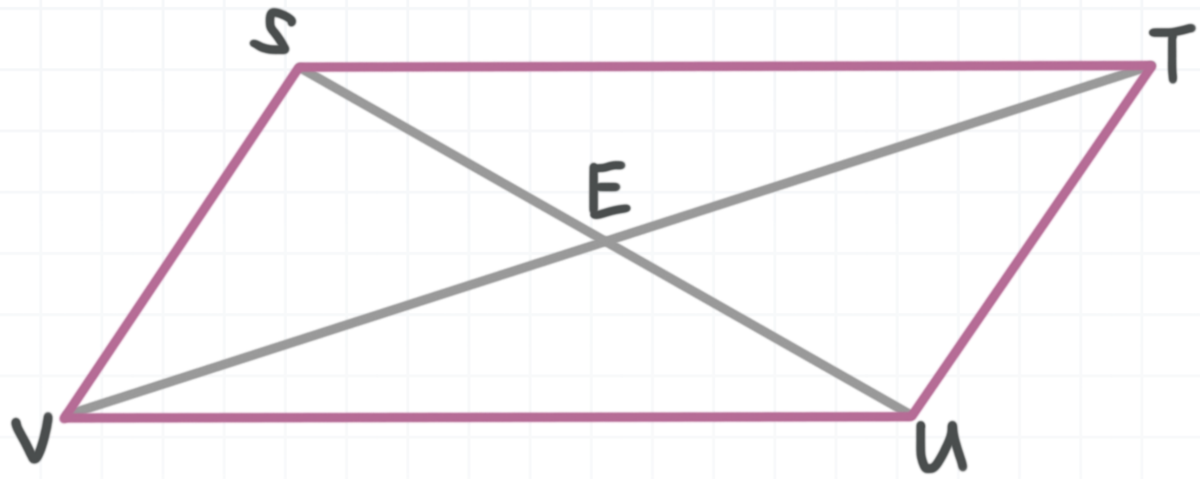
$$y + 57^\circ + 64^\circ = 180^\circ$$

$$y = 59^\circ$$

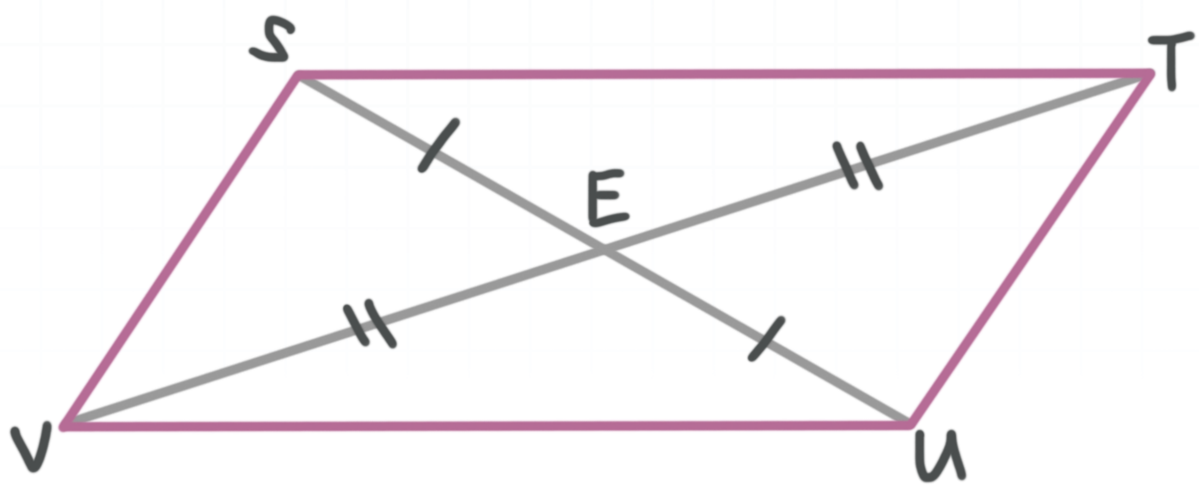
Let's do an example that involves the diagonals of a parallelogram.

Example

The quadrilateral $STUV$ in the figure below is a parallelogram. If $\overline{VT} = 4n + 34$ and $\overline{VE} = 7n - 3$, what is the length of $\overline{ET}$?



We know that the diagonals of a parallelogram bisect each other. Let's add this information into the diagram.



Now we can see the relationships we need. Because the diagonals bisect each other, $\overline{VE} = \overline{ET}$ and the length of $\overline{VE}$ is half that of $\overline{VT}$. We can use what we know to find the length of $\overline{VE}$, and then we'll know the length of $\overline{ET}$ as well.

$$\overline{VE} = \frac{1}{2}(\overline{VT})$$

$$7n - 3 = \frac{1}{2}(4n + 34)$$

$$7n - 3 = 2n + 17$$

$$5n = 20$$

$$n = 4$$

Therefore,

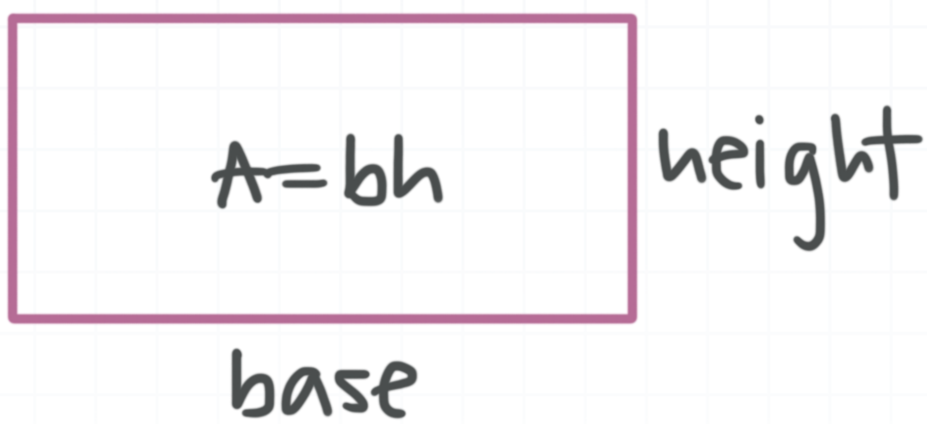
$$\overline{ET} = \overline{VE} = 7n - 3$$

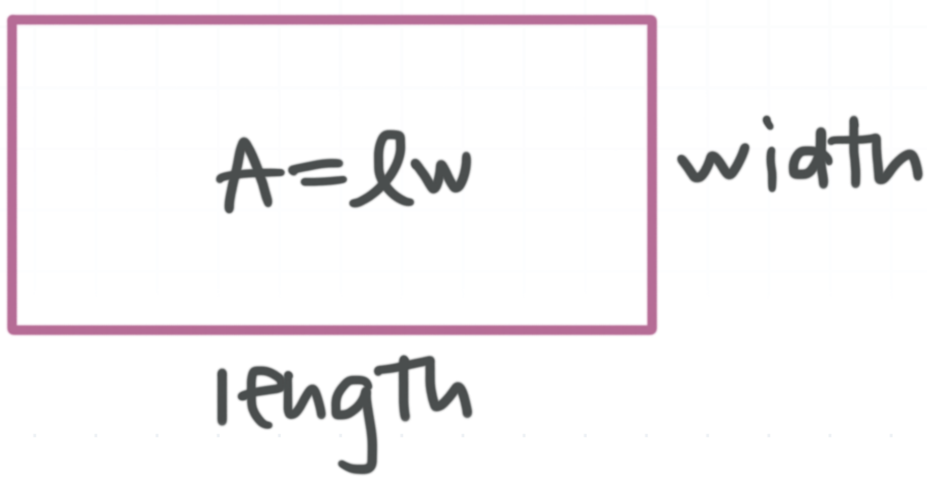
$$\overline{ET} = \overline{VE} = 7(4) - 3$$

$$\overline{ET} = \overline{VE} = 25$$

Area of a rectangle

The area of a rectangle is the product of its base and its height. We can also express the area of a rectangle as the product of its length and its width.





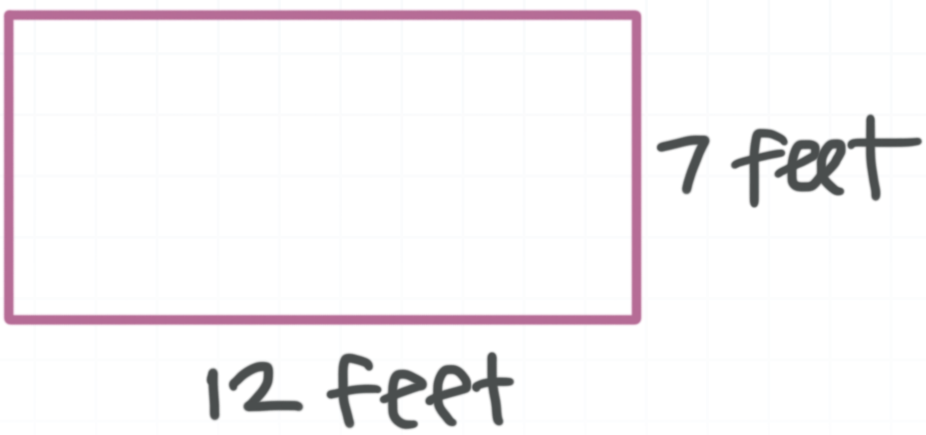
An area is always given in units of length^2 ("length squared"). In other words, if the dimensions (base and height, or length and width) of a rectangle are given in inches, the units for area will be in^2 (also called square inches); if the dimensions are given in centimeters, the units for area will be cm^2 (also called square centimeters).

When we given a dimension in units of feet, we sometimes use a single quotation mark instead of the word "feet" or the abbreviation ft. For example, we could express 6 feet as $6'$. For a dimension in units of inches, we sometimes use a double quotation mark instead of the word inches or the abbreviation in. For example, we could express 37 inches as $37''$.

Let's start by working through an example.

Example

Find the area of the rectangle.



We'll use the formula $A = bh$, where A is the area, b is the base, and h is the height of the rectangle. Plugging in the dimensions of the rectangle, we get

$$A = bh$$

$$A = (12 \text{ ft})(7 \text{ ft})$$

$$A = 84 \text{ ft}^2$$

Let's do one with an additional step.

Example

A rectangular mirror measures 24 inches by 4 feet. What is the area of the mirror?

To find the area of the mirror, you must use the same units for both dimensions. You can convert inches to feet by remembering that there are 12 inches in 1 foot.

$$24 \text{ inches} \cdot \frac{1 \text{ foot}}{12 \text{ inches}} = 2 \text{ feet}$$

Now we can use the fact that the mirror is a rectangle to find its area in square feet. We'll use the formula $A = lw$, where A is the area, l is the length, and w is the width. We don't know which dimension of the mirror is horizontal and which is vertical (and that doesn't matter), so let's let $l = 2$ ft and $w = 4$ ft.

$$A = lw$$

$$A = (2 \text{ ft})(4 \text{ ft})$$

$$A = 8 \text{ ft}^2$$

We could have also found the area in square inches by converting 4 feet to inches.

$$4 \text{ feet} \cdot \frac{12 \text{ inches}}{1 \text{ foot}} = 48 \text{ inches}$$

Then the area in square inches would be

$$A = lw$$

$$A = (24 \text{ in})(48 \text{ in})$$

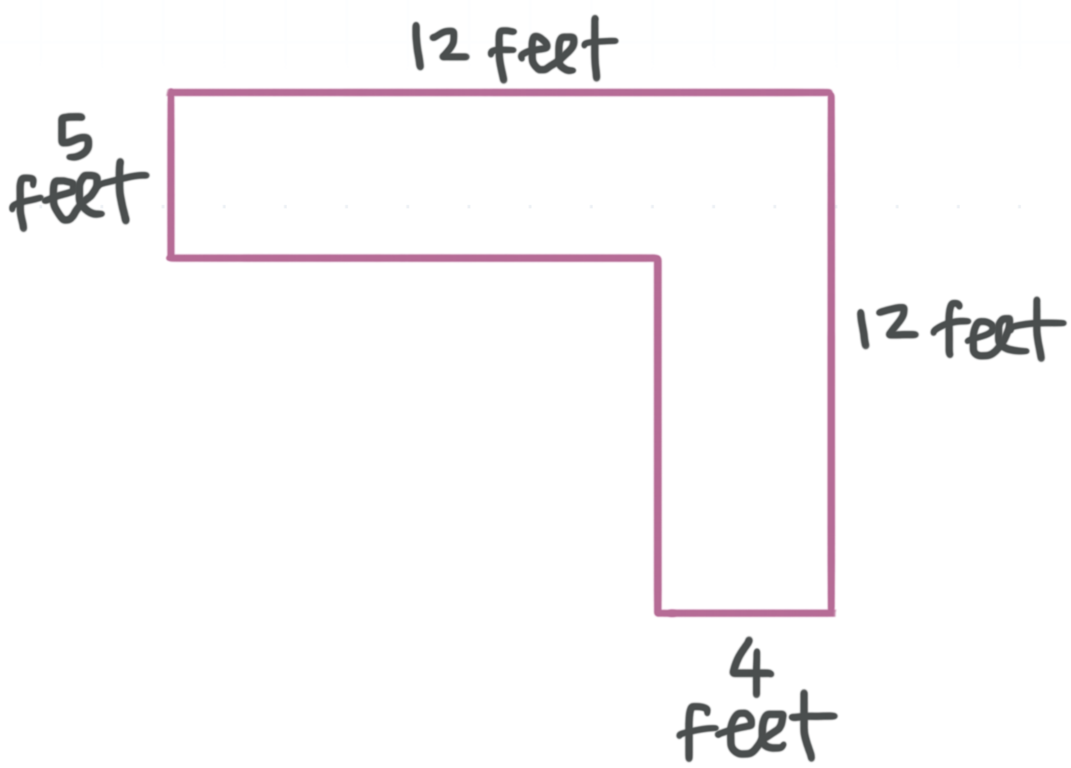
$$A = 1{,}152 \text{ in}^2$$

Area of a rectangle using sums and differences

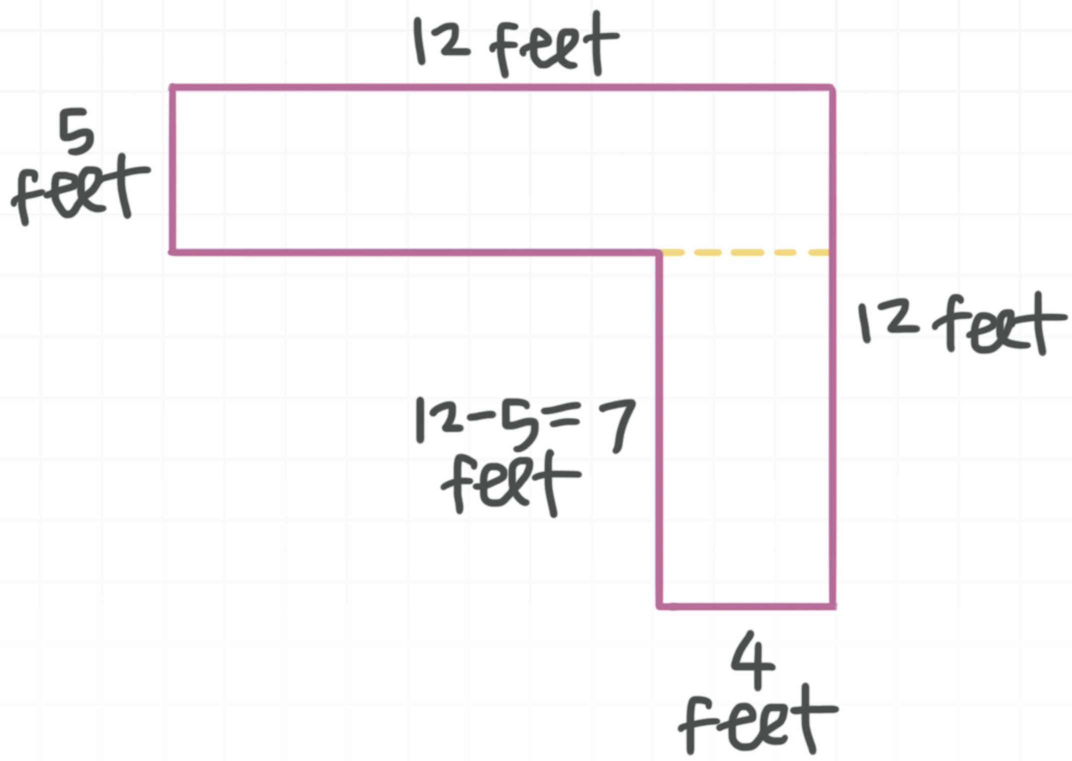
In this lesson we'll look at composite figures made from rectangles and how to find their areas. A **composite figure** is made by combining different shapes. We'll find the area of a composite figure by dividing the composite shape into shapes whose areas we already know how to find.

Area using a sum

Let's look at the composite figure below, which is made of rectangles. We can divide the shape into two rectangles, use the area formula for a rectangle twice to find their individual areas, and then add their areas to get the total area of the figure.



A shape can be divided in more than one way, but no matter how we divide the shape, the value of the area will always be the same. We can divide this figure into two rectangles with a horizontal line.



The height of the entire figure is 12, and the height of the upper rectangle is 5, so we can find the height of the lower rectangle by subtraction: $12-5=7$. Now we know the dimensions of both of the smaller rectangles.

The dimensions of the upper rectangle are 12 and 5, so its area is

$$A=bh$$

$$A=(12 \text{ ft})(5 \text{ ft})$$

$$A=60 \text{ ft}^2$$

The dimensions of the lower rectangle are 4 and 7, so its area is

$$A=bh$$

$$A=(4 \text{ ft})(7 \text{ ft})$$

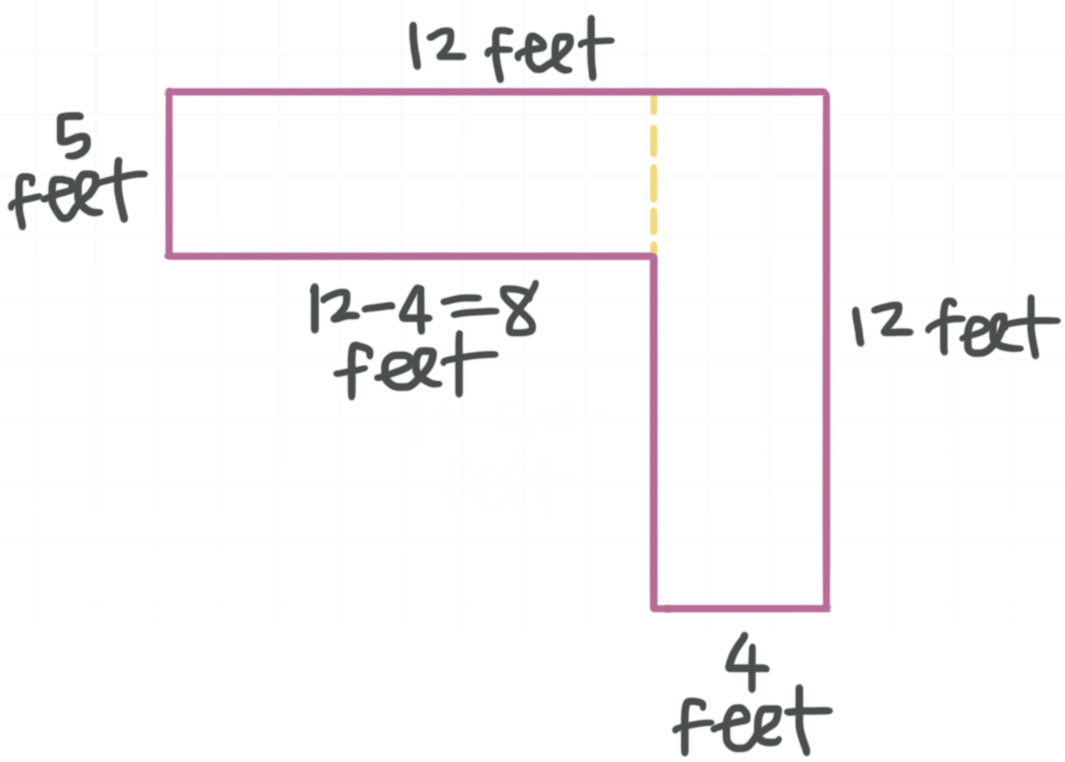
$$A = 28 \text{ ft}^2$$

So the total area of the figure is

$$A = 60 + 28$$

$$A = 88 \text{ ft}^2$$

We could also divide this figure into two rectangles with a vertical line.



The length of the entire figure is 12, and the length of the right rectangle is 4, so we can find the length of the left rectangle by subtraction: $12 - 4 = 8$. Now we know the dimensions of both rectangles.

The dimensions of the left rectangle are 8 and 5, so its area is

$$A = lw$$

$$A = (8 \text{ ft})(5 \text{ ft})$$

$$A = 40 \text{ ft}^2$$

The dimensions of the right rectangle are 4 and 12, so its area is

$$A = lw$$

$$A = (4\text{ ft})(12\text{ ft})$$

$$A = 48\text{ ft}^2$$

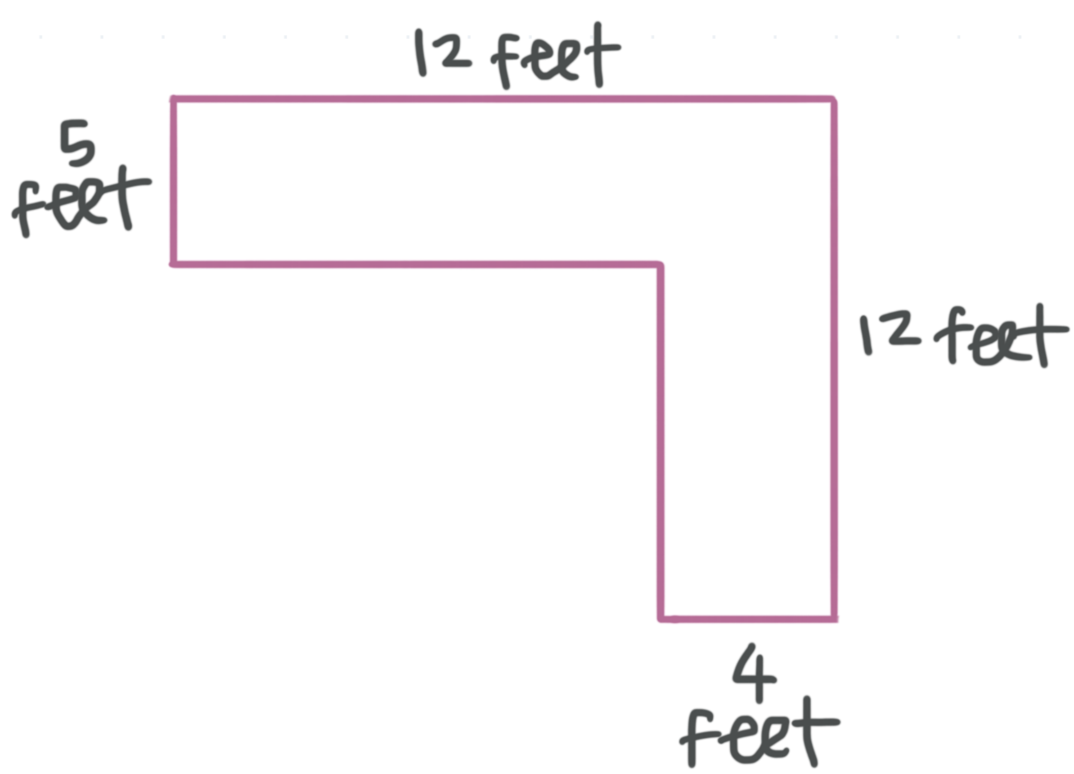
So the total area of the figure is

$$A = 40 + 48$$

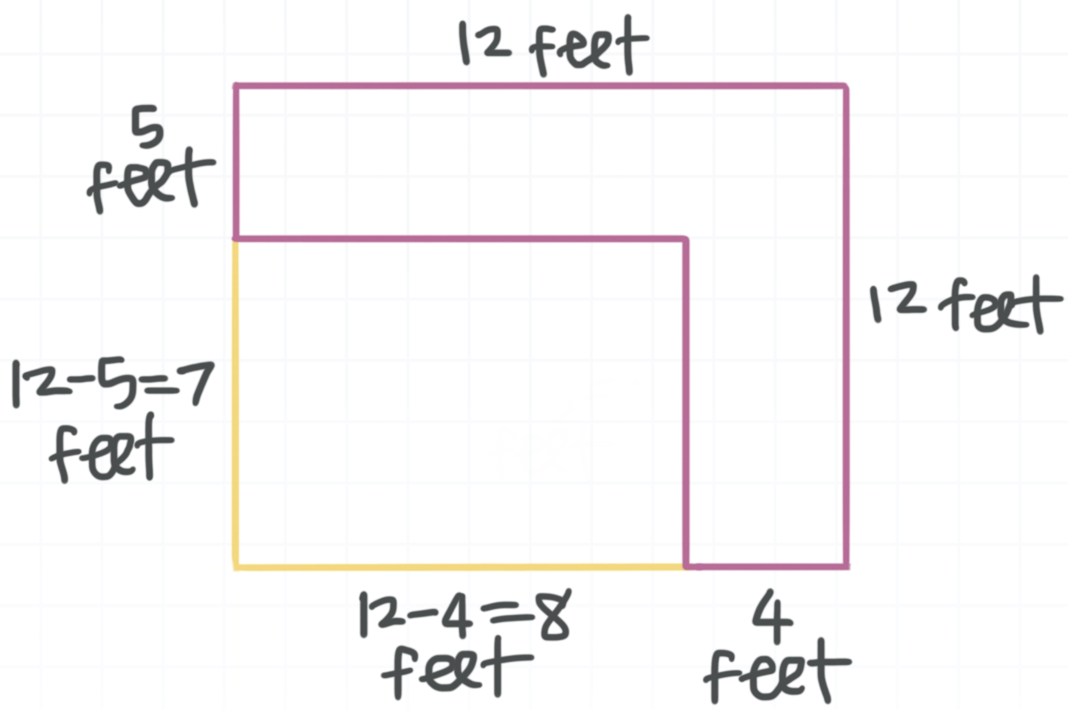
$$A = 88\text{ ft}^2$$

Area using a difference

You can also use a difference to find the area of a composite figure. Let's look at this one again.



We can form a new, large rectangle by drawing a rectangle that fills in the empty space.



The base of the new, large rectangle we formed is 12, so the base of the rectangle we drew to fill in the empty space must be $12-4=8$. The height of the new, large rectangle we formed is 12, so the height of the rectangle we drew to fill in the empty space must be $12-5=7$.

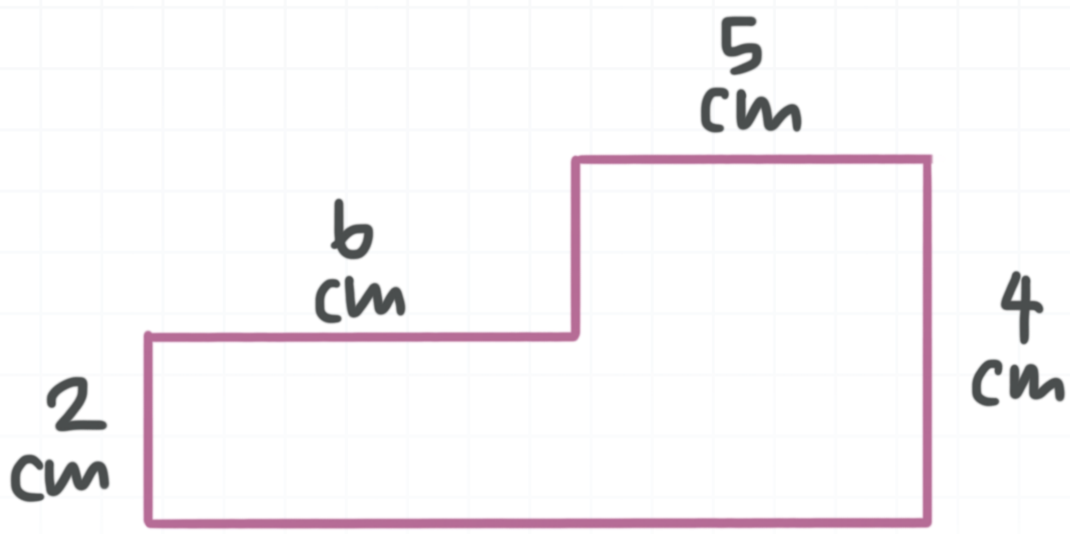
The area of the new, large rectangle we formed (which is actually a square since its base is equal to its height) is $12 \cdot 12 = 144$ ft^2. The area of the rectangle we drew to fill in the empty space is $7 \cdot 8 = 56$ ft^2. So to find the area of the original figure, we can subtract the area of the rectangle we drew to fill in the empty space from the area of the new, large rectangle we formed:

$$A = 144 - 56 = 88 \text{ ft}^2$$

Which is the same area we got when we used sums of areas of two rectangles instead of a difference of areas of two rectangles. Let's do some more examples.

Example

The figure is made by combining rectangles. What is the area of the figure?



The height of the lower rectangle is 2, so we can find the height of the upper rectangle by subtraction: $4-2=2$ cm. The dimensions of the upper rectangle are therefore 5 and 2, so its area is

$$A = bh$$

$$A = (5\text{ cm})(2\text{ cm})$$

$$A = 10\text{ cm}^2$$

We need to add to find the base of the lower rectangle. Its base is made of a horizontal piece of length 6 cm and a horizontal piece of length 5 cm. So the base of the lower rectangle is $6+5=11$ cm. The dimensions of the lower rectangle are therefore 11 and 2, so its area is

$$A = bh$$

$$A = (11 \text{ cm})(2 \text{ cm})$$

$$A = 22 \text{ cm}^2$$

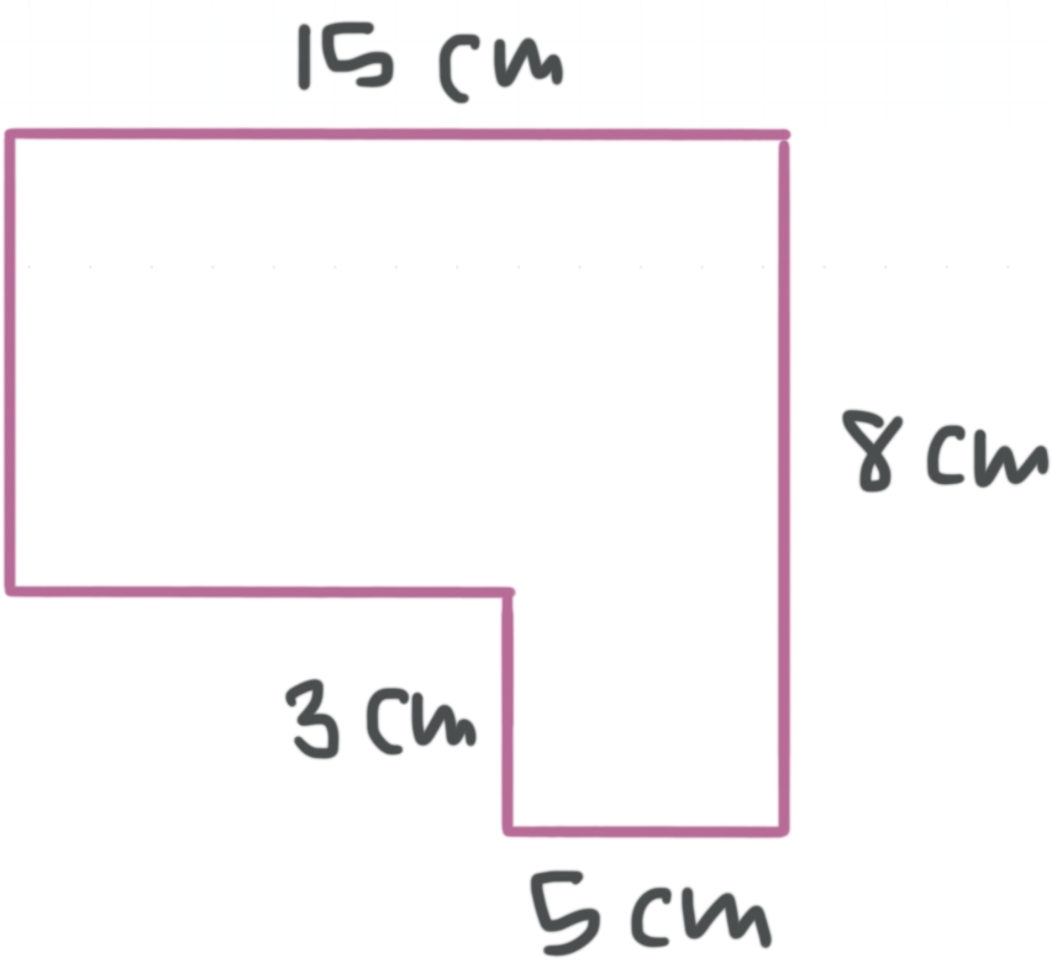
So the total area of the figure is

$$A = 10 + 22 = 32 \text{ cm}^2$$

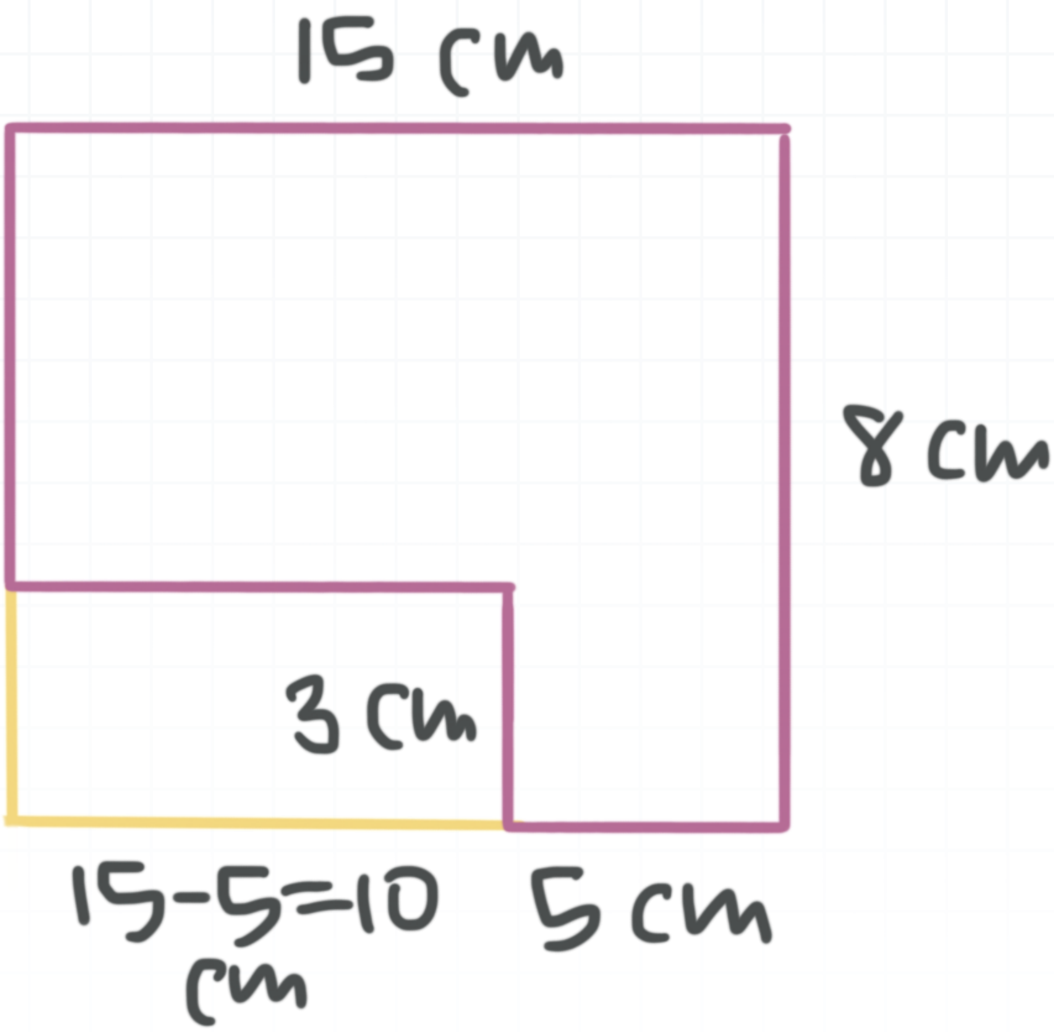
Let's do one more example.

Example

The figure is made by combining rectangles. What is the area of the figure?



Form a new, large rectangle by drawing a rectangle that fills in the empty space.



The height of the new, large rectangle we formed is 8, and its base is 15, so its area is

$$A = bh$$

$$A = (15 \text{ cm})(8 \text{ cm})$$

$$A = 120 \text{ cm}^2$$

The rectangle we drew to fill in the empty space has a height of 3, and we can find its base by subtraction: $15 - 5 = 10$ cm. The dimensions of the rectangle we drew to fill in the empty space are therefore 10 and 3, so its area is

$$A = bh$$

$$A = (10 \text{ cm})(3 \text{ cm})$$

$$A = 30 \text{ cm}^2$$

We see that the area of the original figure is

$$A = bh$$

$$A = 120 - 30$$

$$A = 90 \text{ cm}^2$$

Perimeter of a rectangle

The perimeter of a rectangle is the length of its boundary. We can find the perimeter of a rectangle by finding the sum of the lengths of its four sides. But the lengths of opposite sides of a rectangle are equal, so we'll usually see the formula for the perimeter of a rectangle written as

$$P = 2l + 2w$$

$$P = 2(l + w)$$

where P is the perimeter, l is the length, and w is the width of the rectangle.

Let's do a few examples.

Example

What is the perimeter of the rectangle?

You can find the perimeter by plugging the length and width that we've been given into the formula for the perimeter.

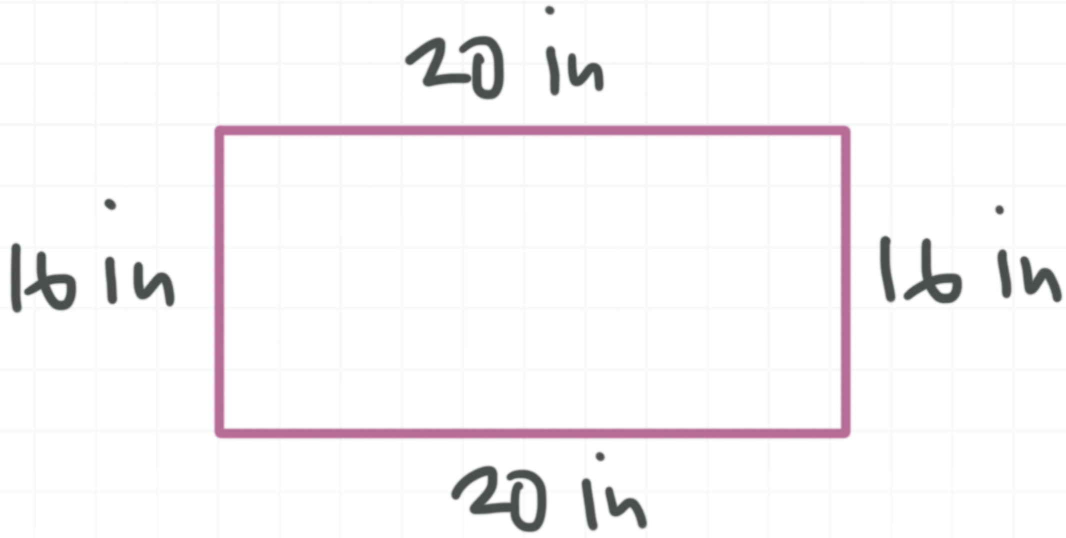
$$P = 2l + 2w$$

$$P = 2(20) + 2(16)$$

$$P = 40 + 32$$

$$P = 72 \text{ in}$$

Remember that opposite sides of a rectangle are equal in length, so you could also find the perimeter by adding the lengths of all four sides.



$$P = 20 + 16 + 20 + 16$$

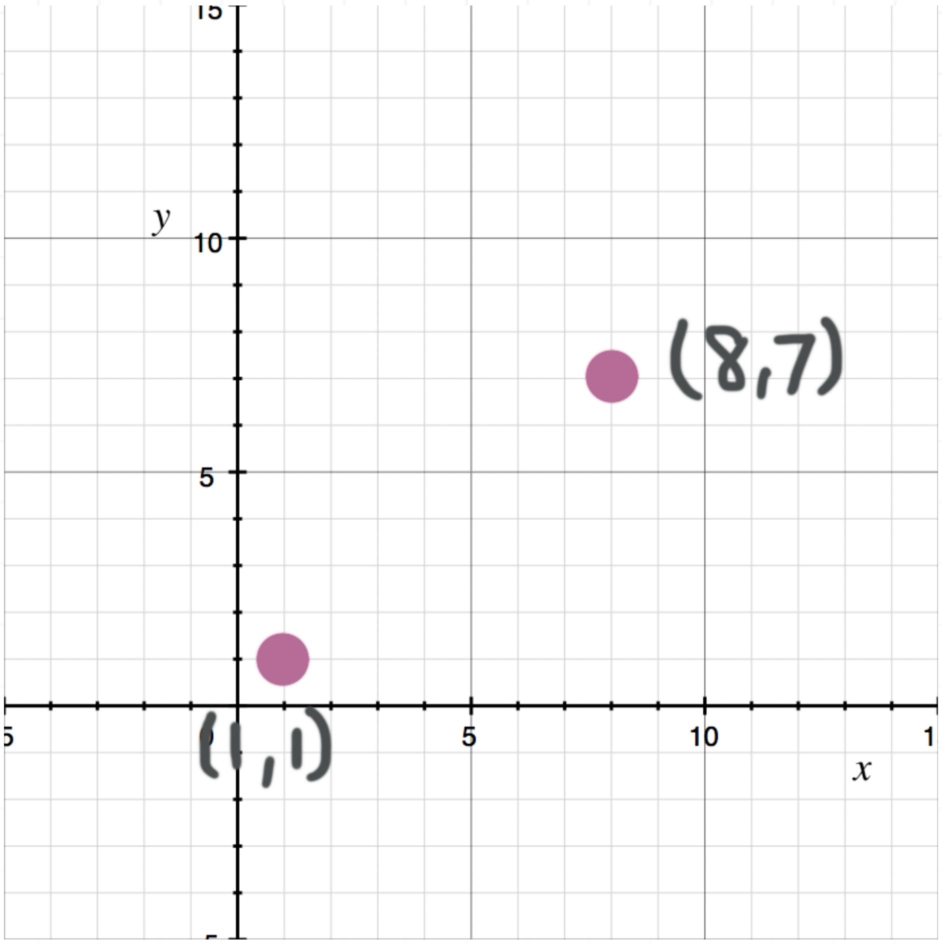
$$P = 72 \text{ in}$$

Sometimes you'll be given the coordinates of some or all of the vertices of a rectangle in a coordinate plane and asked to find the perimeter.

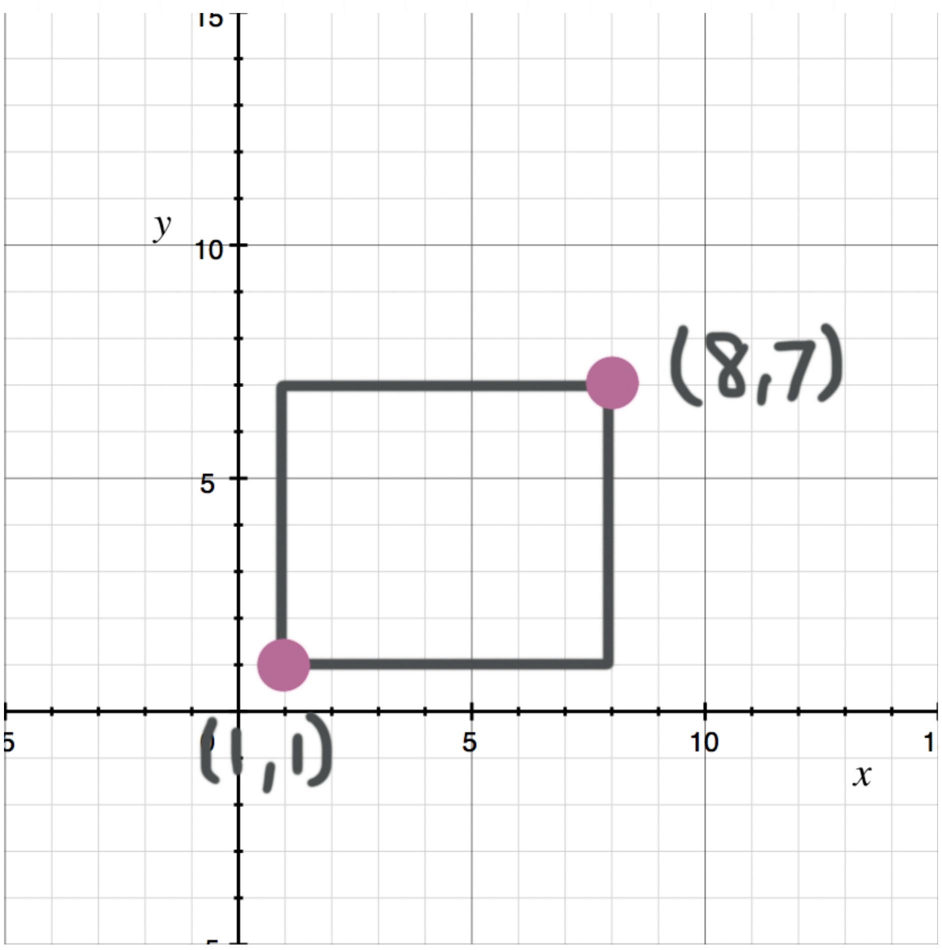
Example

A rectangle has one vertex at $(1,1)$, and the opposite vertex (the vertex that's connected to $(1,1)$ by a diagonal) is at $(8,7)$. If the sides of the rectangle are parallel to the coordinate axes, what is the perimeter of the rectangle?

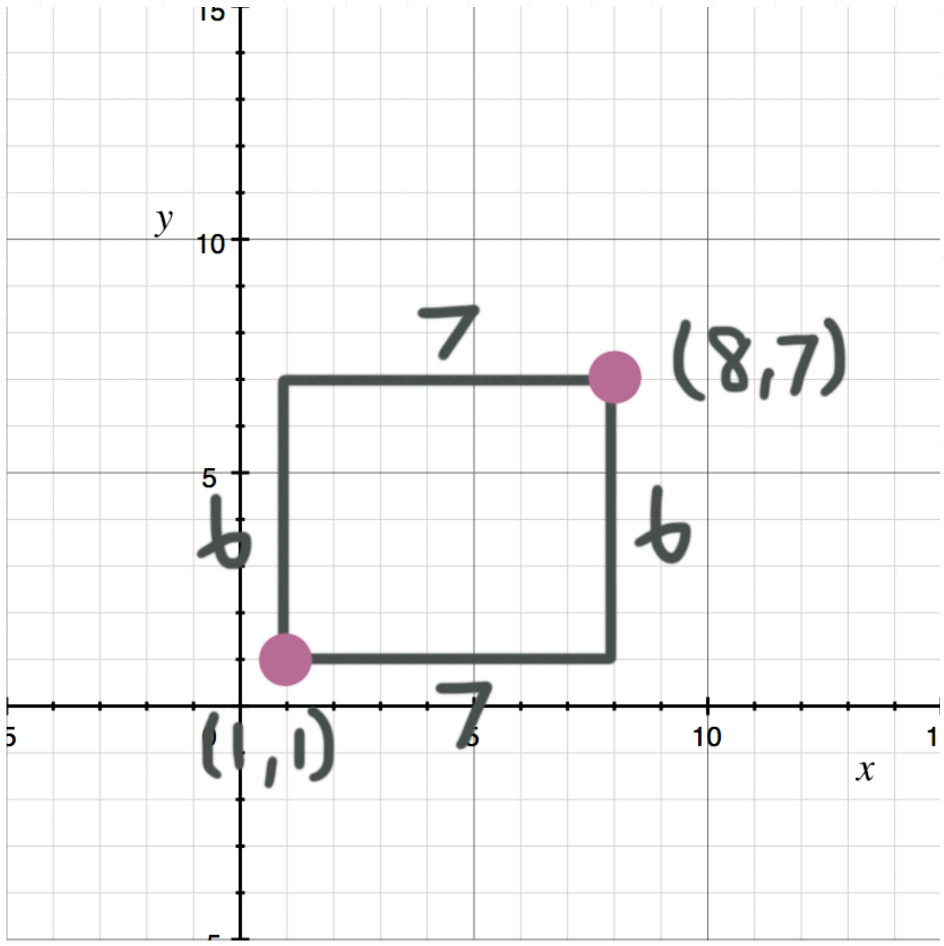
Drawing a sketch of the rectangle on the coordinate plane can help, so start by plotting the points.



Now draw in the rectangle.



Find the length of each side. Since the sides of the rectangle are parallel to the coordinate axes, the length l will be the difference between the x-coordinates of the given points ($l = 8 - 1 = 7$), and the width w will be the difference between their y-coordinates ($w = 7 - 1 = 6$).



Add the lengths of the four sides to find the perimeter.

$$P = 6 + 7 + 6 + 7 = 26$$

Other times we might need to perform some other type of calculation before we can find the perimeter.

Example

What is the perimeter of the rectangle that has an area of 96 and a length of 8?

We need to know the dimensions of the rectangle in order to find its perimeter. We know the area is 96 and the length is 8, so we can use the formula for area of a rectangle and solve for the width.

$$A = lw$$

$$96 = 8w$$

$$w = 12$$

Now we can find the perimeter.

$$P = 2l + 2w$$

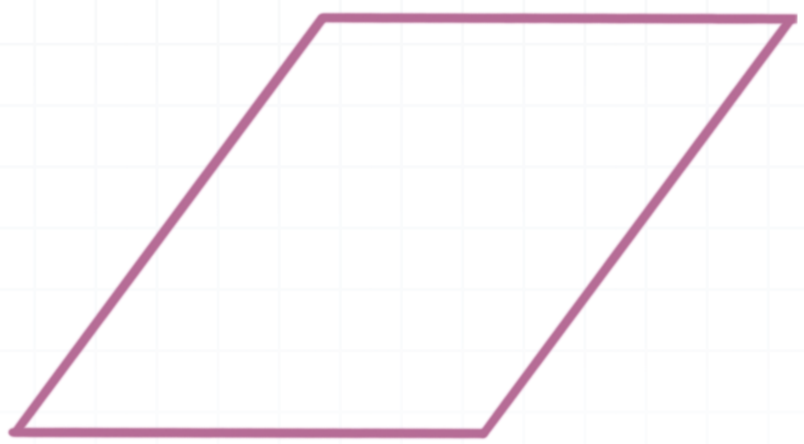
$$P = 2(8) + 2(12)$$

$$P = 16 + 24$$

$$P = 40$$

Area of a parallelogram

In this lesson we'll look at how to find the area of a parallelogram. A parallelogram is a quadrilateral with two pairs of opposite parallel sides.

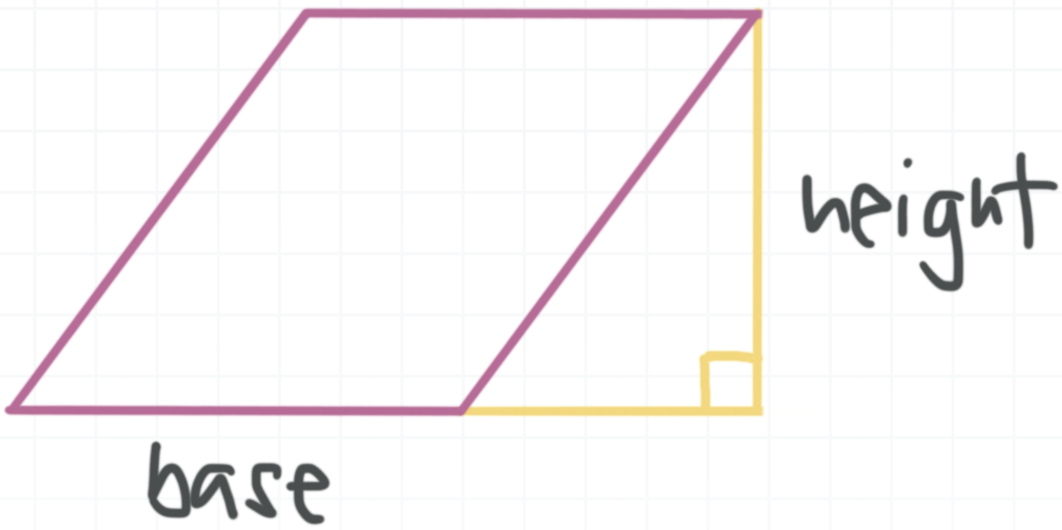


Area of a parallelogram

The area of a parallelogram is found by multiplying the base by the height, so

$$A = bh$$

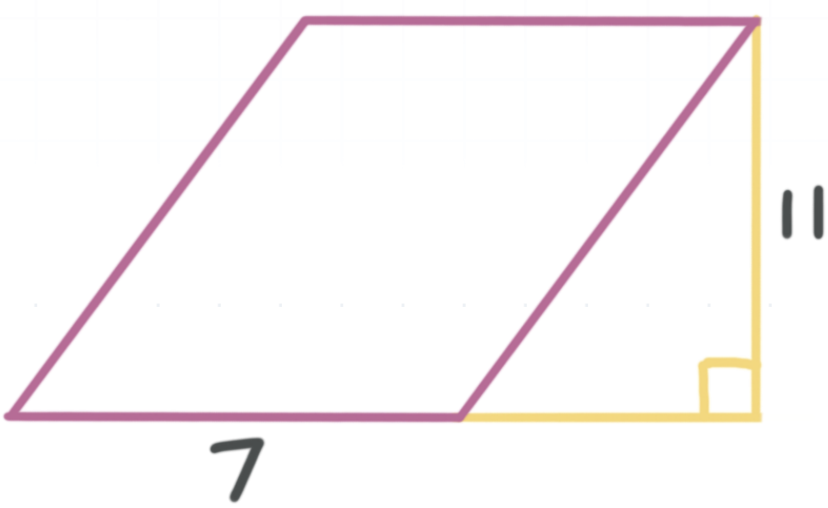
The height of a parallelogram needs to be drawn in and is perpendicular to its base.



The area of a parallelogram is always given in units of length^2 ("length squared"). Let's start by working through an example.

Example

What is the area of the parallelogram?



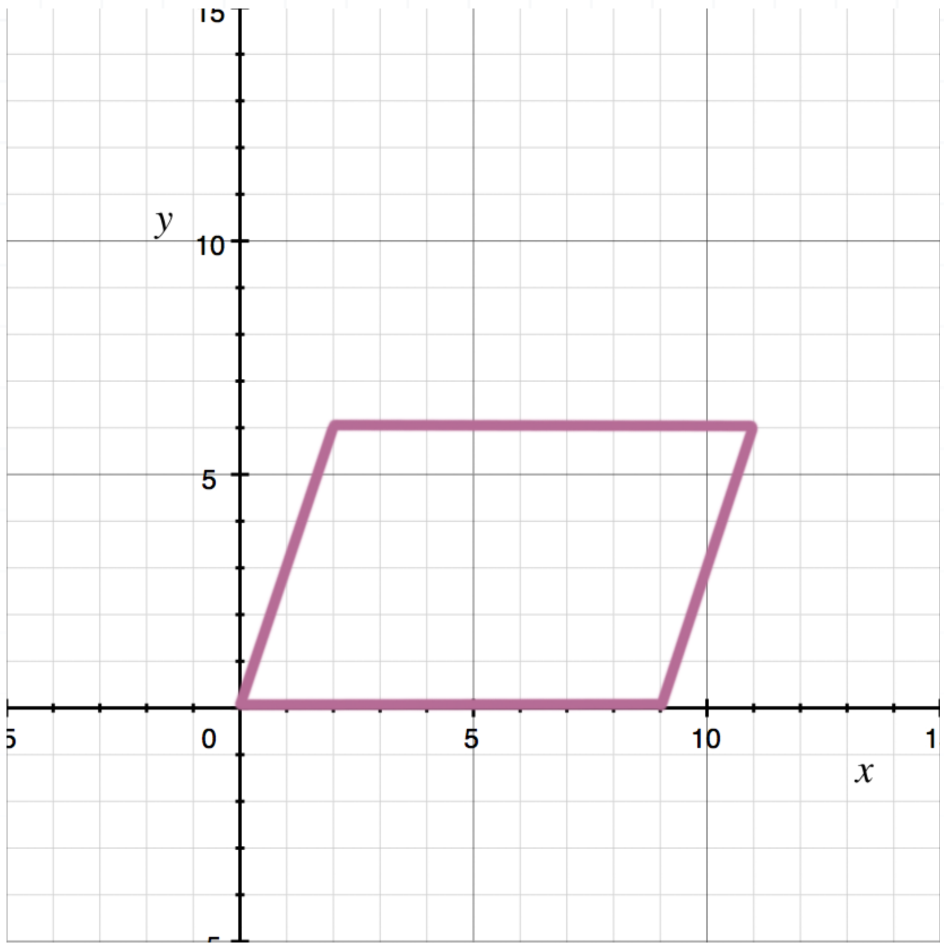
In this parallelogram, the base has a length of 7, and a height of 11, so the area is

$$A = 7 \cdot 11 = 77$$

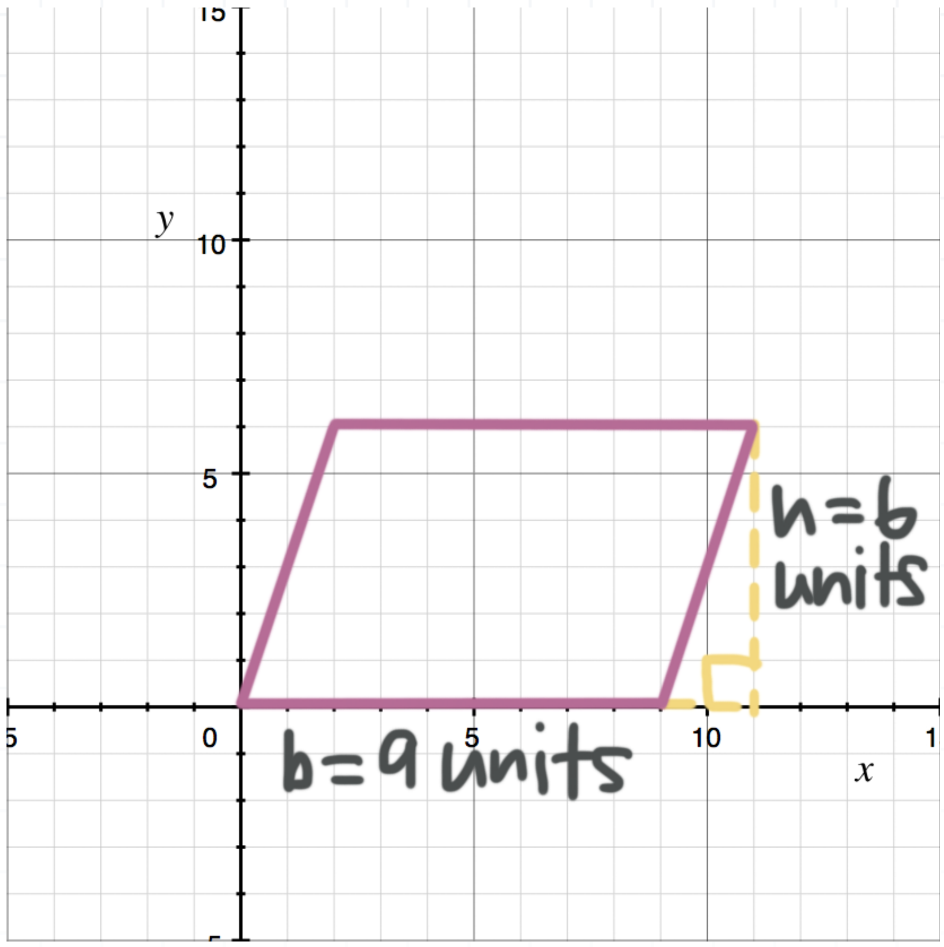
Let's do another example.

Example

What is the area of the parallelogram?



Use the grid (the system of horizontal and vertical lines) in the figure to find the dimensions of the parallelogram.



Now we can use the area formula.

$$A = bh$$

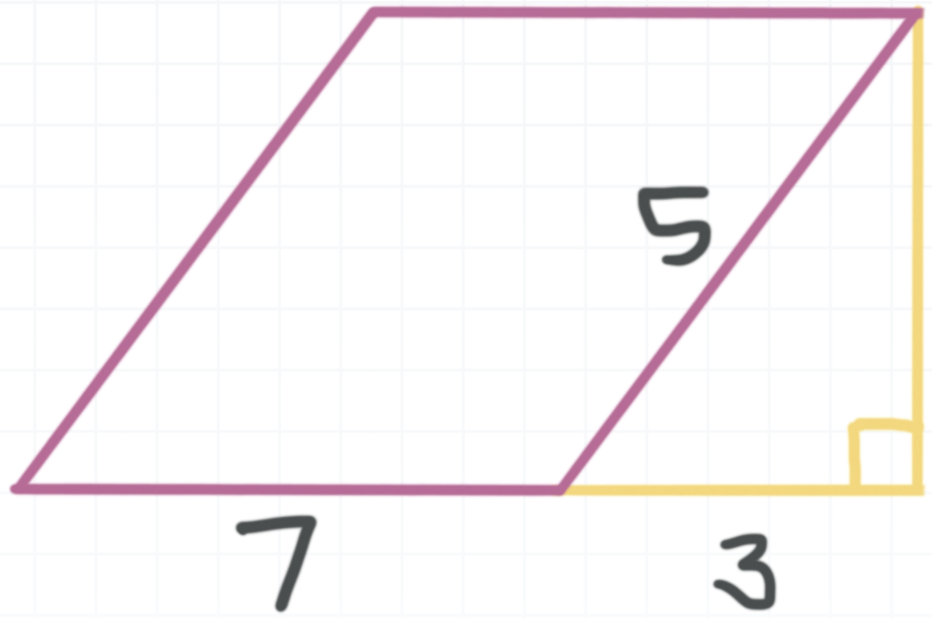
$$A = 9 \cdot 6$$

$$A = 54$$

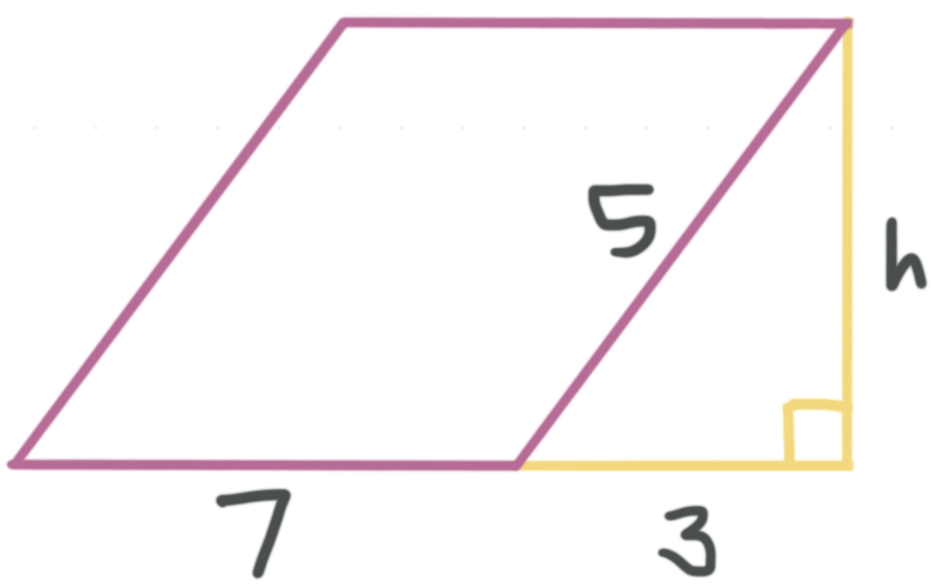
Let's do one with a few more steps.

Example

What is the area of the parallelogram?



The area of a parallelogram is $A = bh$. We know the base of the parallelogram is 7, but we need to find the height. We can see that the yellow lines and the slanted side of the parallelogram form a right triangle, so we can use the Pythagorean theorem to solve for the height. Let's call the height h.



Then we can plug everything we have into the Pythagorean theorem to find h.

$$3^2 + h^2 = 5^2$$

$$9 + h^2 = 25$$

$$h^2 = 16$$

$$h = \pm\sqrt{16}$$

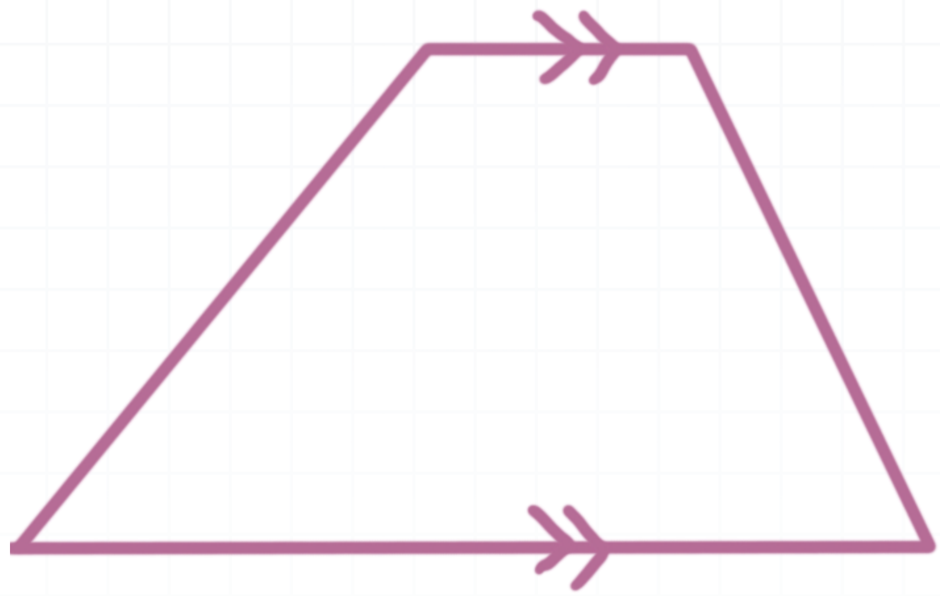
$$h = \pm 4$$

Height can't take on a negative value, so $h = 4$. Now we can say that the area of the parallelogram is

$$A = 7 \cdot 4 = 28$$

Area of a trapezoid

In this lesson we'll look at how to find the area of a trapezoid. A trapezoid is a quadrilateral with exactly one pair of opposite parallel sides.

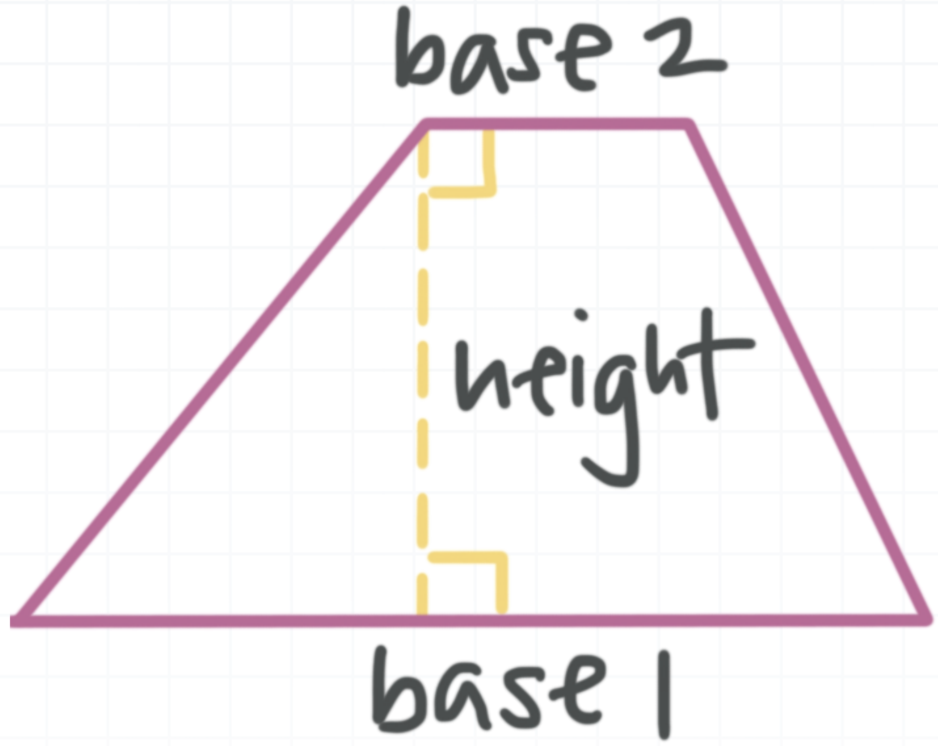


Area of a trapezoid

The area of a trapezoid is given by

$$A = \frac{1}{2}(b_1 + b_2)h$$

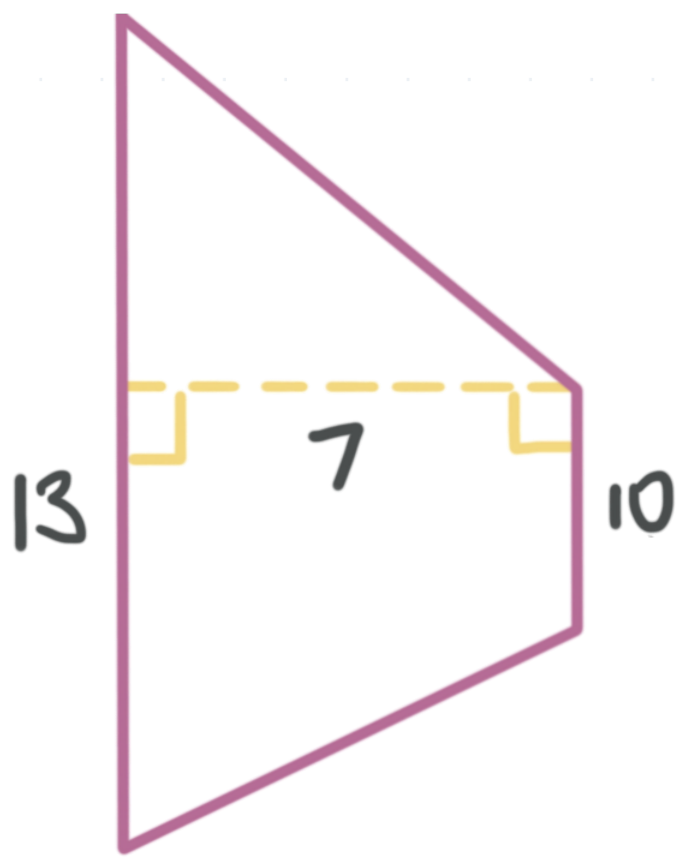
where b_1 and b_2 are the lengths of the parallel sides (which we call the **bases**), and h is the height of the trapezoid (which is perpendicular to both bases). Sometimes you'll need to draw in the height.



The area of a trapezoid is always given in units of length^2 ("length squared"). Let's start by working through an example.

Example

What is the area of the trapezoid?



The bases of a trapezoid are the parallel sides, so this trapezoid has bases of length 13 and 10.

The height of a trapezoid is the length of any line segment that has one endpoint on each base and is perpendicular to both bases, so this trapezoid has a height of 7.

$$A = \frac{1}{2}(b_1 + b_2)h$$

$$A = \frac{1}{2}(13 + 10)(7)$$

$$A = \frac{1}{2}(23)(7)$$

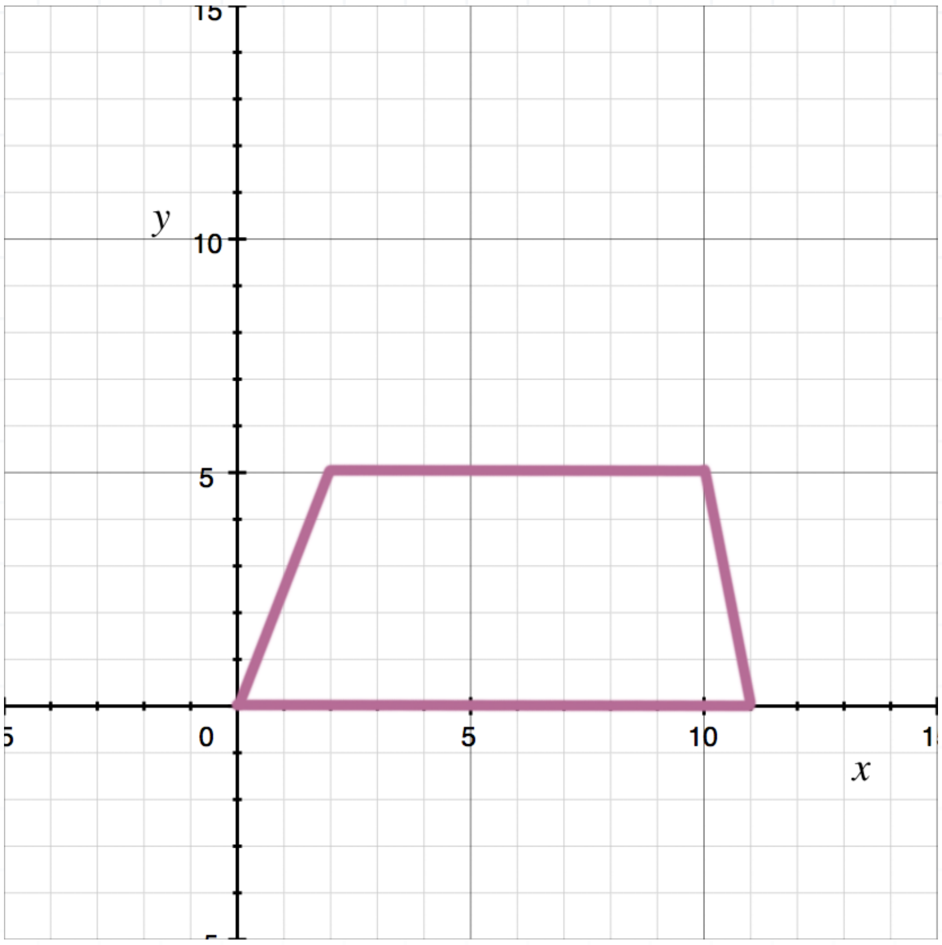
$$A = \frac{1}{2}(161)$$

$$A = 80.5$$

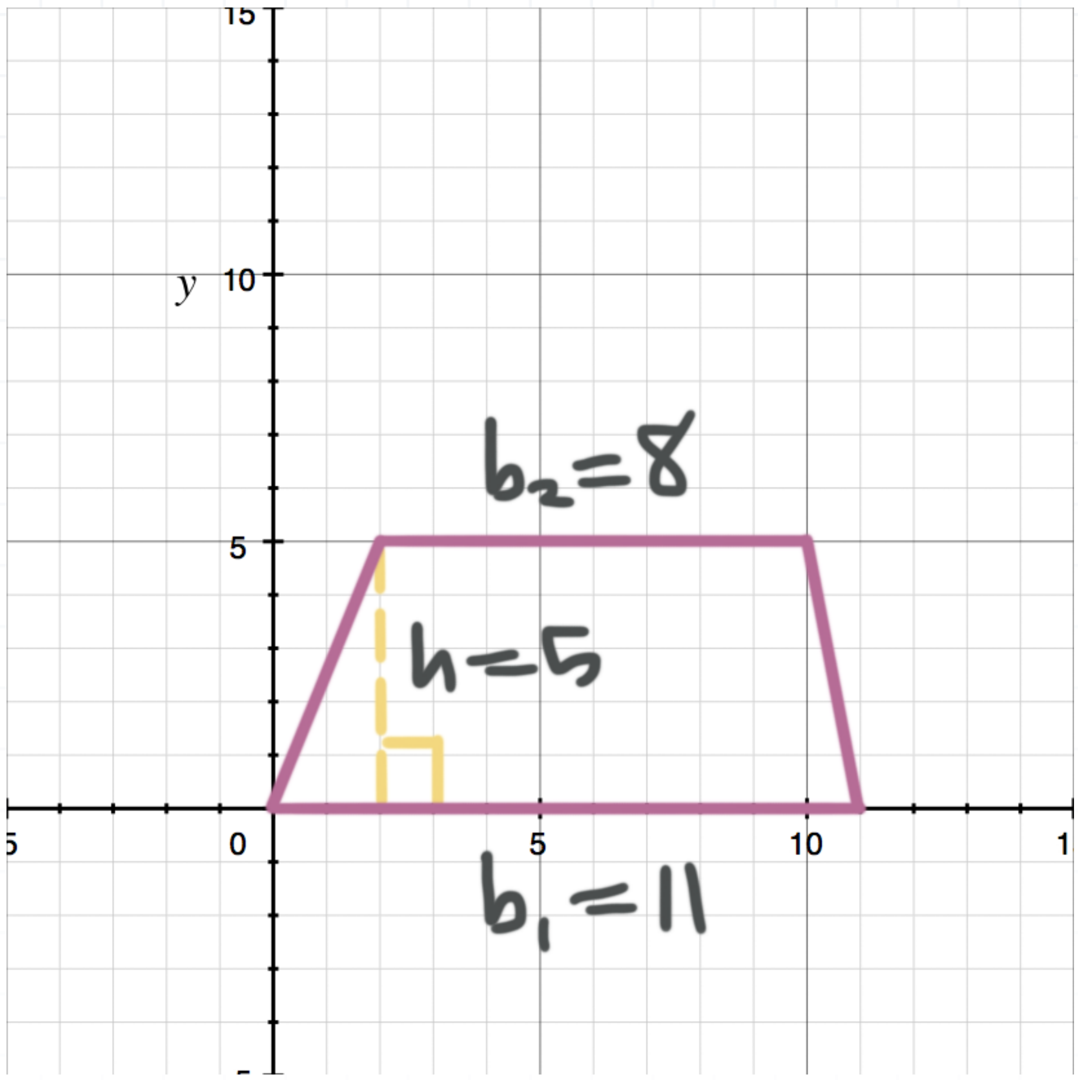
Let's do one more example.

Example

What is the area of the trapezoid?



Use the grid (the system of horizontal and vertical lines) in the figure to find the dimensions of the trapezoid (the lengths of the bases and the height).



Now use the formula for the area of a trapezoid.

$$A = \frac{1}{2}(b_1 + b_2)h$$

$$A = \frac{1}{2}(11 + 8)(5)$$

$$A = 47.5$$

Area of a triangle

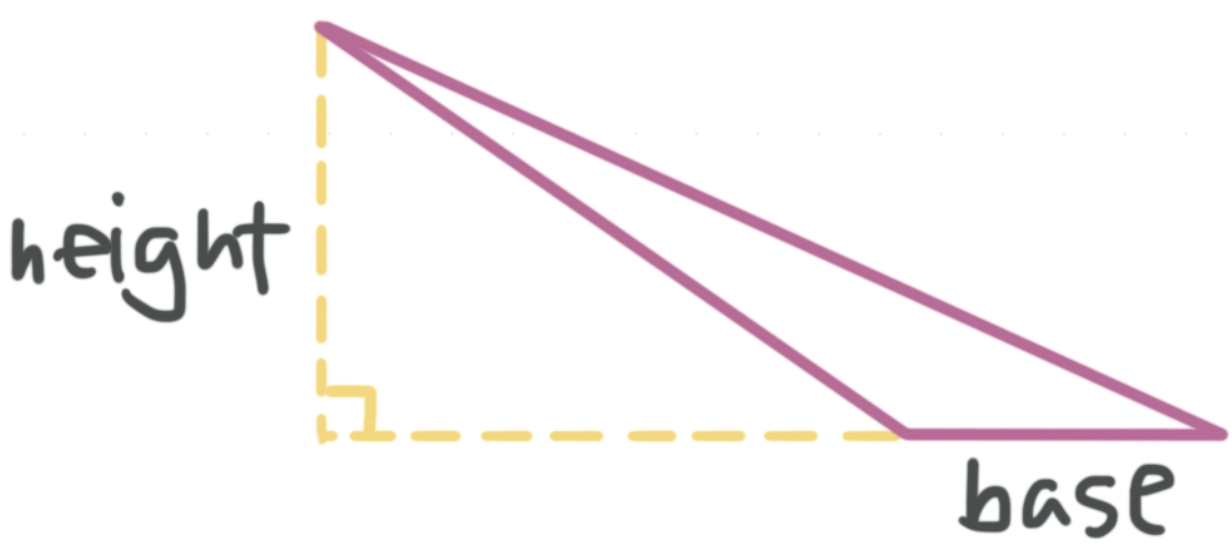
In this lesson we'll look at how to find the area of a triangle, which is equivalent to half of the product of the base and the height.

$$A = \frac{1}{2}bh$$

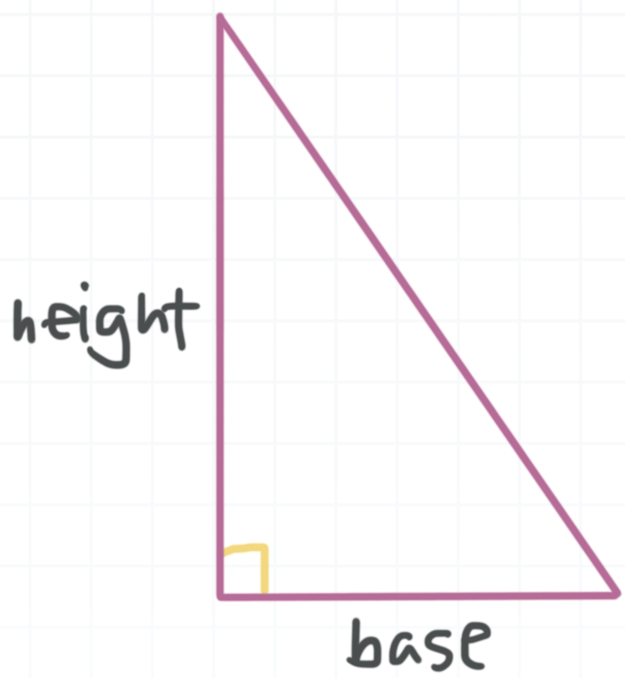
The area is always in units of length^2 ("length squared").

Any side of a triangle can be the base, but once you've chosen the base, the height is drawn from the opposite vertex (the vertex opposite the base) to the side that you're using as the base. The height could look different depending on the type of triangle, but it's always perpendicular to the base (in some cases perpendicular to an extension of the base).

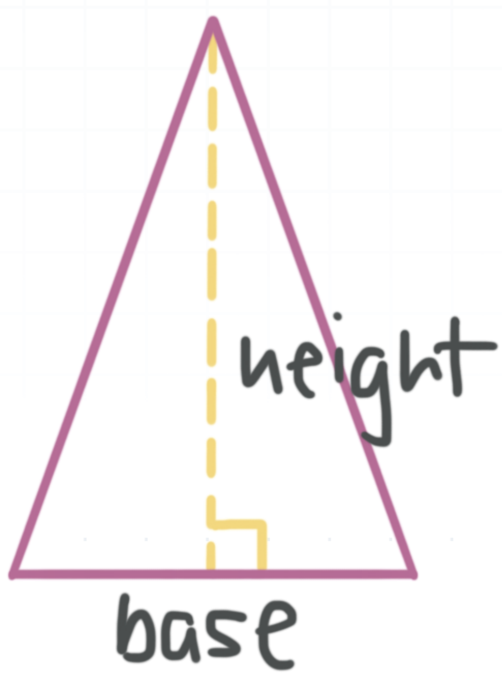
In a scalene triangle, the lengths of all three sides are different.



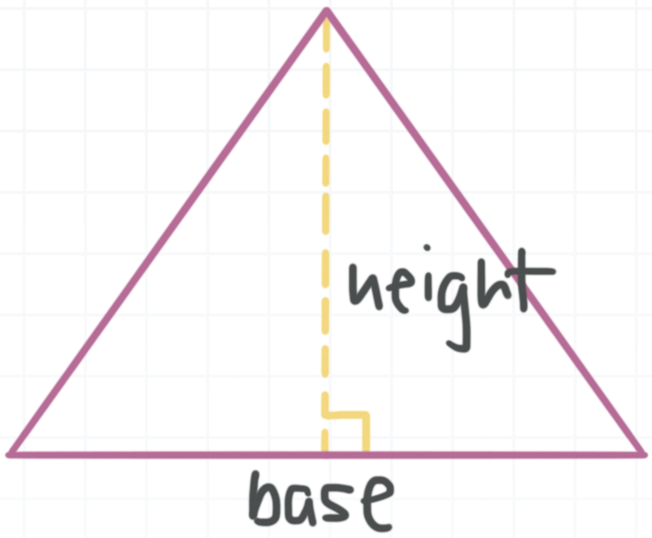
In a right triangle, one of the interior angles is a right (90°) angle.



In an isosceles triangle, the lengths of exactly two sides are equal.



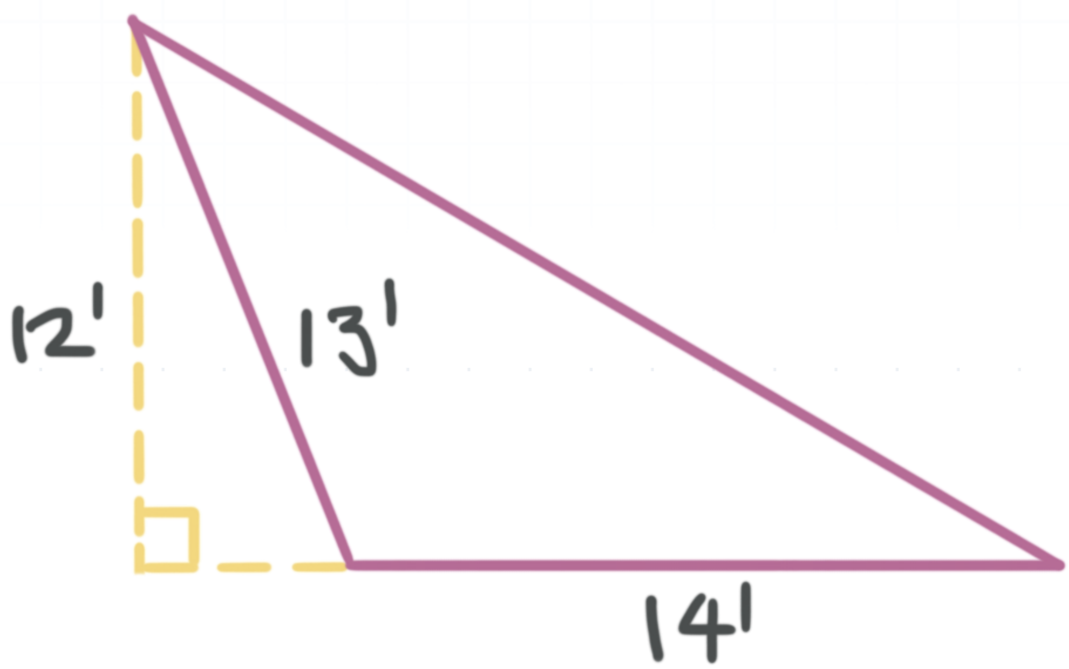
In an equilateral triangle, the lengths of all three sides are equal.



Let's start by working through an example.

Example

Find the area of the triangle.



The area formula for a triangle is

$$A = \frac{1}{2}bh$$

In the diagram, the base of the triangle is 14 feet and the height is 12 feet. Plugging these into the area formula, we get

$$A = \frac{1}{2}(14\text{ ft})(12\text{ ft})$$

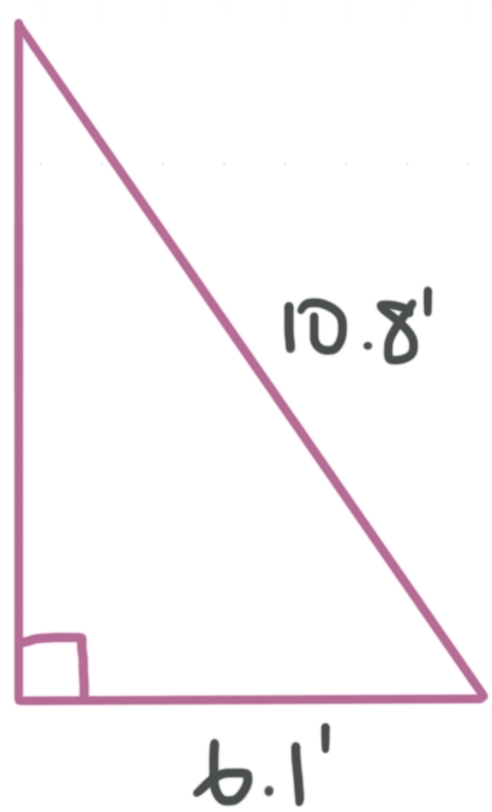
$$A = \frac{1}{2}(168\text{ ft}^2)$$

$$A = 84\text{ ft}^2$$

Let's do one more.

Example

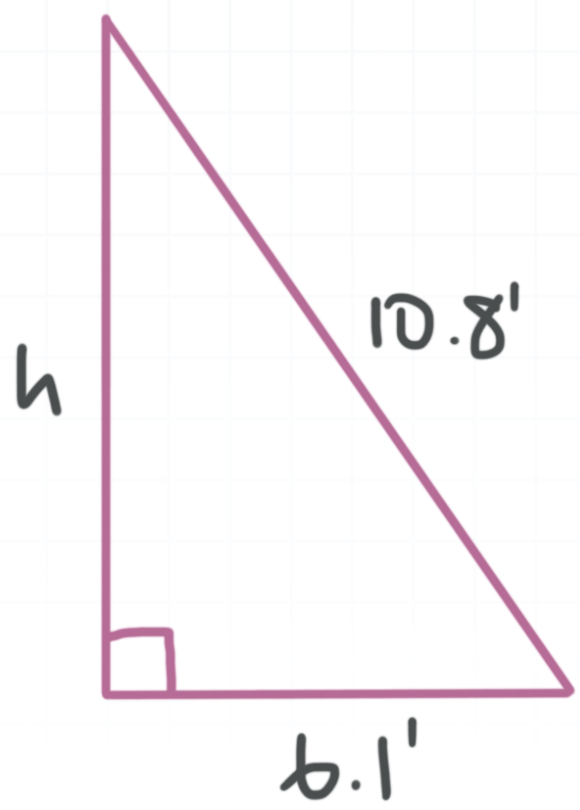
Find the area of the right triangle to the nearest tenth.



The area of a triangle is given by

$$A = \frac{1}{2}bh$$

We can see that the base of the triangle is $6.1'$, but we'll need to use the Pythagorean Theorem to find the height. We'll sketch in the height,



and then plug everything into the Pythagorean Theorem.

$$6.1^2 + h^2 = 10.8^2$$

$$37.21 + h^2 = 116.64$$

$$h^2 = 79.43$$

$$h = \pm\sqrt{79.43}$$

$$h \approx \pm 8.9$$

Since we're looking for height, we need only the positive answer, $h \approx 8.9$. Now we can use the area formula.

$$A \approx \frac{1}{2}(6.1 \text{ ft})(8.9 \text{ ft})$$

$$A = \frac{1}{2}(54.29 \text{ ft}^2)$$

$$A \approx 27.1 \text{ ft}^2$$

Perimeter of a triangle

To find the perimeter of a triangle, we can use the Pythagorean theorem, regardless of whether or not the triangle is a right triangle.

Right triangles

When the triangle *is* right, we only need to know two of the three sides of the triangle. Remember that the Pythagorean theorem tells us that the sum of the squares of the legs, a and b, is equal to the square of the hypotenuse, c.

$$a^2 + b^2 = c^2$$

Which means, given two of the values a, b, and c, we'll be able to use the Pythagorean theorem to solve for the third value that we don't know. Then once we know all three of the values a, b, and c, we can add them to find the perimeter of the triangle.

Example

One leg of a right triangle is 4 meters long, and the triangle's hypotenuse is 5 meters long. Find the perimeter of the triangle.

Let's call the known leg $a = 4$, the unknown leg b, and the hypotenuse $c = 5$. Then substitute into the Pythagorean theorem to find the length of the leg b.

$$a^2 + b^2 = c^2$$

$$4^2 + b^2 = 5^2$$

$$16 + b^2 = 25$$

$$b^2 = 9$$

$$b = 3$$

Now that we know the lengths of all three sides, we add them together to find the triangle's perimeter.

$$P = a + b + c$$

$$P = 4 + 3 + 5$$

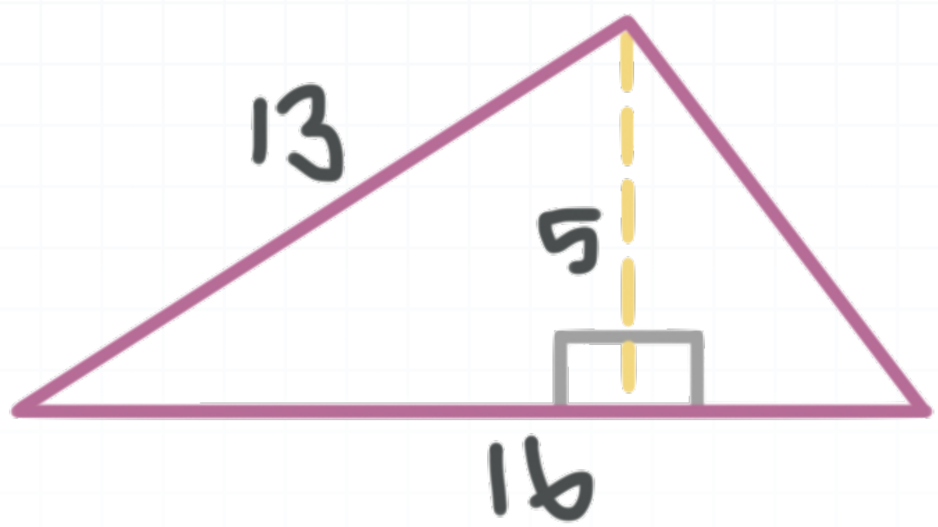
$$P = 12$$

Oblique triangles

Any triangle that isn't a right triangle is an oblique triangle. When a triangle is oblique and we want to find its perimeter, we start by separating the oblique triangle into two right triangles. From there, we find the lengths of sides of the right triangles, and use those lengths to find the perimeter of the oblique triangle.

Example

Find the perimeter of the oblique triangle.



The height of the oblique triangle is 5. We can consider that height as a leg of the right triangle on the left. The hypotenuse of the triangle on the left is 13, so we can use these two side lengths in the Pythagorean theorem to find the length of the unknown leg.

$$a^2 + b^2 = c^2$$

$$5^2 + b^2 = 13^2$$

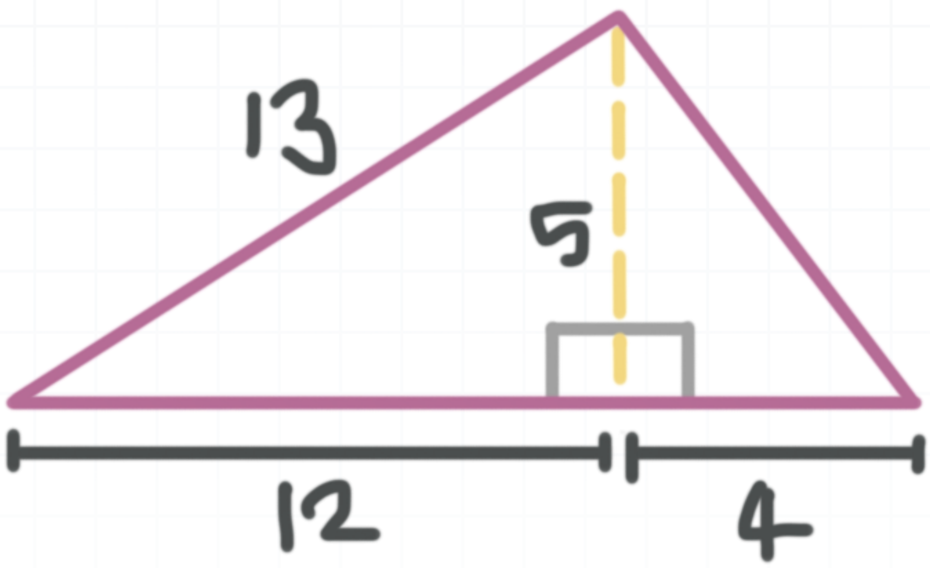
$$25 + b^2 = 169$$

$$b^2 = 144$$

$$b = \sqrt{144}$$

$$b = 12$$

From the figure, the base length of the oblique triangle is 16, but we just found $b = 12$, which means the length of the base leg of the right triangle on the right is $16 - 12 = 4$.



For the right triangle on the right side, the vertical leg is the height $a = 5$, and we just found that the length of the base leg is $b = 4$, so from the Pythagorean theorem the hypotenuse of the triangle on the right is

$$a^2 + b^2 = c^2$$

$$5^2 + 4^2 = c^2$$

$$25 + 16 = c^2$$

$$41 = c^2$$

$$c = \sqrt{41}$$

Therefore, the three side lengths of the oblique triangle are 13, 16, and $\sqrt{41}$, which means its perimeter is

$$P = 13 + 16 + \sqrt{41}$$

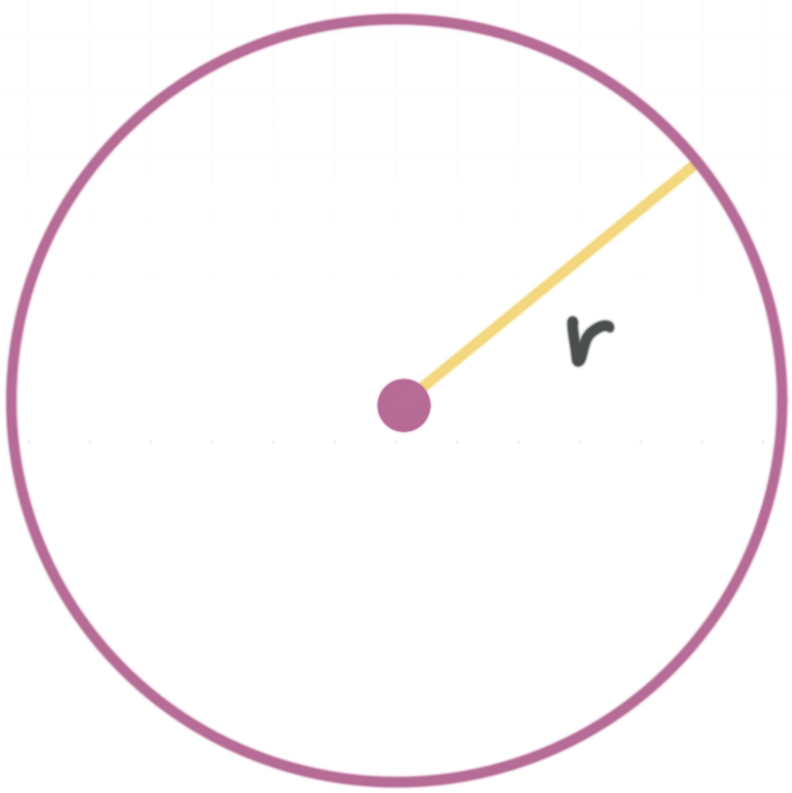
$$P = 29 + \sqrt{41}$$

Area of a circle

In this lesson we'll look at how to use the area formula for a circle. In order to do that, we'll need to start by defining the parts of a circle.

Parts of a circle

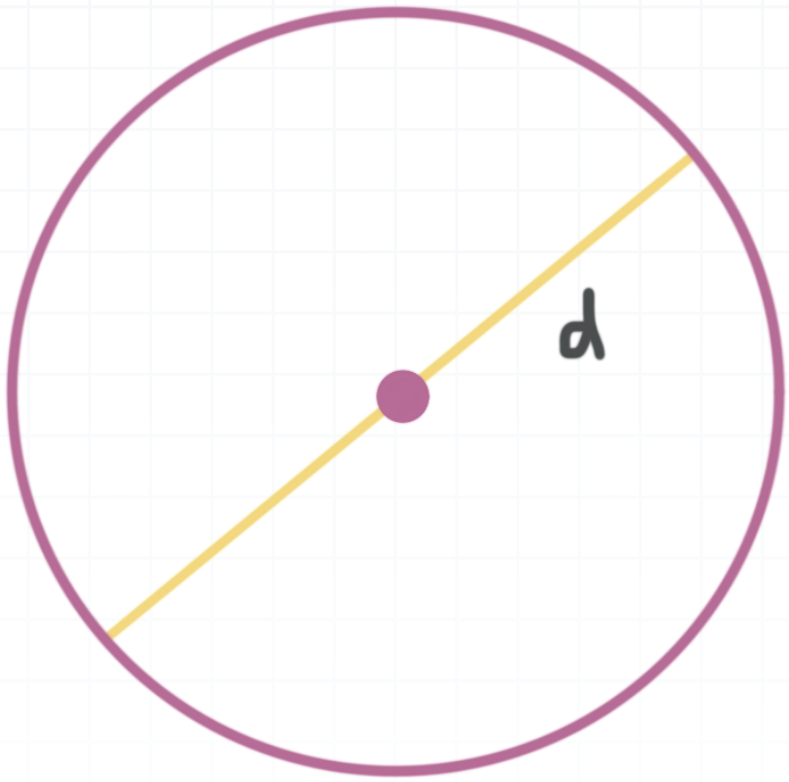
The **radius** of a circle is the length of any line segment from the center of the circle to a point on the circle. We usually use the variable r for the radius.



The **diameter** of a circle is the length of any chord that passes through the center of the circle. We usually use the variable d for the diameter of a circle. The diameter is a chord whose length is equal to twice the radius.

$$d = 2r$$

$$r = \frac{1}{2}d$$



The number π ("**pi**") is a special number - with the lowercase Greek letter pi used as the symbol for it - that describes the relationship between the circumference of a circle (the "length around" a circle) and its diameter:

$$\text{circumference} = \pi d$$

Since the diameter of a circle is twice its radius, we also have

$$\text{circumference} = \pi(2r) = 2\pi r$$

The decimal expansion of π has infinitely many decimal places. Rounded to two decimal places (rounded to the nearest hundredth), its value is 3.14; we write this approximation as $\pi = 3.14$.

Area of a circle

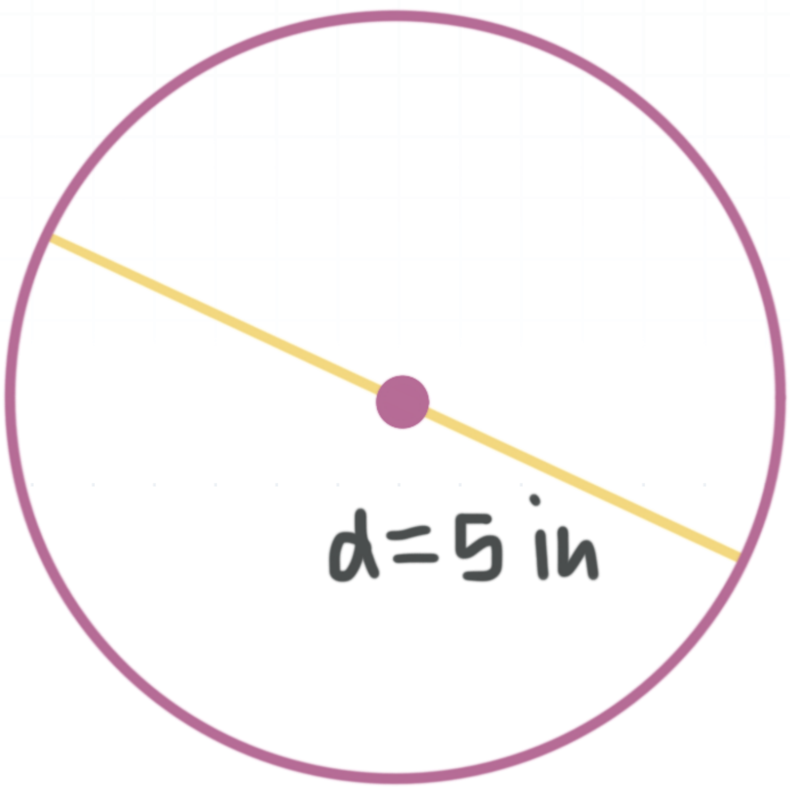
Now that we know the value of π, and the radius of the circle, we can define the area of a circle as the product of π and the square of the radius.

$$A = \pi r^2$$

As with all areas, the area of a circle is in units of length^2 ("length squared"). Let's start by working through an example.

Example

What is the area of the circle? Round your answer to the nearest hundredth.



The formula for the area is

$$A = \pi r^2$$

We're given the diameter, and we need to find the radius. The radius is half of the diameter.

$$r = \frac{1}{2}(5 \text{ in})$$

$$r = 2.5 \text{ in}$$

Now we can use the area formula.

$$A = \pi(2.5 \text{ in})^2$$

$$A = 6.25\pi \text{ in}^2$$

This is the exact answer. We're asked to round the answer to the nearest hundredth, so we'll use the fact that π is about 3.14.

$$A \approx 6.25(3.14) \text{ in}^2$$

$$A \approx 19.63 \text{ in}^2$$

Sometimes you'll be given the area and asked to solve for something else.

Example

What is the radius of a circle with an area of 75π cm^2?

The formula for the area of a circle is $A = \pi r^2$, and the area is 75π cm^2.

$$\pi r^2 = 75\pi \text{ cm}^2$$

$$r^2 = 75 \text{ cm}^2$$

$$r = \sqrt{75 \text{ cm}^2}$$

$$r = \sqrt{75} \text{ cm}$$

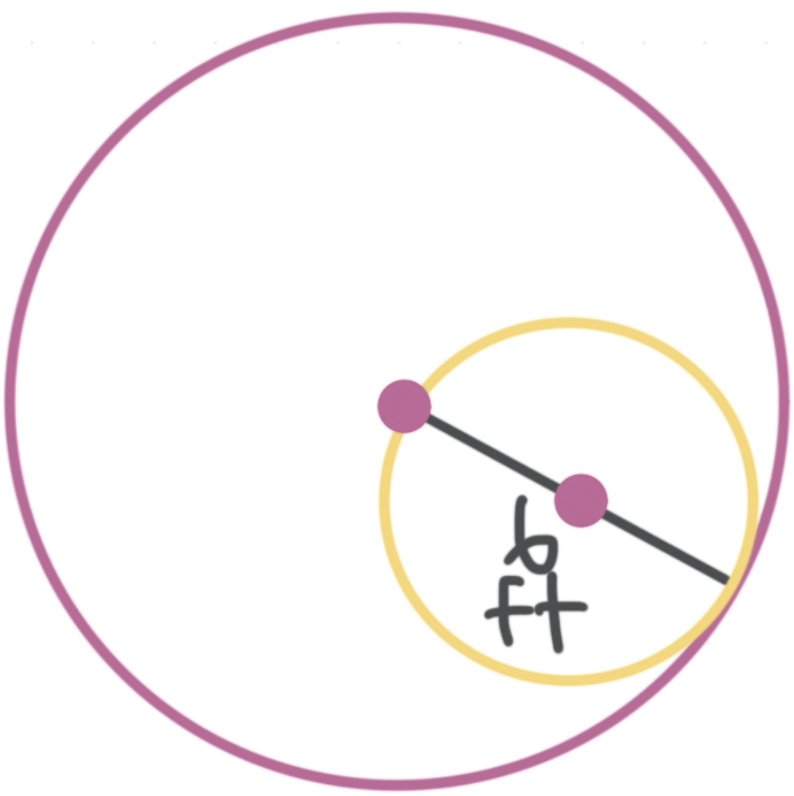
$$r = \sqrt{25} \cdot \sqrt{3} \text{ cm}$$

$$r = 5\sqrt{3} \text{ cm}$$

Sometimes you'll need to find the area of a composite figure that's made of circles.

Example

What is the area of the region that's inside the larger circle, but outside the smaller circle? Leave your answer in terms of π.



We need to find the area of the larger circle and the subtract from it the area of the smaller circle. The formula for the area of a circle is $A = \pi r^2$, so we need to know the radius of each circle.

The radius of the larger circle is 6 feet, so the area of the larger circle is

$$A = \pi \cdot (6 \text{ ft})^2$$

$$A = 36\pi \text{ ft}^2$$

The smaller circle has a diameter of 6 feet, so its radius is $r = 6/2 = 3$ feet. Therefore, the area of the smaller circle is

$$A = \pi \cdot (3 \text{ ft})^2$$

$$A = 9\pi \text{ ft}^2$$

So the area of the region that's inside the larger circle but outside the smaller circle is

$$36\pi \text{ ft}^2 - 9\pi \text{ ft}^2 = 27\pi \text{ ft}^2$$

Circumference of a circle

Remember that the radius of a circle is the length of any line segment from the center of the circle to a point on the circle, and that the diameter is the length of any chord that passes through the center of the circle.

A diameter of a circle is always made of two radii, so $d = 2r$, where d is the diameter and r is the radius. And π is a special number, $\pi \approx 3.14$, that describes the relationship between a circle's circumference and the its diameter.

Circumference of a circle

The circumference of a circle, which we'll call C, is the distance around the circle (its perimeter). We can also think of the circumference as the length of the circle. The circumference can be expressed in terms of the radius or in terms of the diameter:

$$C = 2\pi r$$

$$C = \pi d$$

Let's start by working through an example.

Example

What is the circumference of a circle with a diameter of 10 in?

The formula for the circumference of a circle when we know the circle's diameter is $C = \pi d$. We know the diameter of this circle is 10 inches, so when we plug this into the formula we get

$$C = \pi(10 \text{ in})$$

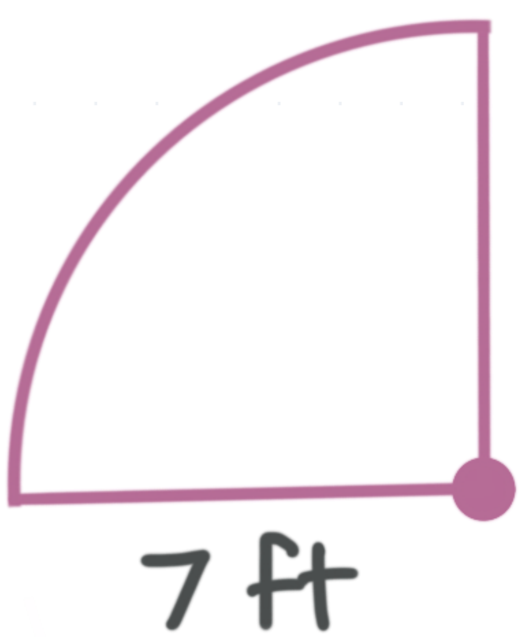
$$C \approx 3.14(10 \text{ in})$$

$$C \approx 31.4 \text{ in}$$

Sometimes you could be asked to find the length of part of a circle.

Example

To the nearest hundredth, what is the length of this quarter-circle (one-fourth of a circle)?



The formula for circumference when you know the radius is $C = 2\pi r$, and we know the radius is 7 feet, so the circumference is

$$C = 2\pi(7 \text{ ft})$$

$$C = 14\pi \text{ ft}$$

We need to divide the circumference by 4 to find the length of the quarter-circle.

$$\text{length of quarter-circle} = \frac{C}{4} = \frac{14\pi \text{ ft}}{4} = 3.5\pi \text{ ft}$$

Notice that if we had been asked to find the perimeter of the figure, we would have needed to add the lengths of the two radii shown in the figure to the length of the quarter-circle. The perimeter would be

$$P = (3.5\pi \text{ ft}) + (7 \text{ ft}) + (7 \text{ ft})$$

$$P \approx 3.5(3.14) \text{ ft} + 14 \text{ ft}$$

$$P \approx 10.99 \text{ ft} + 14 \text{ ft}$$

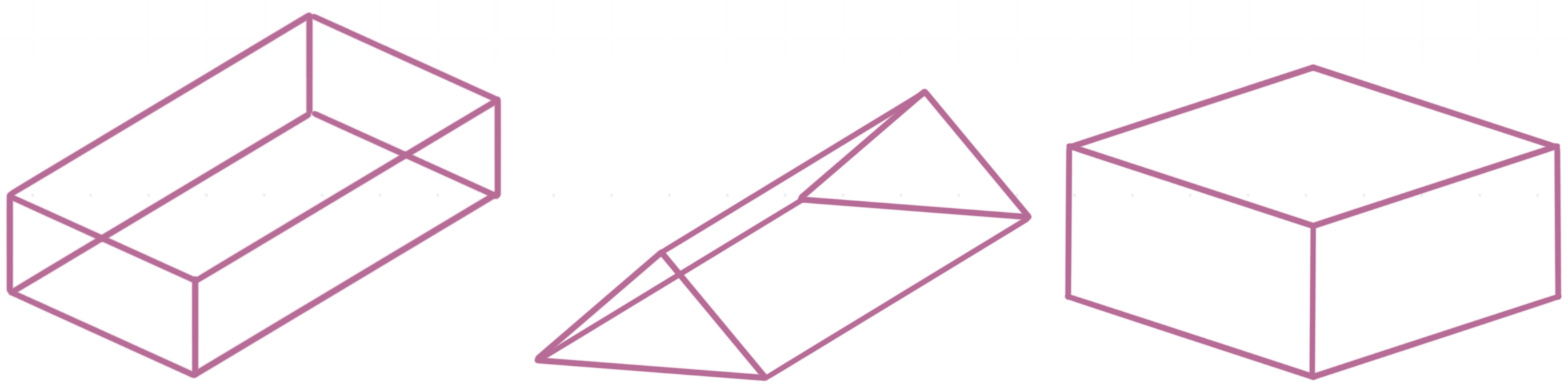
$$P \approx 24.99 \text{ ft}$$

Nets/volume/surface area of prisms

In this lesson we'll look at an introduction to three-dimensional geometric figures, specifically nets, volume, and surface area of prisms.

Prism

A **polyhedron** is a closed three-dimensional shape that's made up of polygons (which are the **faces** of the polyhedron). A **prism** is a polyhedron that has a pair of parallel congruent faces of any shape, and whose other faces are parallelograms. In this lesson, we'll be dealing only with prisms in which all the faces other than the bases are rectangles. Here are some examples of prisms:



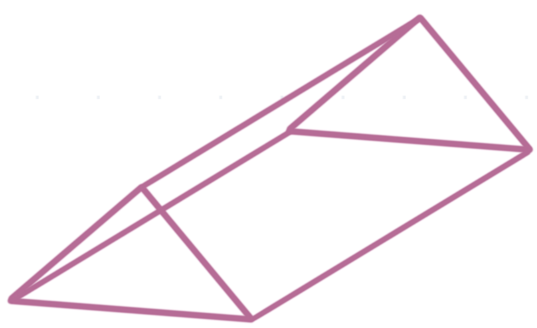
The height of a prism is the length of any line segment that has one endpoint on each base and is perpendicular to both bases. A prism whose bases are triangles is called a **triangular prism**, a prism whose bases are rectangles is called a **rectangular prism**, a prism whose bases are pentagons is called a **pentagonal prism**, etc.

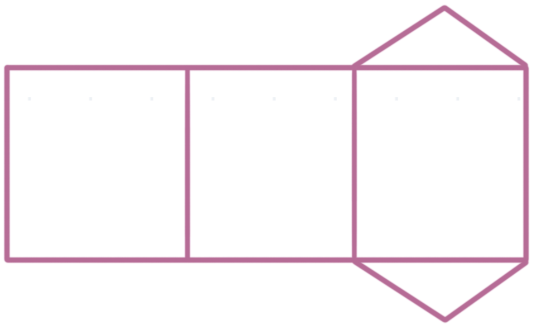
In a rectangular prism in which all the faces (not just the bases) are rectangles, any pair of parallel, congruent rectangles can be used as the bases.

Net

A **net** of a polyhedron is a two-dimensional flattened out version of it. We make the net by cutting the polyhedron along one or more of the edges until we can lay out the whole thing flat in a plane.

Once we have the net of a polyhedron, we should be able to reconstruct it by folding the pieces of the net, using the polygons in the net as faces, and using each line segment in the net as the boundary between some pair of faces of the polyedron. Here is a triangular prism (on the left) and its net (on the right).





Surface area of a rectangular prism

The surface area of any prism (in fact, of any polyhedron) is the sum of the areas of all of its faces. Since the bases of a rectangular prism are rectangles, and all the other faces are rectangles, we already have all the tools we'll need to find the surface area of a rectangular prism.

The areas of the faces of the rectangular prism are given in the table:

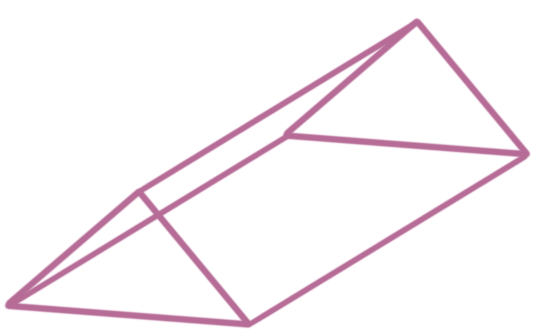
Top and bottom	length x width
Left and right	width x height
Front and back	length x height

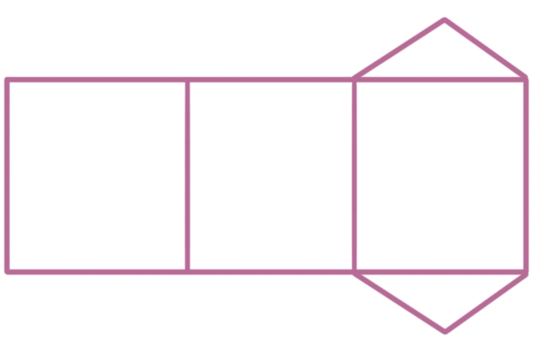
The formula for surface area of a rectangular prism is

$$A = 2lw + 2wh + 2lh$$

Surface area of a prism

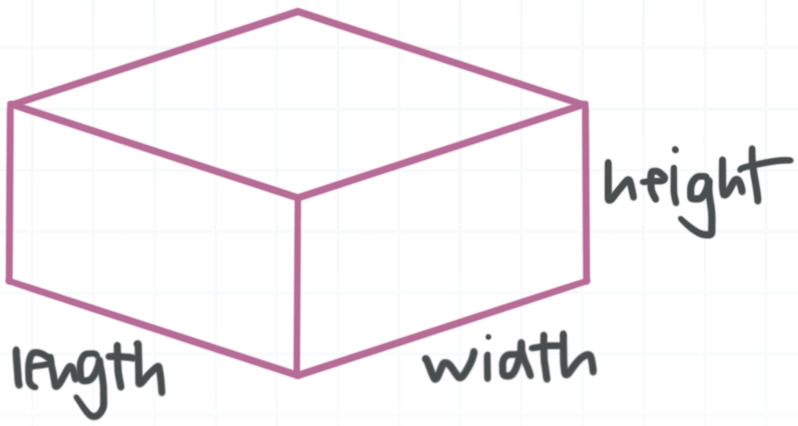
To find the surface area of any prism, it can be helpful to sketch its net, find the area of each shape in the net, and then add those areas. To find the surface area of this triangular prism, which has a pair of (congruent) triangular bases and three rectangular faces, find the areas of the three rectangles and two triangles in this net and add them.





Volume of a rectangular prism

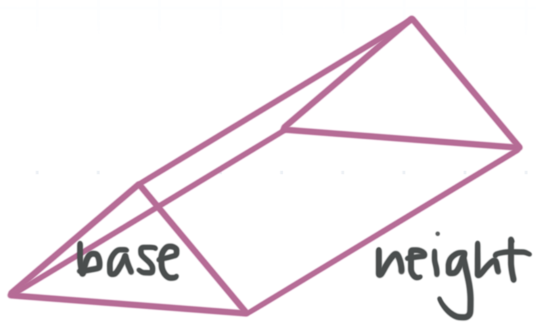
The volume of the type of rectangular prism we're studying is the product of its length, its width, and its height.



$V = lwh$

Volume of a prism

The volume of any prism is the product of its base and its height.



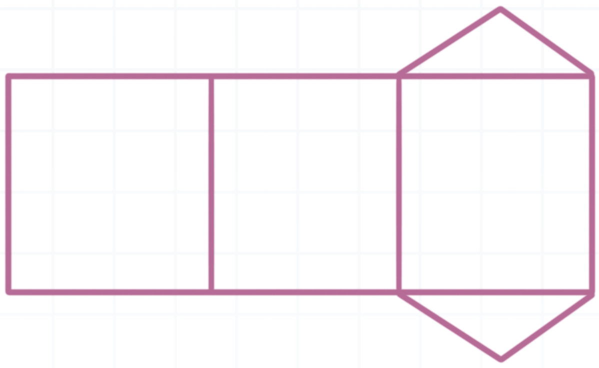
$V =$ (area of base)(height)

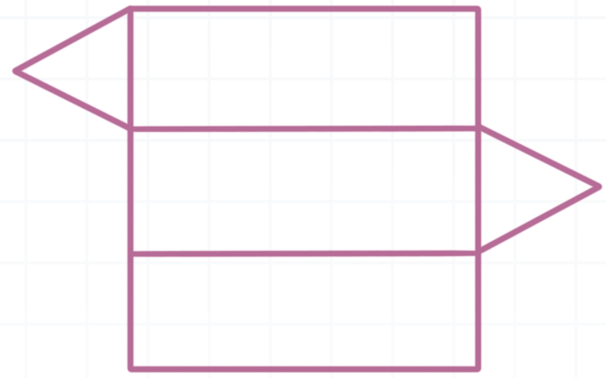
The volume of a triangular prism is the product of the area of one of its bases (one of its triangles) and its height (the length of any line segment that has one endpoint on each base and is perpendicular to both bases).

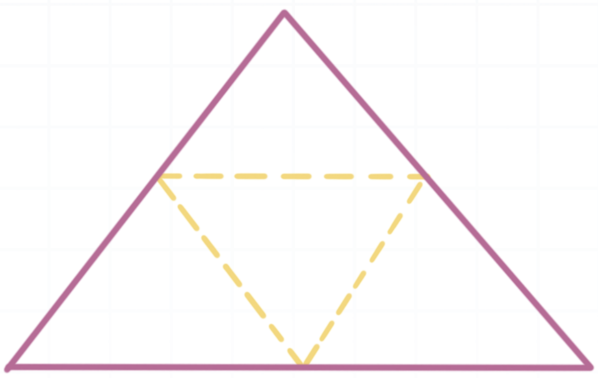
Let's start by working through an example.

Example

Which net does not belong to a prism?

A. 

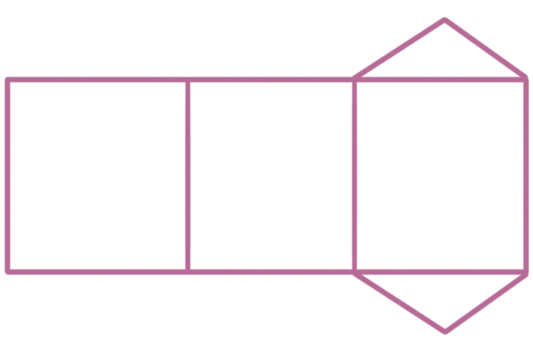
B. 

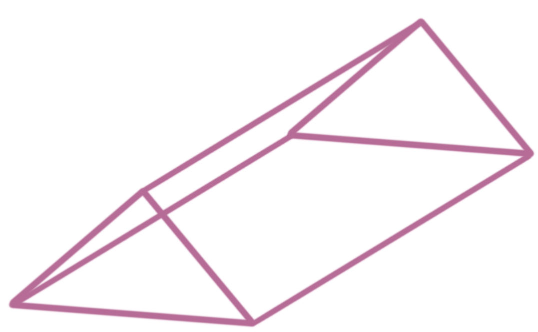
C. 

D. 

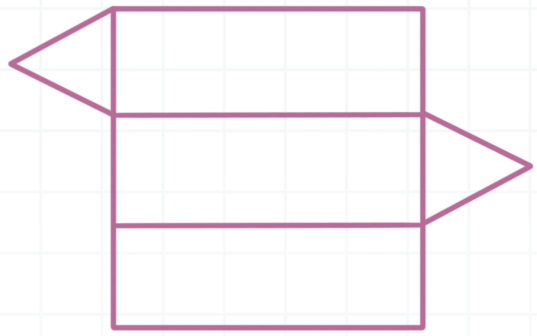
Net C is the net of a triangular pyramid. All of the other nets are nets of prisms, because they have one pair of congruent bases and all the other faces are rectangles.

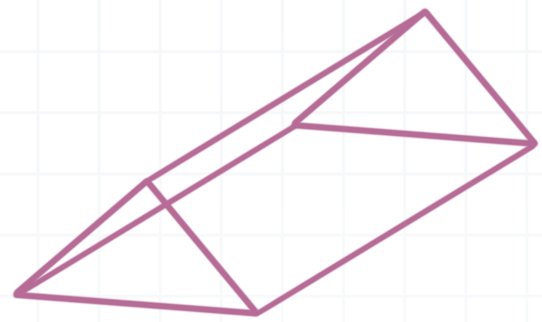
Net A is a net of a triangular prism because it has two triangular bases.



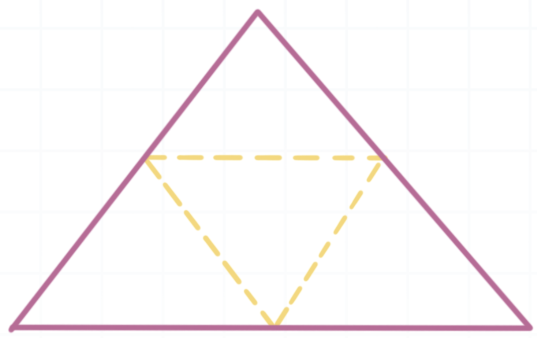


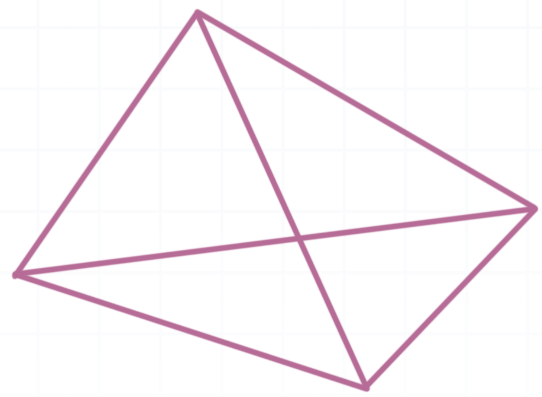
Net B is another example of a net of a triangular prism.



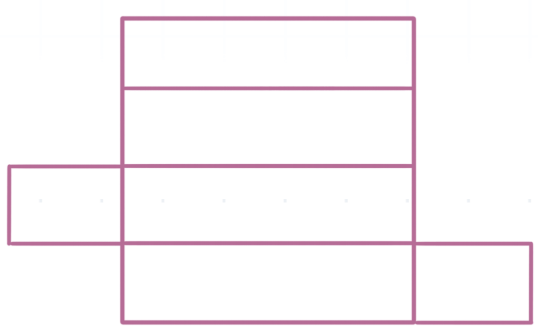


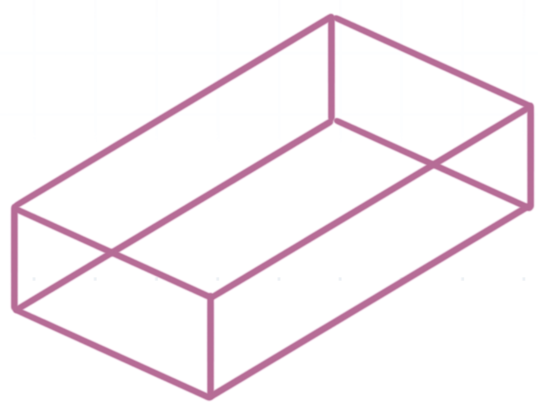
Net C is a net of a triangular pyramid and not a prism.





Net D is a net of a rectangular prism.

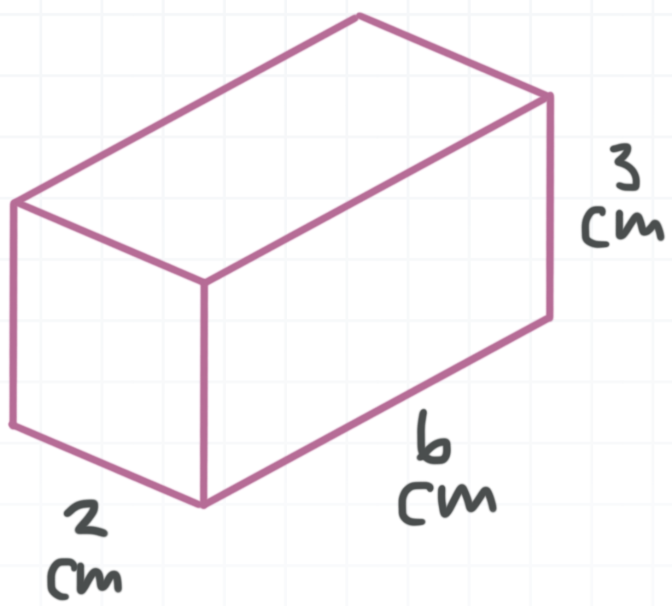




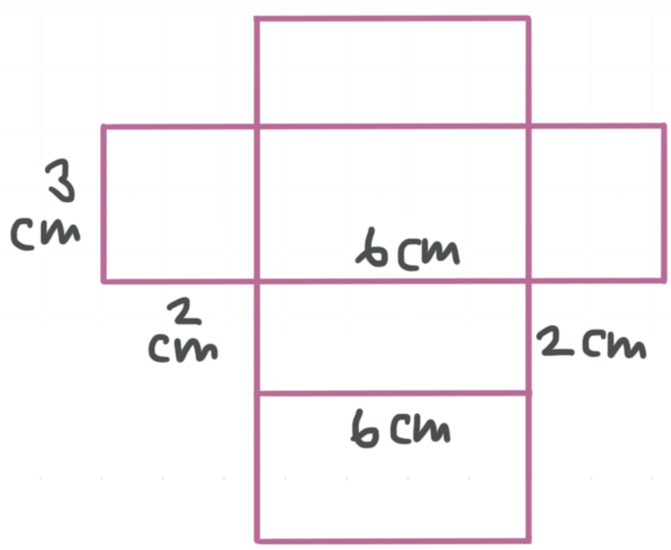
Let’s do an example of surface area.

Example

What is the surface area of the figure?



It can be helpful to draw a net of the figure to calculate the surface area.



Now we can see that we have three pairs of shapes. We can find the area of each and then add them.

$$A = 2[(2 \text{ cm})(3 \text{ cm})] + 2[(3 \text{ cm})(6 \text{ cm})] + 2[(2 \text{ cm})(6 \text{ cm})]$$

$$A = 2[(6 \text{ cm}^2)] + 2[(18 \text{ cm}^2)] + 2[(12 \text{ cm}^2)]$$

$$A = 12 \text{ cm}^2 + 36 \text{ cm}^2 + 24 \text{ cm}^2$$

$$A = 72 \text{ cm}^2$$

You can also think of the surface area of a rectangular box as the sum of the areas of its six sides.

Top and bottom	length x width
Left and right	width x height
Front and back	length x height

We'll use the surface area formula.

$$A = 2lw + 2wh + 2lh$$

Plugging in 6 cm for l, 2 cm for w, and 3 cm for h, we get

$$A = 2[(6\text{ cm})(2\text{ cm})] + 2[(2\text{ cm})(3\text{ cm})] + 2[(6\text{ cm})(3\text{ cm})]$$

$$A = 2[(12\text{ cm}^2)] + 2[(6\text{ cm}^2)] + 2[(18\text{ cm}^2)]$$

$$A = 24\text{ cm}^2 + 12\text{ cm}^2 + 36\text{ cm}^2$$

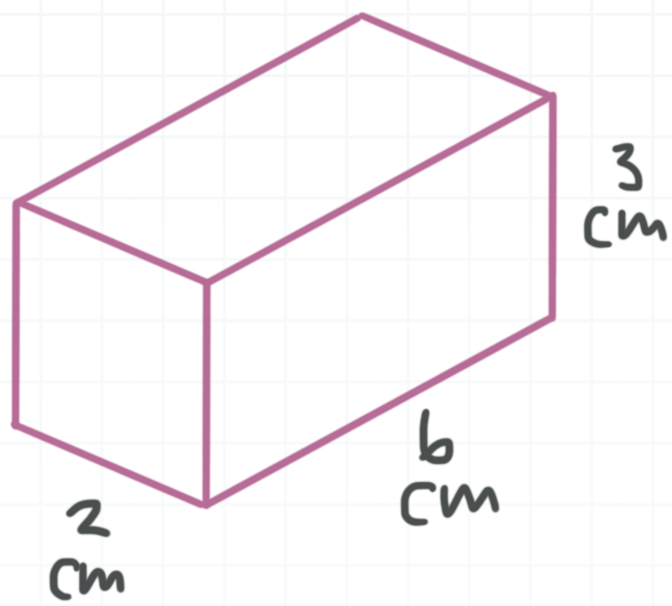
$$A = 72\text{ cm}^2$$

Which is what we got by using the net.

Let's do an example with volume.

Example

What is the volume of the figure?



To find the volume of the rectangular prism, multiply the length, width, and height.

$$V = lwh$$

$$V = (6 \text{ cm})(2 \text{ cm})(3 \text{ cm})$$

$$V = 36 \text{ cm}^3$$

Surface area to volume ratio of prisms

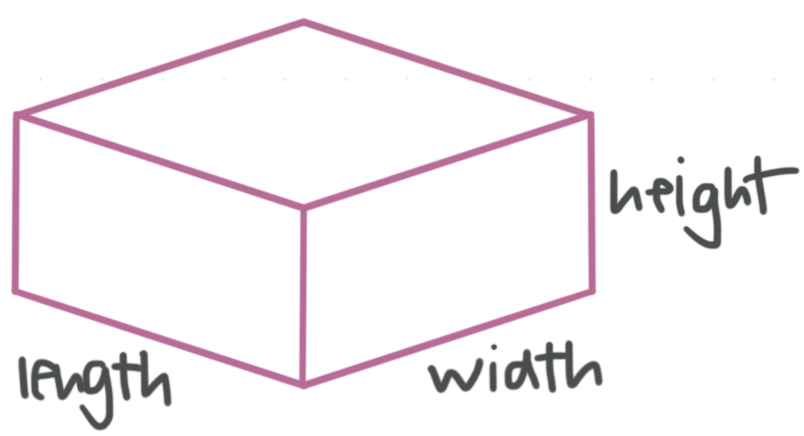
Remember that the formulas for the areas of the faces of a right rectangular prism (a prism in which all the faces are rectangles) are given as

Top and bottom	length x width
Left and right	width x height
Front and back	length x height

The formula for surface area of a rectangular prism (we'll call that area S, for "surface") is

$$S = 2lw + 2wh + 2lh$$

And the volume of a rectangular prism is the product of the length, the width, and the height.



$$V = lwh$$

Surface area to volume ratio

The ratio of the surface area, S, to the volume, V, can be expressed as a fraction S/V, or converted to a decimal. Since area is in units of length^2, and volume is in units of length^3, the ratio S/V will be in units of (length^2)/(length^3) = length^{-1}.

Let's start by working through an example.

Example

Calculate the surface area to volume ratio of a right rectangular prism that measures 4 cm high, 6 cm wide, and 8 cm long. Express your answer as a decimal rounded to the nearest tenth.

Use $S = 2lw + 2wh + 2lh$ to find the surface area.

$$S = 2(8 \text{ cm} \cdot 6 \text{ cm}) + 2(6 \text{ cm} \cdot 4 \text{ cm}) + 2(8 \text{ cm} \cdot 4 \text{ cm})$$

$$S = 2(48 \text{ cm}^2) + 2(24 \text{ cm}^2) + 2(32 \text{ cm}^2)$$

$$S = 96 \text{ cm}^2 + 48 \text{ cm}^2 + 64 \text{ cm}^2$$

$$S = 208 \text{ cm}^2$$

Use $V = lwh$ to find the volume.

$$V = (8 \text{ cm}) \cdot (6 \text{ cm}) \cdot (4 \text{ cm})$$

$$V = 192 \text{ cm}^3$$

Now find the ratio of surface area to volume.

$$\frac{S}{V} = \frac{208\text{ cm}^2}{192\text{ cm}^3} = \frac{13}{12}\text{ cm}^{-1}$$

As a decimal rounded to the nearest tenth, the ratio is 1.1 cm^{-1}.

Let's do another example.

Example

The surface area to volume ratio of a right rectangular prism is $(11/20)\text{ feet}^{-1}$. The volume of the prism is $1{,}140\text{ feet}^3$. What is the prism's surface area?

We know that

$$\frac{S}{V} = \frac{11}{20}\text{ feet}^{-1}$$

and that the volume is $V = 1{,}140\text{ feet}^3$. Therefore,

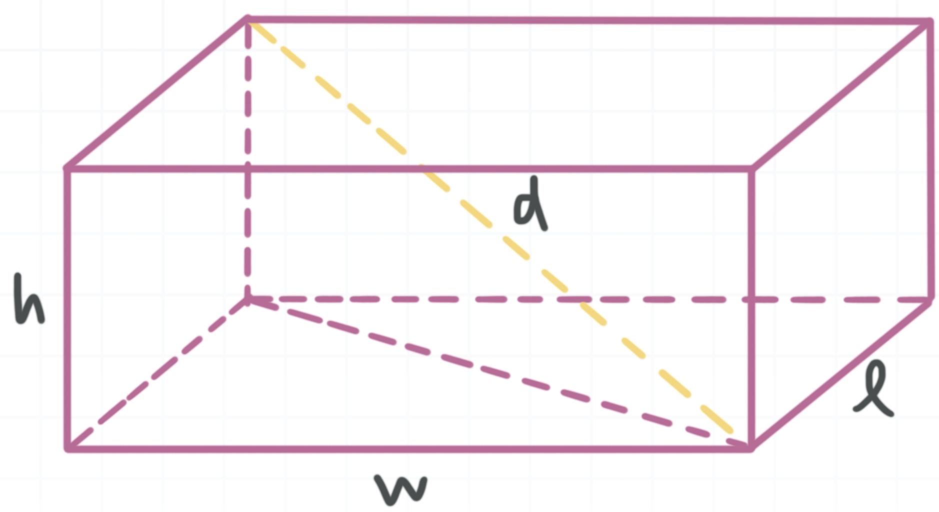
$$\frac{S}{1{,}140\text{ feet}^3} = \frac{11}{20}\text{ feet}^{-1}$$

$$\left(\frac{S}{1{,}140\text{ feet}^3}\right)(1{,}140\text{ feet}^3) = \left(\frac{11}{20}\text{ feet}^{-1}\right)(1{,}140\text{ feet}^3)$$

$$S = 627\text{ feet}^2$$

Diagonal of a right rectangular prism

The diagonal of a right rectangular prism goes from one corner of the prism, across the interior of it, all the way to the opposite corner.



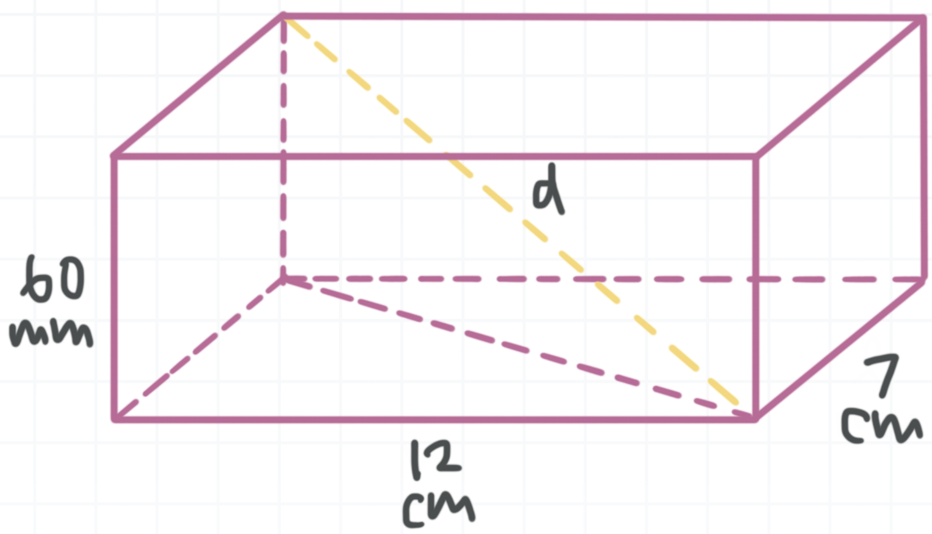
You can find the length of a diagonal of a right rectangular prism using

$$d = \sqrt{l^2 + w^2 + h^2}$$

where d is the length of the diagonal, and l, w, and h are the length, width, and height, respectively. Let's start by working through an example.

Example

What is the length of the diagonal of the right rectangular prism?



Not all of the dimensions here are given in the same units. Change 60 mm to centimeters first.

There are $1,000$ mm in 1 m, and there are 100 cm in 1 m. Since both 100 cm and $1,000$ mm are equal to 1 m, they're equal to each other, so

$$100 \text{ cm} = 1,000 \text{ mm}$$

Dividing both sides of this equation by $1,000$ mm, we get

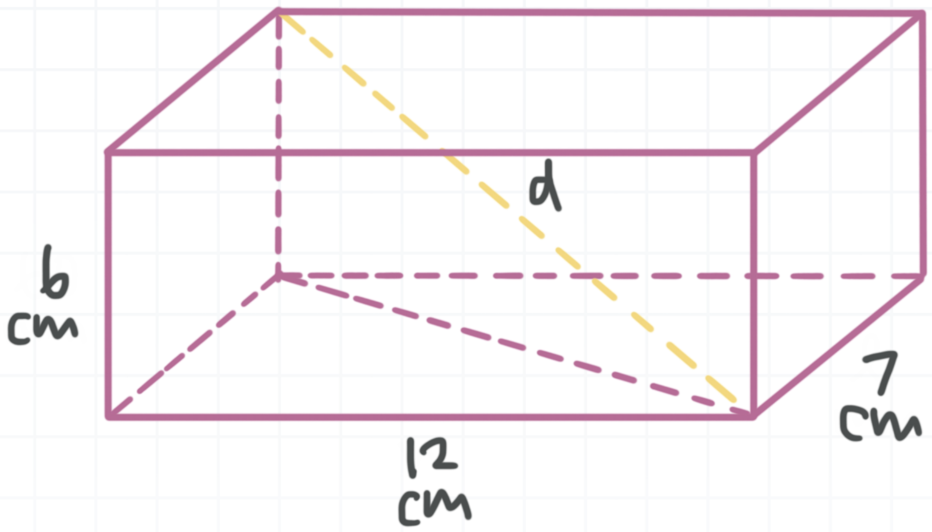
$$\frac{100 \text{ cm}}{1,000 \text{ mm}} = \frac{1,000 \text{ mm}}{1,000 \text{ mm}}$$

$$\frac{1 \text{ cm}}{10 \text{ mm}} = 1$$

Therefore, the conversion factor is going to be $(1 \text{ cm})/(10 \text{ mm})$, so

$$60 \text{ mm} = 60 \text{ mm} \cdot \frac{1 \text{ cm}}{10 \text{ mm}} = 6 \text{ cm}$$

Then the dimensions are



Plugging these into the formula for the diagonal, we get

$$d = \sqrt{l^2 + w^2 + h^2}$$

$$d = \sqrt{(7 \text{ cm})^2 + (12 \text{ cm})^2 + (6 \text{ cm})^2}$$

$$d = \sqrt{49 \text{ cm}^2 + 144 \text{ cm}^2 + 36 \text{ cm}^2}$$

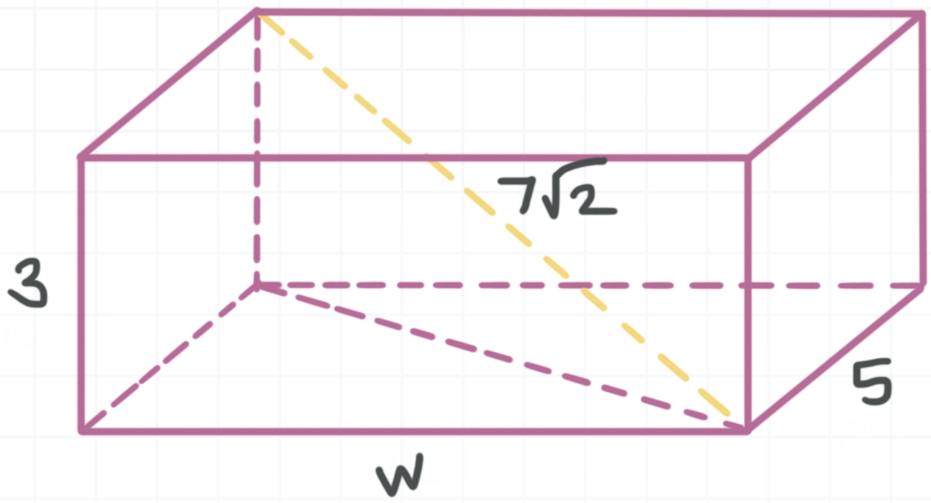
$$d = \sqrt{229 \text{ cm}^2}$$

$$d = \sqrt{229} \text{ cm}$$

Let's try another one.

Example

Find the width of the right rectangular prism.



We just need to plug the dimensions we've been given into the formula for the diagonal.

$$d = \sqrt{l^2 + w^2 + h^2}$$

$$7\sqrt{2} = \sqrt{5^2 + w^2 + 3^2}$$

Manipulate the equation to solve for w, starting with squaring both sides.

$$(7\sqrt{2})^2 = (\sqrt{5^2 + w^2 + 3^2})^2$$

$$(49)(2) = 5^2 + w^2 + 3^2$$

$$98 = 25 + w^2 + 9$$

$$98 = 34 + w^2$$

$$64 = w^2$$

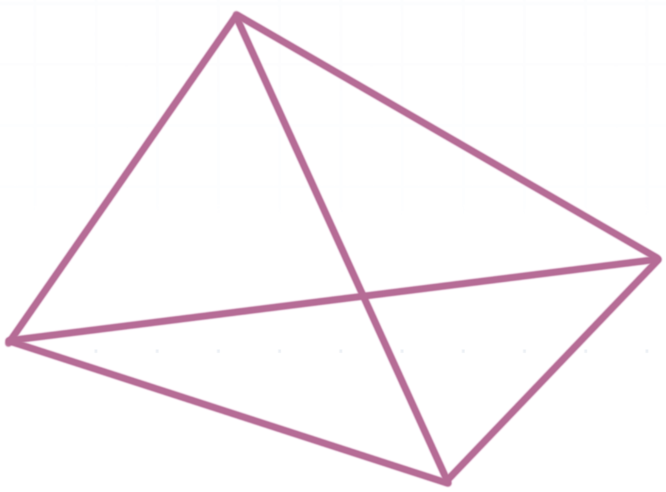
$$8 = w$$

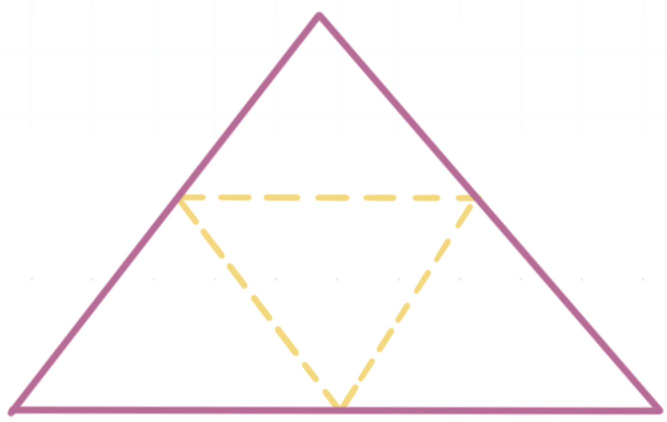
Nets/volume/surface area of pyramids

A **pyramid** has one base, in the shape of any polygon, and the rest of the faces are triangles. Each side of the base coincides with a side of exactly one of the triangular faces, and there's exactly one point of a pyramid at which all of the triangular faces intersect.

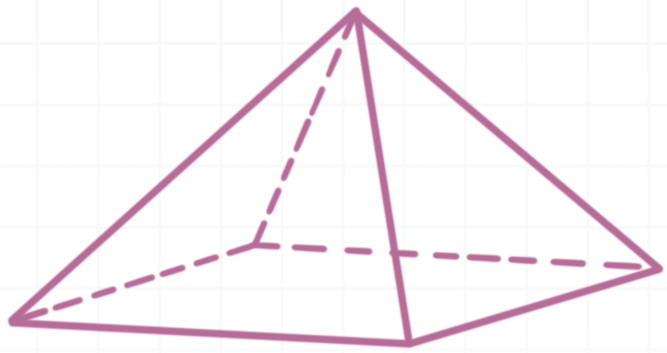
A pyramid is named by the shape of its base. So a **triangular pyramid** is a pyramid with a triangular base, a **rectangular pyramid** is a pyramid with a rectangular base, a **square pyramid** is a pyramid with a square base, etc.

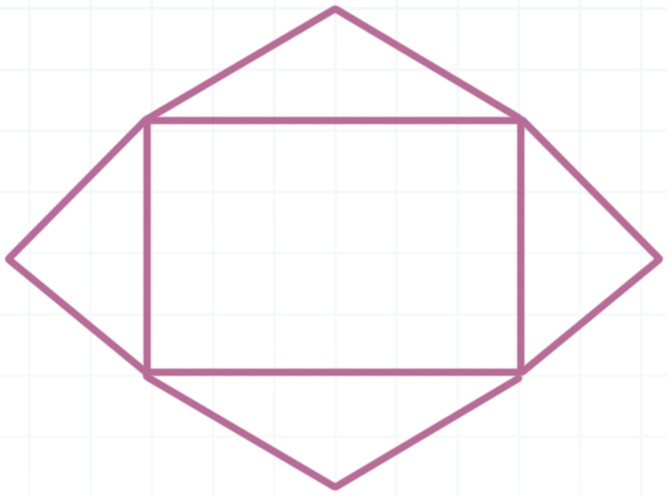
This is a triangular pyramid and its net. The net of a triangular pyramid has three triangular faces and one triangular base.





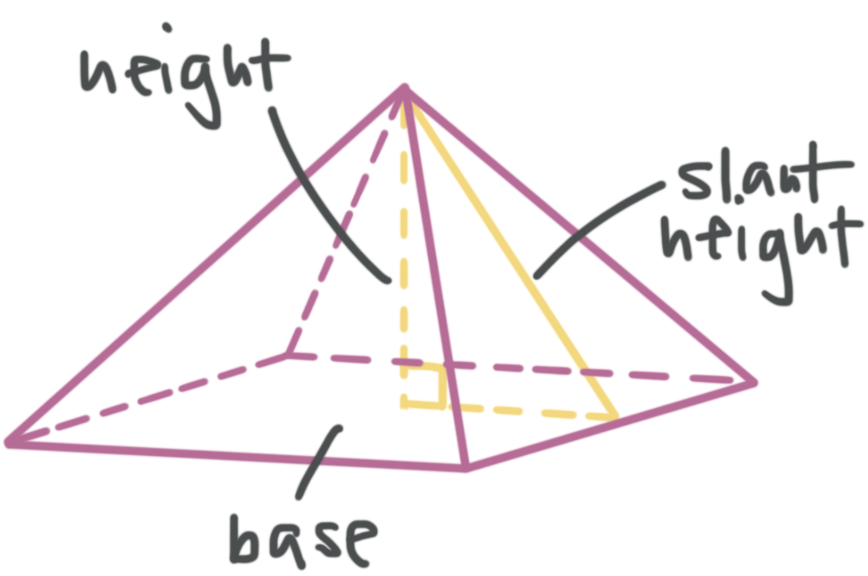
This is a rectangular pyramid and its net. The net of a rectangular pyramid has four triangular faces and one rectangular base.





You'll need to know the names for the different parts of a pyramid. The **apex** of a pyramid is the point at which all the triangular faces intersect. A **right pyramid** (the only kind of pyramid we're dealing with in this lesson) is one in which the line segment from the apex to the center of the base is perpendicular to the base.

The **height** of a right pyramid is the length of that line segment. A **right regular pyramid** is one in which all the triangular faces are congruent. The **slant height** of a right regular pyramid, which is the length of a line segment from the apex of they pyramid to the midpoint of any side of the base, is also called the **lateral height**, and is often represented in formulas with the variable l.



Volume and surface area

The volume of a pyramid is given by

$$V = \frac{1}{3}Bh$$

where V is the volume of the pyramid, B is the area of the base of the pyramid, and h is the height of the pyramid.

A pyramid has a surface area given by

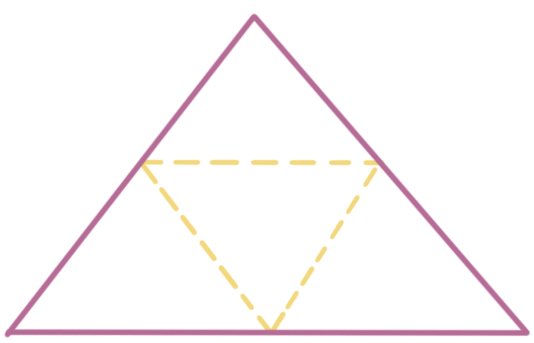
$$S = \frac{1}{2}lp + B$$

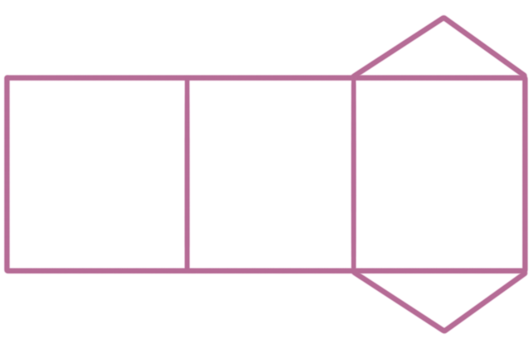
where p is the perimeter of the base, l is the slant height of the pyramid, and B is the area of the base.

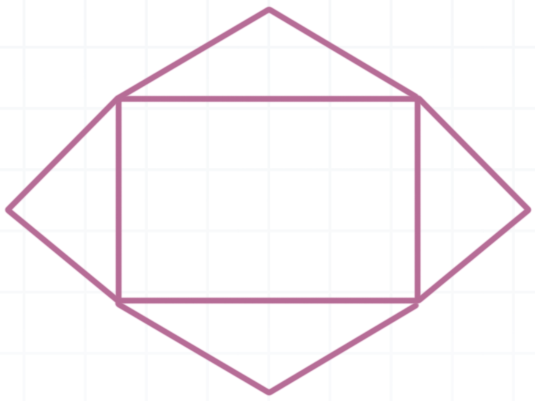
Let's work through a few examples.

Example

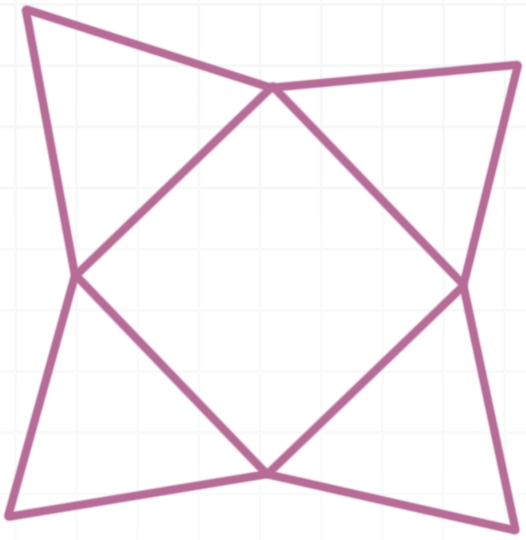
Which net does not belong to a pyramid?

A. 

B. 

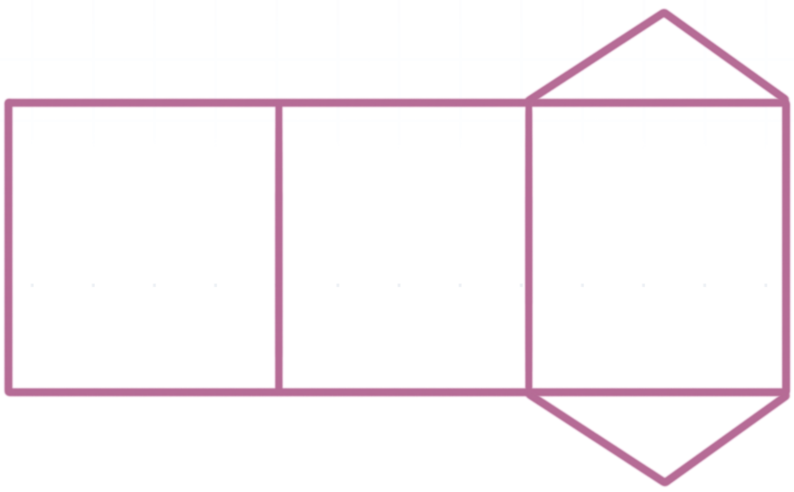


C.



D.

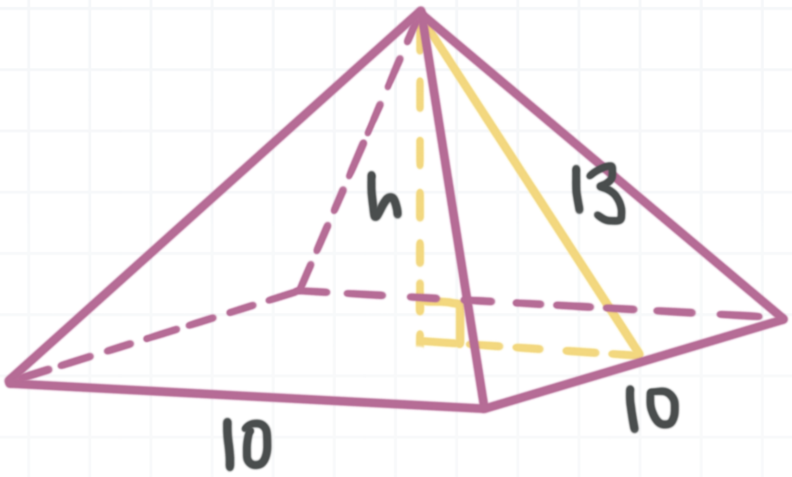
Net B can't be a net of a pyramid, because it has only two triangular faces. The base of a pyramid is a polygon, so the base has at least three sides. Each side of the base of a pyramid coincides with a side of exactly one of the triangular faces, so a pyramid has at least three triangular faces.



Let's look at another example.

Example

What is the volume of the square pyramid, which has a 10×10 base and a lateral height of 13?



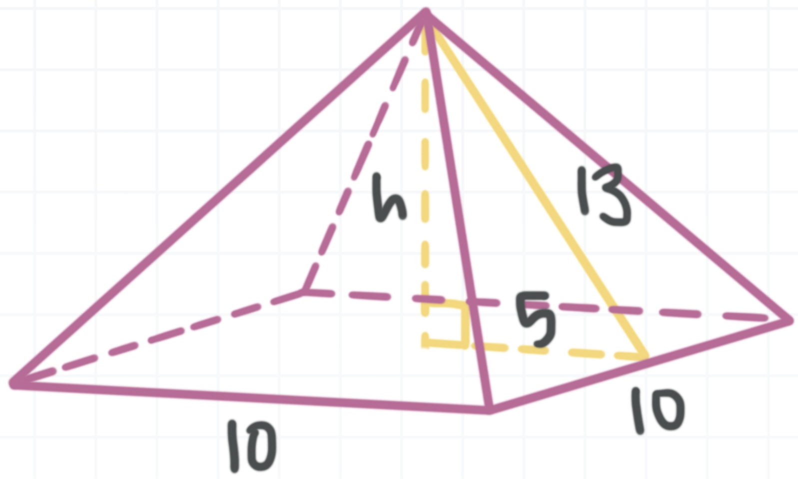
The volume of a pyramid is

$$V = \frac{1}{3}Bh$$

The area of the square base is $B = (10)(10) = 100$, and we need to use the Pythagorean Theorem to find the height of the pyramid.

To find the height h (the length of the line segment from the apex of the pyramid to the center of the base), we can use a right triangle that has that line segment as one of its legs, and where the second leg is a line segment from the center of the base of the pyramid to the midpoint of a side of its base.

The length of a side of the base of this pyramid is 10, so the length of the second leg of the right triangle is 5. Then the length of the hypotenuse of the right triangle is equal to the lateral height l of the pyramid, which is 13.



$$5^2 + h^2 = 13^2$$

$$25 + h^2 = 169$$

$$h^2 = 144$$

$$h = 12$$

Now we'll plug the values of B and h into the volume formula.

$$V = \frac{1}{3}Bh$$

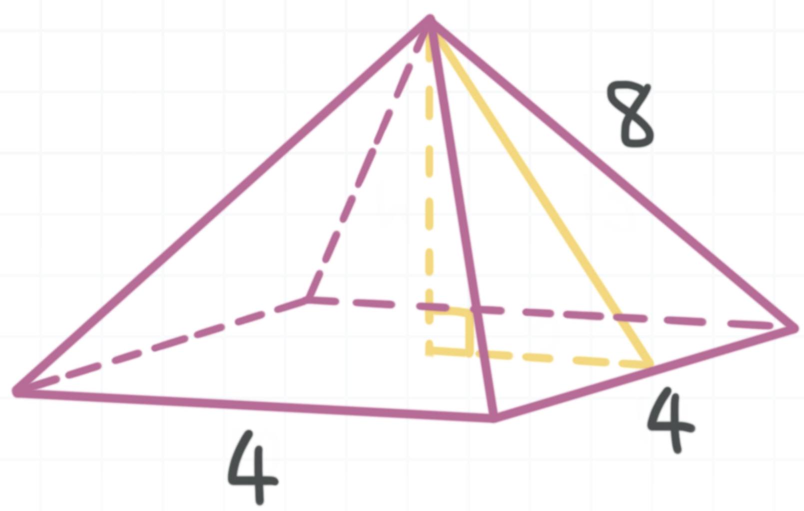
$$V = \frac{1}{3}(100)(12)$$

$$V = 400$$

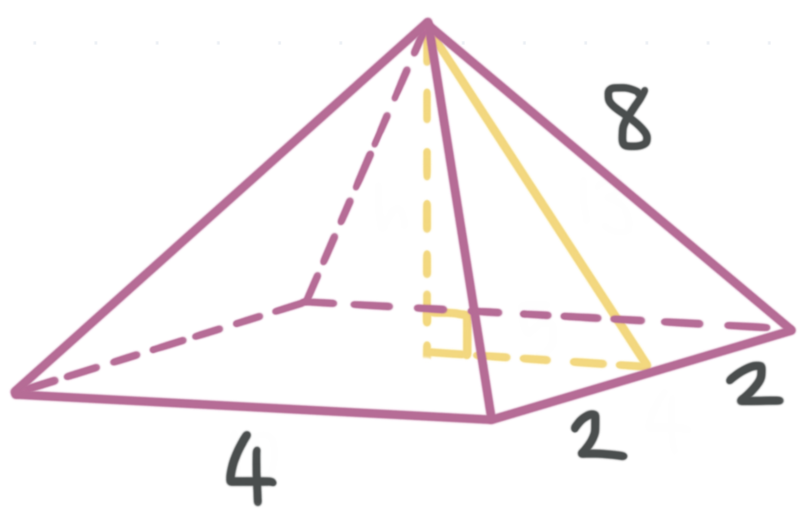
Let's do a problem with surface area.

Example

What is the surface area of a square pyramid with a base of 4 cm by 4 cm if the length of a line segment from the apex of the pyramid to a vertex of its base (the length of an edge of the pyramid) is 8 cm?



First we need to find the lateral height by using the Pythagorean theorem, applied to the right triangle that's laying on the face of the pyramid, formed by the slant height down the center of the face, the slant down the edge of the pyramid, and half of the edge of the base.



$$(2\text{ cm})^2 + l^2 = (8\text{ cm})^2$$

$$4\text{ cm}^2 + l^2 = 64\text{ cm}^2$$

$$l^2 = 60 \text{ cm}^2$$

$$l = \sqrt{60 \text{ cm}^2}$$

$$l = \sqrt{4 \cdot 15 \text{ cm}^2}$$

$$l = 2\sqrt{15} \text{ cm}$$

Since this is a square pyramid, the perimeter of the base is four times the length of a side of the base:

$$p = 4(4 \text{ cm}) = 16 \text{ cm}$$

And the area of the base is

$$B = (4 \text{ cm})^2 = 16 \text{ cm}^2$$

Now we'll plug everything into the surface area formula.

$$S = \frac{1}{2}lp + B$$

$$S = \frac{1}{2}(2\sqrt{15} \text{ cm})(16 \text{ cm}) + 16 \text{ cm}^2$$

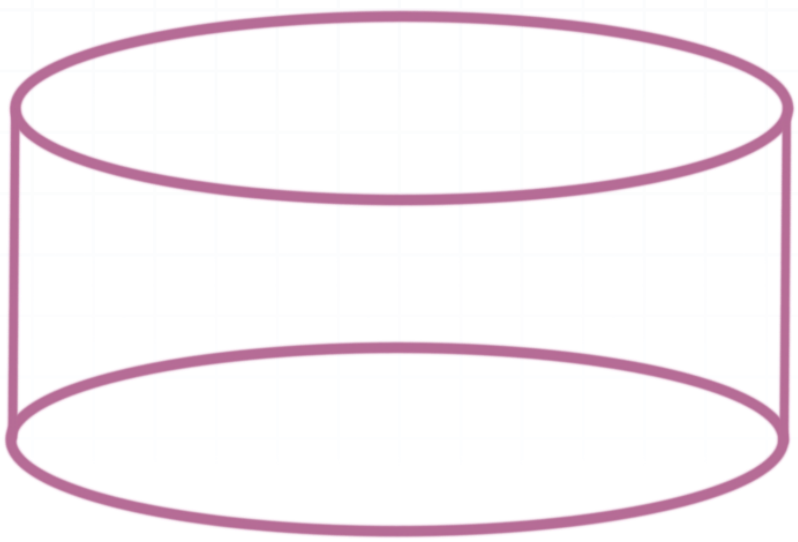
$$S = (16\sqrt{15} + 16) \text{ cm}^2$$

Nets/volume/surface area of cylinders

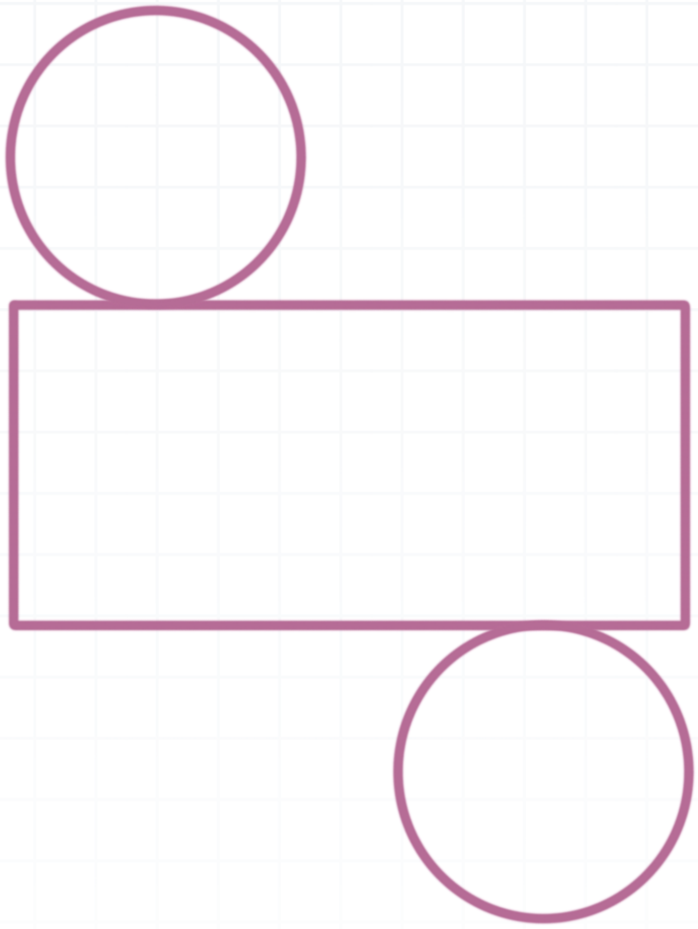
In this lesson we'll look at the nets, volume, and surface area of cylinders.

Cylinders

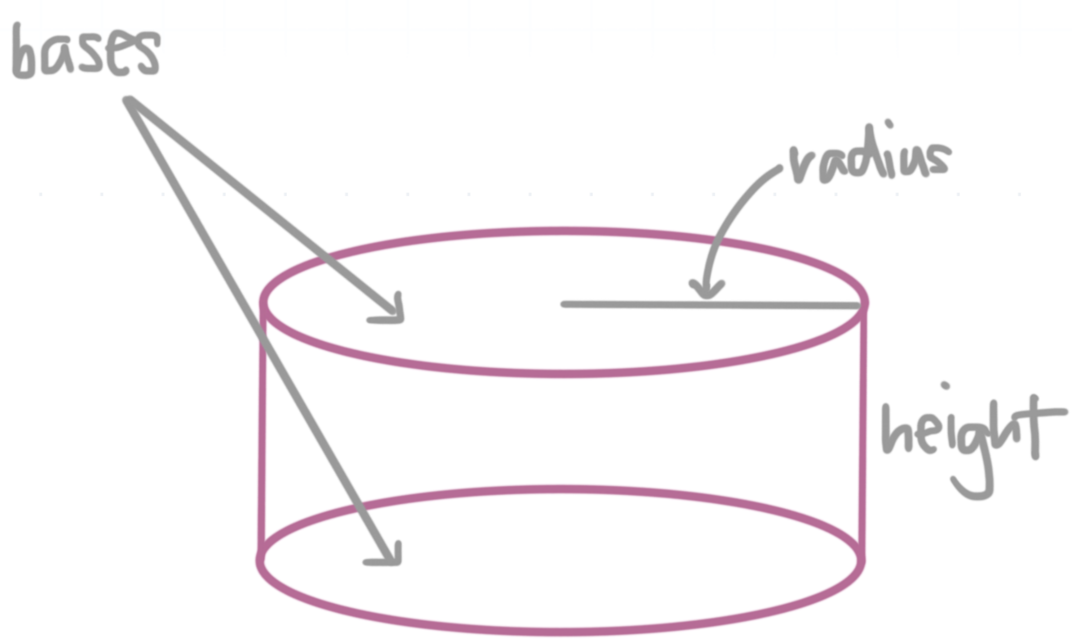
A **right circular cylinder** (the only kind of cylinder we're dealing with in this lesson) has a pair of parallel, congruent circular bases.



The net of a cylinder looks like a rectangle with two circles attached at opposite ends.



We also define a base radius for the cylinder as the radius of the base, and the height of the cylinder as the distance between the bases.



Volume and surface area

The volume of a cylinder is the product of π, the square of the radius, and the height of the cylinder. Sometimes we use the estimated value of $\pi \approx 3.14$, and sometimes we use the symbol π to represent the exact value.

$$V = \pi r^2 h$$

And the surface area of a cylinder is given by

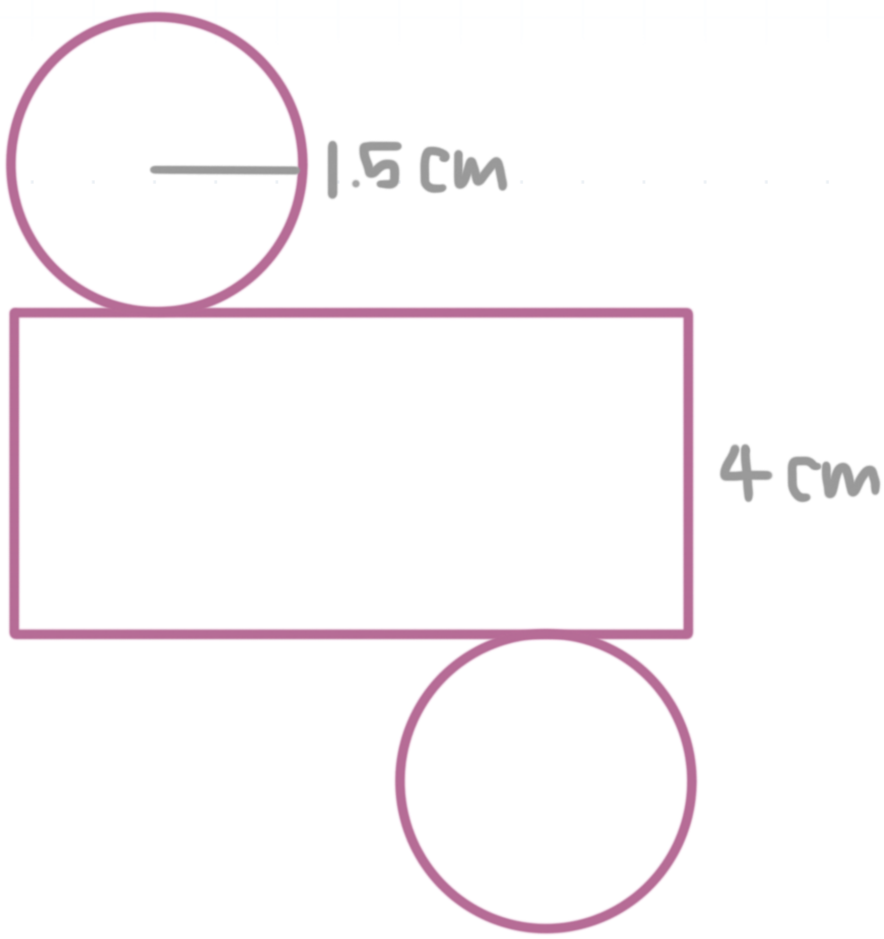
$$S = 2\pi rh + 2\pi r^2$$

where r is the radius of the cylinder and h is the height of the cylinder.

Let's do a few examples.

Example

What is the area of the rectangle shown in the net?



The area of the rectangle in the net of a cylinder is the product of the circumference of the circle (which is the length of the horizontal dimension of the rectangle in this net) and the height of the cylinder (which is the length of the vertical dimension of the rectangle in this net). The circumference of a circle is $C = 2\pi r$, so the area of the rectangle is $A = 2\pi rh$. You'll notice this shows up in the first part of the surface area formula for a cylinder.

$$S = 2\pi rh + 2\pi r^2$$

Area of the rectangle

Area of the circles

Let's go ahead and calculate the area.

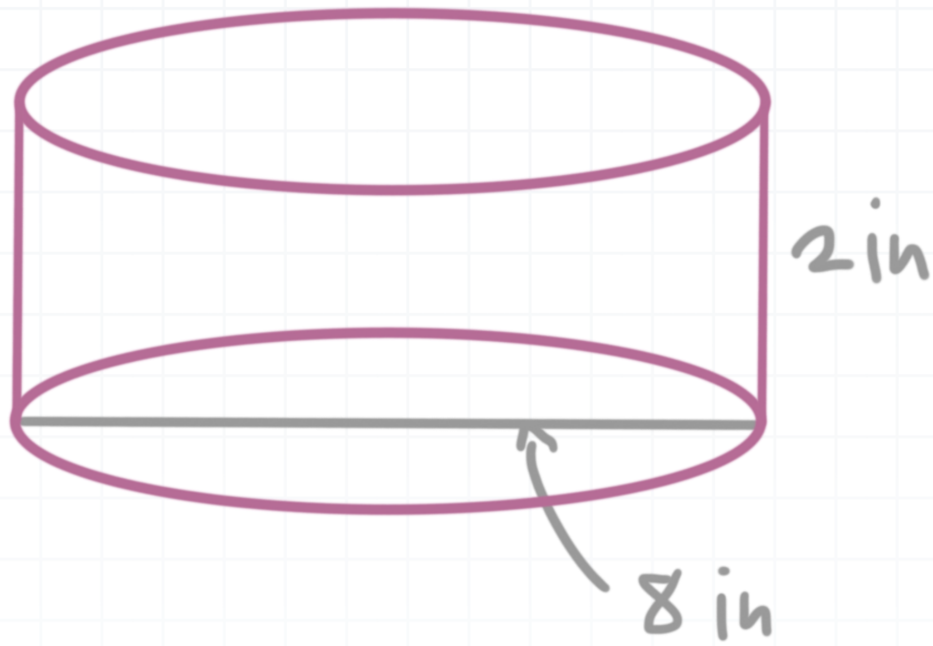
$$A = 2\pi(1.5 \text{ cm})(4 \text{ cm})$$

$$A = 12\pi \text{ cm}^2$$

Let's do a volume problem.

Example

What is the volume of the cylinder, assuming $\pi \approx 3.14$.



Use the formula for volume.

$$V = \pi r^2 h$$

The diameter of the cylinder is 8, so we need to divide it by 2 to get the radius.

$$r = \frac{d}{2} = \frac{8 \text{ in}}{2} = 4 \text{ in}$$

Plugging in the dimensions of the cylinder, we get

$$V \approx 3.14(4 \text{ in})^2(2 \text{ in})$$

$$V \approx 3.14(16 \text{ in}^2)(2 \text{ in})$$

$$V \approx 100.48 \text{ in}^3$$

Let's do a surface area problem.

Example

A cylinder has a radius of 12 ft and a surface area of $1{,}356.48$ ft^2. What is the height of the cylinder, assuming $\pi \approx 3.14$?

We'll plug what we know into the surface area formula.

$$S = 2\pi rh + 2\pi r^2$$

$$1{,}356.48 \text{ ft}^2 = 2(3.14)(12 \text{ ft})(h) + 2(3.14)(12 \text{ ft})^2$$

Now we can solve for the height.

$$1{,}356.48 \text{ ft}^2 = (75.36 \text{ ft})(h) + 904.32 \text{ ft}^2$$

$$452.16 \text{ ft}^2 = (75.36 \text{ ft})(h)$$

$$\frac{452.16 \text{ ft}^2}{75.36 \text{ ft}} = h$$

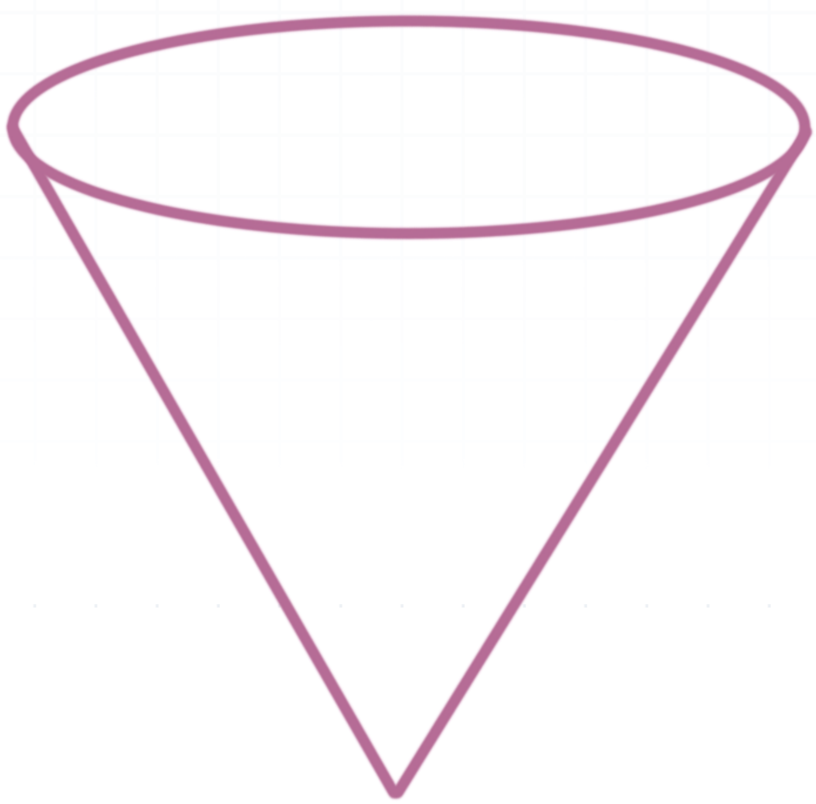
$$h = 6 \text{ ft}$$

Nets/volume/surface area of cones

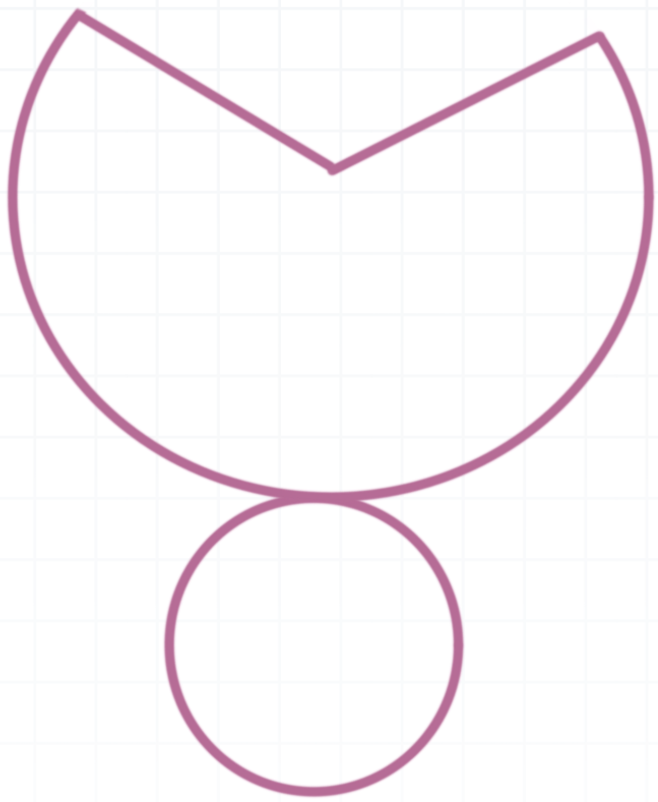
In this lesson we'll look at the nets, volume, and surface area of cones.

Cones

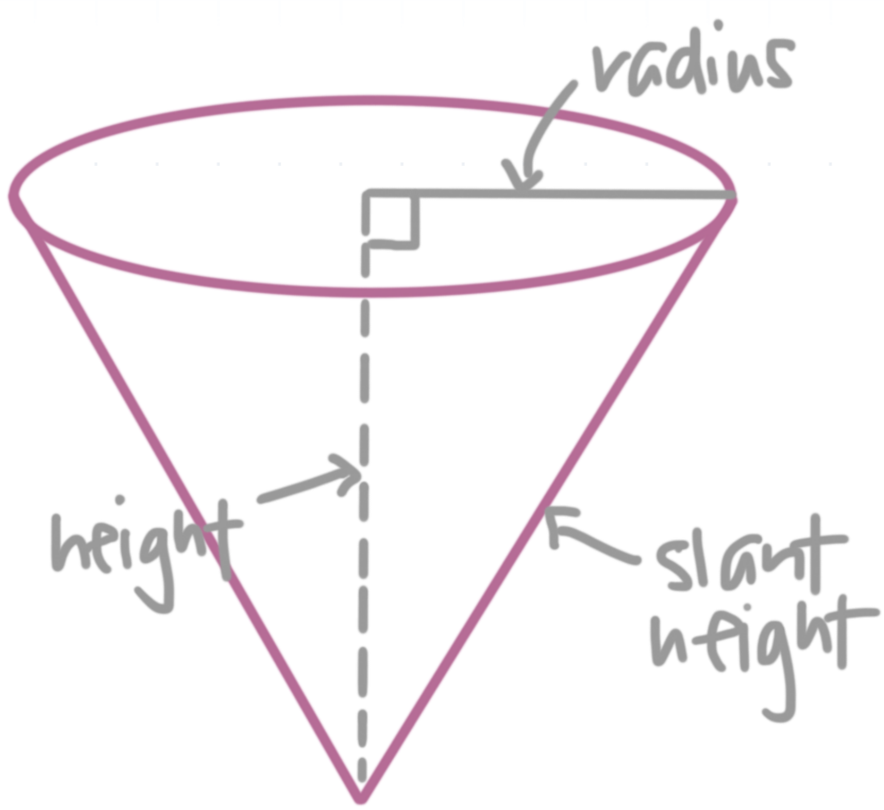
A **right circular cone** (the only kind of cone we're dealing with in this lesson) has a circular base.



The net of a cone consists of a full circle for the base and part of another circle for the wall of the cone.



We'll also need to know the base radius, which is the radius of the base of the cone, the height, which is the distance from the base to the tip of the cone, and the slant height, which is the length along the side of the cone from the base to the tip.



The radius r, height h, and slant height l are all related to one another by the Pythagorean theorem.

$$r^2 + h^2 = l^2$$

Volume and surface area

The volume of a cone is given by

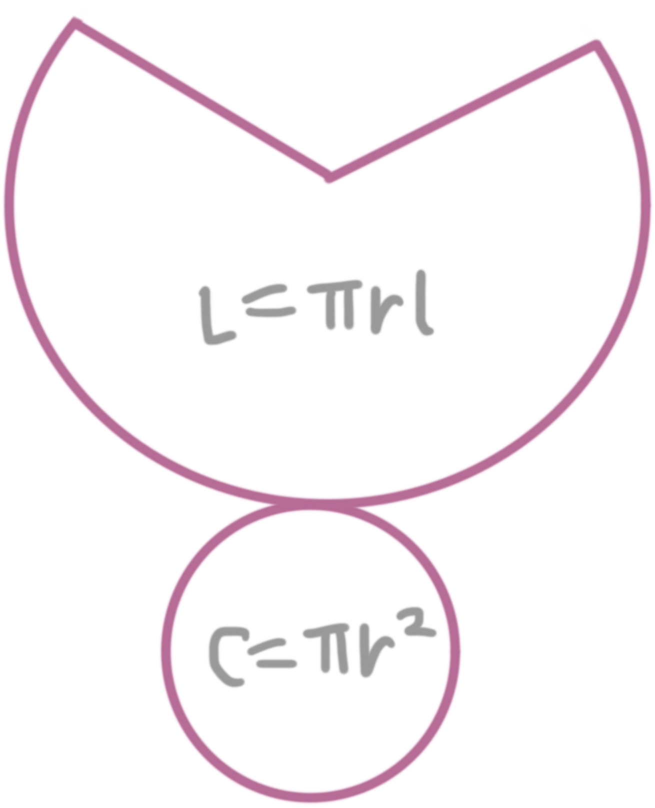
$$V = \frac{1}{3}\pi r^2 h$$

where r is the radius and h is the height of the cone.

The surface area of a cone is given by

$$S = \pi rl + \pi r^2$$

where r is the radius and l is the slant height of the cone.



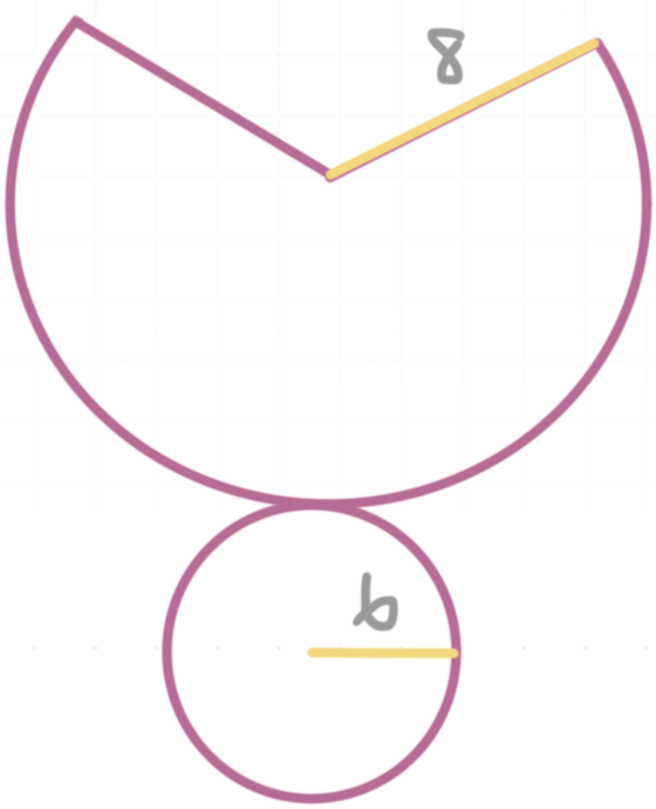
L is called the lateral area, so we can also write the surface area of the cone as

$$S = L + \pi r^2$$

Let's do a few examples.

Example

Find the surface area of the cone that's represented by the net.



The formula for the surface area of a cone is

$$S = \pi r l + \pi r^2$$

In this case, the slant height is $l = 8$ and the radius is $r = 6$. Plugging these into the formula, we get

$$S = \pi(6)(8) + \pi(6)^2$$

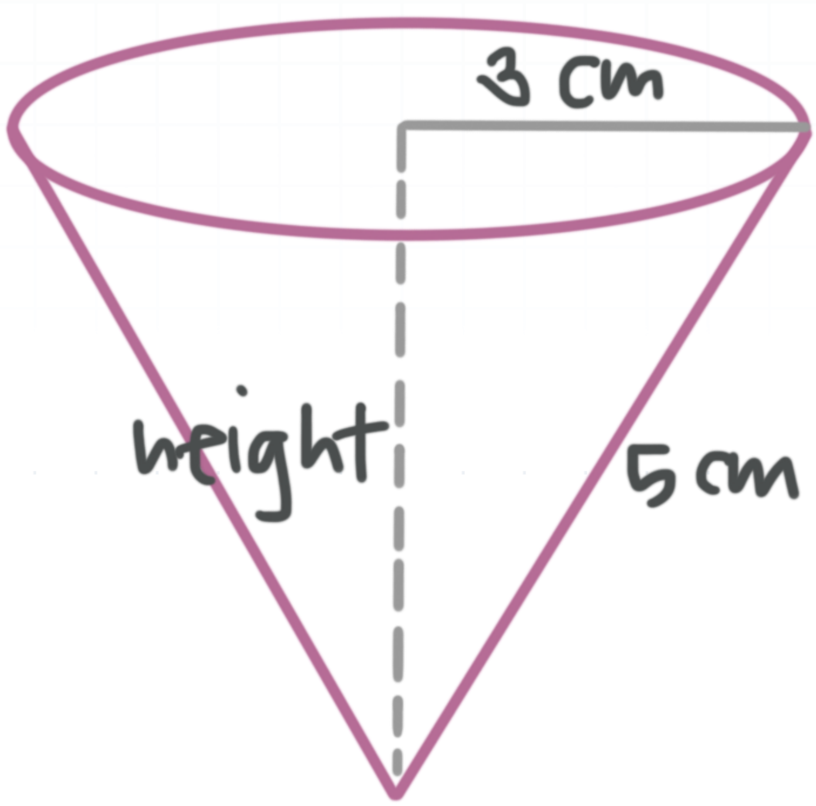
$$S = 48\pi + 36\pi$$

$$S = 84\pi$$

Let's do a few more examples.

Example

What is the volume of the cone?



The formula for volume is

$$V = \frac{1}{3}\pi r^2 h$$

We already know the radius of the cone is 3 cm, and we need to find the height of the cone in order to find the volume. Remember, the radius, height, and slant height are related by the Pythagorean Theorem.

$$r^2 + h^2 = l^2$$

Plugging in, we get

$$(3 \text{ cm})^2 + h^2 = (5 \text{ cm})^2$$

$$9 \text{ cm}^2 + h^2 = 25 \text{ cm}^2$$

$$h^2 = 16 \text{ cm}^2$$

$$h = 4 \text{ cm}$$

Now we can use the volume formula.

$$V = \frac{1}{3}\pi r^2 h$$

$$V = \frac{1}{3}\pi(3 \text{ cm})^2(4 \text{ cm})$$

$$V = \frac{1}{3}\pi(9 \text{ cm}^2)(4 \text{ cm})$$

$$V = 12\pi \text{ cm}^3$$

Let's do one more.

Example

What is the surface area of a cone with a slant height of 16.8 cm and a diameter of 16 cm? Use $\pi = 3.14$.

The formula for the surface area is

$$S = \pi rl + \pi r^2$$

The slant height is $l = 16.8$ cm. We can use the diameter to find the radius.

$$r = \frac{d}{2} = \frac{16 \text{ cm}}{2} = 8 \text{ cm}$$

The radius is $r = 8$. Plug the values of r and l into the formula for surface area.

$$S = \pi rl + \pi r^2$$

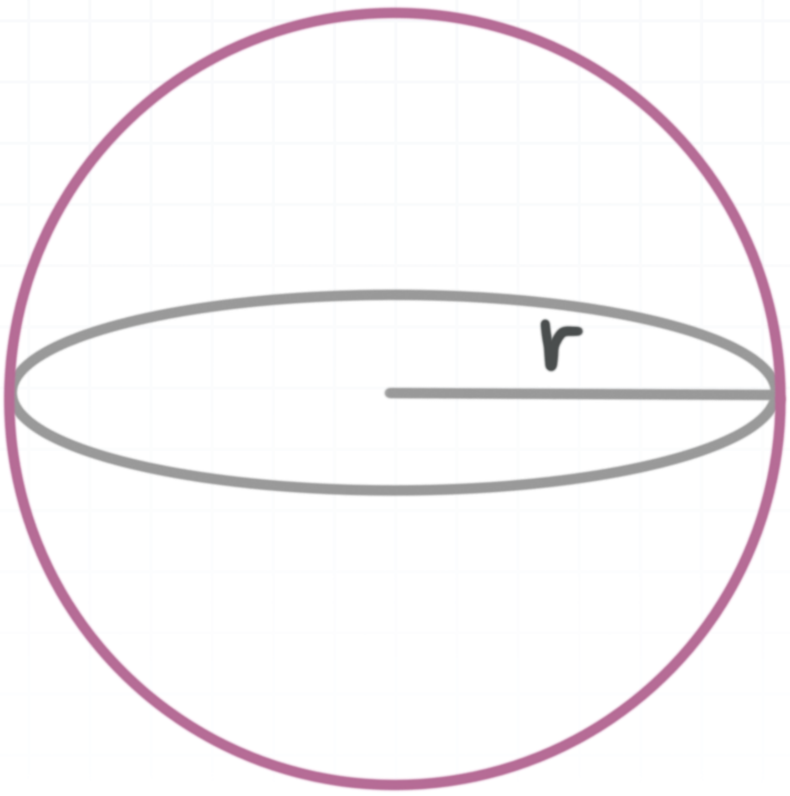
$$S = 3.14(8 \text{ cm})(16.8 \text{ cm}) + 3.14(8 \text{ cm})^2$$

$$S = 422.016 \text{ cm}^2 + 200.960 \text{ cm}^2$$

$$S = 622.976 \text{ cm}^2$$

Volume/surface area of spheres

In this lesson we'll look at the volume and surface area of spheres. A **sphere** is a perfectly round ball; it's the three-dimensional version of a circle.



Volume and surface area

The volume of a sphere is given by

$$V = \frac{4}{3}\pi r^3$$

The symbol π is used for exact answers and $\pi \approx 3.14$ is used for approximate answers.

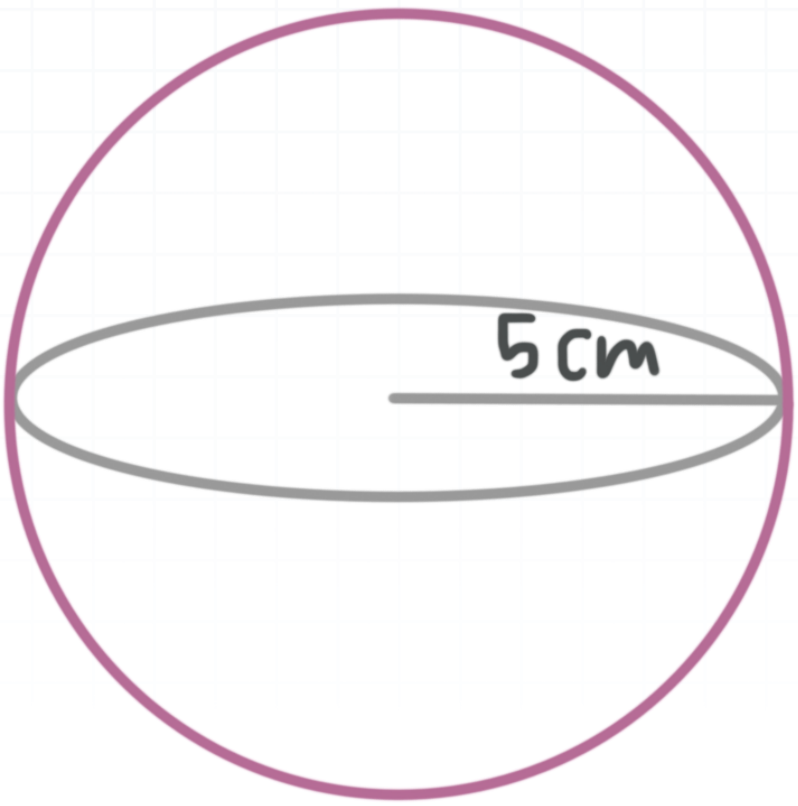
The surface area of a sphere is given by

$$S = 4\pi r^2$$

Let's do a few examples.

Example

What is the surface area of the sphere?



Use the formula for surface area, and plug in the value of the radius.

$$S = 4\pi r^2$$

$$S = 4\pi(5 \text{ cm})^2$$

$$S = 100\pi \text{ cm}^2$$

Let's try one with volume.

Example

What is the volume of a sphere with a diameter of 50 cm?

The formula for volume is

$$V = \frac{4}{3}\pi r^3$$

We're given the diameter, so we need to divide by 2 to get the radius.

$$r = \frac{d}{2} = \frac{50 \text{ cm}}{2} = 25 \text{ cm}$$

Plugging into the formula for volume, we get

$$V = \frac{4}{3}(3.14)(25 \text{ cm})^3$$

$$V = \frac{4}{3}(3.14)(15{,}625 \text{ cm})^3$$

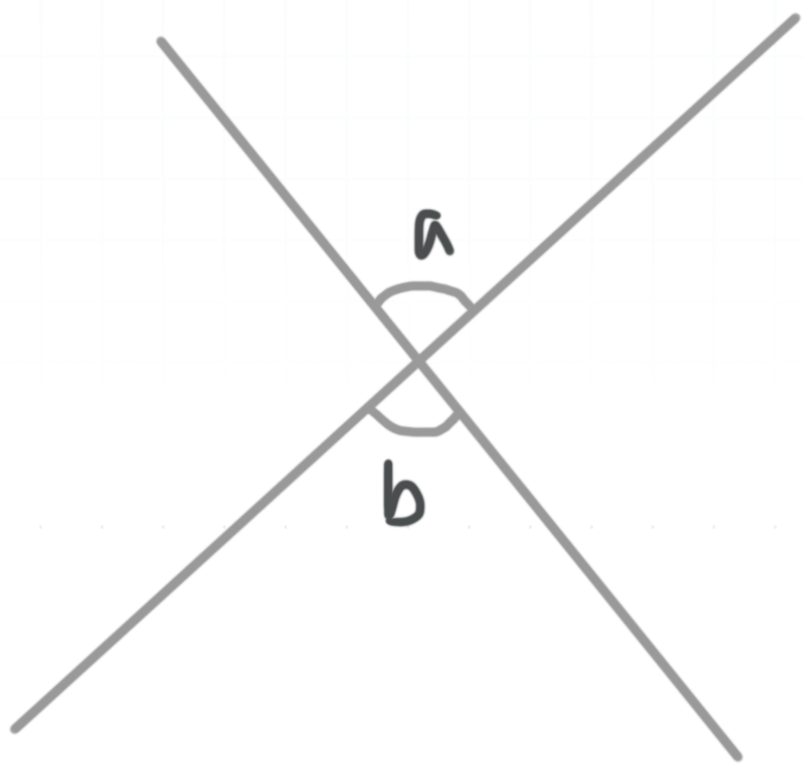
$$V = 65{,}416.67 \text{ cm}^3$$

Congruent angles

In this lesson we'll look at how to use vertical angles to solve problems.

Vertical angles

Vertical angles are a pair of angles that share a vertex and whose rays lie on the same pair of straight lines (but point in opposite directions from the common vertex), like these:

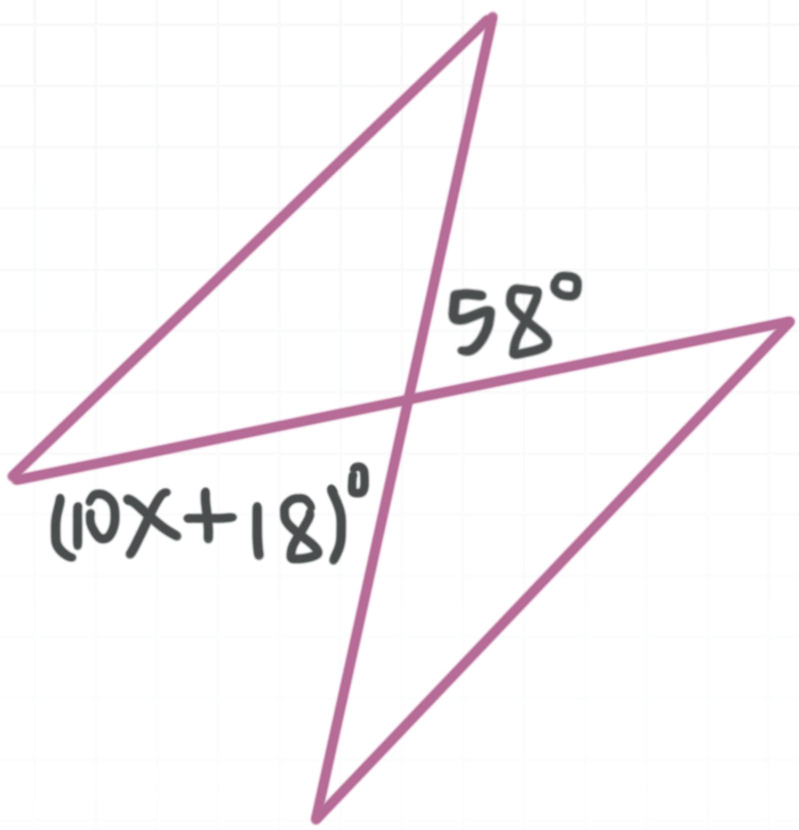


The angles in a vertical angle pair are congruent. Therefore, from the diagram above, $m\angle a = m\angle b$. We can also write this as $\angle a \cong \angle b$. The symbol $\cong$ means "is congruent to." It's used not only to express congruence of angles (any angles, not just vertical angles) but also to express congruence of line segments, triangles, and other geometric figures.

Let's do a few problems so you can get the idea.

Example

Find the value of x.



The angles of measure 58° and $(10x+18)^\circ$ are vertical angles, and are therefore congruent, so we can set their measures equal to each another and solve for the variable.

$$(10x+18)^\circ = 58^\circ$$

$$10x^\circ = 40^\circ$$

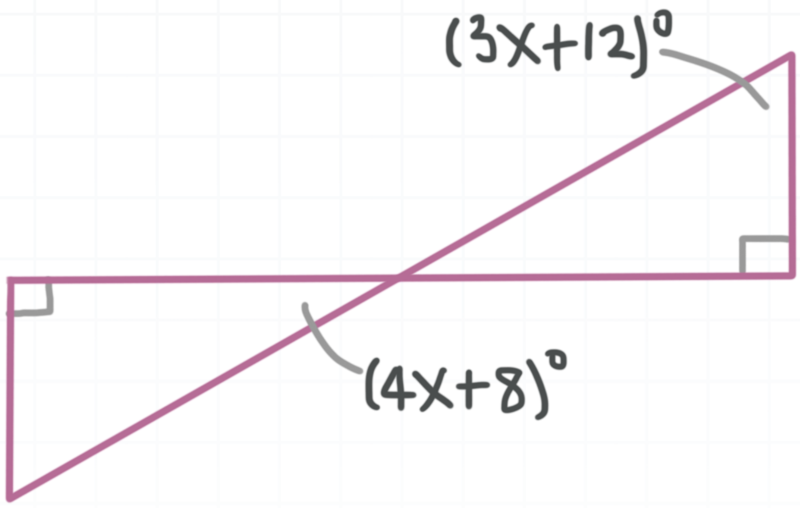
$$x^\circ = 4^\circ$$

$$x = 4$$

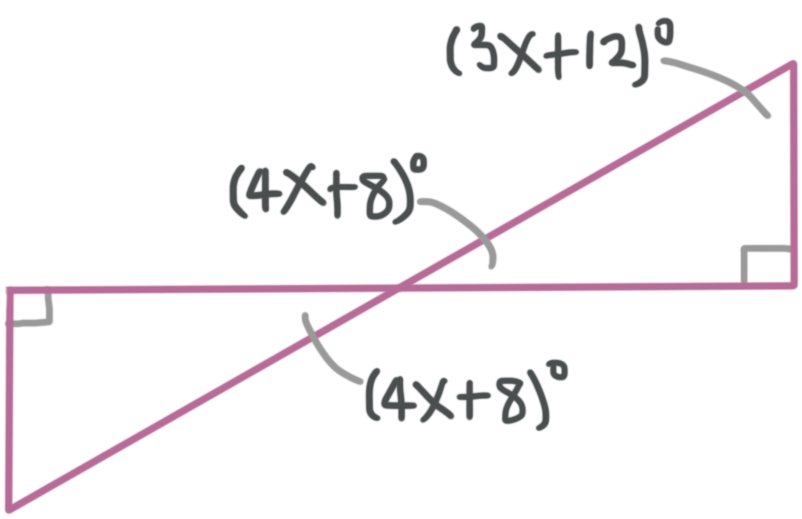
Let's try another one.

Example

Solve for the variable.



In the triangle on the right, one of the angles is a right angle (and therefore has measure 90°), and another angle has measure $(3x+12)^\circ$. The third angle and the angle of measure $(4x+8)^\circ$ in the triangle on the left are a vertical angle pair, so they're congruent. Therefore, the measure of the third angle in the triangle on the right is also $(4x+8)^\circ$.



The measures of the interior angles of a triangle sum to 180°, so we can set up an equation to solve for the variable. From the triangle on the right, we have

$$(4x + 8)^\circ + (3x + 12)^\circ + 90^\circ = 180^\circ$$

$$7x^\circ + 110^\circ = 180^\circ$$

$$7x^\circ = 70^\circ$$

$$x^\circ = 10^\circ$$

$$x = 10$$

Triangle congruence with SSS, ASA, SAS

In this lesson we'll look at how to use triangle congruence theorems to prove that triangles, or parts of triangles, are congruent.

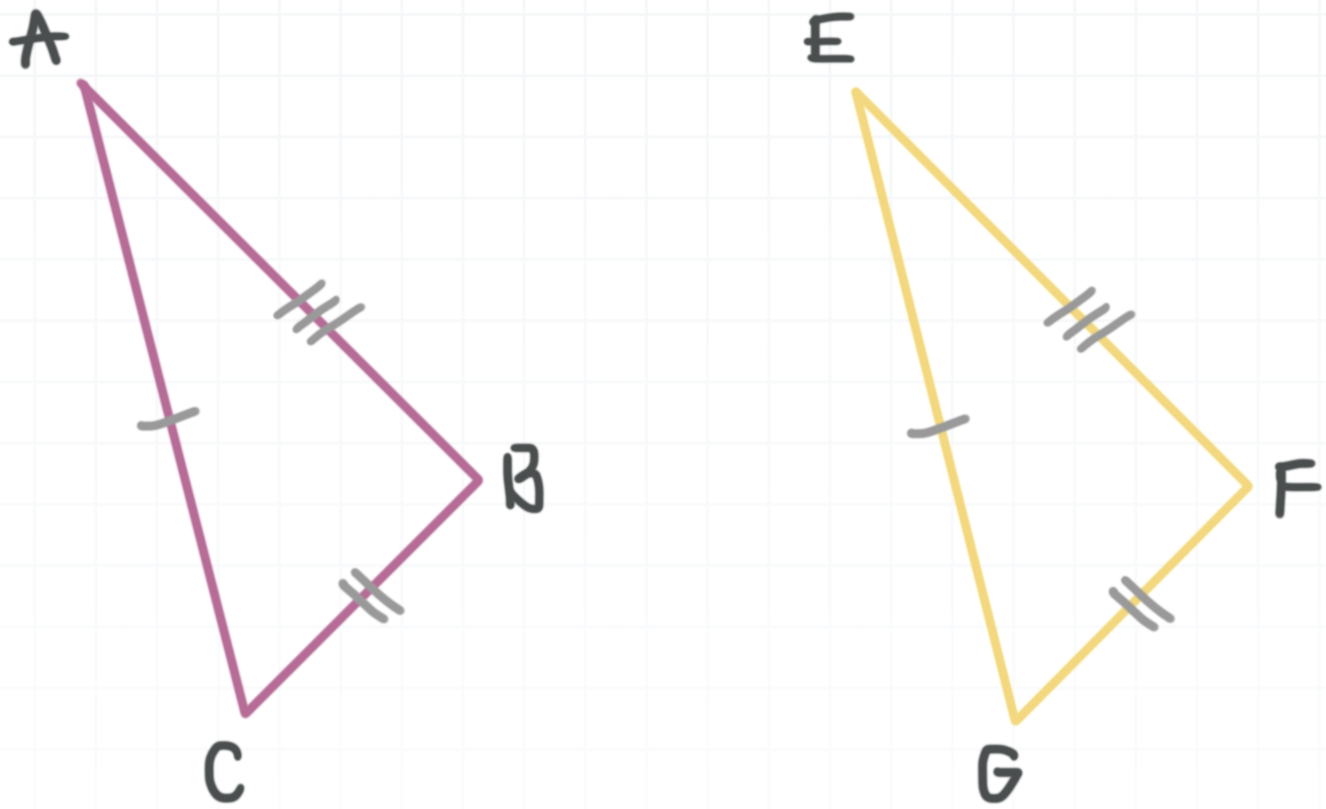
Congruent triangles

A pair of congruent triangles have exactly the same size and shape. That means that we could place one triangle on top of the other in such a way that they're identical, this is, corresponding sides have the same length and corresponding angles have the same measure.

The good news is that to prove that two triangles are congruent, we don't have to show that all three pairs of sides and all three pairs of angles match up. There are some triangle theorems that you can use as a short cut to prove that two triangles are congruent.

Side, side, side (SSS)

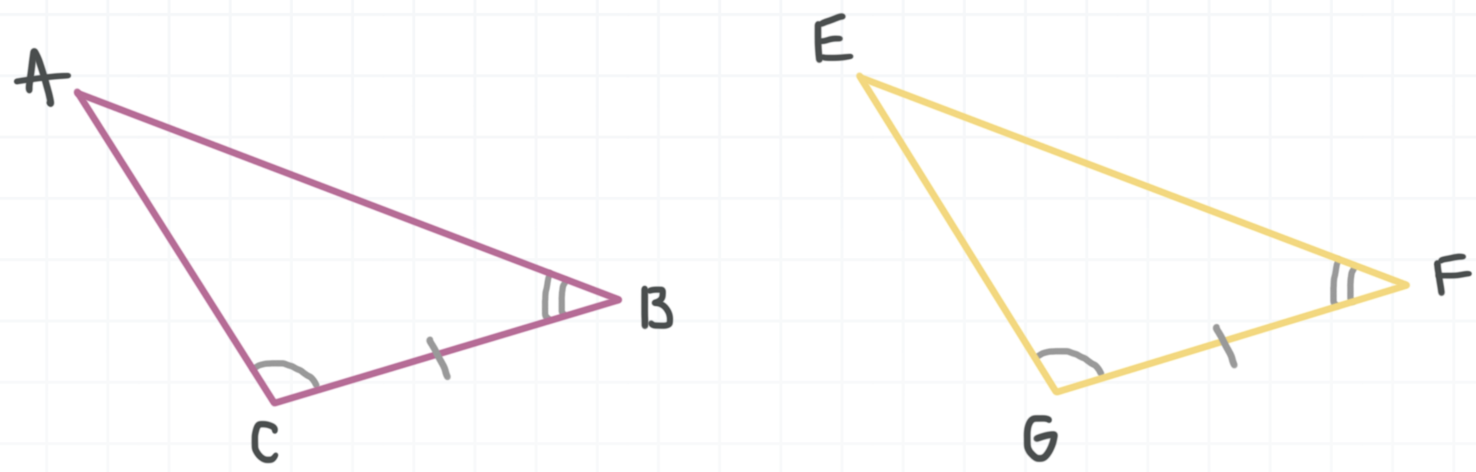
If you can show that all three pairs of sides of two triangles are congruent, then you'll have proven that the triangles are congruent, without needing to check any pair of angles. In the figure below, $\triangle ABC \cong \triangle EFG$ by side, side, side (often called SSS).



Angle, side, angle (ASA)

For “angle, side, angle,” you need to have two pairs of congruent angles and the corresponding pair of “included sides” must be congruent. The **included side** of two angles of a triangle is the side that connects those two angles.

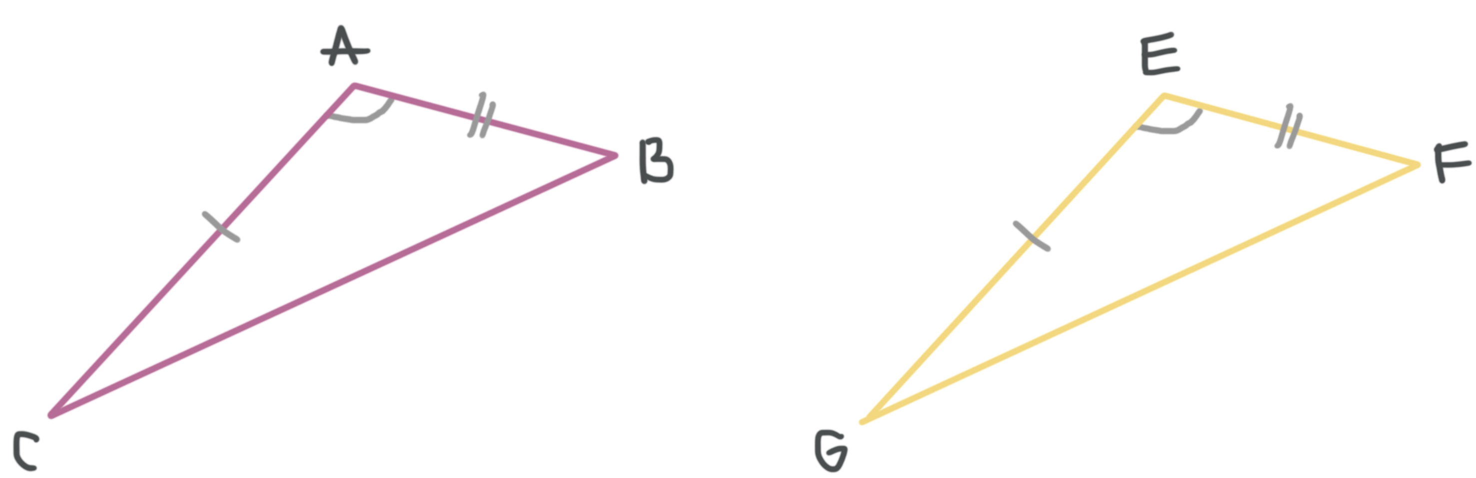
If we can prove that two triangles have these three congruences, then we’ve proven that the triangles are congruent, without checking the third pair of angles or the other two pairs of sides. In the figure below, $\triangle ABC \cong \triangle EFG$ by angle, side, angle (often called ASA): The included side of angles C and B in $\triangle ABC$ is $\overline{BC}$, and the included side of angles G and F in $\triangle EFG$ is $\overline{FG}$.



Side, angle, side (SAS)

For "side, angle, side," we need to have two pairs of congruent sides, and the corresponding pair of included angles must be congruent. The **included angle** of two sides of a triangle is the angle whose vertex is the point of intersection of those two sides.

If we can prove that two triangles have these three congruences, then we've proven that the triangles are congruent, without needing to check the third pair of sides or the other two pairs of angles. In the figure below, $\triangle ABC \cong \triangle EFG$ by side, angle, side (often called SAS): The included angle of sides $\overline{CA}$ and $\overline{AB}$ in $\triangle ABC$ is $\angle A$, and the included angle of sides $\overline{GE}$ and $\overline{EF}$ in $\triangle EFG$ is $\angle E$.



Matching congruent parts

Whenever you state that two triangles are congruent, you must match the letters for corresponding vertices when you name the triangles.

Even if the letters for the vertices of one of the triangles are in alphabetical order, the letters for the corresponding vertices of the other triangle will not necessarily be in alphabetical order. Write the names so that the letters for the vertices are in the same places. Then the letters for the endpoints of pairs of congruent sides will also be in the same places.

If we have a pair of congruent triangles, $\triangle ABC$ and $\triangle DEF$, then the triangle congruency statement $\triangle ABC \cong \triangle DEF$ means that all of the following are true:

> The letters A, B, and C for the vertices of $\triangle ABC$ correspond to the letters D, E, and F, respectively, for the vertices of $\triangle DEF$.
>
> Side $\overline{AB}$ in $\triangle ABC$ is congruent to side $\overline{DE}$ in $\triangle DEF$.
>
> Side $\overline{BC}$ in $\triangle ABC$ is congruent to side $\overline{EF}$ in $\triangle DEF$.
>
> Side $\overline{AC}$ in $\triangle ABC$ is congruent to side $\overline{DF}$ in $\triangle DEF$.

For instance, in the figure above, it would be correct to say that $\triangle ABC \cong \triangle EFG$ because angles A and E are congruent, angles B and F are congruent, and angles C and G are congruent. But it would be incorrect to say $\triangle ABC \cong \triangle GFE$, since this statement doesn't list the letters for the vertices of the triangle on the right (in the figure) in the same order as the letters for the corresponding vertices of the triangle on the left. In other words, we can actually state the congruence of the triangles in any of these ways:

$$\triangle ABC \cong \triangle EFG$$

$$\triangle BCA \cong \triangle FGE$$

$$\triangle CAB \cong \triangle GEF$$

$$\triangle ACB \cong \triangle EGF$$

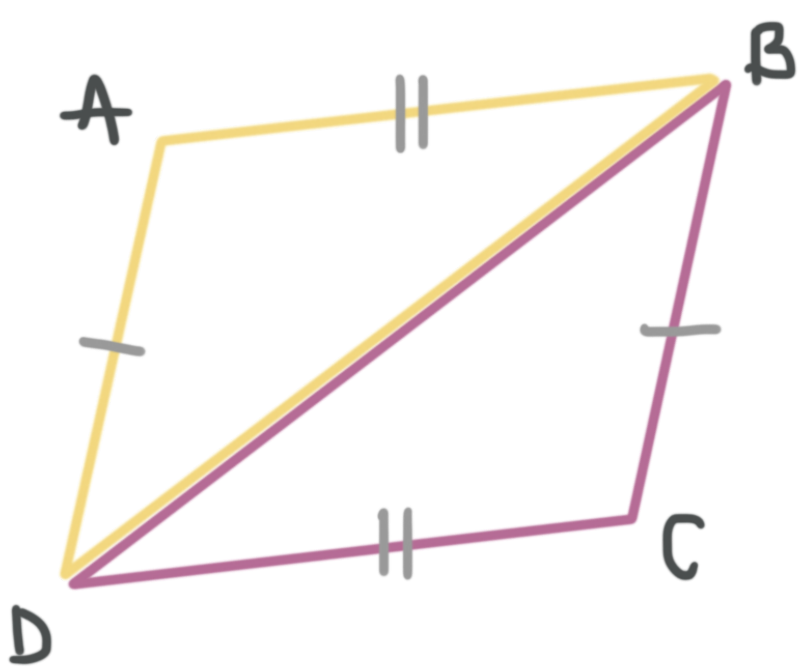
$$\triangle BAC \cong \triangle FEG$$

$$\triangle CBA \cong \triangle GFE$$

Let's start by working through an example.

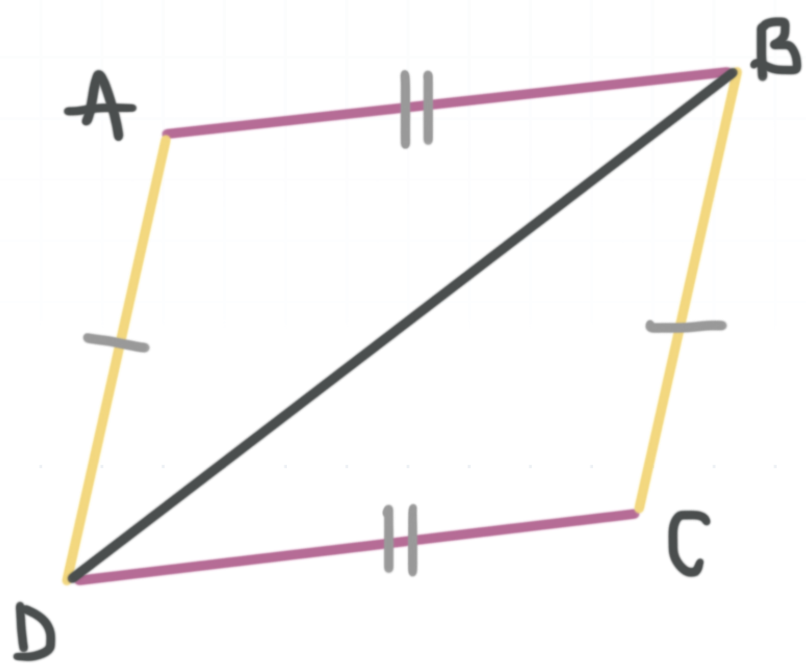
Example

Name the pair of congruent triangles in a triangle congruency statement, and state how we know that the triangles are congruent.



When we look at the two triangles, we see that $\overline{DA} \cong \overline{BC}$ and $\overline{AB} \cong \overline{CD}$. We also know that any line segment is congruent to itself, so $\overline{DB} \cong \overline{DB}$ (this is the **reflexive property**, which we'll use when we do proofs). This means we have three congruent pairs of sides, so we can prove triangle congruence by side, side, side.

In our triangle congruency statement, we need to name the triangles by matching the letters for the corresponding vertices. Sometimes color coding the given congruent parts of the triangles can help you match up the names.



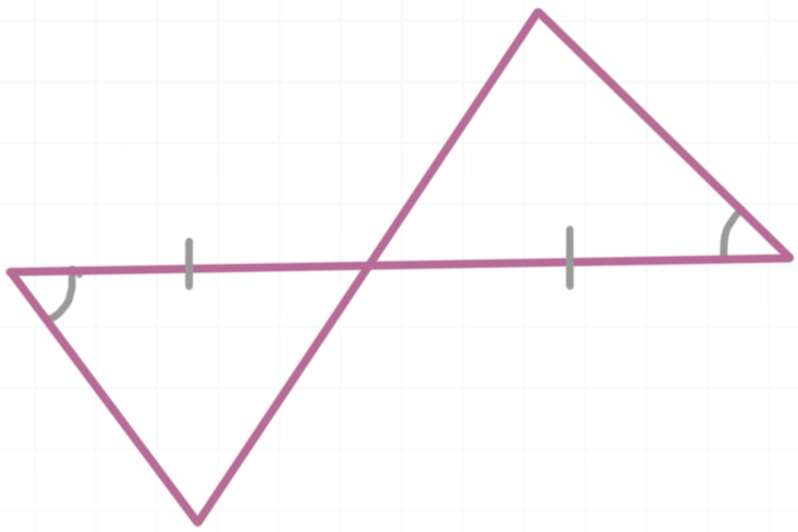
We can then write a triangle congruency statement for these triangles as follows:

$$\triangle DAB \cong \triangle BCD$$

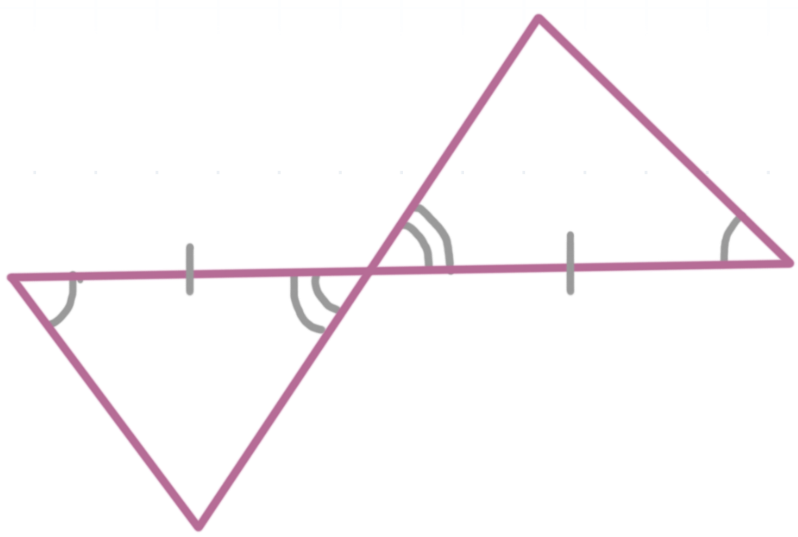
Let's try two more.

Example

State how we know that the triangles are congruent.



In these two triangles, we have a congruent pair of angles and a congruent pair of sides. We also have a pair of vertical angles here:

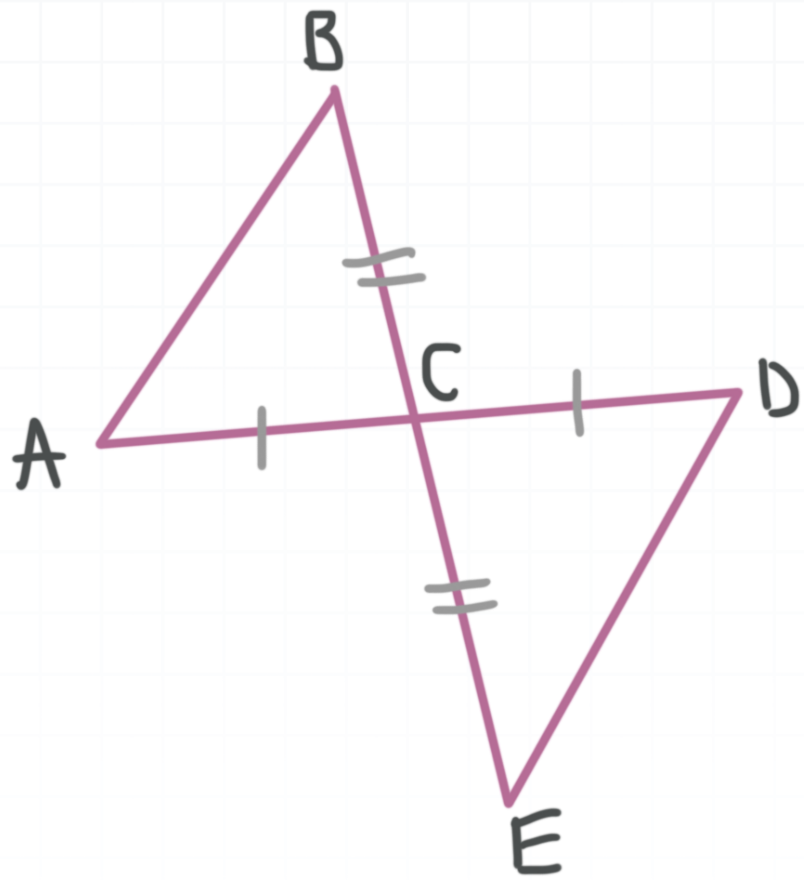


Remember that vertical angles are congruent, so these two triangles are congruent by ASA.

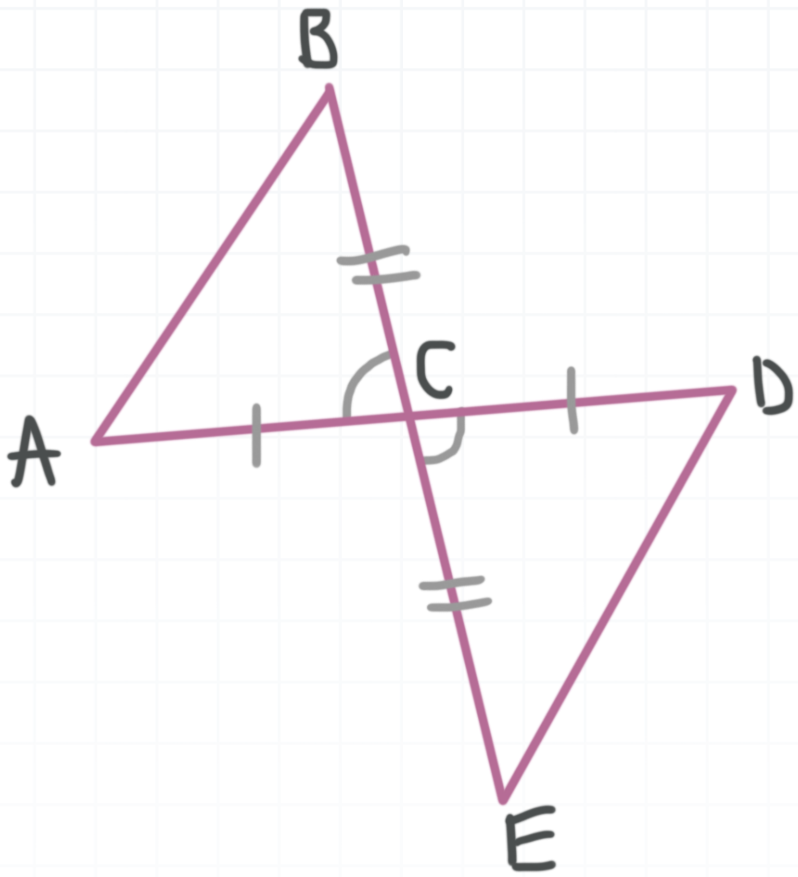
Let's do the last example.

Example

State how we know that the triangles are congruent, and write a triangle congruency statement for them.



The figure tells us that $\overline{AC} \cong \overline{DC}$ and $\overline{BC} \cong \overline{EC}$. We also know that $\angle BCA \cong \angle ECD$ because they are a pair of vertical angles.



This means $\triangle ABC \cong \triangle DEC$ by SAS: The included angle of sides $\overline{AC}$ and $\overline{BC}$ in $\triangle ABC$ is $\angle BCA$, and the included angle of sides $\overline{DC}$ and $\overline{EC}$ in $\triangle DEC$ is $\angle ECD$.

Triangle congruence with AAS, HL

In this lesson we'll look at how to use two more triangle congruence theorems, called angle, angle, side (AAS) and hypotenuse, leg (HL), to show that triangles, or parts of triangles, are congruent.

Congruent triangles

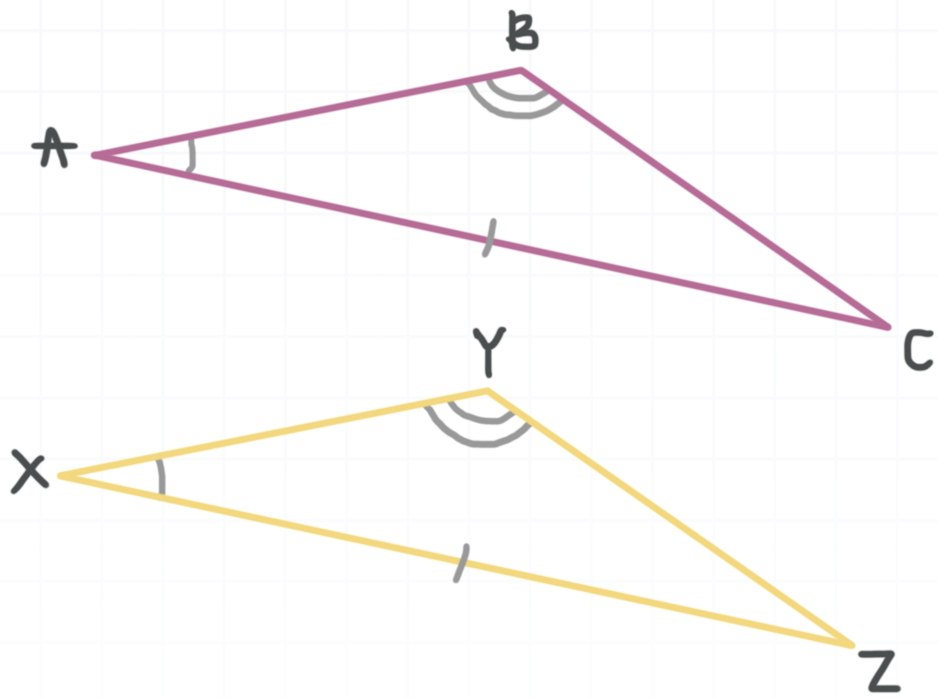
A pair of congruent triangles have exactly the same size and shape. That means that we could place one triangle on top of the other in such a way that they're identical, that is, corresponding sides have the same length and corresponding angles have the same measure.

The good news is that, to prove that the two triangles are congruent, we don't have to show that all three pairs of sides and all three pairs of angles match up. There are some triangle theorems that you can use as a short cut to prove that two triangles are congruent.

Angle, angle, side (AAS)

If you can show that two pairs of angles of two triangles are congruent, and that the side opposite an angle in one of those two pairs of congruent angles is congruent to the side opposite the other angle in that pair, then you've proven that the triangles are congruent by "angle, angle, side" (AAS), without needing to check the third pair of angles or the other two pairs of sides. In the figure below, $\triangle ABC \cong \triangle XYZ$ by angle, angle, side: In

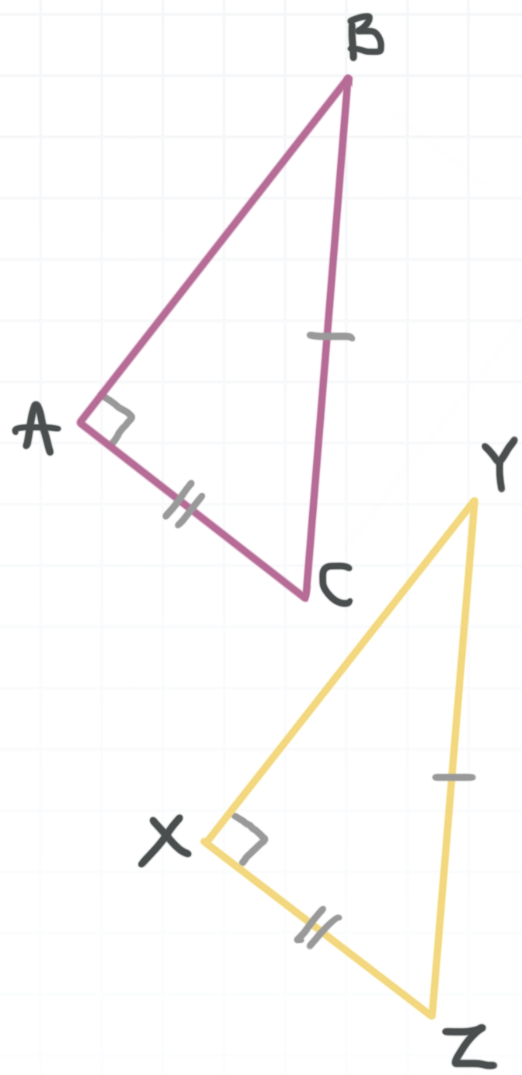
$\triangle ABC$, $\overline{AC}$ is the side opposite $\angle ABC$ (an angle in one of the pairs of congruent angles, namely the pair $\angle ABC$ and $\angle XYZ$), and in $\triangle XYZ$, $\overline{XZ}$ is the side opposite $\angle XYZ$ (the other angle in that congruent pair).



Hypotenuse, leg (HL)

This theorem can be used only with right triangles, so in order to use "hypotenuse, leg" to prove that a pair of triangles are congruent, we need to know before we even begin that both triangles are right triangles. Then we need congruent hypotenuses and a pair of congruent legs.

For instance, in the figure below, $\triangle ABC \cong \triangle XYZ$ by hypotenuse, leg: The hypotenuse of $\triangle ABC$ (side $\overline{BC}$) is congruent to the hypotenuse of $\triangle XYZ$ (side $\overline{YZ}$), and one of the legs of $\triangle ABC$ (side $\overline{AC}$) is congruent to one of the legs of $\triangle XYZ$ (side $\overline{XZ}$).



Be careful

Whenever you state that two triangles are congruent, you must match the letters for corresponding vertices when you name the triangles. Even if the letters for the vertices of one of the triangles are in alphabetical order, the letters for the corresponding vertices of the other triangle will not necessarily be in alphabetical order.

Write the names so that the letters for the vertices are in the same places. Then the letters for the endpoints of pairs of congruent sides will also be in the same places.

If you have a pair of congruent triangles, $\triangle ABC$ and $\triangle DEF$, then the triangle congruency statement $\triangle ABC \cong \triangle DEF$ means that all of the following are true:

The letters A, B, and C for the vertices of $\triangle ABC$ correspond to the letters D, E, and F, respectively, for the vertices of $\triangle DEF$.

Side $\overline{AB}$ in $\triangle ABC$ is congruent to side $\overline{DE}$ in $\triangle DEF$.

Side $\overline{BC}$ in $\triangle ABC$ is congruent to side $\overline{EF}$ in $\triangle DEF$.

Side $\overline{AC}$ in $\triangle ABC$ is congruent to side $\overline{DF}$ in $\triangle DEF$.

For instance, in the figure above, it would be correct to say that $\triangle ABC \cong \triangle XYZ$. But it would be incorrect to say $\triangle ABC \cong \triangle YZX$, since this statement doesn't list the letters for the vertices of the lower triangle (in the figure) in the same order as the letters for the corresponding vertices of the upper triangle. In other words, we could correctly write this congruence statement as any of the following:

$\triangle ABC \cong \triangle XYZ$

$\triangle BCA \cong \triangle YZX$

$\triangle CAB \cong \triangle ZXY$

$\triangle ACB \cong \triangle XZY$

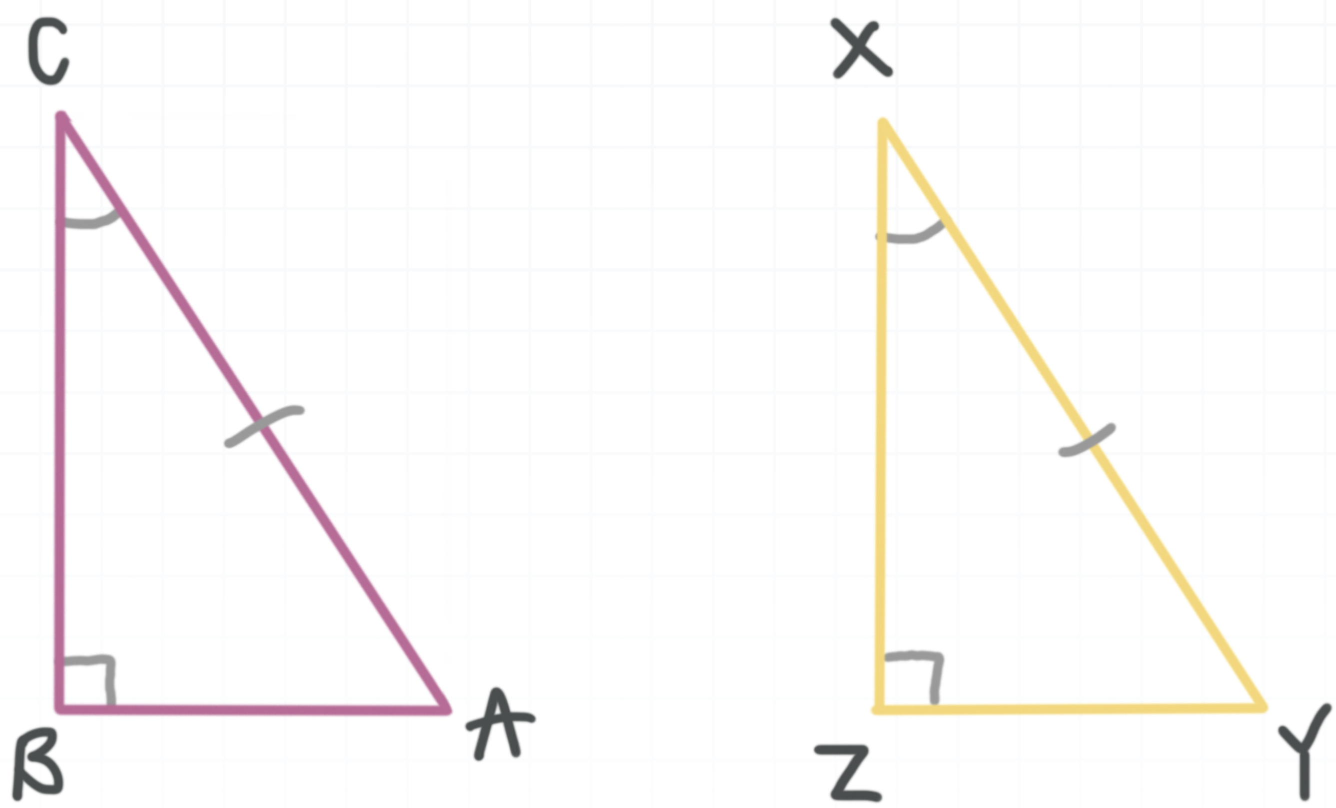
$\triangle BAC \cong \triangle YXZ$

$\triangle CBA \cong \triangle ZYX$

Let's do an example.

Example

Can you prove that the two triangles are congruent? If so, how?



Let's write down what we know.

$$\angle ZXY \cong \angle BCA$$

$$\overline{XY} \cong \overline{CA}$$

$$m\angle YZX = 90^\circ$$

$$m\angle ABC = 90^\circ$$

Because $m\angle YZX = 90^\circ$ and $m\angle ABC = 90^\circ$, we know that

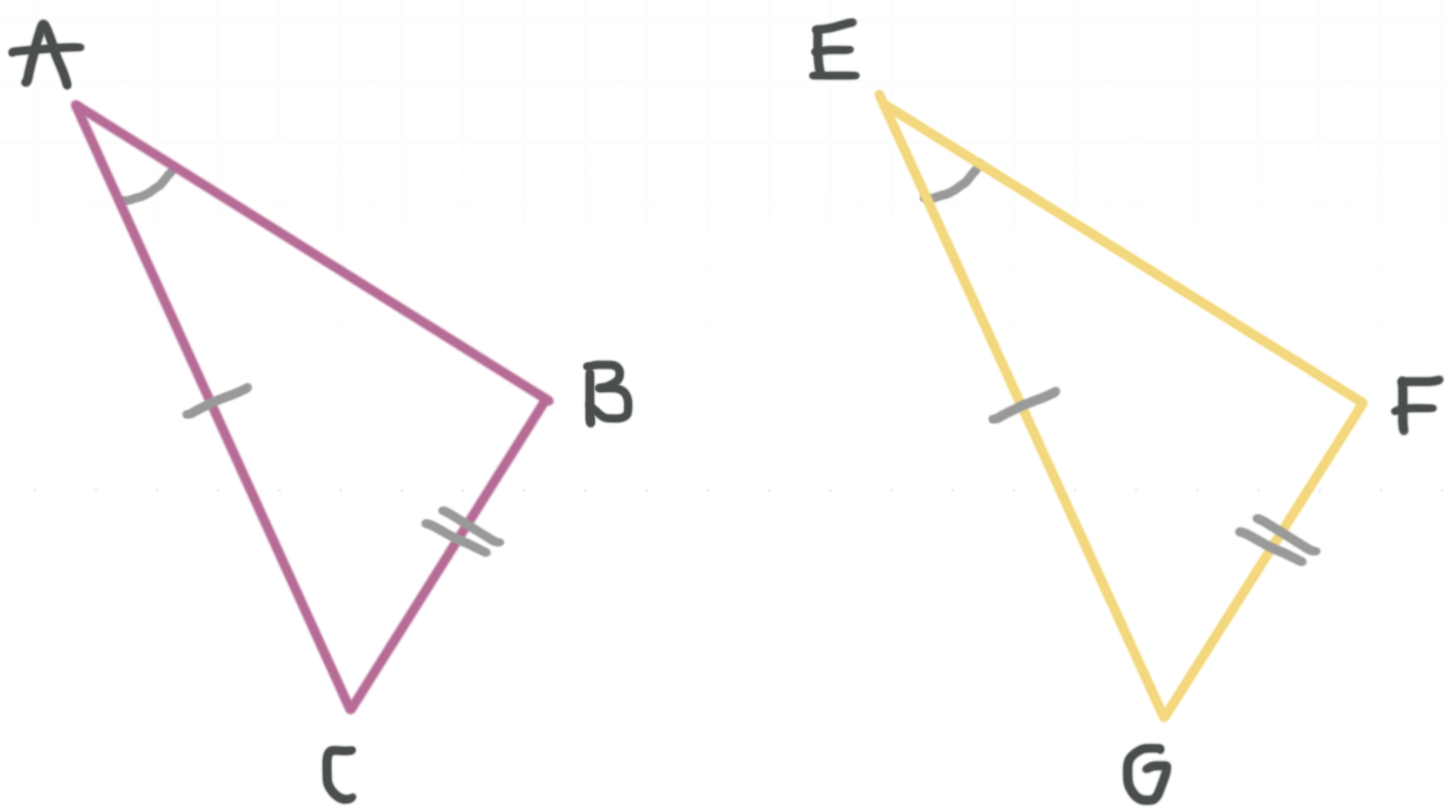
$$\angle YZX \cong \angle ABC$$

Therefore, the triangles are congruent by angle, angle, side: In $\triangle ABC$, $\overline{CA}$ is the side opposite $\angle ABC$ (an angle in one of the two pairs of congruent angles, namely the pair $\angle ABC$ and $\angle YZX$); and in $\triangle XYZ$, $\overline{XY}$ is the side opposite $\angle YZX$ (the other angle in that congruent pair).

Let's try two more.

Example

Can you prove that the two triangles are congruent? If so, how?



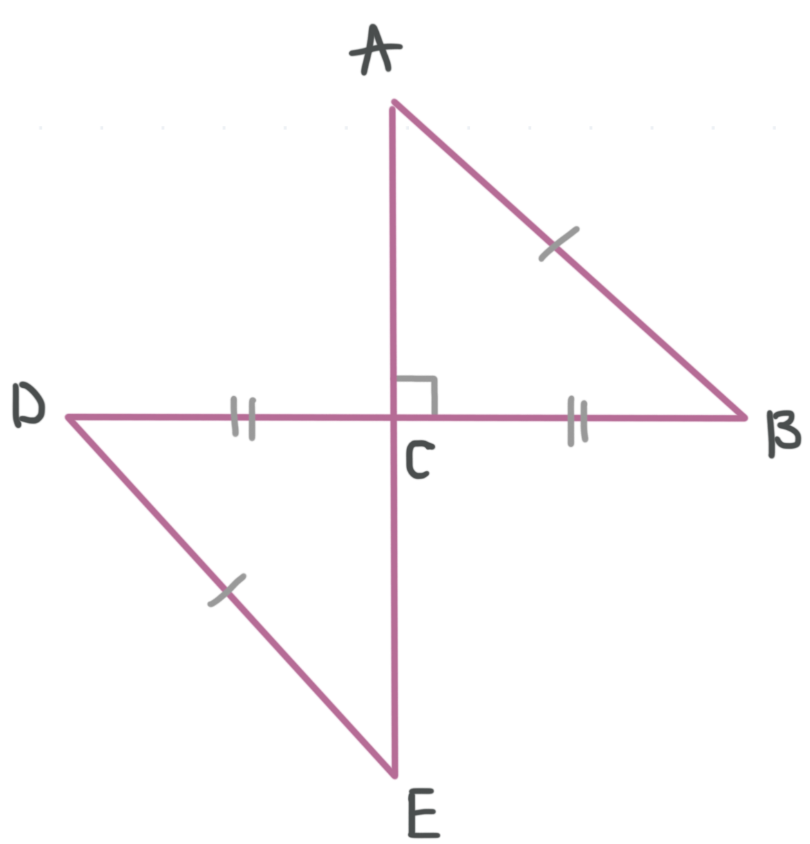
Here's a case where it might be tempting to use hypotenuse, leg because we have two pairs of congruent sides. Unfortunately, we weren't told whether the angles that *look* like right angles actually *are* right angles, so we don't know for sure that these are right triangles, even though they look as if they could be.

We have side, side, angle (where the angles in the pair of congruent angles, $\angle CAB$ and $\angle GEF$, are opposite sides $\overline{BC}$ in $\triangle ABC$ and $\overline{GF}$ in $\triangle EFG$, respectively), but that's not a triangle congruence theorem, so we don't have a way to prove that these triangles are congruent without being given more information.

Let's do the last example.

Example

Can you prove that the two triangles are congruent? If so, state how and write a triangle congruency statement for them.



We know that $\angle BCA \cong \angle DCE$ because they are a pair of vertical angles. Which means that because $\angle BCA$ is a right angle, $\angle DCE$ is a right angle as well. This means that both triangles are right triangles.

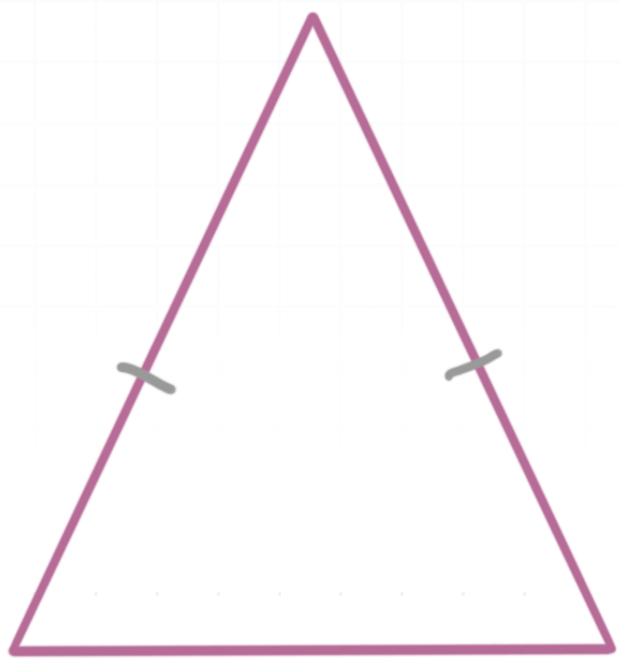
$\overline{ED}$ is the hypotenuse of $\triangle EDC$, and $\overline{AB}$ is the hypotenuse of $\triangle ABC$. According to the diagram, $\overline{ED} \cong \overline{AB}$, so the hypotenuses are congruent. We also have a pair of congruent legs because $\overline{DC} \cong \overline{BC}$, so $\triangle ABC \cong \triangle EDC$ by hypotenuse, leg.

Isosceles triangle theorem

In this lesson we'll look at isosceles triangles and how to use the isosceles triangle theorem to solve problems.

Isosceles triangles

An **isosceles triangle** is a triangle with at least two congruent sides.

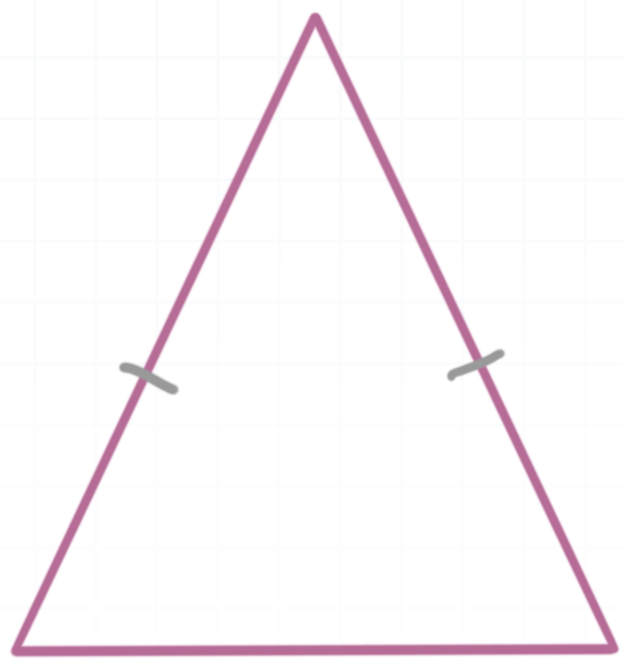


If a triangle is equilateral, then it has three congruent sides, which is therefore considered a special case of an isosceles triangle. The specific case of the equilateral triangle is the reason that the definition of isosceles triangle includes the words "*at least two* congruent sides."

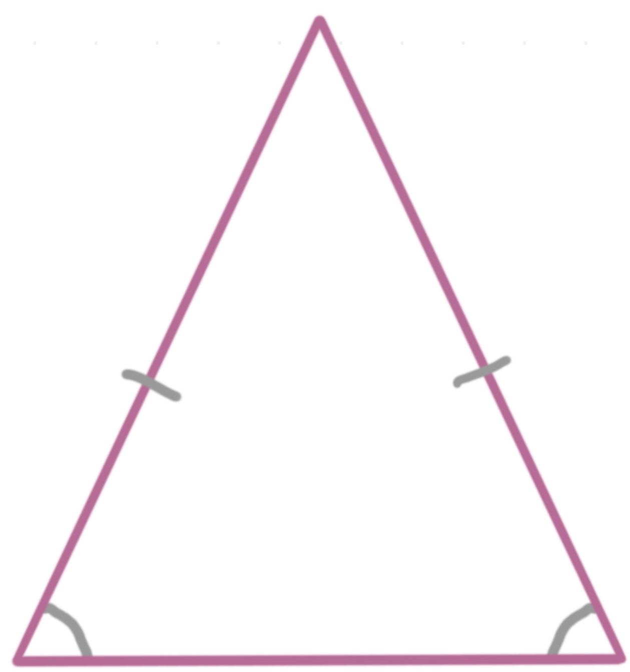
Isosceles triangle theorem

The isosceles triangle theorem says that if two sides of a triangle are congruent, then the angles opposite those sides are congruent.

If a triangle has only two congruent sides (if a triangle is congruent but not equilateral), the angles opposite the congruent sides are sometimes called the **base angles**. So if you know this:



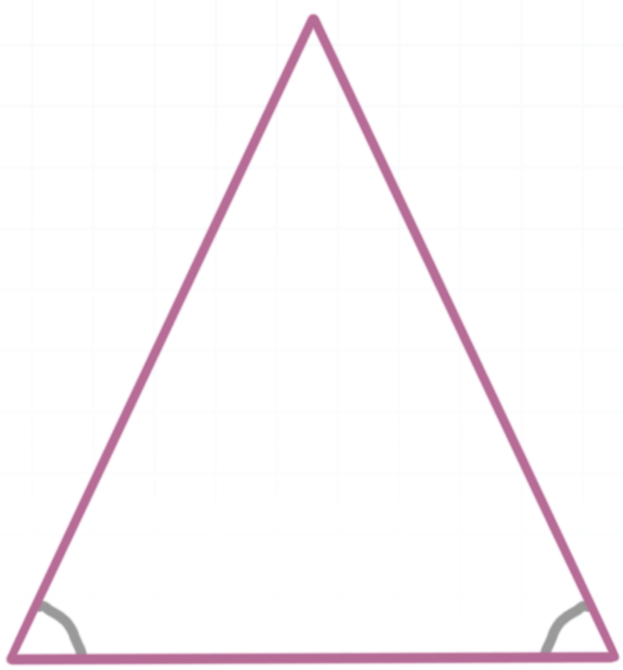
then you'll also know this:



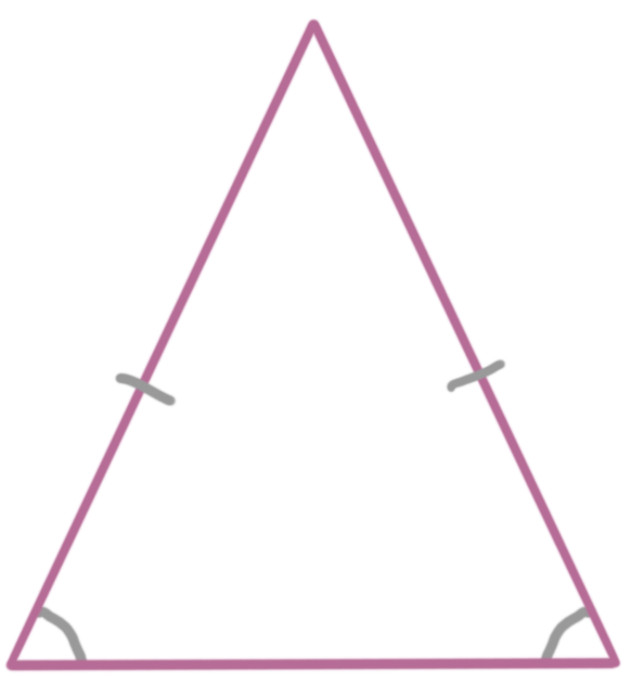
Converse of the isosceles triangle theorem

The converse of the isosceles triangle theorem just turns around the original theorem. It says that, if you know that two angles of a triangle are congruent, then the sides opposite those angles are congruent, which means it's an isosceles triangle.

In other words, if you know this:



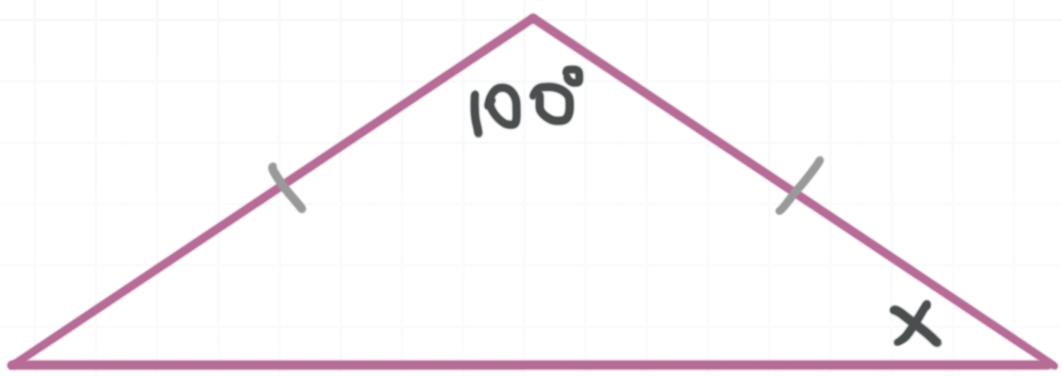
then you also know this:



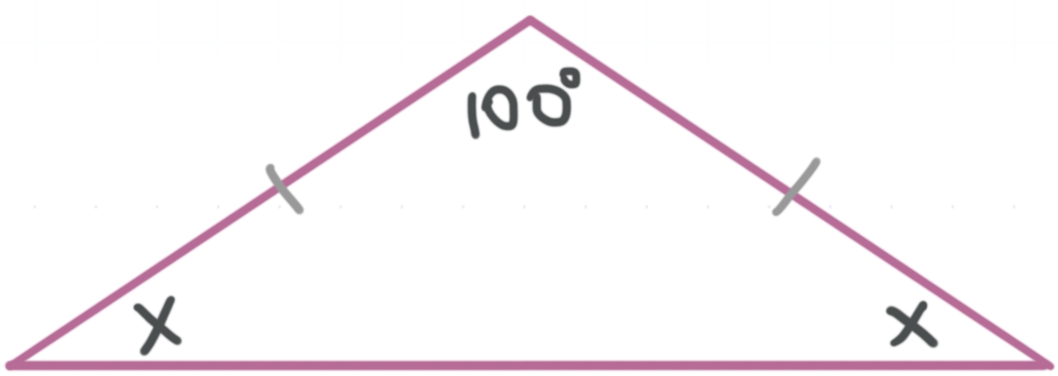
Let's start by working through an example.

Example

What is the value of x?



The triangle is an isosceles triangle, so we know that the angles opposite the congruent sides (the base angles) are congruent.



The measures of the interior angles of a triangle sum to 180°, so we can set up an equation for the sum of the interior angles.

$$x + x + 100^\circ = 180^\circ$$

$$2x + 100^\circ = 180^\circ$$

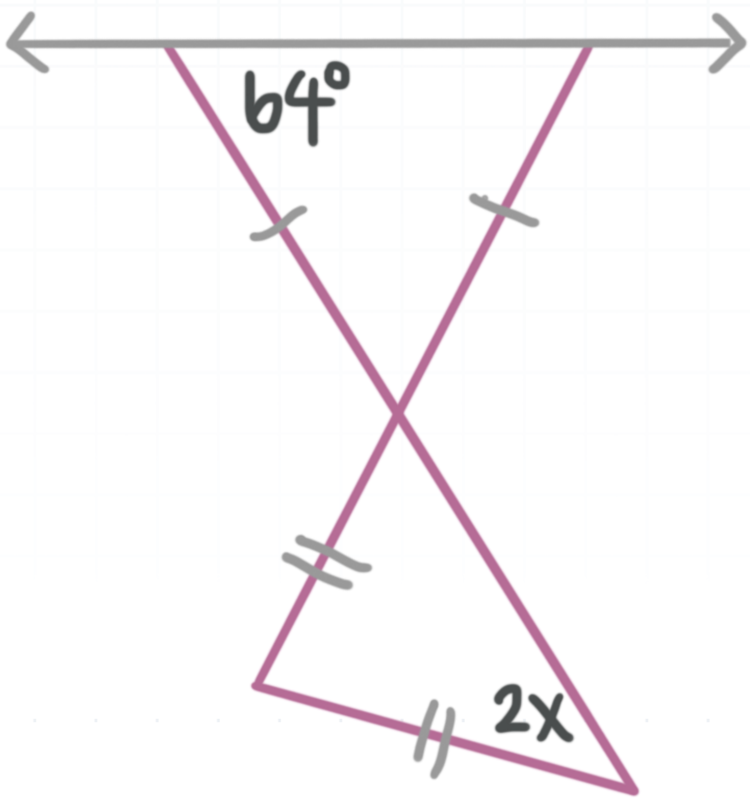
$$2x = 80^\circ$$

$x = 40^\circ$

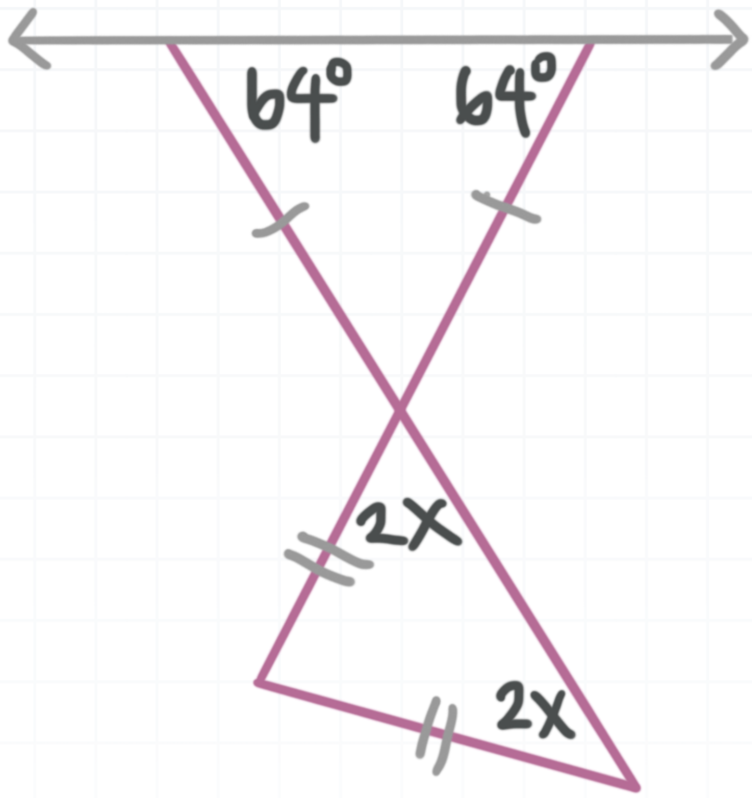
Let's try one more.

Example

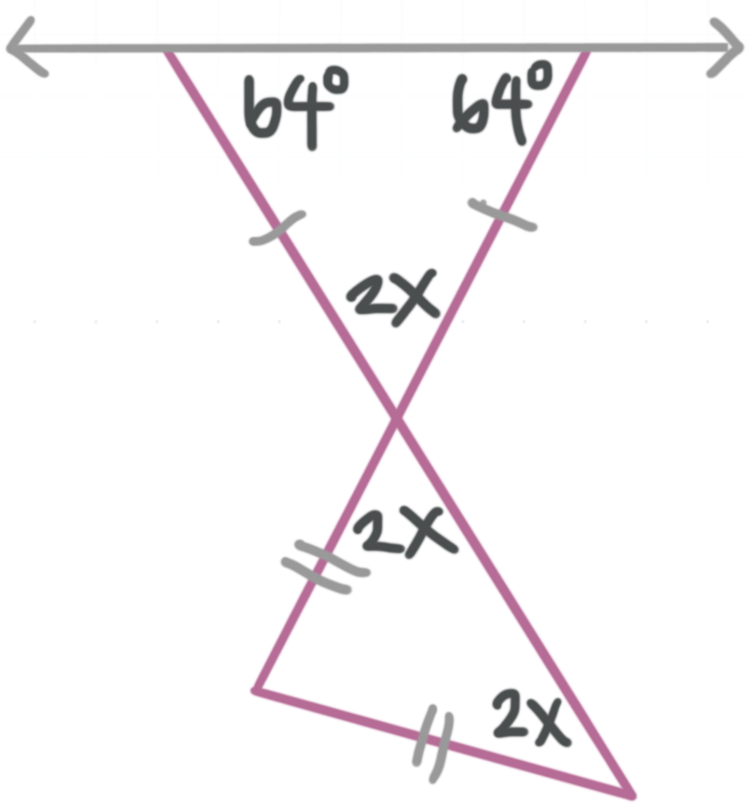
What is the value of x?



Let's use what we know about isosceles triangles to fill in the diagram. Remember that the base angles of an isosceles triangle are congruent.



We also know that vertical angles are congruent, so we can fill in one more angle.



Now we can use the top triangle to solve for the variable by remembering that the measures of the interior angles of a triangle sum to 180°.

$$64^\circ + 64^\circ + 2x = 180^\circ$$

$$128^\circ + 2x = 180^\circ$$

$$2x = 52^\circ$$

$$x = 26^\circ$$

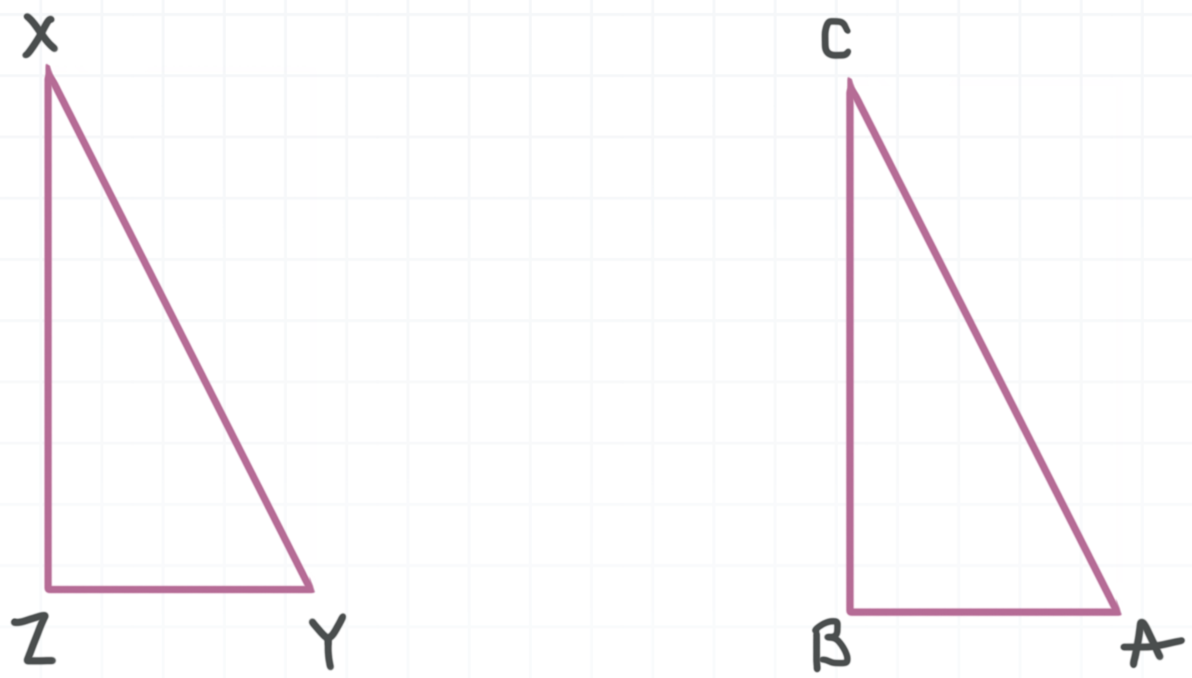
CPCTC

CPCTC stands for "corresponding parts of congruent triangles are congruent."

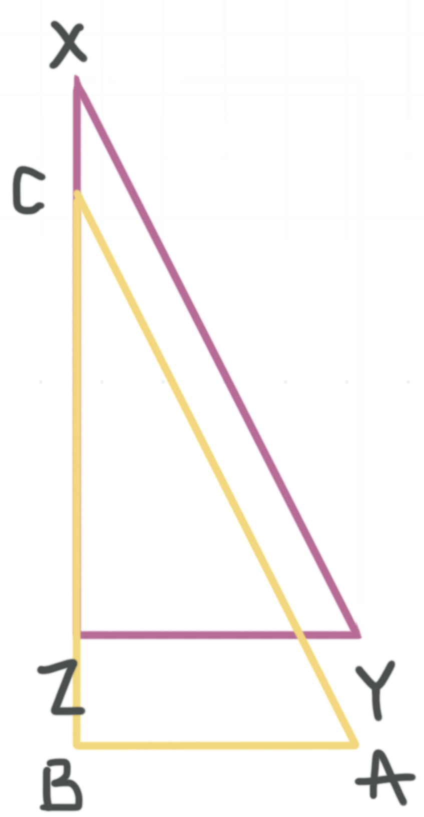
In some of the previous lessons on congruence, we used congruent parts of a pair of triangles to try to prove that the triangles themselves are congruent. CPCTC flips this around, and makes the point that, given two congruent triangles, corresponding parts of those triangles must also be congruent.

In other words, we can place one of the triangles on top of the other in such a way that all three pairs of corresponding sides are congruent and all three pairs of corresponding angles are congruent. Remember that when you state that two triangles are congruent, you need to name them with the letters for corresponding vertices in the same order.

For example, if you know the two triangles below are congruent, you need to match up the letters for their corresponding vertices, and then use those to state the congruences of angles and sides.



In the figure above, we see that the letters A, B, and C for the vertices of $\triangle ABC$ correspond to the letters Y, Z, and X, respectively, for the vertices of $\triangle XYZ$, so these triangles match in this way:



Now that the letters for the vertices are matched up, you can state the congruences of angles and sides.

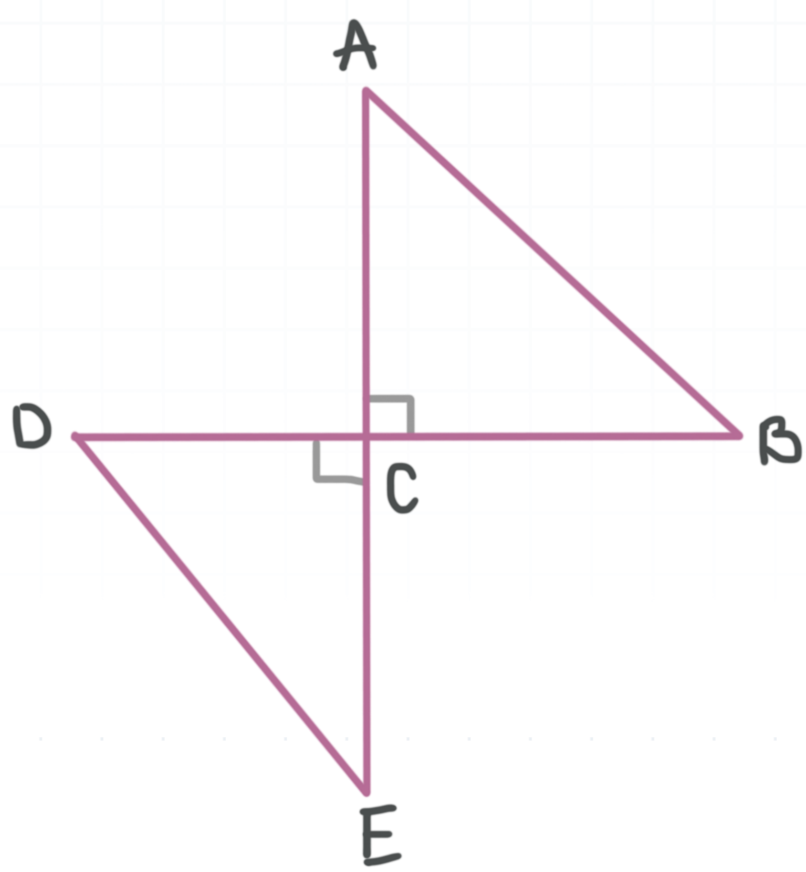
Angle congruences: $\angle A \cong \angle Y$, $\angle B \cong \angle Z$, and $\angle C \cong \angle X$

Side congruences: $\overline{AB} \cong \overline{YZ}$, $\overline{BC} \cong \overline{ZX}$, and $\overline{AC} \cong \overline{YX}$

Now we name the congruent triangles so that the letters for corresponding vertices match up in the triangle congruency statement.

$$\triangle ABC \cong \triangle YZX$$

This would also work if you were given the triangle congruency statement. For instance, say we're given that $\triangle CED \cong \triangle CAB$.



Then from this statement alone (even if we didn't have the figure) we could match up the letters for the corresponding vertices, and then use those to state congruences of sides.

$$\triangle CED \cong \triangle CAB$$

This statement tells us that the letters C, E, and D for the vertices of $\triangle CDE$ correspond to the letters C, A, and B, respectively, for the vertices of $\triangle CAB$. Therefore,

$$\overline{CE} \cong \overline{CA}$$

$$\overline{ED} \cong \overline{AB}$$

$$\overline{CD} \cong \overline{CB}$$

Finally, you can use the figure to state congruences of angles.

$$\angle DCE \cong \angle BCA$$

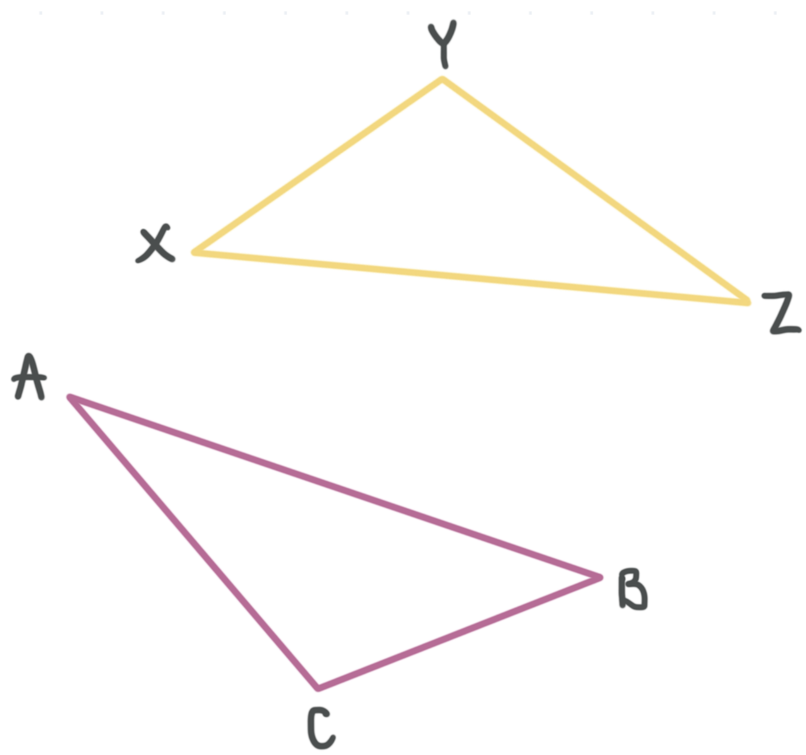
$$\angle CED \cong \angle CAB$$

$$\angle CDE \cong \angle ABC$$

Let's start by working through an example.

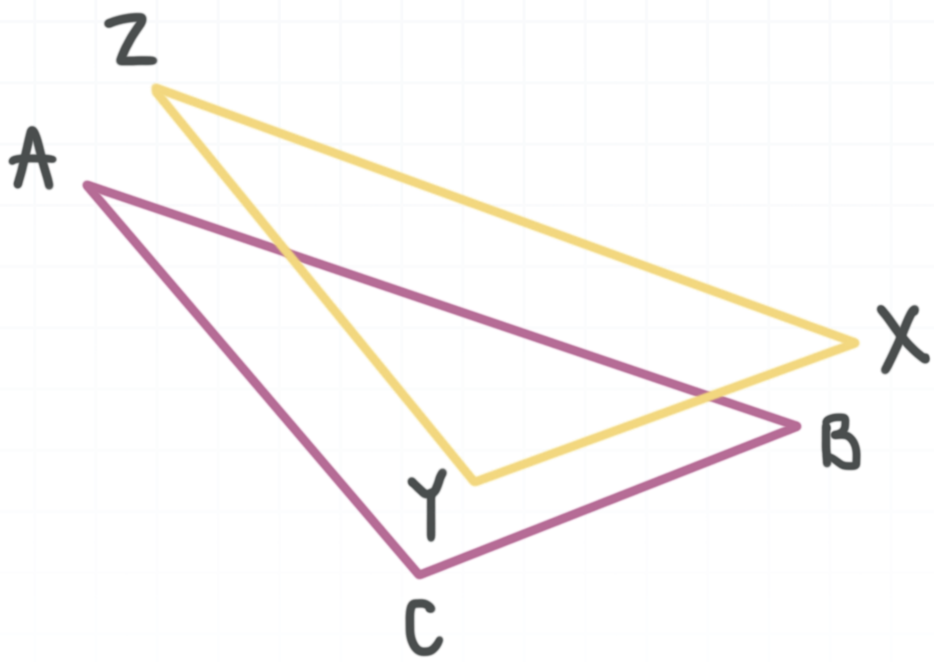
Example

Name the angle in $\triangle ABC$ that corresponds to $\angle YZX$ in $\triangle ZXY$, given that $\triangle ABC \cong \triangle ZXY$.

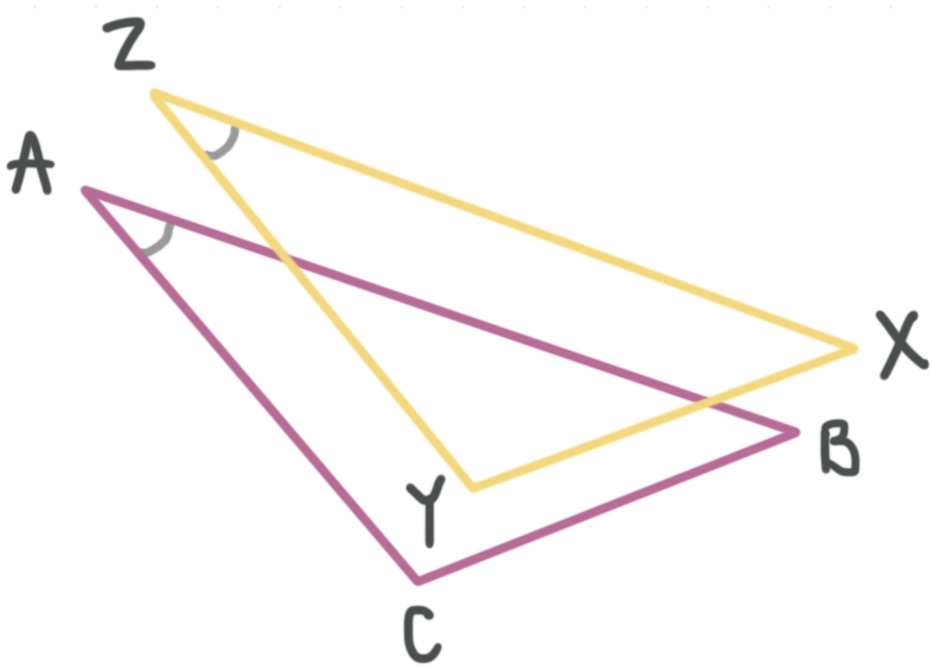


We know that $\triangle ABC \cong \triangle ZXY$, so the letters A, B, and C for the vertices of $\triangle ABC$ correspond to the letters Z, X, and Y, respectively, for the vertices of $\triangle ZXY$.

The triangles match like this:



Now we can find the angle in $\triangle ABC$ that corresponds to $\angle YZX$ in $\triangle ZXY$.

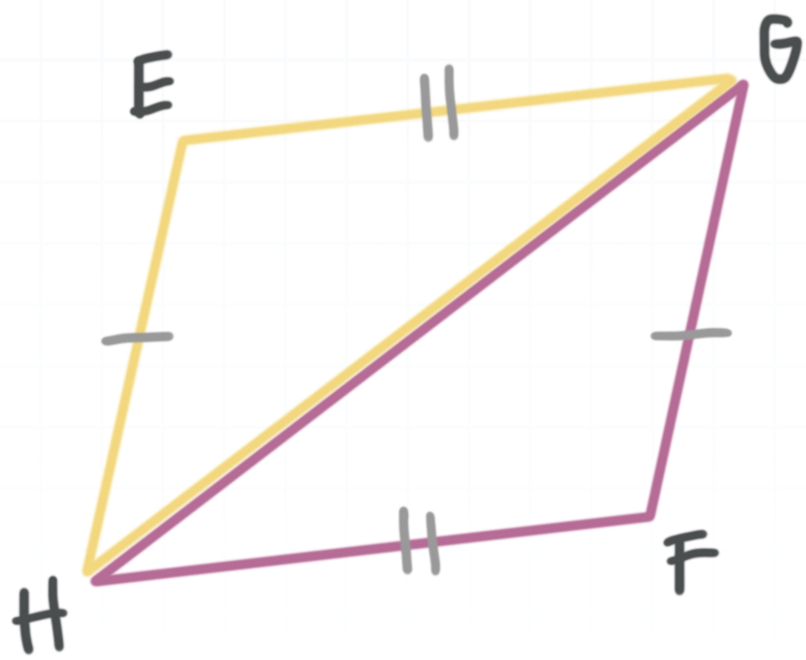


In the figure above, we see that $\angle YZX \cong \angle CAB$.

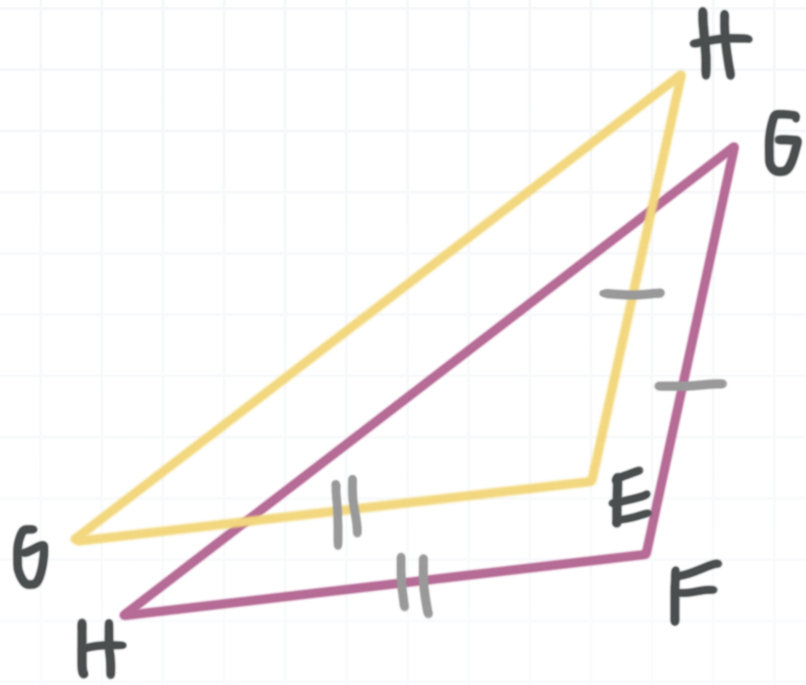
Let's try another example.

Example

These two triangles are congruent by side, side, side. Write congruency statements for the triangles.



Use the congruences of the two indicated pairs of sides to match up the letters for corresponding vertices.



In the figure above, we see that the letters H, E, and G for the vertices of $\triangle EGH$ correspond to the letters G, F, and H, respectively, for the vertices of $\triangle FGH$. Therefore, the congruences of sides are as follows:

$$\overline{HE} \cong \overline{GF}$$

$$\overline{EG} \cong \overline{FH}$$

$$\overline{HG} \cong \overline{GH}$$

From the matched up sides, we can write any of these triangle congruency statements:

$$\triangle HEG \cong \triangle GFH$$

$$\triangle EGH \cong \triangle FHG$$

$$\triangle GHE \cong \triangle HGF$$

$$\triangle HGE \cong \triangle GHF$$

$$\triangle EHG \cong \triangle FGH$$

$$\triangle GEH \cong \triangle HFG$$

As you can see, there's more than one correct way to write the statement. As long as the letters for the corresponding vertices match up, the triangle congruency statement is correct.

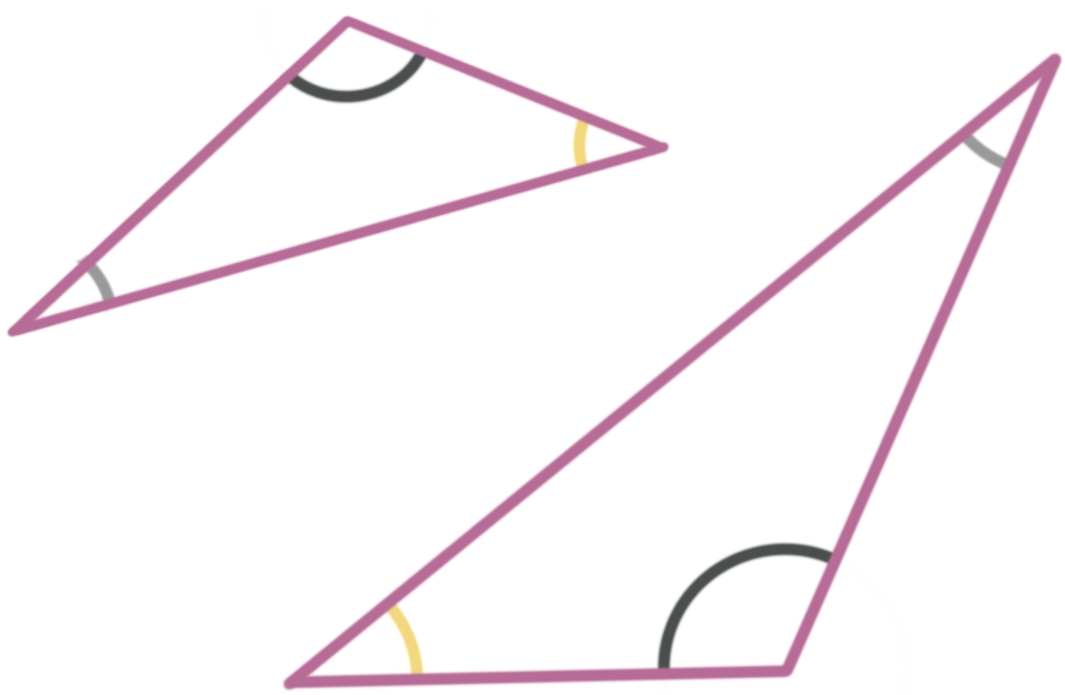
Similar triangles

In this lesson we'll look at similar triangles and how to find an unknown length of a side in one triangle (in a pair of similar triangles) from known lengths of sides in the two triangles.

In a pair of **similar triangles**, all three pairs of angles are congruent, and lengths of corresponding sides are proportional. This means that if you know two triangles are similar, you can use the given information to find an unknown length of a side in one of them from known lengths of sides in the two triangles.

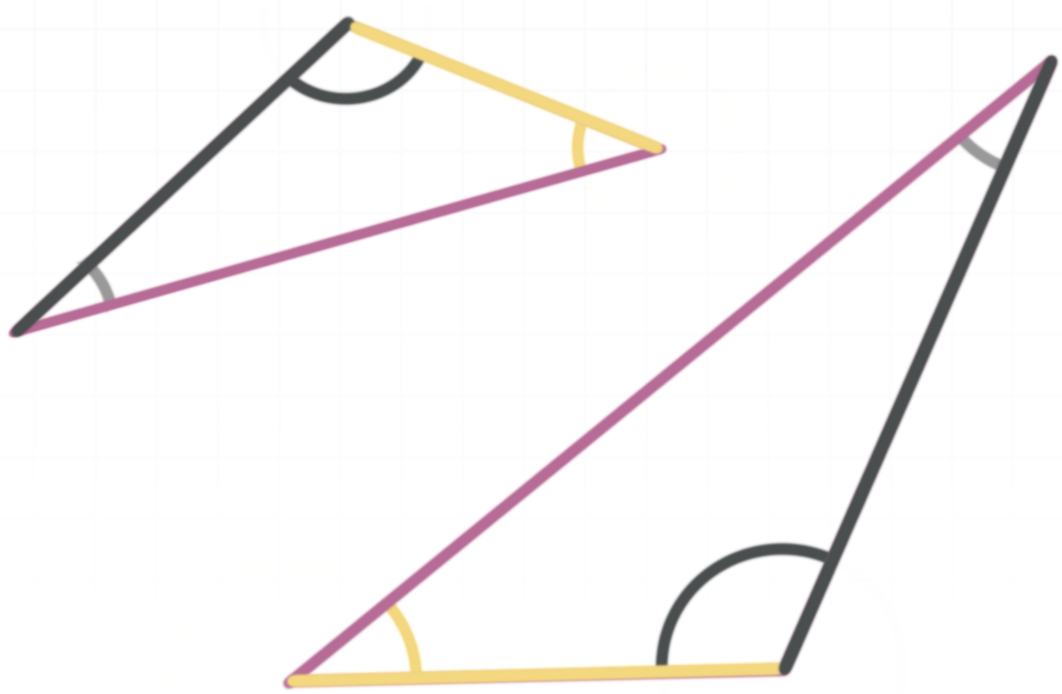
Corresponding angles and sides

In a pair of similar triangles corresponding angles have the same measure. In the pair of similar triangles shown in the figure below, each pair of corresponding angles is in a different color.



And in a pair of similar triangles, the lengths of corresponding sides are proportional. Corresponding sides are the included sides for the same two pairs of corresponding angles.

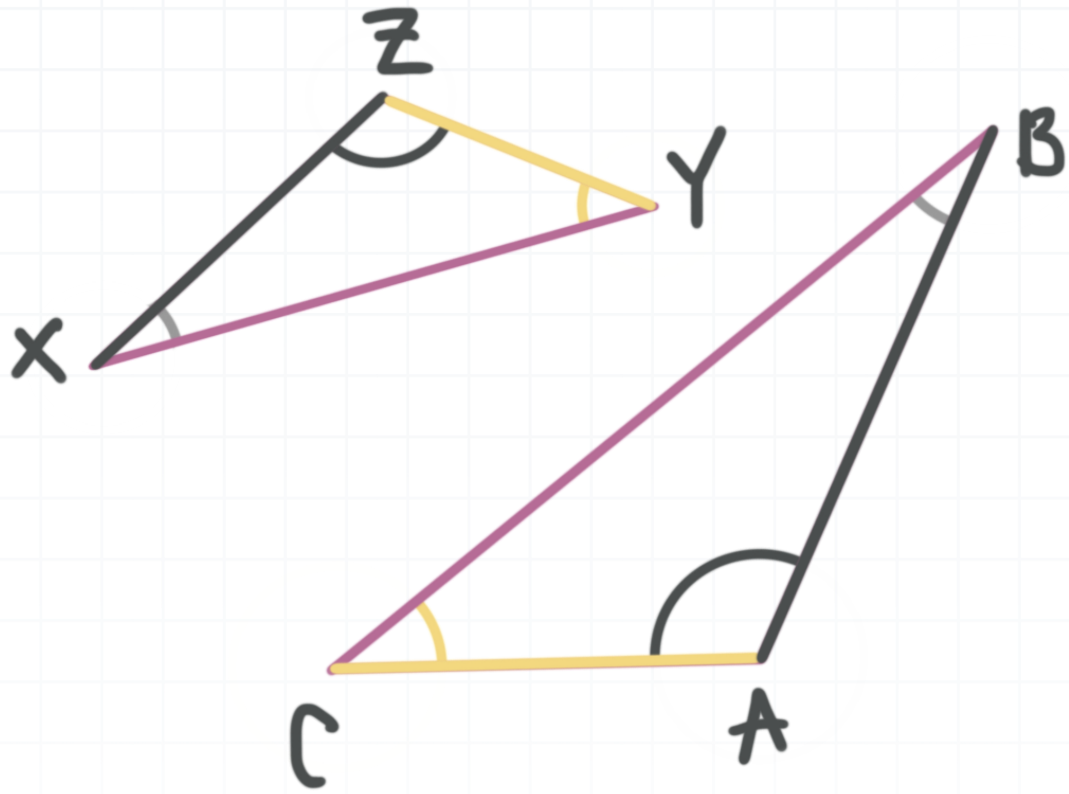
To say that lengths of corresponding sides are proportional means that there is a constant (number) such that the lengths of the sides of one triangle can be found by multiplying the lengths of the sides of the other triangle by that constant.



In the pair of similar triangles in the figure above, each pair of corresponding sides is in a different color.

Naming similar triangles

To state that two triangles are similar, you use the symbol $\sim$. You need to match the letters for the vertices of the first triangle to the letters for the corresponding vertices of the second triangle.

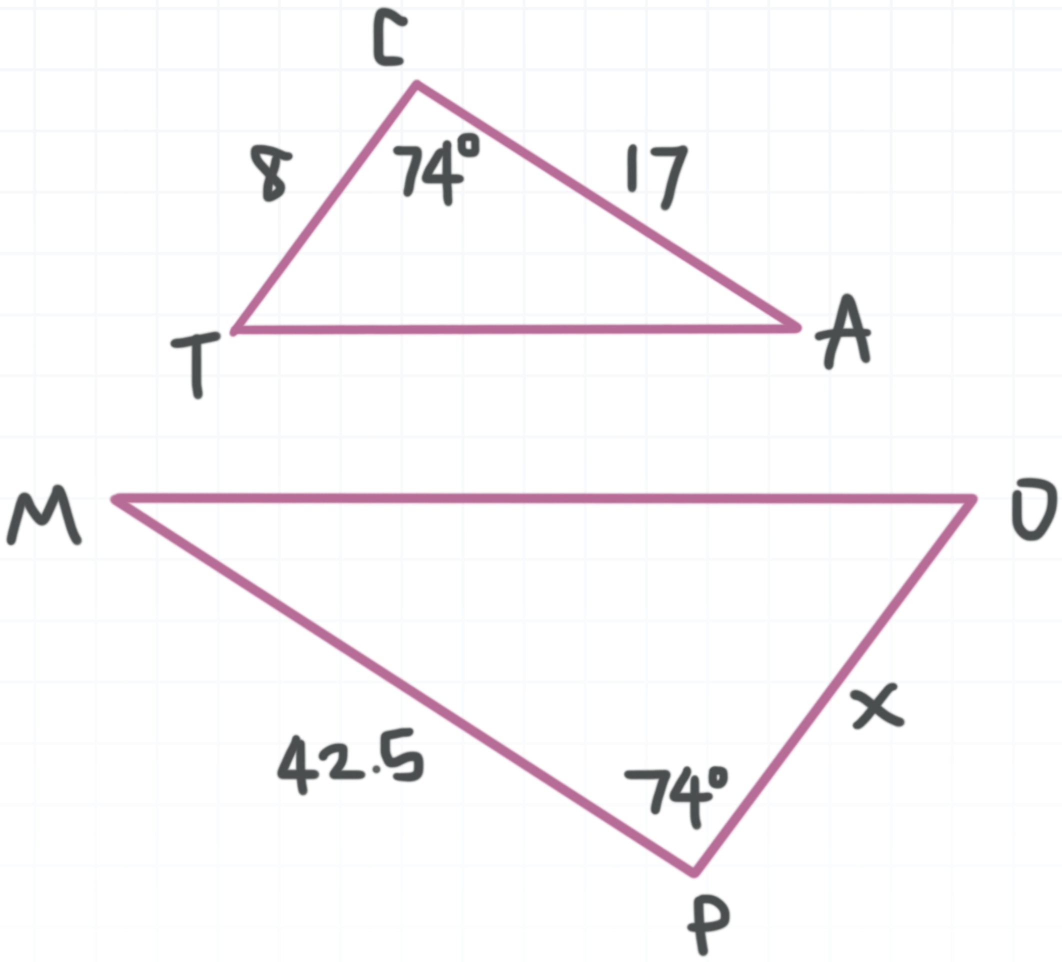


Here, the letters X, Y, and Z for the vertices of the first triangle correspond to the letters B, C, and A, respectively, for the vertices of the second triangle, so we see that $\triangle XYZ \sim \triangle BCA$.

Let's start by working through an example.

Example

The triangles in the figure are similar. Solve for the value of x.



In a pair of similar triangles, lengths of corresponding sides are proportional. The sides of length x and 42.5 in the bottom triangle correspond to the sides of length 8 and 17, respectively, in the top triangle. So we have the following proportion:

$$\frac{x}{8}=\frac{42.5}{17}$$

Cross multiply.

$$17x=8(42.5)$$

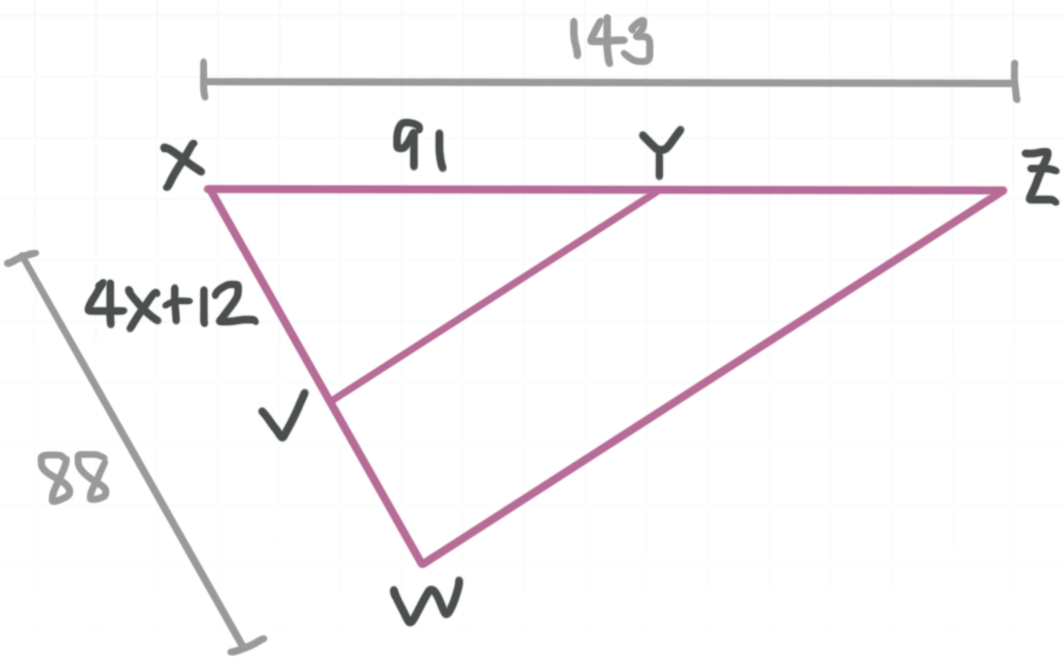
$$17x=340$$

$$x=20$$

Let's try one more.

Example

In the figure, $\triangle XVY \sim \triangle XWZ$. Find the value of x.



In a pair of similar triangles, lengths of corresponding sides are proportional. The sides of length $4x + 12$ and 91 in $\triangle XVY$ correspond to the sides of length 88 and 143, respectively, in $\triangle XWZ$. So we have the following proportion:

$$\frac{4x + 12}{88} = \frac{91}{143}$$

Cross multiply.

$$(4x + 12)143 = 91(88)$$

$$(4x + 12)143 = 8{,}008$$

$$\frac{(4x + 12)143}{143} = \frac{8{,}008}{143}$$

$$4x + 12 = 56$$

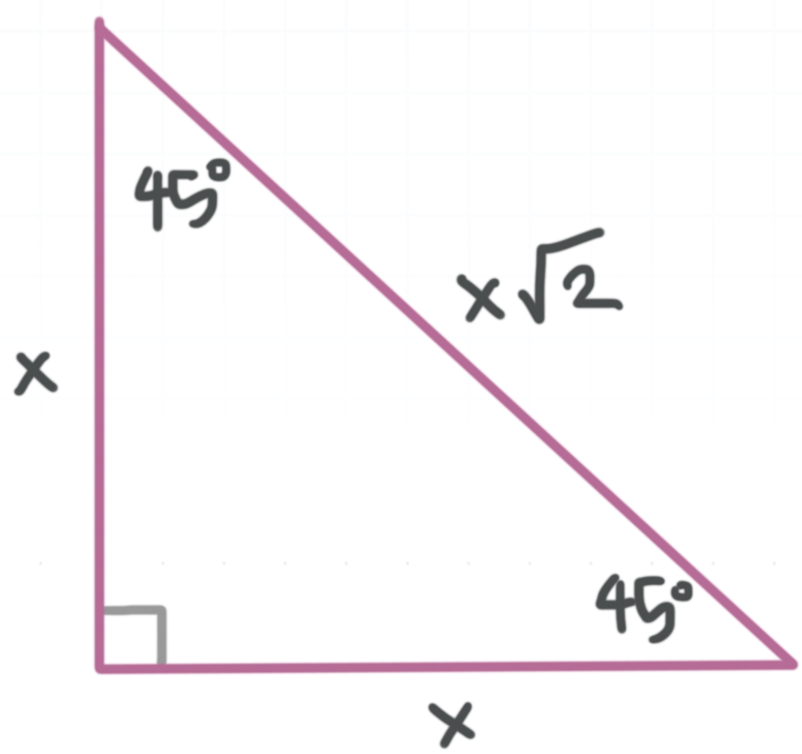
$$4x = 44$$

$$x = 11$$

45-45-90 triangles

A 45-45-90 triangle is a special kind of right triangle, because it's isosceles with two congruent sides and two congruent angles. Since it's a right triangle, the length of the hypotenuse has to be greater than the length of each leg, so the congruent sides are the legs of the triangle.

In this figure, the legs are labeled x, and the hypotenuse is labeled $x\sqrt{2}$, because in a 45-45-90 triangle the ratio of the length of the hypotenuse to the length of each leg is equal to $\sqrt{2}$.

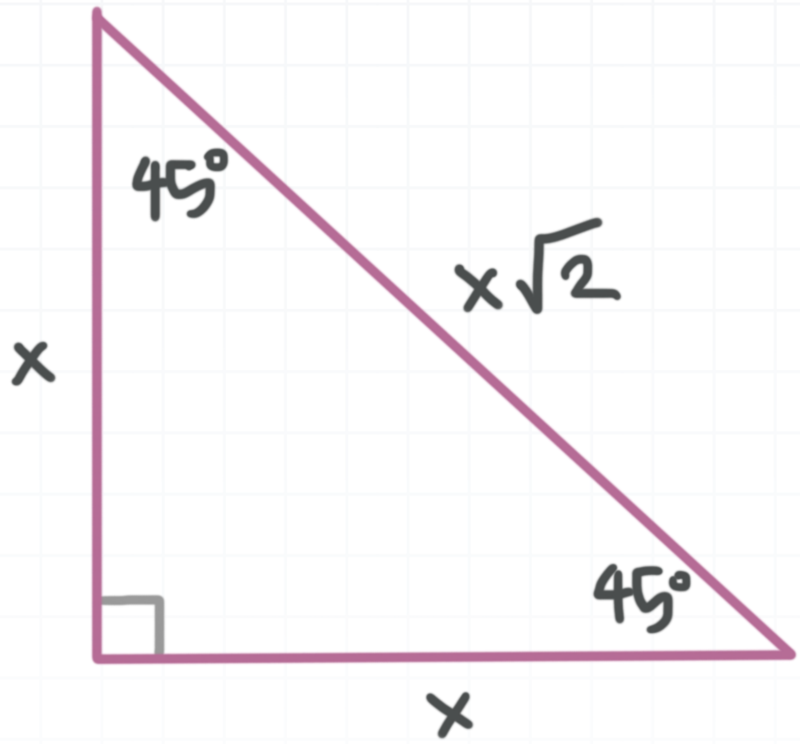


You can use this ratio to find the length of a leg of any 45-45-90 triangle if you know the length of the hypotenuse, or to find the length of the hypotenuse if you know the length of a leg.

Let's start by working through an example.

Example

If $x = 12$, what is the length of the hypotenuse?

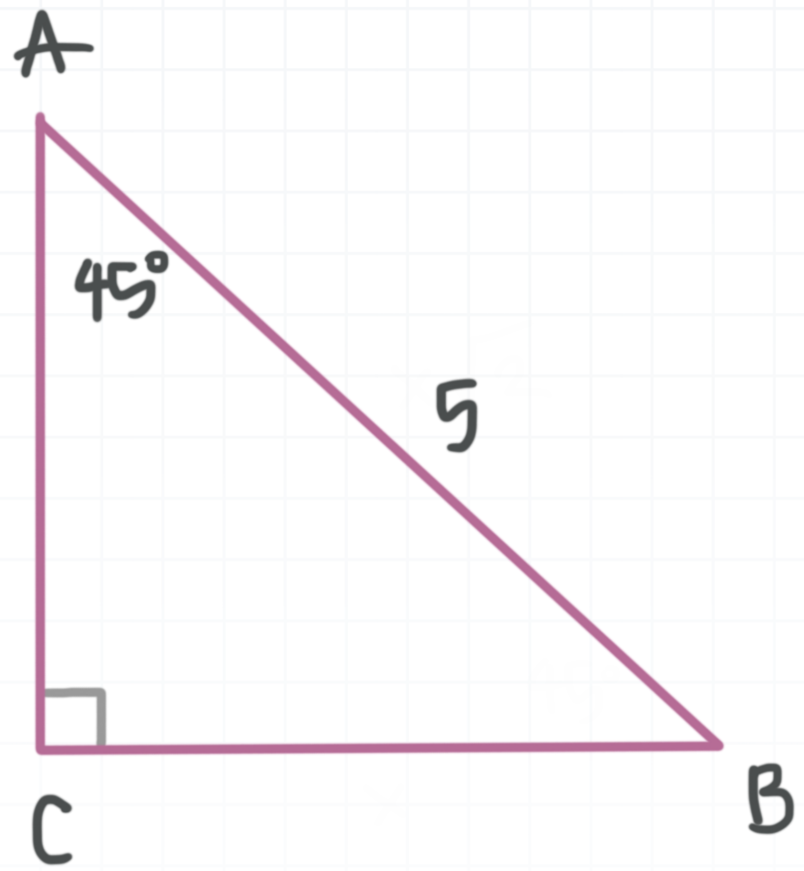


The length of the hypotenuse is $x\sqrt{2}$. We know $x = 12$, so the length of the hypotenuse is $x\sqrt{2}$, or $12\sqrt{2}$.

Let's do another example.

Example

What are the lengths of side $\overline{AC}$ and side $\overline{CB}$?



We can see that $\triangle ABC$ is a 45-45-90 triangle, because $m\angle A = 45^\circ$ and $m\angle C = 90^\circ$, which means the measure of $\angle B$ must be

$$m\angle B = 180^\circ - (m\angle A + m\angle C)$$

$$m\angle B = 180^\circ - (45^\circ + 90^\circ)$$

$$m\angle B = 180^\circ - 135^\circ$$

$$m\angle B = 45^\circ$$

Therefore, angles A and B are congruent, which means that the lengths of the sides opposite angles A and B (sides $\overline{CB}$ and $\overline{AC}$, respectively) are congruent.

So let x be the length of $\overline{AC}$. This tells us that the length of the hypotenuse ($\overline{AB} = 5$) is represented by $x\sqrt{2}$. We want to set up an equation that we can use to find x.

$$5 = x\sqrt{2}$$

$$x = \frac{5}{\sqrt{2}} = \frac{5}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{5\sqrt{2}}{2}$$

This means sides $\overline{AC}$ and $\overline{CB}$ both have length $5\sqrt{2}/2$. We could also have used the Pythagorean theorem to find x.

$$a^2 + b^2 = c^2$$

$$x^2 + x^2 = 5^2$$

$$2x^2 = 25$$

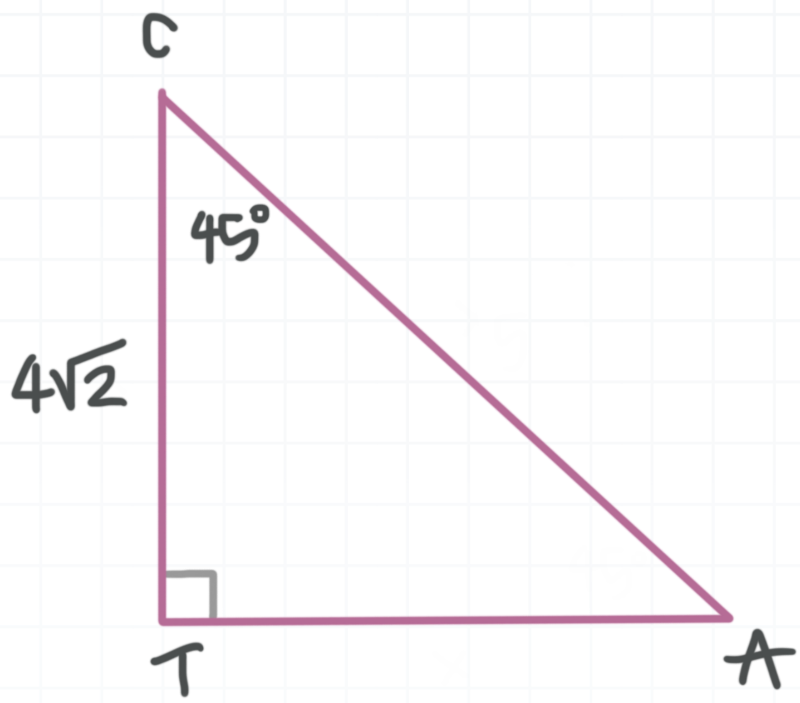
$$x^2 = \frac{25}{2}$$

$$x = \frac{5}{\sqrt{2}} = \frac{5}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{5\sqrt{2}}{2}$$

Let's try another example.

Example

What is the length of side $\overline{CA}$?



Because the measures of two of the interior angles of this right triangle are 45° and 90°, we immediately see that the measure of the remaining interior angle is 45°, so this is a 45-45-90 triangle.

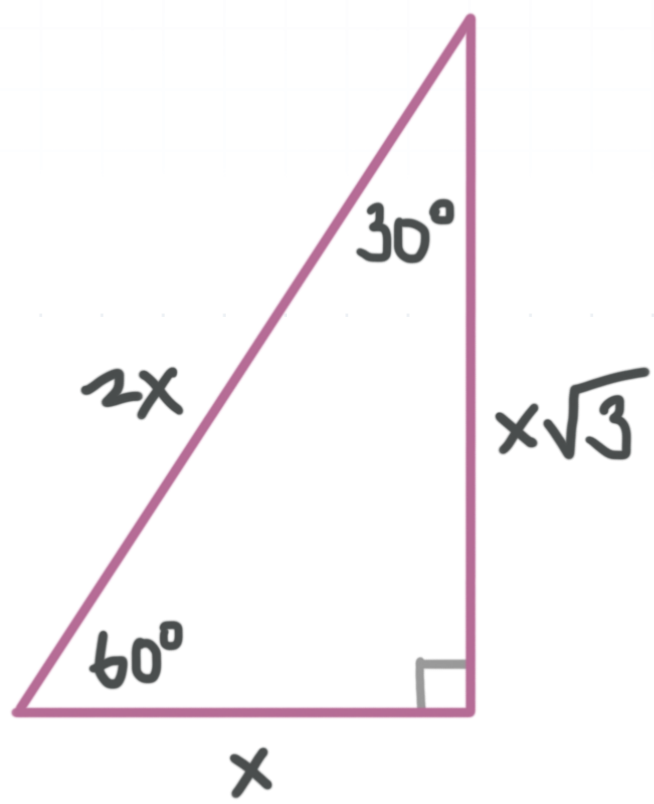
The pattern for the lengths of the sides of a 45-45-90 triangle is x, x, and $x\sqrt{2}$, where x is the length of each leg. In this case, $x = 4\sqrt{2}$. The length of the hypotenuse is $x\sqrt{2}$, so the length of side $\overline{CA}$, which is the hypotenuse, is

$$x\sqrt{2} = (4\sqrt{2})(\sqrt{2}) = 4(\sqrt{2} \cdot \sqrt{2}) = 4(2) = 8$$

30-60-90 triangles

A 30-60-90 is a right triangle in which the measures of the interior angles are 30°, 60°, and 90°, which means it's a scalene right triangle, that is, each side has a different length. Since it's a right triangle, it has a long leg and a short leg, and just like any right triangle, the hypotenuse is the longest side of all.

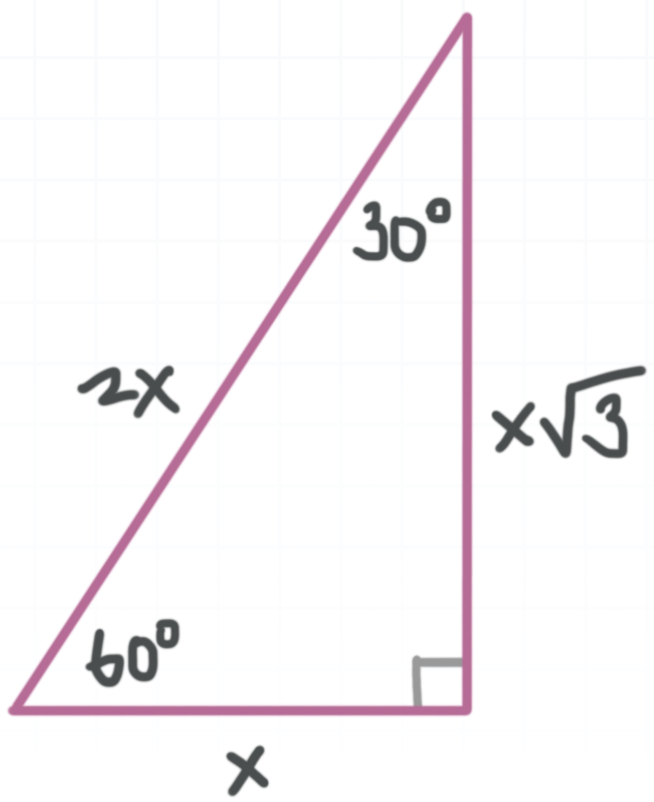
In this 30-60-90 triangle (and in any 30-60-90 triangle), the length of the hypotenuse is twice the length of the short leg, and the ratio of the length of the long leg to the length of the short leg is $\sqrt{3}$. In this figure, the length of the short leg is x, so the length of the long leg is $x\sqrt{3}$ and the length of the hypotenuse is $2x$



We can use these relationships to find the length of a side of any 30-60-90 triangle if we know the length of one of the other sides, provided that we also know which angle is opposite the side of known length. Let's start by working through an example.

Example

If $x = 6$, what are the lengths of the hypotenuse and the long leg?

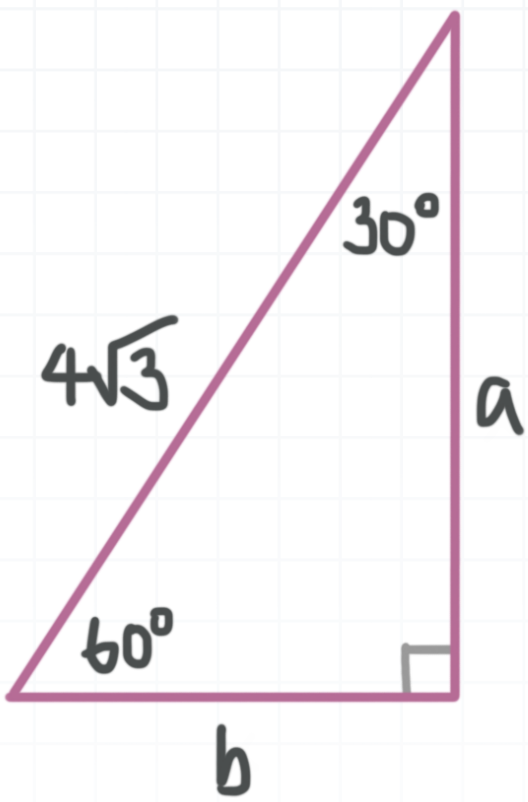


The length of the hypotenuse is $2x$. We know that $x = 6$, so the length of the hypotenuse is $2x = 12$. The length of the long leg is $x\sqrt{3} = 6\sqrt{3}$.

Let's look at another example.

Example

What are the values of a and b?



The pattern for the lengths of the sides of a 30-60-90 triangle is x for the short leg, $x\sqrt{3}$ for the long leg, and $2x$ for the hypotenuse. In this case we know that the length of the hypotenuse is $4\sqrt{3}$, so

$$2x = 4\sqrt{3}$$

$$x = 2\sqrt{3}$$

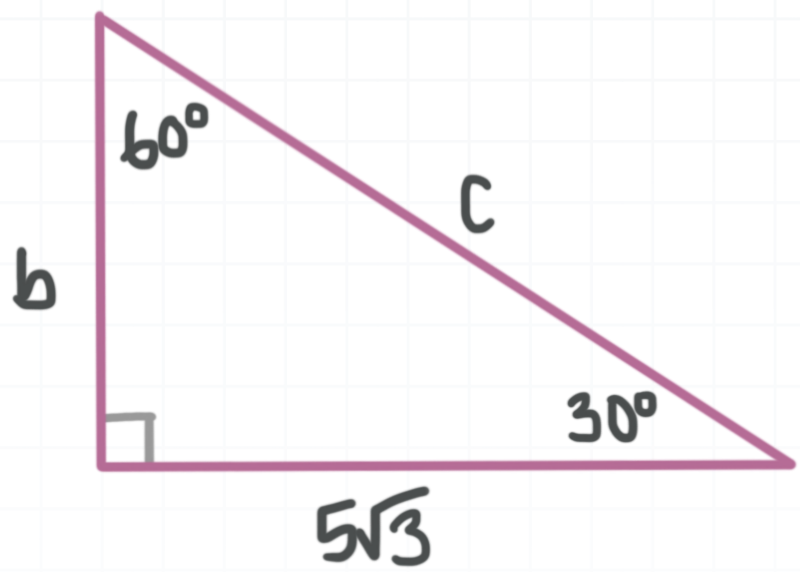
b is the length of the short leg, so $b = x = 2\sqrt{3}$, and a is the length of the long leg, so

$$a = x\sqrt{3} = (2\sqrt{3})(\sqrt{3}) = 2(\sqrt{3} \cdot \sqrt{3}) = 2(3) = 6$$

Let's try another example.

Example

What are the values of b and c?



We know that the length of the long leg is $5\sqrt{3}$, so

$$x\sqrt{3} = 5\sqrt{3}$$

$$x = 5$$

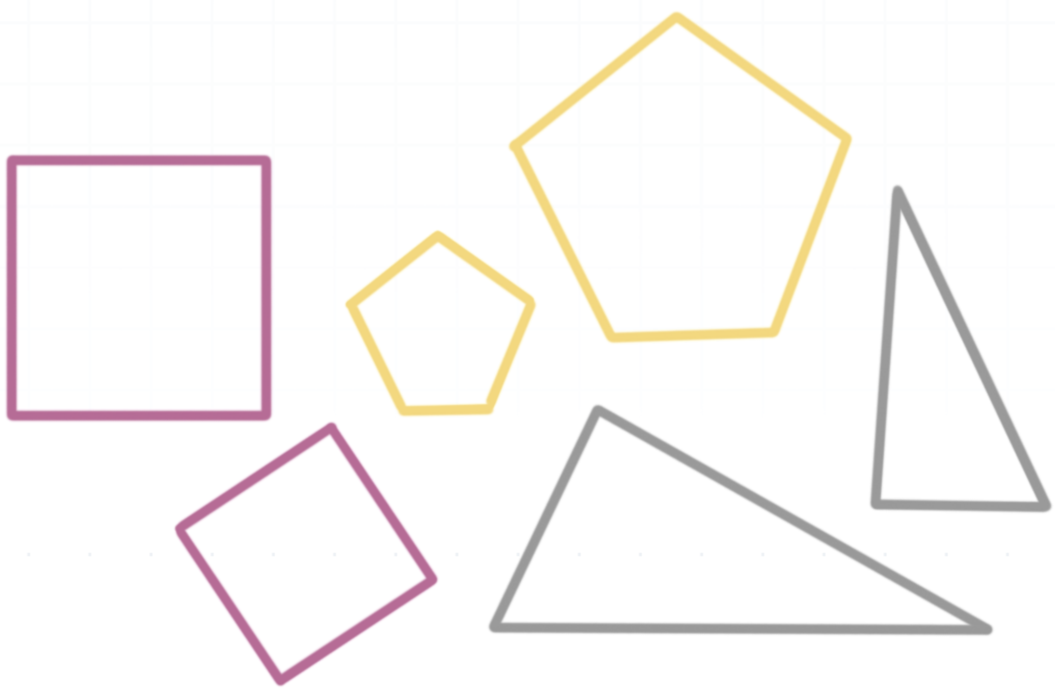
b is the length of the short leg, so $b = x = 5$, and c is the length of the hypotenuse, so

$$c = 2x = 2(5) = 10$$

Triangle similarity theorems

In this lesson we'll look at how to prove that a pair of triangles are similar. In math, the word "similarity" has a very specific meaning. Outside of math, when we say two things are similar, we just mean that they're generally like each other.

But in math, to say two figures are similar means that they have exactly the same shape but different sizes. Here are examples of similar squares, similar pentagons, and similar triangles:



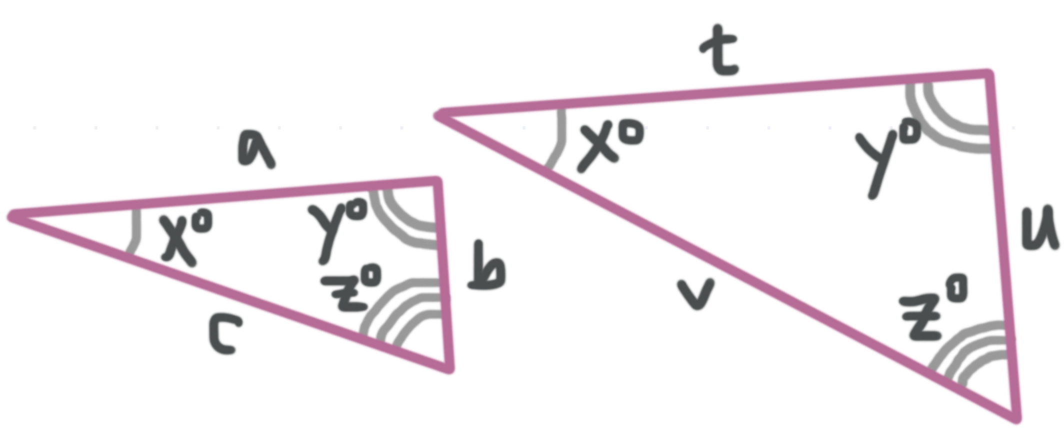
Similar triangles

Similar triangles have the same shape but not the same size. Remember that if two triangles have exactly the same shape and exactly the same size, then we say they're **congruent**. According to one definition of similarity, however, congruence implies similarity; by that definition, congruent triangles are also similar.

In a pair of **similar triangles**, all three pairs of corresponding angles are congruent and the lengths of all three pairs of corresponding sides are proportional. In fact, if all three pairs of corresponding angles are congruent, then the lengths of all three pairs of corresponding sides are automatically proportional, and vice versa. The symbol for similarity is $\sim$, so if we want to say that triangles A and B are similar, we can write that as $A \sim B$.

The triangles below are similar because the corresponding interior angles are congruent, so the lengths of corresponding sides are proportional like this:

$$\frac{a}{t} = \frac{b}{u} = \frac{c}{v}$$



We're going to look at three theorems that allow you to prove that triangles are similar.

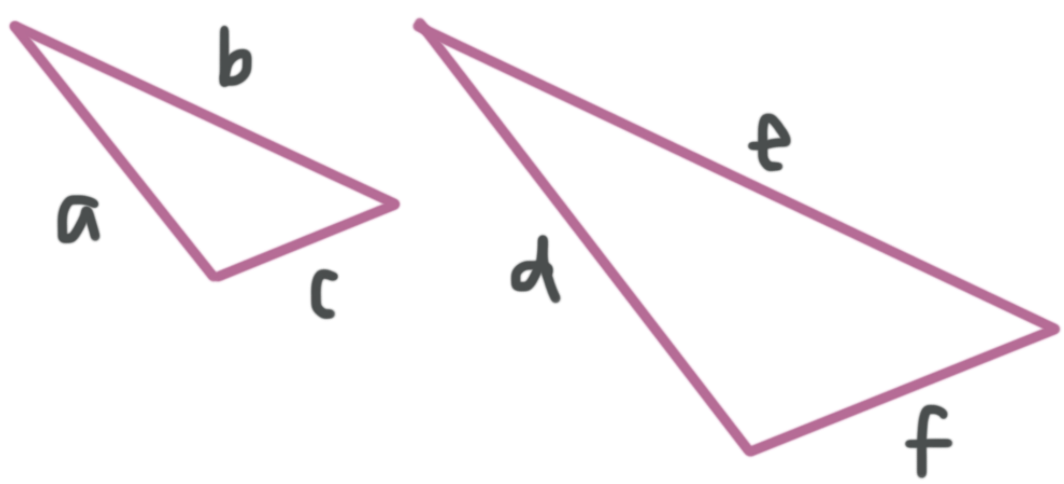
Angle, angle (AA)

If a pair of triangles have two pairs of congruent angles, then the triangles are similar. The reason is that if two pairs of angles are congruent, then the third pair of angles have to be congruent as well because the measures of the interior angles of a triangle always sum to 180°.

Side, side, side (SSS)

If the lengths of all three pairs of sides of a pair of triangles are proportional, then the triangles are similar. The reason is that, if the lengths of all three pairs of sides are proportional, then that forces all three pairs of their interior angles to be congruent.

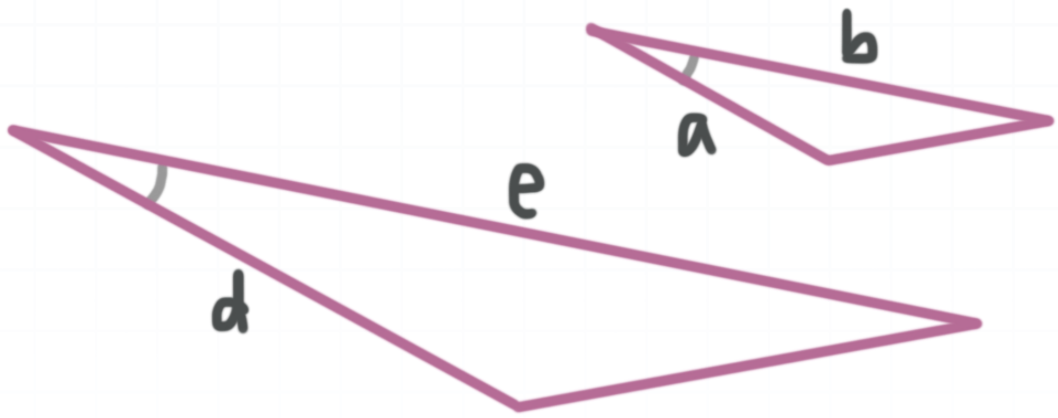
$$\frac{a}{d} = \frac{b}{e} = \frac{c}{f}$$



Side, angle, side (SAS)

If the lengths of two pairs of sides of a pair of triangles are proportional and the corresponding pair of included angles are congruent, then the triangles are similar. Remember that the included angle of two sides of a triangle is the angle whose vertex is the point of intersection of those two sides.

$$\frac{a}{d} = \frac{b}{e}$$

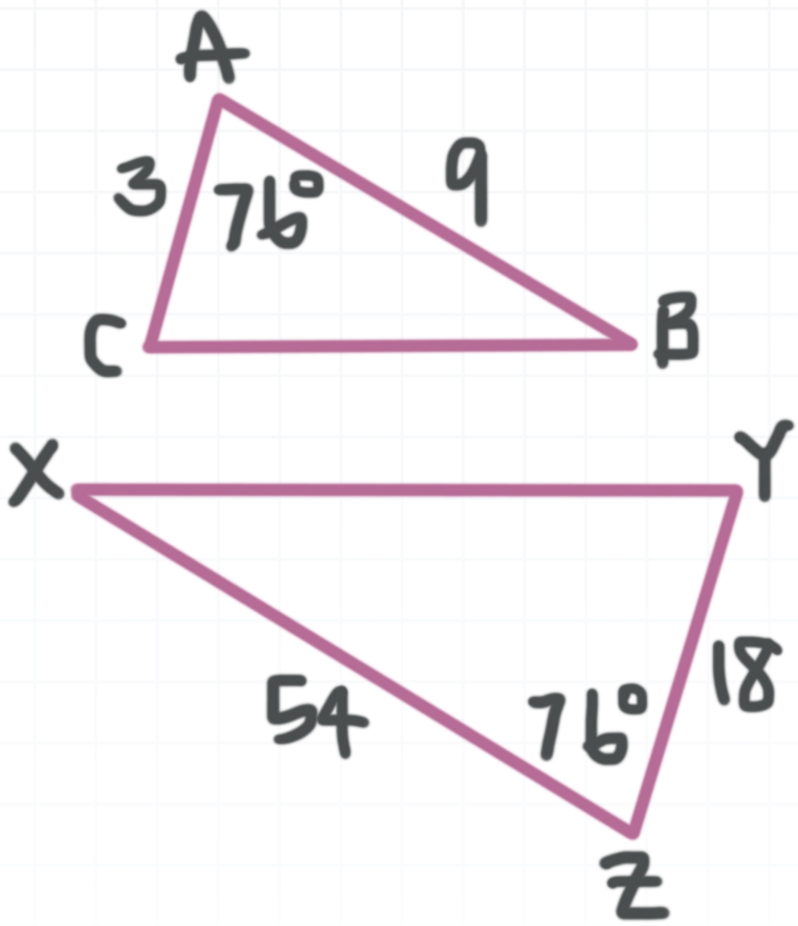


Let's do some practice problems where we use these three theorems to prove similarity of pairs of triangles.

Example

Are the triangles similar? If so, determine which theorem proves that they're similar and complete the similarity statement.

$$\triangle ABC \sim \triangle \underline{\qquad}$$



We know from the figure that $\angle A \cong \angle Z$, because both of those angles have measure 76°. So we have a pair of congruent angles, and we need to see if the lengths of the pair of sides of $\triangle XYZ$ whose included angle has measure 76° (sides $\overline{ZY}$ and $\overline{ZX}$) are proportional to the lengths of the pair of sides of $\triangle ABC$ whose included angle has measure 76° (sides $\overline{AC}$ and $\overline{AB}$, respectively).

$$\frac{\overline{ZY}}{\overline{AC}} = \frac{18}{3} = 6 \qquad\qquad \frac{\overline{ZX}}{\overline{AB}} = \frac{54}{9} = 6$$

We have the same ratio for the lengths of those two pairs of sides.

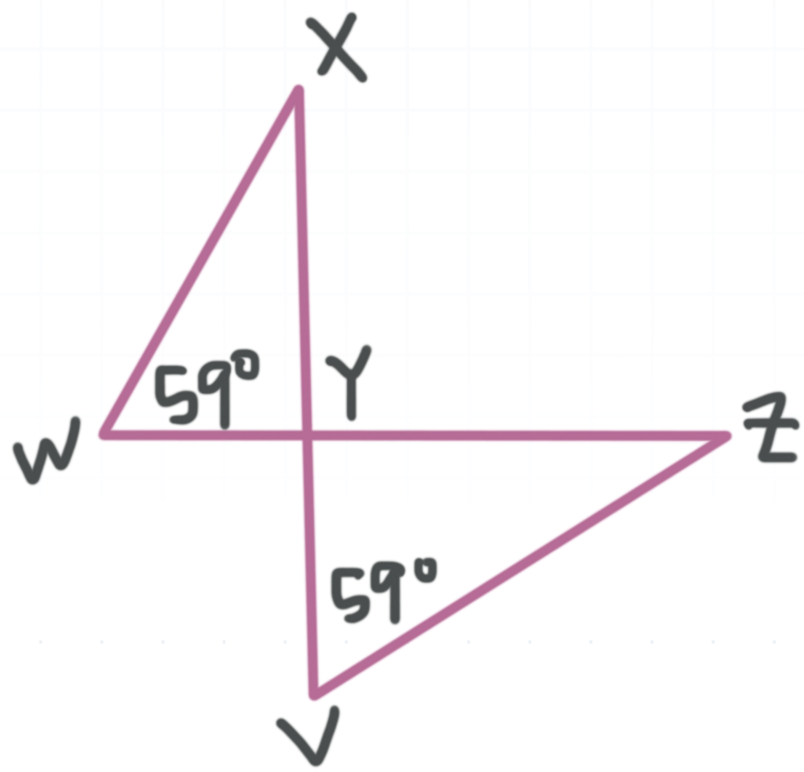
Putting all this together, we know that the triangles are similar by side, angle, side (SAS). When we match up the corresponding parts, we see that the similarity statement is $\triangle ABC \sim \triangle ZXY$.

Let's try another.

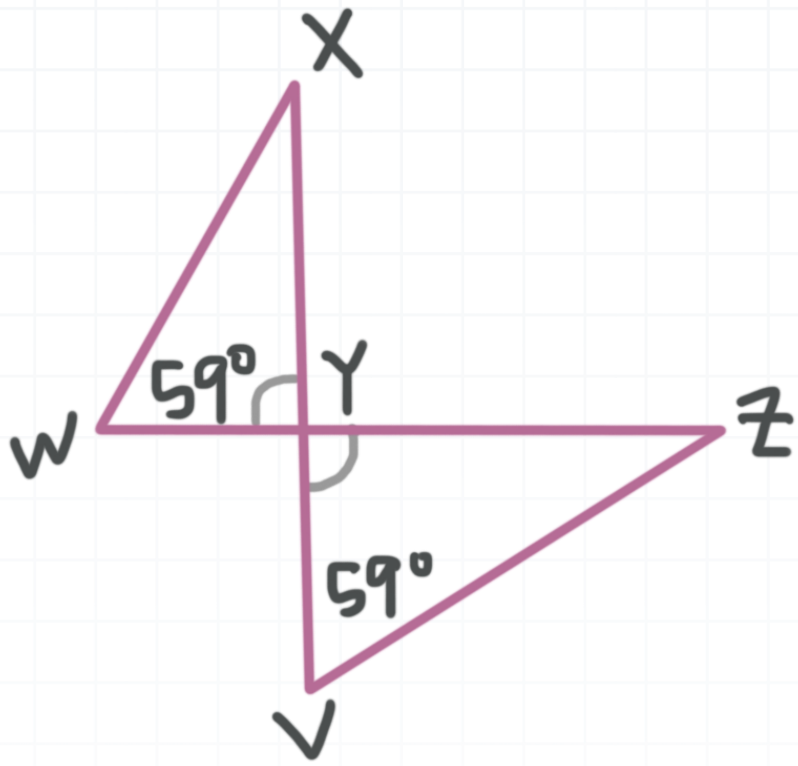
Example

Are the triangles similar? If so, determine which theorem proves that they're similar and complete the similarity statement.

$\triangle WXY \sim \triangle$ ____



We know from the figure that $\angle YWX \cong \angle ZVY$, because both of those angles have measure 59°. We also have a pair of vertical angles at Y, and remember that vertical angles are congruent.



Putting all this together, we know that the triangles are similar by angle, angle (AA). When we match up the corresponding parts, we see that the similarity statement is $\triangle WXY \sim \triangle VZY$.

Triangle side-splitting theorem

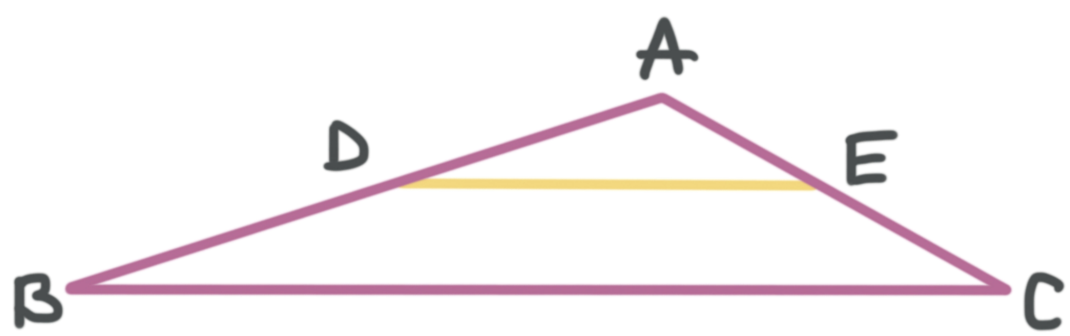
In this lesson we'll look at the triangle side-splitting theorem and how it's used in finding the length of part of a side of a triangle.

Triangle side-splitting theorem

If a line segment intersects two sides of a triangle, and is parallel to the third side of the triangle, then the two sides intersected by the segment are split proportionally (in terms of their lengths).

In the triangle below, segment $\overline{DE}$ is parallel to side $\overline{BC}$, $\overline{DE} \parallel \overline{BC}$, so the segment splits the lengths of sides $\overline{AB}$ and $\overline{AC}$ of the triangle proportionally:

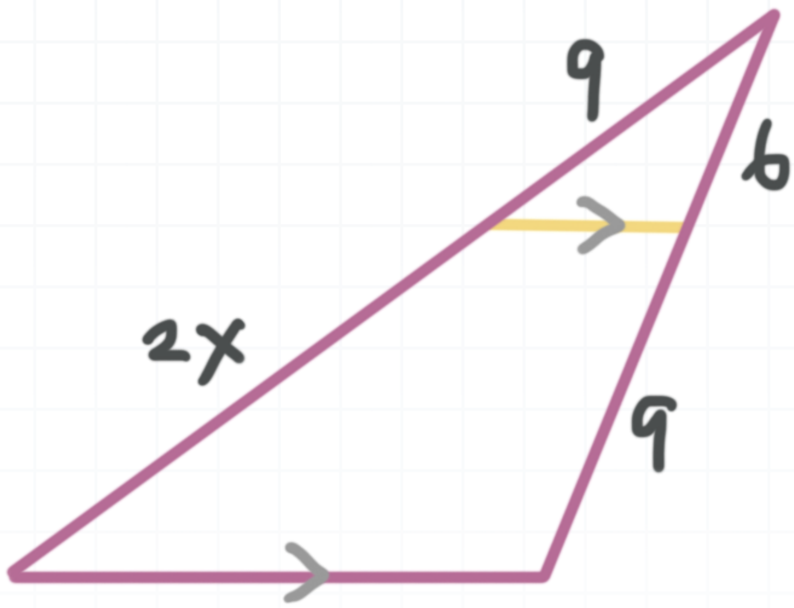
$$\frac{\overline{AD}}{\overline{DB}} = \frac{\overline{AE}}{\overline{EC}}$$



Let's start by working through an example of how to find the length of part of one side of a triangle using a segment that splits two sides of the triangle (including the side in question) proportionally.

Example

In the figure, two sides of the triangle are split by a segment that's parallel to the third side. Find the value of x.



Since the segment is parallel to the third side, we can use the triangle side-splitting theorem to find the value of the variable.

The ratio $9/2x$ has to be equal to $6/9$.

$$\frac{9}{2x} = \frac{6}{9}$$

Cross multiply.

$$9(9) = 6(2x)$$

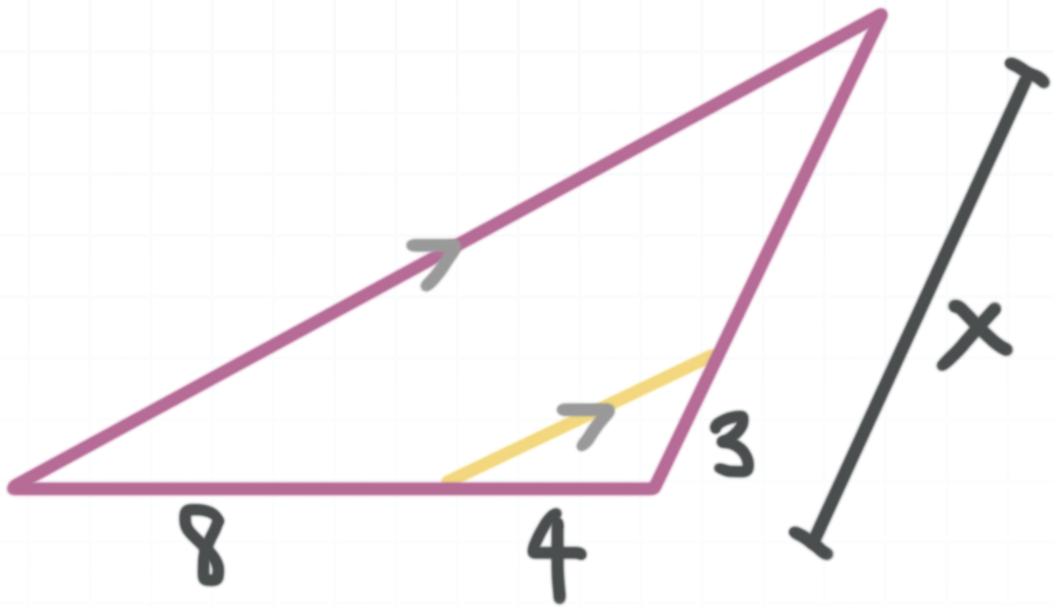
$$81 = 12x$$

$$6.75 = x$$

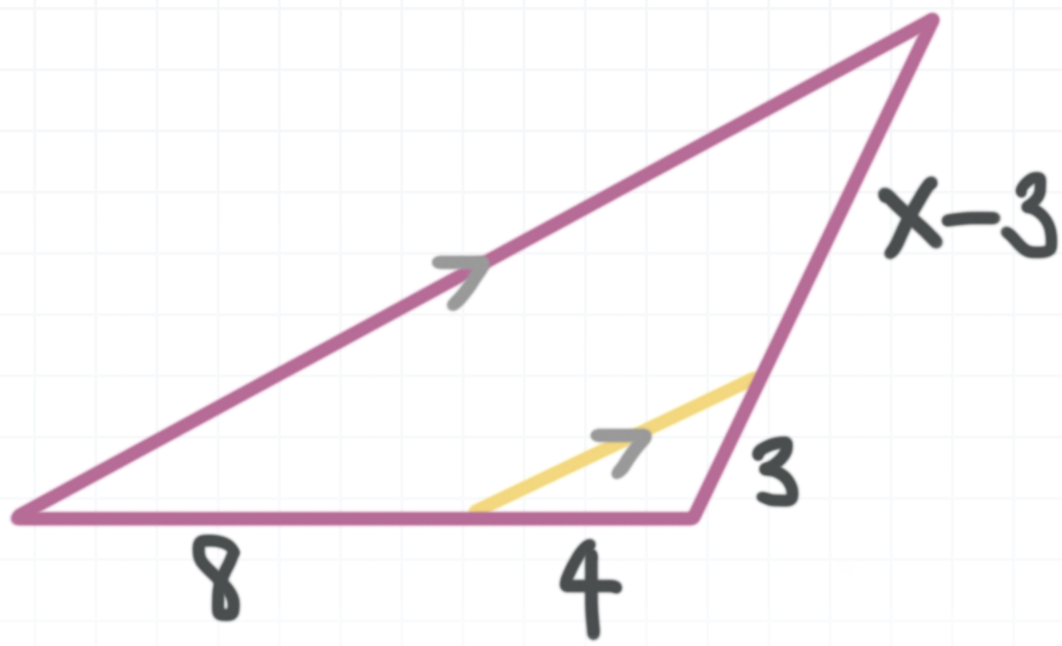
Let's try another one.

Example

In the figure, two sides of the triangle are split by a segment that's parallel to the third side. Find the value of x.



Since the segment is parallel to the third side, we can use the triangle side-splitting theorem to find the value of the variable. We know that the length of one of the sides that are split is x, and that the length of one part of that split side is 3. That means that the length of the other part of that split side is $x-3$.



The ratio $3/(x-3)$ has to be equal to $4/8$.

$$\frac{3}{x-3}=\frac{4}{8}$$

Cross multiply.

$$8(3)=4(x-3)$$

$$24=4x-12$$

$$36=4x$$

$$9=x$$

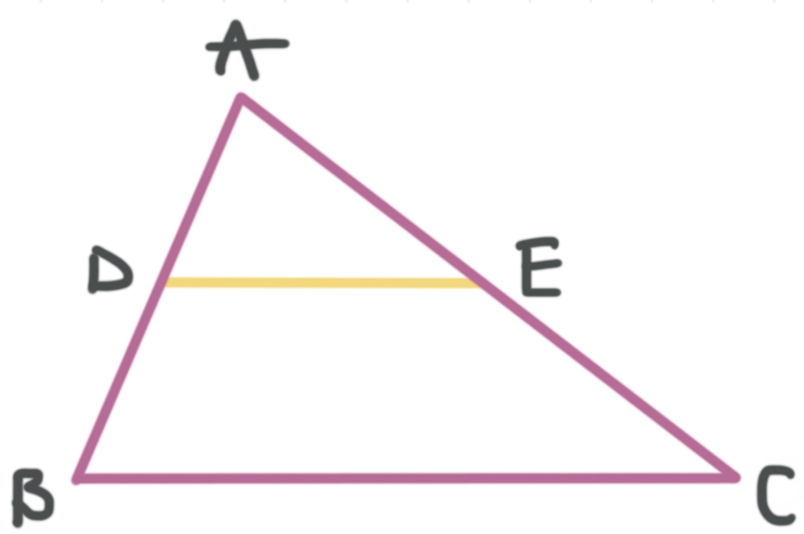
Midsegments of triangles

In this lesson we'll define midsegment of a triangle and show how it's used in finding the length of a side (or part of a side) of a triangle.

Midsegment of a triangle

Like the side-splitting segments we talked about in the previous lesson, a **midsegment** of a triangle is a line segment that intersects two sides of a triangle and is parallel to the third side of the triangle (the side it doesn't intersect).

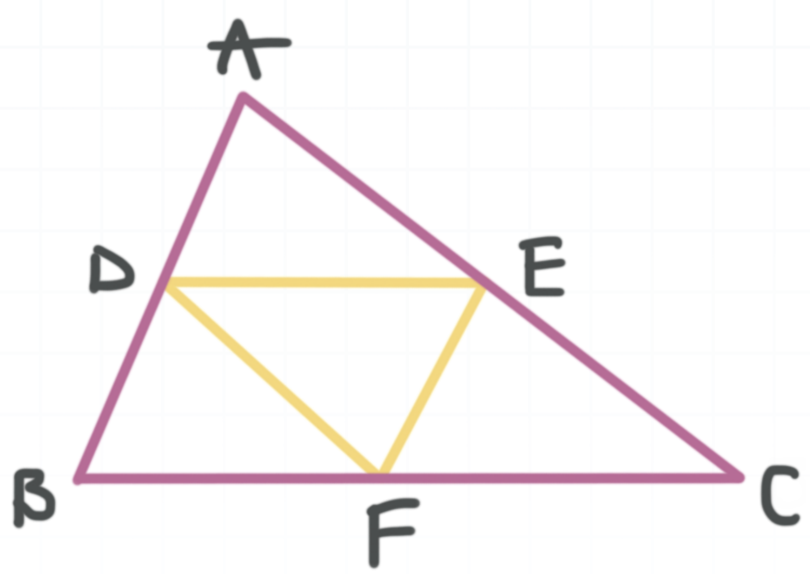
The difference between a midsegment and any other side-splitting segment, is that a midsegment intersects the triangle at the midpoints of the two sides that it splits, so it cuts them in half. So in the figure below, $\overline{DE}$ cuts $\overline{AB}$ and $\overline{AC}$ in half.



Remember that the midpoint of any side of a triangle divides that side into two parts of equal length, which means that $\overline{AD} = \overline{DB}$ and $\overline{AE} = \overline{EC}$.

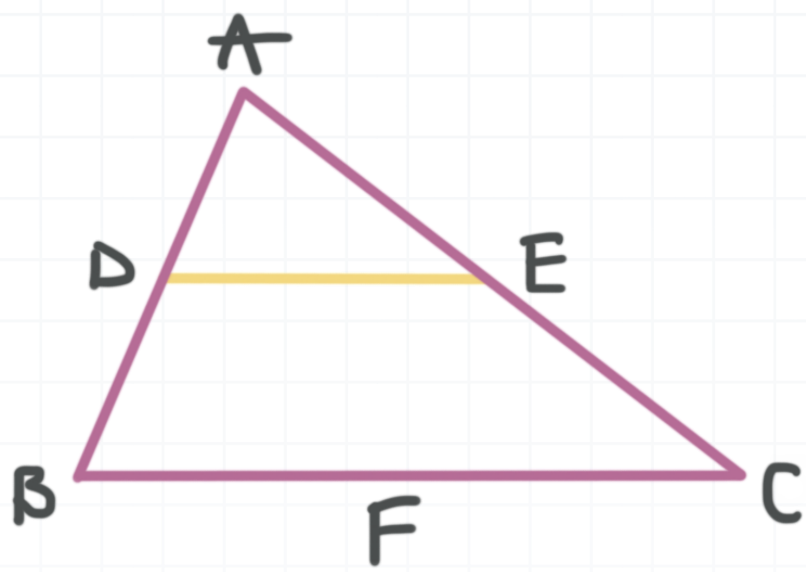
Triangles have three midsegments

Since a triangle has three sides, it also has three midsegments. Each midsegment is parallel to a different side of the triangle. If D is the midpoint of $\overline{AB}$, E is the midpoint of $\overline{AC}$, and F is the midpoint of $\overline{BC}$, then $\overline{DE}$, $\overline{DF}$, and $\overline{EF}$ are all midsegments of triangle ABC.



Midsegment of a triangle theorem

A midsegment of a triangle is parallel to the third side of the triangle (the side that the midsegment doesn't intersect), and the length of the midsegment is equal to half of the length of the third side. This means that if $\overline{DE}$ is a midsegment of this triangle,



then

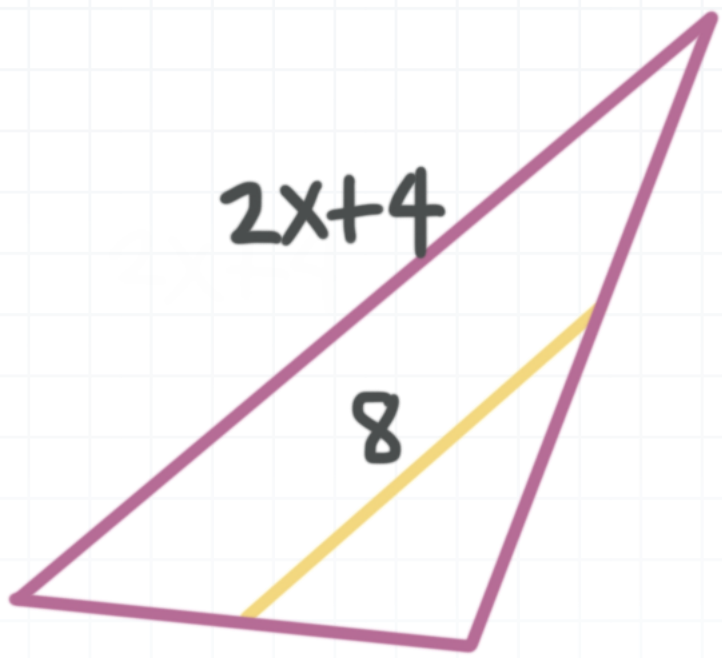
$$\overline{DE} \parallel \overline{BC} \text{ and } \overline{DE} = (1/2)\overline{BC}$$

Then it's also true that, if F is the midpoint of $\overline{BC}$, then $\overline{DE} = \overline{BF} = \overline{FC}$.

Let's work through some examples with the midsegment of a triangle theorem.

Example

If the segment of length 8 is a midsegment of the triangle in the figure, what's the value of x?



Because the segment of length 8 is a midsegment of this triangle, we know that its length is half that of the side of length $2x+4$, so

$$8=\frac{1}{2}(2x+4)$$

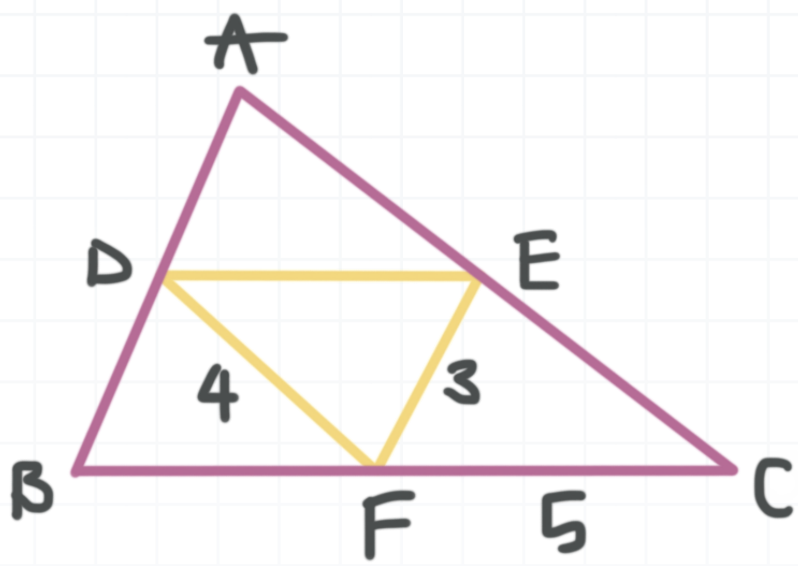
$$8=x+2$$

$$6=x$$

Let's try one with a few more steps.

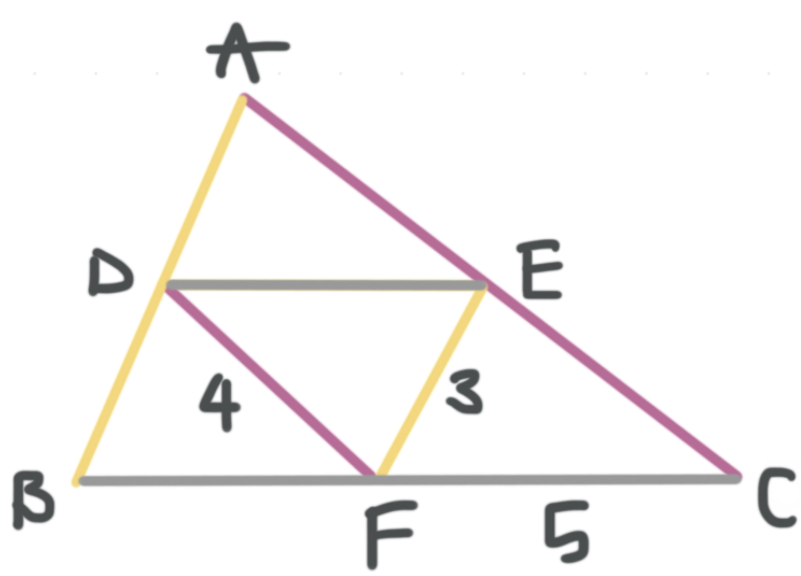
Example

If D is the midpoint of $\overline{AB}$, E is the midpoint of $\overline{AC}$, and F is the midpoint of $\overline{BC}$, find the perimeter of triangle ABC.

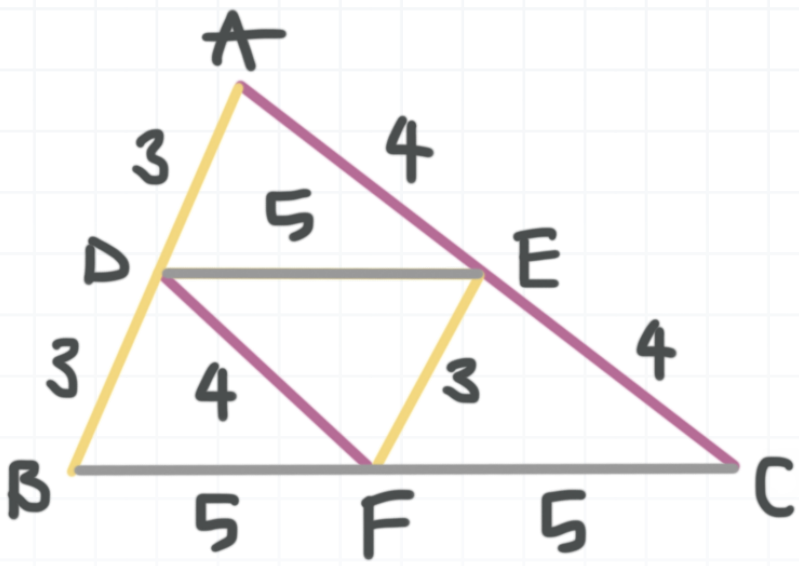


$\overline{DE}$, $\overline{DF}$, and $\overline{EF}$ are all midsegments of triangle ABC, which means we can determine the lengths of all three sides of this triangle by using the fact that the length of a midsegment of a triangle is half the length of the third side of the triangle (the side that the midsegment doesn't intersect).

Let's color each midsegment the same as the corresponding "third side of the triangle."



Now we can fill in what we know.



To find the perimeter, we'll just add the lengths of the two halves of each side of the triangle.

$$P = (3 + 3) + (4 + 4) + (5 + 5)$$

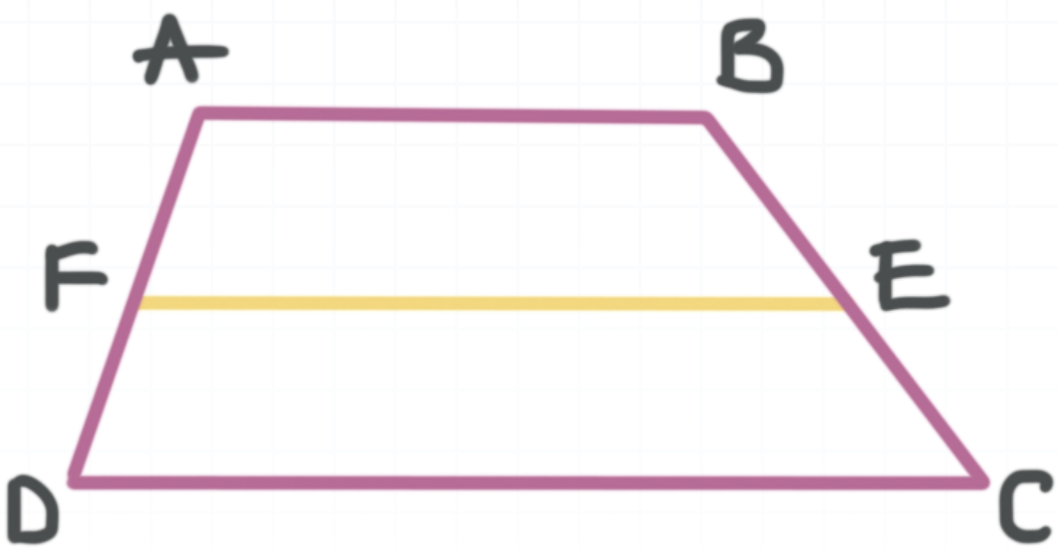
$$P = 6 + 8 + 10$$

$$P = 24$$

Midsegments of trapezoids

The midsegment of a trapezoid is the segment that connects the midpoints of the opposite non-parallel sides of the trapezoid.

If $\overline{AB} \parallel \overline{DC}$, if F is the midpoint of $\overline{AD}$, and if E is the midpoint of $\overline{BC}$, then $\overline{FE}$ is the midsegment of the trapezoid.



The relationship between the length of the midsegment and the lengths of the parallel sides is

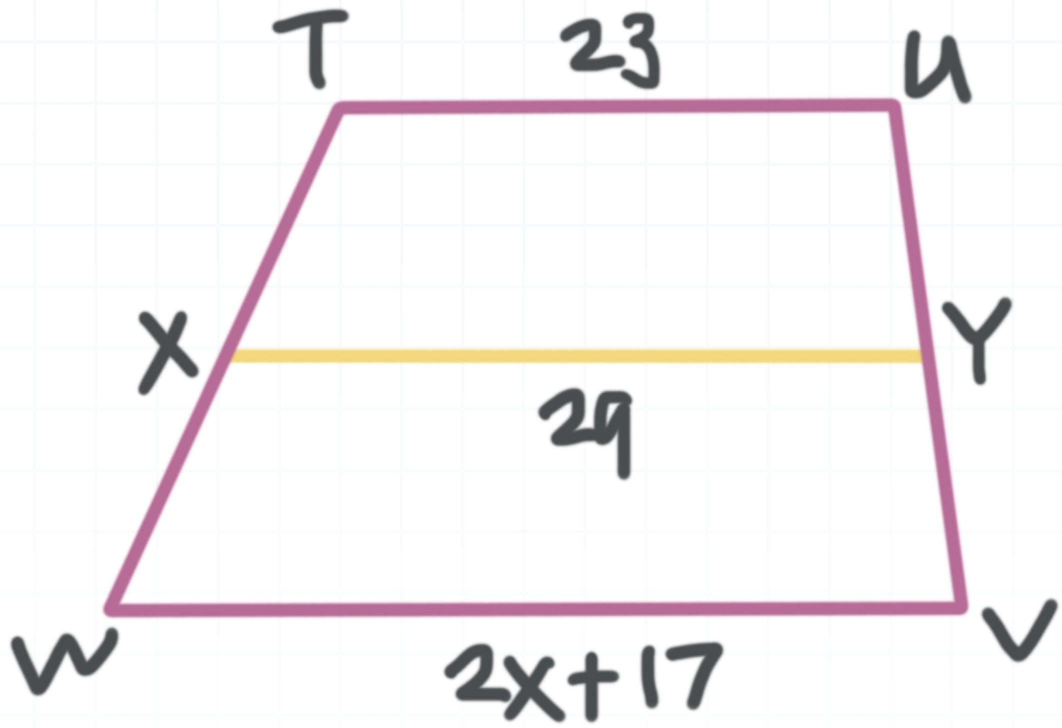
$$\overline{FE} = \frac{1}{2}(\overline{AB} + \overline{DC})$$

The length of the midsegment of a trapezoid is always equal to half of the sum of the lengths of the parallel sides (also called the bases) of the trapezoid.

Let's work through a couple of examples.

Example

In the trapezoid pictured, $\overline{TU} \parallel \overline{WV}$, X is the midpoint of $\overline{TW}$, and Y is the midpoint of $\overline{UV}$. What is the length of $\overline{WV}$?



By definition, $\overline{XY}$ is the midsegment of the trapezoid. Therefore, we know that

$$\overline{XY} = \frac{1}{2}(\overline{TU} + \overline{WV})$$

Let's plug in what we know and then solve for x.

$$29 = \frac{1}{2}[23 + (2x + 17)]$$

$$29 = \frac{1}{2}(40 + 2x)$$

$$29 = 20 + x$$

$$9 = x$$

Therefore,

$$\overline{WV} = 2x + 17$$

$$\overline{WV} = 2(9) + 17$$

$$\overline{WV} = 18 + 17$$

$$\overline{WV} = 35$$

Let's try one with a few more steps.

Example

In the coordinate plane, a trapezoid $XYWZ$ has vertices at $X = (-2,2)$, $Y = (3,2)$, $Z = (-3,-2)$, and $W = (4,-2)$. What is the length of the midsegment of the trapezoid?

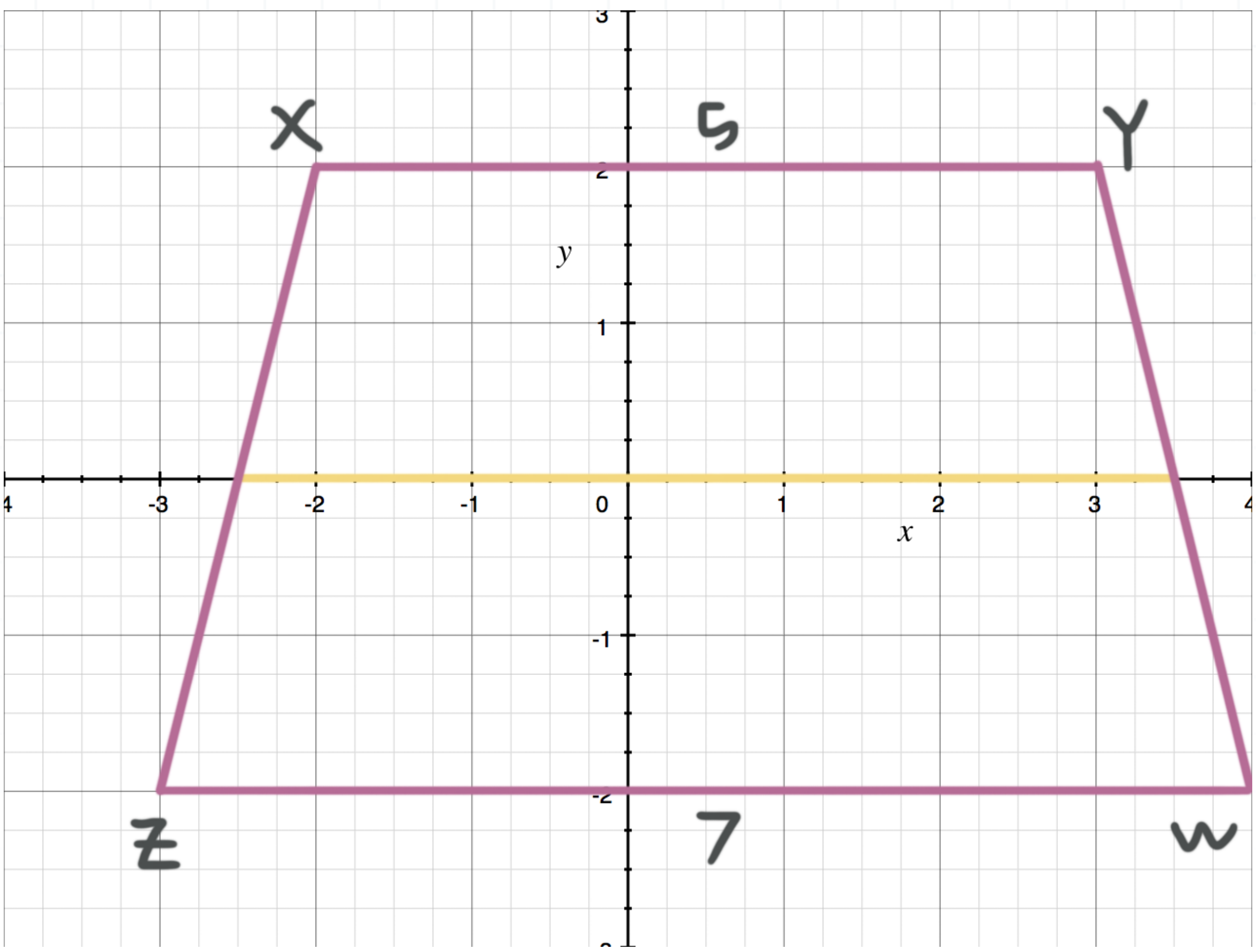
You can plot the trapezoid to identify the parallel sides and find find their lengths.

Notice that the vertices X and Y have the same y-coordinate of 2, and that vertices Z and W have the same y-coordinate of -2. That means that $\overline{XY} \parallel \overline{ZW}$ ($\overline{XY}$ and $\overline{ZW}$ are the parallel sides of the trapezoid). Also, since the y-coordinates of vertices X and Y are equal, the length of $\overline{XY}$ is the difference in their x-coordinates (which are 3 and -2), so

$$\overline{XY} = 3 - (-2) = 5$$

Similarly, since the y-coordinates of vertices Z and W are equal, the length of $\overline{ZW}$ is the difference in their x-coordinates (which are 4 and -3), so

$$\overline{ZW} = 4 - (-3) = 7$$



Remember that the length of the midsegment is equal to half of the sum of the lengths of the parallel sides, so the length of the midsegment is

$$\frac{1}{2}(\overline{XY} + \overline{ZW}) = \frac{1}{2}(5 + 7) = \frac{1}{2}(12) = 6$$

Translating figures in coordinate space

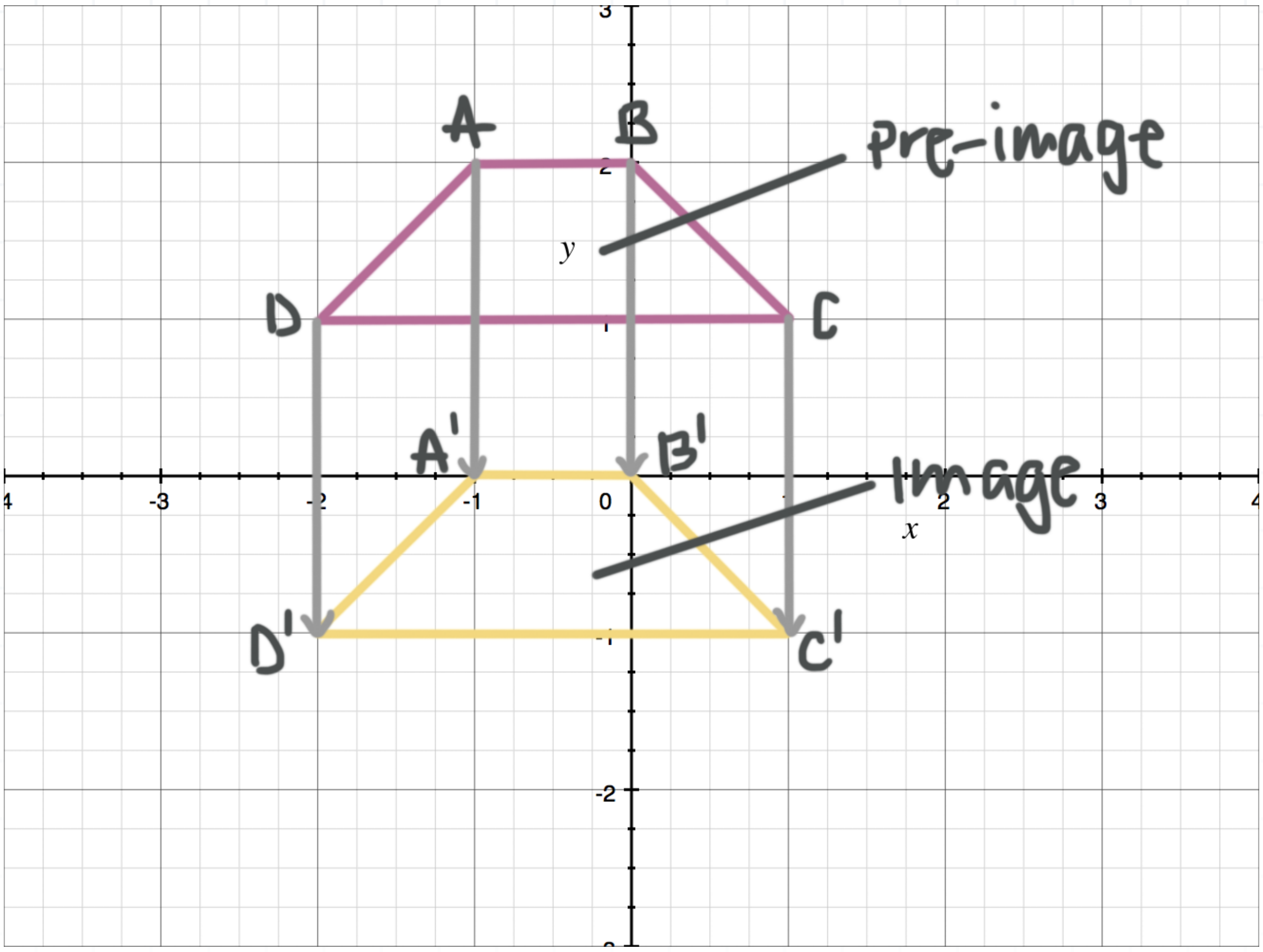
In this lesson we'll look at translation of a figure in a coordinate plane and how to determine where the figure is located after the translation takes place.

A **translation** is a transformation that moves a figure from one location to another. A translation can be thought of as a slide with no rotation. The slide won't change the shape or size of the figure, and because there's no rotation, the orientation won't change either.

The pre-image and image

Before a translation, we have the **pre-image** (the figure in its original location and orientation). Points in the pre-image are usually labeled with capital letters. After the translation, we have the **image** (the figure in its final location and orientation). Points in the image are usually labeled with the same capital letters, plus the prime symbol $'$ after each letter. So if figure $ABCD$ is translated, its image becomes figure $A'B'C'D'$.

In a translation, the image and pre-image are always congruent, because a translation never changes the measures of angles or the lengths of line segments and curves in the figure.



Translation notation

A translation can be vertical, horizontal, or both. Regardless of the direction, a translation can be written in mathematical notation. We'll use a "rule" $T(x, y)$ that express the coordinates of a point in the image in terms of the coordinates (x, y) of the corresponding point in the pre-image.

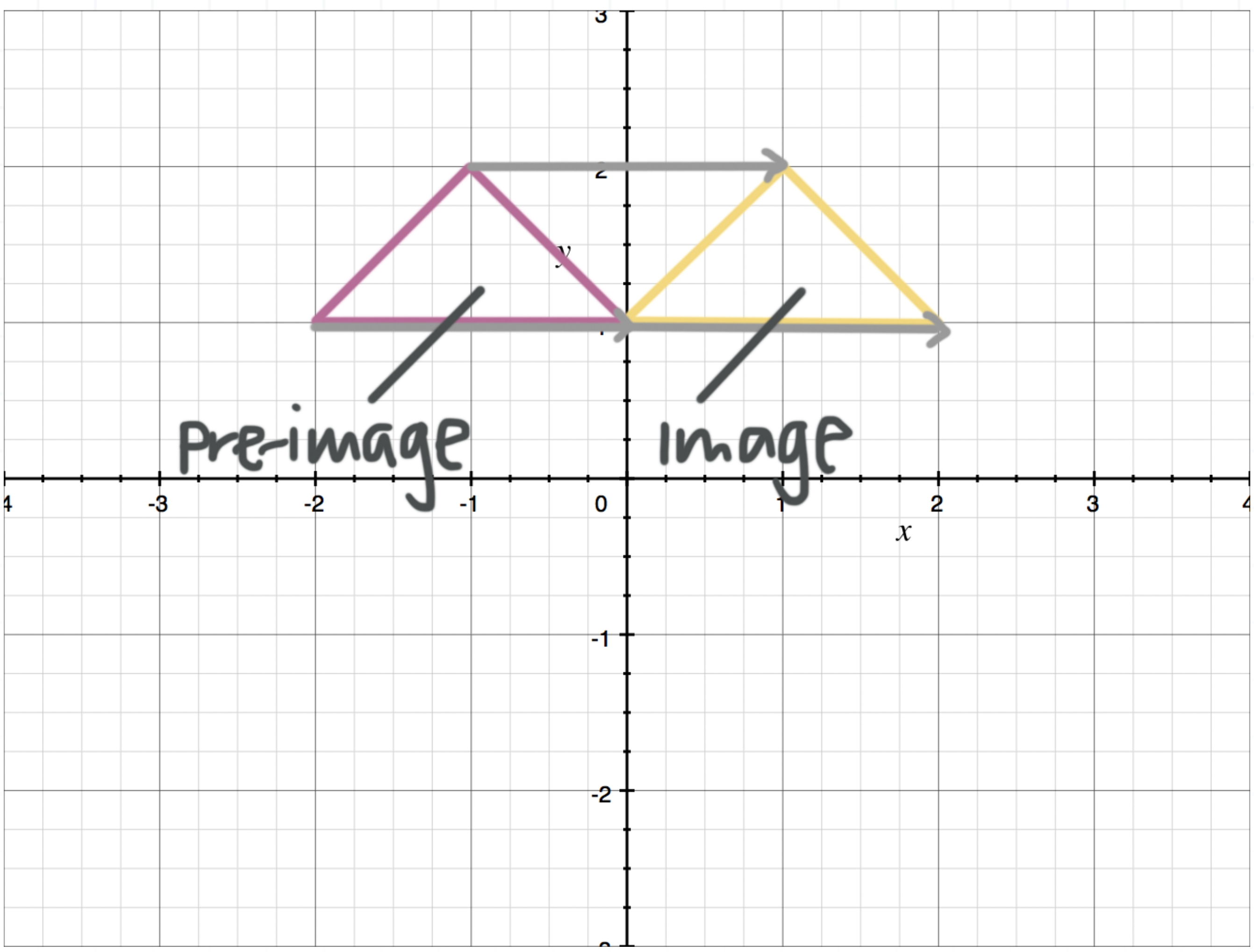
A translation 3 units to the **left**: $T(x, y) = (x - 3, y)$

A translation 2 units to the **right**: $T(x, y) = (x + 2, y)$

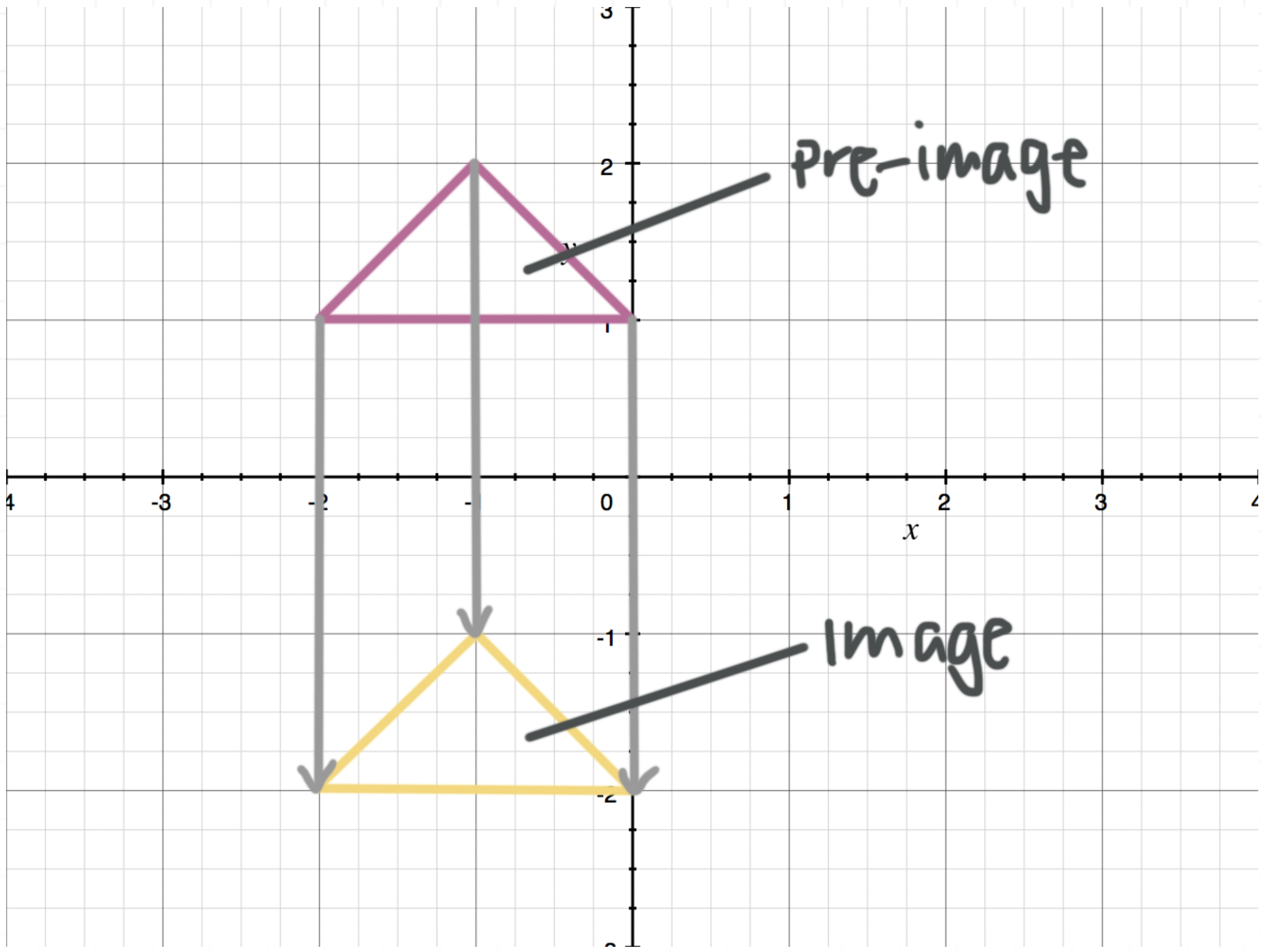
A translation 4 units **down**: $T(x, y) = (x, y - 4)$

A translation 1 unit **up**: $T(x, y) = (x, y + 1)$

In the translation of the triangle in this figure, the pre-image is translated 2 units to the right to get the image, so $T(x, y) = (x + 2, y)$.



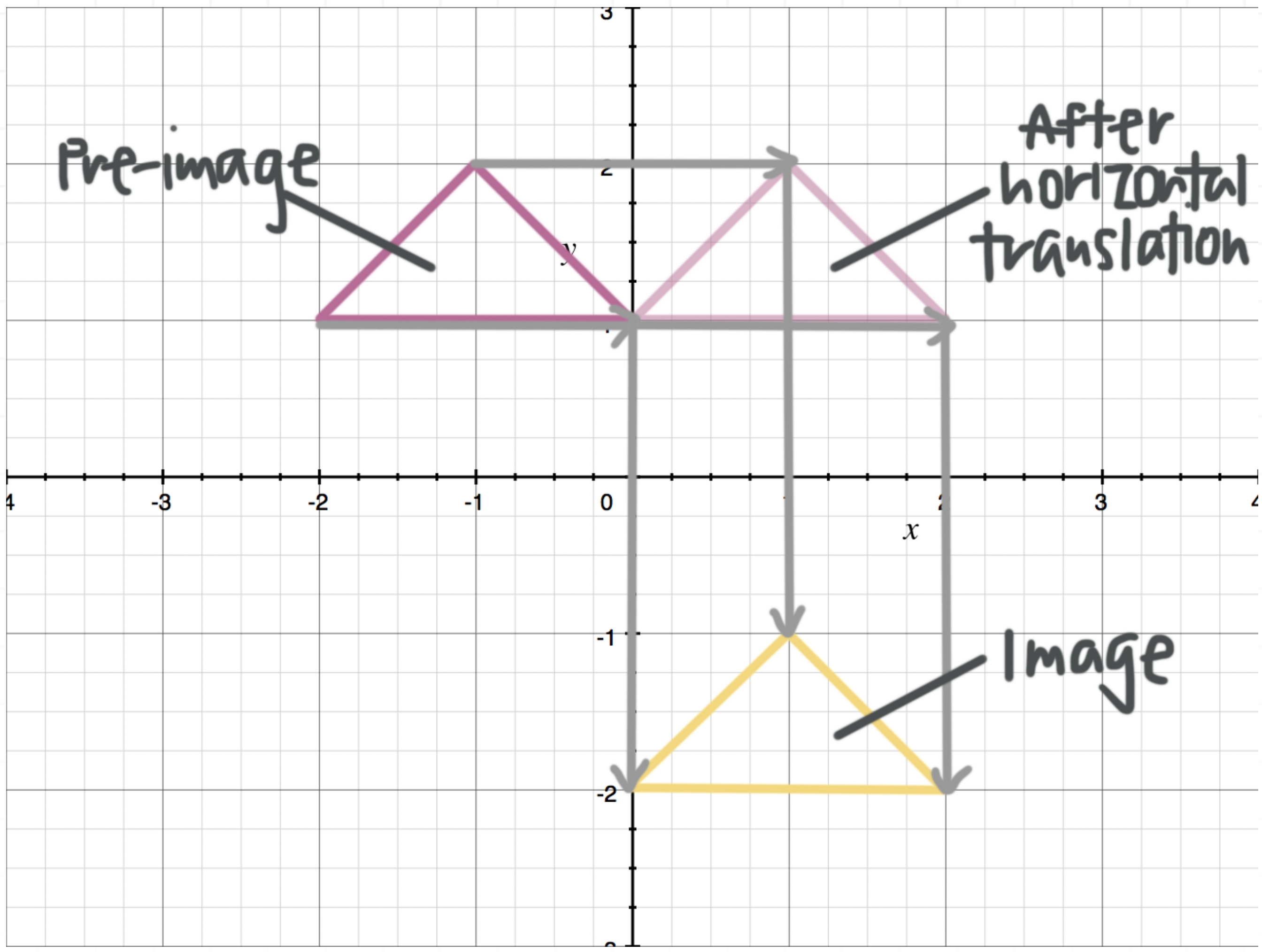
In the translation of the triangle in the next figure, the pre-image is translated 4 units down to get the image, so $T(x, y) = (x, y - 4)$.



Two translations together

We said that we can do a translation that's both horizontal and vertical. If we want to translate a figure 2 units to the right and 4 units down, then the rule for the translation is

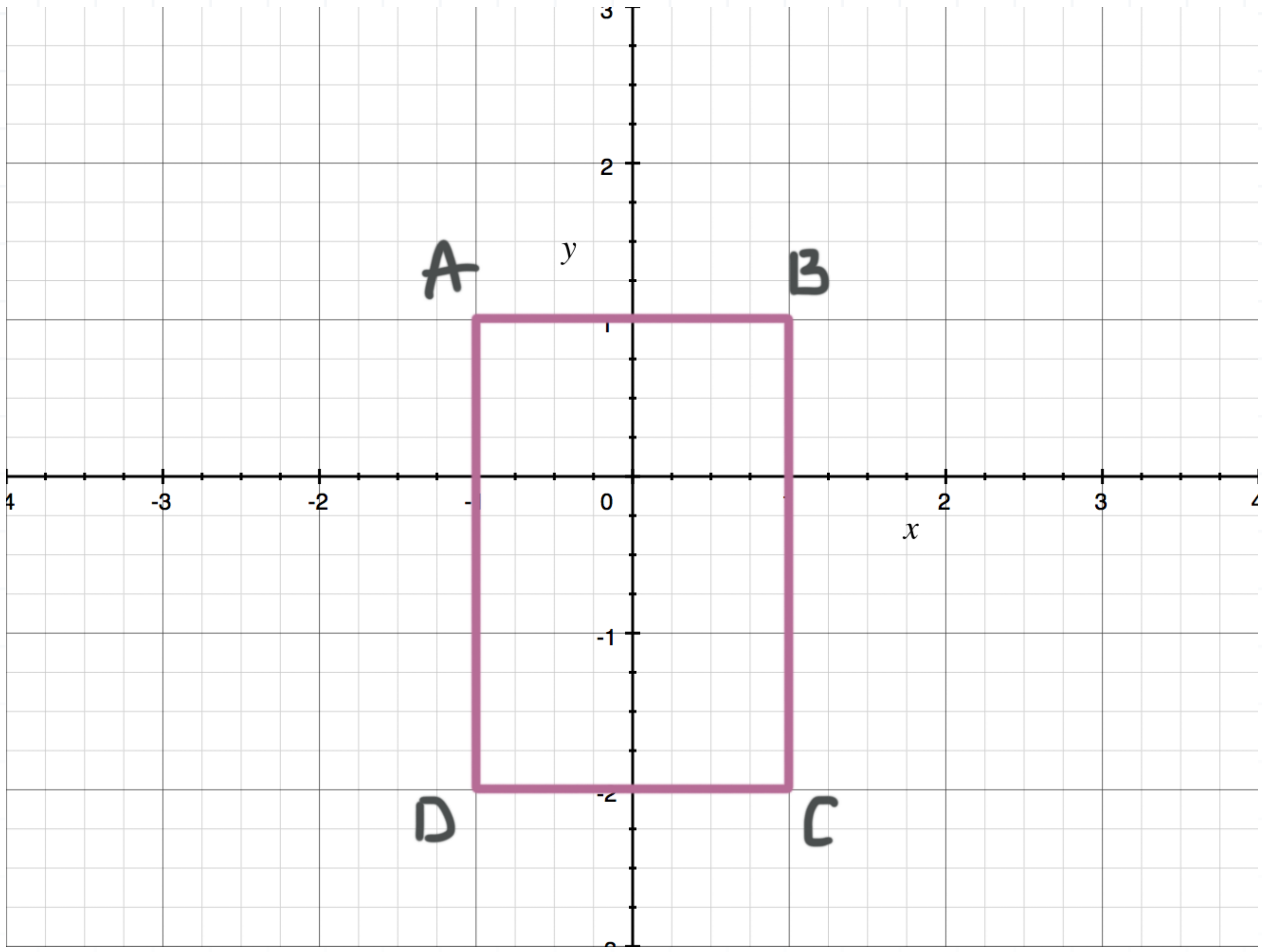
$$T(x, y) = (x + 2, y - 4)$$



Let's start by working through an example.

Example

If rectangle $ABCD$ undergoes the translation described by $T(x, y) = (x - 2, y + 3)$, where will point D' be located?



The translation is $T(x, y) = (x - 2, y + 3)$. The $x - 2$ tells you that the x-coordinate of any point in the image will be 2 less than the x-coordinate of the corresponding point in the pre-image, and that the y-coordinate of any point in the image will be 3 more than the y-coordinate of the corresponding point in the pre-image.

In other words, after the translation the figure will be located 2 units to the left of and 3 units above, its original location. The coordinates of point D are $(-1, -2)$, so the coordinates of point D' are given by

$$T(-1, -2) = (-1 - 2, -2 + 3) = (-3, 1)$$

So $D' = (-3, 1)$.

Let's try another translation problem.

Example

If a translation moves a point A to a point A', write the rule for the translation if $A=(-3,7)$ and $A'=(5,-2)$.

Let's look at what happens to each coordinate.

The x-coordinate: $-3 \to 5$

This means we added 8 because $-3+8=5$.

The y-coordinate: $7 \to -2$

This means we subtracted 9 because $7-9=-2$.

Now we can put this all together to write the rule for the translation.

$$T(x,y)=(x+8,y-9)$$

Rotating figures in coordinate space

In this lesson we'll look at rotation of a figure in a coordinate plane and how to determine the location and orientation of the figure after the rotation takes place.

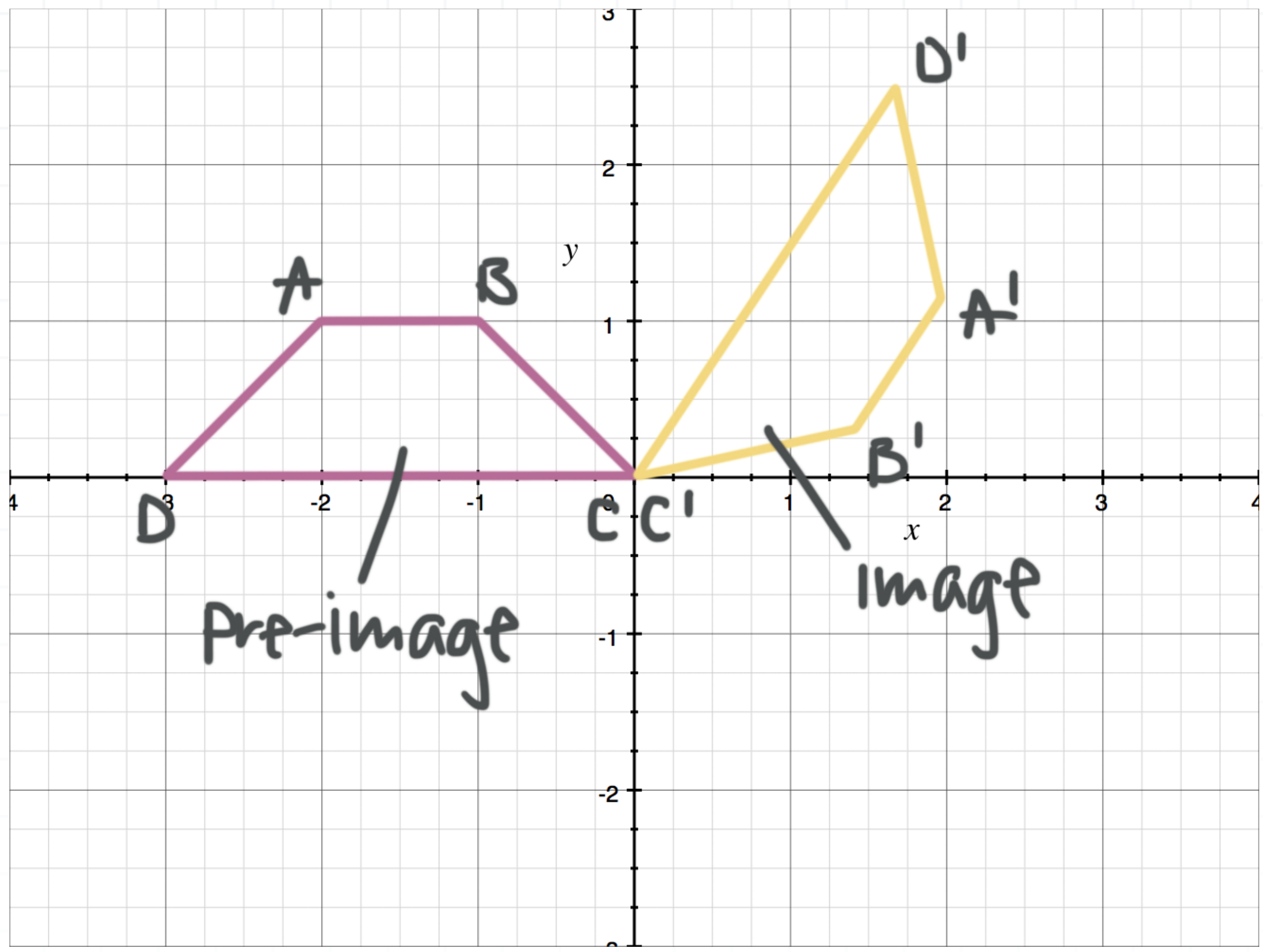
A **rotation** is a type of transformation that turns a figure around a central point, called the **point of rotation**, with no translation of the figure. The point of rotation, which remains fixed (it isn't moved by the rotation), can be inside or outside of the figure.

The pre-image and image

Before a rotation, we have the **pre-image** (the figure in its original location and orientation). Points in the pre-image are usually labeled with capital letters.

After the rotation, we have the **image** (the figure in its final location and orientation). Points in the image are usually labeled with the same capital letters, plus the prime symbol $'$ after each letter. So if figure $ABCD$ is rotated, its image becomes figure $A'B'C'D'$.

In a rotation, the image and pre-image are always congruent, because a rotation never changes the measures of angles or the lengths of line segments and curves in the figure.



Rotating figures

To rotate a figure you need three things:

1. a direction (clockwise or counterclockwise),
2. an angle (the number of degrees through which you're rotating the figure), and
3. the point of rotation

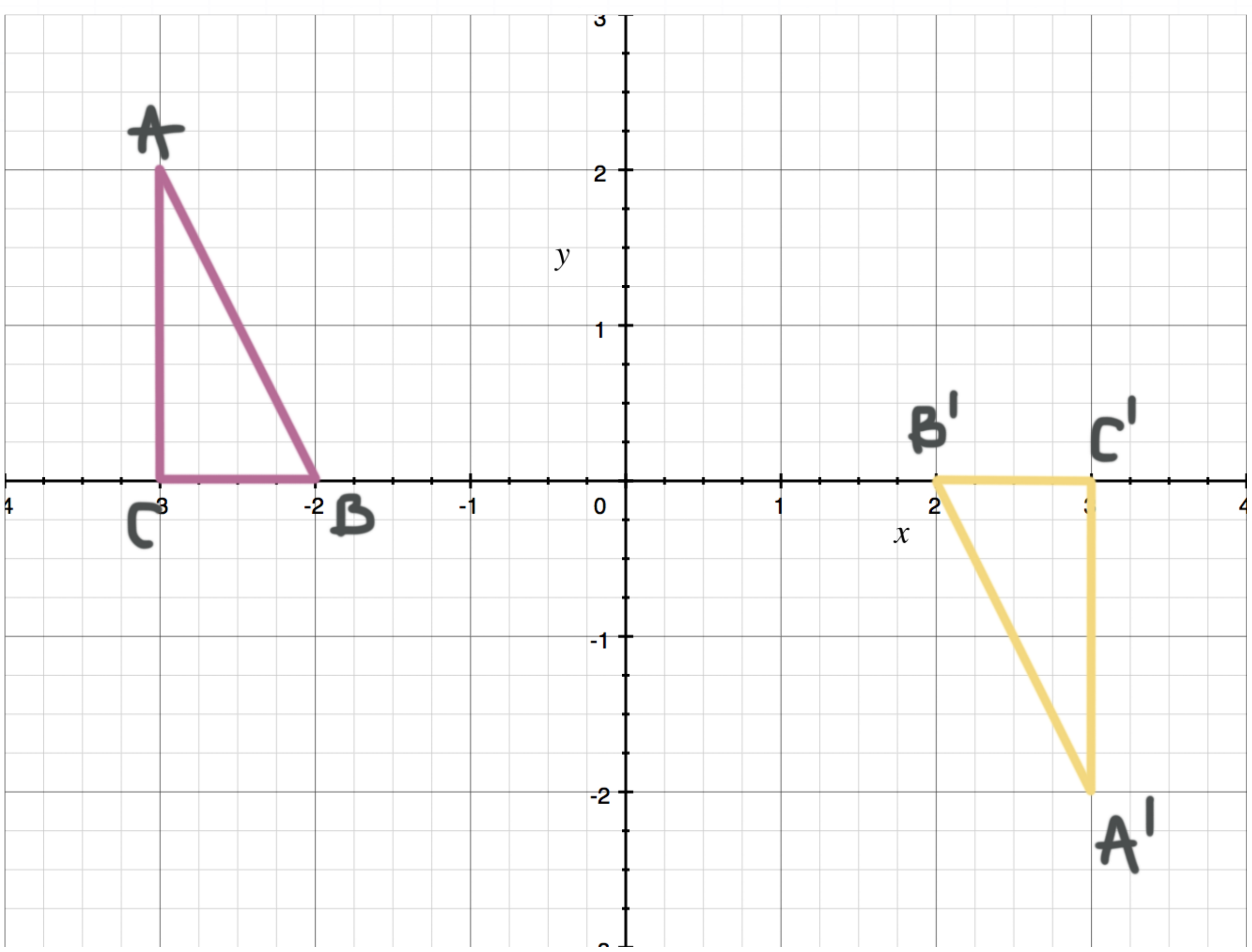
There are some common rotations about the origin, $(0,0)$, that we can easily use rules for:

Angle	Direction	Rule
90 degrees	Clockwise	(x, y) --> (y, -x)
90 degrees	Counterclockwise	(x, y) --> (-y, x)
180 degrees	Clockwise	(x, y) --> (-x, -y)
180 degrees	Counterclockwise	(x, y) --> (-x, -y)
270 degrees	Clockwise	(x, y) --> (-y, x)
270 degrees	Counterclockwise	(x, y) --> (y, -x)

Let's look at some examples.

Example

Write a rule to describe the rotation shown in the graph.



You can visually see that the triangle has been rotated 180° about the origin, but you could also look at the rules to see if this follows any of them. Let's compare the coordinates of the points in the pre-image to the coordinates of the corresponding points in the image.

The points in the pre-image are

$$A = (-3,2)$$

$$B = (-2,0)$$

$$C = (-3,0)$$

The corresponding points in the image are

$$A' = (3,-2)$$

$$B' = (2,0)$$

$$C' = (3,0)$$

Now we compare the points:

$$A = (-3,2) \quad \rightarrow \quad A' = (3,-2)$$

$$B = (-2,0) \quad \rightarrow \quad B' = (2,0)$$

$$C = (-3,0) \quad \rightarrow \quad C' = (3,0)$$

When you compare the coordinates, you can see they're following the rule for a 180° rotation clockwise around the origin.

Angle	Direction	Rule
180 degrees	Clockwise	(x, y) --> (-x, -y)

So the relationship between the coordinates of any point in the pre-image and the coordinates of the corresponding point in the image is

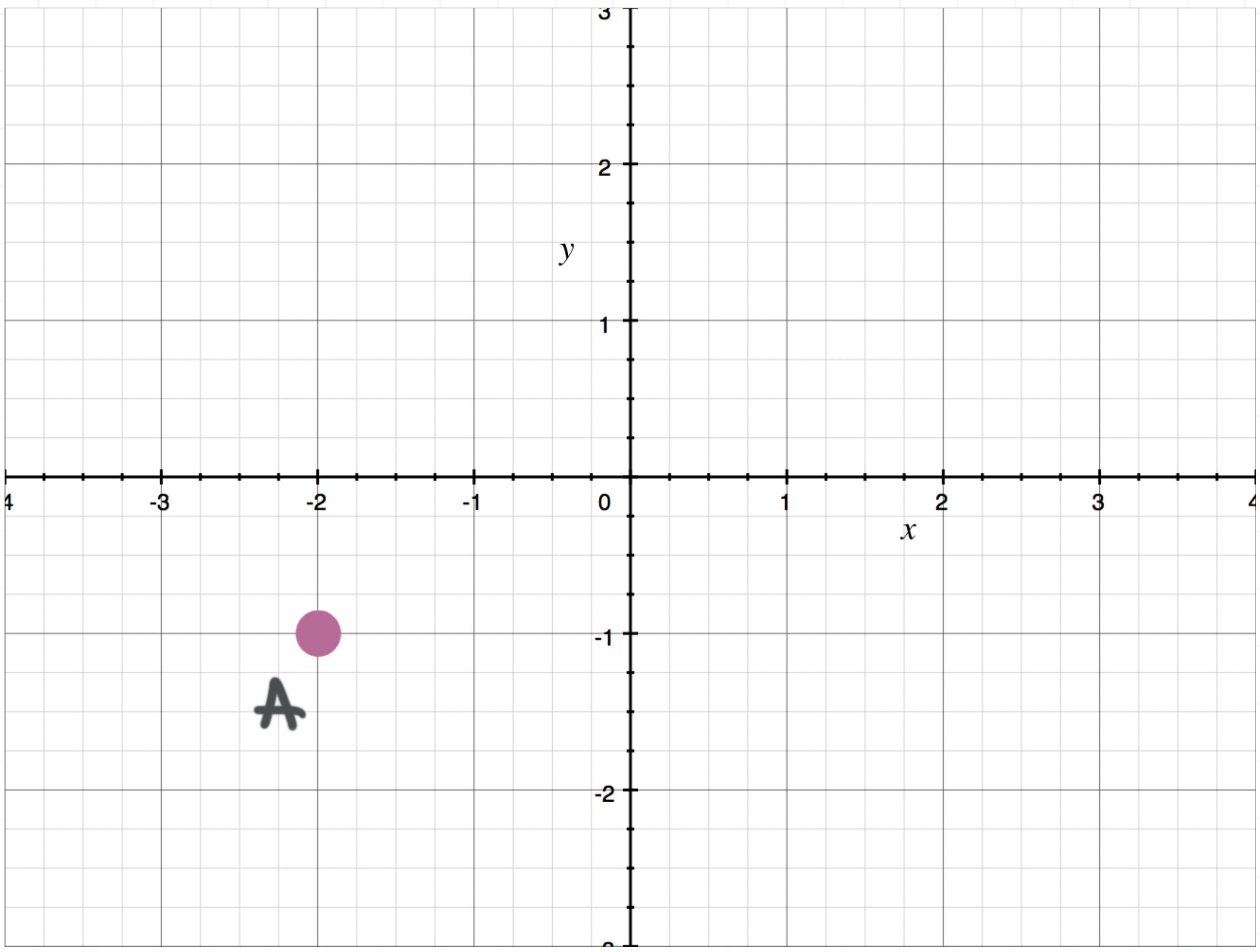
$$(x, y) \to (-x, -y)$$

Since the rule for a counterclockwise rotation of 180° about the origin is the same as the rule for a clockwise rotation of 180° about the origin, you could say either that the figure in this example has been rotated 180° clockwise about the origin or that it's been rotated 180° counterclockwise about the origin.

Let's try another example.

Example

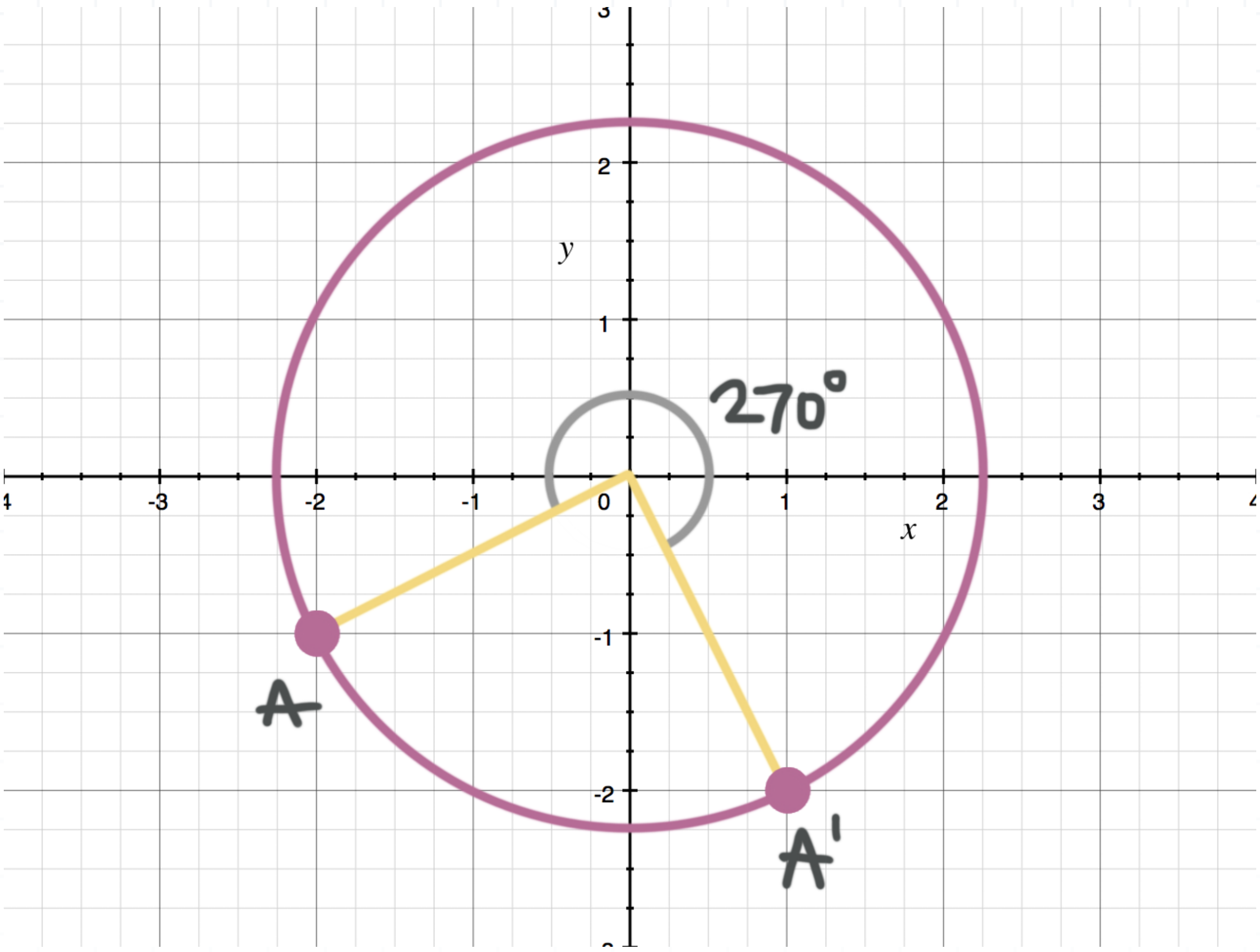
To what point A' will point A be moved in a 270° clockwise rotation around the origin?



Look up the rule for a rotation of 270° clockwise around the origin.

Angle	Direction	Rule
270 degrees	Clockwise	(x, y) --> (-y, x)

Point A is located at $(-2,-1)$, and we use the rule $(x,y)\to(-y,x)$, so $(-2,-1)\to(1,-2)$. The figure shows what happens to point A in a 270° clockwise rotation around the origin: It's moved to point $A'=(1,-2)$.



270°
A
A'
x
y

Reflecting figures in coordinate space

In this lesson we'll look at reflection of a figure in a coordinate plane and how to determine the location and orientation of the figure after the reflection takes place.

A **reflection** is a type of transformation that flips a figure across some line. The line is called the **line of reflection**, or the mirror line. The line of reflection, which remains fixed (the points on the line aren't moved by the reflection), can be horizontal, vertical, or diagonal.

Pre-image/image

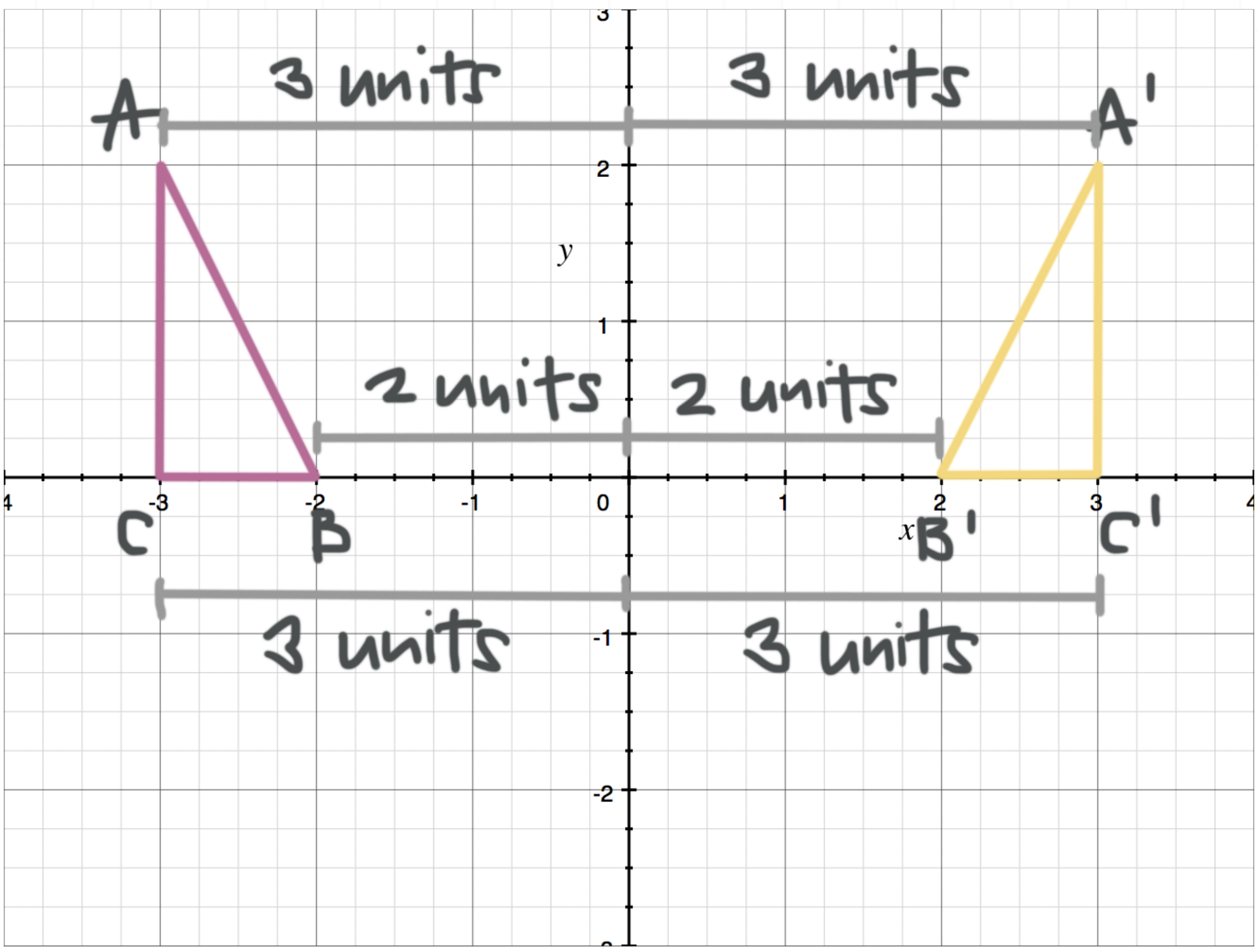
Before a reflection, we have the **pre-image** (the figure in its original location and orientation). Points in the pre-image are usually labeled with capital letters. After the reflection, we have the **image** (the figure in its final location and orientation). Points in the image are usually labeled with the same capital letters, plus the prime symbol $'$ after each letter. So if figure $ABCD$ is reflected, its image becomes figure $A'B'C'D'$.

In a reflection, the image and pre-image are always congruent, because a reflection never changes the measures of angles or the lengths of line segments and curves in the figure.

Reflecting figures

When we reflect a figure across a line (which we also refer to as reflecting a figure *in* that line), the distance of any point in the pre-image from the line of reflection is equal to the distance of the corresponding point in the image from that line.

In this reflection, which is a reflection across the y-axis, each point in the pre-image is at the same distance from the y-axis as the corresponding point in the image.



Notice that when we reflect across the y-axis, the sign of the x-coordinate of any point in the image will be opposite that of the corresponding point in the pre-image, and that the y-coordinate of any point in the image will be equal to that of the corresponding point in the pre-image.

$A = (-3,2)$ $\qquad\qquad$ $A' = (3,2)$

$$B = (-2,0) \quad \rightarrow \quad B' = (2,0)$$

$$C = (-3,0) \qquad\qquad C' = (3,0)$$

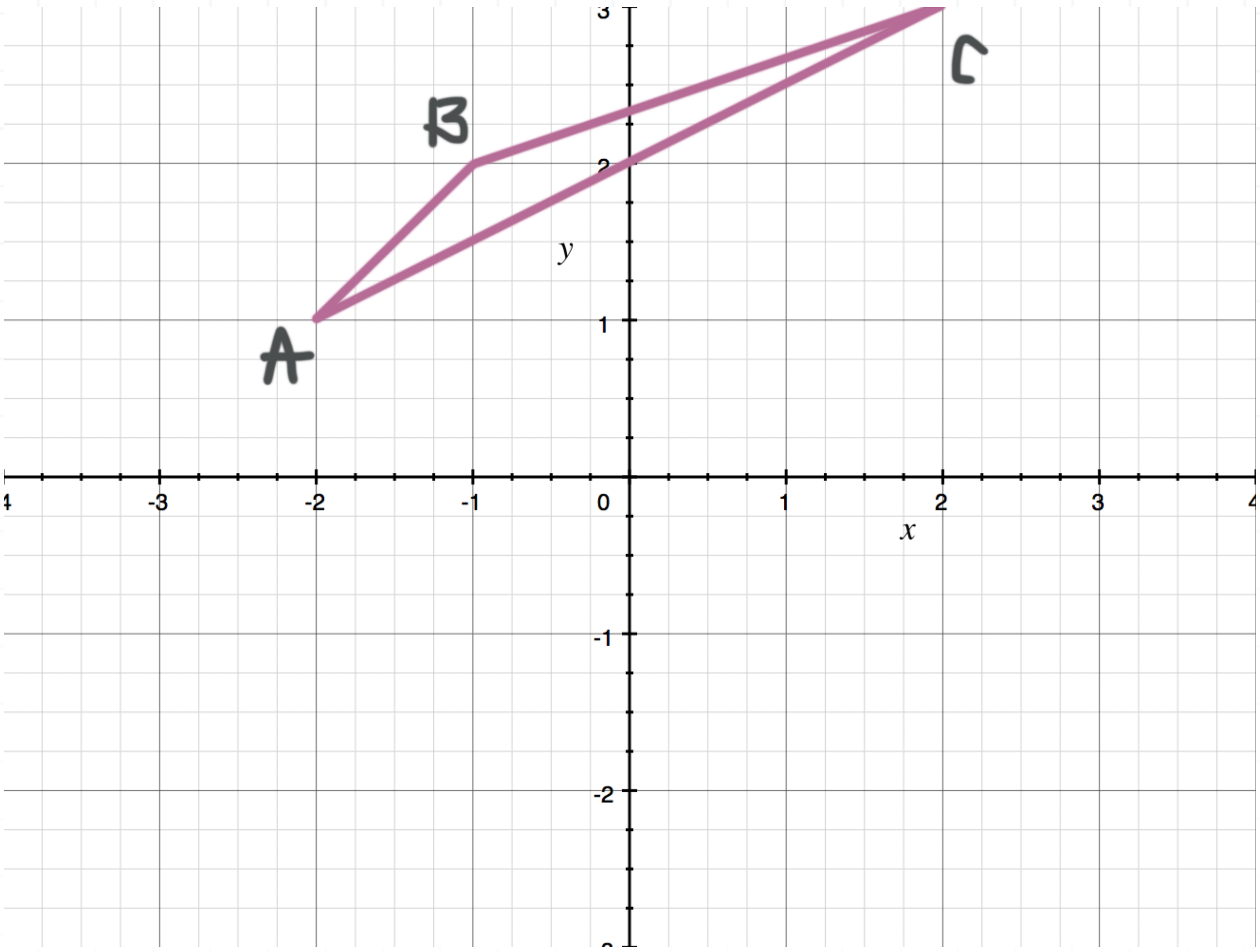
Here are some commonly used reflections and their rules.

Line of reflection	Rule
y-axis	(x, y) to (-x, y) The x-coordinates will change sign.
x-axis	(x, y) to (x,-y) The y-coordinates will change sign.
y=x	(x, y) to (y, x) The x- and y-coordinates will change places.

Let's look at some examples.

Example

Draw $\triangle A'B'C'$, the reflection of $\triangle ABC$ across the x-axis.



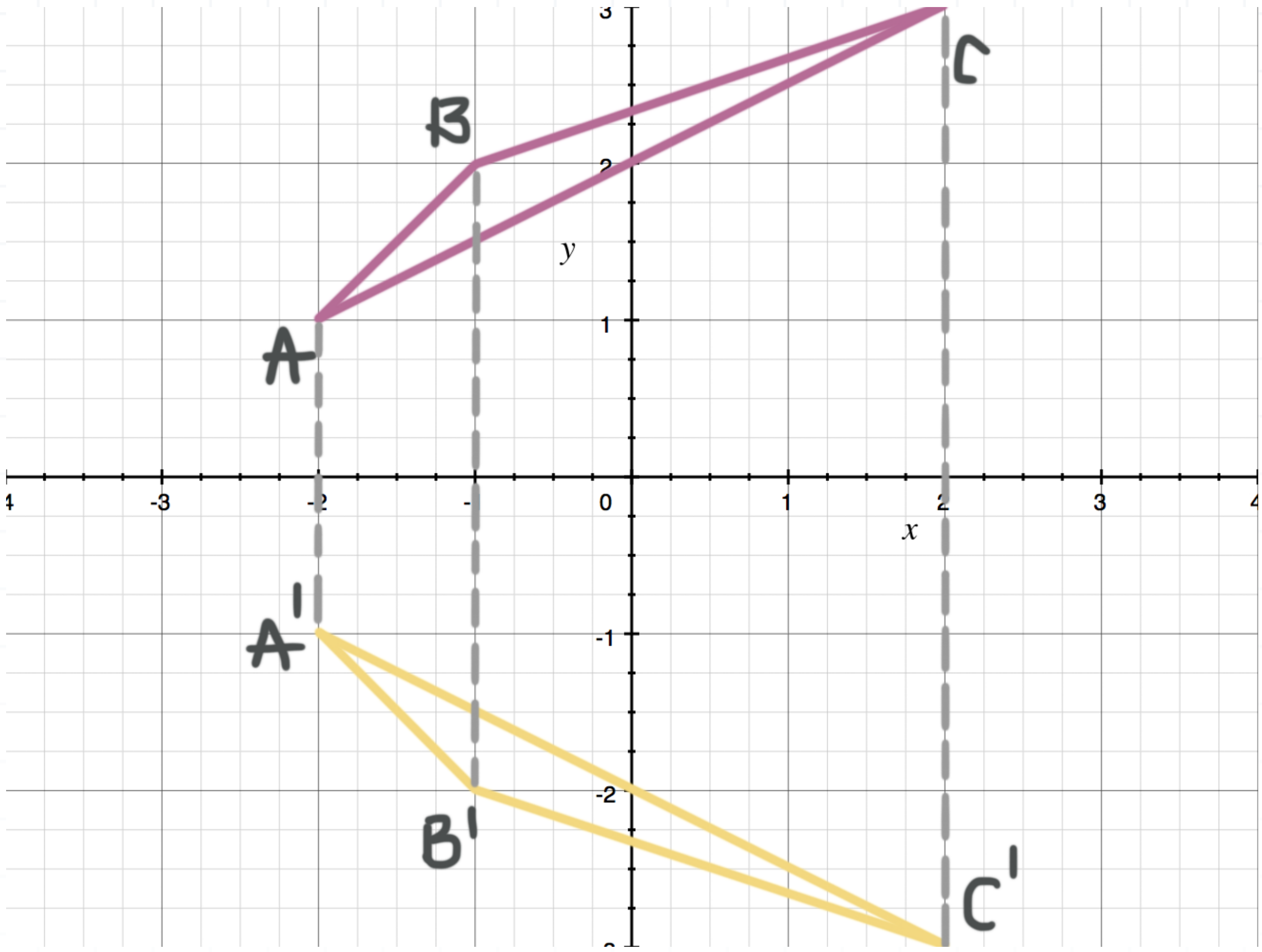
Let's look at one vertex of the triangle at a time.

Point A is 1 unit above the x-axis, so A' will be 1 unit below the x-axis.

Point B is 2 units above the x-axis, so B' will be 2 units below the x-axis.

Point C is 3 units above the x-axis, so C' will be 3 units below the x-axis.

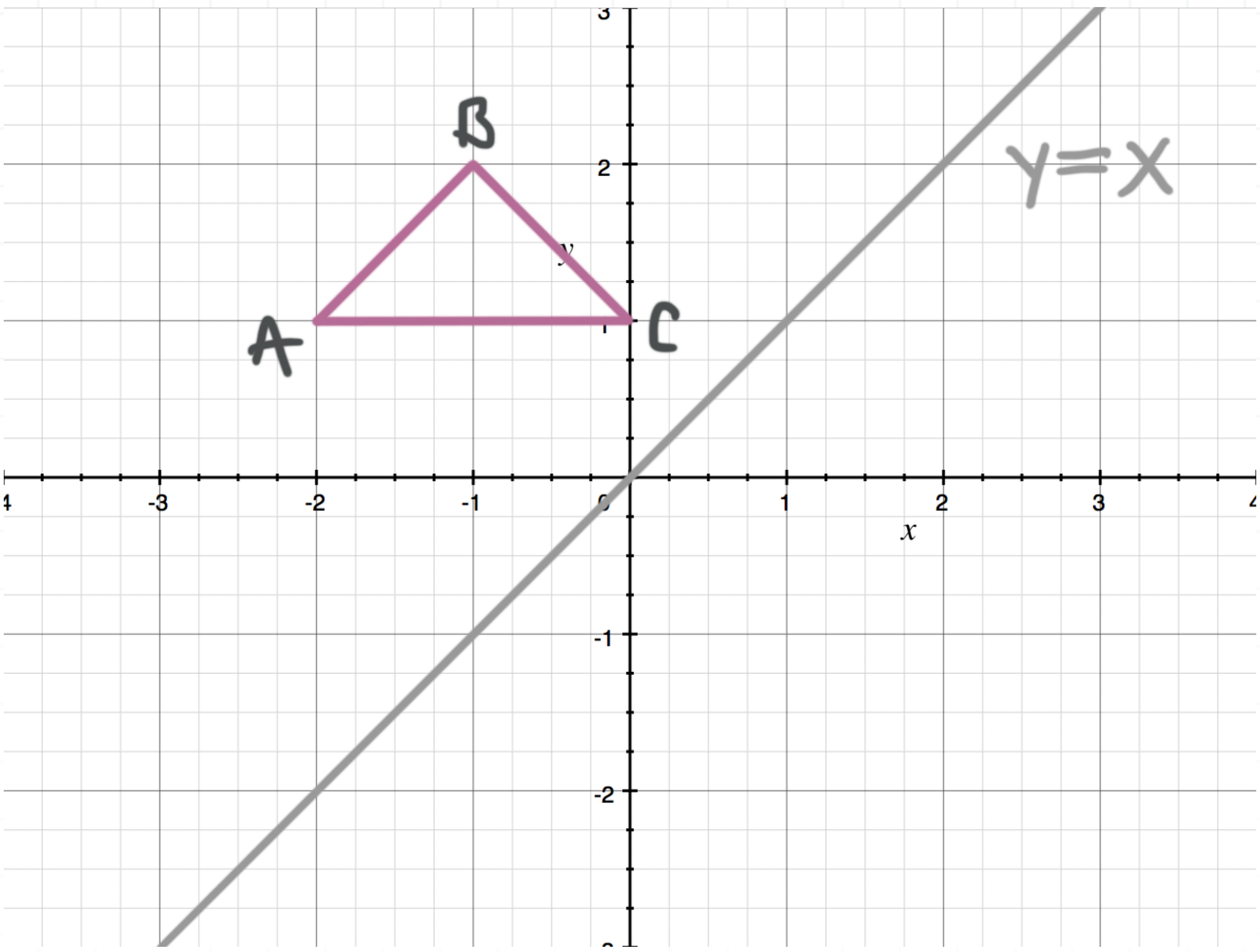
This means that when you reflect $\triangle ABC$ across the x-axis, the y-coordinates will change sign.



Let's look at another example.

Example

Reflect triangle ABC across the line $y = x$.



When you reflect a figure across the line $y = x$, the x- and y-coordinates switch places. Let's write down the coordinates of the vertices of the triangle.

$$A = (-2,1)$$

$$B = (-1,2)$$

$$C = (0,1)$$

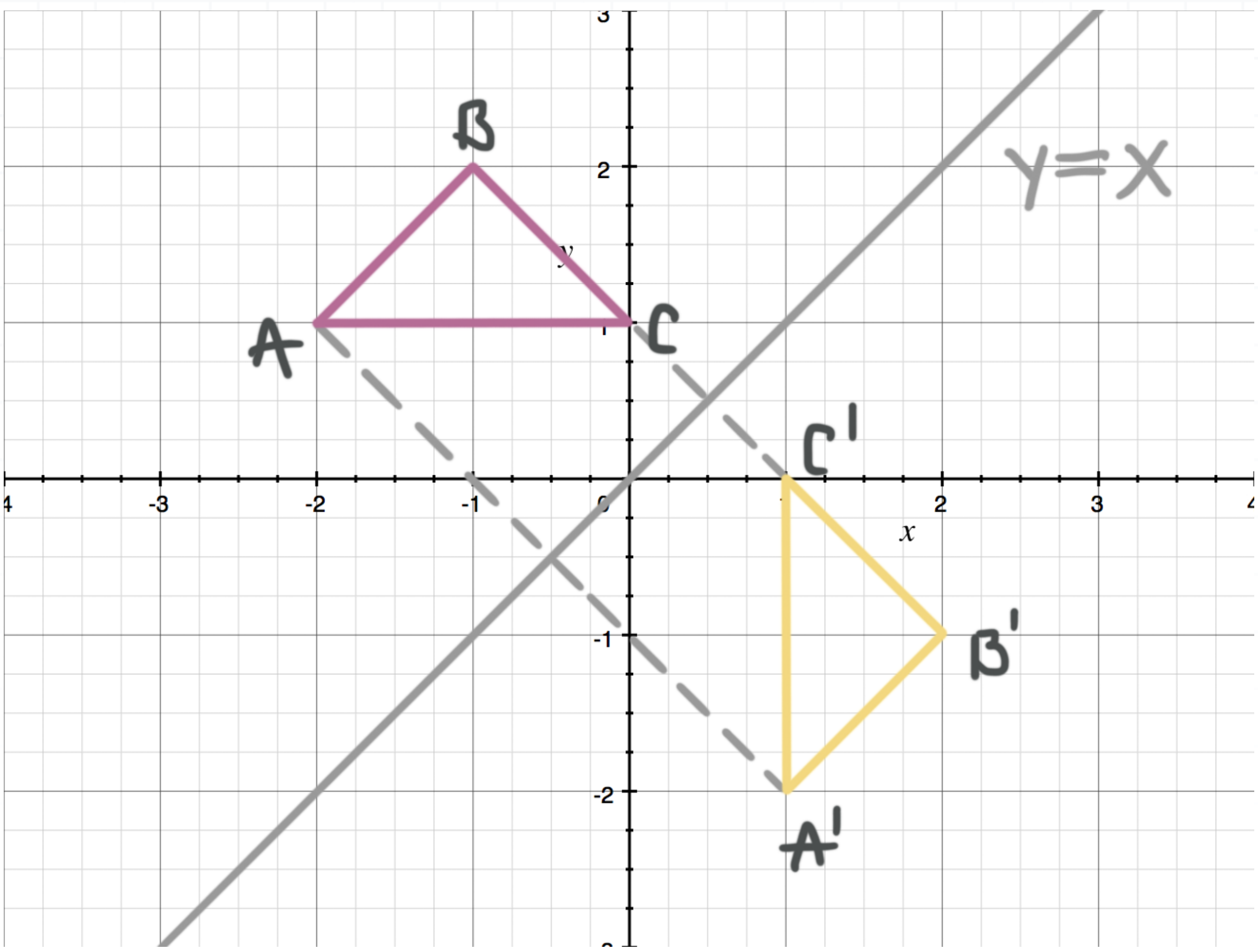
Now to reflect the triangle across the line $y = x$, you switch the x- and y-coordinates of each vertex.

$A = (-2,1)$ $\qquad\qquad$ $A' = (1,-2)$

$B = (-1,2) \quad \rightarrow \quad B' = (2, -1)$

$C = (0,1) \quad \quad C' = (1,0)$

Then you can draw the image.

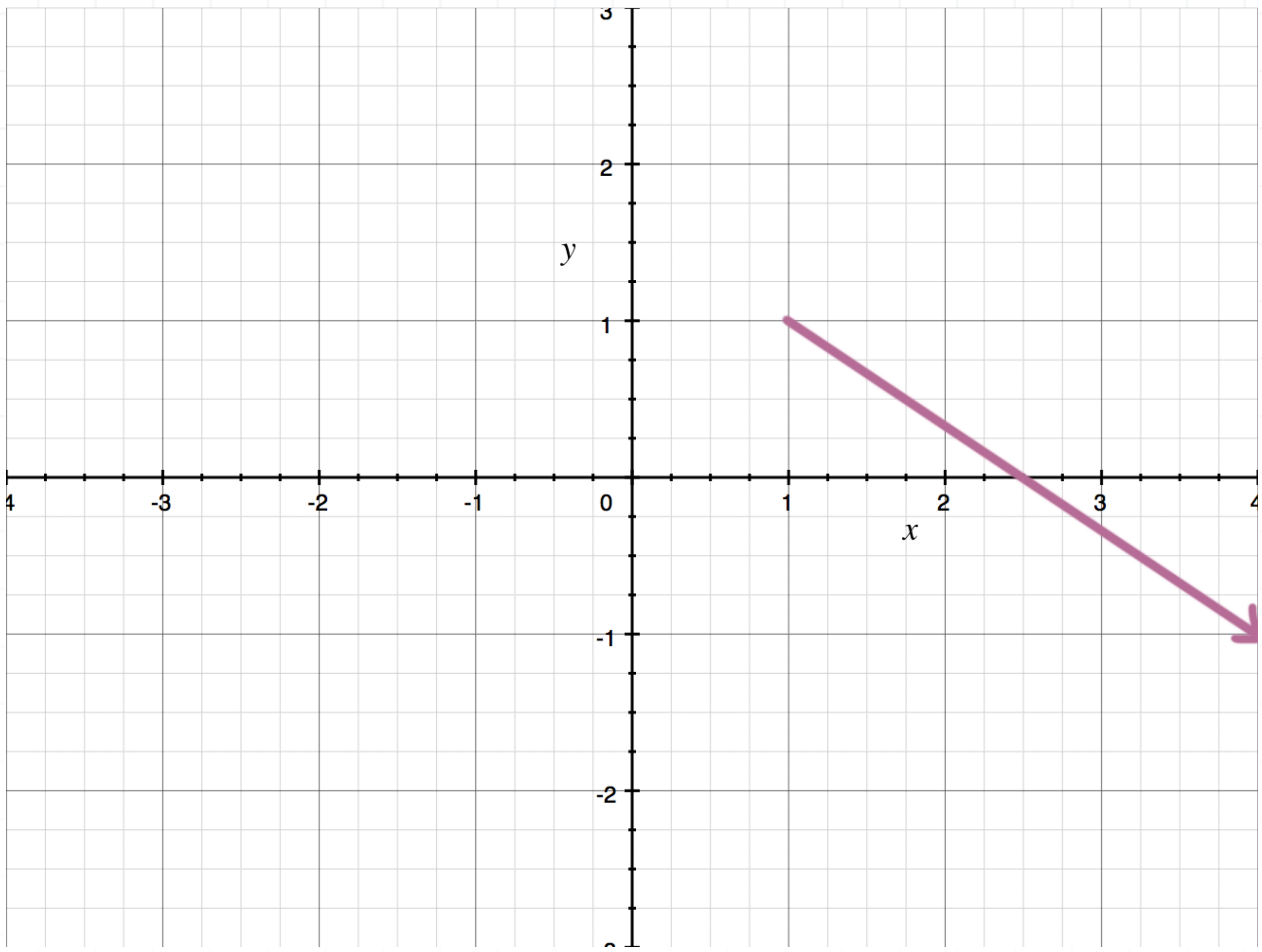


Translation vectors

In this lesson we'll look at how to use translation vectors to indicate a translation of a figure.

A translation vector is a type of quantity that indicates a translation of a figure in the coordinate plane from one location to another. In other words, a translation vector can be thought of as indicating a slide with no rotation. The slide won't change the shape or size of the figure, and because there's no rotation, the orientation won't change either.

A translation vector can be drawn on the coordinate grid or written as $\overrightarrow{v}=\langle a,b\rangle$. For example, a translation vector that indicates translation of a figure 3 units to the right and 2 units down can be represented mathematically as $\overrightarrow{v}=\langle 3,-2\rangle$, or graphically as an arrow, as shown in the figure.



It doesn’t matter where the vector is positioned in the plane. In this figure, the vector starts at $(1,1)$, the location of the “tail” of the arrow, and ends at $(4,-1)$, the location of the “head” of the arrow.

Notice that the difference is the x-coordinates of the “head” and “tail” of the vector is $4-1=3$, and that the difference in their y-coordinates is $(-1)-1=-2$.

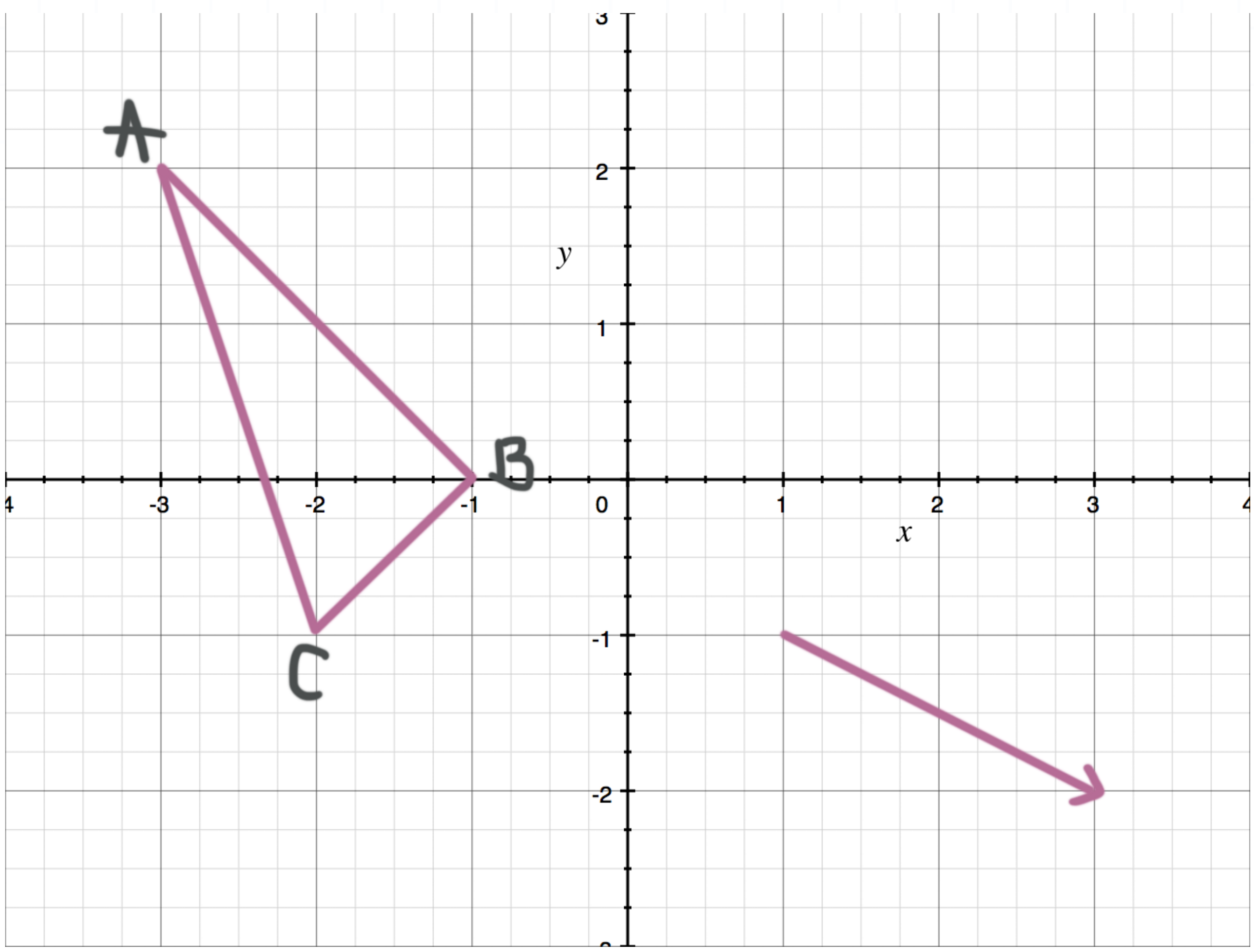
But the initial point and terminal point of the vector are irrelevant. What matters is the length of the vector and the direction in which it points, so all you have to look at is the difference in the x-coordinates of the head and tail of the translation vector and the difference in the y-coordinates of its head and tail.

A translation vector $\overrightarrow{v}=(a,b)$ has two components: a horizontal component a (which is given by the difference in the x-coordinates of its head and tail) and a vertical component b (which is given by the difference in the y-coordinates of its head and tail). By the Pythagorean theorem, the length of a translation vector is equal to $\sqrt{a^2+b^2}$, the square root of the sum of the squares of its horizontal and vertical components. The direction of a translation vectors is given by its slope, which is equal to b/a, the ratio of its vertical component to its horizontal component.

Let's look at some examples.

Example

Use the translation vector shown to find the coordinates of the vertices of triangle $A'B'C'$.



The tail and head of this translation vector are at $(1, -1)$ and $(3, -2)$, respectively, which means that this vector indicates a translation of 2 units to the right (because the difference in the x-coordinates of its head and tail is $3 - 1 = 2$) and 1 unit down (because the difference in the y-coordinates of its head and tail is $(-2) - (-1) = -1$).

We can therefore add 2 to the x-coordinate of each vertex of the triangle in the pre-image, and subtract 1 from the y-coordinate of each vertex of the triangle in the pre-image, to find the coordinates of the vertices of the triangle in the image.

First let's write down the coordinates of the vertices of the triangle in the pre-image ($\triangle ABC$).

$$A = (-3,2)$$

$$B = (-1,0)$$

$$C = (-2, -1)$$

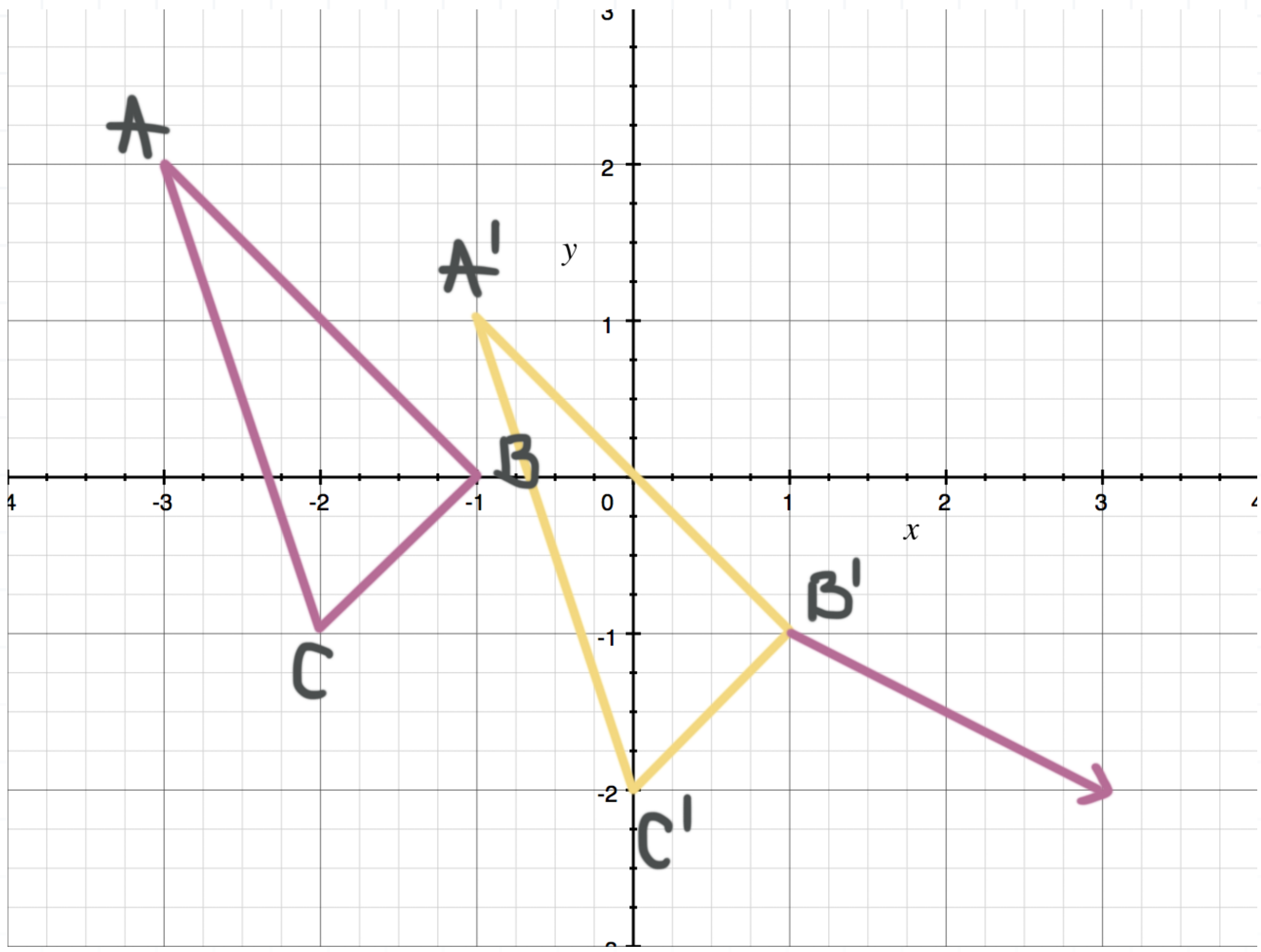
Now we can calculate the coordinates of the vertices of the triangle in the image ($\triangle A'B'C'$).

$$A' = (-3 + 2, 2 - 1) = (-1,1)$$

$$B' = (-1 + 2, 0 - 1) = (1, -1)$$

$$C' = (-2 + 2, -1 - 1) = (0, -2)$$

The pre-image, the translation vector, and the image are shown in the figure below.



Let's look at one more example.

Example

The vertices of $\triangle ABC$ are at $A=(-5,5)$, $B=(-2,5)$, and $C=(-3,0)$. If the triangle is translated by $\vec{v}=\langle -5,-6\rangle$, what are the locations of the vertices of the triangle in the image ($\triangle A'B'C'$)?

The translation vector $\vec{v}=\langle -5,-6\rangle$ means each point is being moved 5 units to the left and 6 units down. So for each vertex of $\triangle ABC$, we'll subtract 5 from its x-coordinate and we'll subtract 6 from its y-coordinate.

Then the vertices of the triangle in the image are

$$A' = (-5 - 5, 5 - 6) = (-10, -1)$$

$$B' = (-2 - 5, 5 - 6) = (-7, -1)$$

$$C' = (-3 - 5, 0 - 6) = (-8, -6)$$

Conditionals and Euler diagrams

In this lesson we'll look at how to write conditional statements and how to draw and interpret Euler diagrams.

Conditional statement

A **conditional statement** is an if/then statement where the "if" part is the hypothesis and comes first, and the "then" part is the conclusion and comes second.

You always write a conditional statement like this:

"If A, then B."

This kind of conditional statement tells you that if A is true, then B must also be true. It says nothing about the truth of B in the case where A is false.

Let's look at an example.

Example

Write the statement as a conditional statement.

"All dogs have fur."

We need to create a hypothesis and a conclusion from the statement “All dogs have fur.”, and write it as an if/then statement. In this statement, the hypothesis is “It’s a dog” and the conclusion is “it has fur”. This makes the conditional statement

“If it’s a dog, then it has fur.”

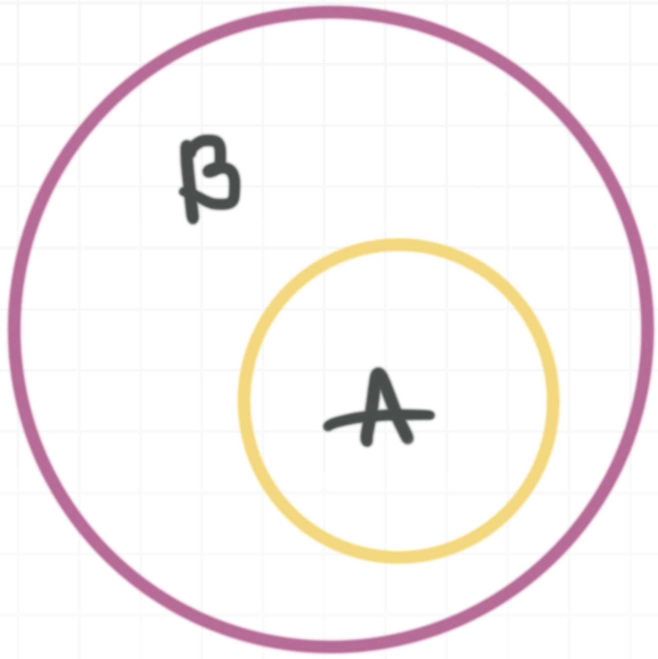
Because we were told that all dogs have fur, we know that if something is a dog, then it must have fur.

Euler diagrams

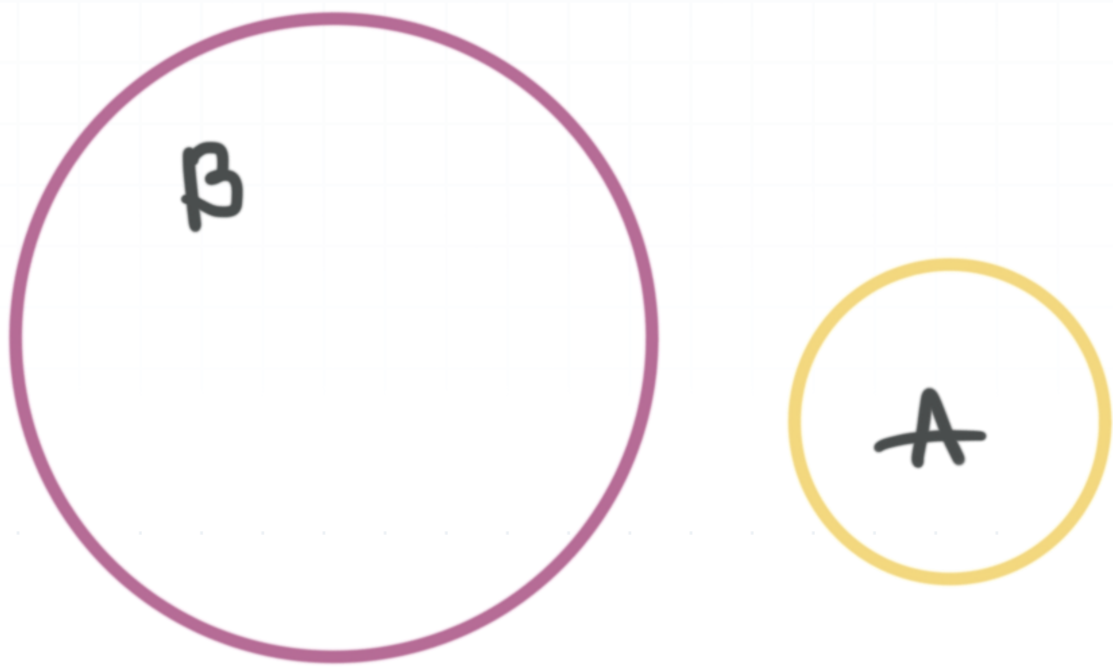
An **Euler diagram** shows the exact relationship described in a conditional statement. It’s different from a Venn diagram, because a Venn diagram would show all of the possibilities, including the ones not described in the conditional statement.

The conditional statement and corresponding Euler diagram often take one of the following two forms:

“If A, then B.”



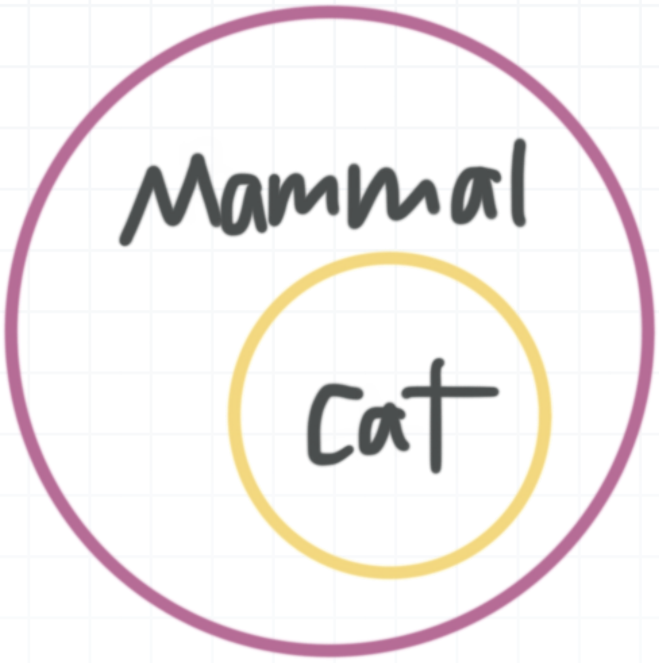
"If A, then not B."



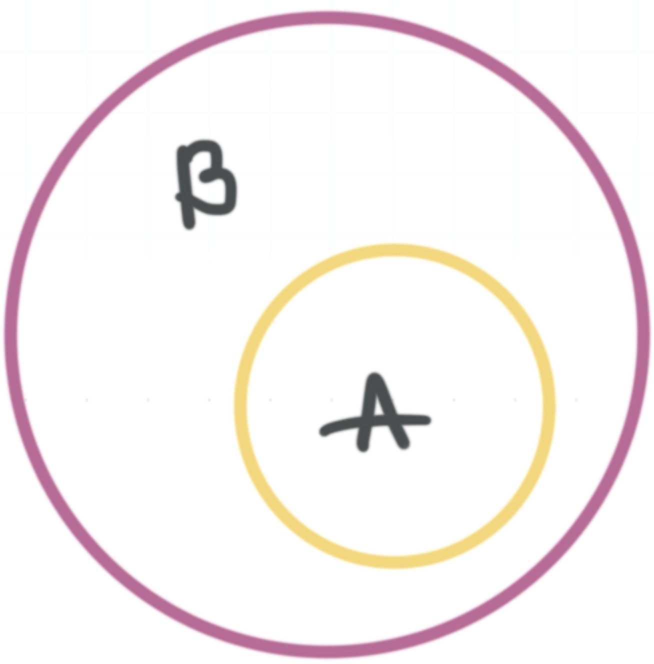
Let's look at an example.

Example

What is the conditional statement represented by the Euler diagram?



The conditional statement will have the form “If A, then B.”



So we replace A and B with the words from the diagram, and we write

“If it’s a cat, then it’s a mammal.”

Let’s tie the ideas of conditional statement and Euler diagram together.

Example

Write the following statement as a conditional statement and draw the corresponding Euler diagram.

"I get my allowance when I do my homework."

Let's think about which action comes first. First, the homework gets done. Then the allowance is received. This makes the conditional statement

"If I do my homework, then I get my allowance."

A represents doing the homework, and B represents getting the allowance, which makes the Euler diagram

Converses of conditionals

In this lesson we'll look at how to write a converse statement from a conditional.

Conditionals and their converses

We learned in the last section that a conditional statement is an if/then statement where the first part is the hypothesis and the second part is the conclusion. They're written like this:

"If A, then B."

The **converse of a conditional statement** switches what goes with the "if" and what goes with the "then." So instead of "If A, then B," the converse says

"If B, then A."

Notice that the converse of a conditional statement is itself a conditional statement.

Let's look at an example.

Example

Write the converse of the statement.

"If it snows, then they cancel school."

The converse of a conditional statement switches what goes with the "if" and what goes with the "then." So we switch the places of "canceling school" and "snow."

Conditional: "If **it snows**, then **they cancel school**."

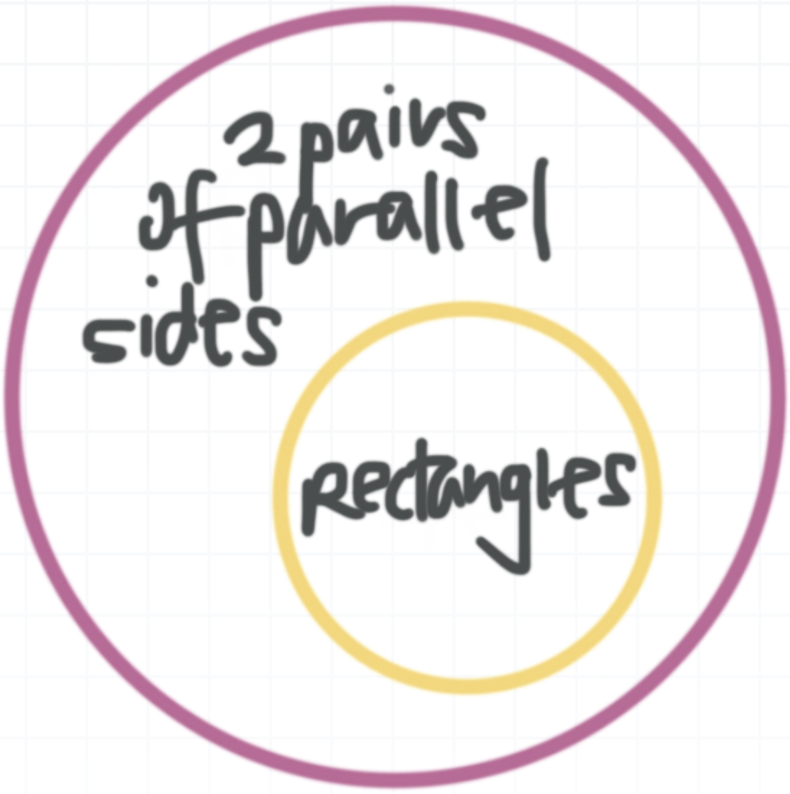
Converse: "If **they cancel school**, then **it snows**."

Notice that the conditional statement makes sense and could possibly be true. But its converse doesn't really make sense, because cancelling school doesn't necessarily mean it will snow.

Let's look at another example.

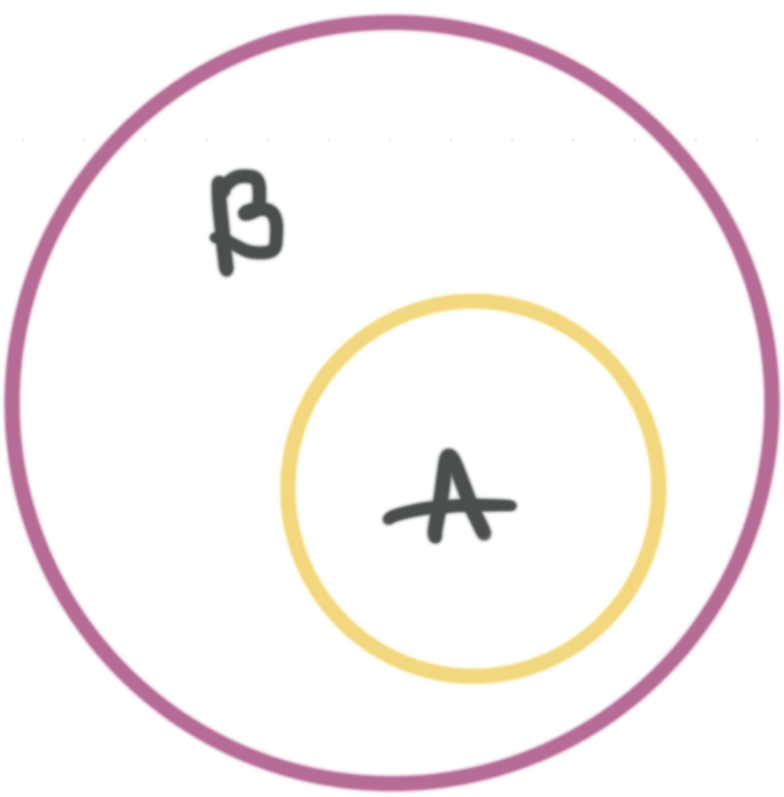
Example

What is the conditional statement represented in the Euler diagram? What is its converse?



This diagram is of the form:

"If A, then B."



This means the conditional statement is:

"If it's a rectangle, then it has two pairs of parallel sides."

For the converse, we switch what goes with the if and what goes with the then.

Conditional: "If **it's a rectangle**, then **it has two pairs of parallel sides**."

Converse: "If **it has two pairs of parallel sides**, then **it's a rectangle**."

Notice that the converse is not always true. A shape that has two pairs of parallel sides could be any parallelogram, not necessarily a rectangle.

Arranging conditionals in a logical chain

In this lesson we'll look at how to use conditionals to write a logic chain of statements.

We already know that a conditional statement is an if/then statement where the first part is the hypothesis and the second part is the conclusion, like this:

"If A, then B."

Logic chains (with conditionals)

A logic chain is something we make by stacking linked conditional statements back-to-back. Here's a logic chain with three conditional statements:

If A, then B.

If B, then C.

If C, then D.

Notice that the conclusion of the first statement, B, is also the hypothesis of the second statement, and that the conclusion of the second statement, C, is also the hypothesis of the third statement. This pattern would be continued if the chain had more than three conditional statements.

If events are listed in a logical chain like this, then you can conclude the following:

If A is true, then C is also true (through B), and

If A is true, then D is also true (through B and C).

Therefore, given a logic chain of conditionals, you can form another conditional (which we'll call the **logical conclusion**) by using the hypothesis of the first conditional in the chain as its hypothesis, and using the conclusion of the last conditional in the chain as its conclusion.

Given the logic chain of three conditionals above (if A, then B; if B, then C; and if C, then D), the logical conclusion is the conditional statement "If A, then D."

Let's look at an example.

Example

What is the missing conditional statement in the three-step logic chain shown here?

1) If juice gets spilled on the table, then it will get sticky.

2)

3) If the ants come, then we will need an exterminator.

Let's think about what creates this pattern:

If A, then B.

If B, then C.

If C, then D.

We can fill this in with what we know.

If **juice gets spilled on the table**, then **it will get sticky**.

If B, then C.

If **the ants come**, then **we'll need an exterminator**.

The hypothesis of the second conditional, B, must be the same as the conclusion of the first conditional (the table getting sticky), and the conclusion of the second conditional, C, must be the same as the hypothesis of the third conditional (the ants coming), so we can now form the second, missing conditional:

If the table is sticky, then the ants will come.

And then we can see the whole logic chain:

If **juice gets spilled on the table**, then **it will get sticky**.

If **the table is sticky**, then **the ants will come**.

If **the ants come**, then **we'll need an exterminator**.

Let's look at another example.

Example

What is the logical conclusion of the logic chain?

1) If I clean my house, then I will invite over some company.

2) If I invite over some company, then I will cook dinner.

3) If I cook dinner, then my kitchen will be dirty.

The parts of this logic chain appear in the following order:

from “If I clean my house”

to “I will invite over some company”

to “I will cook dinner”

to “My kitchen will be dirty.”

To come up with the logical conclusion of this logic chain, we use the first part (cleaning my house) as its hypothesis, and the last part (my kitchen being dirty) as its conclusion.

“If I clean my house, then my kitchen will be dirty.”

W W W . K R I S T A K I N G M A T H . C O M