# Graphing lemniscates

The last shape we'll look at is a figure-eight shape called a "lemniscate." These curves have polar equations in these forms:

$$r^2 = c^2 \sin(2\theta) \qquad r^2 = -c^2 \sin(2\theta)$$

$$r^2 = c^2 \cos(2\theta) \qquad r^2 = -c^2 \cos(2\theta)$$

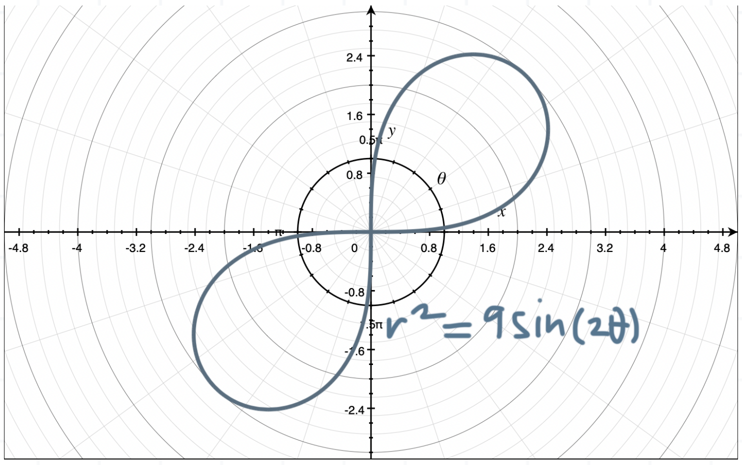
where $c$ is positive constant. Lemniscates are always symmetric around the origin.

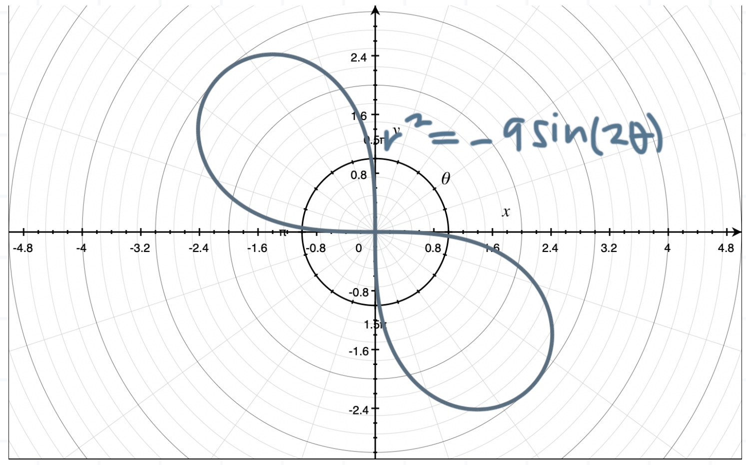
## Properties of lemniscates

The argument of the trigonometric function in the equation of a lemniscate is always $(2\theta)$, which is a reminder that a lemniscate always has two loops. These are almost like the petals in a rose, except that there are always exactly two loops, instead of the varying number of petals that we find in roses.

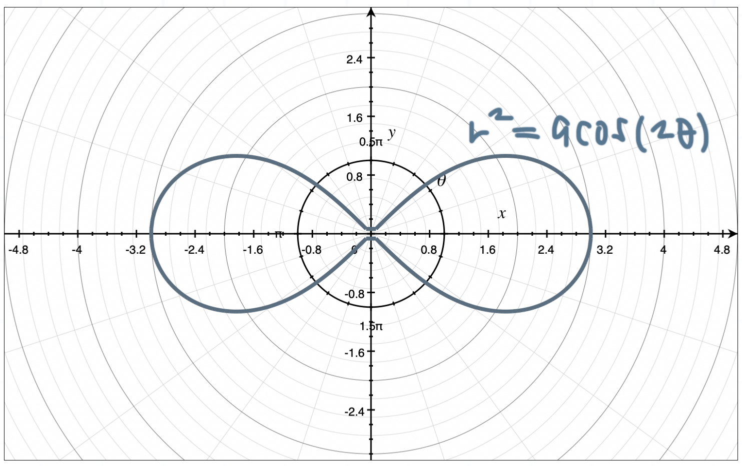
The tip of the two loops of the lemniscate will always lie a distance of $c$ away from the origin. Where the loops themselves lie depends on whether we have a sine lemniscate or cosine lemniscate, and whether the equation includes $c^2$ or $-c^2$.

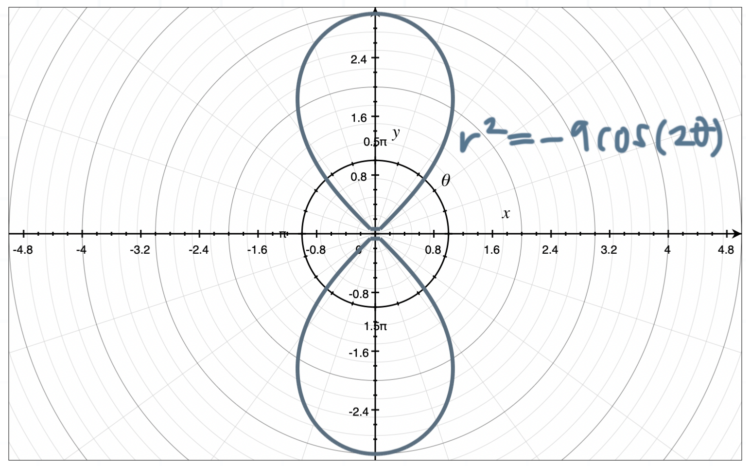
The loops of a sine lemniscate with $c^2$ lie in the first and third quadrants, while the loops of a sine lemniscate with $-c^2$ lie in the second and fourth quadrants.




The loops of a cosine lemniscate with $c^2$ lie along the horizontal axis, while the loops of a cosine lemniscate with $-c^2$ lie along the vertical axis.




## How to sketch lemniscates

Because of the $r^2$ value that we see in lemniscate equations, we can sometimes find angles $\theta$ at which the lemniscate is undefined. For instance, take the lemniscate $r^2 = 9\cos(2\theta)$ that we graphed above. If we try to evaluate this polar equation at $\theta = \pi/2$, we get

$$r^2 = 9\cos\left(2 \cdot \frac{\pi}{2}\right)$$

$$r^2 = 9\cos\pi$$

$$r^2 = 9(-1)$$

$$r^2 = -9$$

$$r = \pm\sqrt{-9}$$

We can't use real numbers to take the square root of a negative value, so we can run into a problem when we try to sketch this polar equation at $\theta = \pi/2$.

Because of this issue, we'll simplify our plan for sketching lemniscates and just use the value of $c^2$, the positive or negative sign in front of the $c^2$, and whether the lemniscate is a sine or cosine lemniscate, in order to sketch the graph. Our plan will be to

1. Identify the lemniscate as a sine or cosine lemniscate.
2. Identify whether the equation begins with a positive sign or negative sign.
3. Determine the value of $c$ (not $c^2$).
4. Use these three facts to sketch the lemniscate.

Let's do an example where we sketch the graph of a cosine lemniscate.

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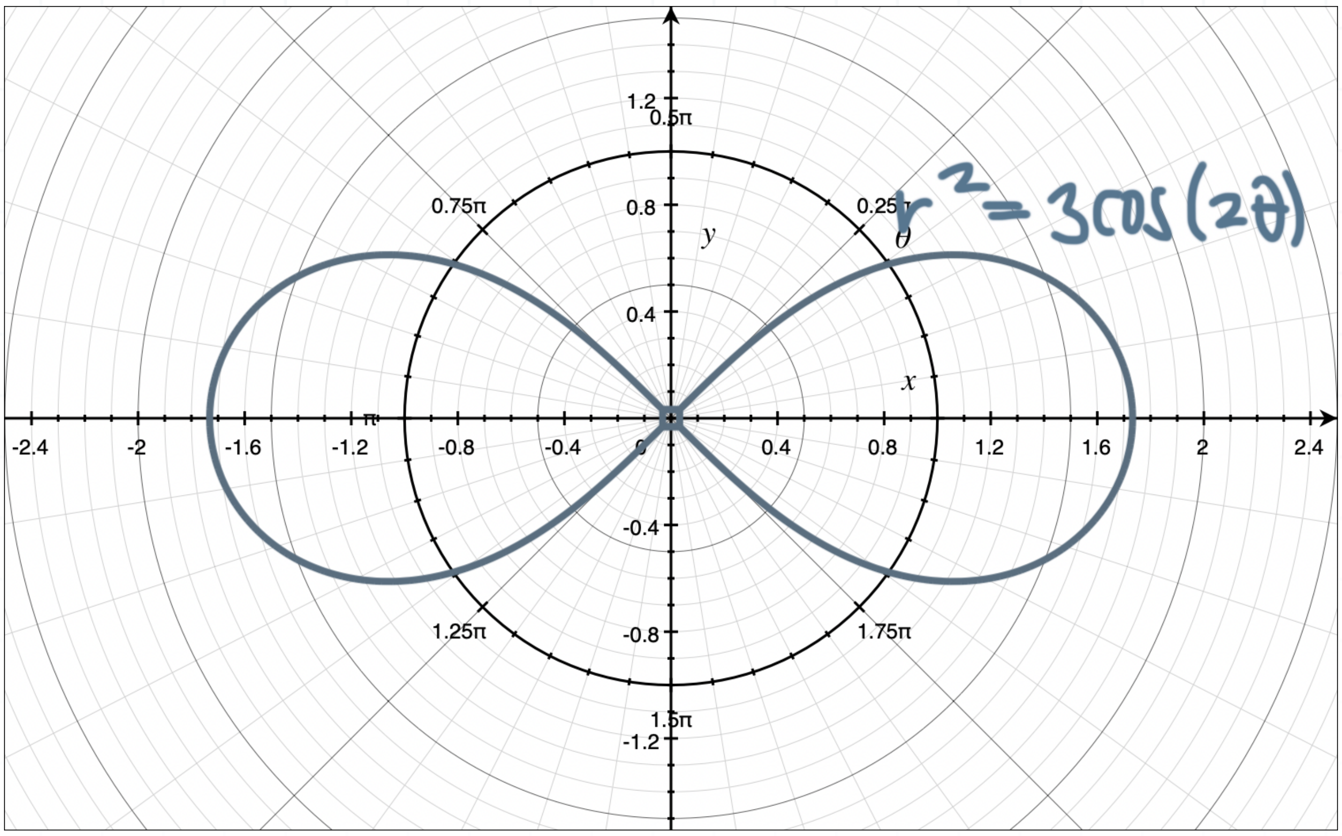
**Example**

Sketch the graph of $r^2 = 3\cos(2\theta)$.

This is a positive cosine lemniscate, which means the two loops will sit along the horizontal axis. And with $c = \sqrt{3} \approx 1.73$, the two loops will extend out a distance of $r \approx 1.73$. Therefore, the graph of the lemniscate will be

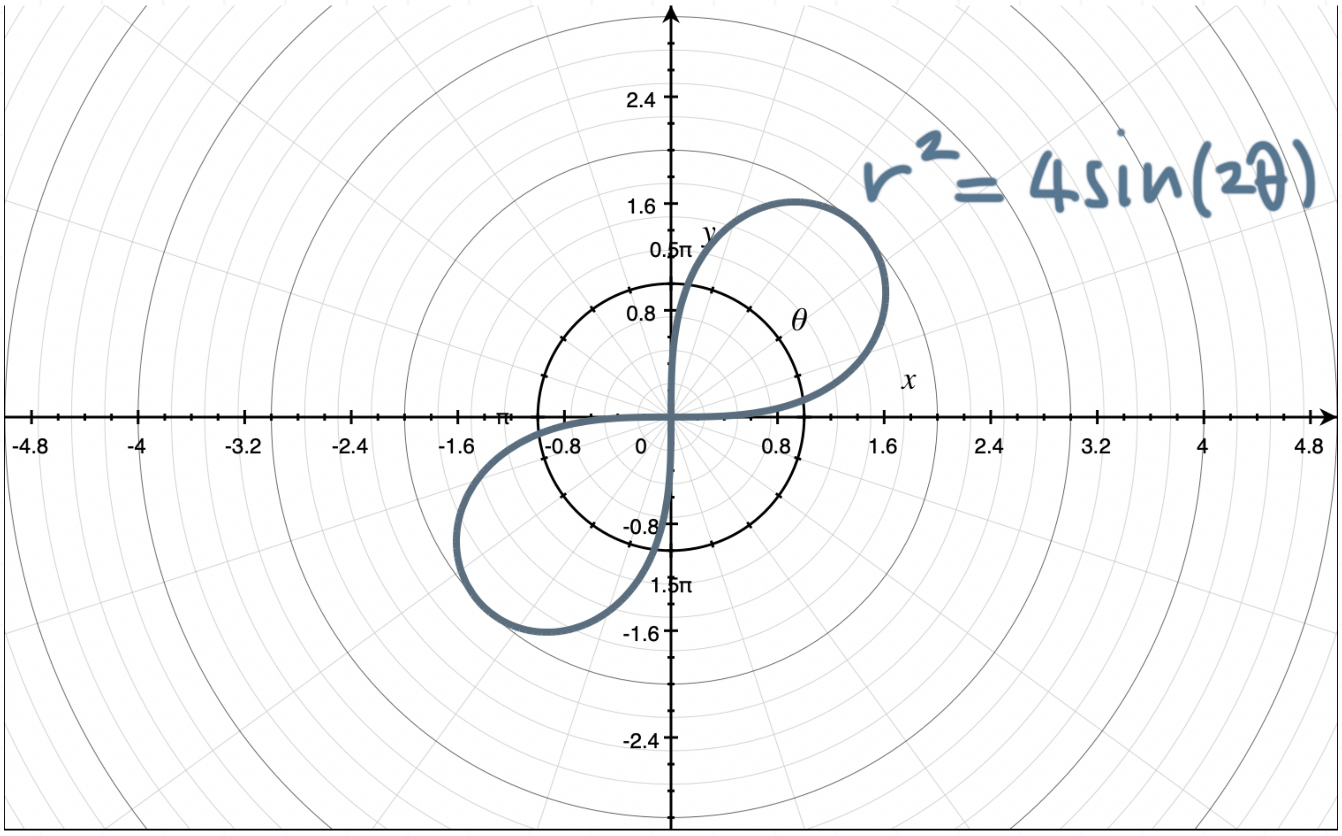



Now let's do an example with a sine lemniscate.

**Example**

Graph the lemniscate $r^2 = 4\sin(2\theta)$.

This is a positive sine lemniscate, which means the two loops will sit in the first and third quadrants. And with $c = \sqrt{4} = 2$, the two loops will extend out a distance of $r = 2$. Therefore, the graph of the lemniscate will be



$r^2 = 4\sin(2\theta)$
2.4
1.6
0.5π
0.8
θ
x
y
π
-4.8
-4
-3.2
-2.4
-1.6
-0.8
0
0.8
1.6
2.4
3.2
4
4.8
-0.8
1.5π
-1.6
-2.4