

Double-angle identities

Very often we'll handle trig functions with an angle of 2θ , instead of just θ , like $\sin 2\theta$. We'll usually want to convert these to get the 2 out of the angle.

But we can't simply say $\sin 2\theta = 2 \sin \theta$; that's just not true. 2θ is the argument of the sine function; we're evaluating the sine function at the angle 2θ , so the \sin and 2θ aren't simply multiplied together such that we can pull the 2 out in front of the sine function.

The way we'll get the 2 out of the argument is by using the double-angle identities, which are actually derived from the sum identities we learned about earlier.

Double-angle identities from the sum identities

To build the double-angle identity for sine, we start with the sum identity for sine, $\sin(\theta + \alpha) = \sin \theta \cos \alpha + \cos \theta \sin \alpha$, and replace α with θ .

$$\sin(\theta + \alpha) = \sin \theta \cos \alpha + \cos \theta \sin \alpha$$

$$\sin(\theta + \theta) = \sin \theta \cos \theta + \cos \theta \sin \theta$$

$$\sin(2\theta) = \sin \theta \cos \theta + \sin \theta \cos \theta$$

$$\sin(2\theta) = 2 \sin \theta \cos \theta$$



Similarly, we build the double-angle identity for cosine by starting with the sum identity for cosine, $\cos(\theta + \alpha) = \cos \theta \cos \alpha - \sin \theta \sin \alpha$, and replacing α with θ .

$$\cos(\theta + \alpha) = \cos \theta \cos \alpha - \sin \theta \sin \alpha$$

$$\cos(\theta + \theta) = \cos \theta \cos \theta - \sin \theta \sin \theta$$

$$\cos 2\theta = \cos^2 \theta - \sin^2 \theta$$

Keep in mind that we'll also often see this double-angle cosine identity rewritten using the Pythagorean identity with sine and cosine. From the Pythagorean identity, we can rewrite the double-angle cosine identity by substituting $\cos^2 \theta = 1 - \sin^2 \theta$,

$$\cos 2\theta = (1 - \sin^2 \theta) - \sin^2 \theta$$

$$\cos 2\theta = 1 - \sin^2 \theta - \sin^2 \theta$$

$$\cos 2\theta = 1 - 2 \sin^2 \theta$$

or by substituting $\sin^2 \theta = 1 - \cos^2 \theta$.

$$\cos 2\theta = \cos^2 \theta - (1 - \cos^2 \theta)$$

$$\cos 2\theta = \cos^2 \theta - 1 + \cos^2 \theta$$

$$\cos 2\theta = 2 \cos^2 \theta - 1$$

And to build the double-angle identity for tangent, we'll again start with the sum identity for tangent and replace α with θ .

$$\tan(\theta + \alpha) = \frac{\tan \theta + \tan \alpha}{1 - \tan \theta \tan \alpha}$$



$$\tan(\theta + \theta) = \frac{\tan \theta + \tan \theta}{1 - \tan \theta \tan \theta}$$

$$\tan(2\theta) = \frac{2 \tan \theta}{1 - \tan^2 \theta}$$

To summarize, the **double-angle identities**, including the alternate forms of the double-angle identity for cosine, are

$$\sin 2\theta = 2 \sin \theta \cos \theta$$

$$\cos 2\theta = \cos^2 \theta - \sin^2 \theta$$

$$\tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta}$$

$$\cos 2\theta = 1 - 2 \sin^2 \theta$$

$$\cos 2\theta = 2 \cos^2 \theta - 1$$

All of these give us a way to change the angle 2θ into the angle θ , which we'll be something we'll want to do all the time in Trigonometry and beyond.

Let's do an example with the double-angle cosine identity.

Example

If θ is an angle in the second quadrant with $\cos \theta = -\sqrt{5}/6$, find $\cos 2\theta$ and $\sin 2\theta$.

By the double-angle identity for cosine,

$$\cos 2\theta = \cos^2 \theta - \sin^2 \theta$$



In order to use the double-angle identity, we first need to find $\sin^2 \theta$. By the basic Pythagorean identity,

$$\sin^2 \theta + \cos^2 \theta = 1$$

$$\sin^2 \theta = 1 - \cos^2 \theta$$

Since $\cos \theta = -\sqrt{5}/6$, we get

$$\sin^2 \theta = 1 - \left(-\frac{\sqrt{5}}{6}\right)^2$$

$$\sin^2 \theta = 1 - \frac{5}{36}$$

$$\sin^2 \theta = \frac{31}{36}$$

Plus, since we know that $\cos \theta = -\sqrt{5}/6$, we can say

$$\cos^2 \theta = \left(-\frac{\sqrt{5}}{6}\right)^2$$

$$\cos^2 \theta = \frac{5}{36}$$

Now to find $\cos 2\theta$, we'll substitute into the double-angle identity for cosine.

$$\cos 2\theta = \cos^2 \theta - \sin^2 \theta$$

$$\cos 2\theta = \frac{5}{36} - \frac{31}{36}$$



$$\cos 2\theta = -\frac{26}{36}$$

$$\cos 2\theta = -\frac{13}{18}$$

To find $\sin 2\theta$ for this same angle θ , we'll use the double-angle identity for sine.

$$\sin 2\theta = 2 \sin \theta \cos \theta$$

We've already found

$$\sin^2 \theta = \frac{31}{36}$$

but we don't yet know the value of $\sin \theta$. Since θ is in the second quadrant, $\sin \theta$ is positive. Therefore,

$$\sin \theta = \sqrt{\frac{31}{36}} = \frac{\sqrt{31}}{\sqrt{36}} = \frac{\sqrt{31}}{6}$$

Now we're ready to apply the double-angle identity for sine.

$$\sin 2\theta = 2 \sin \theta \cos \theta$$

$$\sin 2\theta = 2 \left(\frac{\sqrt{31}}{6} \right) \left(-\frac{\sqrt{5}}{6} \right)$$

$$\sin 2\theta = -\frac{2\sqrt{31}\sqrt{5}}{36}$$



$$\sin 2\theta = -\frac{\sqrt{155}}{18}$$

Let's look at an example where we only know the value of $\sin \theta$ and the quadrant of the angle, and we need to find $\sin 2\theta$ and $\cos 2\theta$.

Example

Find $\sin 2\theta$ and $\cos 2\theta$ for an angle θ which lies in the third quadrant and has $\sin \theta = -3/7$.

By the double-angle identity for sine,

$$\sin 2\theta = 2 \sin \theta \cos \theta$$

We already know that $\sin \theta = -3/7$, but we need to find $\cos \theta$. We'll plug into the Pythagorean identity with sine and cosine.

$$\sin^2 \theta + \cos^2 \theta = 1$$

$$\cos^2 \theta = 1 - \sin^2 \theta$$

$$\cos^2 \theta = 1 - \left(-\frac{3}{7}\right)^2$$

$$\cos^2 \theta = 1 - \frac{9}{49}$$



$$\cos^2 \theta = \frac{40}{49}$$

$$\cos \theta = \pm \sqrt{\frac{40}{49}}$$

$$\cos \theta = \pm \frac{\sqrt{40}}{7}$$

Since θ is in the third quadrant, we know that $\cos \theta$ is negative, so we can ignore the positive value and say

$$\cos \theta = -\frac{\sqrt{40}}{7}$$

Now, substituting $\cos \theta = -\sqrt{40}/7$ and $\sin \theta = -3/7$ into the double-angle identity for sine, we get

$$\sin 2\theta = 2 \sin \theta \cos \theta$$

$$\sin 2\theta = 2 \left(-\frac{3}{7} \right) \left(-\frac{\sqrt{40}}{7} \right)$$

$$\sin 2\theta = \frac{6\sqrt{40}}{49}$$

$$\sin 2\theta = \frac{12\sqrt{10}}{49}$$

By the double-angle identity for cosine, we get

$$\cos 2\theta = \cos^2 \theta - \sin^2 \theta$$



$$\cos 2\theta = \left(-\frac{\sqrt{40}}{7}\right)^2 - \left(-\frac{3}{7}\right)^2$$

$$\cos 2\theta = \frac{40}{49} - \frac{9}{49}$$

$$\cos 2\theta = \frac{31}{49}$$

If we're given just the value of $\tan \theta$ for some angle θ , we can compute the value of $\tan 2\theta$ (even if we don't know the quadrant in which θ lies) by using the double-angle identity for tangent.

Example

Find $\tan 2\theta$ if $\tan \theta = \sqrt{23}$.

By the double-angle identity for tangent,

$$\tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta}$$

$$\tan 2\theta = \frac{2\sqrt{23}}{1 - (\sqrt{23})^2}$$

$$\tan 2\theta = \frac{2\sqrt{23}}{1 - 23}$$



$$\tan 2\theta = \frac{2\sqrt{23}}{-22}$$

$$\tan 2\theta = -\frac{\sqrt{23}}{11}$$

