Calculus 1 Workbook Solutions

Optimization and sketching graphs

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MATH

CRITICAL POINTS AND THE FIRST DERIVATIVE TEST

1. Identify the critical point(s) of the function on the interval $[-3,2]$.

$$f(x)=x^{\frac{2}{3}}(x+2)$$

Solution:

Find $f'(x)$ and the x-values inside the given interval for which $f'(x)=0$ or is undefined.

Rewrite the function.

$$f(x)=x^{\frac{2}{3}}(x+2)$$

$$f(x)=x^{\frac{5}{3}}+2x^{\frac{2}{3}}$$

Find the derivative.

$$f'(x)=\frac{5}{3}x^{\frac{2}{3}}+2\cdot\frac{2}{3}x^{-\frac{1}{3}}$$

$$f'(x)=\frac{5}{3}\sqrt[3]{x^2}+\frac{4}{3\sqrt[3]{x}}$$

When $x=0$, the denominator of the second fraction will be 0, which will make the derivative undefined. The derivative will also be equal to 0:

$$\frac{5}{3}\sqrt[3]{x^2}+\frac{4}{3\sqrt[3]{x}}=0$$

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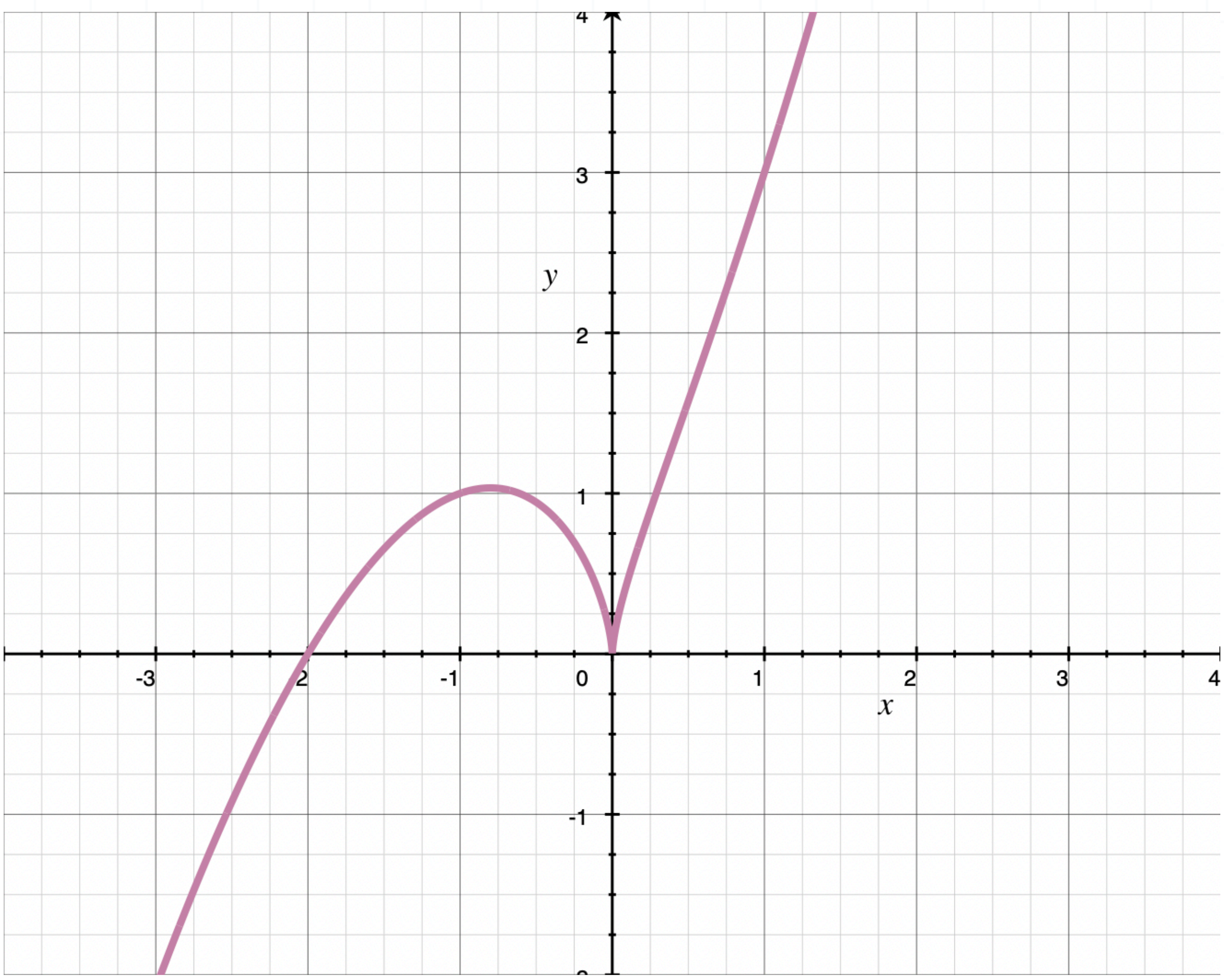
$$\frac{5}{3}\sqrt[3]{x^2} = -\frac{4}{3\sqrt[3]{x}}$$

$$5x = -4$$

$$x = -\frac{4}{5}$$

Because both $x = -4/5$ and $x = 0$ are in the interval $[-3,2]$, the critical numbers are $x = -4/5,\ 0$.



2. Identify the critical point(s) of the function on the interval $[-2,2]$.

$$g(x) = x\sqrt{4-x^2}$$

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Solution:

Find $g'(x)$ and the x-values inside the given interval for which $g'(x) = 0$ or is undefined.

Find the derivative.

$$g'(x) = (1)\sqrt{4 - x^2} + (x)\left(\frac{1}{2}\right)(4 - x^2)^{-\frac{1}{2}}(-2x)$$

$$g'(x) = \sqrt{4 - x^2} - x^2(4 - x^2)^{-\frac{1}{2}}$$

$$g'(x) = \sqrt{4 - x^2} - \frac{x^2}{\sqrt{4 - x^2}}$$

When $x = \pm 2$, the denominator of the second fraction will be 0, which will make the derivative undefined. The derivative will also be equal to 0:

$$\sqrt{4 - x^2} - \frac{x^2}{\sqrt{4 - x^2}} = 0$$

$$\sqrt{4 - x^2} = \frac{x^2}{\sqrt{4 - x^2}}$$

$$4 - x^2 = x^2$$

$$4 = 2x^2$$

$$2 = x^2$$

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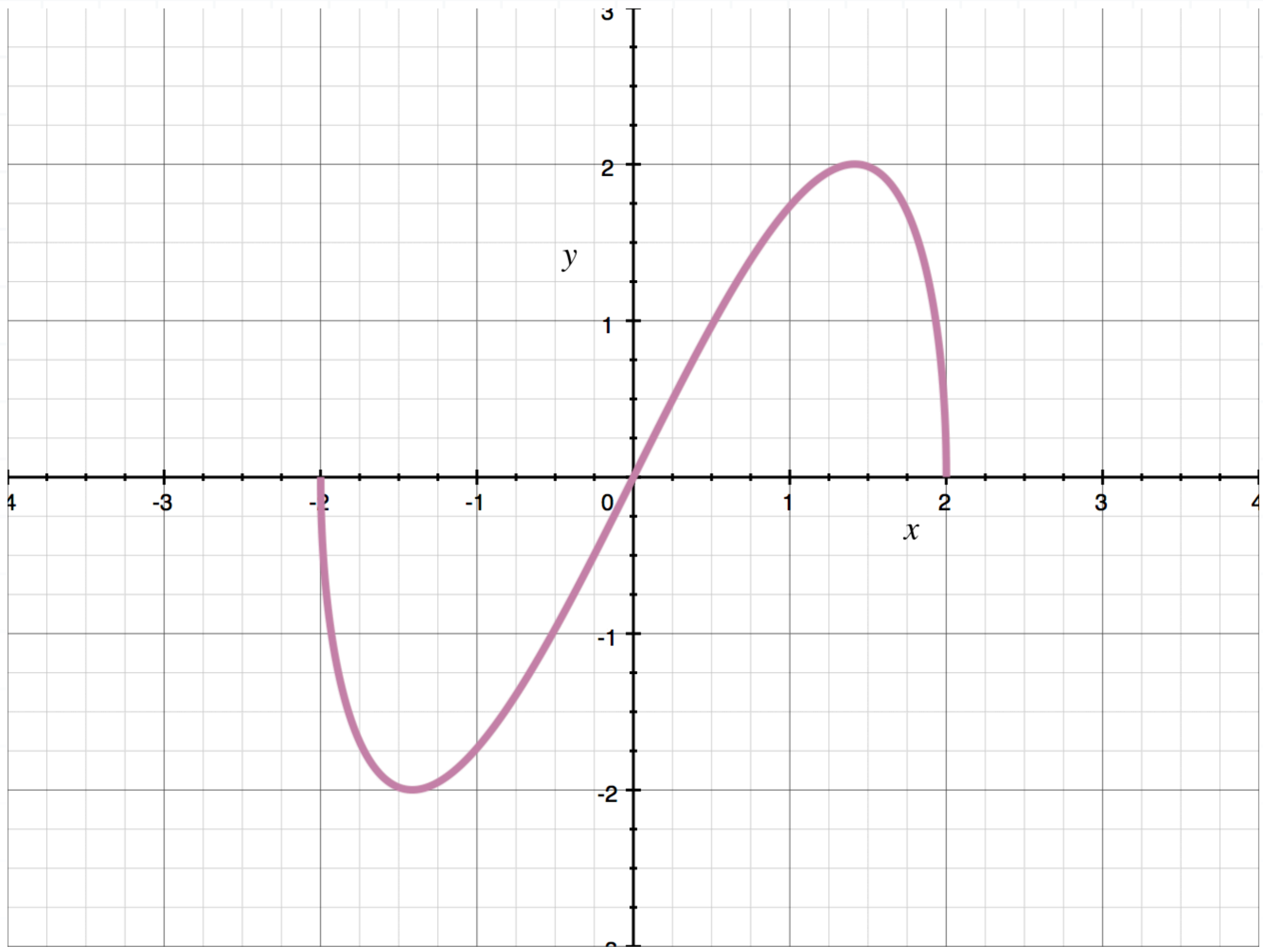
$$x = \pm\sqrt{2}$$

The critical numbers are therefore $x = - \pm \sqrt{2}, \ \pm 2$.



■ 3. Determine the intervals in which the function is increasing and decreasing.

$$f(x) = \frac{5}{4}x^4 - 10x^2$$

Solution:

Find the derivative $f'(x) = 5x^3 - 20x$, then identify the critical points where $f'(x) = 0$ or $f'(x)$ is undefined. The derivative exists everywhere.

$$5x^3 - 20x = 0$$

$$5x(x^2 - 4) = 0$$

$$5x(x + 2)(x - 2) = 0$$

$$x = -2,\ 0,\ 2$$

Determine where $f'(x) > 0$ or $f'(x) < 0$ by selecting a value between each critical number.

Interval	x<-2	-2<x<0	0<x<2	x>2
x	-3	-1	1	3
f'(x)	<0	>0	<0	>0
f(x)	Decreasing	Increasing	Decreasing	Increasing

The graph of $f'(x)$ shows that $f'(x) < 0$ on $(-\infty, -2) \cup (0,2)$ and $f'(x) > 0$ on $(-2,0) \cup (2,\infty)$.

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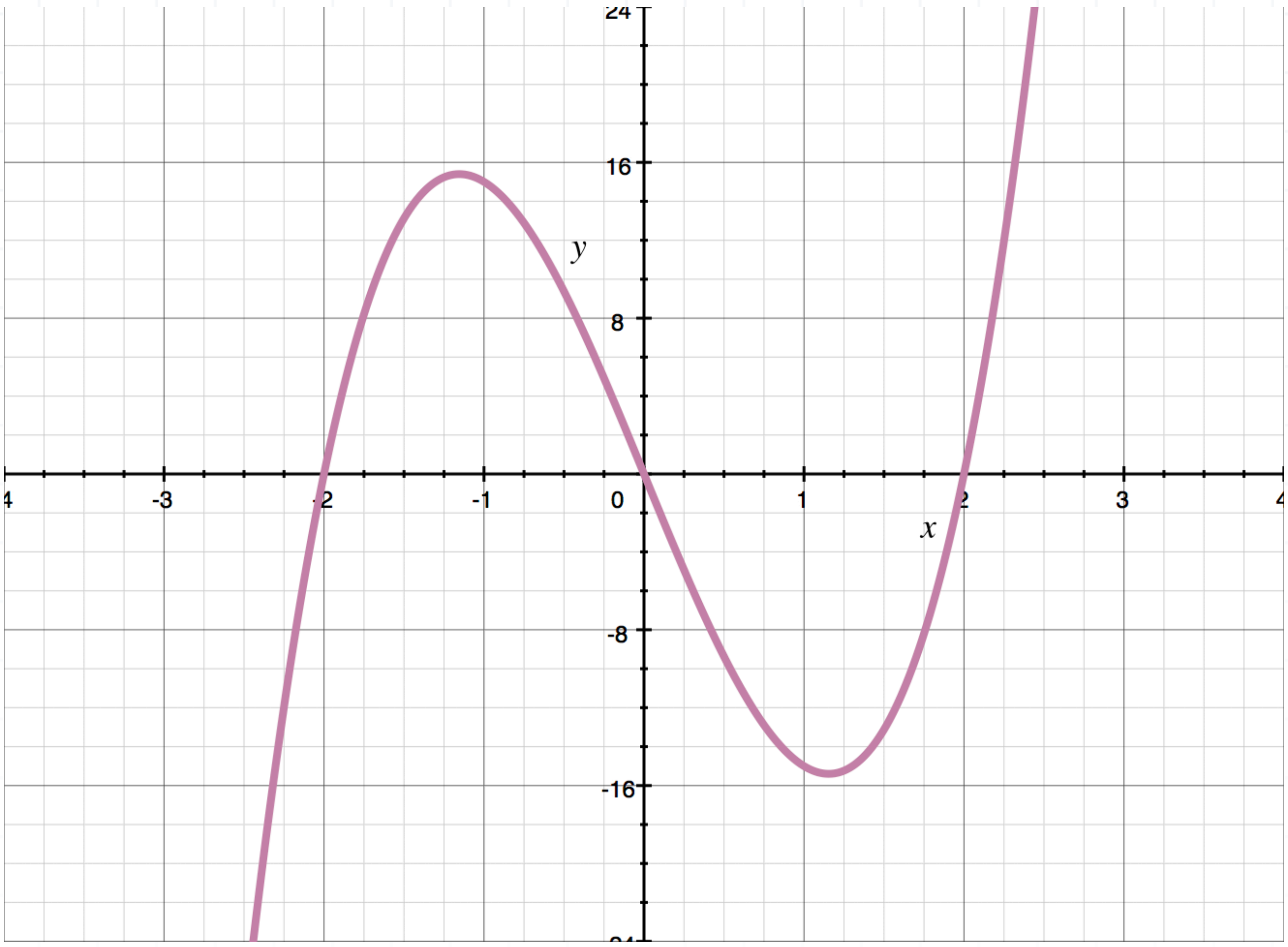

The graph of $f(x)$ shows that the function is decreasing on $(-\infty, -2) \cup (0,2)$ and increasing on $(-2,0) \cup (2,\infty)$.

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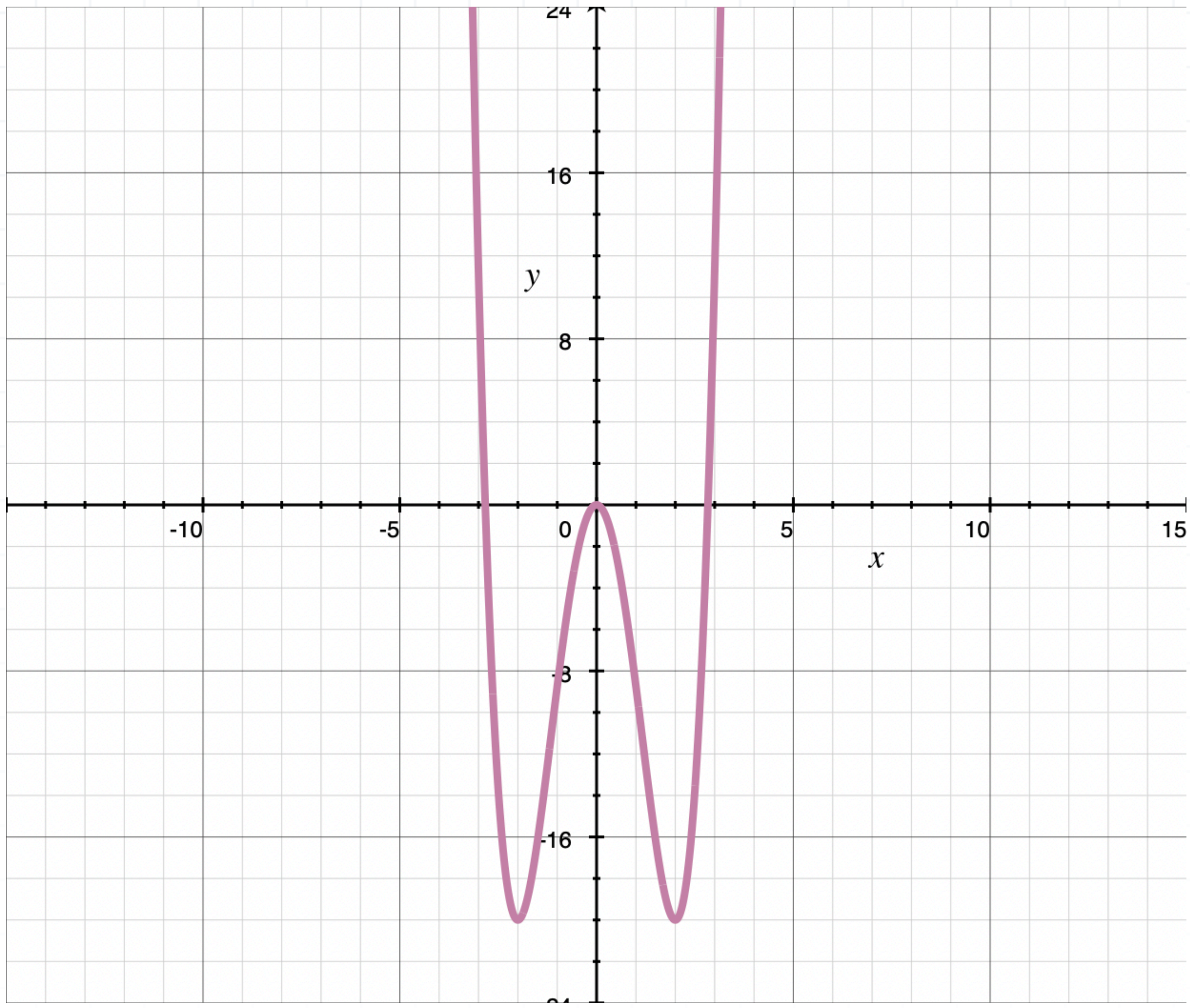

■ 4. Determine the intervals in which the function is increasing and decreasing.

$$f(x) = (4 - 3x)e^x$$

Solution:

Find $f'(x)$ and the x-values inside the given interval for which $f'(x) = 0$ or is undefined.

Find the derivative using product rule.

$$f'(x) = -3e^x + (4 - 3x)e^x$$

$$f'(x) = e^x(-3 + 4 - 3x)$$

$$f'(x) = e^x(1 - 3x)$$

This derivative exists everywhere. The derivative will also be equal to 0 when

$$e^x(1 - 3x) = 0$$

$$1 - 3x = 0$$

$$x = \frac{1}{3}$$

Determine where $f'(x) > 0$ or $f'(x) < 0$ by selecting a value between each critical number.

Interval	x<1/3	x>1/3
x	0	1
f'(x)	>0	<0
f(x)	Increasing	Decreasing

The graph of $f(x)$ shows that the function is decreasing on $(1/3,\infty)$ and increasing on $(-\infty,1/3)$.

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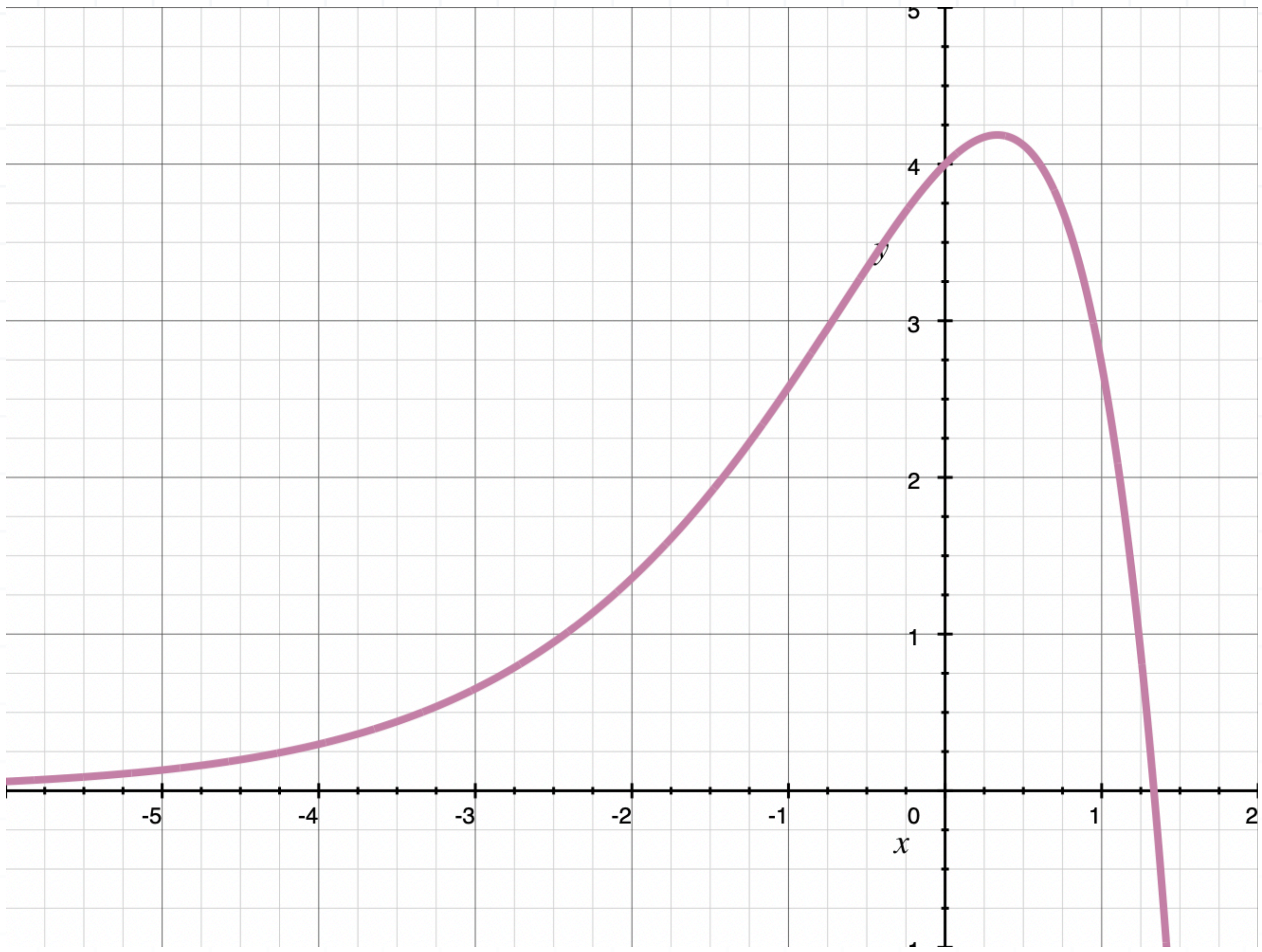

5. Identify the critical point(s) of the function.

$$f(x) = x + 3\ln(2x + 3)$$

Solution:

Find $f'(x)$ and the x-values inside the given interval for which $f'(x) = 0$ or is undefined.

Find the derivative.

$$f'(x) = 1 + \frac{3}{2x + 3} \cdot 2$$

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$$f'(x) = 1 + \frac{6}{2x+3}$$

When $x = -3/2$, the denominator of the second fraction will be 0, which will make the derivative undefined, but $x = -3/2$ is not in the domain of function, so $x = -3/2$ isn't a critical point. The derivative will also be equal to 0 when

$$1 + \frac{6}{2x+3} = 0$$

$$\frac{6}{2x+3} = -1$$

$$6 = -2x - 3$$

$$-2x = 9$$

$$x = -\frac{9}{2}$$

In the same way, $x = -9/2$ is not in the domain of the function, so $x = -9/2$ isn't critical point. Therefore, the function does not have critical points.

■ 6. Find the values a and b such that $f(x) = x^3 + ax^2 + b$ will have a critical point at $(-1,5)$.

Solution:

Since $(-1,5)$ is a critical point, we have

$$5 = (-1)^3 + a(-1)^2 + b$$

$$5 = -1 + a + b$$

$$a + b = 6$$

The derivative is $f'(x) = 3x^2 + 2ax$, so we can identify the critical points where $f'(-1) = 0$. This derivative exists everywhere.

$$3x^2 + 2ax = 0$$

$$3(-1)^2 + 2a(-1) = 0$$

$$3 - 2a = 0$$

$$a = \frac{3}{2}$$

Substitute $a = 3/2$ into $a + b = 6$ and solve for b.

$$\frac{3}{2} + b = 6$$

$$b = \frac{12}{2} - \frac{3}{2}$$

$$b = \frac{9}{2}$$

So the function is

$$f(x) = x^3 + \frac{3}{2}x^2 + \frac{9}{2}$$

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INFLECTION POINTS AND THE SECOND DERIVATIVE TEST

■ 1. Find the inflection points of the function.

$$f(x) = \frac{1}{3}x^3 + x^2$$

Solution:

The function will have inflection points where its second derivative is equal to 0, so we'll find the second derivative,

$$f'(x) = x^2 + 2x$$

$$f''(x) = 2x + 2$$

then set it equal to 0 and solve for x.

$$2x + 2 = 0$$

$$x + 1 = 0$$

$$x = -1$$

So the function has an inflection point at $x = -1$, which means that it's either concave down to the left of $x = -1$ and concave up to the right of $x = -1$, or its concave up to the left of $x = -1$ and concave down to the right of $x = -1$.

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■ 2. For $g(x) = -x^3 + 2x^2 + 3$, find inflection points and identify where the function is concave up and concave down.

Solution:

Find the first and second derivatives.

$$g'(x) = -3x^2 + 4x$$

$$g''(x) = -6x + 4$$

The function has an inflection point when $g''(x) = 0$.

$$-6x + 4 = 0$$

$$-6x = -4$$

$$x = \frac{2}{3}$$

Check values around $x = 2/3$.

Interval	x<2/3	x=2/3	x>2/3
x	-1	2/3	1
g"(x)	+	0	-
Concavity	Up	Inflection	Down

The inflection point is at $x = 2/3$ and $g(2/3) = 97/27$, so the inflection point is $(2/3,97/27)$. The function is concave up on $(-\infty,2/3)$ and concave down on $(2/3,\infty)$.

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3. For $h(x) = x^4 + x^3 - 3x^2 + 2$, find inflection points and identify where the function is concave up and concave down.

Solution:

Find the first and second derivatives.

$$h'(x) = 4x^3 + 3x^2 - 6x$$

$$h''(x) = 12x^2 + 6x - 6 = 6(2x^2 + x - 1) = 6(2x - 1)(x + 1)$$

The function has an inflection point when $h''(x) = 0$.

$$6(2x - 1)(x + 1) = 0$$

$$x = -1, \frac{1}{2}$$

Check values around these inflection points.

Interval	x<-1	x=-1	-1<x<1/2	x=1/2	x>1/2
x	-2	-1	0	1/2	1
h''(x)	+	0	-	0	+
Concavity	Up	Inflection	Down	Inflection	Up

An inflection point is at $x = -1$ and $h(-1) = -1$, so an inflection point is $(-1, -1)$. Another inflection point is at $x = 1/2$ and $h(1/2) = 23/16$, so an inflection point is $(1/2,23/16)$. The function is concave up on $(-\infty,-1)\cup(1/2,\infty)$ and concave down on $(-1,1/2)$.

■ 4. Use the second derivative test to identify the extrema of $f(x)=x^3-12x-2$ as maximum values or minimum values.

Solution:

Find the first and second derivatives.

$$f'(x)=3x^2-12$$

$$f''(x)=6x$$

The function has critical points when $f'(x)=0$.

$$3x^2-12=0$$

$$(x+2)(x-2)=0$$

$$x=-2,\ 2$$

Plug these values into the second derivative.

$$f''(-2)=6(-2)=-12$$

$$f''(2)=6(2)=12$$

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By the second derivative test, the function has a local maximum at $x = -2$. Since $f(-2) = 14$, $(-2,14)$ is a maximum. The function has a local minimum at $x = 2$. Since $f(2) = -18$, $(2,-18)$ is a minimum.

■ 5. Use the second derivative test to identify the extrema of $g(x) = -4xe^{-\frac{x}{2}}$ as maxima or minima.

Solution:

Find the first and second derivatives.

$$g'(x) = -4e^{-\frac{x}{2}} - 4x\left(-\frac{1}{2}\right)e^{-\frac{x}{2}}$$

$$g'(x) = -4e^{-\frac{x}{2}} + 2xe^{-\frac{x}{2}}$$

$$g'(x) = e^{-\frac{x}{2}}(2x-4)$$

and

$$g''(x) = e^{-\frac{x}{2}}(4-x)$$

The function has critical points when $g'(x) = 0$.

$$e^{-\frac{x}{2}}(2x-4) = 0$$

$$2x-4 = 0$$

$$x = 2$$

Plug this value into the second derivative.

$$g''(2) = e^{-\frac{2}{2}}(4 - 2) = \frac{2}{e}$$

The second derivative is positive at $x = 2$, which means the function has a minimum at $(2, -8/e)$.

$$g(2) = -\frac{8}{e}$$

■ 6. Use the second derivative test to identify the extrema of $h(x) = 2x^4 - 4x^2 + 1$ as maximum values or minimum values.

Solution:

Find the first and second derivatives.

$$h'(x) = 8x^3 - 8x$$

$$h''(x) = 24x^2 - 8$$

The function has critical points when $h'(x) = 0$.

$$8x^3 - 8x = 0$$

$$x(x + 1)(x - 1) = 0$$

$$x = -1, 0, 1$$

Plug these values into the second derivative.

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$$h''(-1) = 24(-1)^2 - 8 = 16$$

$$h''(0) = 24(0)^2 - 8 = -8$$

$$h''(1) = 24(1)^2 - 8 = 16$$

By the second derivative test, the function has a local minimum at $x = -1$. Since $h(-1) = -1$, $(-1, -1)$ is a minimum. The function has a local maximum at $x = 0$. Since $h(0) = 1$, $(0,1)$ is a maximum. The function has a local minimum at $x = 1$. Since $h(1) = -1$, $(1, -1)$ is a minimum.

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INTERCEPTS AND VERTICAL ASYMPTOTES

■ 1. Find the x-intercepts and any vertical asymptote(s) of the function.

$$f(x) = \frac{-x^2 + 16x - 63}{x^2 - 2x - 35}$$

Solution:

Factor the numerator and denominator as completely as possible.

$$f(x) = \frac{-(x-7)(x-9)}{(x-7)(x+5)}$$

The denominator is equal to 0 if $x = 7$ or $x = -5$, which means the function has two discontinuities. However, the function simplifies to

$$f(x) = \frac{-(x-9)}{x+5}$$

$$f(x) = \frac{9-x}{x+5}$$

So, the function has an x-intercept at $(9,0)$. Therefore, the function has a removable discontinuity at $x = 7$ and a vertical asymptote at $x = -5$, which means the domain of the function is $(-\infty, -5) \cup (-5,7) \cup (7,\infty)$.

■ 2. Find any vertical asymptote(s) of the function.

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$$g(x) = \frac{x^2 - 3x - 10}{x^2 + x - 2}$$

Solution:

Factor the numerator and denominator as completely as possible.

$$g(x) = \frac{(x-5)(x+2)}{(x-1)(x+2)}$$

The denominator is equal to 0 if $x = -2$ or $x = 1$, which means the function has two discontinuities. However, the function simplifies to

$$g(x) = \frac{x-5}{x-1}$$

Therefore, the function has a removable discontinuity at $x = -2$ and a vertical asymptote at $x = 1$, which means the domain of the function is $(-\infty, -2) \cup (-2,1) \cup (1,\infty)$.

■ 3. Find any vertical asymptote(s) of the function.

$$h(x) = \frac{40 - 27x - 12x^2 - x^3}{9x^2 + 63x - 72}$$

Solution:

Factor the numerator and denominator as completely as possible.

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$$h(x) = \frac{-(x+8)(x+5)(x-1)}{9(x-1)(x+8)}$$

Cancel common factors from the numerator and denominator, then simplify.

$$h(x) = \frac{-(x+5)}{9}$$

$$h(x) = -\frac{x+5}{9}$$

There are no values of x that make this denominator 0, so the function has no vertical asymptotes. But it does have removable discontinuities for the factors we canceled, at $x = -8$ and $x = 1$.

■ 4. Find the y-intercepts and any vertical asymptote(s) of the function.

$$f(x) = \frac{x^2 + -2x - 8}{x^2 - 9x + 20}$$

Solution:

Factor the numerator and denominator as completely as possible.

$$f(x) = \frac{(x+2)(x-4)}{(x-4)(x-5)}$$

Cancel common factors from the numerator and denominator, then simplify.

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$$f(x) = \frac{x+2}{x-5}$$

Therefore, the function has a removable discontinuity at $x = 4$ and a vertical asymptote at $x = 5$. Substitute $x = 0$ into the function,

$$f(x) = \frac{0+2}{0-5} = -\frac{2}{5}$$

So the function has a y-intercept at $(0, -2/5)$.

■ 5. Find any vertical asymptote(s) of the function.

$$g(x) = \ln(x^2 + 5x)$$

Solution:

The logarithmic function is undefined where the argument of the function is equal to zero.

$$x^2 + 5x = 0$$

$$x(x+5) = 0$$

$x = 0$ and $x = -5$

Therefore, the function has vertical asymptotes at $x = -5$ and $x = 0$.

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6. Find any vertical asymptote(s) of the function.

$$h(x) = \sec\left(x + \frac{\pi}{2}\right)$$

Solution:

We know that

$$h(x) = \sec\left(x + \frac{\pi}{2}\right) = \frac{1}{\cos\left(x + \frac{\pi}{2}\right)}$$

The function is undefined whenever the denominator is equal to 0.

$$\cos\left(x + \frac{\pi}{2}\right) = 0$$

The cosine function is equal to 0 when the argument is equal to

$\frac{\pi}{2} + k\pi$, where k is any integer

Therefore, the function has vertical asymptotes at

$$x + \frac{\pi}{2} = \frac{\pi}{2} + k\pi$$

$x = k\pi$, where k is any integer

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HORIZONTAL AND SLANT ASYMPTOTES

■ 1. Find the horizontal asymptote(s) of the function.

$$f(x) = \frac{8x^4 - x^2 + 1}{4x^4 - 1}$$

Solution:

The degree of the numerator is 4, and the degree of the denominator is 4. So the degree of the numerator is equal to the degree of the denominator, which means the ratio of the coefficients on these highest-degree terms is the equation of the horizontal asymptote,

$$y = \frac{8}{4} = 2$$

so the function has a horizontal asymptote at $y = 2$.

■ 2. Find the horizontal asymptote(s) of the function.

$$g(x) = \frac{2x^2 - 5x + 12}{3x^2 - 11x - 4}$$

Solution:

The degree of the numerator is 2, and the degree of the denominator is 2. So the degree of the numerator is equal to the degree of the denominator, which means the ratio of the coefficients on these highest-degree terms is the equation of the horizontal asymptote,

$$y = \frac{2}{3}$$

so the function has the horizontal asymptote at $y = 2/3$.

■ 3. Find the horizontal asymptote(s) of the function.

$$h(x) = \frac{x^3 - x^2 + 6x - 1}{7x^4 - 1}$$

Solution:

The x^3 term is the highest-degree term in the numerator, and the x^4 term is the highest-degree term in the denominator.

Because the degree of the numerator is less than the degree of the denominator, the function has a horizontal asymptote at $y = 0$.

■ 4. Find the slant asymptote of the function.

$$f(x) = \frac{3x^4 - x^3 + x^2 - 4}{x^3 - x^2 + 1}$$

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Solution:

The degree of the numerator is exactly one greater than the degree of the denominator, so the function has a slant asymptote.

Use polynomial long division to rewrite the function as

$$f(x) = 3x + 2 + \frac{3x^2 - 3x - 6}{x^3 - x^2 + 1}$$

The slant asymptote is what we get when we remove the remainder from this rewritten function. If we remove the remainder, we get

$$y = 3x + 2$$

Therefore, the equation of the slant asymptote is $y = 3x + 2$.

■ 5. Find the slant asymptote of the function.

$$g(x) = \frac{8x^2 + 14x - 7}{4x - 1}$$

Solution:

The degree of the numerator is exactly one greater than the degree of the denominator, $2 > 1$, so the function has a slant asymptote.

Use polynomial long division to rewrite the function as

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$$2x + 4 - \frac{3}{4x - 1}$$

The slant asymptote is what we get when we remove the remainder from this rewritten function. If we remove the remainder, we get

$$y = 2x + 4$$

Therefore, the equation of the slant asymptote is $y = 2x + 4$.

■ 6. Determine whether the function has a horizontal asymptote, slant asymptote, or neither.

$$h(x) = \frac{x^4 - x^3 - 8}{x^2 - 5x + 6}$$

Solution:

The degree of the numerator is greater than the degree of the denominator, but we know that the function has a slant asymptote if the degree of the numerator is exactly one greater than the degree of the denominator. So the function doesn't have a horizontal asymptote or a slant asymptote.

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SKETCHING GRAPHS

1. Sketch the graph of the function.

$$f(x) = x^3 - 4x^2 + 8$$

Solution:

First, let's find the y-intercepts by substituting $x = 0$.

$$y = 0^3 - 4(0)^2 + 8 = 8$$

So the function has a y-intercept at $(0,8)$. To find x-intercepts, we'll substitute $y = 0$.

$$0 = x^3 - 4x^2 + 8$$

$$(x - 2)(x^2 - 2x - 4) = 0$$

$$x = 2, 1 - \sqrt{5}, 1 + \sqrt{5}$$

So the function has x-intercepts at $(1 - \sqrt{5}, 0)$, $(1 + \sqrt{5}, 0)$, and $(2,0)$.

Take the derivative, then set it equal to 0 to find critical points.

$$f'(x) = 3x^2 - 8x$$

$$3x^2 - 8x = 0$$

$$x(3x - 8) = 0$$

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$$x = 0, \frac{8}{3}$$

Use the first derivative test to see where $f(x)$ is increasing and decreasing.

Interval	x<0	x=0	0<x<8/3	x=8/3	x>8/3
x	-2	0	1	8/3	4
f'(x)	+	0	-	0	+
Direction	Increasing	Maximum	Decreasing	Minimum	Increasing

We can see that $f(x)$

- increases on the interval $(-\infty,0)$,
- has a local maximum at $x = 0$,
- decreases on the interval $(0,8/3)$,
- has a local minimum at $x = 8/3$, and then
- increases on the interval $(8/3,\infty)$.

Evaluate the function at the extrema.

$$f(0) = (0)^3 - 4(0)^2 + 8 = 8$$

$$f\left(\frac{8}{3}\right) = \left(\frac{8}{3}\right)^3 - 4\left(\frac{8}{3}\right)^2 + 8 = -\frac{40}{27}$$

There's a local maximum at $(0,8)$ and a local minimum at $(8/3, -40/27)$. Now use the second derivative to determine concavity.

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$$f''(x) = 6x - 8$$

$$6x - 8 = 0$$

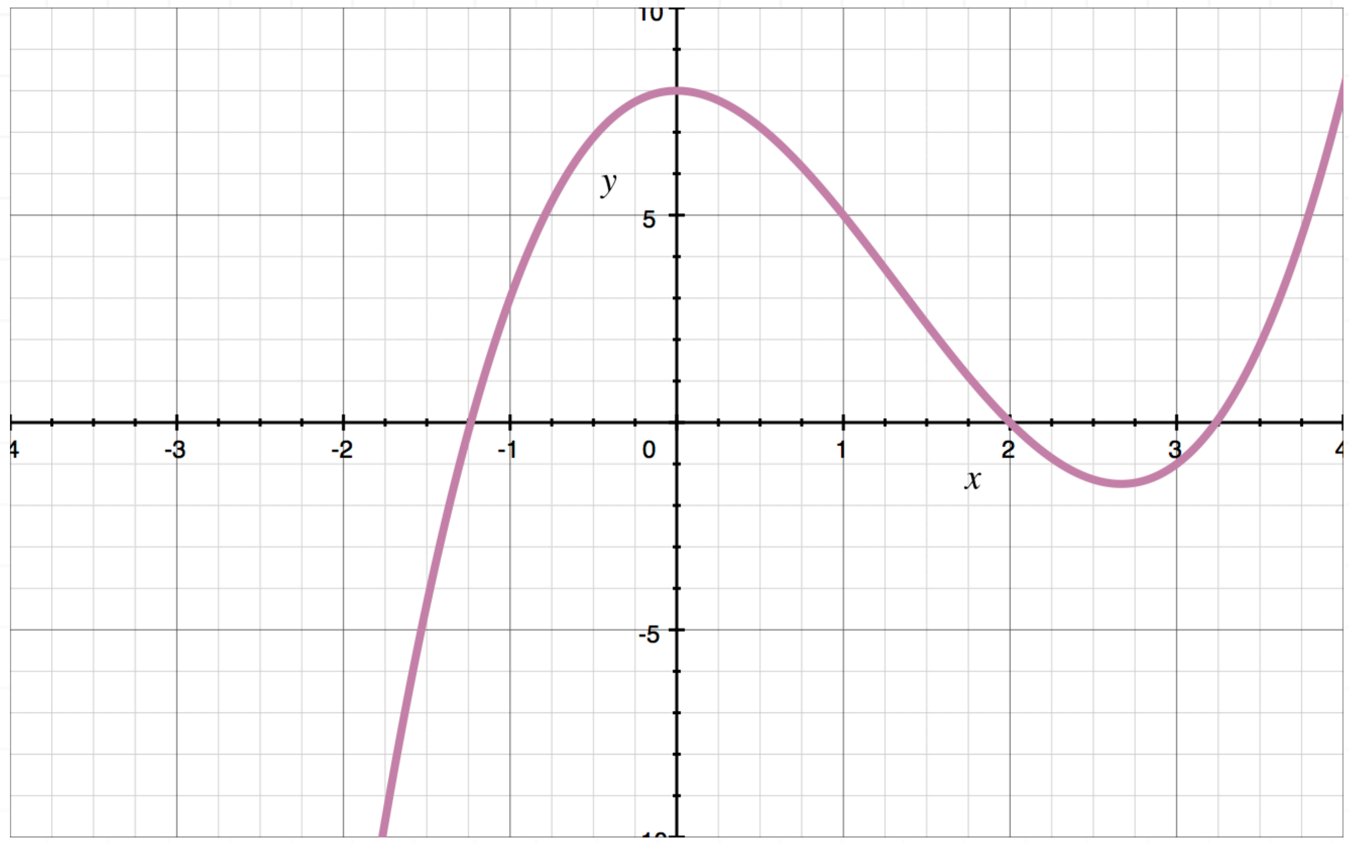
$$x = \frac{4}{3}$$

Test values around the inflection point $x = 4/3$.

Interval	x<4/3	x=4/3	x>4/3
x	0	4/3	3
f''(x)	-	0	+
Concavity	Down	Inflection	Up

We can see that $f(x)$ is concave down on the interval $(-\infty,4/3)$ and concave up on the interval $(4/3,\infty)$. Because $f(4/3) = 88/27$, $f(x)$ has an inflection point at $(4/3,88/27)$. Since $f(x)$ is a polynomial function, its graph has no asymptotes.

Putting all this together, the graph is



■ 2. Sketch the graph of the function.

$$g(x) = \frac{1}{4}x^4 - \frac{1}{3}x^3 - 3x^2 + 1$$

Solution:

First, let's find the y-intercepts by substituting $x = 0$.

$$y = \frac{1}{4}(0)^4 - \frac{1}{3}(0)^3 - 3(0)^2 + 1 = 1$$

So the function has a y-intercept at $(0,1)$. To find x-intercepts, we'll substitute $y = 0$. We get four x-intercepts, but it's not easy to find them.

Take the derivative, then set it equal to 0 to find critical points.

$$g'(x) = x^3 - x^2 - 6x$$

$$x^3 - x^2 - 6x = 0$$

$$x(x-3)(x+2) = 0$$

$$x = -2,\ 0,\ 3$$

Use the first derivative test to see where $g(x)$ is increasing and decreasing.

Interval	x<-2	x=-2	-2<x<0	x=0	0<x<3	x=3	x>3
x	-4	-2	-1	0	2	3	4
g'(x)	-	0	+	0	-	0	+
Direction	Decreasing	Minimum	Increasing	Maximum	Decreasing	Minimum	Increasing

We can see that $g(x)$

- decreases on the interval $(-\infty, -2)$,
- has a local minimum at $x = -2$,
- increases on the interval $(-2,0)$,
- has a local maximum at $x = 0$
- decreases on the interval $(0,3)$
- has a local minimum at $x = 3$, and then
- increases on the interval $(3,\infty)$.

Evaluate the function at the extrema.

$$g(-2) = \frac{1}{4}(-2)^4 - \frac{1}{3}(-2)^3 - 3(-2)^2 + 1 = -\frac{13}{3}$$

$$g(0) = \frac{1}{4}(0)^4 - \frac{1}{3}(0)^3 - 3(0)^2 + 1 = 1$$

$$g(3) = \frac{1}{4}(3)^4 - \frac{1}{3}(3)^3 - 3(3)^2 + 1 = -\frac{59}{4}$$

There's a local minimum at $(-2, -13/3)$, a local maximum at $(0,1)$, and a local minimum at $(3, -59/4)$. Now use the second derivative to determine concavity.

$$g''(x) = 3x^2 - 2x - 6$$

$$3x^2 - 2x - 6 = 0$$

$$x = \frac{2 \pm \sqrt{4 + 72}}{6} = \frac{1 \pm \sqrt{19}}{3}$$

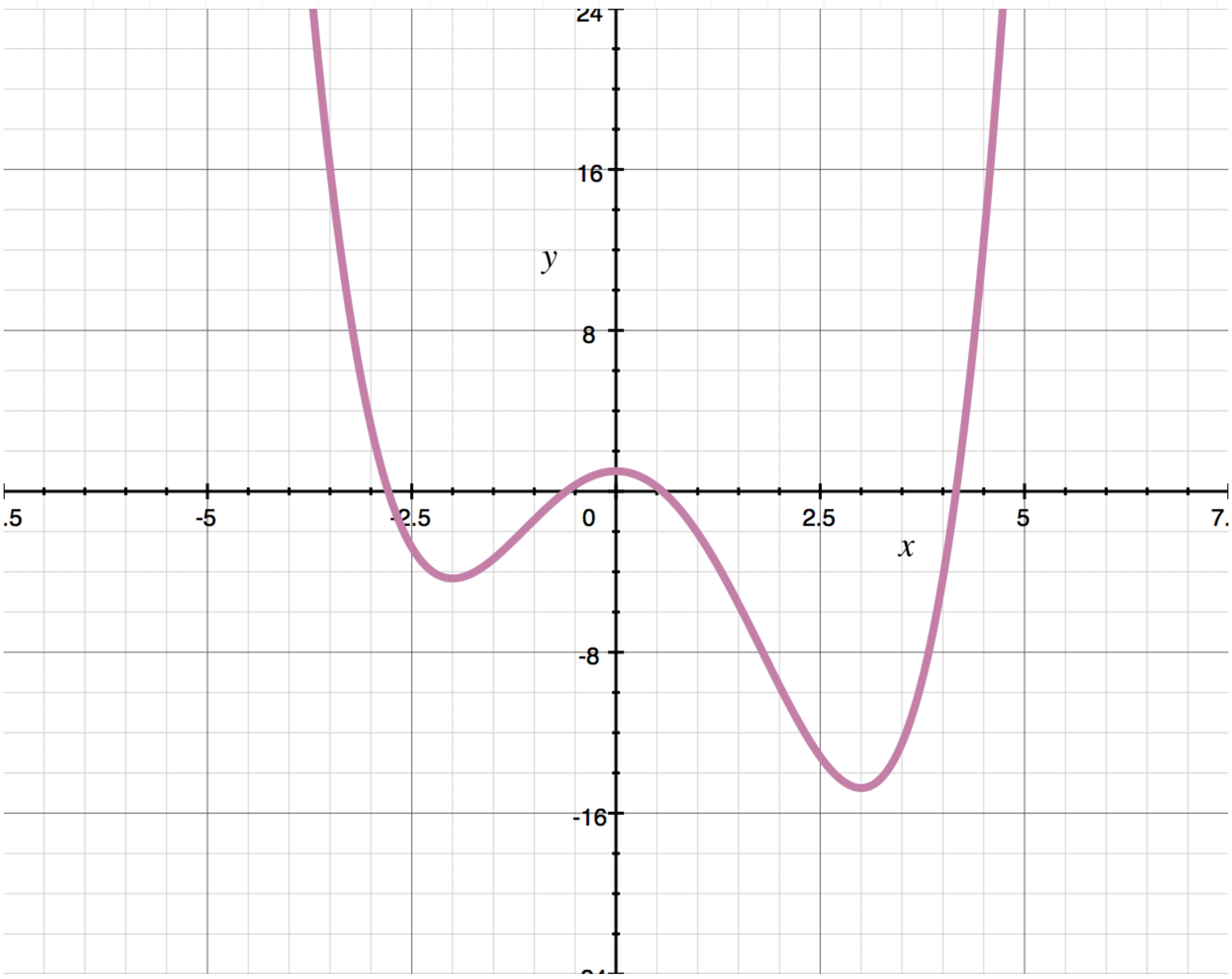
Test values around the inflection points using their approximate values.

Interval	x<-1.12	x=-1.12	-1.12<x<1.79	x=1.79	x>1.79
x	-2	-1.2	1	1.79	4
g"(x)	+	0	-	0	+
Concavity	Up	Inflection	Down	Inflection	Up

We can see that $g(x)$ is concave up on the interval $(-\infty, (1 - \sqrt{19})/3)$, concave down on the interval $((1 - \sqrt{19})/3, (1 + \sqrt{19})/3)$, and concave up on the interval $((1 + \sqrt{19})/3, \infty)$. Because $g((1 - \sqrt{19})/3) \approx -1.9$, $g(x)$ has an inflection point at approximately $(-1.12, -1.9)$. Because

$g((1+\sqrt{19})/3) \approx -7.96$, $g(x)$ has an inflection point at approximately $(1.79, -7.96)$. Since $g(x)$ is a polynomial function, its graph has no asymptotes.

Putting all this together, the graph is



3. Sketch the graph of the function.

$$h(x) = \frac{x^2 + x - 6}{4x^2 + 16x + 12}$$

Solution:

Factor the numerator and denominator, then cancel common factors.

$$h(x) = \frac{x^2 + x - 6}{4x^2 + 16x + 12} = \frac{(x+3)(x-2)}{4(x+3)(x+1)} = \frac{x-2}{4(x+1)}$$

So the domain of the function is all real numbers except $x = -3$ and $x = -1$.

Take the derivative, then set it equal to 0 to find critical points.

$$h'(x) = \frac{12(x^2 + 6x + 9)}{(4x^2 + 16x + 12)^2} = \frac{12(x+3)^2}{[4(x+1)(x+3)]^2} = \frac{12(x+3)^2}{16(x+1)^2(x+3)^2} = \frac{3}{4(x+1)^2}$$

There are no values for which $h'(x) = 0$, so there are no critical points. The derivative is undefined when $x = -1$, but this is not a critical point since it's not in the domain of the function. Now use the second derivative to determine concavity.

$$h''(x) = -\frac{3}{2(x+1)^3}$$

There are no values for which $h''(x) = 0$, but $h''(x)$ is undefined when $x = -1$. Test values around the inflection point $x = -1$.

Interval	x<-1	x=-1	x>-1
x	-3	-1	3
h''(x)	+	DNE	-
Concavity	Up	Inflection	Down

We can see that $h(x)$ is concave up on the interval $(-\infty, -1)$ and concave down on the interval $(-1, \infty)$. Since $h(x)$ is a rational function, we need to look for asymptotes.

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The behavior of the function is dominated by the highest degree terms in the numerator and denominator, which means the horizontal asymptote is

$$\lim_{x \to \pm\infty} \frac{x^2}{4x^2} = \lim_{x \to \pm\infty} \frac{1}{4} = \frac{1}{4}$$

The denominator of $h(x)$ is 0 when $x = -1$, so the function has a vertical asymptote there and the removable discontinuity at $x = -3$. To determine the behavior of the function near $x = -1$, find

$$\lim_{x \to -1^-} h(x) = \infty$$

$$\lim_{x \to -1^+} h(x) = -\infty$$

Lastly, the function crosses the x-axis when the numerator equals 0, which occurs at $x = 2$, and the function crosses the y-axis when $x = 0$, which gives a y-intercept of $(0, -1/2)$.

Putting all this together, the graph is

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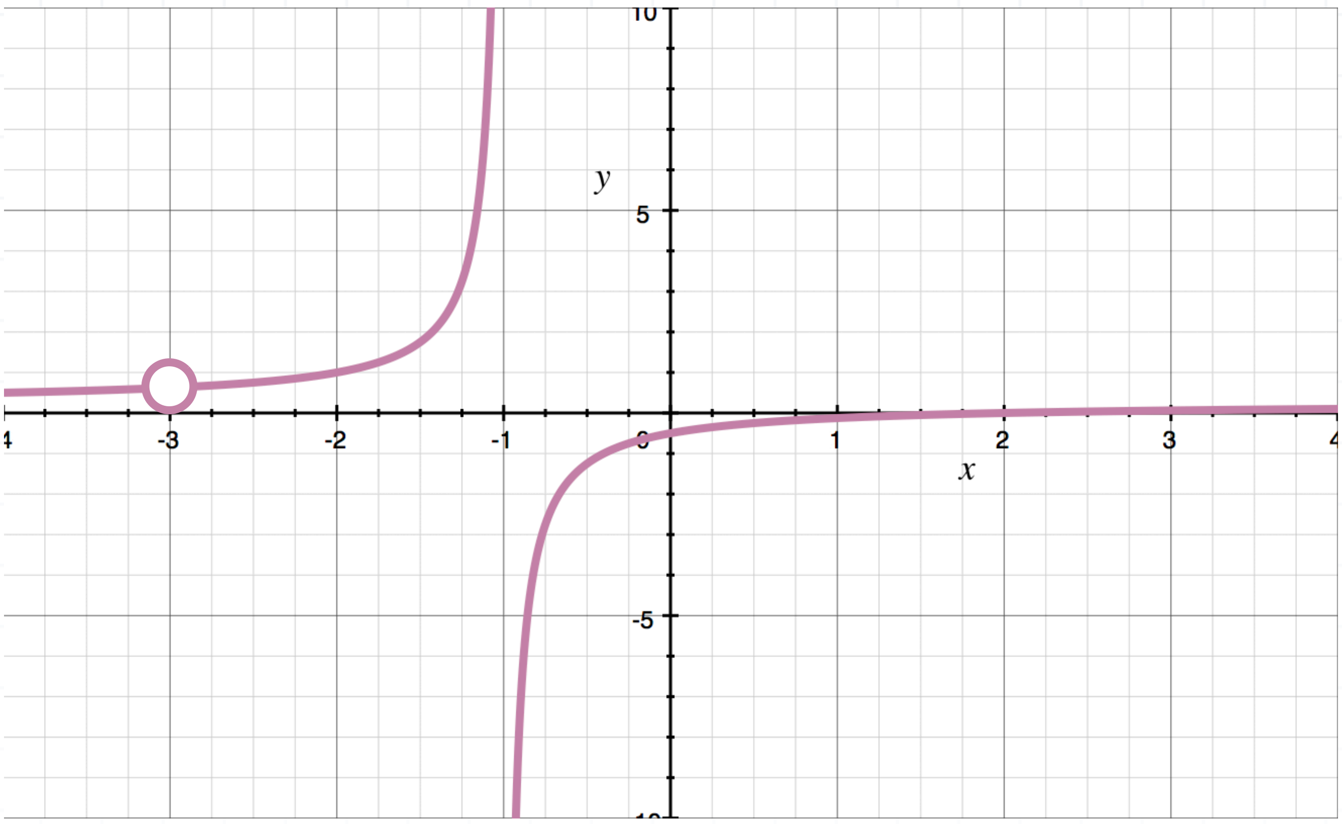

4. Sketch the graph of the function.

$$f(x) = \frac{4}{1 + x^2}$$

Solution:

First, let's find the y-intercepts by substituting $x = 0$.

$$f(0) = \frac{4}{1 + 0^2} = 4$$

So the function has a y-intercept at $(0,4)$. To find x-intercepts, we'll substitute $y = 0$.

$$0 = \frac{4}{1 + x^2}$$

Because there's no value of x that makes this equation true, the function has no x-intercepts.

Take the derivative, then set it equal to 0 to find critical points.

$$f'(x) = -\frac{8x}{(1+x^2)^2}$$

$$-8x = 0$$

$$x = 0$$

Use the first derivative test to see where $f(x)$ is increasing and decreasing.

Interval	x<0	x=0	x>0
x	-1	0	1
f'(x)	+	0	-
Direction	Increasing	Maximum	Decreasing

We can see that $f(x)$

- increases on the interval $(-\infty,0)$,
- has a local maximum at $x = 0$,
- decreases on the interval $(0,\infty)$,

Evaluate the function at the extrema.

$$f(0) = \frac{4}{1+0^2} = 4$$

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There's a local maximum at $(0,4)$. Now use the second derivative to determine concavity.

$$f''(x) = \frac{8(3x^2 - 1)}{(x^2 + 1)^3}$$

$$3x^2 - 1 = 0$$

$$x = \pm \frac{1}{\sqrt{3}}$$

Test values around the inflection point $x = \pm 1/\sqrt{3}$.

Interval	x<-0.6	x=-0.6	-0.6<x<0.6	x=0.6	x>0.6
x	-1	-0.6	0	0.6	1
f''(x)	+	0	-	0	+
Concavity	Up	Inflection	Down	Inflection	Up

We can see that $f(x)$ is concave up on the intervals $(-\infty, -1/\sqrt{3})$ and $(1/\sqrt{3}, \infty)$, and concave down on the interval $(-1/\sqrt{3}, 1/\sqrt{3})$. Because $f(-1/\sqrt{3}) = 3$ and $f(1/\sqrt{3}) = 3$, $f(x)$ has inflection points at $(-1/\sqrt{3}, 3)$ and $(1/\sqrt{3}, 3)$.

The function has no vertical asymptotes, but because the degree of the numerator is less than the degree of the denominator, the function has a horizontal asymptote at $y = 0$.

Putting all this together, the graph is

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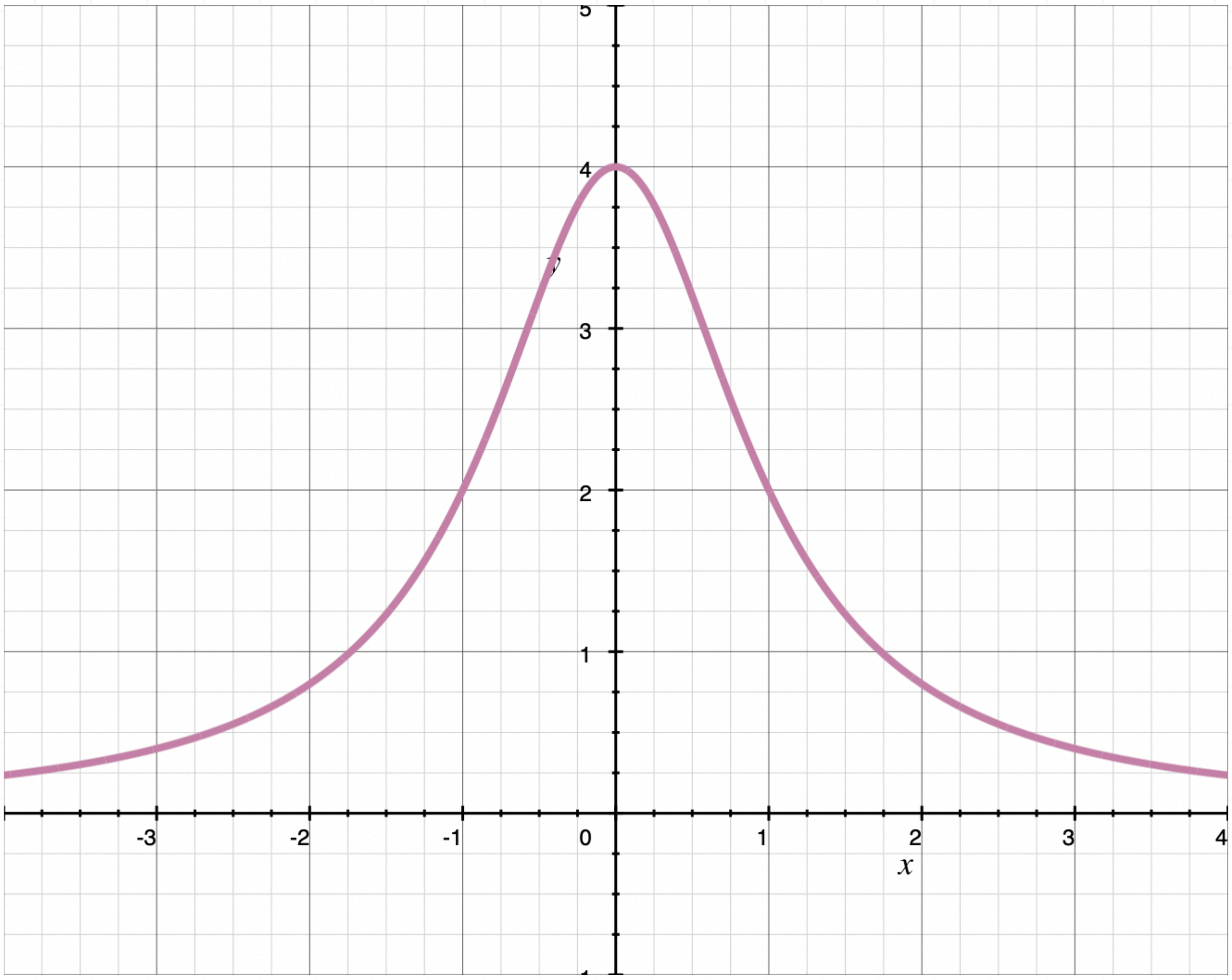

■ 5. Sketch the graph of the function.

$$f(x) = 2x \ln x$$

Solution:

The domain of the function is $x > 0$. The function has no y-intercepts. To find x-intercepts, we'll substitute $y = 0$.

$$0 = 2x \ln x$$

$$x = 1$$

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So the function has an x-intercept at $(1,0)$.

Take the derivative, then set it equal to 0 to find critical points.

$$f'(x) = 2\ln x + 2$$

$$0 = 2\ln x + 2$$

$$\ln x = -1$$

$$x = \frac{1}{e}$$

Use the first derivative test to see where $f(x)$ is increasing and decreasing.

Interval	x<0.37	x=0.37	x>0.37
x	0.1	0.37	1
f'(x)	-	0	+
Direction	Decreasing	Minimum	Increasing

We can see that $f(x)$

- increases on the interval $\left(\frac{1}{e},\infty\right)$,
- has a local minimum at $x = \frac{1}{e}$,
- decreases on the interval $\left(0,\frac{1}{e}\right)$,

Evaluate the function at the extrema.

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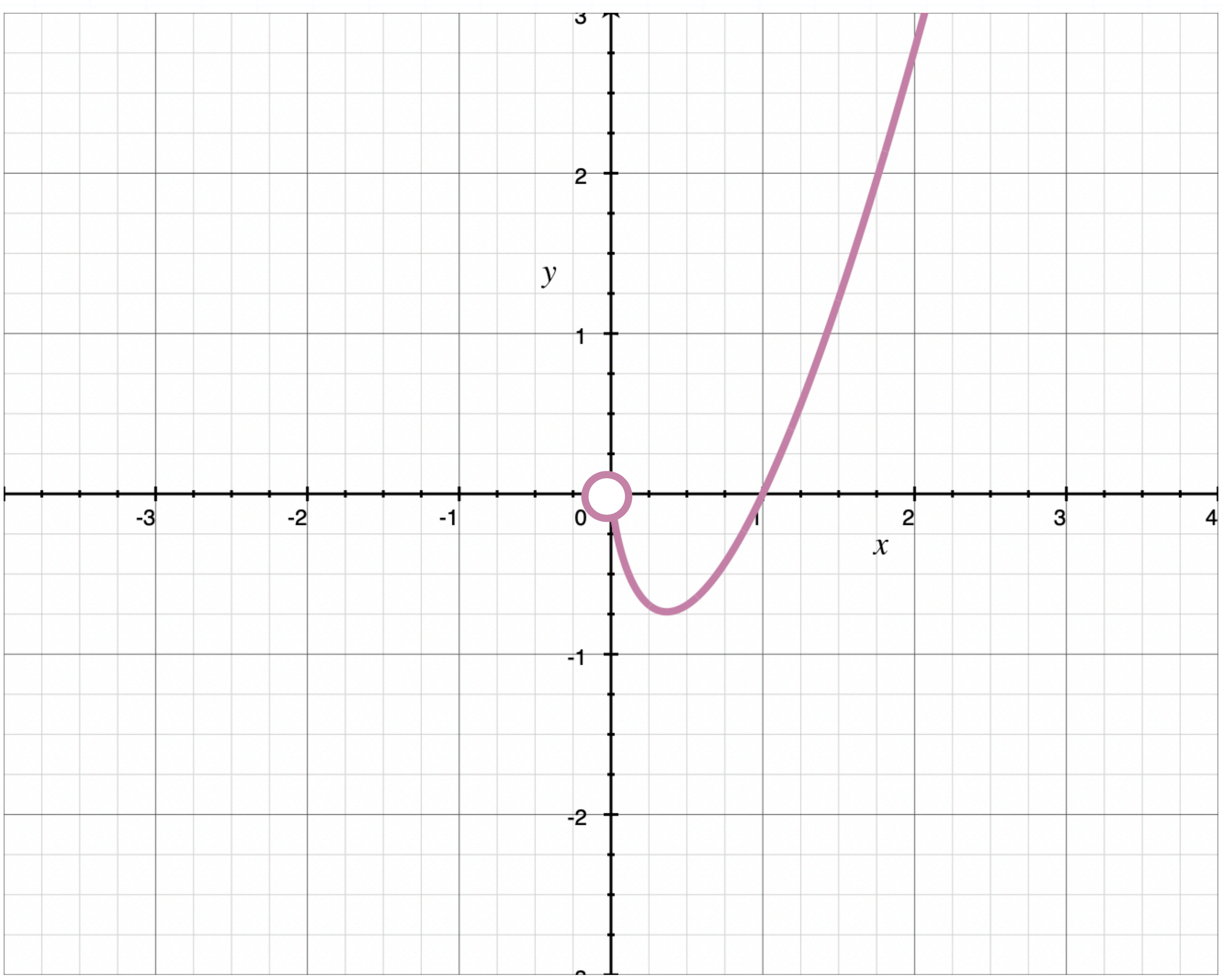
$$f\left(\frac{1}{e}\right) = 2\frac{1}{e}\ln\frac{1}{e} = -\frac{2}{e}$$

There's a local minimum at $(1/e, -2/e)$. Now use the second derivative to determine concavity.

$$f''(x) = \frac{2}{x}$$

$$\frac{2}{x} = 0$$

The function has no inflection points. Since $f''(x) > 0$ for all x in the domain, the function is concave up everywhere. And the function $f(x)$ has no asymptotes. Putting all this together, the graph is



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■ 6. Sketch the graph of the function.

$$f(x) = x^2\sqrt{x+4}$$

Solution:

The domain of the function is $x \geq -4$. First, let's find the y-intercepts by substituting $x = 0$.

$$f(0) = 0^2\sqrt{0+4} = 0$$

So the function has a y-intercepts at $(0,0)$. To find x-intercepts, we'll substitute $y = 0$.

$$0 = x^2\sqrt{x+4}$$

$$x = -4,0$$

So the function has x-intercepts at $(-4,0)$ and $(0,0)$.

Take the derivative, then set it equal to 0 to find critical points.

$$f'(x) = \frac{5x^2 + 16x}{2\sqrt{x+4}}$$

$$0 = \frac{5x^2 + 16x}{2\sqrt{x+4}}$$

$$5x^2 + 16x = 0$$

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$$x = -4, \ -\frac{16}{5}, \ 0$$

Use the first derivative test to see where $f(x)$ is increasing and decreasing.

Interval	-4<x<-16/5	x=-16/5	-16/5<x<0	x=0	x>0
x	-3.5	-16/5	-1	0	1
f'(x)	+	0	-	0	+
Direction	Increasing	Maximum	Decreasing	Minimum	Increasing

We can see that $f(x)$

- increases on the interval $\left(-4,-\frac{16}{5}\right)$ and $(0,\infty)$,
- has a local maximum at $x = -\frac{16}{5}$,
- decreases on the interval $\left(-\frac{16}{5},0\right)$,
- has a local minimum at $x = 0$,

Evaluate the function at the extrema.

$$f(0) = 0^2\sqrt{0+4} = 0$$

$$f\left(-\frac{16}{5}\right) = \left(-\frac{16}{5}\right)^2\sqrt{-\frac{16}{5}+4} = \frac{512}{25\sqrt{5}} \approx 9$$

There's a local maximum at $(-3.2,9)$ and local minimum at $(0,0)$. Now use the second derivative to determine concavity.

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$$f''(x) = \frac{15x^2 + 96x + 128}{4(x+4)^{\frac{3}{2}}}$$

$$0 = \frac{15x^2 + 96x + 128}{4(x+4)^{\frac{3}{2}}}$$

$$15x^2 + 96x + 128 = 0$$

$$x \approx -4.5, \ -1.9$$

Test values around the inflection point $x = -1.9$. The value $x = -4.5$ does not represent an inflection point, because it's not in domain.

Interval	-4<x<-1.9	x=-1.9	x>-1.9
x	-2	-1.9	0
f''(x)	-	0	+
Concavity	Down	Inflection	Up

We can see that $f(x)$ is concave up on the interval $(-1.9,\infty)$ and concave down on the interval $(-4,-1.9)$. Because $f(-1.9) = 5.2$, $f(x)$ has an inflection point at $(-1.9,5.2)$. The function has no asymptotes, so putting all this together, the graph is

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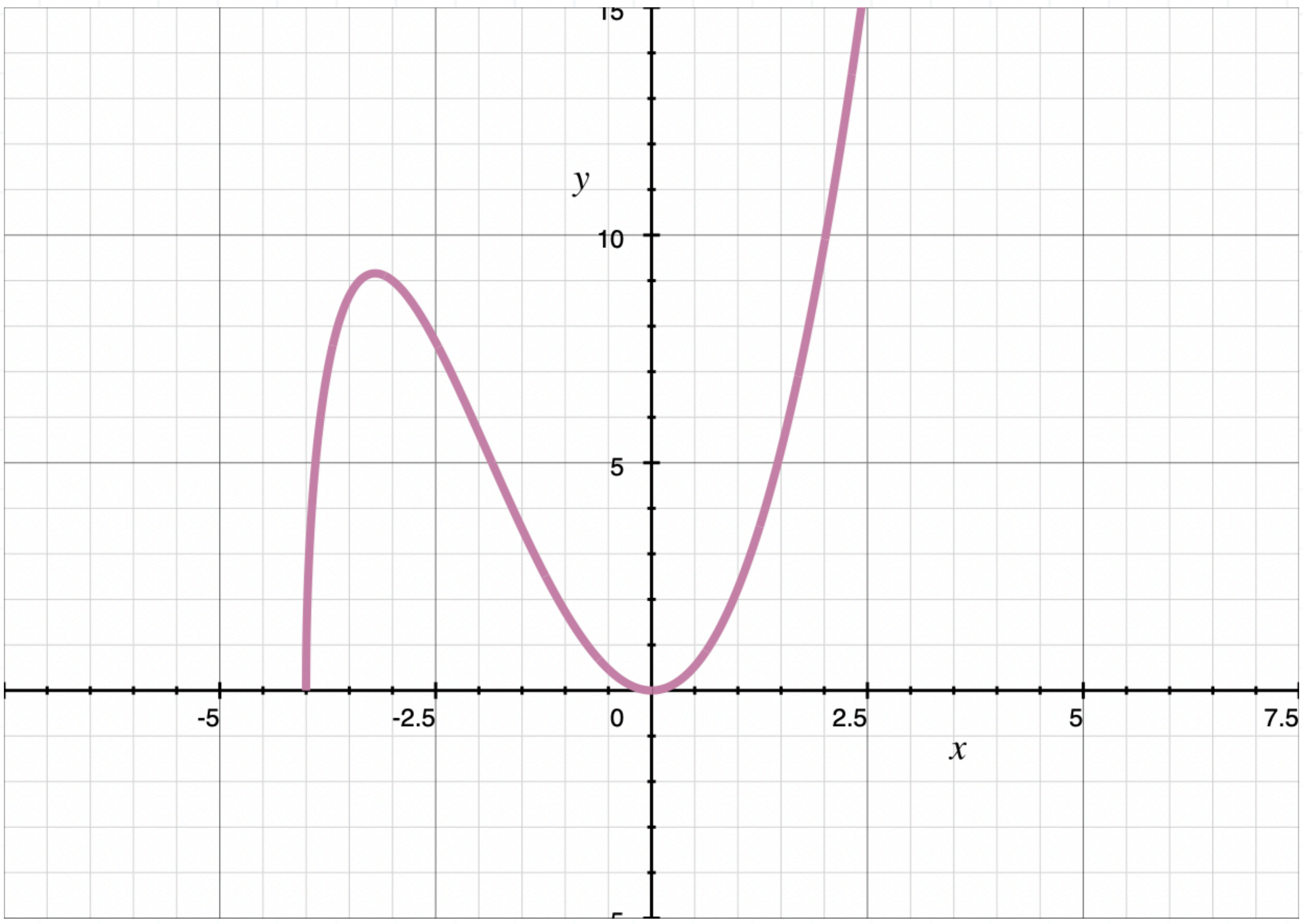

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EXTREMA ON A CLOSED INTERVAL

1. Find the extrema of $f(x) = x^3 - 3x^2 + 5$ over the closed interval $[-3,4]$.

Solution:

Find the critical points of the function.

$$f'(x) = 3x^2 - 6x$$

$$3x^2 - 6x = 0$$

$$3x(x - 2) = 0$$

$$x = 0,\ 2$$

Evaluate the function at the endpoints and at the critical numbers.

For $x = -3$, $\quad (-3)^3 - 3(-3)^2 + 5 = -27 - 27 + 5 = -49$

For $x = 0$, $\quad (0)^3 - 3(0)^2 + 5 = 0 - 0 + 5 = 5$

For $x = 2$, $\quad (2)^3 - 3(2)^2 + 5 = 8 - 12 + 5 = 1$

For $x = 4$, $\quad (4)^3 - 3(4)^2 + 5 = 64 - 48 + 5 = 21$

The results show that $f(x)$ has a global minimum at $(-3, -49)$, a local maximum at $(0,5)$, a local minimum at $(2,1)$, and a global maximum at $(4,21)$.

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■ 2. Find the extrema of $g(x)=\sqrt[3]{2x^2+3}$ over the closed interval $[-1,5]$.

Solution:

Find the critical points of the function.

$$g'(x)=\frac{1}{3}(2x^2+3)^{-\frac{2}{3}}(4x)=\frac{4x}{3\sqrt[3]{(2x^2+3)^2}}$$

$$4x=0$$

$$x=0$$

Evaluate the function at the endpoints and at the critical numbers.

For $x=-1$, $\quad \sqrt[3]{2(-1)^2+3}=\sqrt[3]{5}\approx 1.71$

For $x=0$, $\quad \sqrt[3]{2(0)^2+3}=\sqrt[3]{3}\approx 1.44$

For $x=5$, $\quad \sqrt[3]{2(5)^2+3}=\sqrt[3]{53}\approx 3.76$

The results show that $g(x)$ has a global minimum at $(0,\sqrt[3]{3})$, a local maximum at $(0,\sqrt[3]{5})$, and a global maximum at $(5,\sqrt[3]{53})$.

■ 3. Find the extrema of $h(x)=-4x^3+6x^2-3x-2$ over the closed interval $[-4,6]$.

Solution:

Find the critical points of the function.

$$h'(x) = -12x^2 + 12x - 3$$

$$-12x^2 + 12x - 3 = 0$$

$$-3(4x^2 - 4x + 1) = 0$$

$$-3(2x - 1)(2x - 1) = 0$$

$$x = 1/2$$

Evaluate the function at the endpoints and at the critical numbers.

For $x = -4$, $\quad -4(-4)^3 + 6(-4)^2 - 3(-4) - 2 = 362$

For $x = 1/2$, $\quad -4(1/2)^3 + 6(1/2)^2 - 3(1/2) - 2 = -5/2$

For $x = 6$, $\quad -4(6)^3 + 6(6)^2 - 3(6) - 2 = -668$

The results show that $h(x)$ has a global maximum at $(-4,362)$, a horizontal tangent line at $(1/2, -5/2)$, and a global minimum at $(6, -668)$.

■ 4. Find the extrema of the function over the closed interval $[-1,3]$.

$$f(x) = \frac{x^2}{x^2 + 7}$$

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Solution:

Find the critical points of the function.

$$f'(x) = \frac{14x}{(x^2+7)^2}$$

$$14x = 0$$

$$x = 0$$

Evaluate the function at the endpoints of the interval and at the critical points.

For $x = -1$, $\quad \dfrac{(-1)^2}{(-1)^2+7} = \dfrac{1}{8}$

For $x = 0$, $\quad \dfrac{(0)^2}{(0)^2+7} = 0$

For $x = 3$, $\quad \dfrac{(3)^2}{(3)^2+7} = \dfrac{9}{16}$

The results show that $f(x)$ has a global minimum at $(3,9/16)$, a local maximum at $(-1,1/8)$, and a global maximum at $(0,0)$.

■ 5. Find the extrema of $g(x) = e^{2x^3+4x^2-8x+3}$ over the closed interval $[-4,0]$.

Solution:

Find the critical points of the function.

$$g'(x) = (6x^2 + 8x - 8)e^{2x^3+4x^2-8x+3}$$

$$0 = (6x^2 + 8x - 8)e^{2x^3+4x^2-8x+3}$$

$$(6x^2 + 8x - 8) = 0$$

$$2(3x - 2)(x + 2) = 0$$

$$x = -2, \frac{2}{3}$$

The critical point $x = 2/3$ is outside the interval $[-4,0]$, so we'll ignore it. Evaluate the function at the endpoints of the interval and at the critical points.

For $x = -4$, $\quad e^{2(-4)^3+4(-4)^2-8(-4)+3} = \frac{1}{e^{29}}$

For $x = -2$, $\quad e^{2(-2)^3+4(-2)^2-8(-2)+3} = e^{19}$

For $x = 0$, $\quad e^{2(0)^3+4(0)^2-8(0)+3} = e^3$

The results show that $g(x)$ has a global minimum at $(-4,1/e^{29})$, a local minimum at $(0,e^3)$, and a global maximum at $(-2,e^{19})$.

■ 6. Find the extrema of $h(x) = x - \cos x$ over the closed interval $[0,\pi]$.

Solution:

Find the critical points of the function.

$$f'(x) = 1 + \sin x$$

$$1 + \sin x = 0$$

$$\sin x = -1$$

$$x = -\frac{\pi}{2}, \frac{3\pi}{2}, \ldots$$

All critical points are outside the interval $[0,\pi]$, so we'll ignore them. Evaluate the function at the endpoints of the interval.

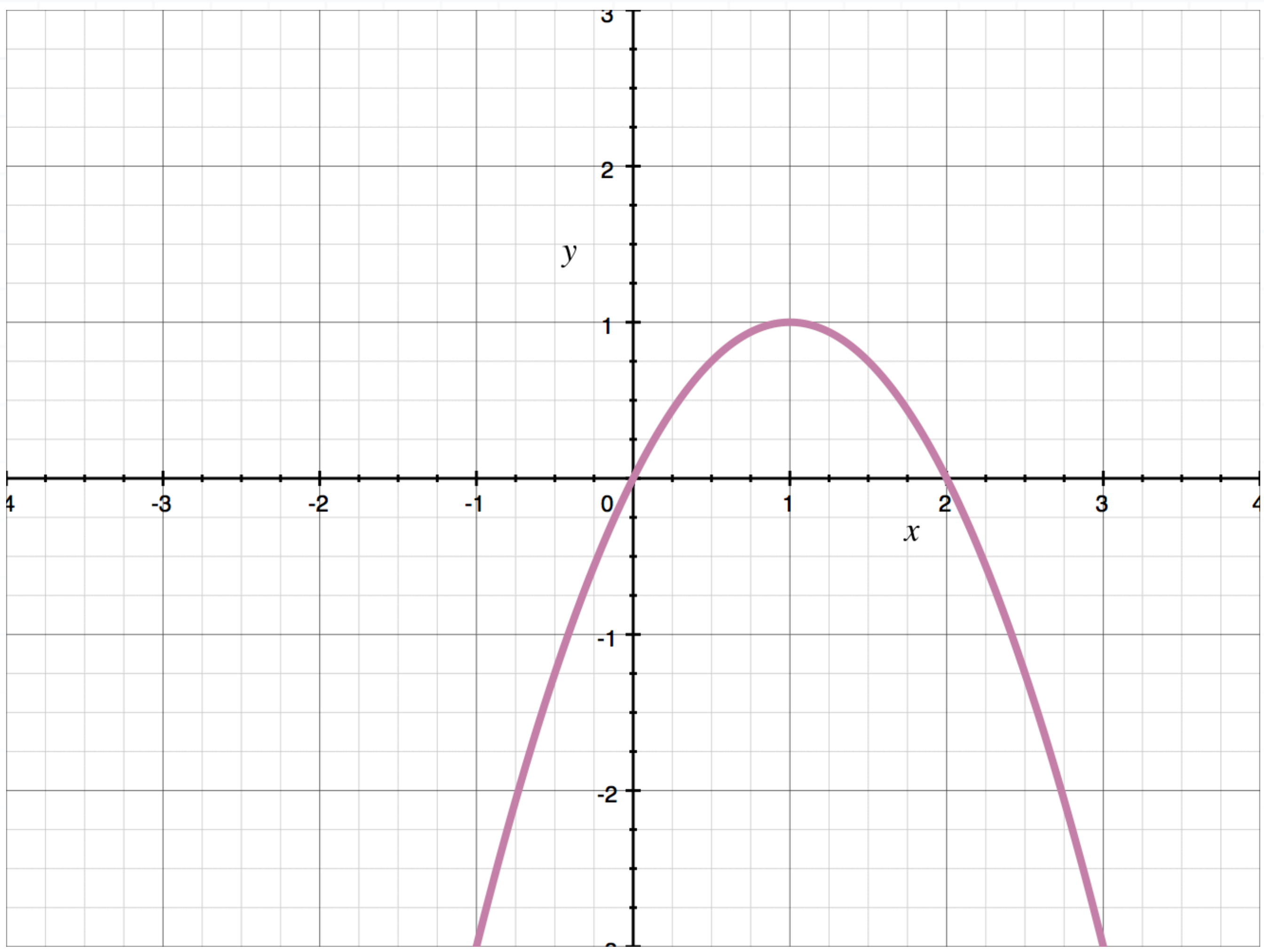
For $x = 0$, $\qquad 0 - \cos 0 = -1$

For $x = \pi$, $\qquad \pi - \cos \pi = \pi + 1$

The results show that $h(x)$ has a global minimum at $(0, -1)$ and a global maximum at $(\pi, \pi + 1)$.

SKETCHING F(X) FROM F'(X)

■ 1. Sketch a possible graph of $f(x)$ given the graph below of $f'(x)$.



Solution:

The graph of $f'(x)$ is below the x-axis on the intervals $(-\infty,0)$ and $(2,\infty)$, which means the function $f(x)$ has a negative slope and is decreasing on these intervals.

Additionally, the graph of $f'(x)$ is above the x-axis on the interval $(0,2)$, which means the function $f(x)$ has a positive slope and is increasing on this interval.

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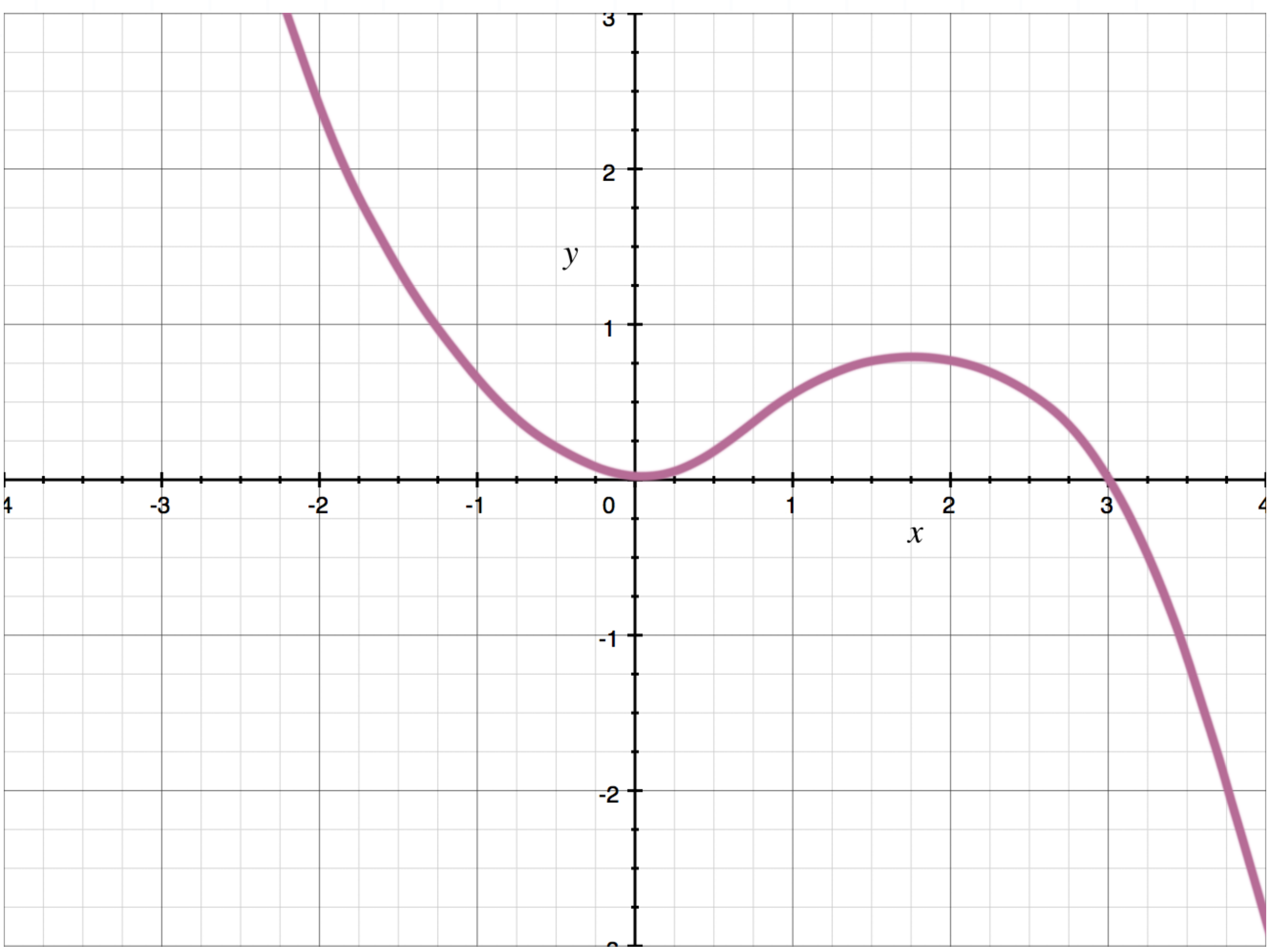
The graph of $f'(x)$ passes through the x-axis and changes sign from negative to positive at $x = 0$, which means that the graph of $f(x)$ has a minimum value at $x = 0$, and the graph of $f'(x)$ passes through the x-axis and changes sign from positive to negative at $x = 2$, which means that the graph of $f(x)$ has a maximum value at $x = 2$.

The graph of $f'(x)$ has a maximum value at $x = 1$, and its slope changes from positive to negative at that point. This means that the graph of $f(x)$ is concave up to the left of $x = 1$, has an inflection point at $x = 1$, and is concave down to the right of $x = 1$.

Putting these facts together, and based on the "assumption" that $f(x)$ contains the point $(0,0)$, this is a possible graph of $f(x)$:



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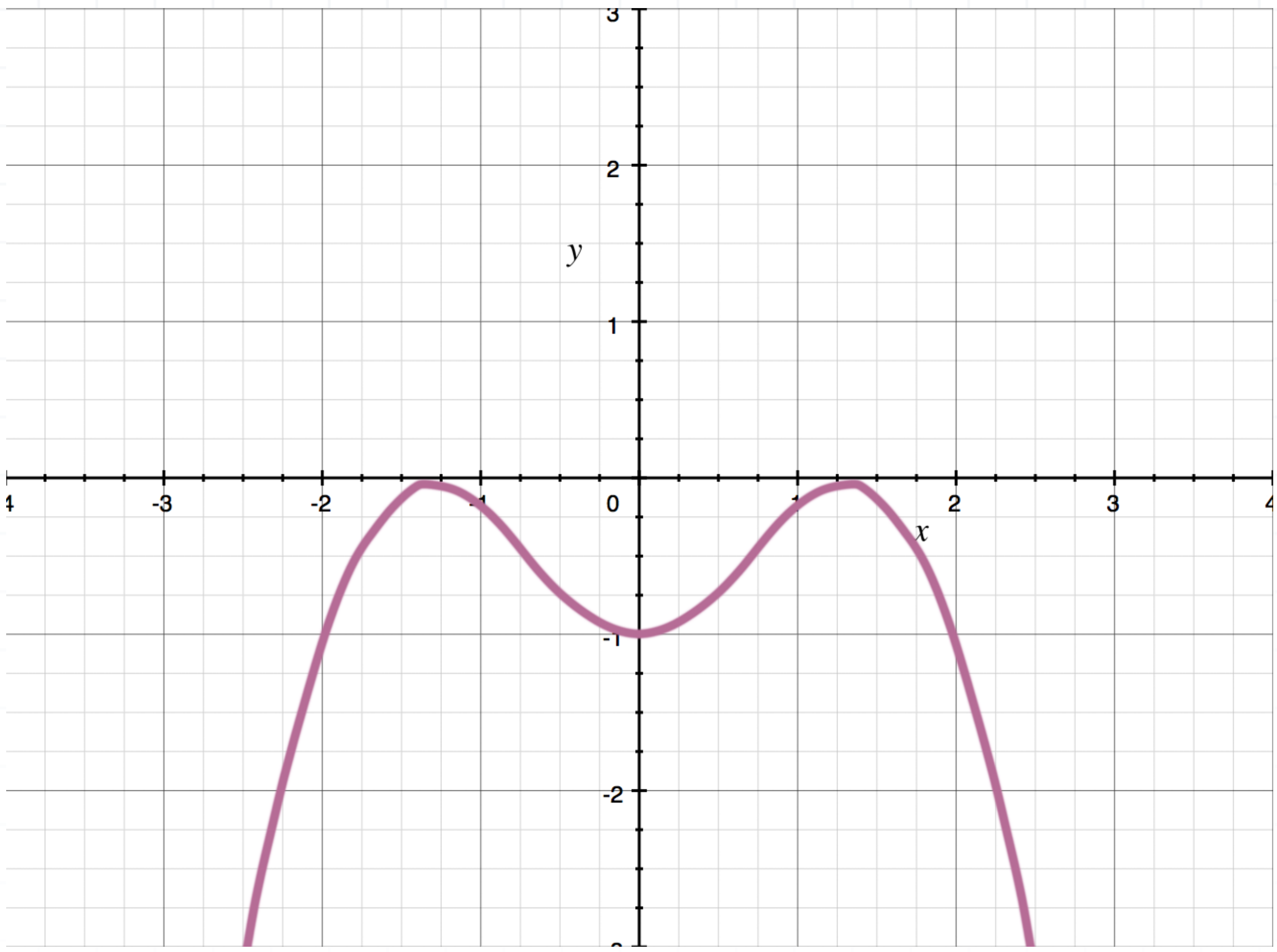
■ 2. Sketch a possible graph of $g'(x)$ given the graph below of $g(x)$.



Solution:

The graph of $g(x)$ has a positive slope on the intervals $(-\infty, -1.5)$ and $(0,1.5)$. Since $g'(x)$ is the derivative of $g(x)$, the graph of $g'(x)$ is above the x-axis on these intervals.

The graph of $g(x)$ has a negative slope on the intervals $(-1.5,0)$ and $(1.5,\infty)$. Since $g'(x)$ is the derivative of $g(x)$, the graph of $g'(x)$ is below the x-axis on these intervals.

The graph of $g(x)$ has a maximum value at $x=-1.5$ and $x=1.5$ and its slope is 0, so the graph of $g'(x)$ passes through the x-axis and changes sign from positive to negative at $x=-1.5$ and $x=1.5$.

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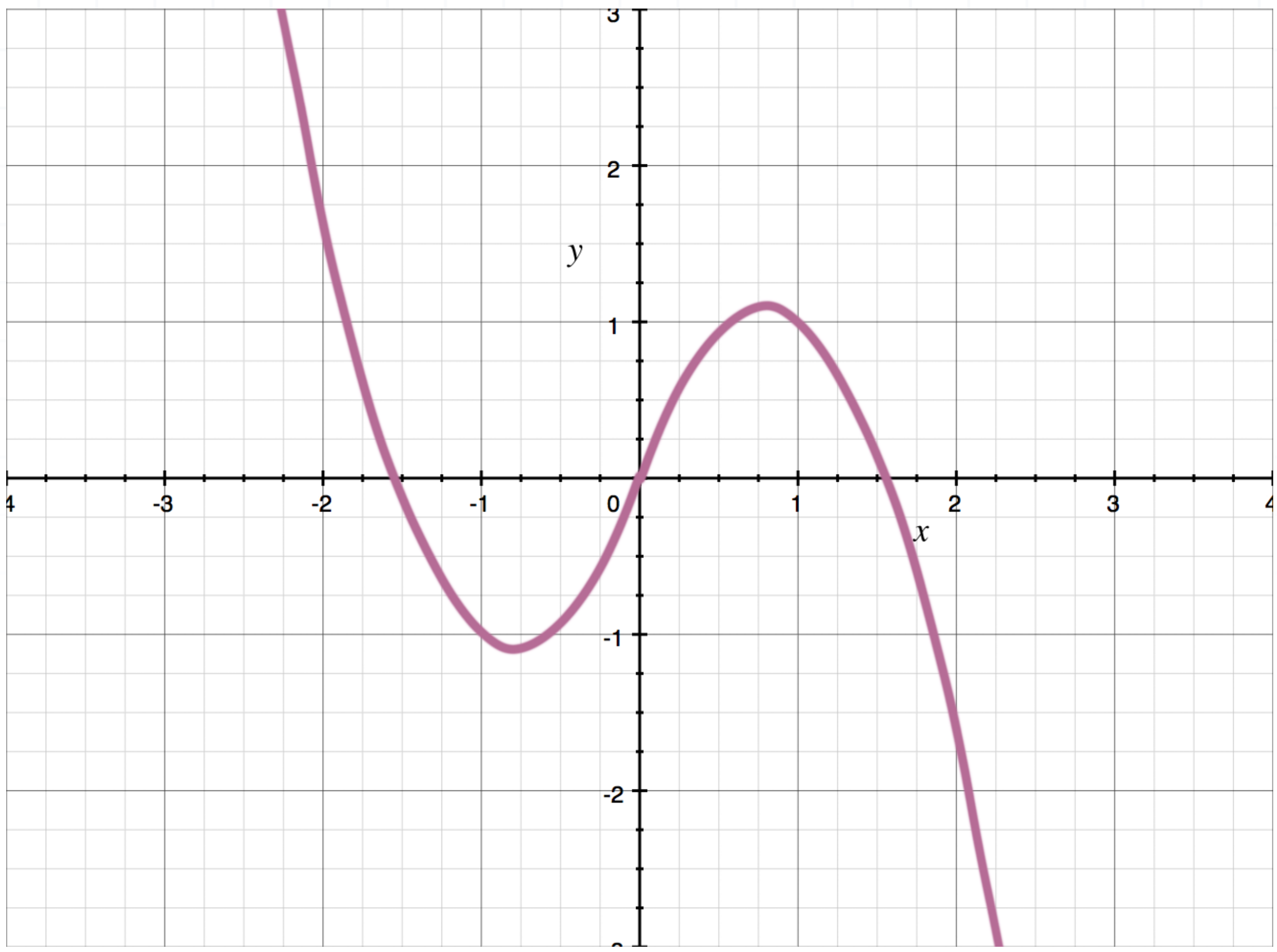
The graph of $g(x)$ has a minimum value at $x = 0$, and its slope changes from negative to positive at that point. This means that the graph of $g'(x)$ passes through the x-axis at $x = 0$, and changes from negative to positive.

It appears that the graph of $g(x)$ has an inflection point at $x = -0.75$ and $x = 0.75$, so the graph of $g'(x)$ has extrema at those points. Putting these facts together, this is a possible graph of $g'(x)$:



■ 3. Sketch a possible graph of $h(x)$ given the graph below of $h'(x)$.

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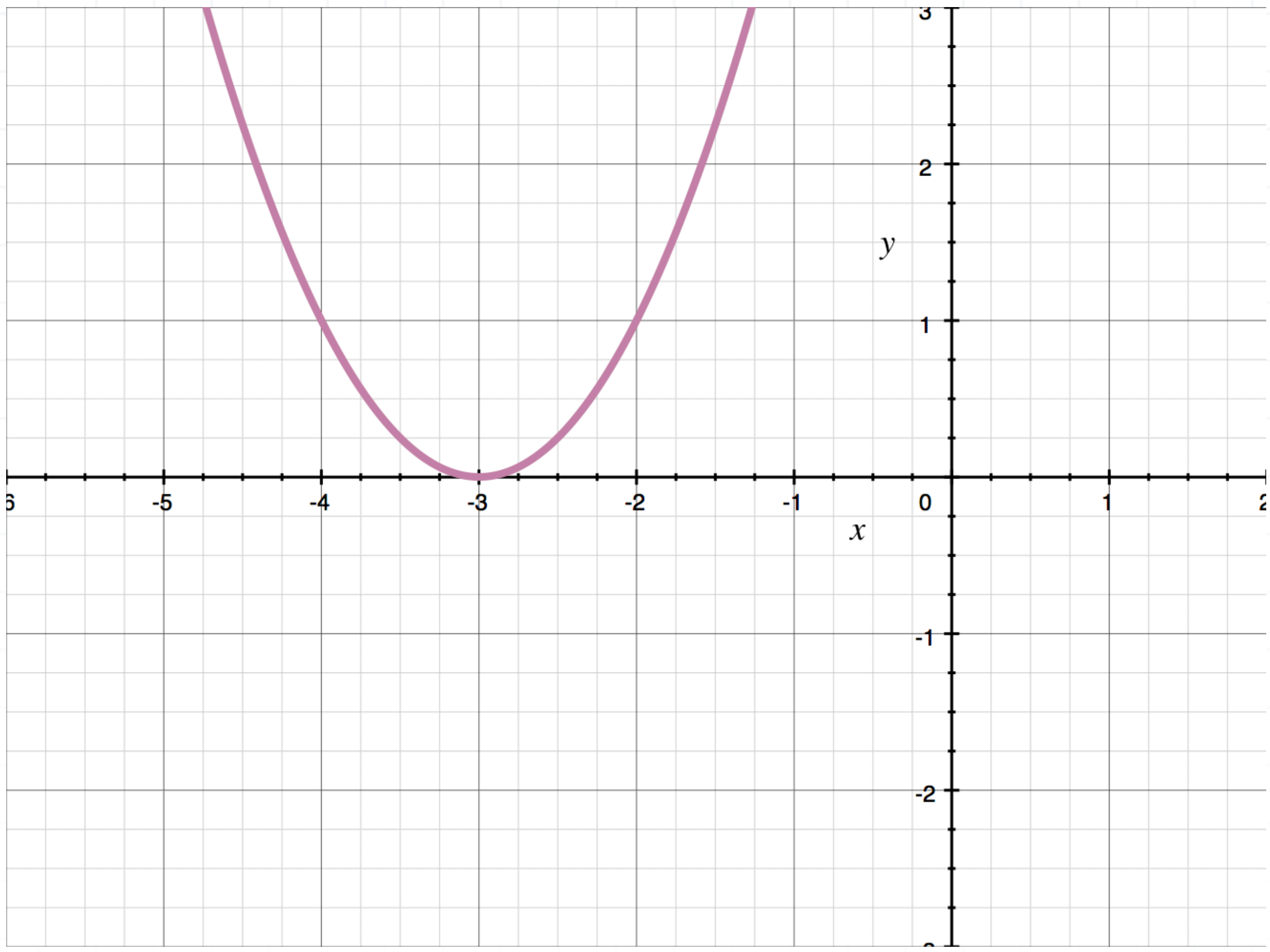

Solution:

The graph of $h'(x)$ is above the x-axis on the intervals $(-\infty, -3)$ and $(-3,\infty)$, which means the function $h(x)$ has positive slopes and is increasing on these intervals. Since we're only excluding the single point $x = -3$, that means the function is essentially increasing everywhere.

The graph of $h'(x)$ is on the x-axis at $x = -3$, which means the function $h(x)$ has a horizontal tangent at $x = -3$ and is increasing on both sides of this point.

The graph of $h'(x)$ has a minimum value at $x = -3$, and its slope changes from positive to negative at that point. This means that the graph of $h(x)$ is concave down to the left of $x = -3$, has an inflection point at $x = -3$, and is

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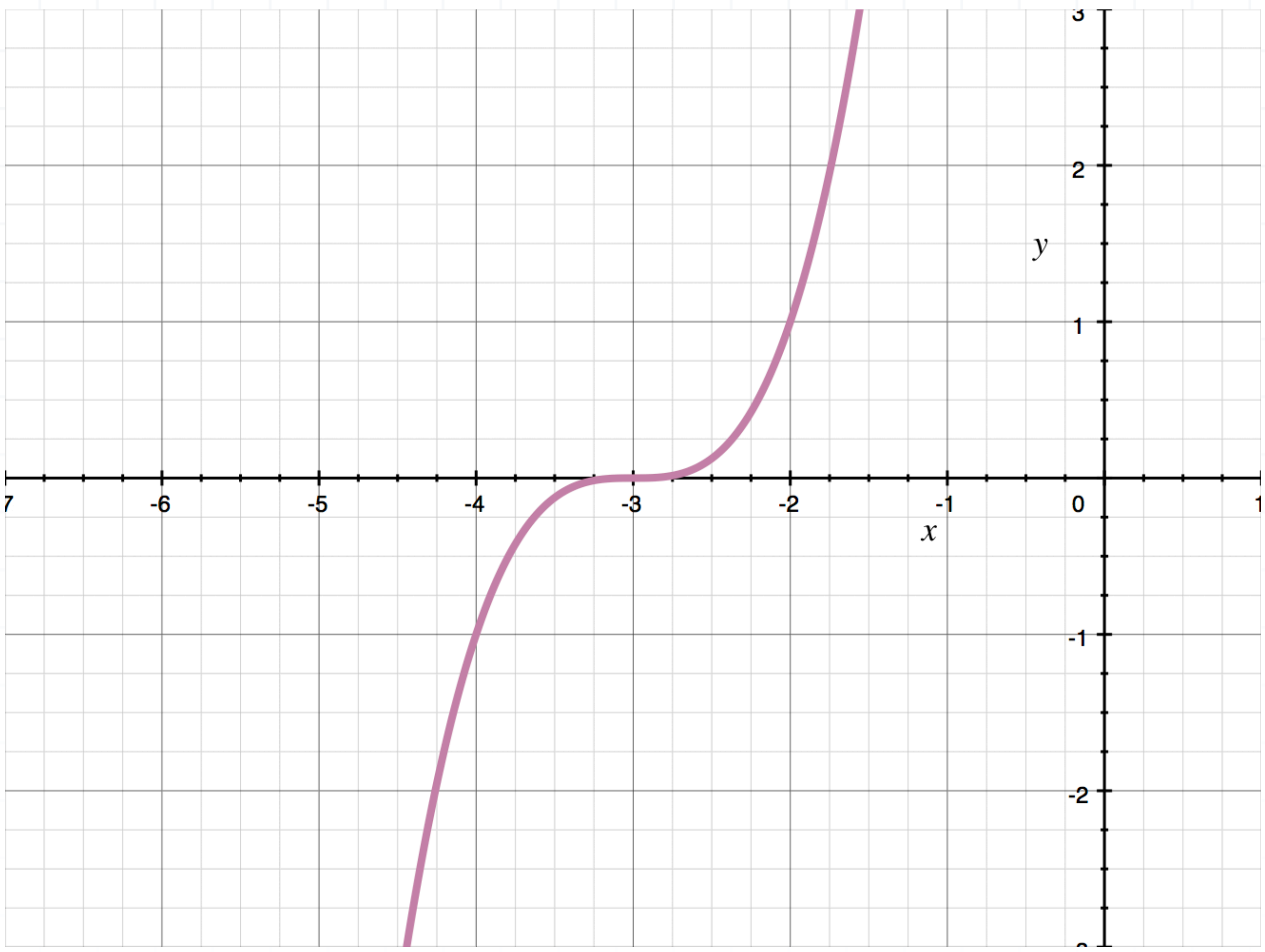
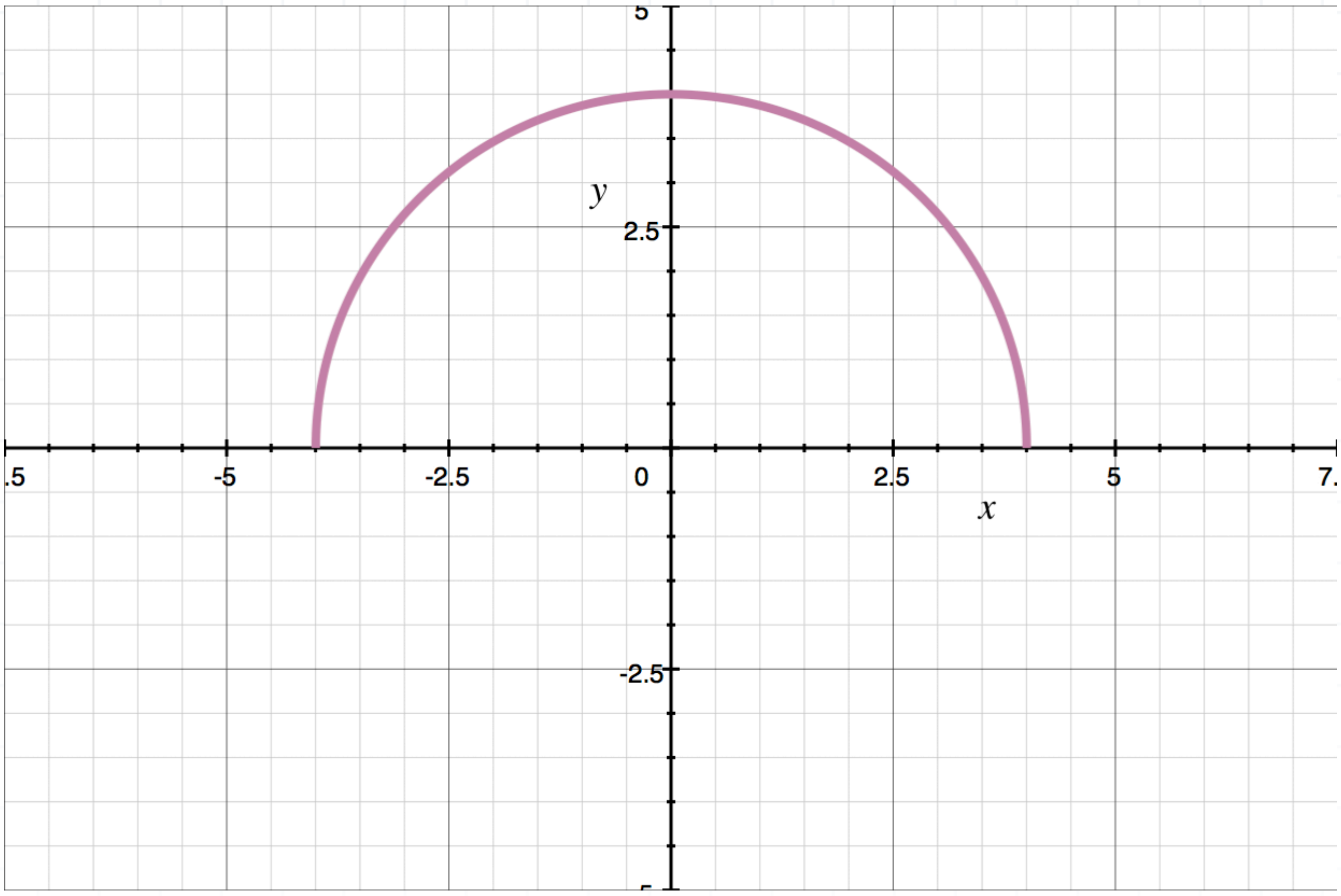
concave up to the right of $x = -3$. Putting these facts together, this is a possible graph of $h(x)$:



■ 4. Sketch a possible graph of $f'(x)$ given the graph below of $f(x)$.



Solution:

The graph of $f(x)$ has a positive slope on the interval $(-4,0)$. Since $f'(x)$ is the derivative of $f(x)$, the graph of $f'(x)$ is above the x-axis on this interval.

The graph of $f(x)$ has a negative slope on the interval $(0,4)$. Since $f'(x)$ is the derivative of $f(x)$, the graph of $f'(x)$ is below the x-axis on this interval.

The graph of $f(x)$ has a maximum value at $x = 0$ and its slope is 0, so the graph of $f'(x)$ passes through the x-axis and changes sign from positive to negative at $x = 0$.

The graph of $f(x)$ has no inflection points so the graph of $f'(x)$ has no extrema in the interval $(-4,4)$. Putting these facts together, this is a possible graph of $f'(x)$:

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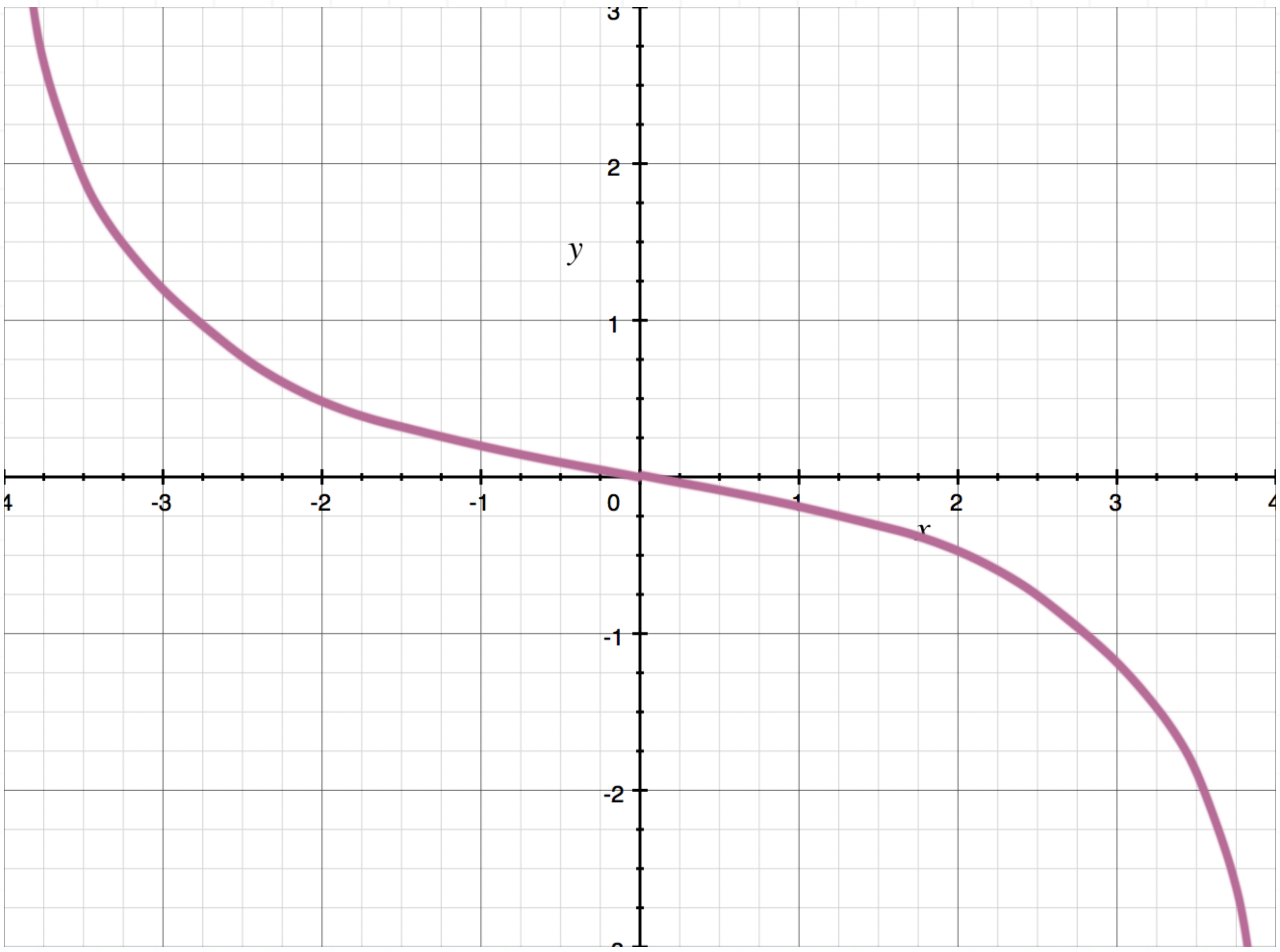

■ 5. Sketch a possible graph of $f(x)$ given the graph below of $f'(x)$.

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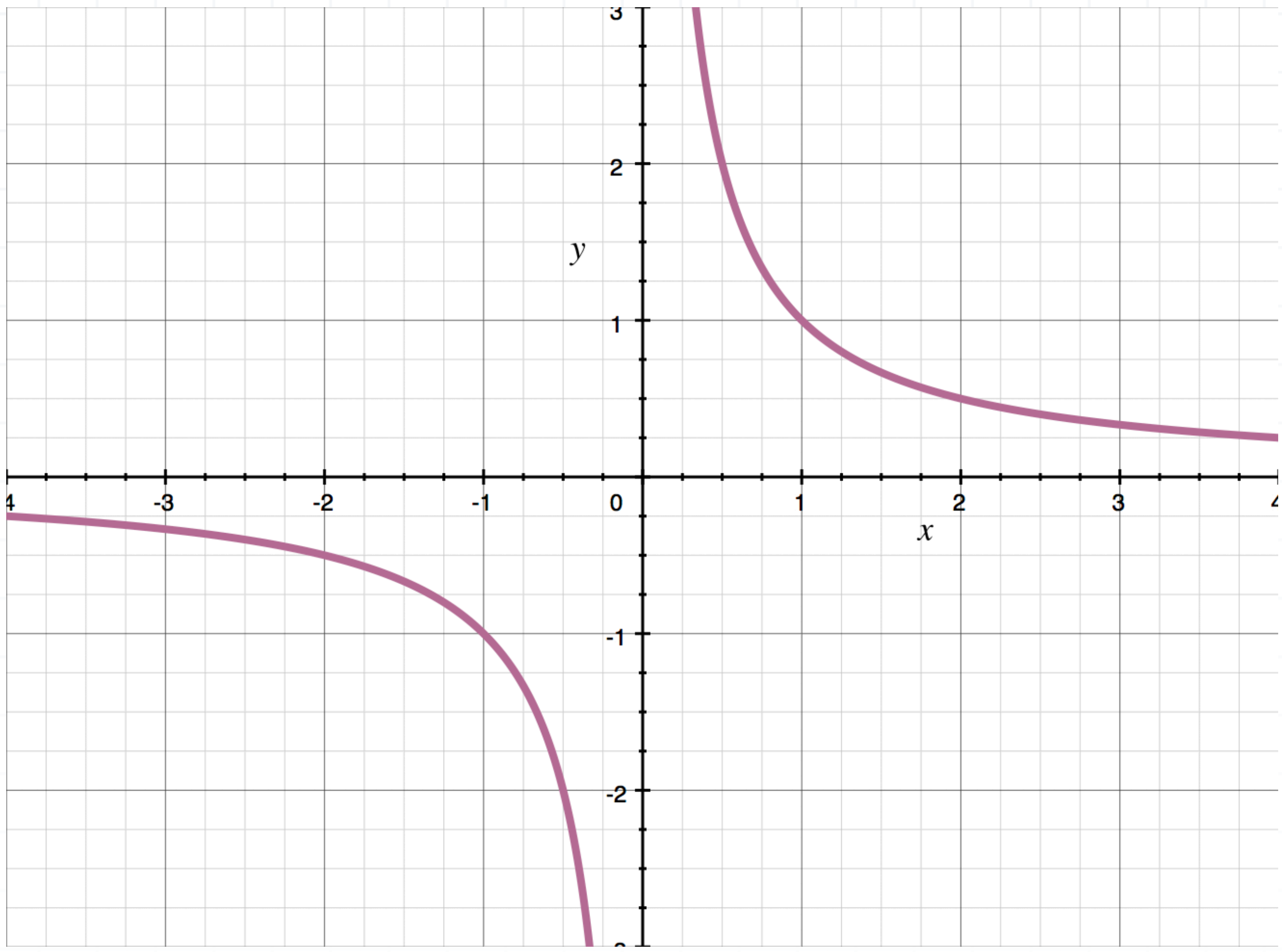

Solution:

The graph of $f'(x)$ is below the x-axis on the interval $(-\infty,0)$, which means the function $f(x)$ has a negative slope and is decreasing on this interval.

The graph of $f'(x)$ is above the x-axis on the interval $(0,\infty)$, which means the function $f(x)$ has a positive slope and is increasing on this interval.

The graph of the $f'(x)$ does not pass through the x-axis, which means that the graph of $f(x)$ does not have any extrema.

The slope of the graph of $f'(x)$ is negative on $(-\infty,0)$ and $(0,\infty)$. This means that the graph of $f(x)$ is concave down to the left and to the right of the y-axis.

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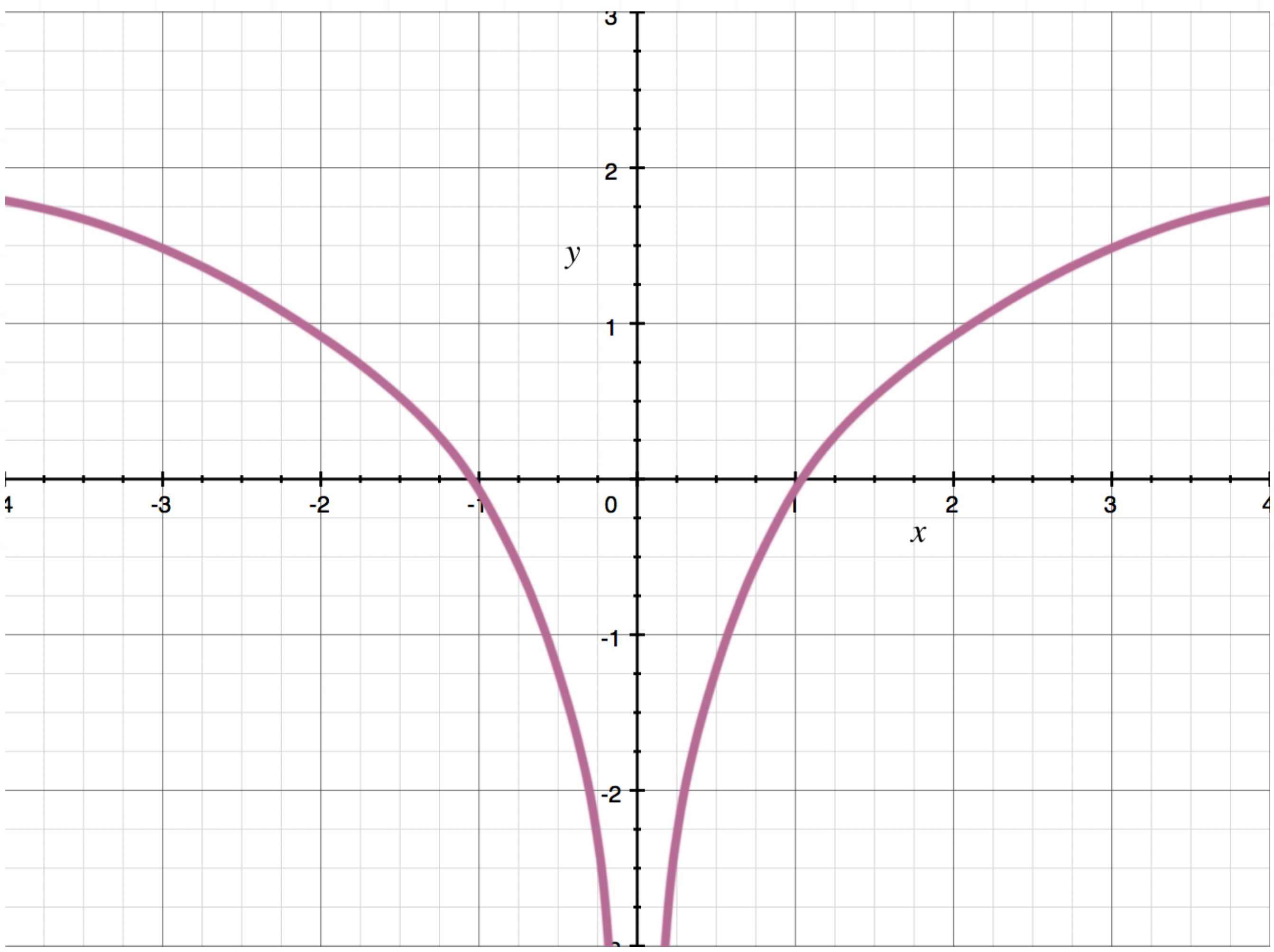
The graph of $f'(x)$ has these limits:

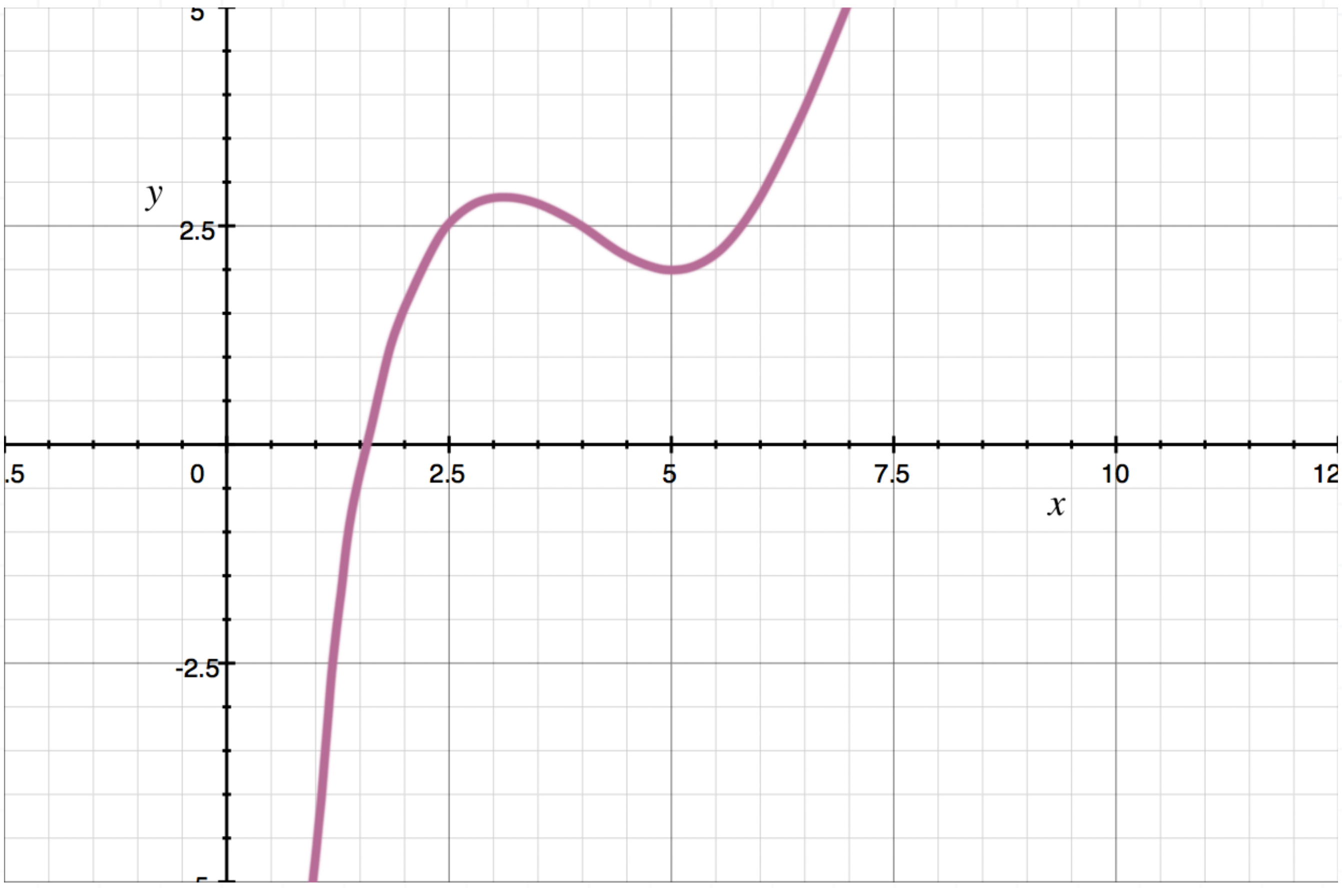
$$\lim_{x\to 0^-} f'(x) = -\infty$$

$$\lim_{x\to 0^+} f'(x) = \infty$$

This means the graph of $f(x)$ has an asymptote on the y-axis. Putting these facts together, this is a possible graph of $f(x)$:



■ 6. Sketch a possible graph of $g'(x)$ and $g''(x)$ given the graph below of $g(x)$.



Solution:

The graph of $g(x)$ has a positive slope on the intervals $(-\infty,3)$ and $(5,\infty)$. Since $g'(x)$ is the derivative of $g(x)$, the graph of $g'(x)$ is above the x-axis on these intervals.

The graph of $g(x)$ has a negative slope on the interval the $(3,5)$. Since $g'(x)$ is the derivative of $g(x)$, the graph of $g'(x)$ is below the x-axis on this interval.

The graph of $g(x)$ has a maximum value at $x=3$ and its slope is 0, so the graph of $g'(x)$ passes through the x-axis and changes sign from positive to negative at $x=3$.

The graph of $g(x)$ has a minimum value at $x=5$, and its slope changes from negative to positive at that point. This means that the graph of $g'(x)$ passes through the x-axis at $x=5$, and changes from negative to positive.

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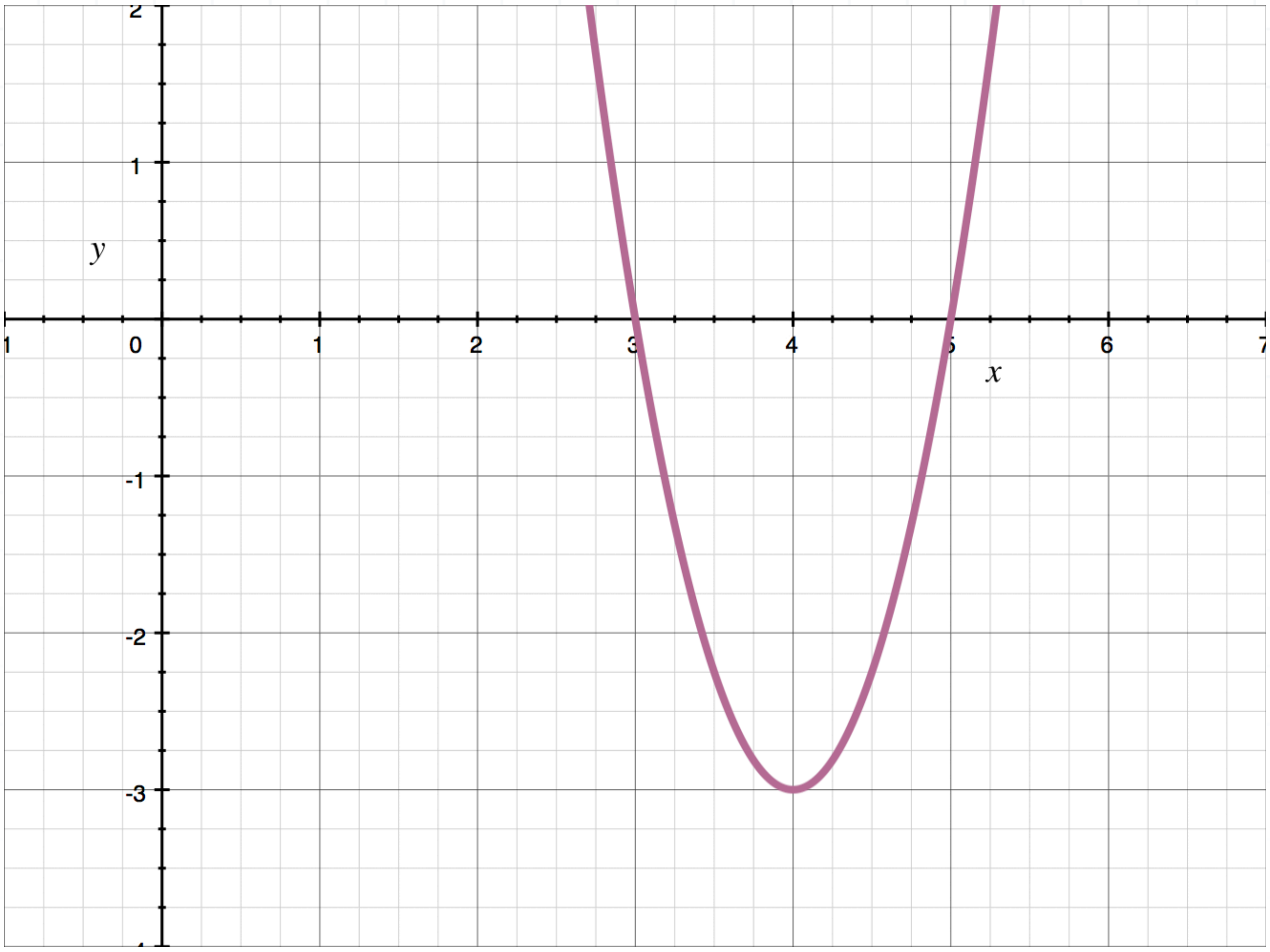
It appears that the graph of $g(x)$ has an inflection point at $x = 4$, so the graph of $g'(x)$ has extrema at $x = 4$. Putting these facts together, this is a possible graph of $g'(x)$:

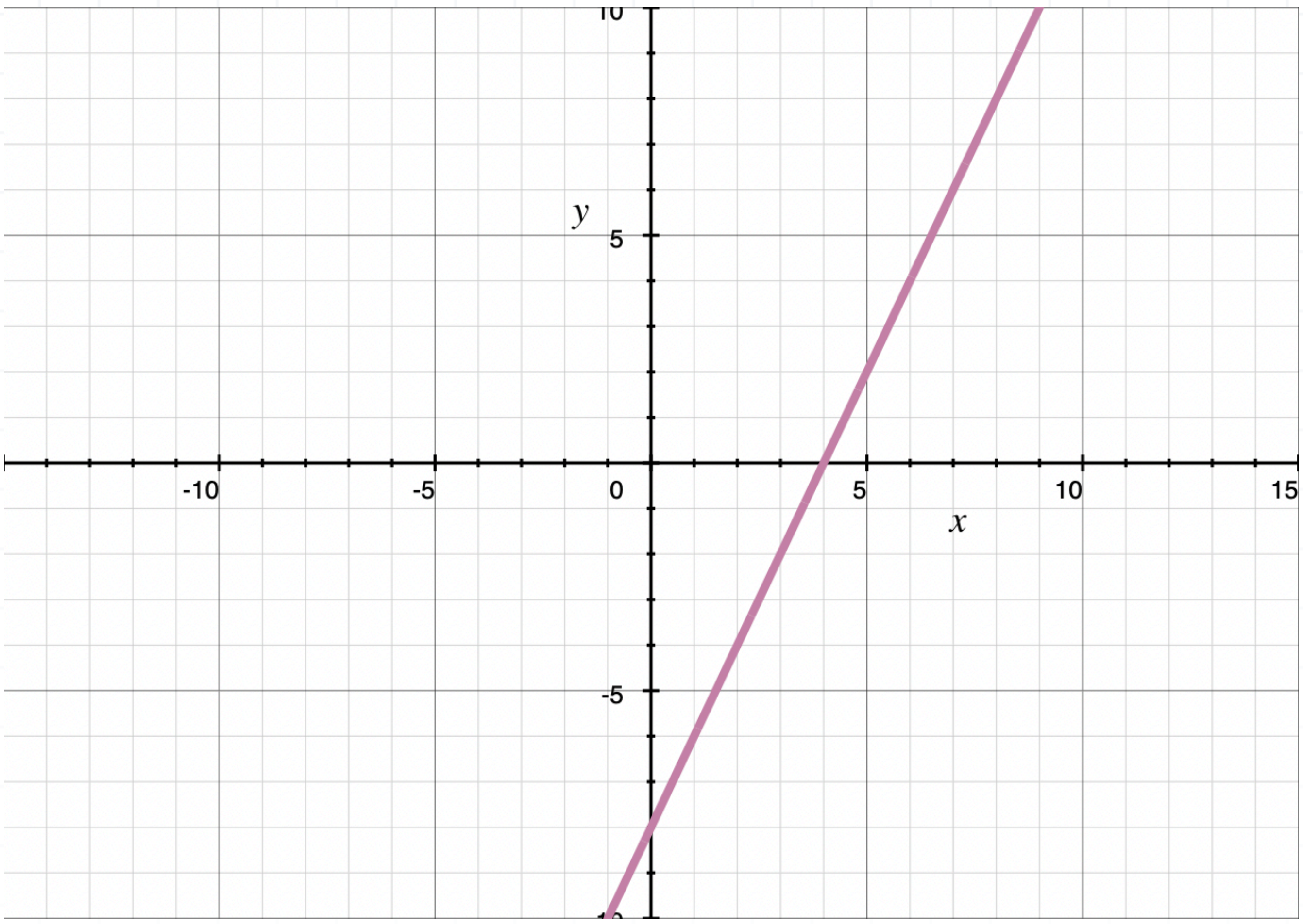


The graph of $g(x)$ has an inflection point at $x = 4$, so the graph of $g''(x)$ has an x-intercept at $x = 4$.

The graph of $g(x)$ is concave down on the interval $(-\infty,4)$, so the graph of $g''(x)$ is above the x-axis on this interval. The graph of $g(x)$ is concave up on the interval $(4,\infty)$, so the graph of $g''(x)$ is below the x-axis on this interval.

The graph of $g'(x)$ is concave up on the interval $(-\infty,\infty)$, so the graph of $g''(x)$ is increasing on this interval.

Putting these facts together, this is a possible graph of $g''(x)$:



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