Trigonometry Notes

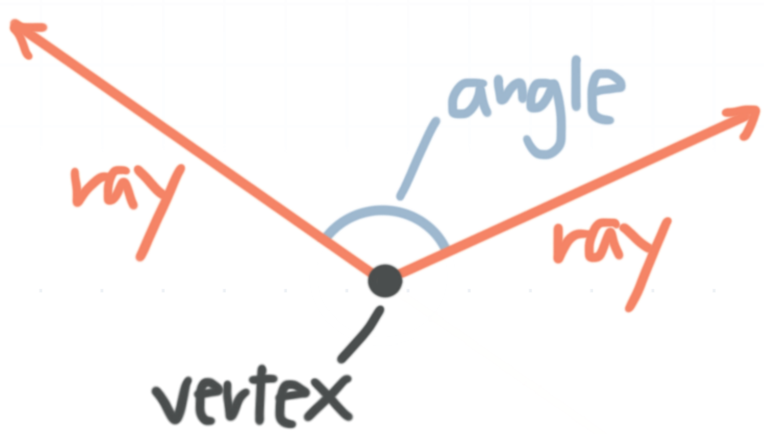
krista king
MATH

Naming angles

We can think about an angle as a wedge, like a piece of pie. Each side of an angle is bounded by a **ray**, which is a line that's infinitely long in only one direction. So a ray has an endpoint on one end, and goes off infinitely on the other end.

Building an angle

We get an **angle** when we put two rays together, with their endpoints at the same spot. That becomes the corner of the angle, which is the **vertex** of the angle.



Notice that every angle has an interior area, but that there's also the area outside the angle. When we cut a pie, we have the area of the piece we cut, but we also still have the area of the rest of the pie.

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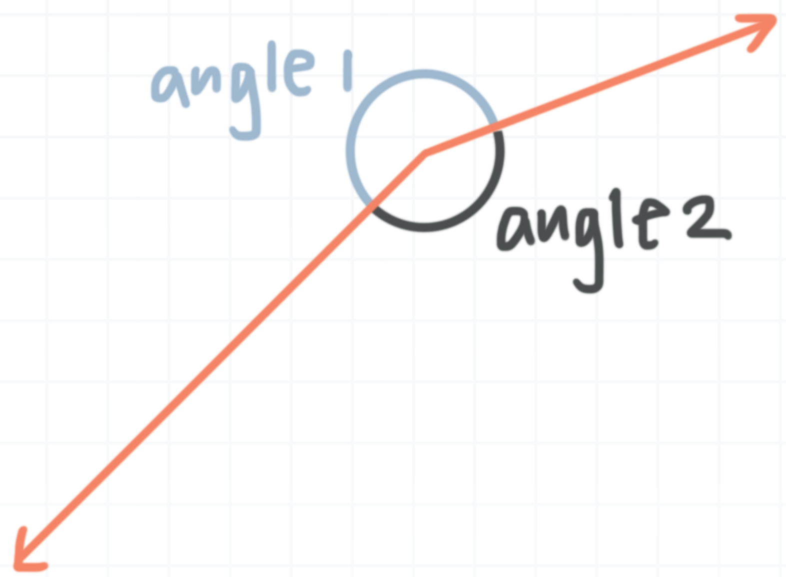

In other words, an angle is really just a section of a complete circle. In the figure above, angle 1 and angle 2 add together to form a full circle.

Types of angles

There are different systems to measure angles. The ones we'll typically use are degrees and radians, and we'll talk about both systems in much more depth later on. For now, we just want to know that a complete circle is made of 360 degrees, which we write as 360°, or of 2π radians, which we write as 2π. Remember that π is a constant; it's equivalent to about $\pi \approx 3.14$.

We usually use the Greek letter θ (theta) for the measure of an angle, but we can also use α, β, γ, or any other variable.

So let's take a moment to think about the 360° of a full circle. A full circle is always $\theta = 360^\circ$, which means a quarter circle is $\theta = 360^\circ/4 = 90^\circ$, and 90° angles are called **right angles**. A half circle is $\theta = 360^\circ/2 = 180^\circ$, and 180° angles are called **straight angles**.

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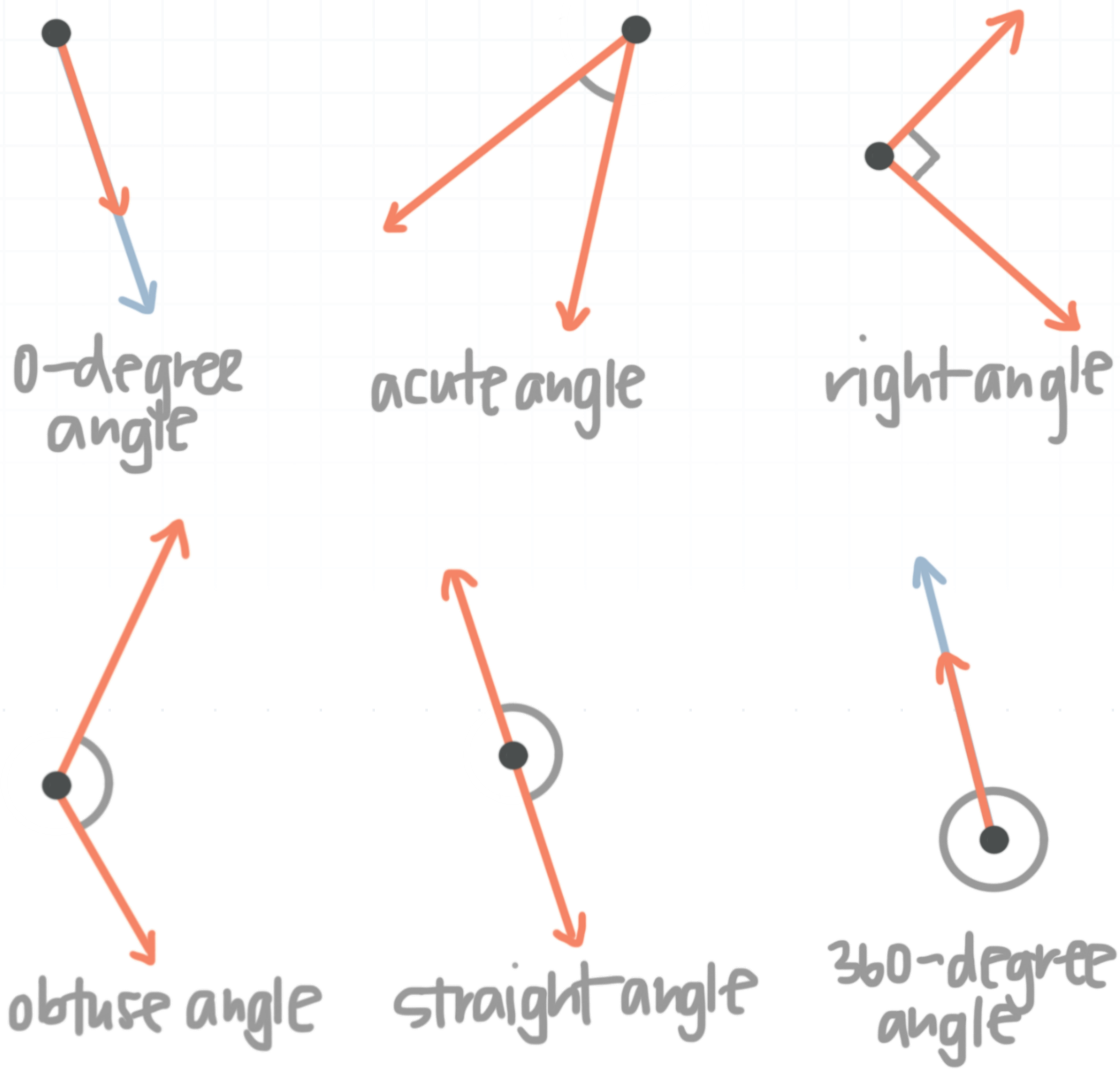
Any angle that's less than a quarter circle, or $0^\circ < \theta < 90^\circ$, is an **acute angle**, and any angle that's greater than a quarter circle but less than a half circle, or $90^\circ < \theta < 180^\circ$, is an **obtuse angle**. Angles which are more than 180° but less than 360° are **reflex angles**.

When the rays of the angle overlap (there's no area in between the rays), then the angle is $\theta = 0^\circ$.



In this figure we've drawn the angles in order from smallest measure to largest measure, and we can also summarize them that way in a table:

Angle in degree	**Angle name**
$\theta = 0^\circ$	0° or zero angle
$0^\circ < \theta < 90^\circ$	Acute angle
$\theta = 90^\circ$	Right angle
$90^\circ < \theta < 180^\circ$	Obtuse angle
$\theta = 180^\circ$	Straight angle
$\theta = 360^\circ$	360° or complete angle

We can do the same thing in radians. As we go further in math, we'll actually use radians more often than degrees. It's really common to see radian angles given in terms of π, like $\pi/3$, $3\pi/2$, or 1.6π. Here's the same chart for radian angles as the one we made for degree angles.

Angle in radians	**Angle name**
$\theta = 0$	0 or zero angle
$0 < \theta < \pi/2$	Acute angle
$\theta = \pi/2$	Right angle
$\pi/2 < \theta < \pi$	Obtuse angle
$\theta = \pi$	Straight angle
$\theta = 2\pi$	2π or complete angle

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It’s a little easier to see the relationship in degree angles, because we can see that 180 is half of 360, and that 90 is one quarter of 360.

But the same math applies to radian angles. Half of 2π is

$$\frac{1}{2}(2\pi) = \frac{2\pi}{2} = \pi$$

so $\theta = \pi$ is the straight angle. And a quarter of 2π is

$$\frac{1}{4}(2\pi) = \frac{2\pi}{4} = \frac{\pi}{2}$$

so $\theta = \pi/2$ is a right angle.

Let’s look at an example where we classify a few angles.

Example

Name each kind of angle.

1. $\theta = 37^\circ$

2. $\theta = 3\pi/4$

3. $\theta = 90^\circ$

4. $\theta = \pi$

Because the angle $\theta = 37^\circ$ is between 0° and 90°, it’s an acute angle measured in degrees.

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Because the angle $\theta = 3\pi/4$ is between $\pi/2$ and π, it's an obtuse angle measured in radians.

The angle $\theta = 90^\circ$ is a right angle measured in degrees.

And the angle $\theta = \pi$ is a straight angle measured in radians.

Radian angles aren't always defined in terms of π. For instance, we might encounter an angle of 4.0 radians. When this is the case, we can always convert the angle into one in terms of π.

To convert 4.0, we'll divide 4.0 by $\pi \approx 3.14$, which gives us about 1.27, so $4.0 \approx 1.27\pi$. Therefore, 4.0 radians and 1.27π are equivalent angles.

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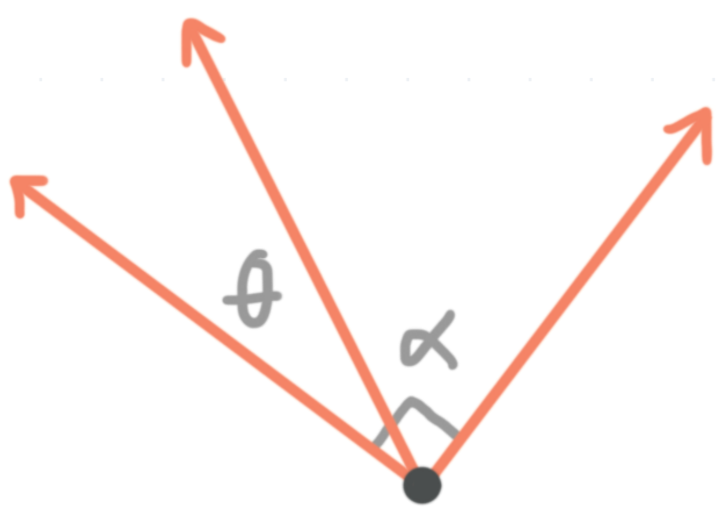
Complementary and supplementary angles

In the last lesson we looked at different types of angles, including right angles, which have measure 90° or $\pi/2$, and straight angles, which have measure 180° or π.

In this lesson, we want to define complementary angle pairs, which have a specific relationship to right angles, and supplementary angle pairs, which have a specific relationship to straight angles.

Complementary angles

Complementary angles are two angles that sum to 90° or $\pi/2$. In other words, if we line up their vertices and match up one ray from each angle, together they form a right angle.



The angles θ and α are complementary because they form a right angle. If we add up their measures, we'll get a total angle of $\theta + \alpha = 90^\circ$ or $\theta + \alpha = \pi/2$.

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Let's do an example where we find an angle that's complementary to another angle we've been given. If we're given an angle in degrees, we want to find its complement in degrees, and if we're given an angle in radians, we want to find its complement in radians.

Example

Find the angle θ that's complementary to 37°.

The angle that's complementary to 37° is whatever angle we have to add to 37° in order to get 90°. Therefore, we can set up an equation where 37° and θ sum to 90°.

$$37^\circ + \theta = 90^\circ$$

$$\theta = 90^\circ - 37^\circ$$

$$\theta = 53^\circ$$

So 37° and 53° degrees are complementary angles because they sum to 90° and form a right angle.

We can find complementary angles in radians, too. For example, the angle that's complementary to $\pi/6$ is

$$\frac{\pi}{6} + \theta = \frac{\pi}{2}$$

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$$\theta = \frac{\pi}{2} - \frac{\pi}{6}$$

Since 6 is the least common multiple of 2 and 6, we'll multiply the first fraction by $3/3$ to create common denominators.

$$\theta = \frac{\pi}{2}\left(\frac{3}{3}\right) - \frac{\pi}{6}$$

$$\theta = \frac{3\pi}{6} - \frac{\pi}{6}$$

$$\theta = \frac{2\pi}{6}$$

$$\theta = \frac{\pi}{3}$$

So $\pi/6$ and $\pi/3$ are complementary angles because they sum to $\pi/2$ and form a right angle.

Supplementary angles

While complementary angles sum to 90° or $\pi/2$ and form a right angle, **supplementary angles** are two angles that sum to 180° or π and form a straight angle.

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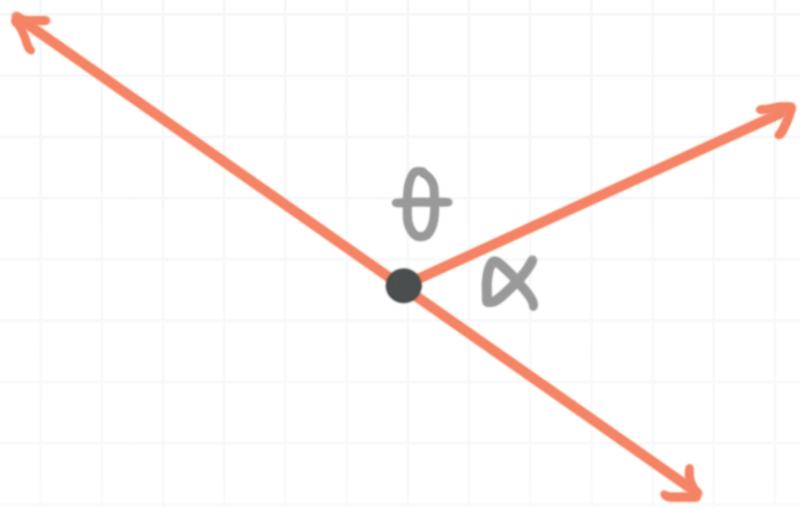

The angles θ and α are supplementary because they form a straight angle. If we add up their measures, we'll get a total angle of $\theta + \alpha = 180^\circ$ or $\theta + \alpha = \pi$.

Let's do an example where we find an angle that's supplementary to another angle we've been given. If we're given an angle in degrees, we want to find its supplement in degrees, and if we're given an angle in radians, we want to find its supplement in radians.

Example

Find the angle θ that's supplementary to $\pi/4$.

The angle that's supplementary to $\pi/4$ is whatever angle we have to add to $\pi/4$ in order to get π. Therefore,

$$\frac{\pi}{4} + \theta = \pi$$

$$\theta = \pi - \frac{\pi}{4}$$

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Find a common denominator.

$$\theta = \pi\left(\frac{4}{4}\right) - \frac{\pi}{4}$$

$$\theta = \frac{4\pi}{4} - \frac{\pi}{4}$$

$$\theta = \frac{3\pi}{4}$$

So $\pi/4$ and $3\pi/4$ are supplementary angles because they sum to π and form a straight angle.

We can find supplementary angles in degrees, too. For example, the angle that's supplementary to 48° is

$$48^\circ + \theta = 180^\circ$$

$$\theta = 180^\circ - 48^\circ$$

$$\theta = 132^\circ$$

So 48° and 132° are supplementary angles because they sum to 180° and form a straight angle.

Let's do one slightly more complicated example.

Example

Find the angle θ that's twice the supplement to $\pi/3$.

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If two angles are supplementary, they sum to 180° or π radians. The angle that's supplementary to $\pi/3$, which we'll call α, is therefore

$$\frac{\pi}{3} + \alpha = \pi$$

$$\alpha = \pi - \frac{\pi}{3}$$

Find a common denominator.

$$\alpha = \pi\left(\frac{3}{3}\right) - \frac{\pi}{3}$$

$$\alpha = \frac{3\pi}{3} - \frac{\pi}{3}$$

$$\alpha = \frac{2\pi}{3}$$

We were asked to find the angle that's twice as big as α, so we'll multiply through this equation by 2 in order to find 2α

$$2\alpha = 2\left(\frac{2\pi}{3}\right)$$

$$2\alpha = \frac{4\pi}{3}$$

The angle $4\pi/3$ is twice the supplement to $\pi/3$.

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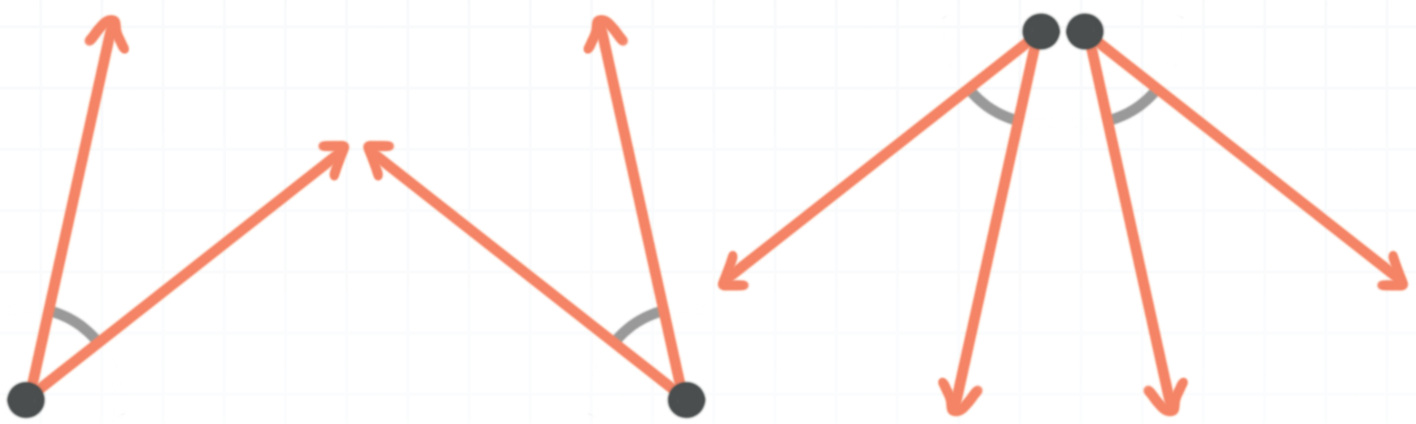
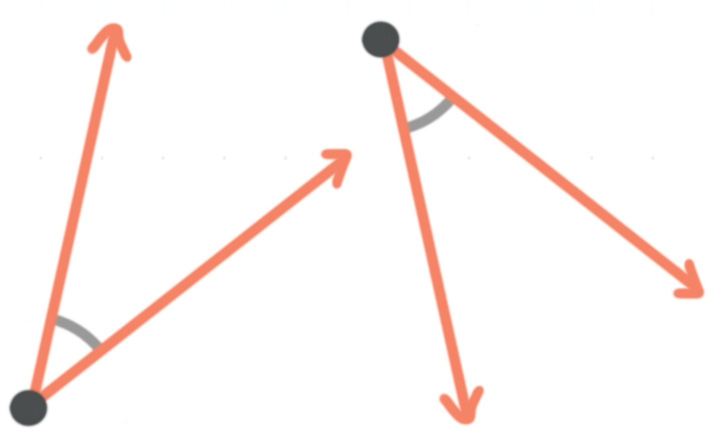
Positive and negative angles

Angles don't necessarily have to have the same orientation in order to have the same measure. For instance, even though they're positioned different ways, each of these angles have the same measure:



If we sketch angles without any kind of standard orientation, it can be difficult for us to visually compare the angles. After all, how are we supposed to know that these two angles



are really the same? One of them might be a few degrees narrower or wider than the other, but it's really hard for us to tell, since the angles are oriented differently.

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To solve this problem, we normally prefer to sketch angles in **standard position**, which means that we align the angle's initial side with the positive direction of the x-axis, placing its vertex at the origin.

Positive and negative angles

Previously, we hadn't distinguished between the two sides of the angle. We only looked at the degree or radian measure between the two rays. But if we define one side as the **initial side**, the side where the angle begins (the ray on the positive direction of the x-axis), and the other as the **terminal side**, the side where the angle ends, then we can distinguish between positive and negative angles.

The angle is measured by the amount of rotation from the initial side to the terminal side. Starting from the initial side, we have a **positive angle** when we have to rotate counterclockwise to get to the terminal side. We have a **negative angle** when we have to rotate clockwise to get to the terminal side.

If we remember the four quadrants of the Cartesian coordinate system,

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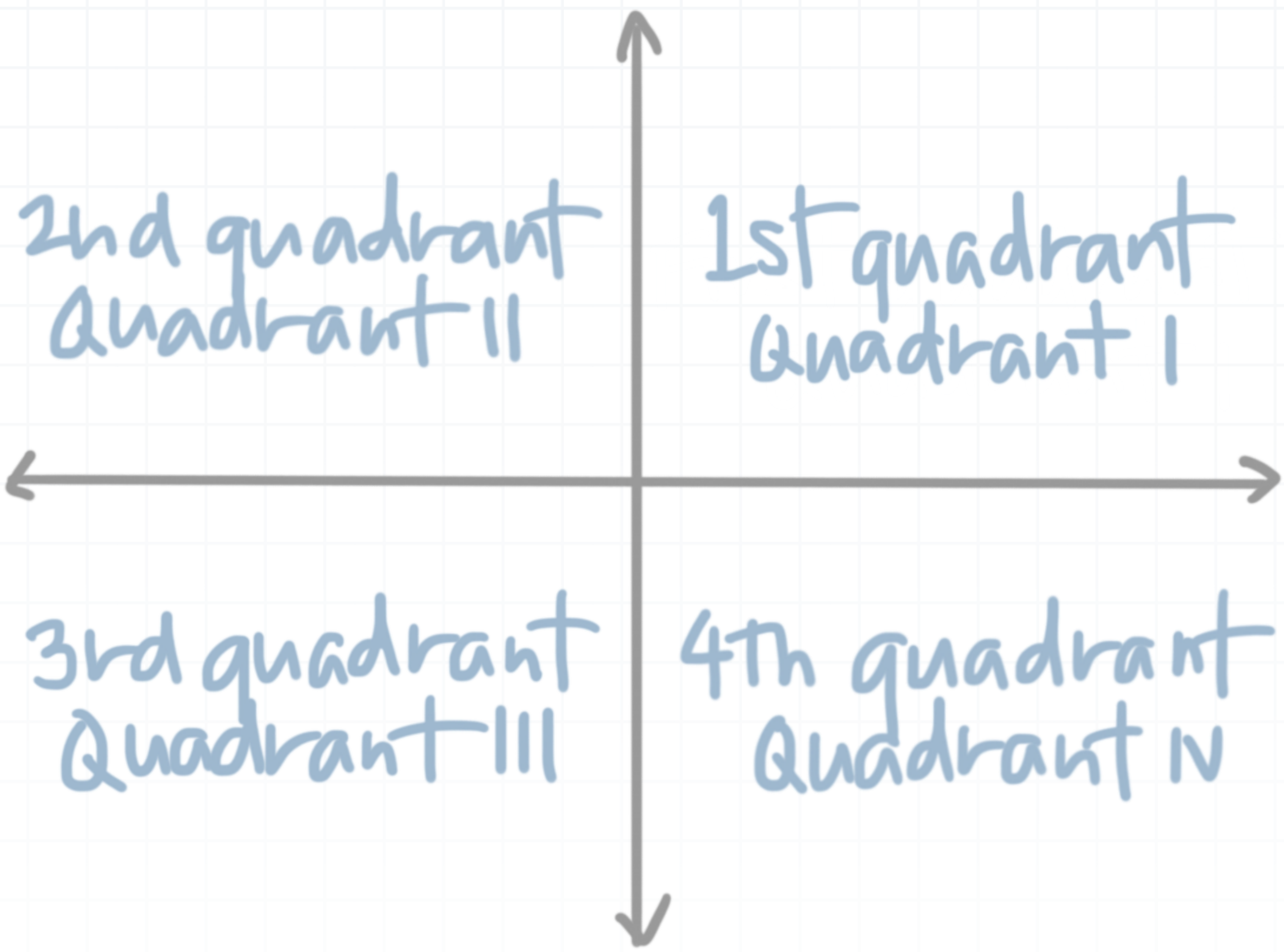
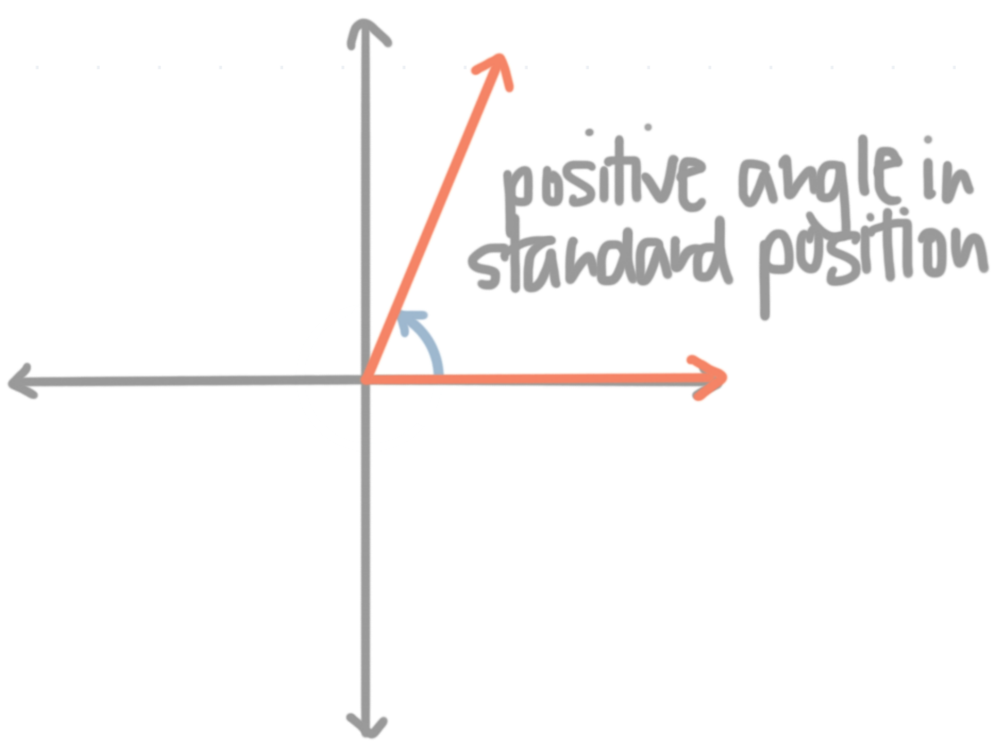


then we can say that a positive angle in standard position has its initial side on the positive direction of the x-axis, and opens up toward the first quadrant,



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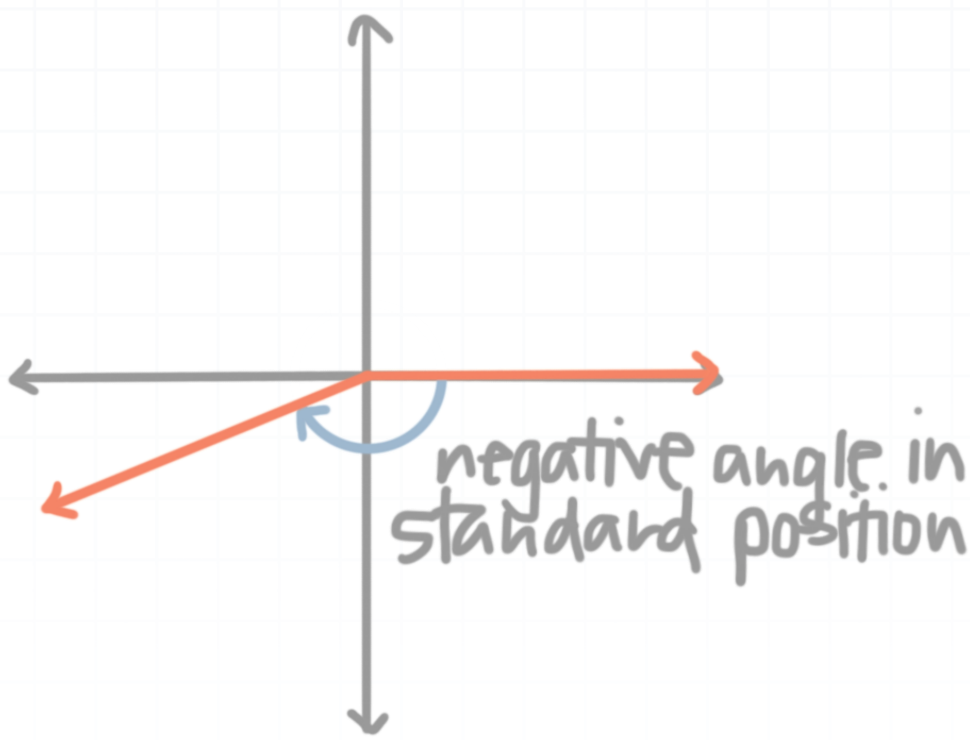
because this kind of angle is a counterclockwise rotation from the positive direction of the x-axis.

On the other hand, a negative angle in standard position has its initial side on the positive direction of the x-axis, and opens up toward the fourth quadrant,



because this kind of angle is a clockwise rotation from the positive direction of the x-axis.

Let's look at how to sketch a positive angle in standard position.

Example

Sketch a 110° angle in standard position.

Since the angle is larger than 90° but smaller than 180°, it's obtuse. We put the initial side along the positive direction of the x-axis, and since the angle

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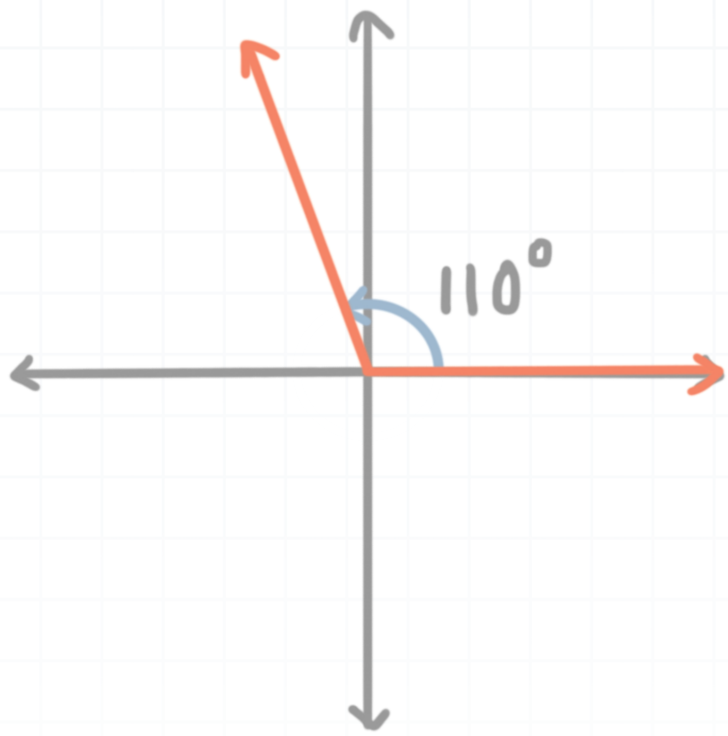
is positive, open up counterclockwise, into the first quadrant, then past the positive direction of the y-axis (which is at 90°) until we get to 110°.



Let's do an example with a negative angle in radians.

Example

Sketch $-\pi/3$ in standard position.

Since $\pi/3$ is less than $\pi/2$, it's an acute angle. We'll put the initial side along the positive direction of the x-axis, and since the angle is negative, open up clockwise, into the fourth quadrant, until we get to $-\pi/3$.

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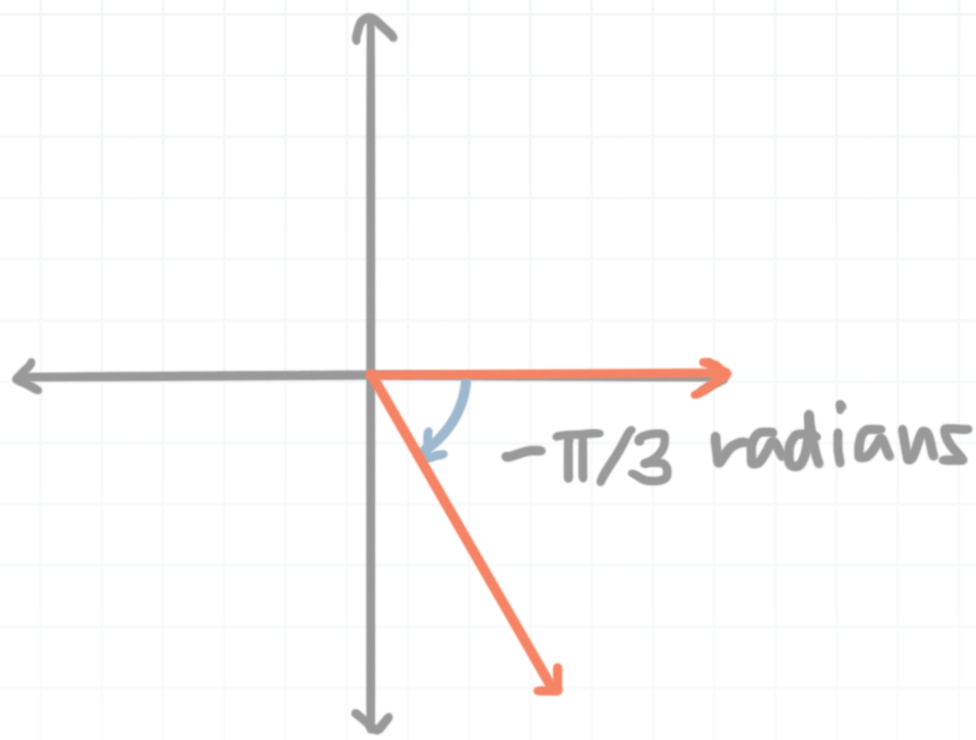

Up to now we've been talking about angles that are less than one full rotation around a circle. A full circle is 360° or 2π radians, and we very often handle angles that are smaller than one full rotation around a circle.

But we can also have angles that are greater than one full rotation around a circle. For example, the angle 600° is more than a full rotation, because $600^\circ > 360^\circ$. Because $600^\circ - 360^\circ = 240^\circ$, the angle 600° just means that we're rotating a full 360°, but then continuing on another 240°.

Let's do an example so that we can see what it looks like to sketch one of these larger angles.

Example

Sketch 420° in standard position.

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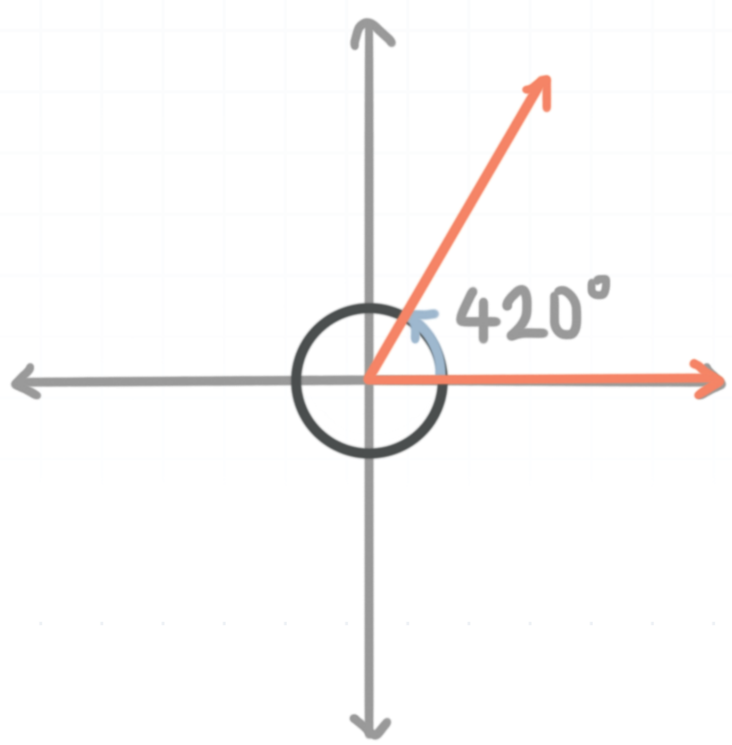
Since $360^\circ < 420^\circ$, the angle 420° is more than one full rotation. We'll find out how much more by finding the difference between the angles.

$$420^\circ - 360^\circ = 60^\circ$$

So to sketch the angle, we'll put the initial side along the positive direction of the x-axis. Then we'll rotate counterclockwise, toward the first quadrant, and rotate one full rotation, all the way around the circle, but then an additional 60°. Because 60° would normally land us in the first quadrant, we'll land in the first quadrant for the 420° angle as well.



The angle looks like a normal 60°, but in the figure, the dark gray arc shows the first 360°, and the blue arc shows the extra 60° of the rotation.

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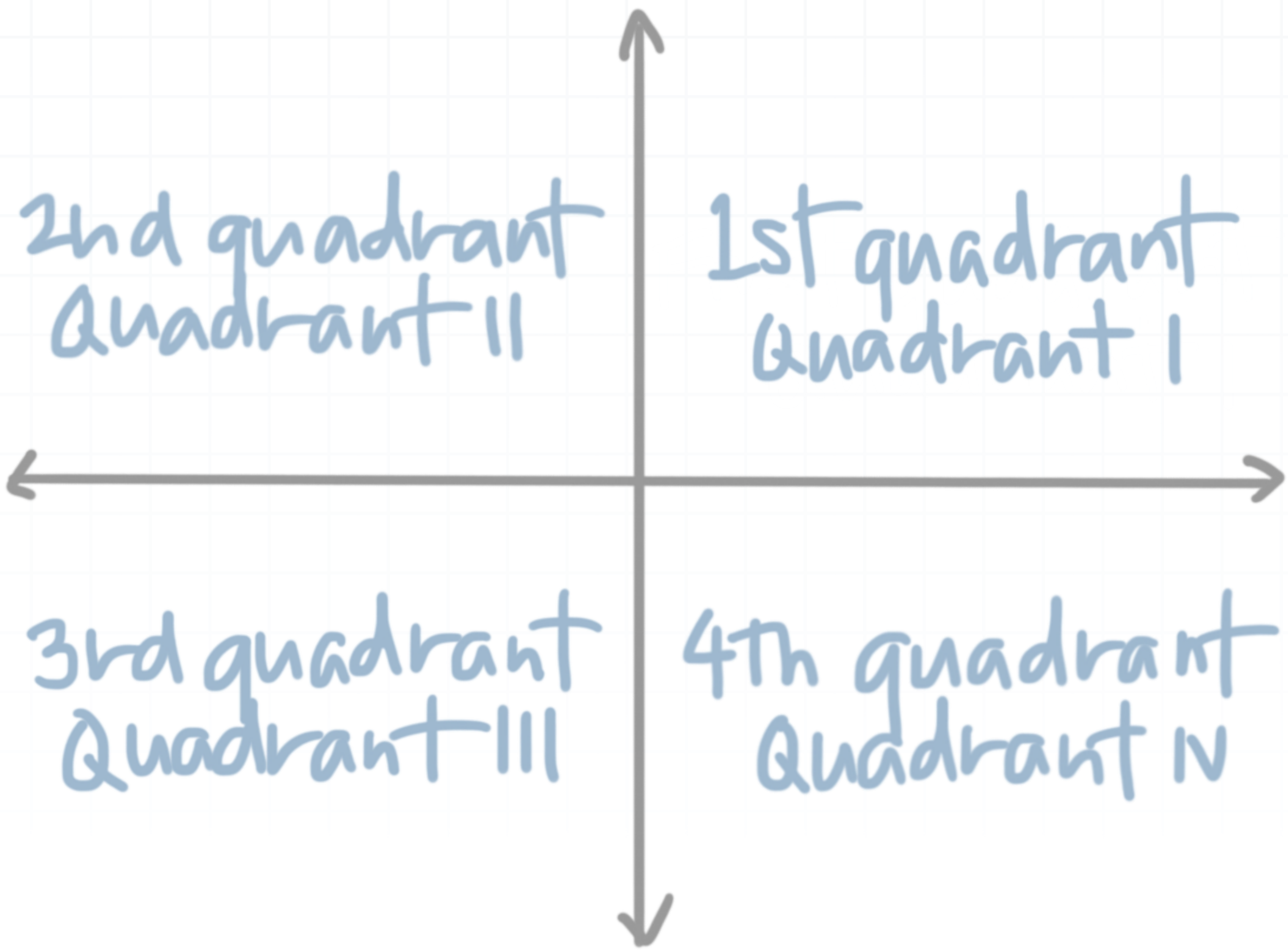
Quadrant of the angle

In the last lesson, we briefly touched on the four quadrants of the Cartesian coordinate system.



In this lesson, we'll go in depth a little more with each of the four quadrants and the axes that divide them.

The quadrant in which the angle lies

When you hear that an angle is "in the third quadrant" that means that when you sketch it in standard position, the terminal side is somewhere in the third quadrant. Or when an angle "lies in the fourth quadrant," that

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means its terminal side falls somewhere in the fourth quadrant when it's sketched in standard position.

When the terminal side of an angle falls exactly on one of the axes, it's called a **quadrantal angle**, and it's technically not in any quadrant since the axes aren't in a quadrant.

Remember that an angle in standard position is always positioned with its initial side along the positive direction of the x-axis, so think about angles as "starting" along the positive direction of the x-axis. That means any zero-angle will also have its terminal side on the positive direction of the x-axis.

From there, we rotate counterclockwise, with positive rotation, toward the positive y-axis, then the negative x-axis, then the negative y-axis, and finally back around to the positive x-axis. Every axis is separated from the previous by 90° or $\pi/2$. So the angles associated with each axis are

Axis	Degrees	Radians
Positive x-axis	0°	0
Positive y-axis	90°	$\pi/2$
Negative x-axis	180°	π
Negative y-axis	270°	$3\pi/2$
Positive x-axis	360°	2π

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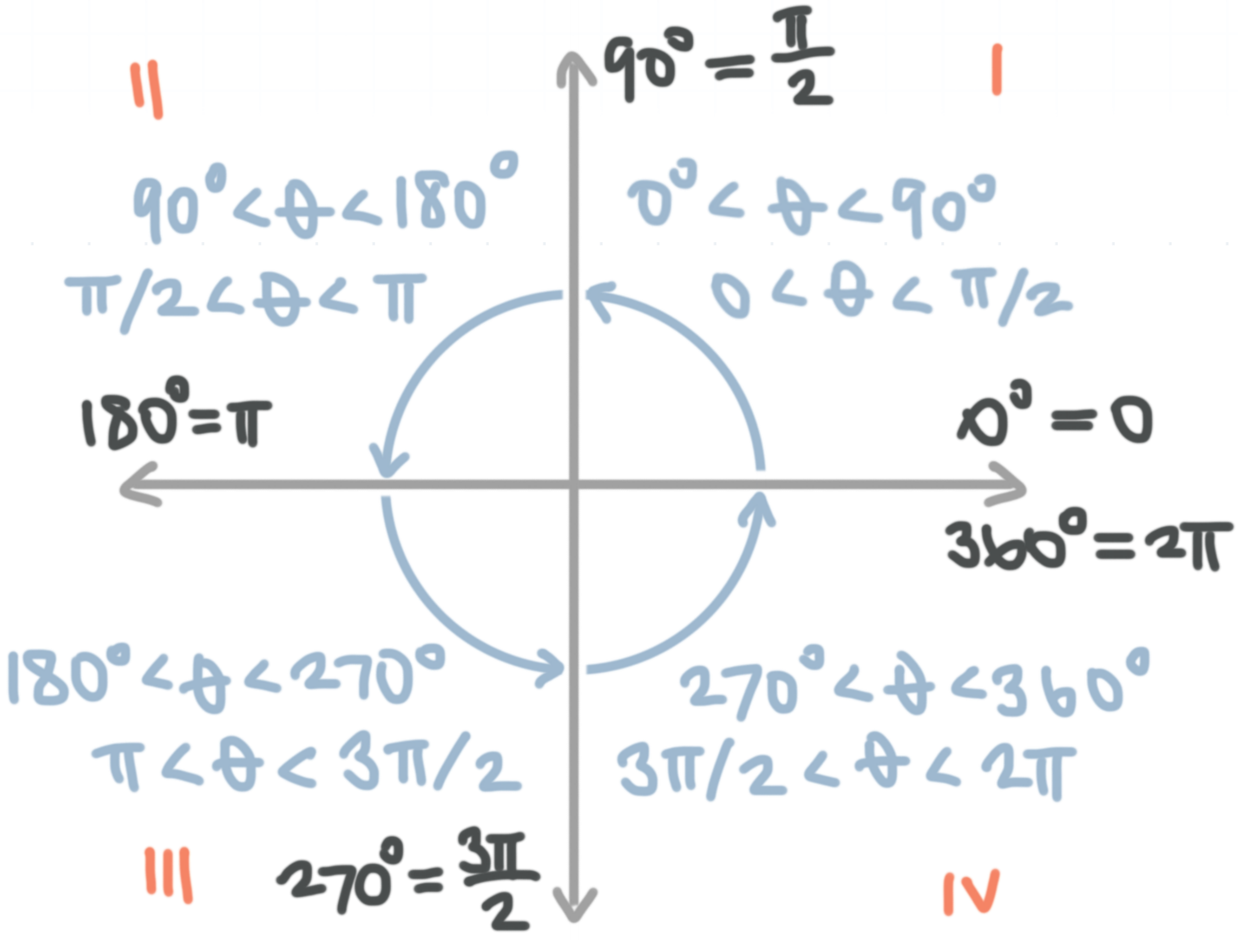
Given these values, we know the angle measures that divide each quadrant, which means we can give an inequality that defines all the angles contained within each quadrant:

Quadrant	**Degrees**	**Radians**
First	$0^\circ < \theta < 90^\circ$	$0 < \theta < \pi/2$
Second	$90^\circ < \theta < 180^\circ$	$\pi/2 < \theta < \pi$
Third	$180^\circ < \theta < 270^\circ$	$\pi < \theta < 3\pi/2$
Fourth	$270^\circ < \theta < 360^\circ$	$3\pi/2 < \theta < 2\pi$

We can illustrate these quadrantal angles and the interval of angles which are defined in each quadrant.



We can also define a similar set of negative angles and inequalities. We start from the positive x-axis and rotate clockwise, with negative rotation, toward the negative y-axis, then the negative x-axis, then the positive y-axis, and finally back around to the positive x-axis. So the angles associated with each axis are

Axis	**Degrees**	**Radians**
Positive x-axis	0°	0
Negative y-axis	-90°	$-\pi/2$
Negative x-axis	-180°	$-\pi$
Positive y-axis	-270°	$-3\pi/2$
Positive x-axis	-360°	-2π

And then we can give an inequality that defines all the negative angles contained within each quadrant:

Quadrant	**Degrees**	**Radians**
Fourth	$-90^\circ < \theta < 0^\circ$	$-\pi/2 < \theta < 0$
Third	$-180^\circ < \theta < -90^\circ$	$-\pi < \theta < -\pi/2$
Second	$-270^\circ < \theta < -180^\circ$	$-3\pi/2 < \theta < -\pi$
First	$-360^\circ < \theta < -270^\circ$	$-2\pi < \theta < -3\pi/2$

We'll also illustrate these negative angles in terms of the four quadrants.

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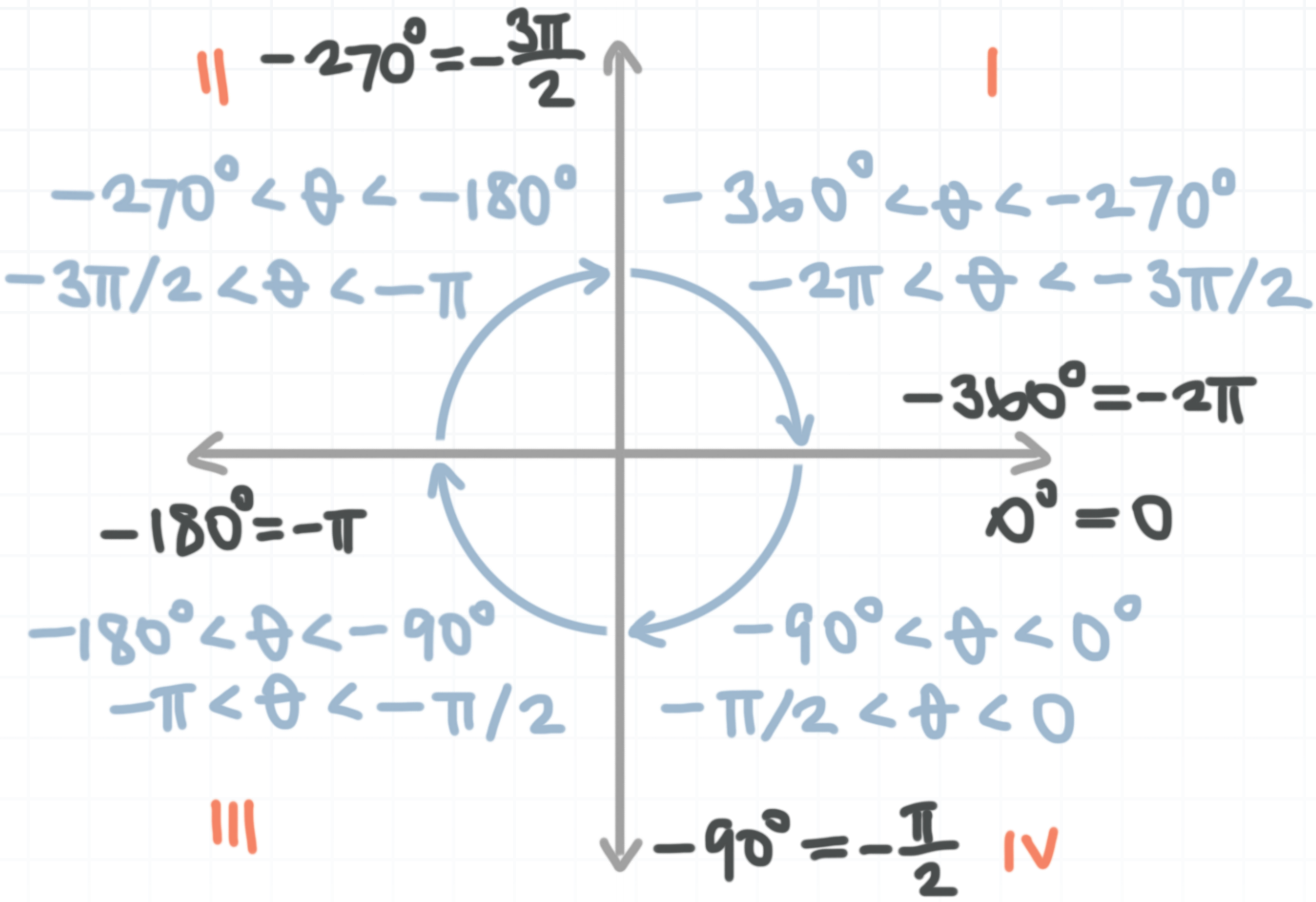

Let's work through a few examples of how to use this information to determine the quadrant in which an angle is located.

Example

Determine the quadrant in which $\theta = 283^\circ$ is located.

We just need to compare the angle against the inequalities we set up to define each quadrant.

$$270^\circ < 283^\circ < 360^\circ$$

The angle $\theta = 283^\circ$ lies in the fourth quadrant, because it's a larger angle than 270°, but a smaller angle than 360°. We know that 270° is along the

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negative y-axis and that 360° is along the positive x-axis, so $\theta = 283^\circ$ falls between them in the fourth quadrant.

Let's look at an example with a negative radian angle that measures more than one full rotation.

Example

In which quadrant is $-(33/5)\pi$ located?

Remember that, in radians, one full rotation is 2π. So to determine how many full rotations are included in $-(33/5)\pi$, divide $-(33/5)\pi$ by 2π.

$$\frac{-\frac{33\pi}{5}}{2\pi} = -\frac{33\pi}{5} \cdot \frac{1}{2\pi} = -\frac{33\pi}{10\pi} = -3.3$$

This tells us that $-(33/5)\pi$ includes 3 full rotations in the negative direction, plus an additional 0.3 rotations in the negative direction. We just need to figure out how much is 0.3 of 2π.

$$0.3(2\pi) = 0.6\pi$$

So, from the starting point of the positive direction of the x-axis, we complete 3 full rotations in the negative direction, which gets us back to the same starting point, and then we rotate an additional 0.6π in the negative direction, which is further of a rotation than $-0.5\pi = -\pi/2$, but not

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as far as $-1.0\pi = -\pi$. Which means the terminal side of the angle will land in the third quadrant.

So we can say that $-(33/5)\pi$ is located in the third quadrant.

Now let's deal with an angle in radians whose measure isn't given in terms of π.

Example

In which quadrant is 21.9 radians located?

To put the angle in terms of π, we'll divide 21.9 by $\pi \approx 3.14$ to get

$$\frac{21.9}{3.14} = 6.97$$

so 21.9 radians $\approx 6.97\pi$. This angle is outside the interval $[0,2\pi)$ that represents one full rotation. We know that 6π represents 3 full rotations, so we know the angle is three full rotations in the positive direction, and then an additional 0.97π rotations in the positive direction.

Now all we need to do is figure out the quadrant of 0.97π. Since 0.97 is more than $1/2$, but less than 1, it means that 0.97π is more than $\pi/2$, but less than π, which leaves us in the second quadrant.

Therefore, 21.9 radians is in the second quadrant.

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Degrees, radians, and DMS

We've already introduced both degrees and radians as two different ways to measure angles. In this lesson, we'll look at **DMS** (degrees, minutes, seconds) as a third angle-measurement system, and then we'll talk about how to convert between all three systems.

The DMS system

In the same way that we can express angles in degrees or in radians, we can also express them in DMS (degrees, minutes, seconds).

In degrees, we already know that one full rotation is 360°. What that really means is that we're splitting up one full circle into three-hundred-sixty 1° angles.

In DMS, each of those 1° angles represents the "**degree**" part of degrees-minutes-seconds. If we then zoom in and divide that single 1° angle into 60 parts, each of those parts is the "**minute**" part of DMS, and then if we zoom in even more and divide each of those minutes into another 60 parts, each of those parts is the "**second**" part of DMS.

Despite the fact that we use the words "minutes" and "seconds," DMS angle measurement actually has nothing to do with time other than the fact that there are 60 seconds in a minute and 60 minutes in a degree, in the same way that there are 60 seconds in a minute and 60 minutes in an hour when we're talking about time.

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As an example, a DMS angle of 36 degrees 40 minutes 7 seconds, will be written as $36^\circ 40' 7''$. In other words, we use $''$ for seconds, $'$ for minutes, and, just like in degree measurement, $^\circ$ for degrees.

Converting between degrees, radians, and DMS

Degrees and radians

Remember that one complete circle is measured with either 360° or 2π radians, so $360^\circ = 2\pi$. Which means that if we want to convert an angle from degrees to radians, we multiply it by $\pi/180^\circ$. For instance, we convert 45° into radians like this:

$$45^\circ \left(\frac{\pi}{180^\circ}\right)$$

$$\frac{45\pi}{180}$$

$$\frac{\pi}{4}$$

So $45^\circ = \pi/4$. And to convert from radians to degrees, we flip the fraction and multiply the radian angle by $180^\circ/\pi$. For instance, we convert $3\pi/2$ into degrees like this:

$$\frac{3\pi}{2}\left(\frac{180^\circ}{\pi}\right)$$

$$\frac{3 \cdot 180^\circ}{2}$$

$$270^\circ$$

Let's convert another angle from degrees to radians.

Example

Convert 68° to radians.

Since we're converting from degrees to radians, we'll multiply by $\pi/180^\circ$.

$$68^\circ\left(\frac{\pi}{180^\circ}\right)$$

$$\frac{68\pi}{180}$$

$$\frac{17\pi}{45}$$

So 68° is equivalent to $17\pi/45$ radians.

Let's show how to multiply a radian angle into a degree angle by multiplying by $180^\circ/\pi$.

Example

Convert $4\pi/5$ to degrees.

Since we're converting from radians to degrees, we'll multiply by $180^\circ/\pi$.

$$\frac{4\pi}{5}\left(\frac{180^\circ}{\pi}\right)$$

$$\frac{4(180^\circ)}{5}$$

$$\frac{720^\circ}{5}$$

$$144^\circ$$

There are certain angles that we commonly use in Trigonometry, and it's helpful to memorize their values in both degrees and radians.

Degrees	**Radians**	**Location**
0°	0	Positive horizontal axis
30°	$\pi/6$	1st quadrant
45°	$\pi/4$	1st quadrant
60°	$\pi/3$	1st quadrant
90°	$\pi/2$	Positive vertical axis
120°	$2\pi/3$	2nd quadrant
135°	$3\pi/4$	2nd quadrant
150°	$5\pi/6$	2nd quadrant

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180°	π	Negative horizontal axis
210°	$7\pi/6$	3rd quadrant
225°	$5\pi/4$	3rd quadrant
240°	$4\pi/3$	3rd quadrant
270°	$3\pi/2$	Negative vertical axis
300°	$5\pi/3$	4th quadrant
315°	$7\pi/4$	4th quadrant
330°	$11\pi/6$	4th quadrant
360°	2π	Positive horizontal axis

We can sketch out these angles in a circle centered at the origin.

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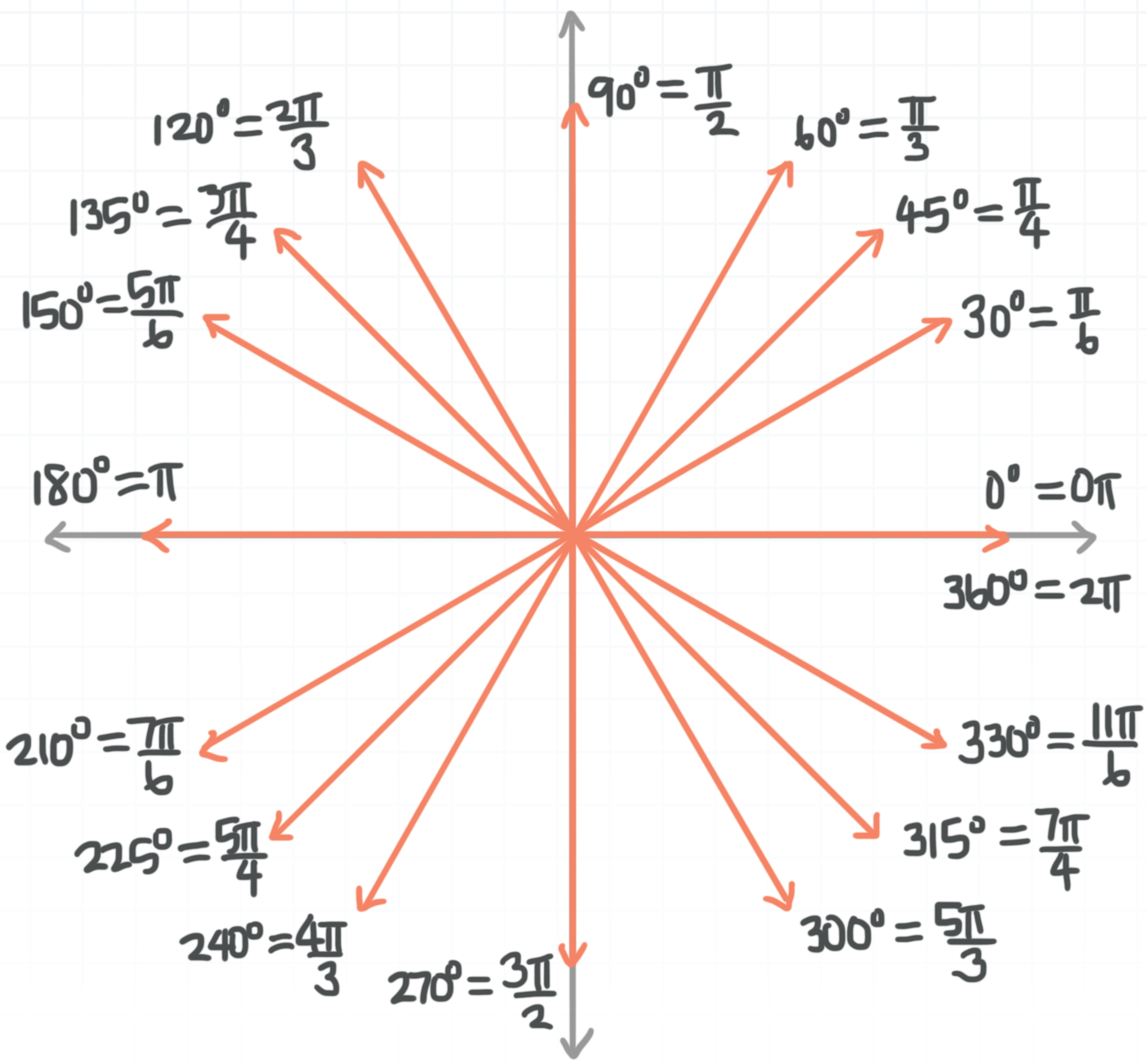

This image is the beginning of the “unit circle” which is a really important topic that we’ll cover later on in the course. For now, we can just use this diagram to visualize the locations of these common angles.

Degrees and DMS

If we’re given an angle in degrees that’s an integer, then the angle will look exactly the same in DMS, because the degrees part of DMS is equivalent to the the way we measure angles with degrees. So an angle of 72° in degrees is also 72° in DMS.

But if we’re given a non-integer angle in degrees, like 72.75°, then converting to DMS requires us to consider minutes and seconds. The angle

72.75° says that we have 72° and 0.75 of one more degree. Remember that each individual degree can be divided into 60 minutes, so 72.75° tells us that we have 0.75 of 60 minutes. Then to find out how many minutes we have, we multiply the decimal number by $60'$.

$$0.75(60')$$

$$45'$$

So 72.75° in degrees converts to $72^\circ 45' 0''$, or just $72^\circ 45'$, in DMS. If we'd had 72.7525° in degrees instead, we'd have ended up with a non-zero seconds part. We'd first identify that the 72 converts to 72°. Then we'd take the decimal 0.7525 and multiply by $60'$ to find minutes.

$$0.7525(60')$$

$$45.15'$$

Then we'd identify that the 45 converts to $45'$, and take the 0.15 and multiply by $60''$ to find seconds.

$$0.15(60'')$$

$$9''$$

Then the angle 72.7525° in degrees is equivalent to the angle $72^\circ 45' 9''$.

Of course, we can work backwards through this same process to convert from DMS to degrees. Given the DMS-angle $72^\circ 45' 9''$, we divide $9''$ by $60'$ to get

$$\frac{9''}{60'}$$

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$$0.15'$$

Then the angle $72^\circ 45' 9''$ becomes $72^\circ 45.15'$ Then we divide $45.15'$ by 60° to get

$$\frac{45.15'}{60^\circ}$$

$$0.7525^\circ$$

Then the angle $72^\circ 45.15'$ becomes 72.7525°, and we've finished converting from DMS back to degrees.

Let's do a few more conversions between degrees and DMS before we talk about converting between radians and DMS.

Example

Convert from degrees to DMS.

$$149.3^\circ$$

The angle in degrees is 149.3°, so the degrees part in DMS is 149°. All we have to do is convert 0.3° to minutes and seconds. First, we'll convert 0.3° to minutes, and then if we get a decimal for the minutes, convert the remaining part to seconds.

There are $60'$ in 1°, so we can multiply 0.3° by $60'/1^\circ$ to convert the degrees to minutes.

$$0.3^\circ \left(\frac{60'}{1^\circ} \right)$$

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$$(0.3(60))'$$

$$18'$$

We’ve found that 0.3° converts to $18'$. Since 18 is an integer, there’s nothing left to convert to seconds, so the angle in DMS is $149^\circ 18'$.

Let's try converting from DMS to degrees.

Example

Express $85^\circ 31' 22''$ in degrees.

We'll convert the seconds part first. We need to convert $22''$ from seconds to minutes. We know that $1' = 60''$, so we’ll multiply $22''$ by $1'/60''$ in order to cancel the seconds and be left with just minutes.

$$22''\left(\frac{1'}{60''}\right)$$

$$\left(\frac{22}{60}\right)'$$

$$\left(\frac{11}{30}\right)'$$

Then the total minutes in $85^\circ 31' 22''$ is

$$\left(31+\frac{11}{30}\right)'$$

$$\left(31\left(\frac{30}{30}\right)+\frac{11}{30}\right)'$$

$$\left(\frac{930}{30}+\frac{11}{30}\right)'$$

$$\left(\frac{941}{30}\right)'$$

To convert this value for minutes into degrees, we'll multiply by $1^\circ/60'$ in order to cancel the minutes and be left with just degrees.

$$\left(\frac{941}{30}\right)'\left(\frac{1^\circ}{60'}\right)$$

$$\left(\frac{941}{30(60)}\right)^\circ$$

$$\left(\frac{941}{1{,}800}\right)^\circ$$

Putting this together with the 85° from the original angle, we get approximately

$$\left(85+\frac{941}{1{,}800}\right)^\circ$$

$$(85+0.5228)^\circ$$

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$$85.5228^\circ$$

Let's look at how to convert an angle from DMS to degrees when the seconds part is a decimal number.

Example

Convert $74^\circ 10' 3.6''$ to degrees.

We'll convert the seconds part first. We need to convert $3.6''$ from seconds to minutes. We know that $1' = 60''$, so we'll multiply $3.6''$ by $1'/60''$ in order to cancel the seconds and be left with just minutes.

$$3.6''\left(\frac{1'}{60''}\right)$$

$$\left(\frac{3.6}{60}\right)'$$

$$0.06'$$

Then the total minutes in $74^\circ 10' 3.6''$ is

$$(10 + 0.06)'$$

$$10.06'$$

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To convert this value for minutes into degrees, we'll multiply by $1^\circ/60'$ in order to cancel the minutes and be left with an approximate value for degrees.

$$10.06'\left(\frac{1^\circ}{60'}\right)$$

$$\left(\frac{10.06}{60}\right)^\circ$$

$$0.1677^\circ$$

Putting this together with the 74° from the original angle, we get approximately

$$(74 + 0.1677)^\circ$$

$$74.1677^\circ$$

Radians and DMS

To convert from radians to DMS, we'll first convert from radians to degrees, and then to DMS. Similarly, to convert the other way from DMS to radians, we'll convert the DMS angle to a degree angle, and then convert the degree angle to a radian angle.

radians → degrees → DMS

DMS → degrees → radians

We can summarize all the conversions we've learned into a table.

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	To radians	To degrees	To DMS
From radians	-	Multiply by 180°/π	Convert through degrees first
From degrees	Multiply by π/180°	-	Convert fraction of a degree to minutes, then fraction of a minute to a second
From DMS	Convert through degrees first	Convert seconds to minutes, then minutes to degrees	-

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Coterminal angles

If we sketch two angles in standard position, they're **coterminal** if their terminal sides lie on top of each other. In other words, if both angles finish up at the same place, then they're coterminal.

Coterminal angles will always differ by 360° or 2π radians. So to find a coterminal angle, we just add or subtract 360° or 2π as many times as we want to. For instance, let's say we want to find angles that are coterminal with 45°. Adding 360° one, two, and three times to 45° gives three angles that are all coterminal with 45°, and therefore all coterminal with one another:

$$45^\circ + 360^\circ = 405^\circ$$

$$45^\circ + 2(360^\circ) = 765^\circ$$

$$45^\circ + 3(360^\circ) = 1{,}125^\circ$$

We could also subtract 360° once, twice, and three times to find three more angles that are coterminal with 45°, coterminal with each other, and coterminal with the three positive we just found:

$$45^\circ - 360^\circ = -315^\circ$$

$$45^\circ - 2(360^\circ) = -675^\circ$$

$$45^\circ - 3(360^\circ) = -1{,}035^\circ$$

And this pattern continues indefinitely in both the positive and negative directions.

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We can also do this with radians. Instead of adding or subtracting some multiple of 360°, we'd add or subtract any multiple of 2π. For example, all of these angles are coterminal with $\pi/6$:

$$\frac{\pi}{6} + 2\pi = \frac{13\pi}{6}$$

$$\frac{\pi}{6} + 2(2\pi) = \frac{25\pi}{6}$$

$$\frac{\pi}{6} + 3(2\pi) = \frac{37\pi}{6}$$

...

and

$$\frac{\pi}{6} - 2\pi = -\frac{11\pi}{6}$$

$$\frac{\pi}{6} - 2(2\pi) = -\frac{23\pi}{6}$$

$$\frac{\pi}{6} - 3(2\pi) = -\frac{35\pi}{6}$$

...

Let's do an example where we find a few positive coterminal angles.

Example

Find the three smallest positive angles that are coterminal with 67°.

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We can add or subtract 360° to find coterminal angles for 67°. If we were to subtract any multiple of 360°, we'd get a negative angle, but we were asked for only positive angles. Therefore, we need to add multiples of 360° in order to find the angles we need.

$$67^\circ + 1(360^\circ) = 67^\circ + 360^\circ = 427^\circ$$

$$67^\circ + 2(360^\circ) = 67^\circ + 720^\circ = 787^\circ$$

$$67^\circ + 3(360^\circ) = 67^\circ + 1{,}080^\circ = 1{,}147^\circ$$

These are the three smallest positive coterminal angles for 67°.

Let's do an example with radian angles.

Example

Find the two smallest positive angles that are coterminal with $-3\pi/2$.

We can add or subtract 2π to find coterminal angles for $-3\pi/2$. If we were to subtract any multiple of 2π, we'd get another negative angle, but we were asked for only positive angles. Therefore, we need to add multiples of 2π in order to find the angles we need.

$$-\frac{3\pi}{2} + 1(2\pi) = -\frac{3\pi}{2} + \frac{4\pi}{2} = \frac{\pi}{2}$$

$$-\frac{3\pi}{2} + 2(2\pi) = -\frac{3\pi}{2} + \frac{8\pi}{2} = \frac{5\pi}{2}$$

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These are the two smallest positive coterminal angles for $-3\pi/2$.

Now we'll try an example with negative rotations.

Example

Find four negative angles that are coterminal with $6\pi/5$.

In order to find negative angles, we'll need to subtract multiples of 2π.

$$\frac{6\pi}{5} - 1(2\pi) = \frac{6\pi}{5} - \frac{10\pi}{5} = -\frac{4\pi}{5}$$

$$\frac{6\pi}{5} - 2(2\pi) = \frac{6\pi}{5} - \frac{20\pi}{5} = -\frac{14\pi}{5}$$

$$\frac{6\pi}{5} - 3(2\pi) = \frac{6\pi}{5} - \frac{30\pi}{5} = -\frac{24\pi}{5}$$

$$\frac{6\pi}{5} - 4(2\pi) = \frac{6\pi}{5} - \frac{40\pi}{5} = -\frac{34\pi}{5}$$

Let's do two quick examples with angles given in DMS so that we know how to handle those as well.

Example

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Find the angle α that's coterminal with $150^\circ 17' 49''$, if we make two full positive rotations around the origin.

To find coterminal angles for DMS angles, we do the same thing we did with angles given in degrees, and we just carry the minutes and seconds along with us.

Since we were asked to make two full positive rotations from $150^\circ 17' 49''$ to find α, we can say that α is

$$\alpha = 150^\circ 17' 49'' + 2(360^\circ)$$

$$\alpha = 150^\circ 17' 49'' + 720^\circ$$

$$\alpha = (150 + 720)^\circ 17' 49''$$

$$\alpha = 870^\circ 17' 49''$$

In the example we just did, the original angle was positive, and we rotated in the positive direction. So the signs of the angle and the rotation matched; they were both positive.

Things get little more complicated when the signs are different (when the angle is positive and we rotate in the negative direction, or when the angle is negative and we rotate in the positive direction). Let's look at an example like that now.

Example

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Find the angle α that's coterminal with $16^\circ 20' 42''$ if we make three full negative rotations around the origin.

Since we were asked to make three full negative rotations from $16^\circ 20' 42''$ to find α, we can say that α is

$$\alpha = (16^\circ + 20' + 42'') - 3(360^\circ)$$

$$\alpha = 16^\circ + 20' + 42'' - 1{,}080^\circ$$

$$\alpha = (16^\circ - 1{,}080^\circ) + 20' + 42''$$

$$\alpha = -1{,}064^\circ + 20' + 42''$$

The reason we separated the degrees, minutes, and seconds from each in this example, but kept them together in the last example, is because for DMS angles, the three parts all must be positive, or all must be negative. Otherwise, if the signs are mixed, then part of the angle is rotating in the positive direction, while the other is rotating in the negative direction, and we don't want that.

To make sure all the signs match, we can calculate degrees first like we did here, and find that we have a negative value for degrees, but then we need to make both the minutes and seconds negative as well.

To make the minutes part negative, we'll borrow -1° from the $-1{,}064^\circ$, and combine that -1° with the $20'$ by using the fact that $1^\circ = 60'$.

$$\alpha = -1{,}063^\circ + (-1^\circ) + 20' + 42''$$

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$$\alpha = -1{,}063^\circ + (-60') + 20' + 42''$$

$$\alpha = -1{,}063^\circ + (-40') + 42''$$

To make the seconds part negative, we'll borrow $-1'$ from the $-40'$, and combine that $-1'$ with the $42''$ by using the fact that $1' = 60''$.

$$\alpha = -1{,}063^\circ + (-39') + (-1') + 42''$$

$$\alpha = -1{,}063^\circ + (-39') + (-60'') + 42''$$

$$\alpha = -1{,}063^\circ + (-39') + (-18'')$$

Now that all three parts are negative, we can write the coterminal angle as $\alpha = -1{,}063^\circ 39' 18''$. The negative sign in front implies that the entire angle is negative, because the negative sign applies to all three parts (degrees, minutes, and seconds) of the DMS angle.

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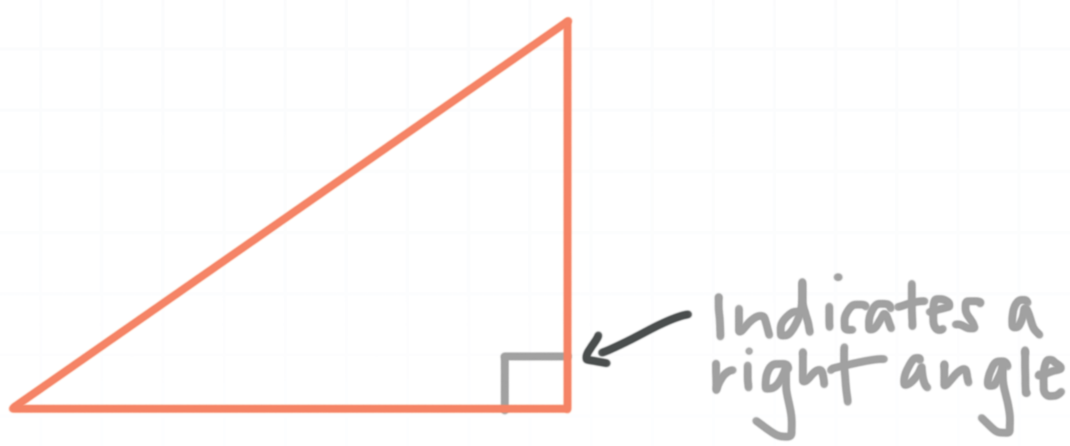
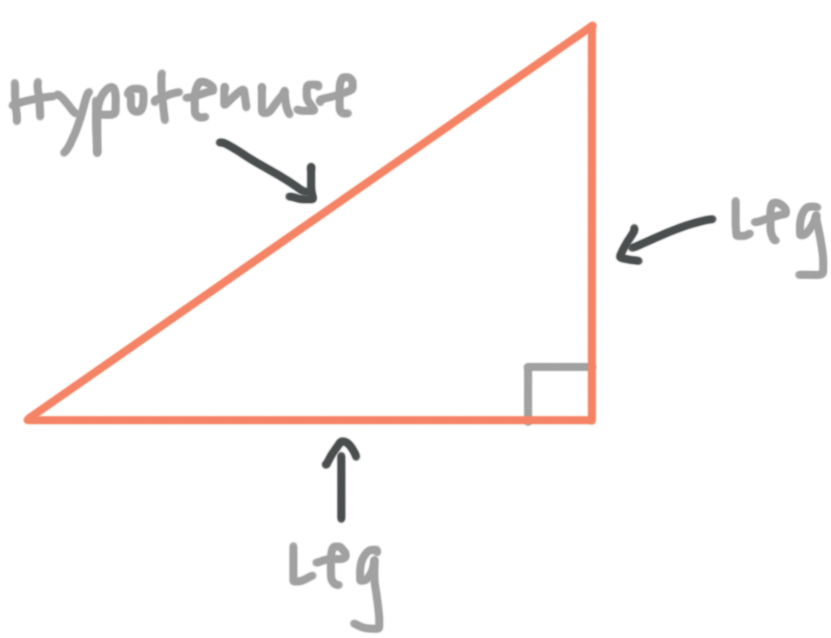
Sine, cosine, and tangent

Remember previously that we talked about different kinds of angles, including right angles, which were angles that measured exactly 90° or $\pi/2$ radians.

Based on this definition of a right angle, if we say that a triangle is a **right triangle**, that means the triangle includes exactly one 90° interior angle. We indicate a right angle with a little square.



The side opposite the right angle is the **hypotenuse**, and it will always be the longest side. The other two sides are the legs.



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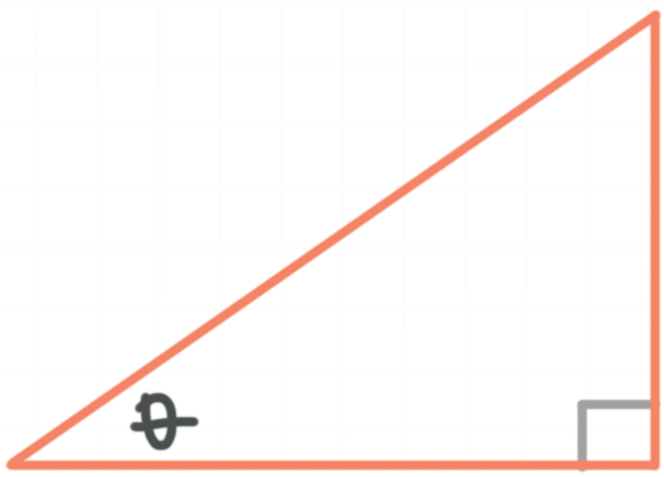
The first three trigonometric functions

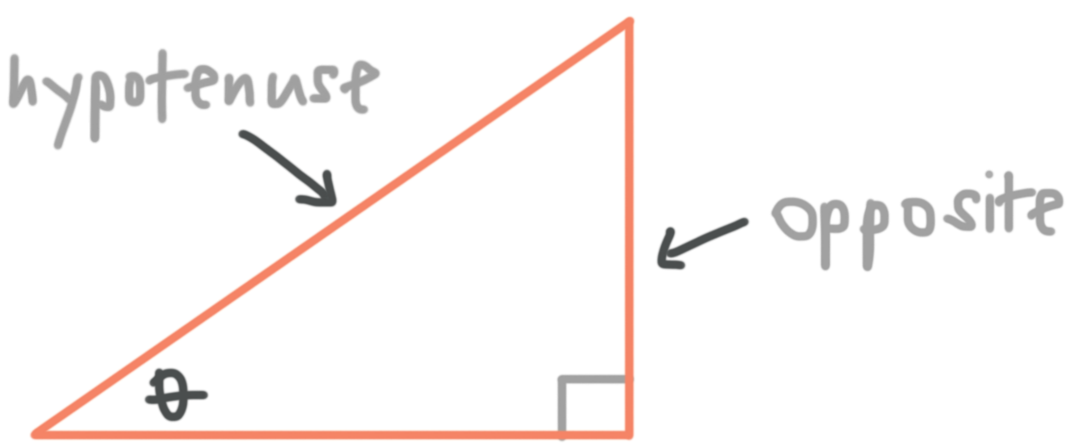
The word “**trigonometry**” literally means “the study of triangles.” And remember that a function like $y = x^2$ is an equation that gives the relationship between the variables x and y. So if we bring those ideas together, we can say that a **trigonometric function** is a function that gives the relationship between different parts of a triangle.

In fact, there are six trigonometric functions, and we can easily define the first three using the parts of a right triangle. In a right triangle like this,



where we show the right angle, and define the angle θ as one of the other angles, then the sine of that angle θ is equivalent to the length of the side opposite the angle θ, divided by the length of the hypotenuse.

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$$



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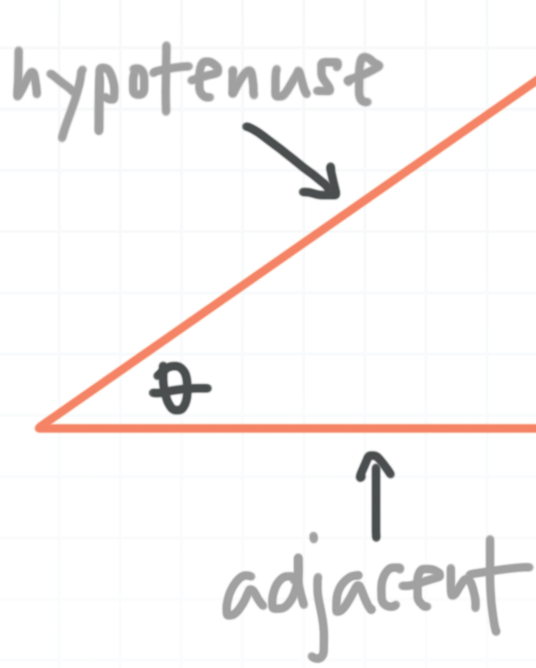
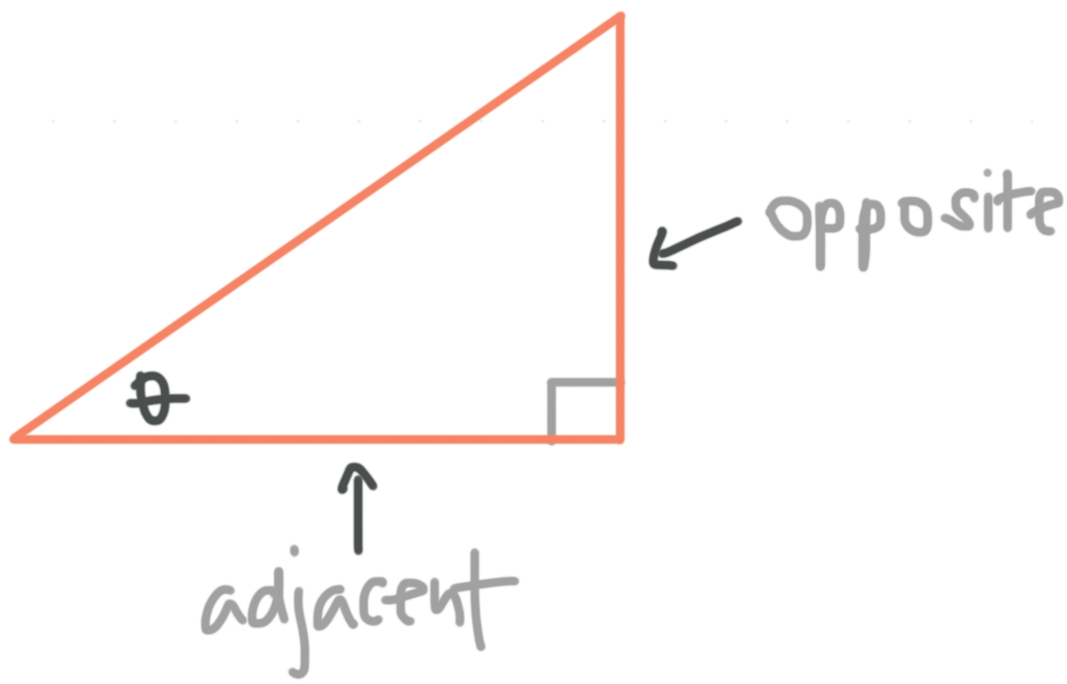
The cosine of that angle θ is equivalent to the length of the side adjacent to the angle θ, divided by the length of the hypotenuse.



$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

And the tangent of that angle θ is equivalent to the length of the side opposite the angle θ, divided by the length of the side adjacent to the angle θ.



$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

Be careful here. These first three trig functions, sine, cosine, and tangent, which we abbreviate as $\sin$, $\cos$, and $\tan$, are functions, just like a function

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$f(x)$. When we're given $f(x)$, it doesn't indicated that f is multiplied by x; instead it means that f is a function of x, or that we can plug x into f. In the same way, $\sin\theta$ doesn't mean that $\sin$ is multiplied by θ; instead it means that $\sin$ is a function of θ, or that we can plug θ into $\sin$.

To remember the definition of these three trig functions, remember SOH-CAH-TOA.

SOH: sine, opposite, hypotenuse

CAH: cosine, adjacent, hypotenuse

TOA: tangent, opposite, adjacent

We can also place this same right triangle in the coordinate plane with the angle θ at the origin. If we sketch out a circle around the triangle, then the hypotenuse becomes the radius of the circle, and we can call the three sides of the triangle x, y, and r, where x is always the adjacent side, y is always the opposite side, and r is always the hypotenuse.

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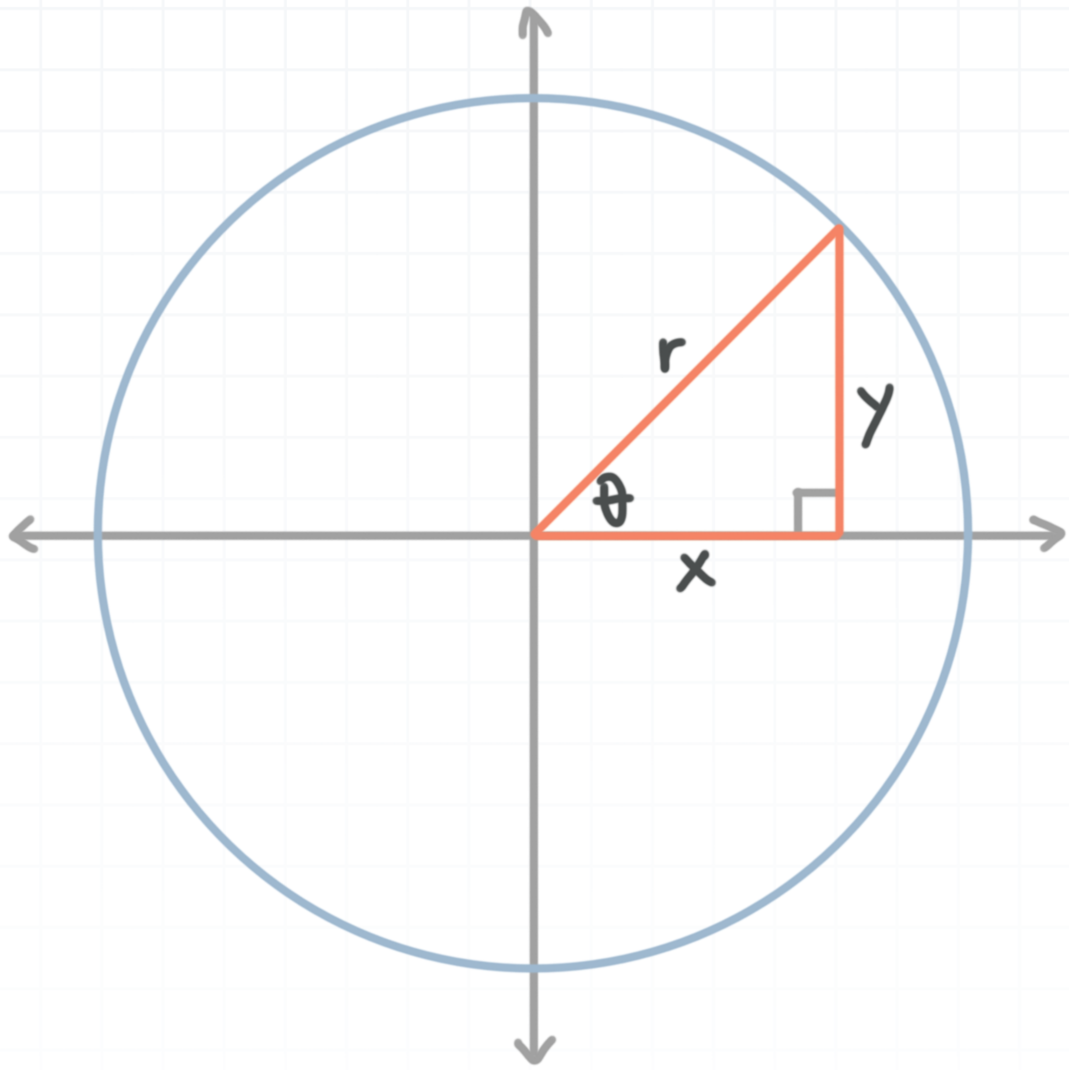

In this context, we can also define the first three trig functions as

$$\sin\theta = \frac{y}{r}$$

$$\cos\theta = \frac{x}{r}$$

$$\tan\theta = \frac{y}{x}$$

Let's calculate these first three trig functions for a particular triangle.

Example

Find the values of the sine, cosine, and tangent functions for θ.

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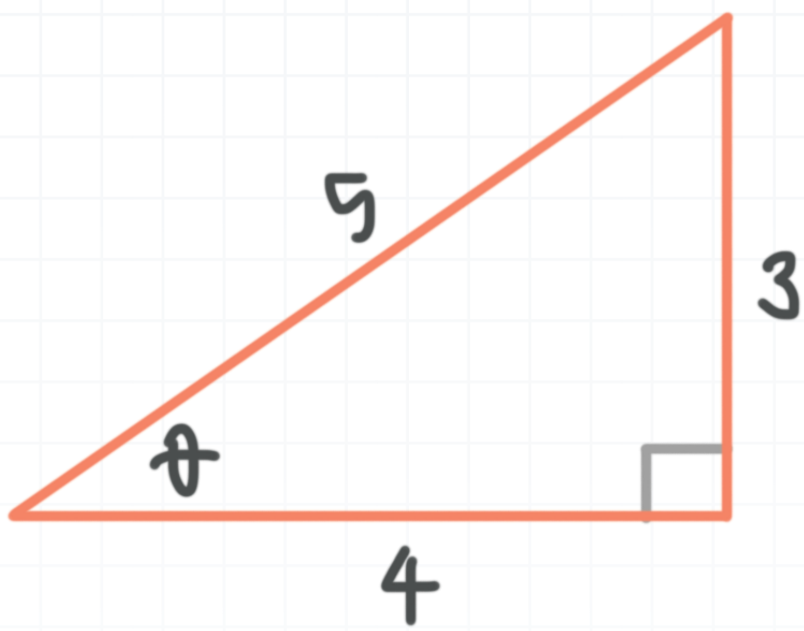

Given the position of the angle θ in the right triangle, the length of the opposite side is 3, the length of the adjacent side is 4, and the length of the hypotenuse is 5.

Then the values of sine, cosine, and tangent for the angle are

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{3}{5}$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{4}{5}$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{3}{4}$$

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Cosecant, secant, cotangent, and the reciprocal identities

In the last lesson we defined the first three of the six trig functions as the sine, cosine, and tangent functions:

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

In this lesson, we'll define the other three of the six trig functions as cosecant, secant, and cotangent. We abbreviate these three functions as $\csc$, $\sec$, and $\cot$. These three functions are the reciprocals of the first three trig functions.

- Sine is the reciprocal of cosecant, and vice versa
- Cosine is the reciprocal of secant, and vice versa
- Tangent is the reciprocal of cotangent, and vice versa

Remember that the reciprocal of a fraction is what we get when we flip the fraction upside down. So the reciprocal of a/b is b/a. Therefore, in terms of side lengths, we can define these three new trig functions as

$$\csc\theta = \frac{\text{hypotenuse}}{\text{opposite}}$$

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$$\sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}}$$

$$\cot\theta = \frac{\text{adjacent}}{\text{opposite}}$$

Notice how these three are just the reciprocals of $\sin$, $\cos$, and $\tan$.

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \csc\theta = \frac{\text{hypotenuse}}{\text{opposite}}$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}}$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} \qquad \cot\theta = \frac{\text{adjacent}}{\text{opposite}}$$

We can also define cosecant, secant, and cotangent in terms of x, y, and r, and they'll of course still be the reciprocals of sine, cosine, and cotangent.

$$\sin\theta = \frac{y}{r} \qquad \csc\theta = \frac{r}{y}$$

$$\cos\theta = \frac{x}{r} \qquad \sec\theta = \frac{r}{x}$$

$$\tan\theta = \frac{y}{x} \qquad \cot\theta = \frac{x}{y}$$

Let's do an example where we calculate $\csc$, $\sec$, and $\cot$ for a particular triangle.

Example

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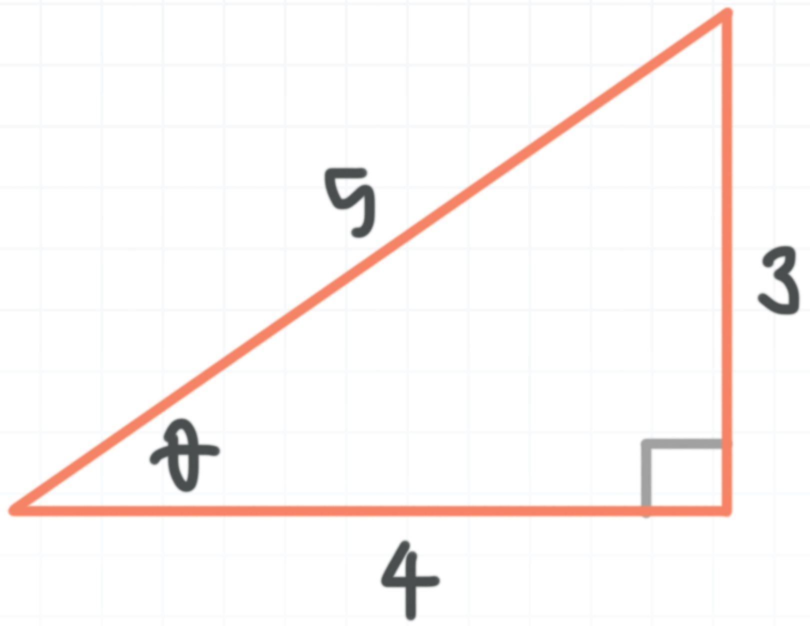
Find the values of the cosecant, secant, and cotangent functions for θ.



Given the position of the angle θ in the right triangle, the length of the opposite side is 3, the length of the adjacent side is 4, and the length of the hypotenuse is 5.

Then the values of cosecant, secant, and cotangent for the angle are

$$\csc\theta = \frac{\text{hypotenuse}}{\text{opposite}} = \frac{5}{3}$$

$$\sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{5}{4}$$

$$\cot\theta = \frac{\text{adjacent}}{\text{opposite}} = \frac{4}{3}$$

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The reciprocal identities

Throughout trigonometry, we'll frequently work with trigonometric identities, which are simply relationships between different trig functions. The identity set we'll talk about here is the set of reciprocal identities.

Of course, these are the reciprocal relationships we've just defined that relate sine, cosine, and tangent to cosecant, secant, and cotangent.

$$\sin\theta = \frac{1}{\csc\theta} \qquad \csc\theta = \frac{1}{\sin\theta}$$

$$\cos\theta = \frac{1}{\sec\theta} \qquad \sec\theta = \frac{1}{\cos\theta}$$

$$\tan\theta = \frac{1}{\cot\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

With these reciprocal identities, we can start to use the value of one trig function to find the value of another trig function.

Example

Given the value of $\sec\theta$, find the value of $\cos\theta$.

$$\sec\theta = \frac{\sqrt{2}}{2}$$

The cosine and secant functions are related to each other by the reciprocal identities, so we can substitute the value of $\sec\theta$ into the reciprocal identity for $\cos\theta$.

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$$\cos\theta = \frac{1}{\sec\theta}$$

$$\cos\theta = \frac{1}{\frac{\sqrt{2}}{2}}$$

Remember that dividing by a fraction is equivalent to multiplying by its reciprocal. So because we're dividing by $\sqrt{2}/2$, we can simplify by instead multiplying by $2/\sqrt{2}$.

$$\cos\theta = 1 \cdot \frac{2}{\sqrt{2}}$$

$$\cos\theta = \frac{2}{\sqrt{2}}$$

We never want to leave an answer with a root in the denominator. It's standard practice to "rationalize the denominator" in order to get roots out of the denominator. So we'll multiply both the numerator and denominator by $\sqrt{2}$. This is equivalent to multiplying by 1, so we're not changing the value of the fraction.

$$\cos\theta = \frac{2}{\sqrt{2}}\left(\frac{\sqrt{2}}{\sqrt{2}}\right)$$

$$\cos\theta = \frac{2\sqrt{2}}{\sqrt{2}\sqrt{2}}$$

$$\cos\theta = \frac{2\sqrt{2}}{2}$$

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$$\cos\theta = \sqrt{2}$$

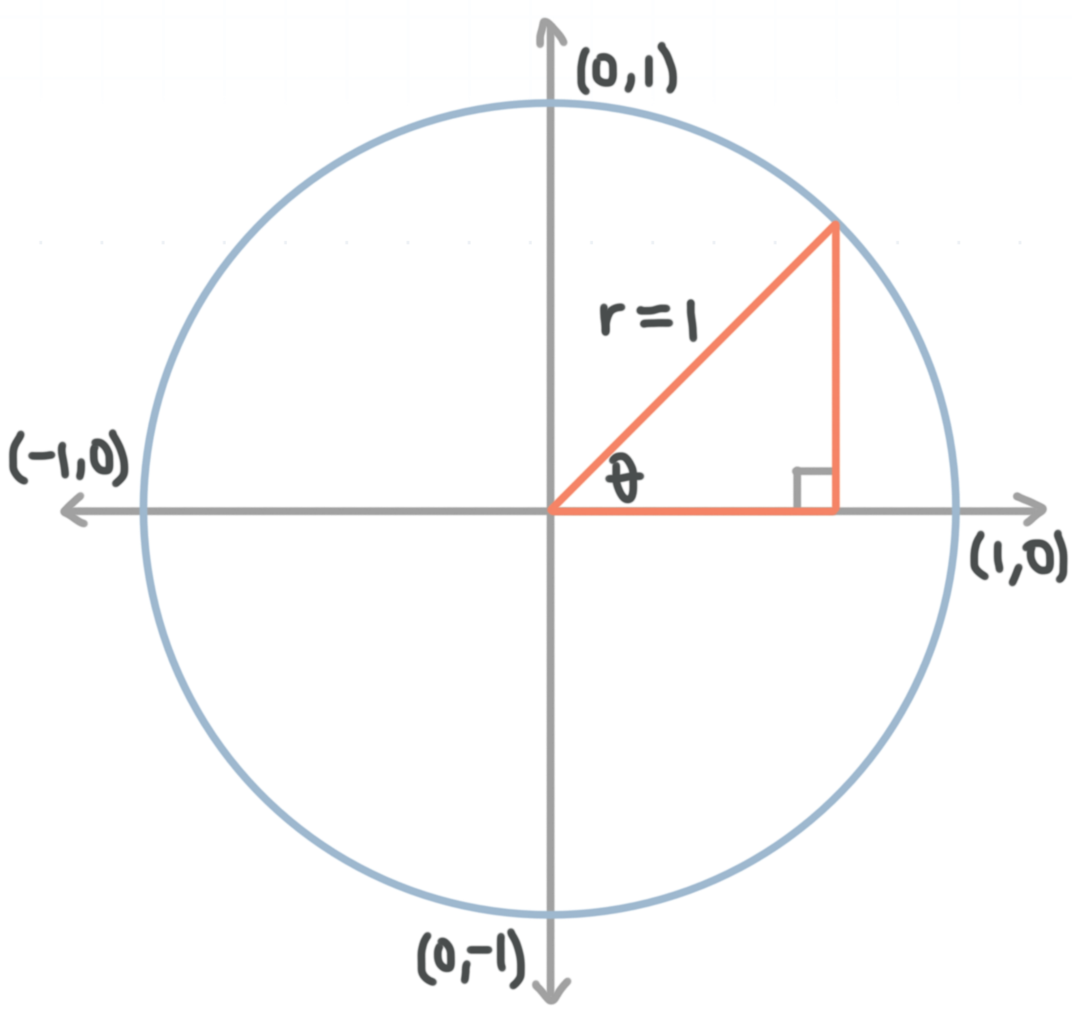
The quotient identities

We've now defined all six trig functions, and we've talked about the reciprocal relationships between them. The reciprocal identities show us that sine and cosecant are reciprocals, cosine and secant are reciprocals, and tangent and cotangent are reciprocals.

In this lesson, we want to build on those relationships in order to define two quotient identities.

The quotient identities

Let's look again at our right triangle in the first quadrant.



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Notice that we've got the triangle intersecting the circle with radius 1. We know the radius is 1 because the circle passes through both $(1,0)$ along the x-axis and $(0,1)$ along the y-axis. And if the radius is 1, that means the hypotenuse of the triangle is also 1, since the hypotenuse forms a radius of the circle.

When we defined the six trig functions earlier, we said that

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$$

but in this particular circle, we already know the hypotenuse is $r = 1$. So we could rewrite this definition of sine as

$$\sin\theta = \frac{\text{opposite}}{1}$$

$$\sin\theta = \text{opposite}$$

In the same way, we said before that

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

but in this particular circle, the hypotenuse is $r = 1$, so we can again rewrite this definition of cosine as

$$\cos\theta = \frac{\text{adjacent}}{1}$$

$$\cos\theta = \text{adjacent}$$

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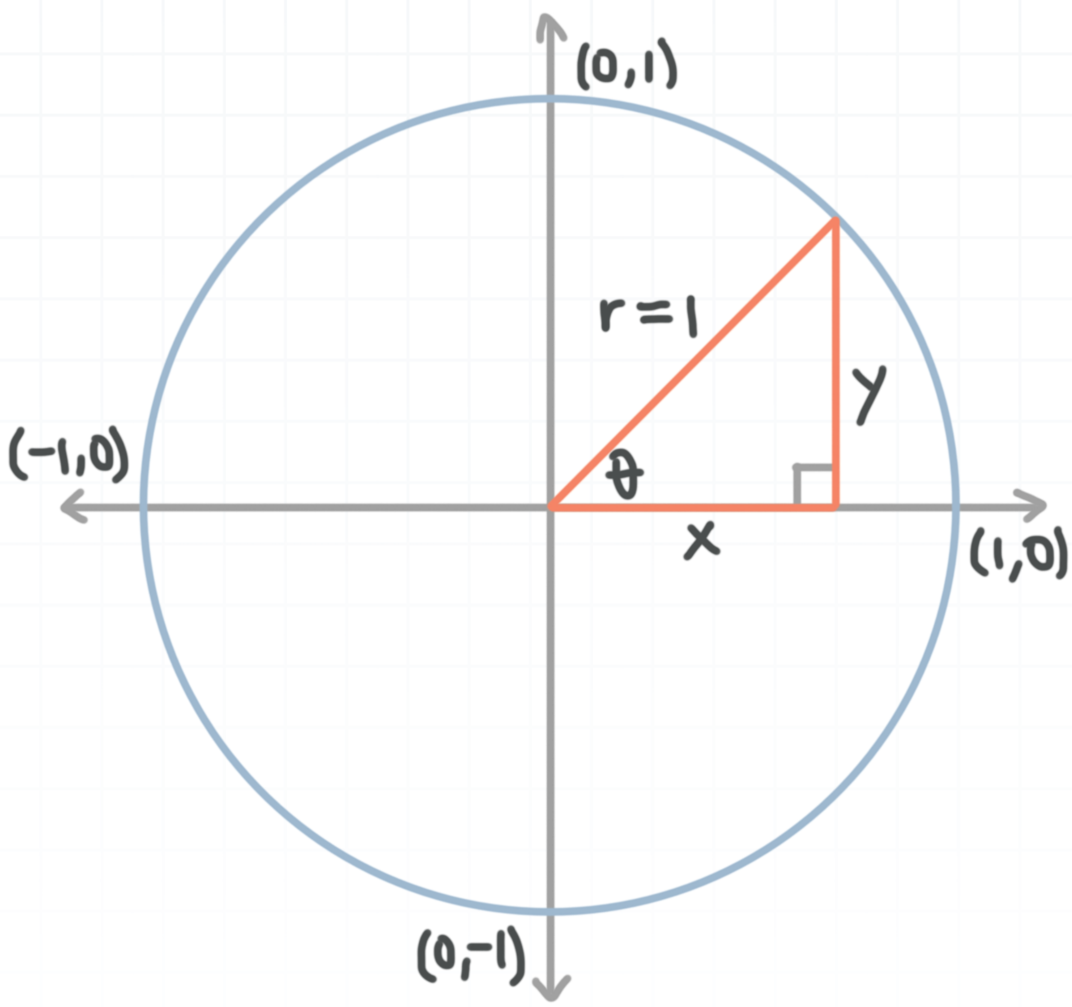
Then, if we define the horizontal leg of the triangle as having length x, and the vertical leg of the triangle as having length y,



then we get definitions for sine and cosine (when the radius of the circle is 1) of

$\sin\theta =$ opposite

$\sin\theta = y$

$y = \sin\theta$

and

$\cos\theta =$ adjacent

$\cos\theta = x$

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$$x = \cos\theta$$

This is actually a really important point that we'll build on throughout Trigonometry. In a circle with radius $r = 1$, we realize now that we can always define the x value of the coordinate point along the circle with cosine of the angle, $x = \cos\theta$, and we can always define the y value of the coordinate point along the circle with sine of the angle, $y = \sin\theta$.

Now remember how we defined the six trig functions earlier in terms of x, y, and r:

$$\sin\theta = \frac{y}{r} \qquad \csc\theta = \frac{r}{y}$$

$$\cos\theta = \frac{x}{r} \qquad \sec\theta = \frac{r}{x}$$

$$\tan\theta = \frac{y}{x} \qquad \cot\theta = \frac{x}{y}$$

But we just concluded that $x = \cos\theta$ and $y = \sin\theta$. If we substitute these values, along with $r = 1$, into these six formulas, we get

$$\sin\theta = \frac{\sin\theta}{1} \qquad \csc\theta = \frac{1}{\sin\theta}$$

$$\cos\theta = \frac{\cos\theta}{1} \qquad \sec\theta = \frac{1}{\cos\theta}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

If we simplify these equations, we get

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$$\sin\theta = \sin\theta \qquad \csc\theta = \frac{1}{\sin\theta}$$

$$\cos\theta = \cos\theta \qquad \sec\theta = \frac{1}{\cos\theta}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

The top four equations aren't interesting to us. The equations for $\sin\theta$ and $\cos\theta$ tell us nothing, and the equations for $\csc\theta$ and $\sec\theta$ are the reciprocal identities that we already know. But the bottom two equations, the ones for $\tan\theta$ and $\cot\theta$, are new to us.

In fact, these are the two quotient identities that we wanted to introduce in this lesson. They tell us that for any right triangle, tangent of the angle is always equivalent to the quotient of sine and cosine of the same angle, and that cotangent of the angle is always equivalent to the quotient of cosine and sine of the same angle.

We'll use these two quotient identities all the time throughout trigonometry, so it's important that we have them memorized.

Let's use the quotient identities to find the tangent and cotangent of an angle.

Example

Find tangent and cotangent of the angle θ.

$$\sin\theta = \frac{1}{2}$$

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$$\cos\theta = \frac{\sqrt{3}}{2}$$

We can find tangent and cotangent of θ just by plugging these sine and cosine values into the quotient identities.

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{1}{2}\cdot\frac{2}{\sqrt{3}} = \frac{2}{2\sqrt{3}} = \frac{1}{\sqrt{3}}$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \frac{\sqrt{3}}{2}\cdot\frac{2}{1} = \frac{2\sqrt{3}}{2} = \sqrt{3}$$

We'll rationalize the denominator for the value of tangent.

$$\tan\theta = \frac{1}{\sqrt{3}}\left(\frac{\sqrt{3}}{\sqrt{3}}\right) = \frac{\sqrt{3}}{\sqrt{3}\sqrt{3}} = \frac{\sqrt{3}}{3}$$

So the values of tangent and cotangent for the same angle θ are

$$\tan\theta = \frac{\sqrt{3}}{3}$$

$$\cot\theta = \sqrt{3}$$

Lastly, remember how we saw that tangent and cotangent were reciprocals of one another when we learned about the reciprocal

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identities. We see that reciprocal relationship represented here as well in the quotient identities, since tangent is sine/cosine, and cotangent is the reciprocal of that, cosine/sine.

So in the last example, instead of calculating both tangent and cotangent of the angle using the quotient identities, we could instead have first calculated tangent using the quotient identity, and then found cotangent as tangent's reciprocal. Or we could have calculated cotangent using the quotient identity, and then found tangent as cotangent's reciprocal.

Both processes get us to the same, correct values for the tangent and cotangent of the angle.

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The Pythagorean identities

So far we've defined the six trig functions and introduced the reciprocal and quotient identities that relate the trig functions to one another.

In this lesson we'll look at the Pythagorean identities, which are another set of three identities that relate the trig functions to each other. We'll look at exactly how to prove each of them, but here they are for quick reference:

$$\sin^2\theta + \cos^2\theta = 1$$

$$1 + \cot^2\theta = \csc^2\theta$$

$$1 + \tan^2\theta = \sec^2\theta$$

We call them "Pythagorean identities" because they're derived from the Pythagorean theorem.

The Pythagorean theorem

Remember from Geometry that the Pythagorean theorem tells us that, for any right triangle, the sum of the squares of the side lengths is equal to the square of the length of the hypotenuse.

In other words, if we call the legs of the right triangle a and b, and the hypotenuse c, then the Pythagorean theorem says

$$a^2 + b^2 = c^2$$

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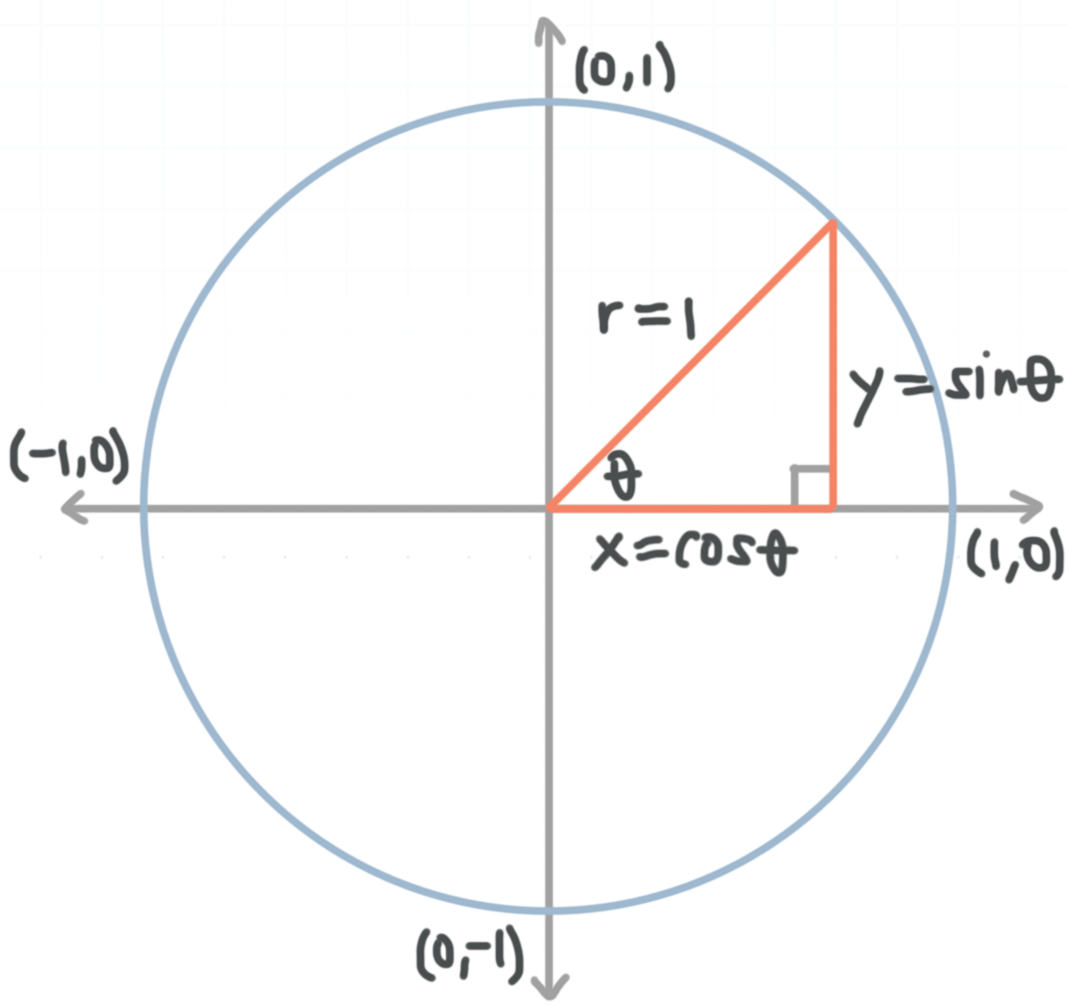
What we want to do now is rewrite this theorem in terms of trig functions, instead of simple side lengths. Doing so will give us the three Pythagorean identities.

The Pythagorean identity with sine and cosine

Remember in the last lesson that we put a right triangle in the first quadrant inside a circle with radius $r = 1$, and used that to define the side lengths of the triangle as $x = \cos\theta$ and $y = \sin\theta$.



If we start with the Pythagorean theorem, then we can substitute $a = x = \cos\theta$, $b = y = \sin\theta$, and $c = r = 1$.

$$a^2 + b^2 = c^2$$

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$$(\sin\theta)^2 + (\cos\theta)^2 = 1^2$$

$$\sin^2\theta + \cos^2\theta = 1$$

This is the first of the three Pythagorean identities. We can derive the other two from this first one.

The Pythagorean identity with cosecant and cotangent

To find the second Pythagorean identity, we can start with the first identity, $\sin^2\theta + \cos^2\theta = 1$, and divide through both sides of the equation by $\sin^2\theta$.

$$\frac{\sin^2\theta}{\sin^2\theta} + \frac{\cos^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta}$$

$$1 + \frac{\cos^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta}$$

$$1 + \left(\frac{\cos\theta}{\sin\theta}\right)^2 = \left(\frac{1}{\sin\theta}\right)^2$$

The quotient identity for tangent lets us simplify the fraction on the left,

$$1 + (\cot\theta)^2 = \left(\frac{1}{\sin\theta}\right)^2$$

$$1 + \cot^2\theta = \left(\frac{1}{\sin\theta}\right)^2$$

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and the reciprocal identity for cosecant lets us simplify the fraction on the right.

$$1 + \cot^2\theta = (\csc\theta)^2$$

$$1 + \cot^2\theta = \csc^2\theta$$

This is our second Pythagorean identity. Sometimes, we'll subtract $\cot^2\theta$ from both sides and see it written as

$$\csc^2\theta - \cot^2\theta = 1$$

The Pythagorean identity with secant and tangent

To find the third Pythagorean identity, we can start with the first identity, $\sin^2\theta + \cos^2\theta = 1$, and divide through both sides of the equation by $\cos^2\theta$.

$$\sin^2\theta + \cos^2\theta = 1$$

$$\frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$

$$\frac{\sin^2\theta}{\cos^2\theta} + 1 = \frac{1}{\cos^2\theta}$$

$$\left(\frac{\sin\theta}{\cos\theta}\right)^2 + 1 = \left(\frac{1}{\cos\theta}\right)^2$$

The quotient identity for tangent lets us simplify the fraction on the left,

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$$(\tan\theta)^2 + 1 = \left(\frac{1}{\cos\theta}\right)^2$$

$$\tan^2\theta + 1 = \left(\frac{1}{\cos\theta}\right)^2$$

and the reciprocal identity for secant lets us simplify the fraction on the right.

$$\tan^2\theta + 1 = (\sec\theta)^2$$

$$\tan^2\theta + 1 = \sec^2\theta$$

$$1 + \tan^2\theta = \sec^2\theta$$

This is our third Pythagorean identity. Sometimes, we'll subtract $\tan^2\theta$ from both sides and see it written as

$$\sec^2\theta - \tan^2\theta = 1$$

Of course, just like any of the other trig identities we've already learned (reciprocal and quotient identities), and just like the trig identities we'll learn later, these Pythagorean identities relate the trig functions to one another. With the Pythagorean identities in particular, we can

- find sine given cosine, or vice versa,
- find cosecant given cotangent, or vice versa, and
- find secant given tangent, or vice versa.
- find cosecant or secant given sine and cosine

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Let's do an example.

Example

Find $\cos\theta$.

$$\sin\theta = \frac{\sqrt{2}}{2}$$

The Pythagorean identity that relates sine and cosine will let us calculate cosine of the angle. We'll substitute the value of sine of the angle into the identity.

$$\sin^2\theta + \cos^2\theta = 1$$

$$\left(\frac{\sqrt{2}}{2}\right)^2 + \cos^2\theta = 1$$

Simplify the left side of the equation, and then get $\cos^2\theta$ by itself.

$$\frac{2}{4} + \cos^2\theta = 1$$

$$\frac{1}{2} + \cos^2\theta = 1$$

$$\cos^2\theta = 1 - \frac{1}{2}$$

$$\cos^2\theta = \frac{1}{2}$$

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Take the square root of both sides in order to solve for $\cos\theta$. When we add in a square root like this, we have to add in a $\pm$ on the right to indicate that we could get both a positive and negative solution.

$$\sqrt{\cos^2\theta} = \pm\sqrt{\frac{1}{2}}$$

$$\cos\theta = \pm\frac{\sqrt{1}}{\sqrt{2}}$$

$$\cos\theta = \pm\frac{1}{\sqrt{2}}$$

Rationalize the denominator.

$$\cos\theta = \pm\frac{1}{\sqrt{2}}\left(\frac{\sqrt{2}}{\sqrt{2}}\right)$$

$$\cos\theta = \pm\frac{\sqrt{2}}{\sqrt{2}\sqrt{2}}$$

$$\cos\theta = \pm\frac{\sqrt{2}}{2}$$

So, what is the answer we just found in the example problem actually telling us?

It's saying that for a circle with radius 1, sine of the angle (or the length of the vertical leg of the triangle), will be

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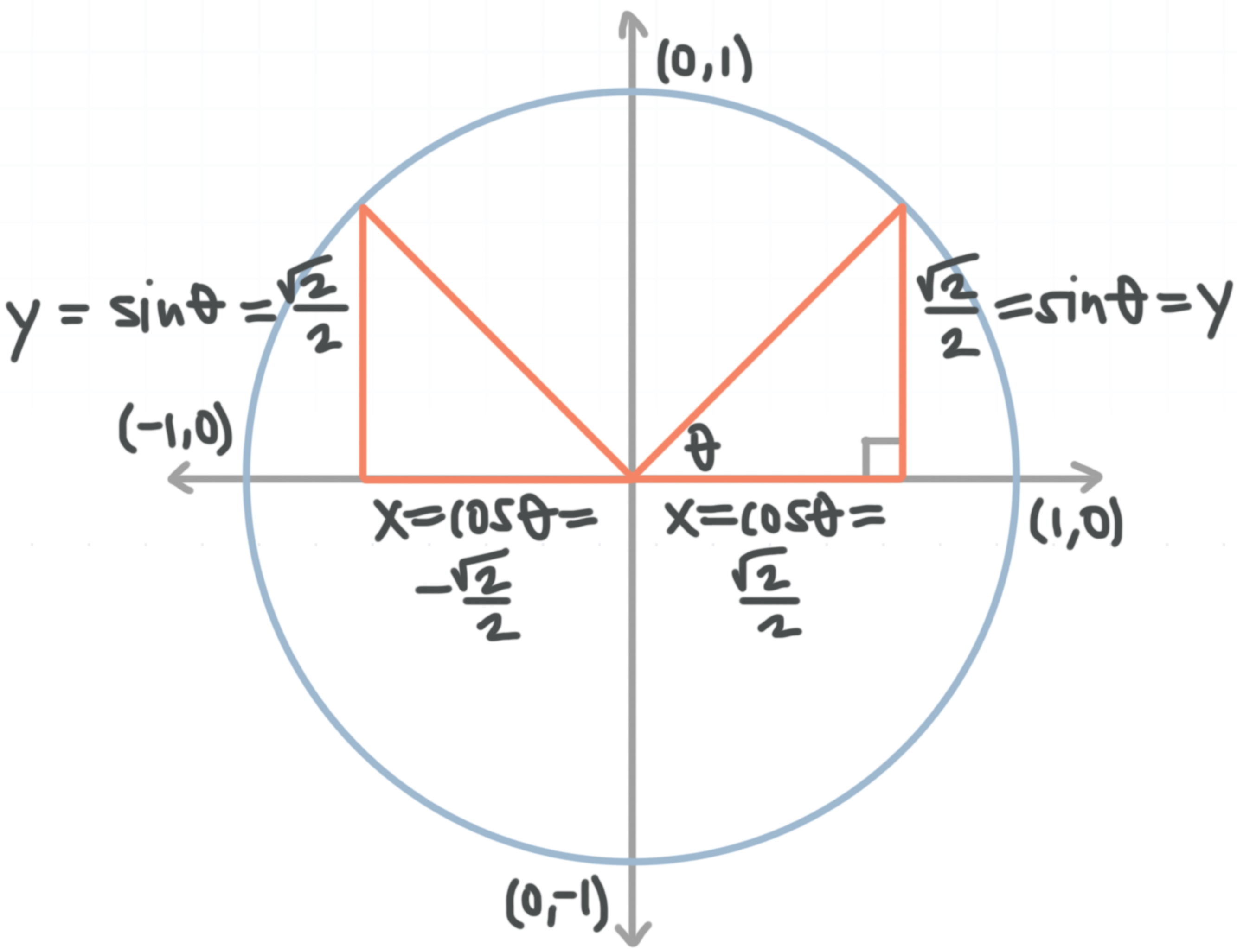
$$y = \sin\theta = \frac{\sqrt{2}}{2}$$

when cosine of the angle (or the length of the horizontal leg of the triangle), is

$$x = \cos\theta = \pm\frac{\sqrt{2}}{2}$$

We can actually see these values in action if we sketch out two triangles in the coordinate plane.



The triangle in the first quadrant shows us the length of the horizontal leg as $x = \cos\theta = \sqrt{2}/2$, and the length of the vertical leg as $y = \sin\theta = \sqrt{2}/2$. And the triangle in the second quadrant shows us the length of the

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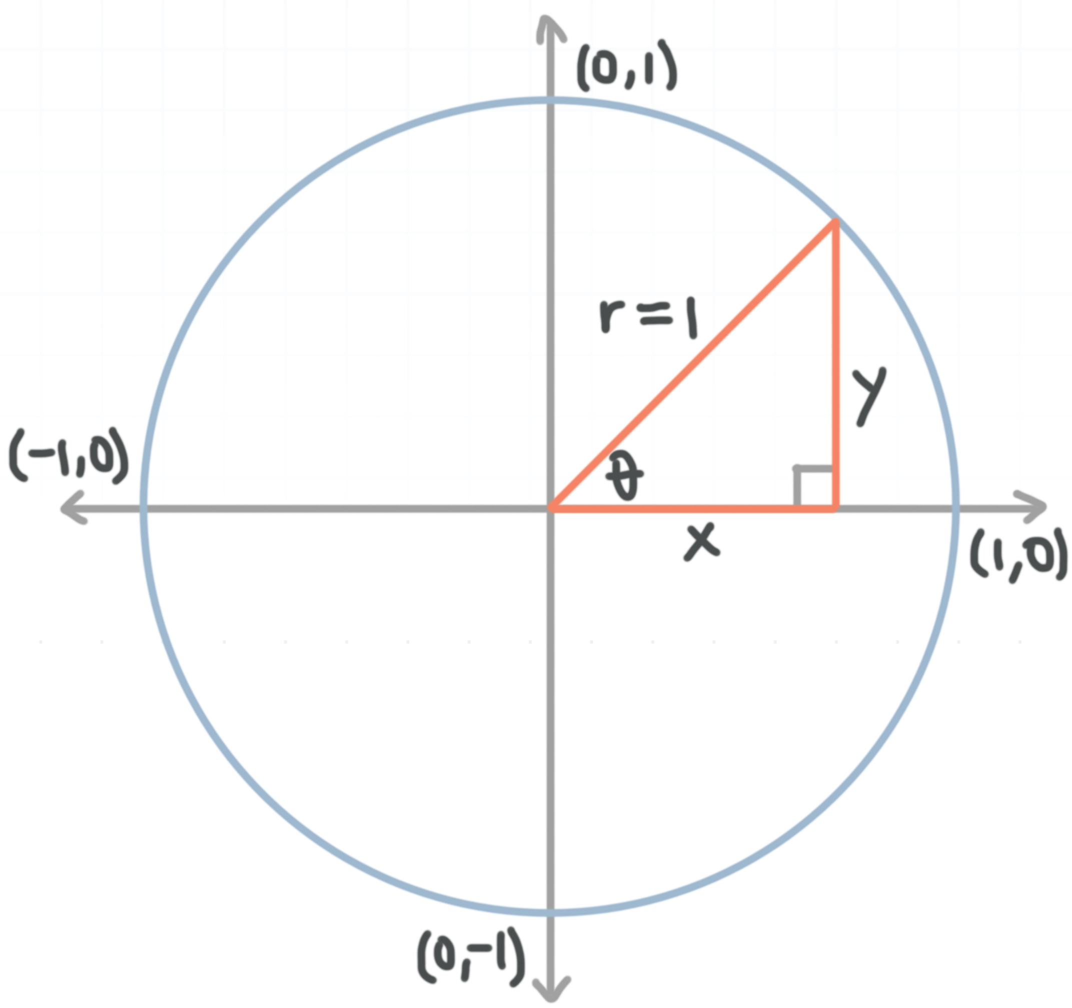
horizontal leg as $x = \cos\theta = -\sqrt{2}/2$, and the length of the vertical leg as $y = \sin\theta = \sqrt{2}/2$.

Signs by quadrant

When we know the value of one of the six trig functions for a particular angle, and we know the quadrant where the angle is located, we'll always be able to find the values of the other five trig functions for the same angle.

Remember what it looked like when we placed the right triangle in the coordinate plane:



Putting the right triangle in the coordinate plane let us define the six trig functions in terms of the horizontal leg x, the vertical leg y, and the hypotenuse as the radius of the circle r.

$$\sin\theta = \frac{y}{r} \qquad\qquad \csc\theta = \frac{r}{y}$$

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$$\cos\theta = \frac{x}{r} \qquad \sec\theta = \frac{r}{x}$$

$$\tan\theta = \frac{y}{x} \qquad \cot\theta = \frac{x}{y}$$

When we define the trig functions this way in terms of x, y, and r, we need to realize that r represents the radius, which means it's a distance, which means its value is always positive. But x and y are signed based on the quadrant of the angle.

When the triangle, and therefore the angle θ, lies in the first quadrant like in the image above, then x and y are positive. But the signs on x and y are different for the other three quadrants.

Quadrant	Sign on x	Sign on y	Sign on r
I	+	+	+
II	-	+	+
III	-	-	+
IV	+	-	+

We can also see this visually when we sketch a triangle in each quadrant.

In quadrant I, x, y, and r are all positive.

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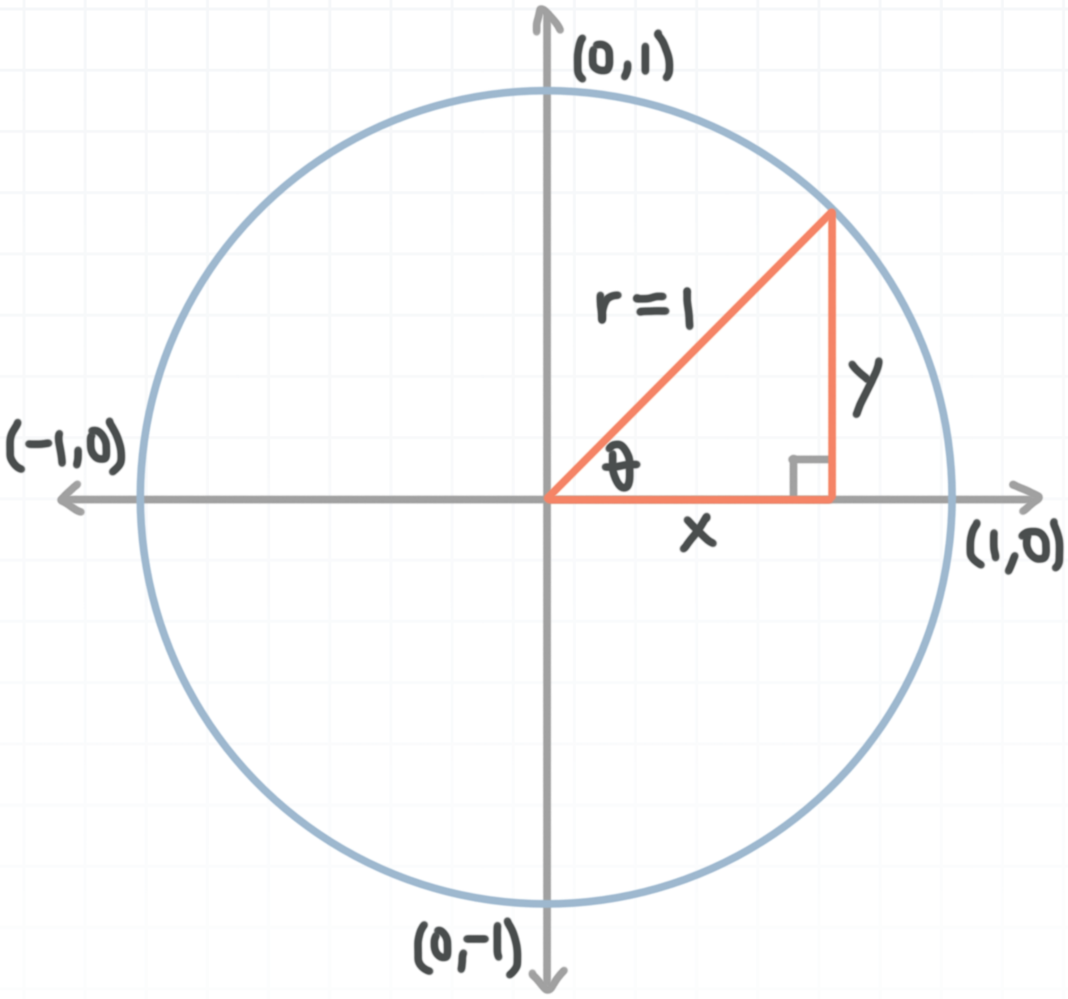
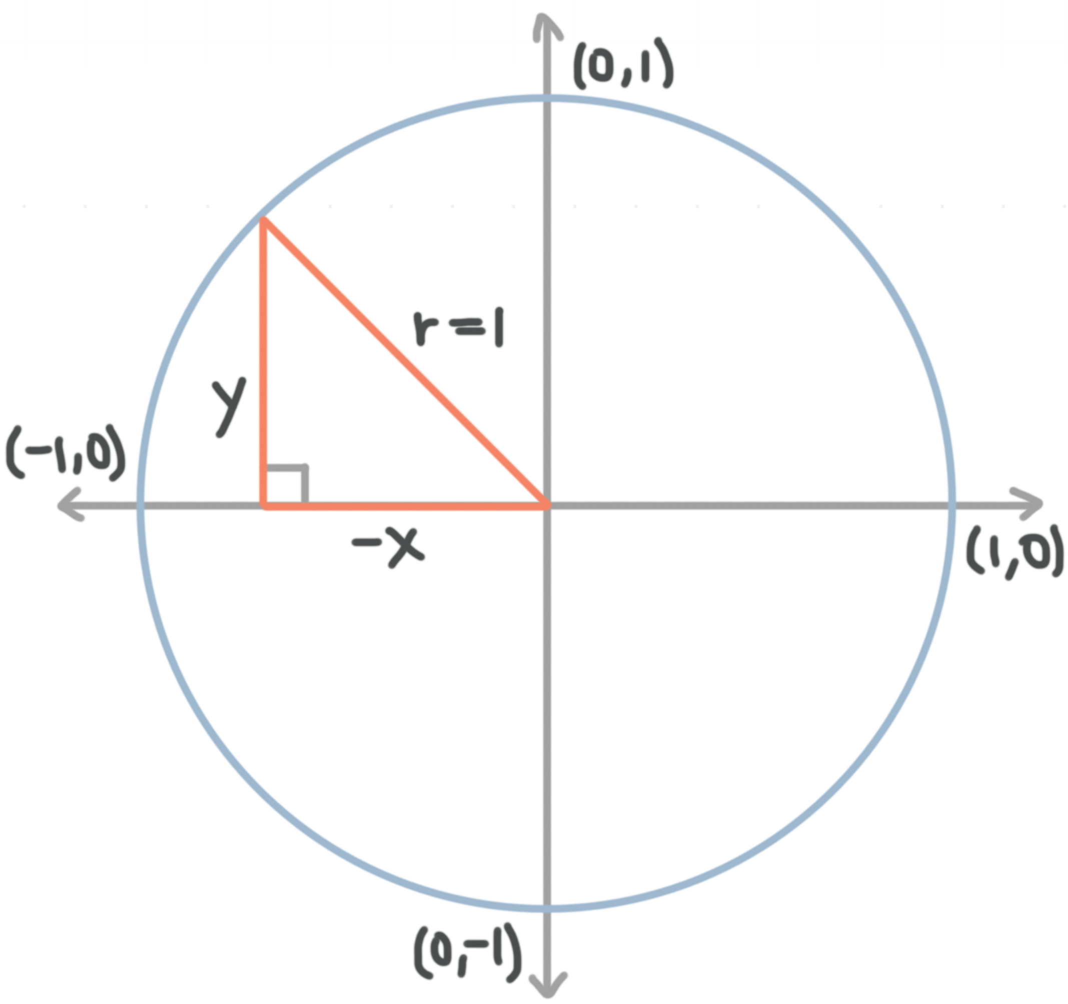


In quadrant II, x is negative, and y and r are positive.



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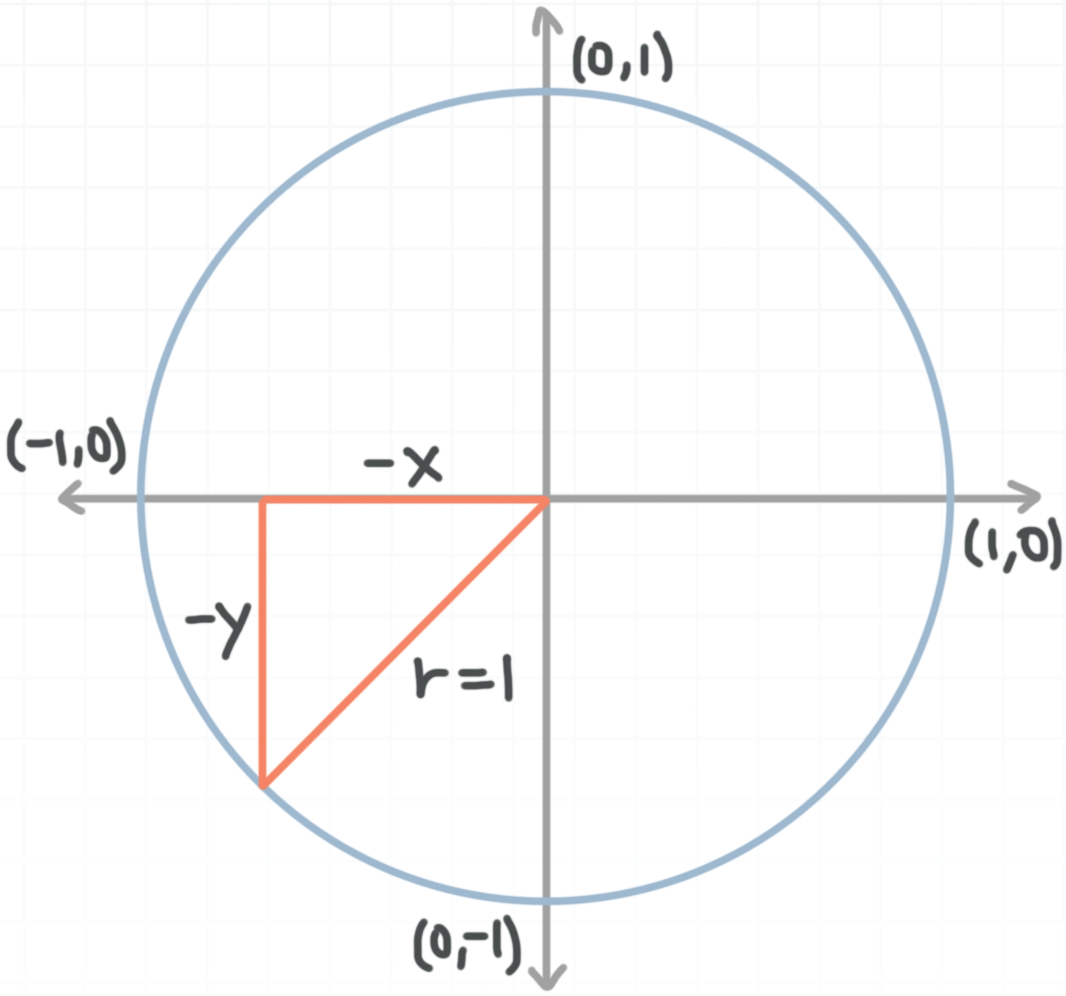
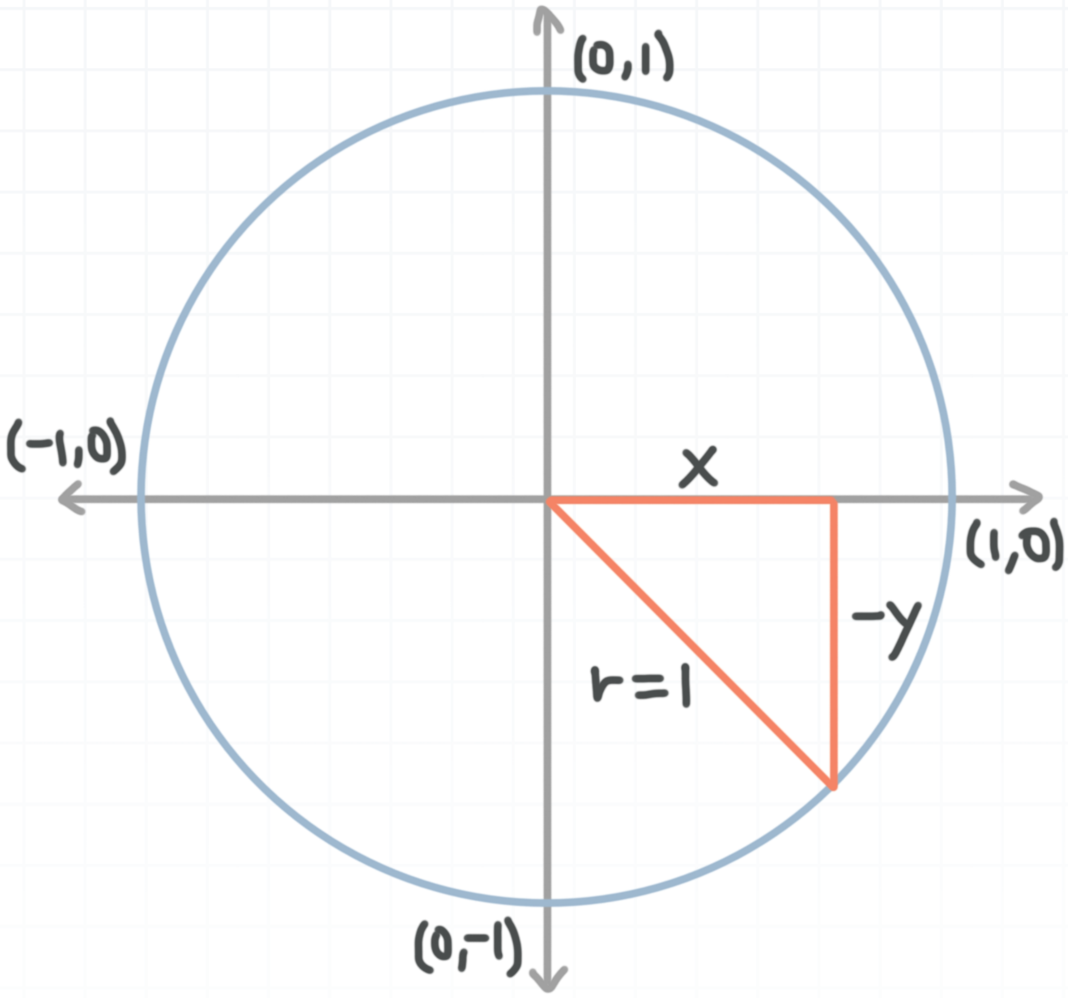
In quadrant III, x and y are negative and r is positive.



In quadrant IV, x is positive, y is negative, and r is positive.



Signs of the trig functions

Now that we understand the signs of x, y, and r in each quadrant, we can plug these signs into formulas in terms of these variables for the six trig functions. For instance, in quadrant I, the signs of the six trig functions are

$$\sin\theta = \frac{y}{r} = \frac{+}{+} = + \qquad \csc\theta = \frac{r}{y} = \frac{+}{+} = +$$

$$\cos\theta = \frac{x}{r} = \frac{+}{+} = + \qquad \sec\theta = \frac{r}{x} = \frac{+}{+} = +$$

$$\tan\theta = \frac{y}{x} = \frac{+}{+} = + \qquad \cot\theta = \frac{x}{y} = \frac{+}{+} = +$$

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In other words, all six of the trig functions are positive in the first quadrant. We can summarize the signs of all six trig functions in each quadrant in another table.

	I	II	III	IV
sin	+	+	-	-
csc	+	+	-	-
cos	+	-	-	+
sec	+	-	-	+
tan	+	-	+	-
cot	+	-	+	-

Notice how the signs match for sine and cosecant, for cosine and secant, and for tangent and cotangent. That makes sense, since each of those pairs are reciprocals of one another (which we saw previously when we talked about the reciprocal identities).

If we sketched out these signs in the coordinate plane, we'd see that all six trig functions are positive in quadrant I, that sine and cosecant are positive in quadrant II, that tangent and cotangent are positive in quadrant III, and that cosine and secant are positive in quadrant IV.

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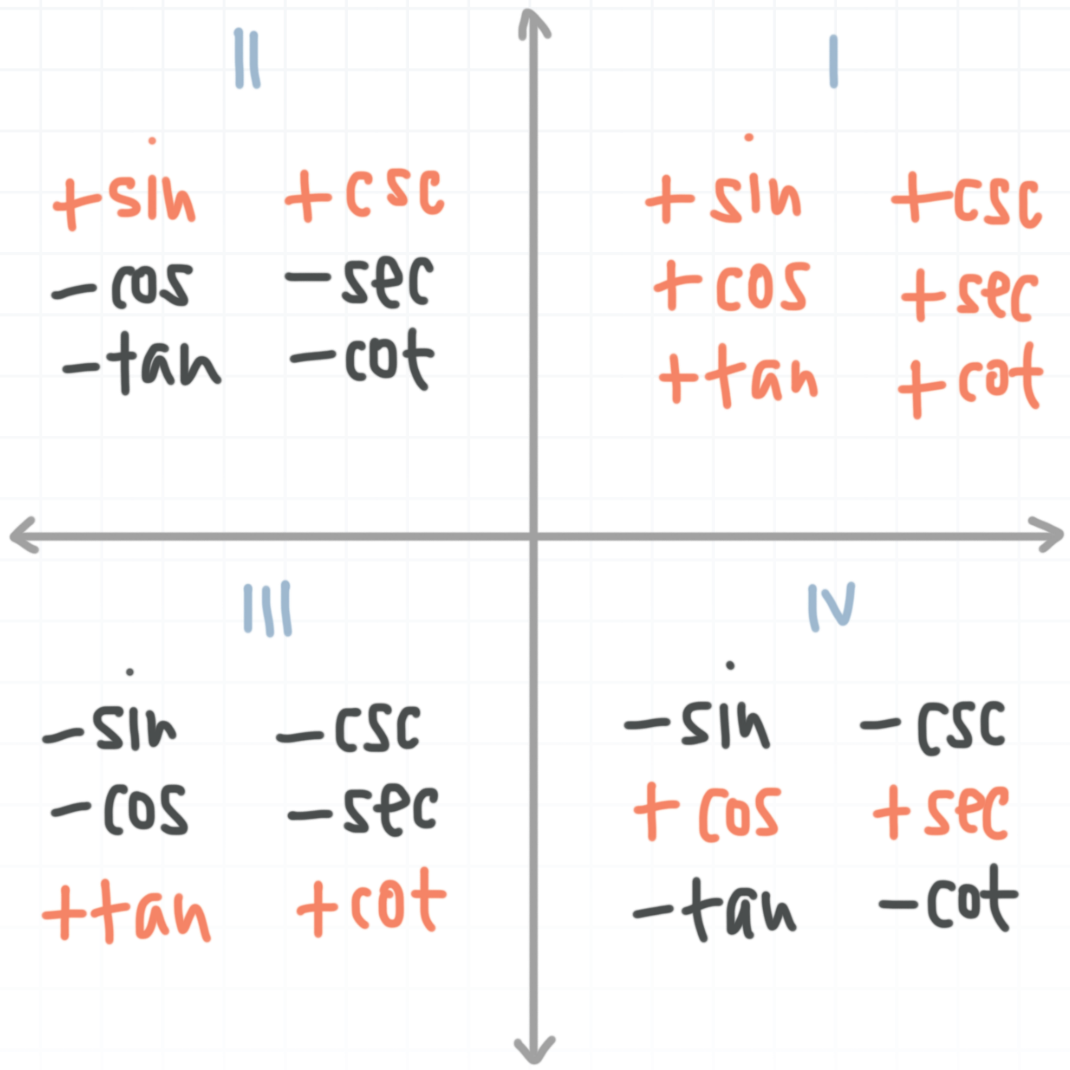

Let's do an example where we use one trig function and the angle's quadrant in order to figure out the values of the other five trig functions.

Example

For an angle θ in the third quadrant with $\sec\theta = -2.53$, find the values of the other five trig functions at θ.

We can immediately use the reciprocal identity to find cosine of the angle.

$$\cos\theta = \frac{1}{\sec\theta} = \frac{1}{-2.53} \approx -0.395$$

If we rewrite the Pythagorean identity for sine and cosine,

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$$\sin^2\theta + \cos^2\theta = 1$$

$$\sin^2\theta = 1 - \cos^2\theta$$

then we can plug the value we just found for $\cos\theta$ into this equation in order to find sine of the angle.

$$\sin^2\theta \approx 1 - (-0.395)^2$$

$$\sin^2\theta \approx 1 - 0.156$$

$$\sin^2\theta \approx 0.844$$

$$\sin\theta \approx \pm\sqrt{0.844}$$

$$\sin\theta \approx \pm 0.919$$

Since the sine of any angle in the third quadrant is negative, we know $\sin\theta \approx -0.919$. Then we can use the reciprocal identity to find cosecant of the angle.

$$\csc\theta = \frac{1}{\sin\theta} \approx \frac{1}{-0.919} \approx -1.09$$

Now we'll plug the values of $\sin\theta$ and $\cos\theta$ into the quotient identity to find tangent of the angle.

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \approx \frac{-0.919}{-0.395} \approx 2.33$$

Use the reciprocal identity to find cotangent of the angle.

$$\cot\theta = \frac{1}{\tan\theta} \approx \frac{1}{2.33} \approx 0.430$$

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Let’s summarize the values we found for all six trig function of the angle θ in the third quadrant whose secant was given as $\sec\theta = -2.53$.

$$\sin\theta \approx -0.919 \qquad \csc\theta \approx -1.09$$

$$\cos\theta \approx -0.395 \qquad \sec\theta = -2.53$$

$$\tan\theta \approx 2.33 \qquad \cot\theta \approx 0.430$$

To double-check ourselves, we can confirm that the sign of each of these trig functions matches the sign table we made earlier.

	I	II	III	IV
sin			-	
csc			-	
cos			-	
sec			-	
tan			+	
cot			+	

Let’s do an example where we’re starting with the tangent function instead of the secant function.

Example

For an angle θ in the fourth quadrant whose tangent is -6.79, find the values of the other five trig functions.

Use the value of $\tan\theta$ in the Pythagorean identity with tangent and secant to find the secant of the angle.

$$1+\tan^2\theta=\sec^2\theta$$

$$1+(-6.79)^2=\sec^2\theta$$

$$1+46.10\approx\sec^2\theta$$

$$\sec^2\theta\approx 47.10$$

$$\sec\theta=\pm\sqrt{47.10}$$

$$\sec\theta=\pm 6.863$$

The secant of any angle in the fourth quadrant is positive, so $\sec\theta=6.86$. Since we have tangent, we can use the reciprocal identity to find the cotangent.

$$\cot\theta=\frac{1}{\tan\theta}$$

$$\cot\theta=\frac{1}{-6.79}\approx-0.15$$

We'll use the value of $\cot\theta$ in the Pythagorean identity with cotangent and cosecant to find the cosecant of the angle.

$$1+\cot^2\theta=\csc^2\theta$$

$$1+(-0.147)^2=\csc^2\theta$$

$$1+0.022\approx\csc^2\theta$$

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$$\csc\theta = \pm\sqrt{1.022}$$

$$\csc\theta = \pm\, 1.01$$

The cosecant of any angle in the fourth quadrant is negative, so $\csc\theta = -\,1.01$.

And now that we have secant and cosecant, we can use the reciprocal identities to find cosine and sine.

$$\sin\theta = \frac{1}{\csc\theta} \approx \frac{1}{-1.01} \approx -\,0.99$$

$$\cos\theta = \frac{1}{\sec\theta} \approx \frac{1}{6.86} \approx 0.15$$

Let's summarize the values we found for all six trig function of the angle θ in the fourth quadrant whose tangent was given as $\tan\theta = -\,6.79$.

$$\sin\theta \approx -\,0.99 \qquad \csc\theta \approx -\,1.01$$

$$\cos\theta \approx 0.15 \qquad \sec\theta \approx 6.86$$

$$\tan\theta = -\,6.79 \qquad \cot\theta \approx -\,0.15$$

To double-check ourselves, we can confirm that the sign of each of these trig functions matches the sign table we made earlier.

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	I	II	III	IV
sin				-
csc				-
cos				+
sec				+
tan				-
cot				-

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When the trig functions are undefined

In the last lesson, we saw that we could find the values of all six trig functions, with only the value of one of them and the quadrant of the angle.

But not every one of the six trig functions is necessarily defined at every angle. The angles where we'll run into trouble will always be the quadrantal angles, where either sine or cosine of the angle is 0.

When sine and cosine are 0

Remember that we learned previously that $\cos\theta$ represents the x-value in a coordinate point, and that $\sin\theta$ represents a y-value in a coordinate point. That means that, for a circle centered at the origin with radius 1, $\sin\theta = 0$ at $(1,0)$ and $(-1,0)$, which are the two points where the circle intersects the horizontal axis.

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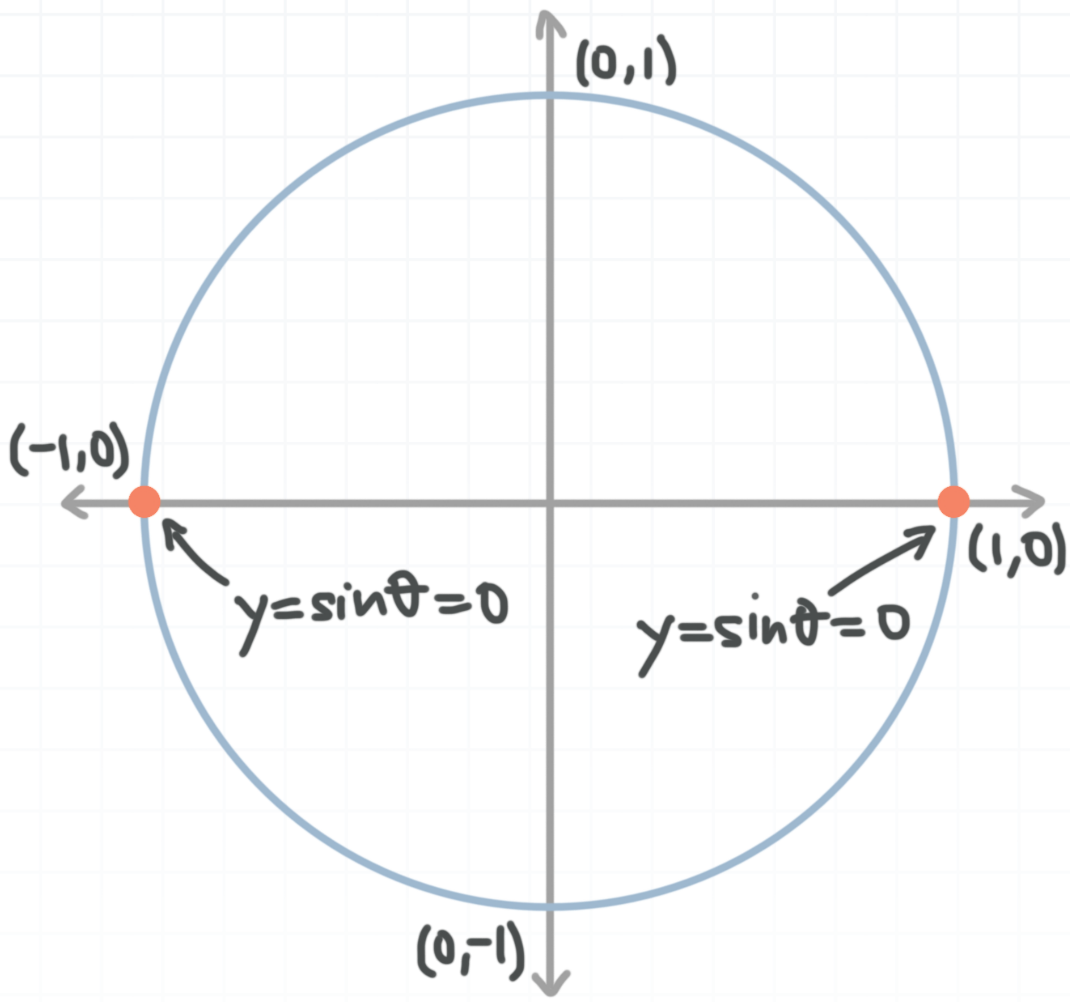

Likewise, $\cos\theta = 0$ at $(0,1)$ and $(0, -1)$, which are the two points where the circle intersects the vertical axis.

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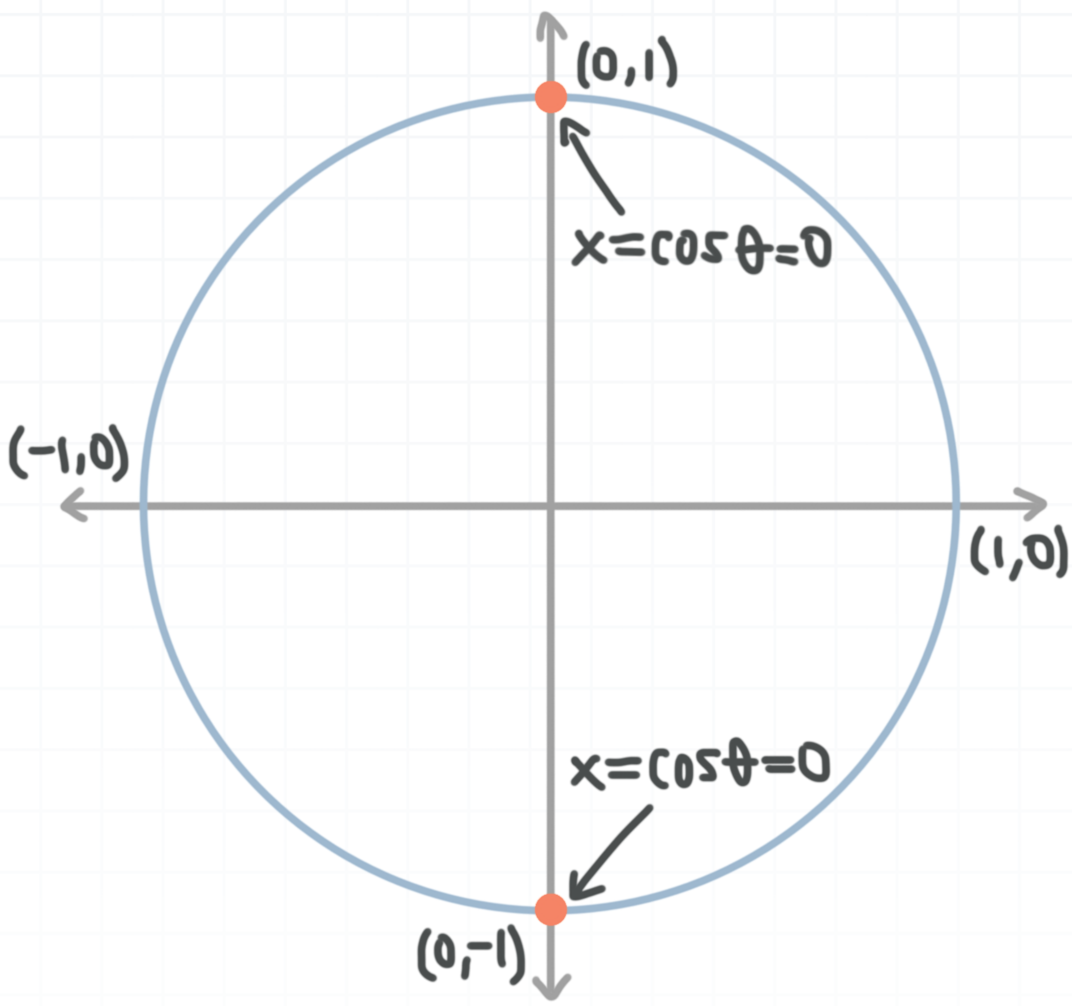

Remember also that, for angles sketched in standard position, these points along the major axes represent angles in degrees of $\theta = 0^\circ,\ 90^\circ,\ 180^\circ,\ 270^\circ,\ 360^\circ$ or angles in radians of $\theta = 0,\ \pi/2,\ \pi,\ 3\pi/2,\ 2\pi$, and all other angles that are coterminal with those sets.

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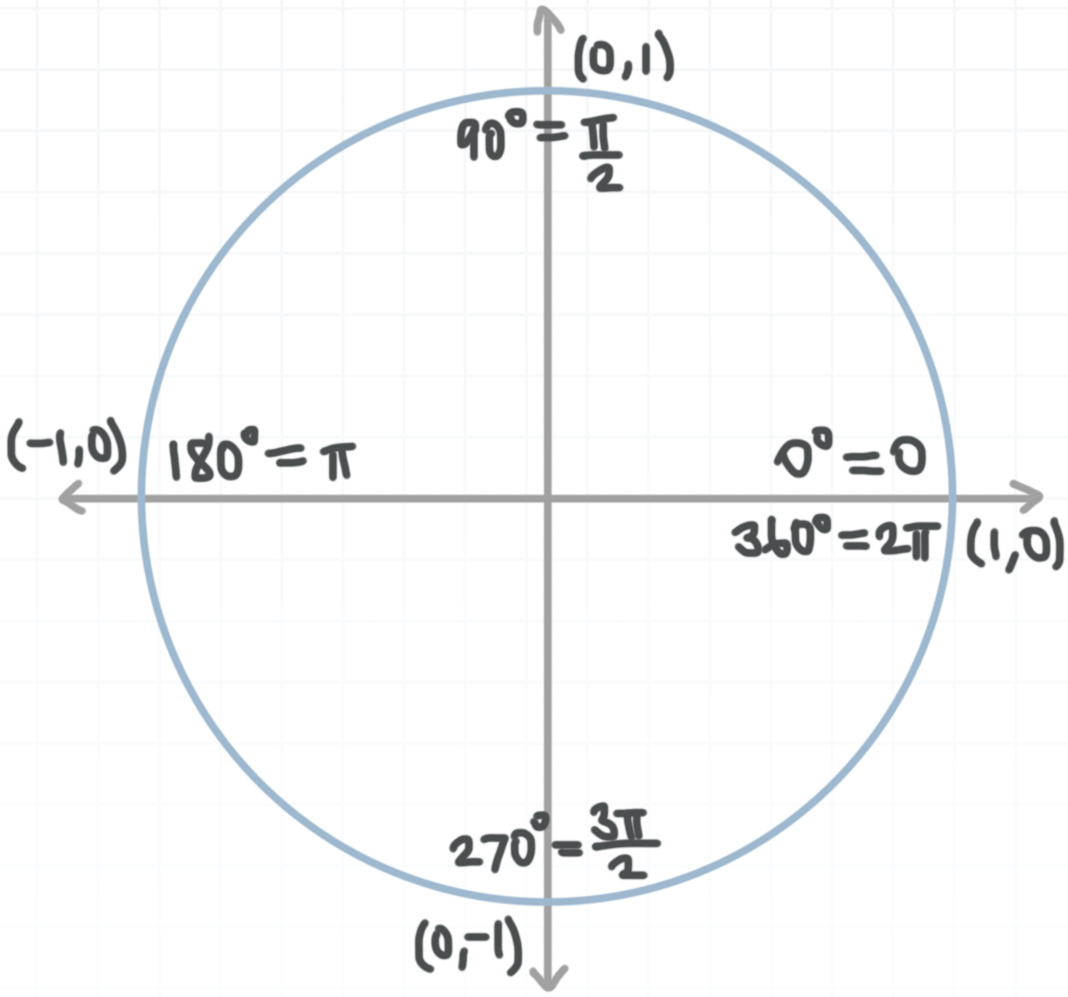

In other words, we'll only have undefined trig functions at the quadrantal angles.

When the trig functions are undefined

The only question now is, which trig functions are undefined at which quadrantal angles? Well, if we remember the reciprocal identities for cosecant and secant, as well as the quotient identities for tangent and cotangent,

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

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and we remember the fact that the value of a denominator can never be 0, then we can say that

- $\csc\theta$ and $\cot\theta$ are undefined when $\sin\theta = 0$

- $\sec\theta$ and $\tan\theta$ are undefined when $\cos\theta = 0$

Since $\sin\theta = 0$ only along the x-axis, and $\cos\theta = 0$ only along the y-axis, we could also write this as

- $\csc\theta$ and $\cot\theta$ are undefined for angles on the x-axis (anything coterminal with $\theta = 0$ or $\theta = \pi$)

- $\sec\theta$ and $\tan\theta$ are undefined for angles on the y-axis (anything coterminal with $\theta = \pi/2$ or $\theta = 3\pi/2$)

Let's summarize these in a table.

	sin	**csc**	**cos**	**sec**	**tan**	**cot**
0°=0	0	Undefined	1	1	0	Undefined
90°=π/2	1	1	0	Undefined	Undefined	0
180°=π	0	Undefined	-1	-1	0	Undefined
270°=3π/2	-1	-1	0	Undefined	Undefined	0
360°=2π	0	Undefined	1	1	0	Undefined

Let's look at an example where we're asked to figure out which trig functions are undefined for a particular angle.

Example

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Find the values of all six trig functions at $\theta = 3\pi/2$, and say whether or not any of them are undefined at that angle.

The angle $\theta = 3\pi/2$ falls on the negative y-axis. So in a circle centered at the origin with radius 1,

$$y = \sin\frac{3\pi}{2} = -1$$

$$x = \cos\frac{3\pi}{2} = 0$$

Use the reciprocal identities to find cosecant and secant of the angle.

$$\csc\frac{3\pi}{2} = \frac{1}{\sin\frac{3\pi}{2}} = \frac{1}{-1} = -1$$

$$\sec\frac{3\pi}{2} = \frac{1}{\cos\frac{3\pi}{2}} = \frac{1}{0}$$

Use the quotient identities to find tangent and cotangent of the angle.

$$\tan\frac{3\pi}{2} = \frac{\sin\frac{3\pi}{2}}{\cos\frac{3\pi}{2}} = \frac{-1}{0}$$

$$\cot\frac{3\pi}{2} = \frac{\cos\frac{3\pi}{2}}{\sin\frac{3\pi}{2}} = \frac{0}{-1} = 0$$

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So to summarize, because we got a 0 value in the denominator of the secant and tangent values, we know these two trig functions are undefined at $\theta = 3\pi/2$. The other four trig functions are defined.

$$\sin\theta = -1 \qquad \csc\theta = -1$$

$$\cos\theta = 0 \qquad \sec\theta = \text{undefined}$$

$$\tan\theta = \text{undefined} \qquad \cot\theta = 0$$

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The unit circle

Up to this point, we've talked a lot about the value of the trig functions as they relate to a circle with radius 1 that's centered at the origin. We've just been using the circle to define the trig functions, but now we really want to focus on defining the circle itself.

The unit circle

In fact, this special circle with center at the origin and radius 1 has a special name: it's called the **unit circle**. We call it the unit circle because its radius is 1 unit long. Of course, that means the unit circle intersects the positive x-axis at $(1,0)$, the positive y-axis at $(0,1)$, the negative x-axis at $(-1,0)$, and the negative y-axis at $(0,-1)$.

Remember that the standard equation of a circle centered at (h,k) with radius r is $(x-h)^2+(y-k)^2=r^2$. Because the unit circle is centered at the origin $(0,0)$ and has radius $r=1$, its equation is

$$(x-0)^2+(y-0)^2=1^2$$

$$x^2+y^2=1$$

We already know the values of the six trig functions along the circle at the quadrantal angles along the major axes:

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	sin	csc	cos	sec	tan	cot
0°=0	0	Undefined	1	1	0	Undefined
90°=π/2	1	1	0	Undefined	Undefined	0
180°=π	0	Undefined	-1	-1	0	Undefined
270°=3π/2	-1	-1	0	Undefined	Undefined	0
360°=2π	0	Undefined	1	1	0	Undefined

But there are lots of other special values along the circle. Because of how often we'll use the unit circle throughout Trigonometry and beyond into other more advanced math classes, the more familiar we can be with the values around this circle, the better.

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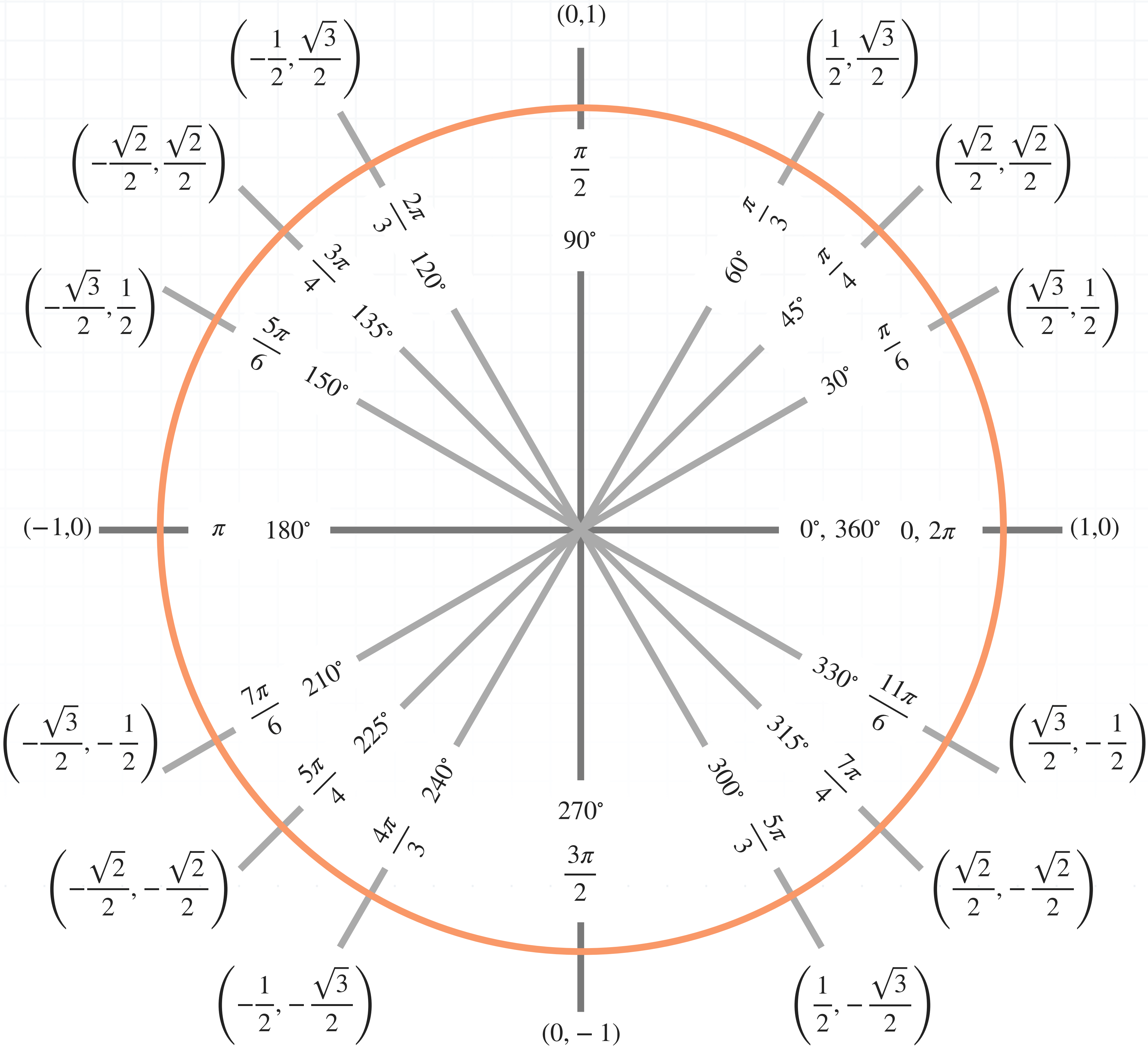

Three sets of information in the unit circle

Notice that the circle really includes three sets of information:

1. Angles in degrees

2. Angles in radians

3. Coordinate points

The degree angles show us that the special points along the circle are at 30° increments: $0^\circ, 30^\circ, 60^\circ, 90^\circ, \dots$, and 45° increments: $0^\circ, 45^\circ, 90^\circ, 135^\circ, \dots$.

The radian angles show us that the special points along the circle are at $\pi/6$ increments: $0, \pi/6, \pi/3, \pi/2, \dots$, and $\pi/4$ increments: $0, \pi/4, \pi/2, 3\pi/4, \dots$.

The coordinate points also have a pattern. Of course, we already know the points at the quadrantal angles, but all of the other points are fractions with a 2 in the denominator. And if we look in the first quadrant we can see that the numerators of the x-values are $\sqrt{3}, \sqrt{2}, \sqrt{1} = 1$, and that the numerators of the y-values are the opposite: $\sqrt{1} = 1, \sqrt{2}, \sqrt{3}$. The other three quadrants follow this same pattern.

Remembering these patterns can help us know these values without having to actually memorize the full circle.

Finding the values of all six trig functions

Remember that the sine function of an angle is represented by the y-value of the coordinate point, and that the cosine function of an angle is represented by the x-value. Which means we can think of each coordinate point along this circle as $(x, y) = (\cos\theta, \sin\theta)$.

For example, at the angle $\theta = 30^\circ = \pi/6$, the coordinate point along the unit circle is

$$\left(\frac{\sqrt{3}}{2},\frac{1}{2}\right)$$

Which means we know right away that, at $\theta = 30^\circ = \pi/6$,

$$\sin 30^\circ = \sin\frac{\pi}{6} = \frac{1}{2}$$

$$\cos 30^\circ = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$$

Of course, now that we have sine and cosine, we can use the quotient identity to find tangent of $\theta = 30^\circ = \pi/6$,

$$\tan 30^\circ = \tan\frac{\pi}{6} = \frac{\sin\frac{\pi}{6}}{\cos\frac{\pi}{6}} = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{1}{2}\cdot\frac{2}{\sqrt{3}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

and then we can use the reciprocal identities to find cosecant, secant, and cotangent of $\theta = 30^\circ = \pi/6$.

$$\csc 30^\circ = \csc\frac{\pi}{6} = \frac{1}{\sin\frac{\pi}{6}} = \frac{1}{\frac{1}{2}} = \frac{2}{1} = 2$$

$$\sec 30^\circ = \sec\frac{\pi}{6} = \frac{1}{\cos\frac{\pi}{6}} = \frac{1}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$

$$\cot 30^\circ = \cot\frac{\pi}{6} = \frac{1}{\tan\frac{\pi}{6}} = \frac{1}{\frac{\sqrt{3}}{3}} = \frac{3}{\sqrt{3}} = \sqrt{3}$$

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Let's do an example at another angle in a different quadrant.

Example

Use the unit circle to find the values of the six trig functions at $\theta = 5\pi/4$.

Looking at the unit circle, we know that sine of $\theta = 5\pi/4$ is the y-value of the coordinate point at that angle, and that cosine of $\theta = 5\pi/4$ is the x-value of the coordinate point at that angle.

$$\sin\frac{5\pi}{4} = -\frac{\sqrt{2}}{2}$$

$$\cos\frac{5\pi}{4} = -\frac{\sqrt{2}}{2}$$

Use the quotient identity to find tangent.

$$\tan\frac{5\pi}{4} = \frac{\sin\frac{5\pi}{4}}{\cos\frac{5\pi}{4}} = \frac{-\frac{\sqrt{2}}{2}}{-\frac{\sqrt{2}}{2}} = 1$$

Then use the reciprocal identities to find cosecant as the reciprocal of sine, secant as the reciprocal of cosine, and cotangent as the reciprocal of tangent.

$$\csc\frac{5\pi}{4} = \frac{1}{\sin\frac{5\pi}{4}} = \frac{1}{-\frac{\sqrt{2}}{2}} = -\sqrt{2}$$

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$$\sec \frac{5\pi}{4} = \frac{1}{\cos \frac{5\pi}{4}} = \frac{1}{-\frac{\sqrt{2}}{2}} = -\sqrt{2}$$

$$\cot \frac{5\pi}{4} = \frac{1}{\tan \frac{5\pi}{4}} = \frac{1}{1} = 1$$

Let's do another example with an angle at a $\pi/6$-increment.

Example

Find the values of all six trig functions at $\theta = \pi/3$.

From the unit circle, we know that

$$\sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

$$\cos \frac{\pi}{3} = \frac{1}{2}$$

The quotient identity for tangent gives

$$\tan \frac{\pi}{3} = \frac{\sin \frac{\pi}{3}}{\cos \frac{\pi}{3}} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \frac{\sqrt{3}}{2} \cdot \frac{2}{1} = \sqrt{3}$$

and the reciprocal identities for cosecant, secant, and cotangent give

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$$\csc\frac{\pi}{3}=\frac{1}{\sin\frac{\pi}{3}}=\frac{1}{\frac{\sqrt{3}}{2}}=1\cdot\frac{2}{\sqrt{3}}=\frac{2}{\sqrt{3}}=\frac{2\sqrt{3}}{3}$$

$$\sec\frac{\pi}{3}=\frac{1}{\cos\frac{\pi}{3}}=\frac{1}{\frac{1}{2}}=2$$

$$\cot\frac{\pi}{3}=\frac{1}{\tan\frac{\pi}{3}}=\frac{1}{\sqrt{3}}=\frac{\sqrt{3}}{3}$$

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Negative angles and angles more than one rotation

In the last lesson, we looked at how to find the values of all six trig functions at some positive angle along the unit circle.

In this lesson, we want to look at the values of trig functions for negative angles in the unit circle, and at the values of trig functions for angles outside of one full rotation, either positive or negative.

The key here is to realize that the value of a trig function is the same for any set of coterminal angles. For instance, $\pi/4$ and $9\pi/4$ are coterminal angles. Which means the value of sine is the same at both angles, the value of cosine is the same at both angles, the value of tangent is the same at both angles, etc.

$$\sin\frac{\pi}{4} = \sin\frac{9\pi}{4} = \frac{\sqrt{2}}{2} \qquad \csc\frac{\pi}{4} = \csc\frac{9\pi}{4} = \sqrt{2}$$

$$\cos\frac{\pi}{4} = \cos\frac{9\pi}{4} = \frac{\sqrt{2}}{2} \qquad \sec\frac{\pi}{4} = \sec\frac{9\pi}{4} = \sqrt{2}$$

$$\tan\frac{\pi}{4} = \tan\frac{9\pi}{4} = 1 \qquad \cot\frac{\pi}{4} = \cot\frac{9\pi}{4} = 1$$

Because the value of the trig functions are equivalent for coterminal angles, if we're given a negative angle, or any angle (positive or negative) outside of one full rotation, we just need to find the positive angle inside $[0^\circ,360^\circ)$ in degrees or $[0,2\pi)$ in radians that's coterminal with it, and then find the values of the trig functions at that coterminal angle.

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Trig functions at negative angles

Let's think about the angle -30°. If we start along the positive horizontal axis in the unit circle, and rotate 30° in the *clockwise* direction (negative direction), we'll arrive at 330°. At that angle,

$$\sin(-30^\circ) = \sin(330^\circ) = -\frac{1}{2}$$

$$\cos(-30^\circ) = \cos(330^\circ) = \frac{\sqrt{3}}{2}$$

Given the sine and cosine of $\theta = -30^\circ$, we can use reciprocal identities to find cosecant and secant of the angle,

$$\sin(-30^\circ) = \sin(330^\circ) = -\frac{1}{2} \qquad \csc(-30^\circ) = \csc(330^\circ) = -2$$

$$\cos(-30^\circ) = \cos(330^\circ) = \frac{\sqrt{3}}{2} \qquad \sec(-30^\circ) = \sec(330^\circ) = \frac{2\sqrt{3}}{3}$$

and then use the quotient identities to find tangent and cotangent.

$$\sin(-30^\circ) = \sin(330^\circ) = -\frac{1}{2} \qquad \csc(-30^\circ) = \csc(330^\circ) = -2$$

$$\cos(-30^\circ) = \cos(330^\circ) = \frac{\sqrt{3}}{2} \qquad \sec(-30^\circ) = \sec(330^\circ) = \frac{2\sqrt{3}}{3}$$

$$\tan(-30^\circ) = \tan(330^\circ) = -\frac{\sqrt{3}}{3} \qquad \cot(-30^\circ) = \cot(330^\circ) = -\sqrt{3}$$

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What this shows is that the values of the six trig functions are equivalent at -30° and 330°. And of course, the reason this is true is because -30° and 330° are coterminal, and they're coterminal because they differ by 360°.

What we want to be able to do now is what we just did with -30°: we want to be able to take any angle *outside* of $[0^\circ,360^\circ)$ or $[0,2\pi)$ and find its coterminal angle that falls *inside* $[0^\circ,360^\circ)$ or $[0,2\pi)$. That way, we'll be able to use our unit circle to find values of the six trig functions at that coterminal angle.

And as we've seen by converting -30° to 330°, we can find an angle within $[0^\circ,360^\circ)$ or $[0,2\pi)$ that's coterminal with a negative angle simply by adding multiples of 360° or 2π to the negative angle until we have an angle within $[0^\circ,360^\circ)$ or $[0,2\pi)$.

For instance, given -300°, we can add 360° to find a coterminal angle.

$$-300^\circ + 360^\circ$$

$$60^\circ$$

Or given -420°, we'll add 360° twice to find a coterminal angle. Adding 360° only once doesn't put us within $[0^\circ,360^\circ)$, which is why we add 360° twice.

$$-420^\circ + 2(360^\circ)$$

$$-420^\circ + 720^\circ$$

$$300^\circ$$

Let's do another example with an angle that's more than one full rotation.

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Example

Find the angle in the interval $[0^\circ,360^\circ)$ that's coterminal with -539°.

Let $\theta=-539^\circ$, and let α be the angle that lies in the interval $[0^\circ,360^\circ)$ and is coterminal with θ. To find α, let's add 360° to $\theta=-539^\circ$ until we get to an angle that lies in the interval $[0^\circ,360^\circ)$.

$$-539^\circ+360^\circ=-179^\circ$$

$$-179^\circ+360^\circ=181^\circ$$

This 181° angle lies within $[0^\circ,360^\circ)$ and is coterminal with $\theta=-539^\circ$.

Trig functions at angles of more than one full rotation

In the process of talking about negative angles, we've already looked at some angles that are more than one full rotation in the negative direction.

But none of the angles we've tackled so far have been very many rotations away from the interval $[0^\circ,360^\circ)$ or $[0,2\pi)$. This won't always be the case. If we're further away from one full rotation, it helps to start by dividing the given angle by either 360° or 2π.

Let's work through an example.

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Example

Find the angle in the interval $[0,2\pi)$ that's coterminal with $\theta = -61\pi/4$.

To find the number of full rotations included in $\theta = -61\pi/4$, we'll divide the angle by 2π.

$$\frac{-\frac{61\pi}{4}}{2\pi}$$

$$-\frac{61\pi}{4} \cdot \frac{1}{2\pi}$$

$$-\frac{61\pi}{8\pi}$$

$$-7.625$$

So $\theta = -61\pi/4$ is 7 full rotations in the negative direction, and then an additional 0.625 of one more rotation in the negative direction. So to find a coterminal angle, we'll get rid of the 7 full rotations by adding $7(2\pi)$ to the angle.

$$-\frac{61\pi}{4} + 7(2\pi)$$

$$-\frac{61\pi}{4} + 14\pi$$

$$-\frac{61\pi}{4} + \frac{56\pi}{4}$$

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$$-\frac{5\pi}{4}$$

Now we have an angle that's less than one full rotation, but we'd still like to find a positive coterminal angle that's less than one full rotation. So we'll add 2π one more time.

$$-\frac{5\pi}{4}+2\pi$$

$$-\frac{5\pi}{4}+\frac{8\pi}{4}$$

$$\frac{3\pi}{4}$$

Therefore, we can say that $3\pi/4$ is coterminal with $\theta=-61\pi/4$.

Evaluating trig functions at these angles

As we've said, the value of a trig function is equivalent for coterminal angles. So, referencing the last example, we found that $\theta=-61\pi/4$ is coterminal with the angle $3\pi/4$.

Therefore, if we were asked to find the values of all six trig functions at $\theta=-61\pi/4$, we'd simply find the coterminal angle $3\pi/4$, and then evaluate the trig functions at $3\pi/4$, by pulling values from the unit circle. The values we get will be the same as the values of the trig functions at $\theta=-61\pi/4$.

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$$\sin\frac{3\pi}{4}=\sin\left(-\frac{61\pi}{4}\right)=\frac{\sqrt{2}}{2} \qquad \csc\frac{3\pi}{4}=\csc\left(-\frac{61\pi}{4}\right)=\sqrt{2}$$

$$\cos\frac{3\pi}{4}=\cos\left(-\frac{61\pi}{4}\right)=-\frac{\sqrt{2}}{2} \qquad \sec\frac{3\pi}{4}=\sec\left(-\frac{61\pi}{4}\right)=-\sqrt{2}$$

$$\tan\frac{3\pi}{4}=\tan\left(-\frac{61\pi}{4}\right)=-1 \qquad \cot\frac{3\pi}{4}=\cot\left(-\frac{61\pi}{4}\right)=-1$$

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Coterminal angles in a particular interval

We've been looking at finding the coterminal angle within the interval that defines one positive full rotation, namely, $[0^\circ, 360^\circ)$ or $[0, 2\pi)$. But we can actually look for coterminal angles in any interval.

Usually we'll do this by setting up an inequality. To do that, we need to first realize that, given an angle θ, an angle α will be coterminal with θ when $\alpha = \theta + n(360^\circ)$ for any n. In radians, we'd of course express this as $\alpha = \theta + n(2\pi)$.

Let's do an example with an angle in degrees so that we can see how this process works.

Example

Find the angle in the interval $(-900^\circ, -540^\circ]$ that's coterminal with 247°.

The interval $(-900^\circ, -540^\circ]$ is a full 360° rotation. Notice how, because we have a parenthesis around the -900° and a bracket around the -540°, it means that the angle -900° exactly isn't included in the interval, but the angle -540° exactly *is* included.

We'll let $\theta = 247^\circ$, and then we'll say that α is the coterminal angle that lies within $(-900^\circ, -540^\circ]$. Then we can say

$$-900^\circ < \alpha \leq -540^\circ$$

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But since α is coterminal with θ, we substitute $\alpha = \theta + n(360^\circ)$ into the inequality.

$$-900^\circ < \theta + n(360^\circ) \leq -540^\circ$$

$$-900^\circ < 247^\circ + n(360^\circ) \leq -540^\circ$$

$$-1{,}147^\circ < n(360^\circ) \leq -787^\circ$$

$$-3.19 < n \leq -2.19$$

Remember, n has to be an integer, which means $n = -3$. And therefore, to find α, we'll substitute $n = -3$ into $\alpha = \theta + n(360^\circ)$.

$$\alpha = 247^\circ + (-3)(360^\circ)$$

$$\alpha = 247^\circ - 1{,}080^\circ$$

$$\alpha = -833^\circ$$

Let's do another example, but this time with a radian angle.

Example

Find the angle in the interval $[-\pi, \pi)$ that's coterminal with $56\pi/3$.

We'll let $\theta = 56\pi/3$ and α be the angle within $[-\pi, \pi)$ that's coterminal with θ. We'll use $\alpha = \theta + n(2\pi)$ and solve for the value of n that makes α lie in that interval.

$$-\pi \leq \alpha < \pi$$

$$-\pi \leq \theta + n(2\pi) < \pi$$

$$-\pi \leq \frac{56\pi}{3} + n(2\pi) < \pi$$

$$-\frac{59\pi}{3} \leq n(2\pi) < -\frac{53\pi}{3}$$

$$-9.83 \leq n < -8.83$$

Because n has to be an integer, we know $n = -9$. To find α, we'll substitute $n = -9$ into $\theta + n(2\pi)$.

$$\alpha = \frac{56\pi}{3} + (-9)(2\pi)$$

$$\alpha = \frac{56\pi}{3} - \frac{54\pi}{3}$$

$$\alpha = \frac{2\pi}{3}$$

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Reference angles

We've talked about coterminal angles a couple of times now. First, we looked at the set of coterminal angles in general for an angle θ. And we said that the set of all possible coterminal angles of θ could be defined by

$$\alpha = \theta + n(360^\circ) \text{ or } \alpha = \theta + n(2\pi)$$

for any integer n. Second, we looked at how to find a coterminal angle in a particular interval. It's true that we usually want the coterminal angle that's both positive, and within one full rotation $[0^\circ, 360^\circ)$ or $[0, 2\pi)$, but other times we'll want to find a coterminal angle within some other specified angle interval.

The reference angle

And now that we're familiar with coterminal angles, and how to find them for both positive and negative angles, and for angles that are within one full rotation and outside of one full rotation, we want to turn our attention toward a closely related concept: the reference angle.

A **reference angle** for an angle θ in standard position is the positive acute angle formed by the x-axis and the terminal side of θ. Because reference angles are always positive and always acute, that means reference angles will always measure between 0° and 90°, or between 0 and $\pi/2$ radians.

If β is the reference angle for θ, we can sketch examples of θ and β in each quadrant.

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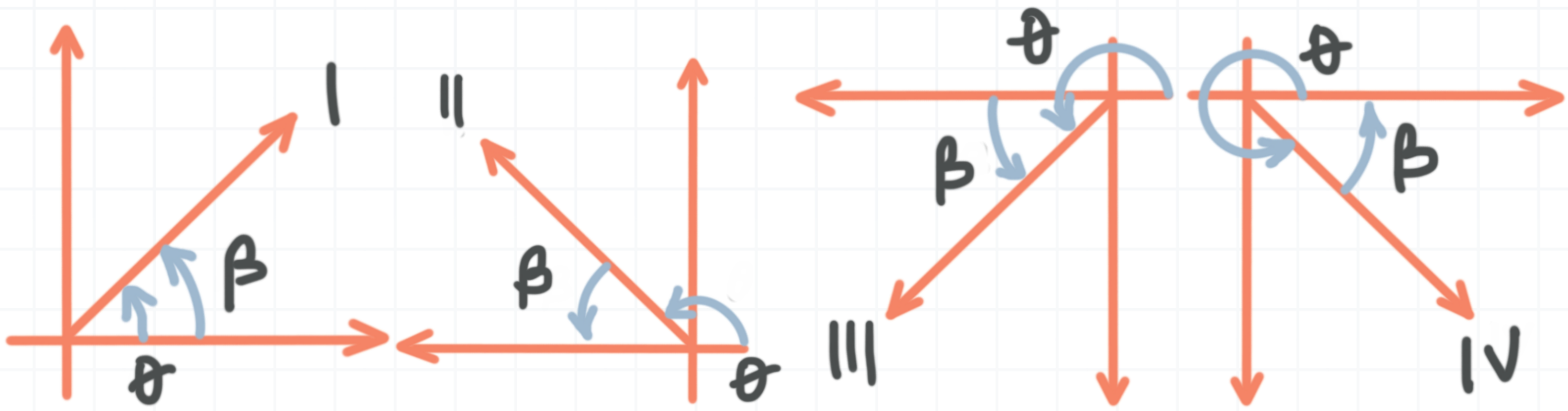

As we can see from the figure, the measure of the reference angle will depend on the quadrant of the angle. For example, for an angle in quadrant II, $\beta = 180^\circ - \theta$ in degrees or $\beta = \pi - \theta$ in radians. But for an angle in quadrant III, $\beta = \theta - 180^\circ$ in degrees or $\beta = \theta - \pi$ in radians.

Let's summarize these reference angle formulas in a table.

θ's quadrant	**β in radians**	**β in degrees**
I	$\beta = \theta$	$\beta = \theta$
II	$\beta = \pi - \theta$	$\beta = 180^\circ - \theta$
III	$\beta = \theta - \pi$	$\beta = \theta - 180^\circ$
IV	$\beta = 2\pi - \theta$	$\beta = 360^\circ - \theta$

Notice that all of the θ angles in the figure are positive angles (they rotate in the positive, counterclockwise direction). In order to use the equations in the table above to find the reference angle, we need θ to be positive. If we have an angle θ that's negative, then we need to first find the positive coterminal angle, and then use that positive angle to find the reference angle.

Let's do an example where we find a reference angle in radians.

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Example

What is the reference angle for $\theta = 2\pi/3$?

The angle $\theta = 2\pi/3$ is in the second quadrant, which means the reference angle β is

$$\beta = \pi - \theta$$

$$\beta = \pi - \frac{2\pi}{3}$$

$$\beta = \frac{3\pi}{3} - \frac{2\pi}{3}$$

$$\beta = \frac{\pi}{3}$$

Let's do an example with an angle in degrees.

Example

What is the reference angle for $\theta = -750^\circ$?

The angle $\theta = -750^\circ$ is two full rotations of 360° in the negative direction, and then an extra 30° in the negative direction, which means the angle is

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coterminal with $\theta = -30^\circ$. We want to convert this to a positive angle, which we can do by adding the negative angle to 360°.

$$\alpha = 360^\circ + (-30^\circ)$$

$$\alpha = 360^\circ - 30^\circ$$

$$\alpha = 330^\circ$$

So the angle $\theta = -750^\circ$ is coterminal with $\theta = -30^\circ$, which is coterminal with $\alpha = 330^\circ$. Now that we have a positive coterminal angle, we can find the reference angle.

Since $\alpha = 330^\circ$ is in the fourth quadrant, the reference angle β is

$$\beta = 360^\circ - \alpha$$

$$\beta = 360^\circ - 330^\circ$$

$$\beta = 30^\circ$$

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Symmetry across axes

When we first looked at the unit circle, we considered angles in, and points along the circle in all four quadrants. With so many values, trying to memorize the entire circle could be difficult.

But it turns out that if we only know the values in the first quadrant, then we can actually use symmetry and reference angles to build out all the values in the other three quadrants.

The unit circle with reference angles and symmetry

If we start with a unit circle that includes only the values in the first quadrant,

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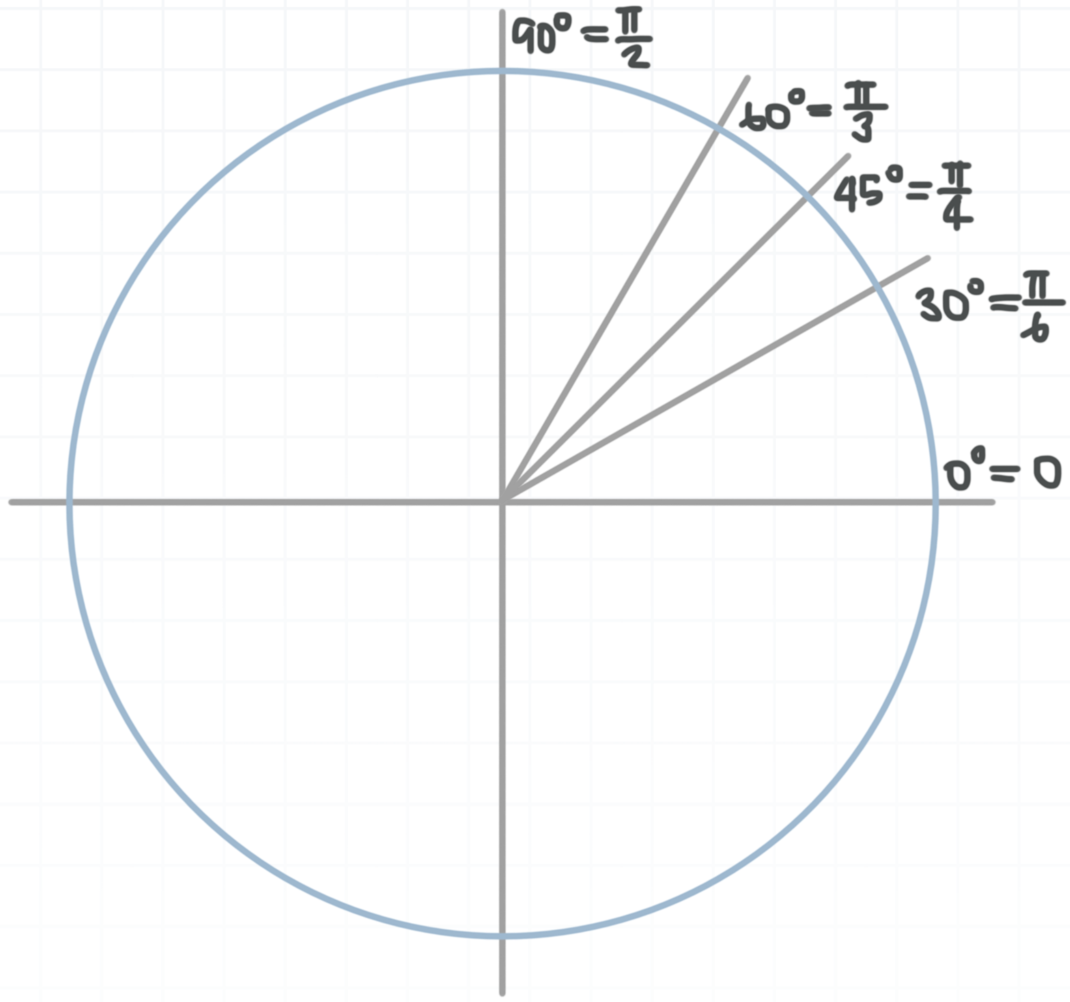
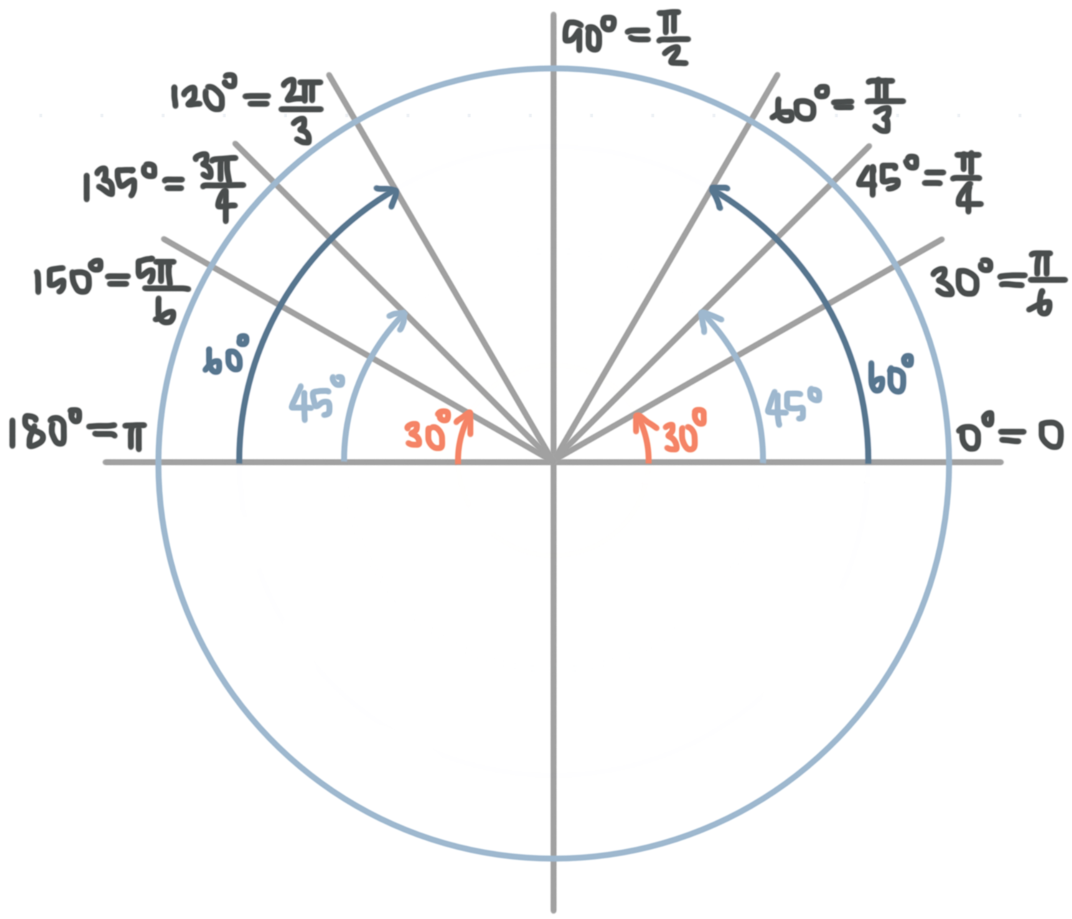


and then we add angles in the second quadrant that have the same reference angles as those in the first quadrant,



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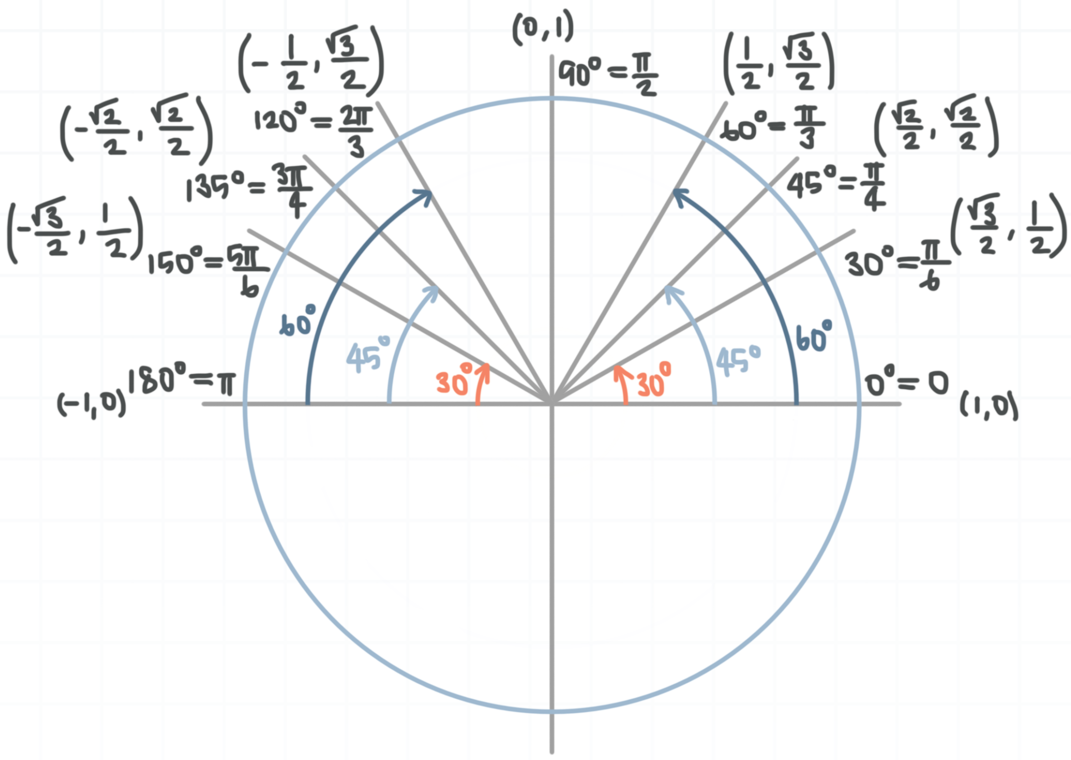
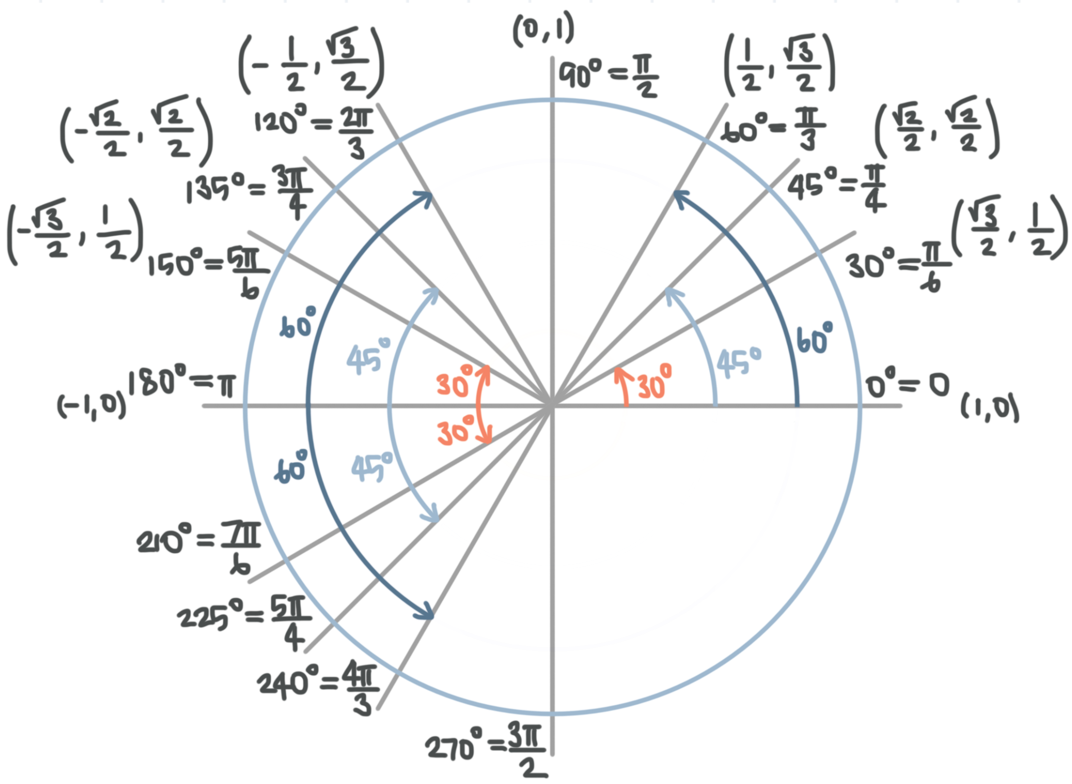
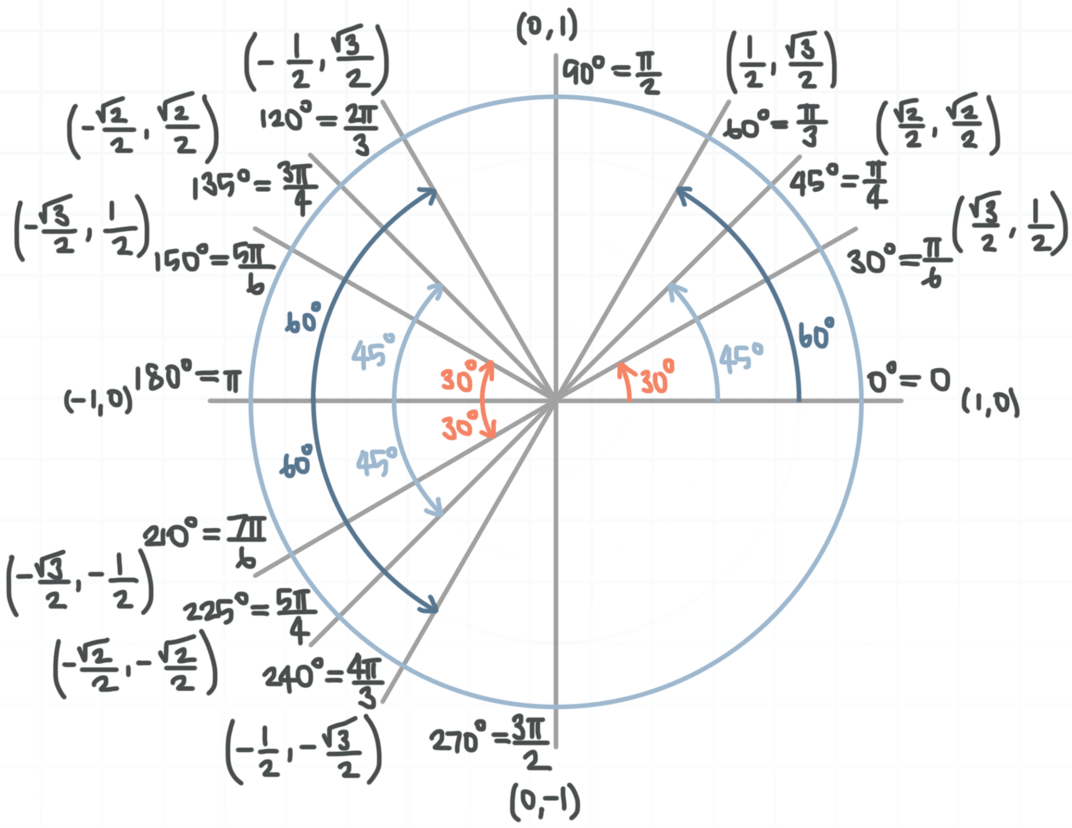
then we know the coordinate points in the second quadrant by symmetry with the angles in the first quadrant. The y-values stay the same and the x-values get multiplied by -1.



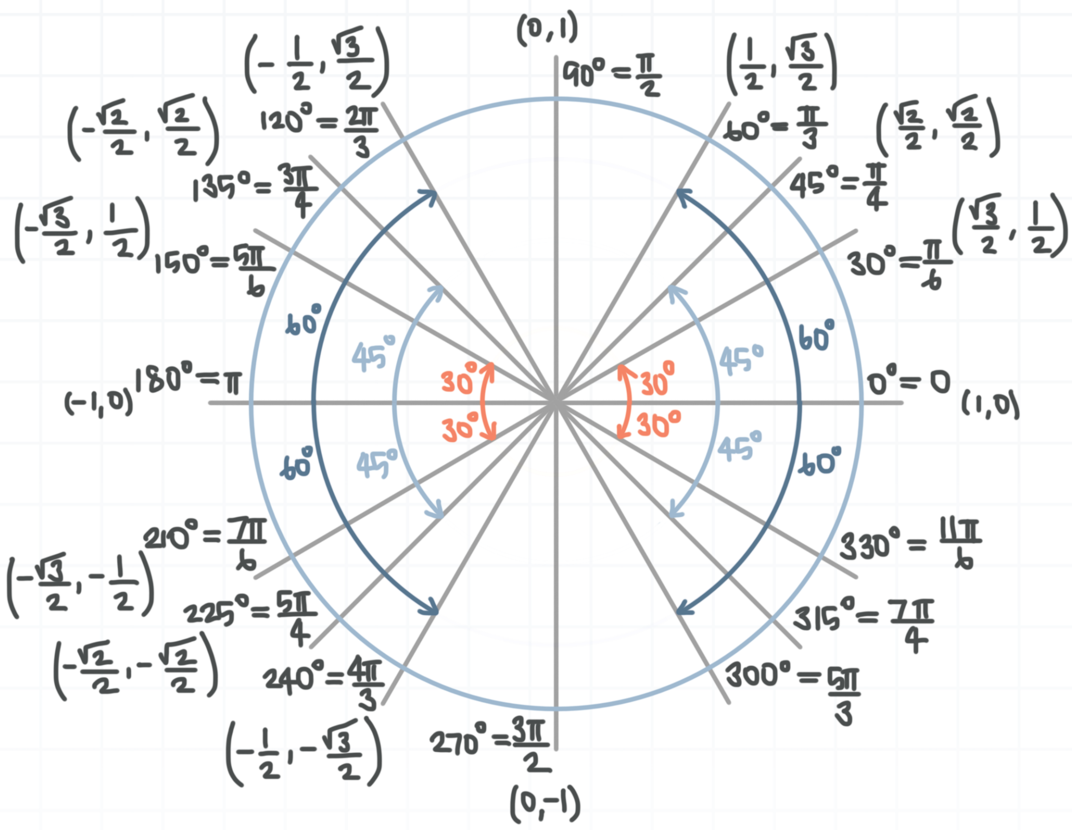
Once we have the second quadrant complete, we add angles into the third quadrant that have the same reference angles as those in the second quadrant.



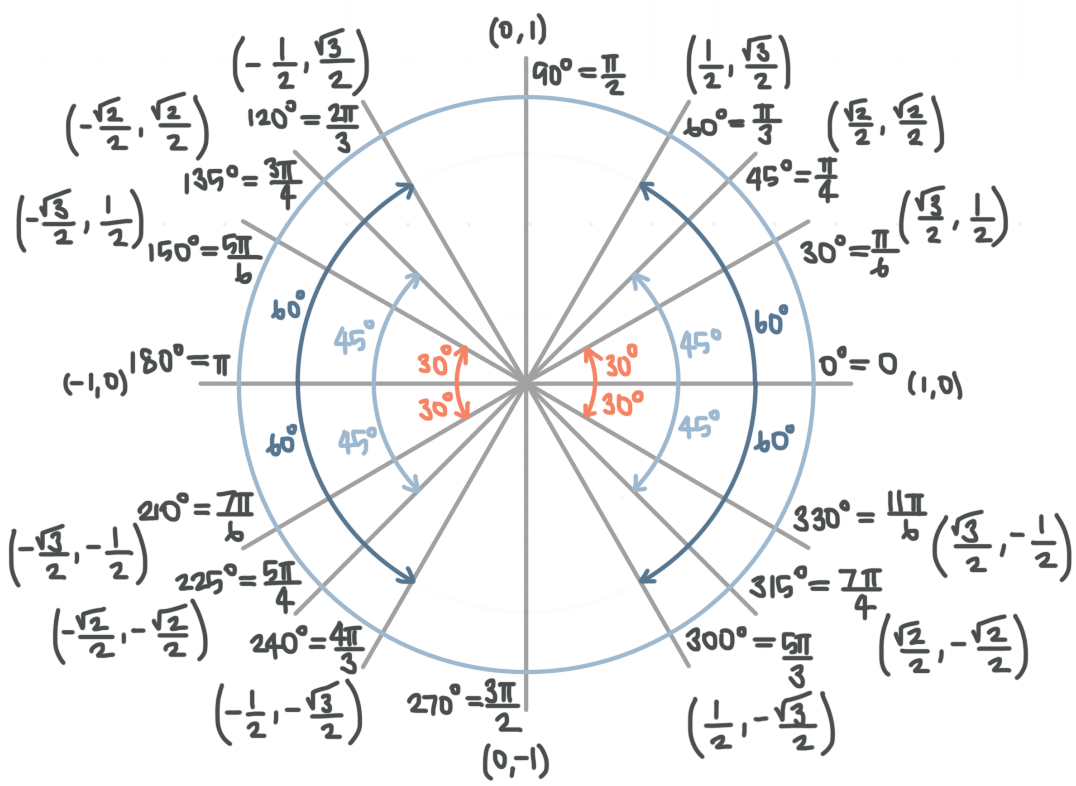
Then we know the coordinate points in the third quadrant by symmetry with the angles in the second quadrant. The x-values stay the same and the y-values get multiplied by -1.



Finally, once we have the third quadrant complete, we add angles into the fourth quadrant that have the same reference angles as those in the third quadrant.



Then we know the coordinate points in the fourth quadrant by symmetry with the angles in the third quadrant. The y-values stay the same and the x-values get multiplied by -1.



Symmetry for the values of the trig functions

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The question now is, “What do these reflections tell us about the values of the trig functions?”

Well, because sine of an angle is given by the y-value of the coordinate point, when we reflect an angle across the y-axis, the value of y won’t change, which means the value of sine won’t change. The value of x (or cosine) just gets multiplied by -1.

Similarly, because cosine of an angle is given by the x-value of the coordinate point, when we reflect an angle across the x-axis, the value of x won’t change, which means the value of cosine won’t change. The value of y (or sine) just gets multiplied by -1.

If we reflect over both axes (over the origin), then both the y-and x-values, and therefore both the sine and cosine values, get multiplied by -1.

Let’s work through an example where we need to use symmetry to find sine and cosine for a particular angle.

Example

If θ is an angle in the first quadrant such that $\cos\theta = 4/7$, find $\cos(\theta + \pi)$ and $\sin(\theta + \pi)$.

Remember that $\pi = 180^\circ$, so adding π radians to an angle is like adding 180°. Furthermore, adding that value to the angle is equivalent to reflecting over the y-axis and then reflecting over the x-axis.

So since θ is in the first quadrant, $\theta + \pi$ is in the third quadrant.

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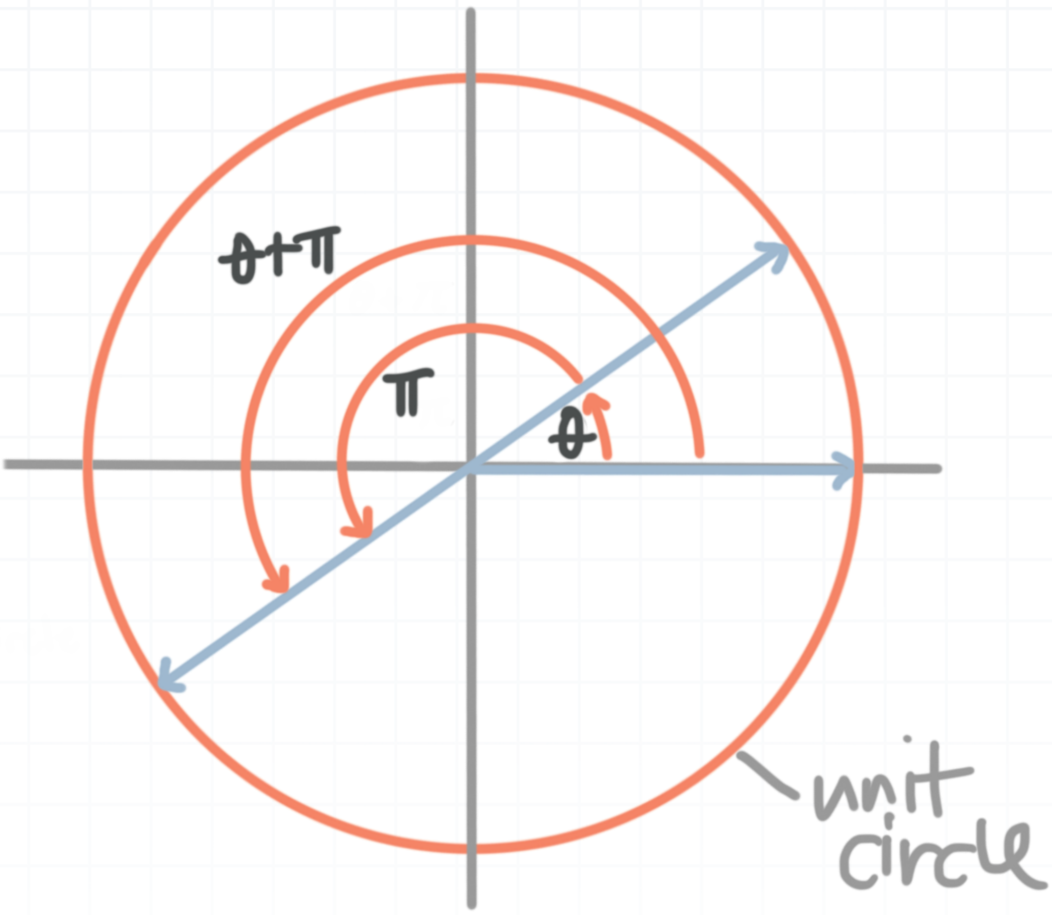

By symmetry, we know that adding π radians to θ puts us at the same x-value, but negative, because we're in the third quadrant instead of the first. In other words, adding π reflects the angle across the y-axis and then across the x-axis. So since $\cos\theta = 4/7$, we know that

$$\cos(\theta + \pi) = -\frac{4}{7}$$

To find $\sin\theta$, we'll substitute $\cos\theta = 4/7$ into the Pythagorean identity for sine and cosine.

$$\sin^2\theta + \cos^2\theta = 1$$

$$\sin^2\theta + \left(\frac{4}{7}\right)^2 = 1$$

$$\sin^2\theta + \frac{16}{49} = 1$$

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$$\sin^2\theta = 1 - \frac{16}{49}$$

$$\sin^2\theta = \frac{49}{49} - \frac{16}{49}$$

$$\sin^2\theta = \frac{33}{49}$$

$$\sin\theta = \pm\sqrt{\frac{33}{49}}$$

$$\sin\theta = \pm\frac{\sqrt{33}}{7}$$

Since θ is in the first quadrant, $\sin\theta$ is positive, so

$$\sin\theta = \frac{\sqrt{33}}{7}$$

By symmetry, we know that adding π radians to θ puts us at the same y -value, but negative, because we're in the third quadrant instead of the first. In other words, adding π reflects the angle across the y-axis and then across the x-axis. So

$$\sin(\theta + \pi) = -\frac{\sqrt{33}}{7}$$

Therefore, based on only the fact that $\cos\theta = 4/7$, and using a little symmetry, we know that

$$\cos(\theta + \pi) = -\frac{4}{7}$$

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$$\sin(\theta + \pi) = -\frac{\sqrt{33}}{7}$$

Let's do another example.

Example

Let θ be an angle in the fourth quadrant such that $\sin\theta = -0.381$, and determine the values of $\sin(\theta - 3\pi)$ and $\cos(\theta - 3\pi)$.

First find $\cos\theta$ by plugging $\sin\theta = -0.381$ into the Pythagorean identity for sine and cosine.

$$\sin^2\theta + \cos^2\theta = 1$$

$$(-0.381)^2 + \cos^2\theta = 1$$

$$\cos^2\theta = 1 - (-0.381)^2$$

$$\cos^2\theta \approx 1 - 0.145$$

$$\cos^2\theta \approx 0.855$$

$$\cos\theta \approx \pm 0.925$$

We've been given that θ is in the fourth quadrant, so $\cos\theta$ is positive and $\cos\theta \approx 0.925$. Since θ is in the fourth quadrant, we know x is positive and y is negative, so $\cos\theta \approx 0.925$ and $\sin\theta = -0.381$.

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When we subtract 3π from the angle θ, that takes us one full rotation of -2π in the negative direction, and brings us back to the fourth quadrant, and then takes us another $-\pi$ rotation into the second quadrant. In the second quadrant, and by symmetry, our positive x-value becomes a negative x-value, and our negative y-value becomes a positive y-value. Therefore

$$\cos(\theta - 3\pi) \approx -0.925$$

$$\sin(\theta - 3\pi) = 0.381$$

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Even-odd identities

In this lesson, we're going to formalize all of the symmetry we learned in the last lesson into trig identities. We've already learned about the reciprocal, quotient, and Pythagorean identities, and now we're going to introduce the even-odd identities.

Even and odd functions

As a reminder from Algebra, every function can be classified as an even function, an odd function, or neither an even nor odd function.

In **even functions**, we replace x everywhere in the function with $-x$, and the function doesn't change. The function $f(x) = x^2$ would be an example, because we can replace x with $-x$ and get

$$f(x) = (-x)^2$$

$$f(x) = x^2$$

After we simplified, we got back to the original function, which means the function didn't change when we substituted $-x$. Therefore, it's an even function.

In **odd functions**, we replace x everywhere with $-x$ and end up with the original function multiplied by -1. The function $f(x) = x^3$ is an example, because we can replace x with $-x$ and get

$$f(x) = (-x)^3$$

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$$f(x) = -x^3$$

$$f(x) = -(x^3)$$

After we simplified, we got back to the original function multiplied by -1. Therefore, it's an odd function.

In other words, the equation $f(-x) = f(x)$ holds true for even functions, and the equation $f(-x) = -f(x)$ holds true for odd functions.

The even-odd identities

Remember we said in the last lesson when we talked about symmetry that the value of x, and therefore the value of cosine, stayed the same when we reflected across the x-axis. That means we were essentially substituting $-\theta$ for θ, and coming out with the same cosine value. We could write that as

$$\cos(-\theta) = \cos\theta$$

Notice how this matches our rule for even functions, $f(-x) = f(x)$, which tells us that cosine is an even function.

We also said that the value of y, and therefore the value of sine, was multiplied by -1 when we reflected across the x-axis. That means we were essentially substituting $-\theta$ for θ, and coming out with the same sine value, but multiplied by -1. We could write that as

$$\sin(-\theta) = -\sin\theta$$

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Notice how this matches our rule for odd functions, $f(-x) = -f(x)$, which tells us that sine is an odd function.

Then the quotient identity for tangent tells us that $\tan(-\theta)$ will be

$$\tan(-\theta) = \frac{\sin(-\theta)}{\cos(-\theta)} = \frac{-\sin\theta}{\cos\theta} = -\tan\theta$$

Then we use the reciprocal identities to find cosecant, secant, and cotangent at $(-\theta)$, and we get the full set of even-odd identities.

$$\sin(-\theta) = -\sin\theta \qquad \csc(-\theta) = -\csc\theta$$

$$\cos(-\theta) = \cos\theta \qquad \sec(-\theta) = \sec\theta$$

$$\tan(-\theta) = -\tan\theta \qquad \cot(-\theta) = -\cot\theta$$

Looking at these identities and comparing them to the even function $f(-x) = f(x)$ and the odd function $f(-x) = -f(x)$, we can say that

- cosine and secant are even functions
- sine, cosecant, tangent, and cotangent are odd functions

With these identities in hand, let's look at how we can use them to find values along the unit circle.

Example

Find the values of $\cos(-\pi/3)$ and $\sin(-\pi/3)$.

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Since the cosine function is even,

$$\cos\left(-\frac{\pi}{3}\right)=\cos\frac{\pi}{3}=\frac{1}{2}$$

Since the sine function is odd,

$$\sin\left(-\frac{\pi}{3}\right)=-\sin\frac{\pi}{3}=-\frac{\sqrt{3}}{2}$$

Let's do an example with an angle outside of one full rotation.

Example

Find the values of $\cos(-27\pi/4)$ and $\sin(-27\pi/4)$.

Since the cosine function is even and the sine function is odd, we can say

$$\cos\left(-\frac{27\pi}{4}\right)=\cos\frac{27\pi}{4}$$

$$\sin\left(-\frac{27\pi}{4}\right)=-\sin\frac{27\pi}{4}$$

Then to find the value of cosine of this positive angle, we'll get the coterminal angle for $27\pi/4$ by dividing it by 2π.

$$\frac{\frac{27\pi}{4}}{2\pi}$$

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$$\frac{27\pi}{4} \cdot \frac{1}{2\pi}$$

$$\frac{27\pi}{8\pi}$$

$$3.375$$

So $27\pi/4$ is three full rotations, plus 0.375 of another rotation. Three full rotations is $3(2\pi) = 6\pi$, so we'll rewrite the angle as

$$\frac{27\pi}{4}$$

$$\frac{24\pi}{4} + \frac{3\pi}{4}$$

$$6\pi + \frac{3\pi}{4}$$

Therefore, $27\pi/4$ is coterminal with $3\pi/4$, so

$$\cos\frac{27\pi}{4} = \cos\frac{3\pi}{4} = -\frac{\sqrt{2}}{2}$$

$$-\sin\frac{27\pi}{4} = -\sin\frac{3\pi}{4} = -\frac{\sqrt{2}}{2}$$

Finally, let's do an example where we use the even-odd identities to find the values of all six trig functions at a negative degree angle.

Example

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Find the values of all six trig functions at -750°.

If we take out two full 360° rotations from 750°, we're left with 30°, which means -750° is coterminal with -30°.

$$-750^\circ + 2(360^\circ) = -750^\circ + 720^\circ = -30^\circ$$

The cosine and secant functions are even, so -30° is the same as 30°, and we can say

$$\cos(-750^\circ) = \cos(30^\circ) = \frac{\sqrt{3}}{2}$$

$$\sec(-750^\circ) = \sec(30^\circ) = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$

The sine, cosecant, tangent, and cotangent functions are odd, so

$$\sin(-750^\circ) = \sin(-30^\circ) = -\sin(30^\circ) = -\frac{1}{2}$$

$$\csc(-750^\circ) = \csc(-30^\circ) = -\csc(30^\circ) = -\frac{2}{1} = -2$$

$$\tan(-750^\circ) = \tan(-30^\circ) = -\tan(30^\circ) = -\frac{\sin(30^\circ)}{\cos(30^\circ)} = -\frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = -\frac{1}{2}\left(\frac{2}{\sqrt{3}}\right) = -\frac{\sqrt{3}}{3}$$

$$\cot(-750^\circ) = \cot(-30^\circ) = -\cot(30^\circ) = -\frac{\cos(30^\circ)}{\sin(30^\circ)} = -\frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = -\frac{\sqrt{3}}{2}\left(\frac{2}{1}\right) = -\sqrt{3}$$

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The set of all possible angles

Previously we defined the set of angles coterminal with θ as

$\alpha = \theta + n(360^\circ)$ or $\alpha = \theta + n(2\pi)$

where n is any integer. We've also said that, from what we understand about the symmetry of x- and y-values across axes, and from the even-odd identities, sine and cosine and the other trig functions can have the same value at different angles.

Given all that, in this lesson we want to take a brief moment to realize that the angles θ that satisfy a trig equation will always include both

1. the complete set of coterminal angles, and

2. the complete set of coterminal angles at any other angle that satisfies the equation.

The easiest way to see how to find the complete solution to a trig equation is to work through an example, so let's do one with a cosine equation.

Example

Solve the equation $\cos\theta = 0$ for all possible values of θ.

We know that the cosine function represents the x-value in a coordinate point, so we might recognize that $\cos\theta = 0$ is telling us $x = 0$. From there,

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we know along the unit circle that $x = 0$ at $\theta = 90^\circ$ and $\theta = 270^\circ$, so we may be tempted to give the solution as that set of two angles.

But we can't forget to account for all the angles that are coterminal with these two! We can think of the coterminal angles in two sets, one for $\theta = 90^\circ$ and one for $\theta = 270^\circ$. Don't forget to include both positive and negative angles.

$$90^\circ,\ 90^\circ \pm 360^\circ,\ 90^\circ \pm 720^\circ,\ \dots$$

$$270^\circ,\ 270^\circ \pm 360^\circ,\ 270 \pm 720^\circ,\ \dots$$

If we use interval notation, we could express these two sets together as

$$\theta = \{90^\circ \pm (360^\circ)k:\ k = 0,1,2,\dots\} \cup \{270^\circ \pm (360^\circ)k:\ k = 0,1,2,\dots\}$$

$$\theta = \{90^\circ + (360^\circ)k:\ k \in \mathbb{Z}\} \cup \{270^\circ + (360^\circ)k:\ k \in \mathbb{Z}\}$$

where $\mathbb{Z}$ is the set of all integers. But since we know that $270^\circ = 90^\circ + 180^\circ$, we can rewrite this as

$$\theta = \{90^\circ + (180^\circ)k:\ k \in \mathbb{Z}\}$$

This notation is telling us that the set of all possible solutions for θ is given by 90°, plus or minus any k number of full 360° rotations, together with (∪) 270°, plus or minus any k number of full 360° rotations. Defining k as any whole number, together with the ± sign, let's us express the infinite set of coterminal angles in a really compact way.

Because 90° is $\pi/2$ and 270° is $3\pi/2$, we could also express the complete solution set in radians as

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$$\left\{\frac{\pi}{2} \pm 2\pi k: \ k = 0,1,2,\ldots\right\} \cup \left\{\frac{3\pi}{2} \pm 2\pi k: \quad k = 0,1,2,\ldots\right\}$$

$$\left\{\frac{\pi}{2} + 2\pi k: \ k \in \mathbb{Z}\right\} \cup \left\{\frac{3\pi}{2} + 2\pi k: \quad k \in \mathbb{Z}\right\}$$

But since we know that $\frac{3\pi}{2} = \frac{\pi}{2} + \pi$, we can rewrite this as

$$\theta = \left\{\frac{\pi}{2} + \pi k: \ k \in \mathbb{Z}\right\}$$

Let's look at another example with cosine, but this time we'll have a value that doesn't fall on one of the major axes.

Example

Solve the equation $\cos\theta = \sqrt{2}/2$ for all possible values of θ.

From the unit circle, we know that

$$\cos 45^\circ = \frac{\sqrt{2}}{2}$$

So 45° is one solution, but we need to include all other angles that are coterminal with 45°.

$$\{45^\circ \pm (360^\circ)k: \ k = 0,1,2,\ldots\}$$

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Remember in the last example that $\cos\theta = 0$ at both 90° and 270°. Similarly in this example, we have to realize that $\cos\theta = \sqrt{2}/2$ is also true at $\theta = 315^\circ$, since cosine represents the x-value from the coordinate point, and x is positive in both the first (45°) and fourth (315°) quadrants.

The complete set of angles that are coterminal with $\theta = 315^\circ$ is

$$\{315^\circ \pm (360^\circ)k: \ k = 0,1,2,\ldots\}$$

Combining our results, we say that the complete set of possible solutions to $\cos\theta = \sqrt{2}/2$ is

$$\theta = \{45^\circ \pm (360^\circ)k: \ k = 0,1,2,\ldots\} \cup \{315^\circ \pm (360^\circ)k: \ k = 0,1,2,\ldots\}$$

$$\theta = \{45^\circ + (360^\circ)k: \ k \in \mathbb{Z}\} \cup \{315^\circ + (360^\circ)k: \ k \in \mathbb{Z}\}$$

or in radians,

$$\left\{\frac{\pi}{4} \pm 2\pi k: \ k = 0,1,2,\ldots\right\} \cup \left\{\frac{7\pi}{4} \pm 2\pi k: \ k = 0,1,2,\ldots\right\}$$

$$\left\{\frac{\pi}{4} + 2\pi k: \ k \in \mathbb{Z}\right\} \cup \left\{\frac{7\pi}{4} + 2\pi k: \ k \in \mathbb{Z}\right\}$$

Let's do an example where the value of cosine is negative.

Example

Solve the equation $\cos\theta = -\sqrt{3}/2$ for all possible values of θ.

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The equation is telling us that the value of cosine is negative, which means the value of the x-coordinate is negative. Therefore, the solution will be limited to values of θ in the second and third quadrants.

From the unit circle, we know that

$$\cos 150^\circ = -\frac{\sqrt{3}}{2}$$

So 150° is one solution, but we need to include all other angles that are coterminal with 150°.

$$\{150^\circ \pm (360^\circ)k: \ k = 0,1,2,\ldots\}$$

We can find the other set of angles by symmetry. If 150° gives us one set of angles, and 150° is 30° "above" the x-axis, then the other set of angles is at 30° "below" the x-axis, so the other angle is $(180+30)^\circ = 210^\circ$. Which means the set

$$\{210^\circ \pm (360^\circ)k: \ k = 0,1,2,\ldots\}$$

also satisfies the equation. Combining these results, the complete solution set is

$$\theta = \{150^\circ \pm (360^\circ)k: \ k = 0,1,2,\ldots\} \cup \{210^\circ \pm (360^\circ)k: \ k = 0,1,2,\ldots\}$$

$$\theta = \{150^\circ + (360^\circ)k: \ k \in \mathbb{Z}\} \cup \{210^\circ + (360^\circ)k: \ k \in \mathbb{Z}\}$$

or in radians,

$$\left\{\frac{5\pi}{6} \pm 2\pi k: \ k = 0,1,2,\ldots\right\} \cup \left\{\frac{7\pi}{6} \pm 2\pi k: \ k = 0,1,2,\ldots\right\}$$

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$$\left\{\frac{5\pi}{6}+2\pi k:\ k \in \mathbb{Z}\right\} \cup \left\{\frac{7\pi}{6}+2\pi k:\ k \in \mathbb{Z}\right\}$$

Now let's do some examples with the sine function, starting with a positive value for sine.

Example

Solve the equation $\sin\theta = 1/2$ for all possible values of θ.

The value of sine is positive, which means the y-coordinate is positive, and therefore that angles which satisfy the equation are found in the first and second quadrants.

From the unit circle, we know that

$$\sin 30^\circ = \frac{1}{2}$$

So 30° is one solution, but we need to include all other angles that are coterminal with 30°.

$$\{30^\circ \pm (360^\circ)k:\ k = 0,1,2,\ldots\}$$

But angles in the first and second quadrant have equivalent sine values, which means 150° is also a solution, as well as all angles that are coterminal with 150°:

$$\{150^\circ \pm (360^\circ)k:\ k = 0,1,2,\ldots\}$$

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Combining these results, the complete solution set is

$$\theta = \{30^\circ \pm (360^\circ)k: \ k = 0,1,2,\dots\} \cup \{150^\circ \pm (360^\circ)k: \ k = 0,1,2,\dots\}$$

$$\theta = \{30^\circ + (360^\circ)k: \ k \in \mathbb{Z}\} \cup \{150^\circ + (360^\circ)k: \ k \in \mathbb{Z}\}$$

or in radians,

$$\left\{\frac{\pi}{6} \pm 2\pi k: \ k = 0,1,2,\dots\right\} \cup \left\{\frac{5\pi}{6} \pm 2\pi k: \ k = 0,1,2,\dots\right\}$$

$$\left\{\frac{\pi}{6} + 2\pi k: \ k \in \mathbb{Z}\right\} \cup \left\{\frac{5\pi}{6} + 2\pi k: \ k \in \mathbb{Z}\right\}$$

Finally, we'll look at the solution set when sine has a negative value.

Example

Solve the equation $\sin\theta = -\sqrt{3}/2$ for all possible values of θ.

The value of sine is negative, which means the y-coordinate is negative, and therefore that angles which satisfy the equation are found in the third and fourth quadrants.

From the unit circle, we know that

$$\sin 240^\circ = -\frac{\sqrt{3}}{2}$$

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So 240° is one solution, but we need to include all other angles that are coterminal with 240°.

$$\{240^\circ \pm (360^\circ)k: \ k = 0,1,2,\dots\}$$

But angles in the third and fourth quadrant have equivalent sine values, which means 300° is also a solution, as well as all angles that are coterminal with 300°:

$$\{300^\circ \pm (360^\circ)k: \ k = 0,1,2,\dots\}$$

Combining these results, the complete solution set is

$$\theta = \{240^\circ \pm (360^\circ)k: \ k = 0,1,2,\dots\} \cup \{300^\circ \pm (360^\circ)k: \ k = 0,1,2,\dots\}$$

$$\theta = \{240^\circ + (360^\circ)k: \ k \in \mathbb{Z}\} \cup \{300^\circ + (360^\circ)k: \ k \in \mathbb{Z}\}$$

or in radians,

$$\left\{\frac{4\pi}{3} \pm 2\pi k: \ k = 0,1,2,\dots\right\} \cup \left\{\frac{5\pi}{3} \pm 2\pi k: \ k = 0,1,2,\dots\right\}$$

$$\left\{\frac{4\pi}{3} + 2\pi k: \ k \in \mathbb{Z}\right\} \cup \left\{\frac{5\pi}{3} + 2\pi k: \ k \in \mathbb{Z}\right\}$$

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Points not on the unit circle

Up to now, we've been hyper-focused on points along the unit circle. But we also want to be able to deal with points that don't fall exactly on this perfect circle with radius $r = 1$.

The good news is that we can still use the unit circle to help us find sine and cosine of an angle that's defined by a point off of the unit circle.

Using a point off of the unit circle

Let's plot a point B that's sitting somewhere outside the unit circle. We want to find the value of sine and cosine at the angle that B forms, but B isn't on the unit circle, which means we can't use the unit circle to find values of x and y.

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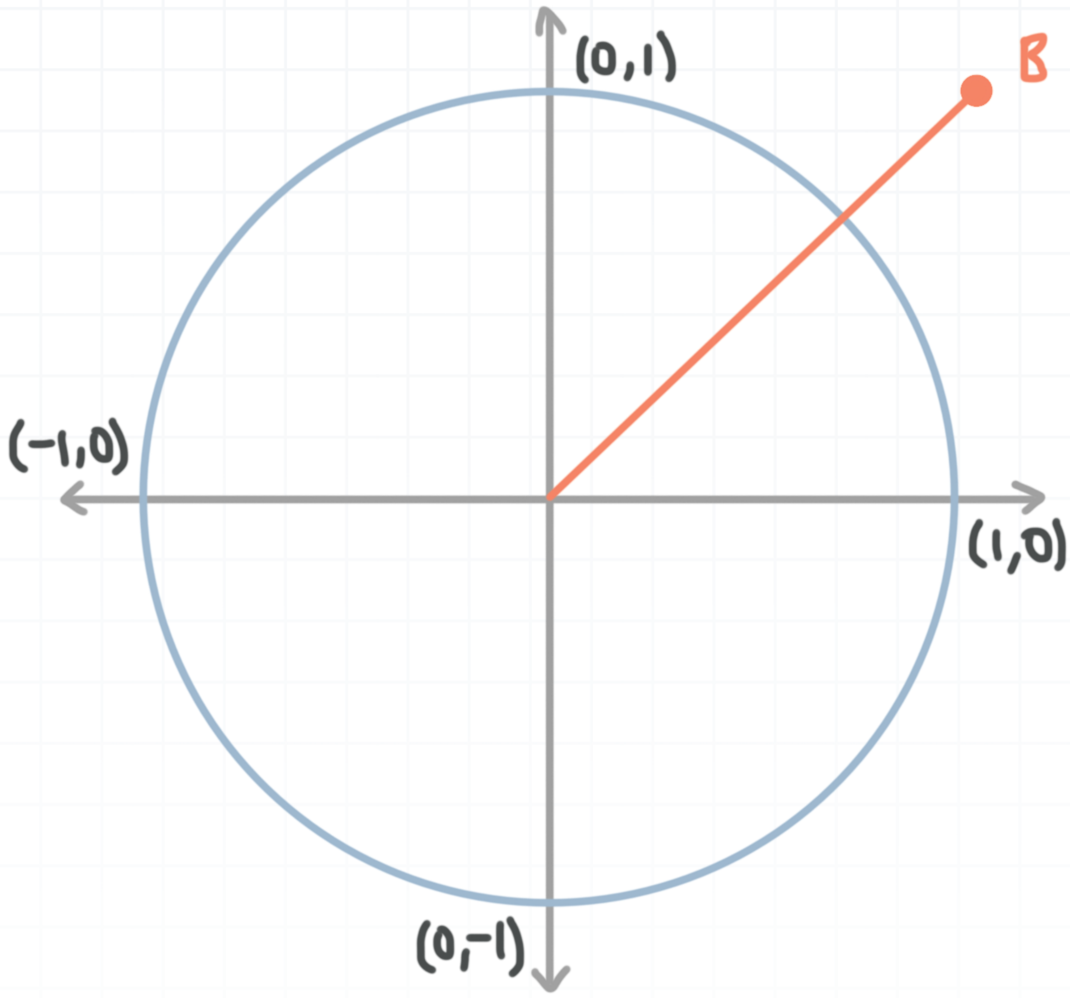

To figure this out, we'll label a point D that's on the terminal side of the angle θ created by B, and where D is on the unit circle. That means the coordinates of D are $(x, y) = (\cos\theta, \sin\theta)$, and because D is on the unit circle, $\overline{AD} = 1$.

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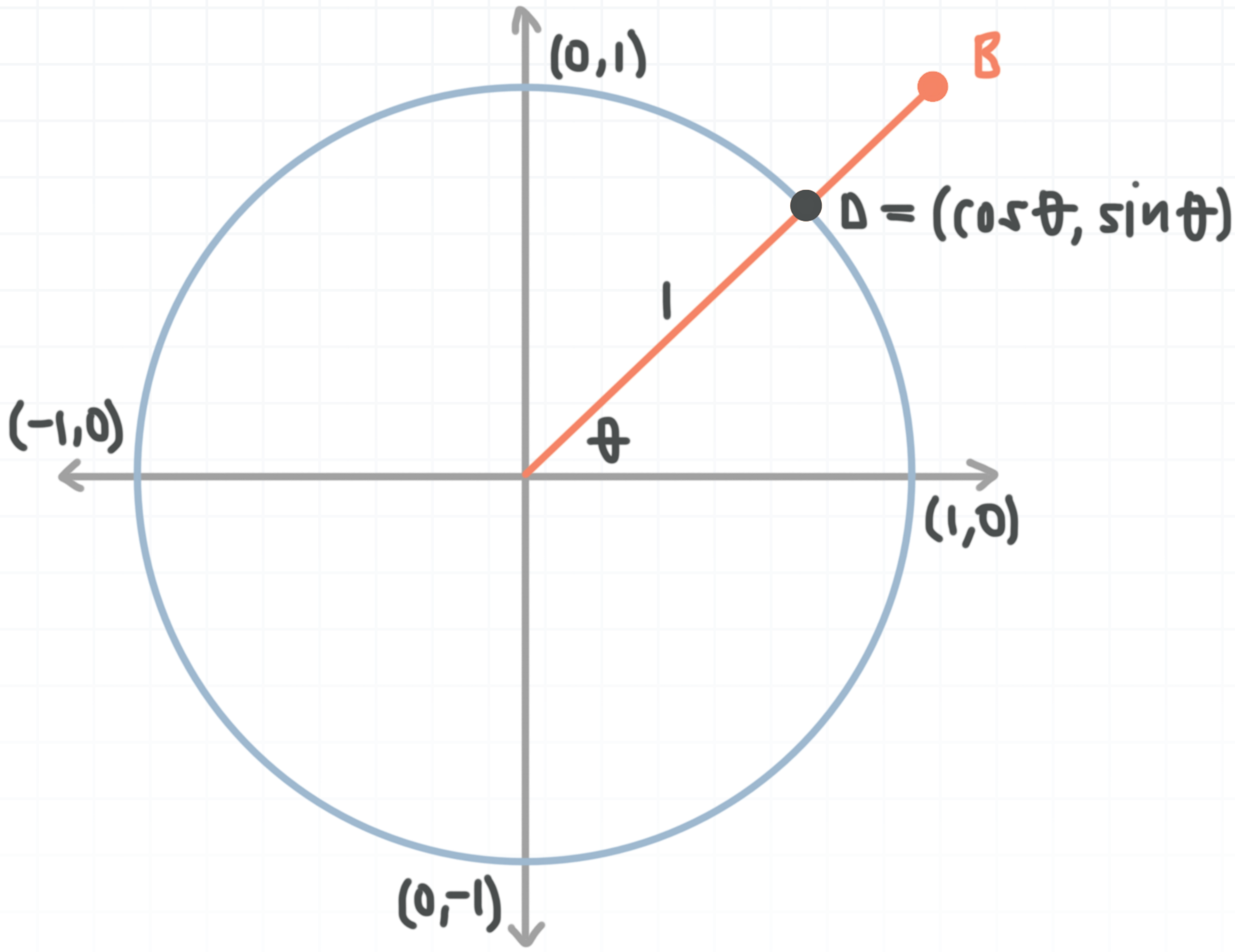

Next we'll connect B and D to the x-axis with vertical lines, and label new points, C and E, where we intersect the x-axis. Point C is at some generic $(x,0)$. Because E is directly below D, it has the same x-coordinate as D, which means it's sitting at $(\cos\theta,0)$.

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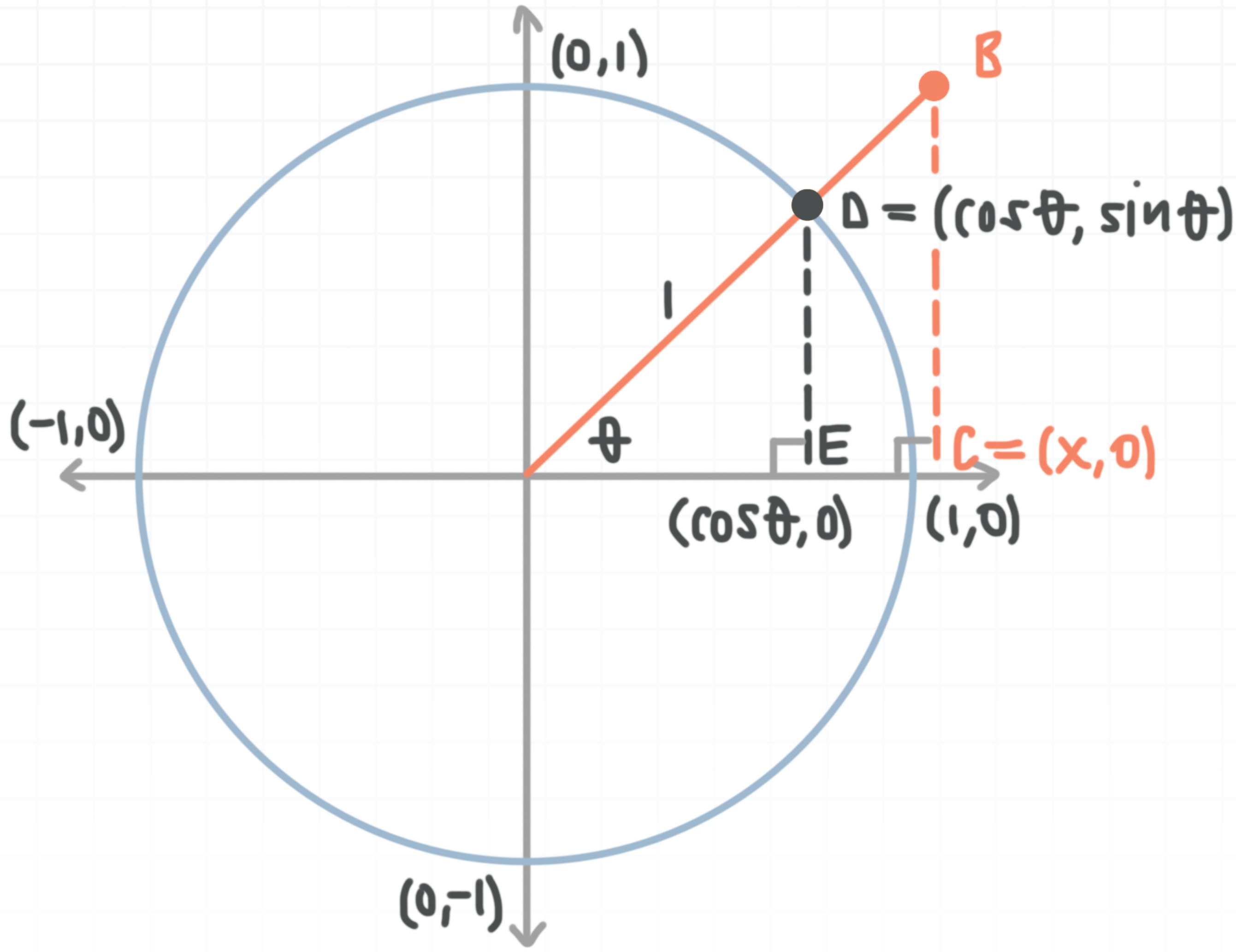

Then we realize that we have two right triangles in this figure: triangle ABC and triangle ADE.

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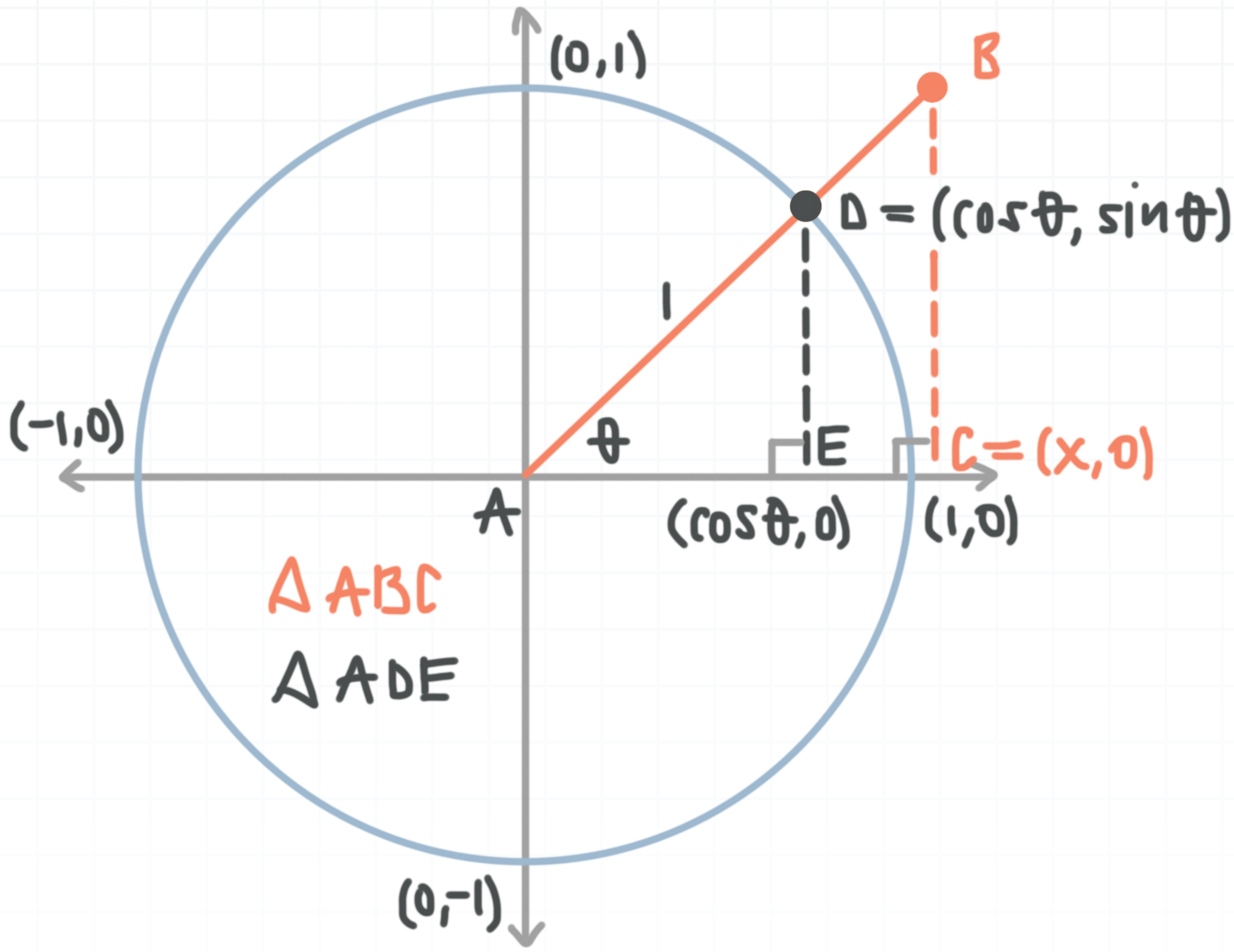

The full triangle and the triangle within the unit circle are **similar triangles**, which means they have equal angles,

$$\angle A = \angle A$$

$$\angle B = \angle D$$

$$\angle C = \angle E$$

and proportional side lengths.

$$\frac{a = \overline{BC}}{\overline{DE}} = \frac{b = \overline{AC}}{\overline{AE}} = \frac{c = \overline{AB}}{\overline{AD}}$$

For the small triangle ADE within the unit circle, we can define sine and cosine as

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{\overline{DE}}{1} = \overline{DE}$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{\overline{AE}}{1} = \overline{AE}$$

Furthermore, for the large triangle ABC, we've defined B at (x, y), which means we can define sine and cosine as

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{y}{c = \overline{AB}}$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{x}{c = \overline{AB}}$$

And we can find the value of $c = \overline{AB}$ using the **distance formula** from Algebra,

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

which gives the distance between two points, (x_1, y_1) and (x_2, y_2). The side $c = \overline{AB}$ is defined between the origin $(0,0)$ and B at (x, y), so the distance formula gives

$$d = \sqrt{(x - 0)^2 + (y - 0)^2}$$

$$d = \sqrt{x^2 + y^2}$$

So for the large triangle ABC, the formulas for sine and cosine become

$$\sin\theta = \frac{y}{c} = \frac{y}{\sqrt{x^2 + y^2}}$$

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$$\cos\theta = \frac{x}{c} = \frac{x}{\sqrt{x^2+y^2}}$$

These formulas hold for angles in all four quadrants, so let's look at some examples.

Example

Find sine and cosine of an angle whose terminal side contains the point $(3,4)$.

Substitute $(x, y) = (3,4)$ into the formulas for sine and cosine.

$$\sin\theta = \frac{y}{\sqrt{x^2+y^2}} = \frac{4}{\sqrt{3^2+4^2}} = \frac{4}{\sqrt{9+16}} = \frac{4}{\sqrt{25}} = \frac{4}{5}$$

$$\cos\theta = \frac{x}{\sqrt{x^2+y^2}} = \frac{3}{\sqrt{3^2+4^2}} = \frac{3}{\sqrt{9+16}} = \frac{3}{\sqrt{25}} = \frac{3}{5}$$

So even though $(3,4)$ isn't on the unit circle, we're able to use formulas for sine and cosine to find the values of those trig functions at the angle created by $(3,4)$.

Let's try an example with an angle in the second quadrant, and this time we'll find the values of all six trig functions.

Example

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Find all six trig functions of an angle whose terminal side contains the point $(-2,3)$.

Substitute $(x, y) = (-2,3)$ into the formulas for sine and cosine.

$$\sin\theta = \frac{y}{\sqrt{x^2+y^2}} = \frac{3}{\sqrt{(-2)^2+3^2}} = \frac{3}{\sqrt{4+9}} = \frac{3}{\sqrt{13}} = \frac{3\sqrt{13}}{13}$$

$$\cos\theta = \frac{x}{\sqrt{x^2+y^2}} = \frac{-2}{\sqrt{(-2)^2+3^2}} = \frac{-2}{\sqrt{4+9}} = \frac{-2}{\sqrt{13}} = -\frac{2\sqrt{13}}{13}$$

Use the quotient identity to find tangent.

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\frac{3\sqrt{13}}{13}}{-\frac{2\sqrt{13}}{13}} = \frac{3\sqrt{13}}{13}\left(-\frac{13}{2\sqrt{13}}\right) = -\frac{39\sqrt{13}}{26\sqrt{13}} = -\frac{3}{2}$$

Then use the reciprocal identities to find cosecant, secant, and cotangent.

$$\csc\theta = \frac{13}{3\sqrt{13}} = \frac{\sqrt{13}}{3}$$

$$\sec\theta = -\frac{13}{2\sqrt{13}} = -\frac{\sqrt{13}}{2}$$

$$\cot\theta = -\frac{2}{3}$$

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So even though $(-2,3)$ isn't on the unit circle, we're able to use formulas for sine and cosine to find the values of those trig functions at the angle created by $(-2,3)$, and then use our other trig identities to find the values of the other four trig functions at the same angle.

$$\sin\theta = \frac{3\sqrt{13}}{13} \qquad \csc\theta = \frac{\sqrt{13}}{3}$$

$$\cos\theta = -\frac{2\sqrt{13}}{13} \qquad \sec\theta = -\frac{\sqrt{13}}{2}$$

$$\tan\theta = -\frac{3}{2} \qquad \cot\theta = -\frac{2}{3}$$

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Solving right triangles

When we talk about "completing a right triangle," or "solving a right triangle" we mean that we're going to try to find all three side lengths, and all three interior angle measures.

The good news is that, once we find a few of these values in the triangle, it gets really easy to find the rest of them.

Formulas for solving right triangles

There are two formulas we'll use all the time during this process:

1. $m\angle A + m\angle B + m\angle C = 180^\circ$

2. $a^2 + b^2 = c^2$

The first formula tells us that the three interior angles of any triangle will always sum to 180°. Of course, in a right triangle, one of the angles is 90°, so the other two angles of a right triangle will always sum to 90° as well.

The second formula is the Pythagorean theorem, which tells us that the sum of the squares of the leg lengths, $a^2 + b^2$, is equal to the square of the length of the hypotenuse, c^2. So a and b are the legs and c is the hypotenuse (which is always the longest side, and the side opposite the right angle).

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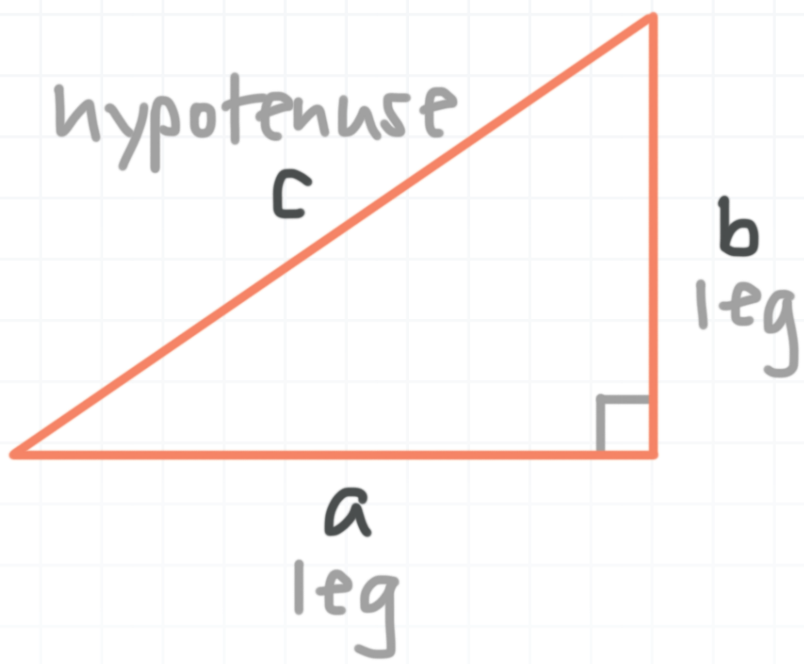

Our general strategy here will be to position the right triangle within the unit circle, but then extend its sides past the unit circle (if the hypotenuse is longer than 1 unit) in order to sketch out the full triangle.

Then we'll treat the full triangle and the triangle within the unit circle as similar triangles.

Let's walk through an example where we're given one angle and the length of one leg.

Example

A right triangle has a leg with length 3. The angle opposite that leg is 40°. Find the measures of all three interior angles and the lengths of all three sides.

It's a good idea to draw a picture of the triangle before we start. Based on what we know, we have

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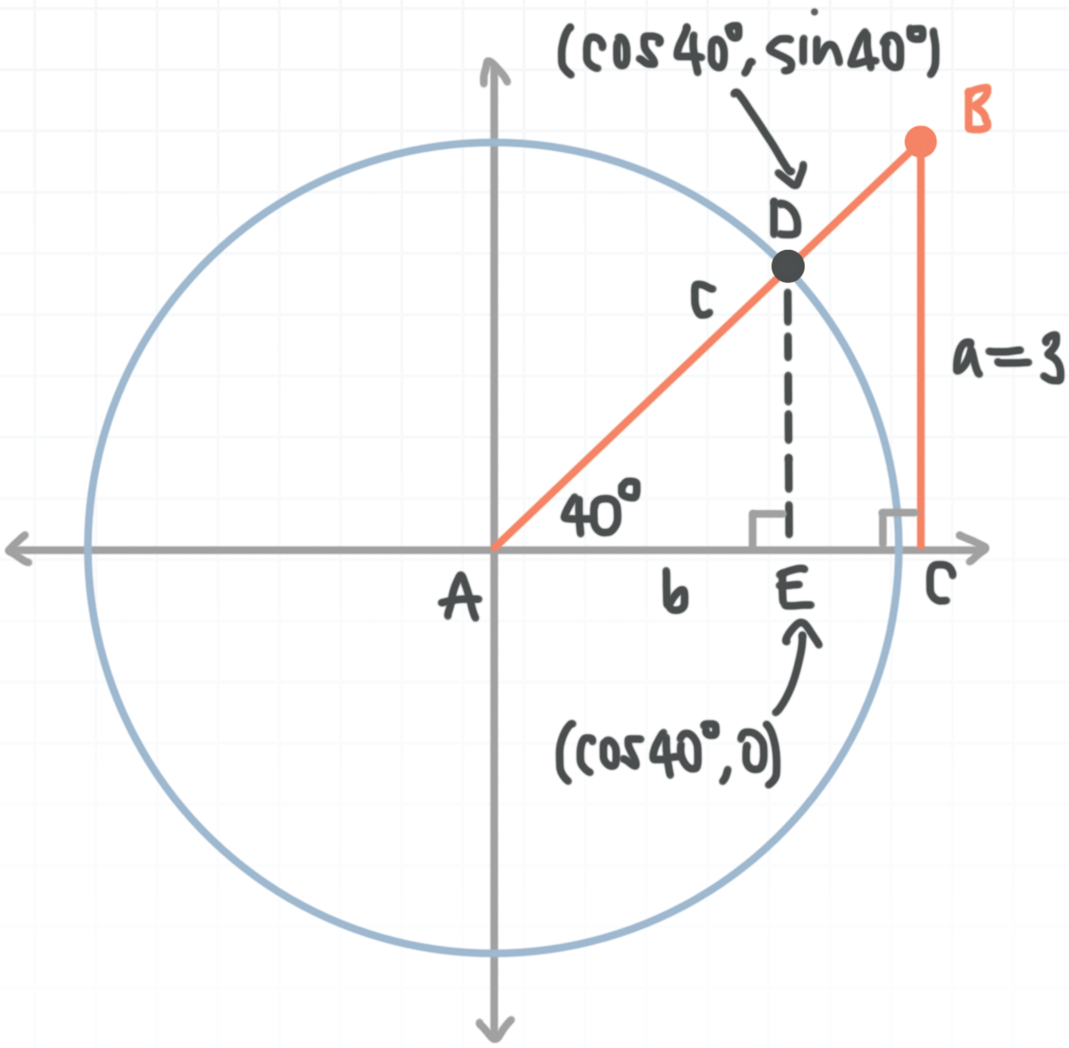

We could just as easily have said $b = 3$ instead of $a = 3$, and therefore, $B = 40^\circ$ instead of $A = 40^\circ$. But it doesn't matter which leg we pick, as long as we make sure that the leg and the 40° angle are opposite each other. It's always helpful to put the right angle out at C so that the hypotenuse can extend out from the origin toward B.

Because we're dealing with a right triangle, we know we have a 90° angle and the 40° angle we were given. So the angle at B must be

$$B = 180^\circ - 90^\circ - 40^\circ$$

$$B = 50^\circ$$

We know the length of side a is $a = 3$, so all we need now is the length of the other two sides. Notice in the diagram that we sketched in $\overline{DE}$ in order

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to form the small triangle ADE. We do that in order to set up similar triangles. We assume that D is on the unit circle, such that $\overline{AD}$ has length 1. Then triangles ADE and ABC are similar, and we can say

$$\angle DAE = \angle BAC = 40^\circ$$

$$\angle ADE = \angle ABC = 50^\circ$$

$$\angle AED = \angle ACB = 90^\circ$$

Furthermore, corresponding sides of similar triangles are proportional, so

$$\frac{a}{\overline{DE}} = \frac{b}{\overline{AE}} = \frac{c}{\overline{AD}}$$

$$\frac{3}{\overline{DE}} = \frac{b}{\overline{AE}} = \frac{c}{1}$$

$$\frac{3}{\overline{DE}} = \frac{b}{\overline{AE}} = c$$

Here's where our sine and cosine functions come in. Since D is on the unit circle, its coordinates (x, y) are $(\cos 40^\circ, \sin 40^\circ)$. Also, the coordinates (x, y) of E are $(\cos 40^\circ, 0)$. Therefore, we can use these coordinates to find the lengths of $\overline{DE}$ and $\overline{AE}$.

$\overline{DE} = \sin 40^\circ - 0 = \sin 40^\circ$ (Since $\overline{DE}$ is perpendicular to the x-axis, we can always find its length as the absolute value of the difference of the y-coordinates of points E and D.)

$\overline{AE} = \cos 40^\circ - 0 = \cos 40^\circ$ (Since $\overline{AE}$ is perpendicular to the y-axis, we can always find its length as the absolute value of the difference of the x-coordinates of points A and E.)

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So the proportion becomes

$$\frac{3}{\sin 40^\circ} = \frac{b}{\cos 40^\circ} = c$$

To find b (the length of $\overline{AC}$), we can use

$$\frac{3}{\sin 40^\circ} = \frac{b}{\cos 40^\circ}$$

$$\frac{3}{\sin 40^\circ}(\cos 40^\circ) = b$$

$$b \approx \frac{3}{0.643}(0.766)$$

$$b \approx 3.57$$

To find c (the length of $\overline{AB}$), we can use

$$\frac{3}{\sin 40^\circ} = c$$

$$c \approx \frac{3}{0.643}$$

$$c \approx 4.67$$

We've solved the right triangle, and we can say that the side lengths are $a = 3$, $b \approx 3.57$, and $c \approx 4.67$, and the angle measures are $A = 40^\circ$, $B = 50^\circ$, and $C = 90^\circ$.

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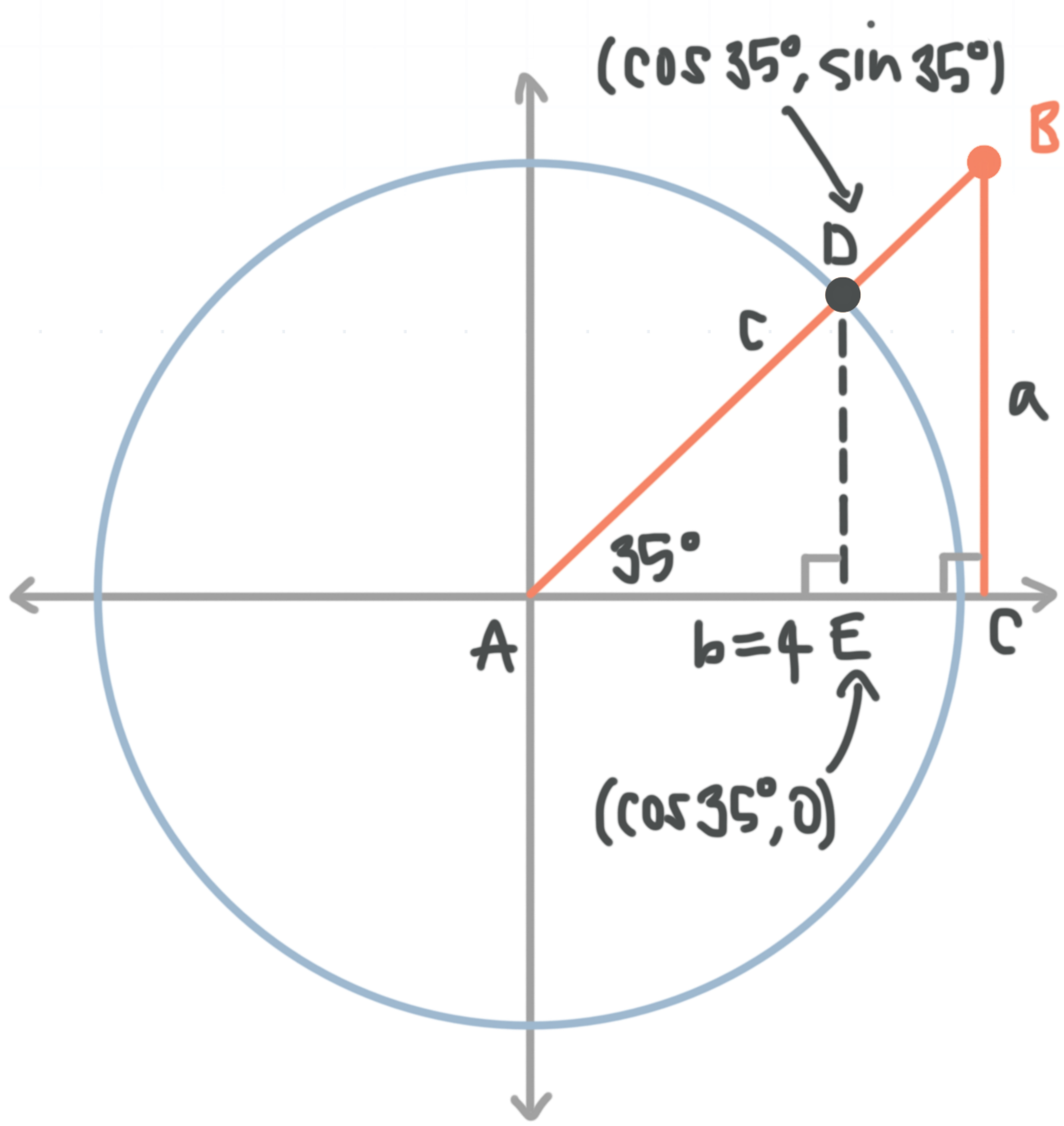
In the last example we had the length of the leg opposite the known angle. Let's do another example where we have the length of the leg adjacent to the known angle.

Example

A right triangle has one leg with length 4 and an interior angle opposite the other leg that measures 35°. Complete the triangle.

We can say $b = 4$ and $A = 35^\circ$, or $a = 4$ and $B = 35^\circ$. Either will work, but we'll use the same orientation as the last example. Let's also sketch in the unit circle and put D on the unit circle.



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With $C = 90^\circ$ and $A = 35^\circ$, B must be

$$B = 180^\circ - 90^\circ - 35^\circ$$

$$B = 55^\circ$$

Because triangles ABC and ADE are similar, we know

$$\angle DAE = \angle BAC = 35^\circ$$

$$\angle ADE = \angle ABC = 55^\circ$$

$$\angle AED = \angle ACB = 90^\circ$$

Furthermore, corresponding sides of similar triangles are proportional, so

$$\frac{a}{\overline{DE}} = \frac{b}{\overline{AE}} = \frac{c}{\overline{AD}}$$

$$\frac{a}{\overline{DE}} = \frac{4}{\overline{AE}} = \frac{c}{1}$$

Point D is again on the unit circle, and its coordinates (x, y) are $(\cos 35^\circ, \sin 35^\circ)$. Also, the coordinates (x, y) of E are $(\cos 35^\circ, 0)$. Therefore, we can use these coordinates to find the lengths of $\overline{DE}$ and $\overline{AE}$.

$$\overline{DE} = \sin 35^\circ - 0 = \sin 35^\circ$$

$$\overline{AE} = \cos 35^\circ - 0 = \cos 35^\circ$$

So the proportion becomes

$$\frac{a}{\sin 35^\circ} = \frac{4}{\cos 35^\circ} = c$$

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To find a (the length of $\overline{BC}$), we can use

$$\frac{a}{\sin 35^\circ} = \frac{4}{\cos 35^\circ}$$

$$a = \frac{4}{\cos 35^\circ}(\sin 35^\circ)$$

$$a \approx \frac{4}{0.819}(0.574)$$

$$a \approx 2.80$$

To find c (the length of $\overline{AB}$), we can use

$$\frac{4}{\cos 35^\circ} = c$$

$$c \approx \frac{4}{0.819}$$

$$c \approx 4.88$$

We've solved the right triangle, and we can say that the side lengths are $a \approx 2.80$, $b = 4$, and $c \approx 4.88$, and the angle measures are $A = 35^\circ$, $B = 55^\circ$, and $C = 90^\circ$.

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Angles of elevation and depression

Right triangles can be used to solve all kinds of real-world problems. Now that we know how to solve right triangles, in this lesson we'll look at a really common application, angles of elevation and depression.

Angle of elevation

Think about an **angle of elevation** as a positive angle in standard position. The angle of elevation is always measured from the horizontal side of the angle along the positive side of the x-axis, *up to* the terminal side of the angle.

To translate this to the real world, let's imagine that we're standing on level ground and looking up at a bird in the air. If we place ourselves at A and the bird at B, then we can call C the point on the ground directly below the bird, then we have a right triangle in which θ is the angle of elevation.

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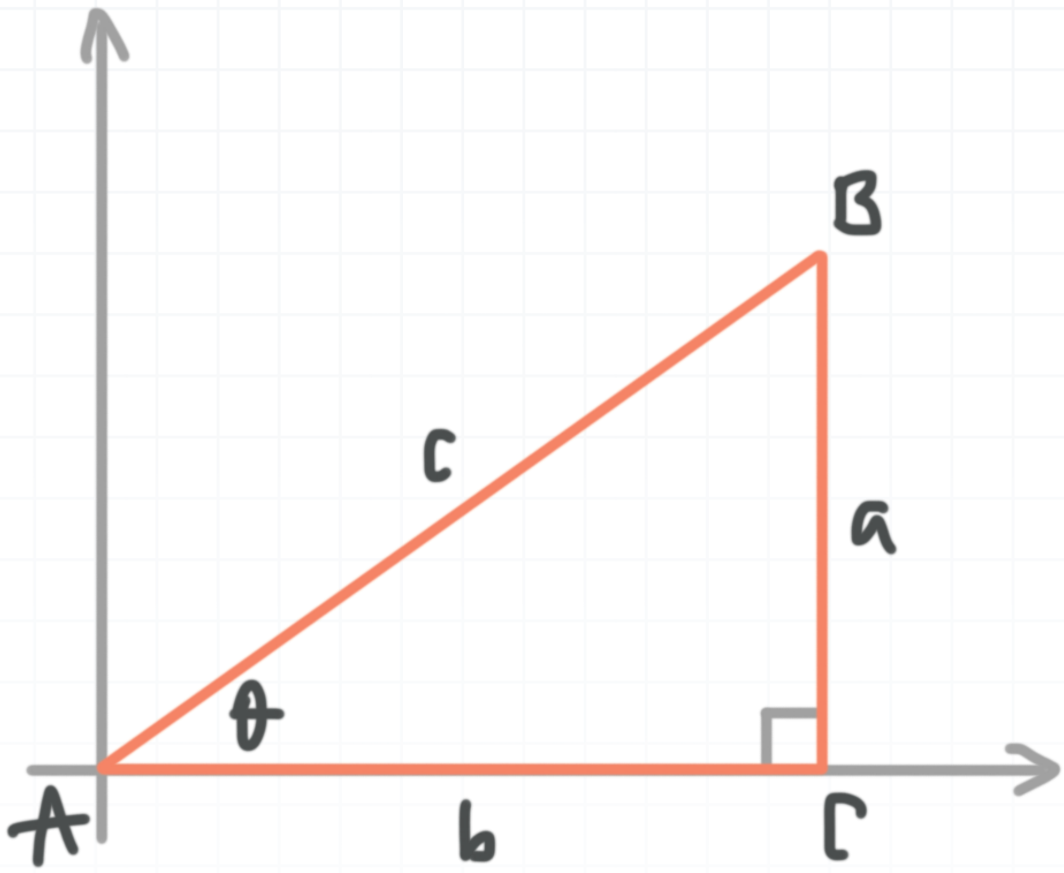

With the figure set up this way, we can now use trigonometry to find the distance from us to the bird, or from the bird to the ground, both of which might be interesting values that we'd want to know.

To help us solve for some of these values, we can always draw the unit circle over the top of the triangle, in order to form a new triangle ADE, and then use the fact that triangles ADE and ABC are similar.

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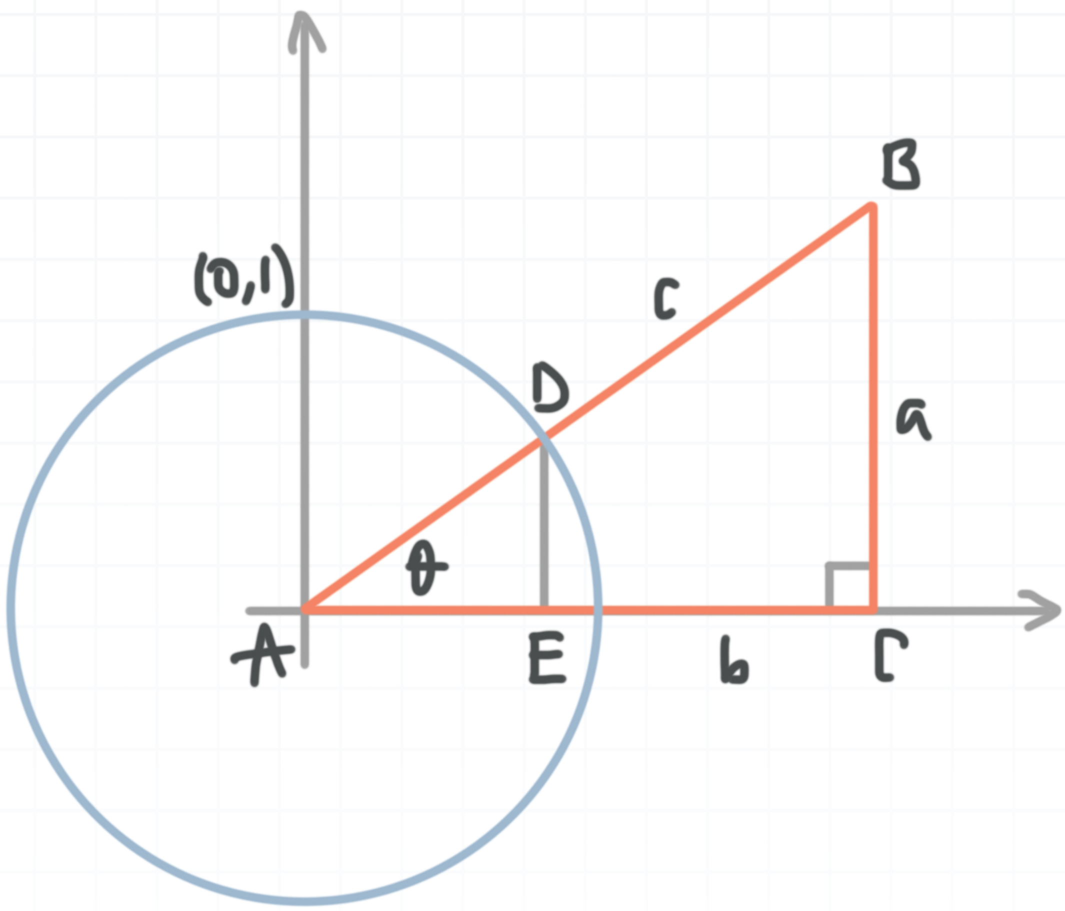

Because they're similar triangles, of course we can set up a proportion of the side lengths,

$$\frac{a}{\overline{DE}} = \frac{b}{\overline{AE}} = \frac{c}{1}$$

$$\frac{a}{\overline{DE}} = \frac{b}{\overline{AE}} = c$$

Since D is on the unit circle, its coordinates are $(\cos\theta, \sin\theta)$. The coordinates of E are $(\cos\theta, 0)$ and the coordinates of A are $(0,0)$. Therefore, the lengths of $\overline{DE}$ and $\overline{AE}$ will always be

$$\overline{DE} = \sin\theta - 0 = \sin\theta$$

$$\overline{AE} = \cos\theta - 0 = \cos\theta$$

Then the proportion becomes

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$$\frac{a}{\overline{DE}} = \frac{b}{\overline{AE}} = c$$

$$\frac{a}{\sin\theta} = \frac{b}{\cos\theta} = c$$

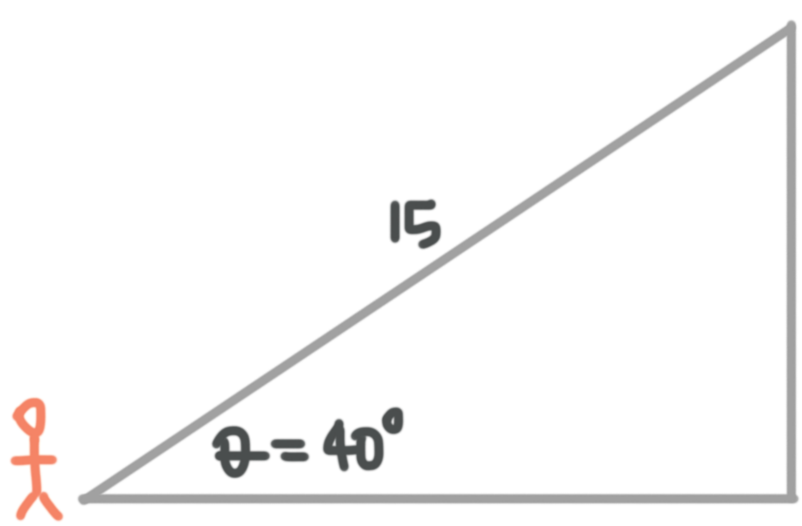
and we should be able to use this equation to calculate the lengths of the sides of the larger triangle.

Let's do a full example where we're standing on the ground and looking at a bird in the air.

Example

If the angle of elevation of a bird with respect to the point where we're standing is 40°, and the distance between us and the bird is 15 feet, find the height of the bird above the ground and our distance from the point on the ground that's directly below the bird.

Let's sketch out the situation.



Since the bird is 15 feet away, $c = 15$ feet. The height of the bird above the ground is a, so using

$$\frac{a}{\sin\theta} = \frac{b}{\cos\theta} = c$$

we get

$$a = 15(\sin 40^\circ)$$

$$a \approx 15(0.643)$$

$$a \approx 9.65 \text{ feet}$$

Our distance from the point on the ground below the bird is b, so

$$b = 15(\cos 40^\circ)$$

$$b \approx 15(0.766)$$

$$b \approx 11.5 \text{ feet}$$

Next let's consider an example where we know the angle of elevation to the top of a building and the height of the building.

Example

If the angle of elevation from where we're standing on the ground to the top of an 80-foot building is 65°, what's the distance between us and the top of the building, and how far do we have to walk to reach the building?

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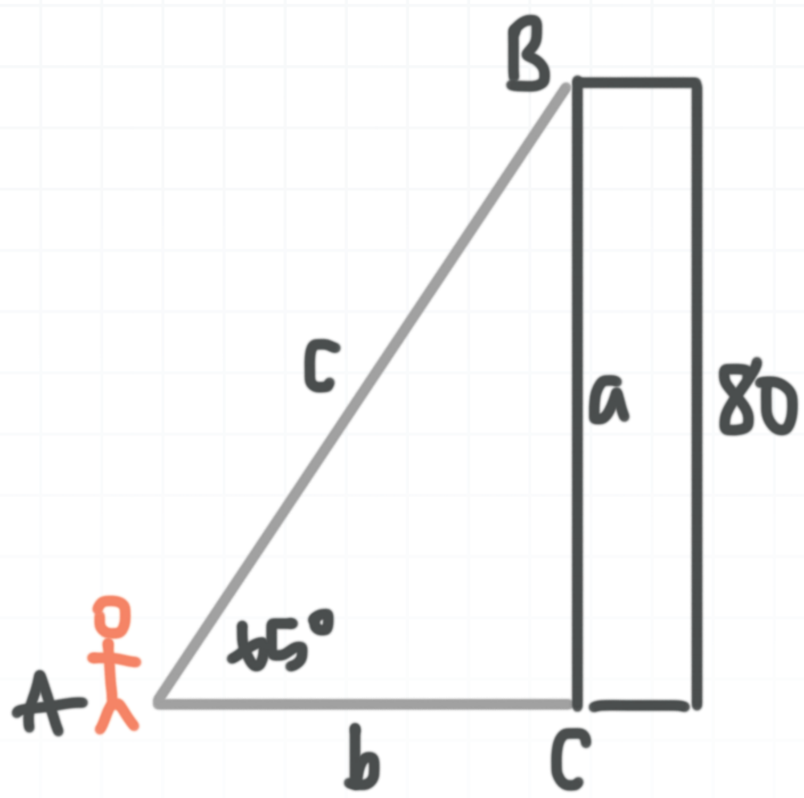
Let's sketch out the situation.



In this case, $\theta = 65^\circ$ and $a = 80$ feet. The distance between A and B is c, and the distance from A to the bottom of the building is b. If we start with

$$\frac{a}{\sin\theta} = \frac{b}{\cos\theta} = c$$

then to find b, we can use

$$\frac{a}{\sin\theta} = \frac{b}{\cos\theta}$$

Multiplying both sides by $\cos\theta$, we get

$$b = \frac{a}{\sin\theta}(\cos\theta)$$

$$b \approx \frac{80}{0.906}(0.423)$$

$b \approx 37.4$ feet

Now find c.

$$c = \frac{a}{\sin\theta}$$

$$c \approx \frac{80}{0.906}$$

$c \approx 88.3$ feet

Therefore, the distance between us and the top of the building is about 88.3 feet, and the distance between us and the bottom of the building (the distance we'd have to walk to reach it) is about 37.4 feet.

Angle of depression

An **angle of depression**, in contrast to an angle of elevation, is like a negative angle in standard position.

The angle of depression is always measured from the horizontal side of the angle along the positive side of the x-axis, *down to* the terminal side of the angle.

To translate this to the real world, let's imagine that we're standing on the top of a building and looking down at a coin on the ground. If we place ourselves at A and the coin at B, then we can call C the point in the air

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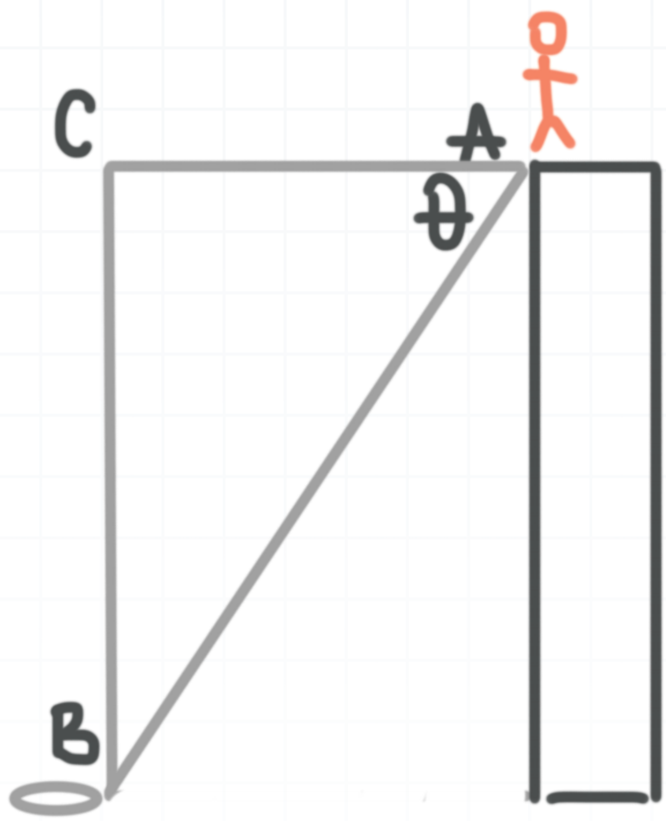
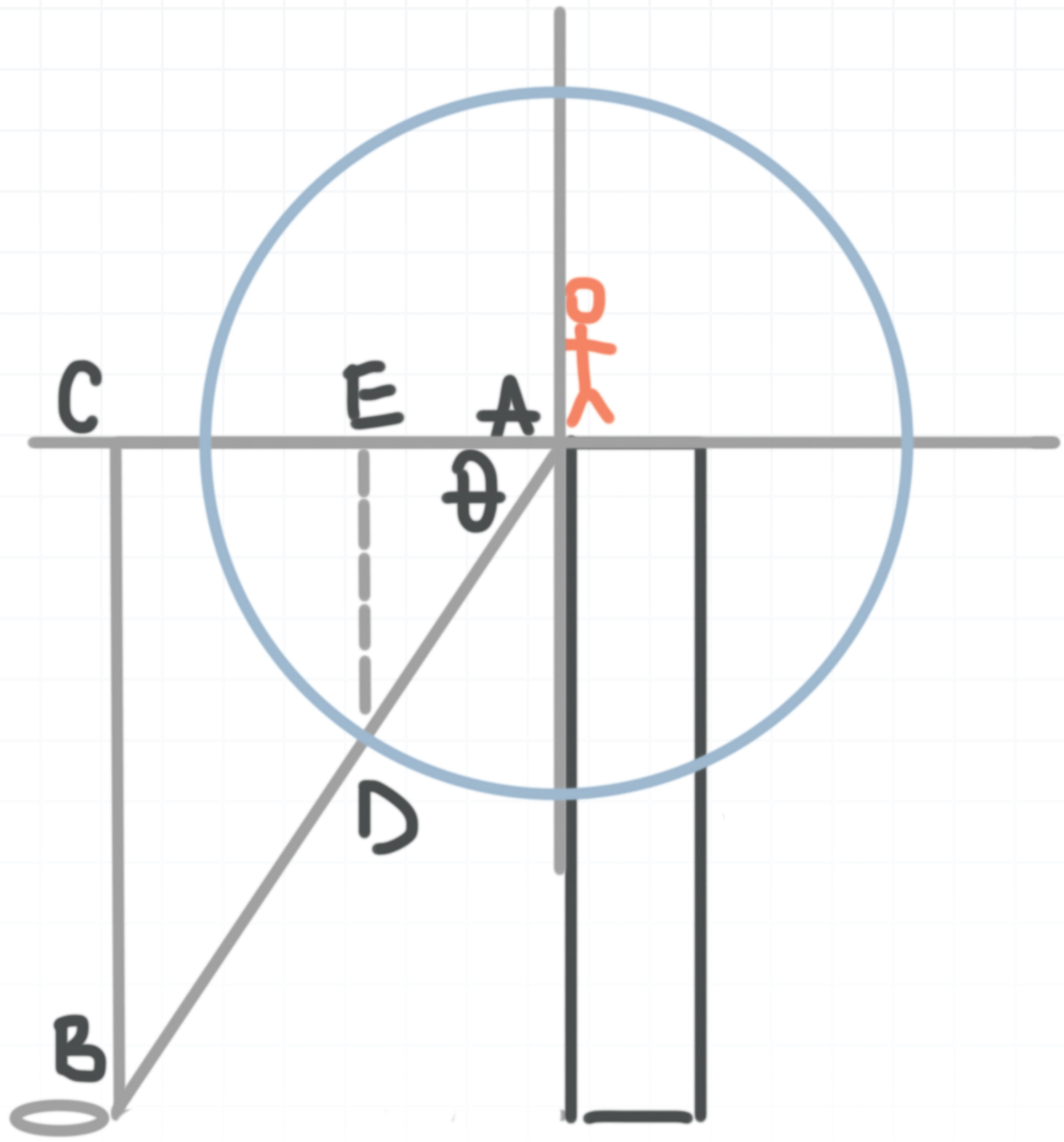
directly above the coin, then we have a right triangle in which θ is the angle of depression.



With the figure set up this way, we can now use trigonometry to find the distance from us to the coin, or the perfectly vertical distance between us and the coin, both of which might be interesting values that we'd want to know.

To help us solve for some of these values, we can always draw the unit circle over the top of the triangle, in order to form a new triangle ADE, and then use the fact that triangles ADE and ABC are similar.



Because they're similar triangles, of course we can set up the same proportion of the side lengths that we did for the angle of elevation triangle.

$$\frac{a}{\overline{DE}} = \frac{b}{\overline{AE}} = \frac{c}{1}$$

$$\frac{a}{\overline{DE}} = \frac{b}{\overline{AE}} = c$$

Since D is on the unit circle, its coordinates are $(\cos\theta, \sin\theta)$. The coordinates of E are $(\cos\theta,0)$ and the coordinates of A are $(0,0)$. Therefore, the lengths of $\overline{DE}$ and $\overline{AE}$ will still always be,

$$\overline{DE} = \sin\theta - 0 = \sin\theta$$

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$$\overline{AE} = \cos\theta - 0 = \cos\theta$$

so the proportion becomes

$$\frac{a}{\overline{DE}} = \frac{b}{\overline{AE}} = c$$

$$\frac{a}{\sin\theta} = \frac{b}{\cos\theta} = c$$

and we should be able to use this equation to calculate the lengths of the sides of the larger triangle.

Let's look at an example where we use the angle of depression.

Example

We've climbed a tree and just spotted a squirrel on the ground. The angle of depression from us to the squirrel is 57°, and the distance between us is 19 feet. How far are we from the squirrel, both vertically and horizontally?

The angle of depression is $\theta = 57^\circ$ and the distance from us to the squirrel is $c = 19$ feet.

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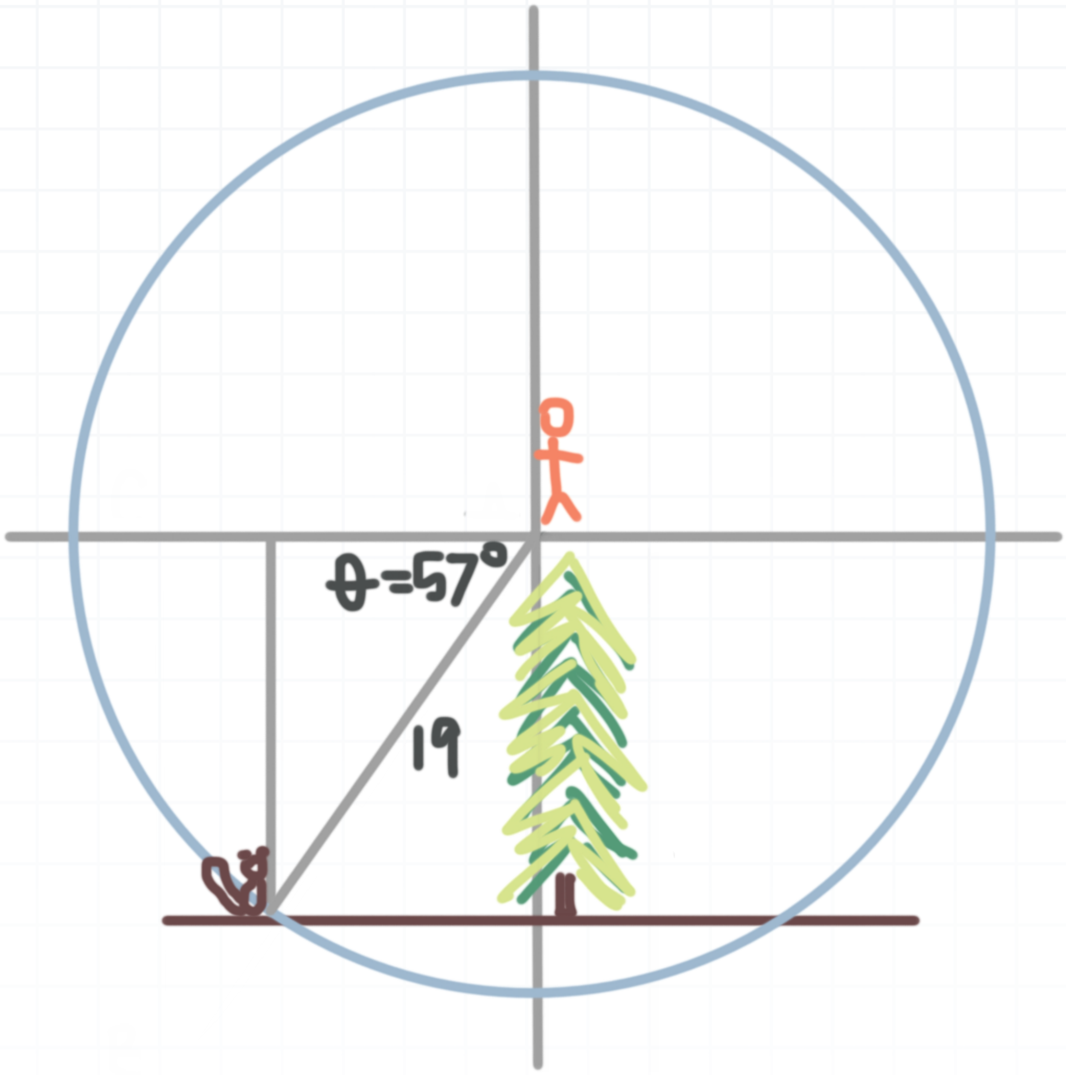

The vertical distance between us in the tree and the squirrel on the ground is a, so

$$\frac{a}{\sin\theta} = c$$

$$\frac{a}{\sin 57^\circ} = 19$$

$$a = 19(\sin 57^\circ)$$

$$a \approx 19(0.839)$$

$a \approx 15.9$ feet

The horizontal distance from us to the squirrel is b, so

$$\frac{b}{\cos\theta} = c$$

$$\frac{b}{\cos 57^\circ} = 19$$

$$b = 19(\cos 57^\circ)$$

$$b \approx 19(0.545)$$

$b \approx 10.4$ feet

So your vertical distance from us in the three to the squirrel on the ground is 15.9 feet, and the horizontal distance from us to the squirrel is 10.4 feet.

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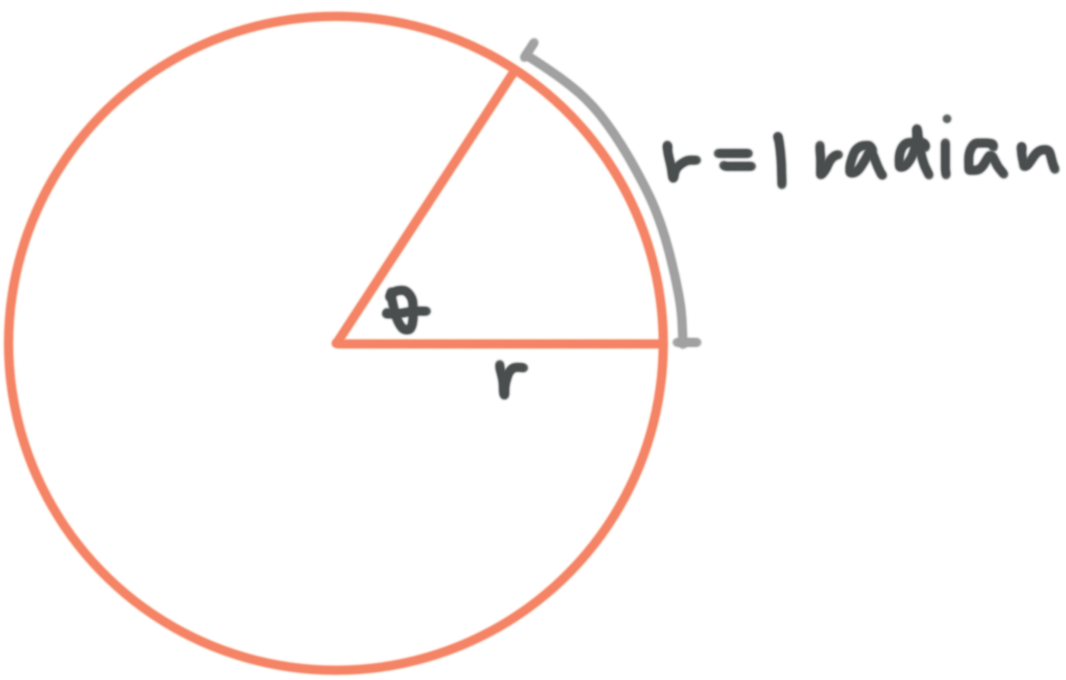
Radians and arc length

We've talked about different units for measuring angles, including degrees, DMS (degrees, minutes, seconds), and radians.

And when we introduced those units, we already defined a degree: it's one 360th of one full rotation. Similarly, in a DMS system, a degree is one 360th of one full rotation, a minute is one 60th of a degree, and a second is one 60th of a minute.

Radian measure

But we never really defined a radian, so let's do that now. Technically, an angle that has its vertex at the center of a circle and that intercepts an arc of the circle equal in length to the circle's radius has an angle measure of **one radian**.



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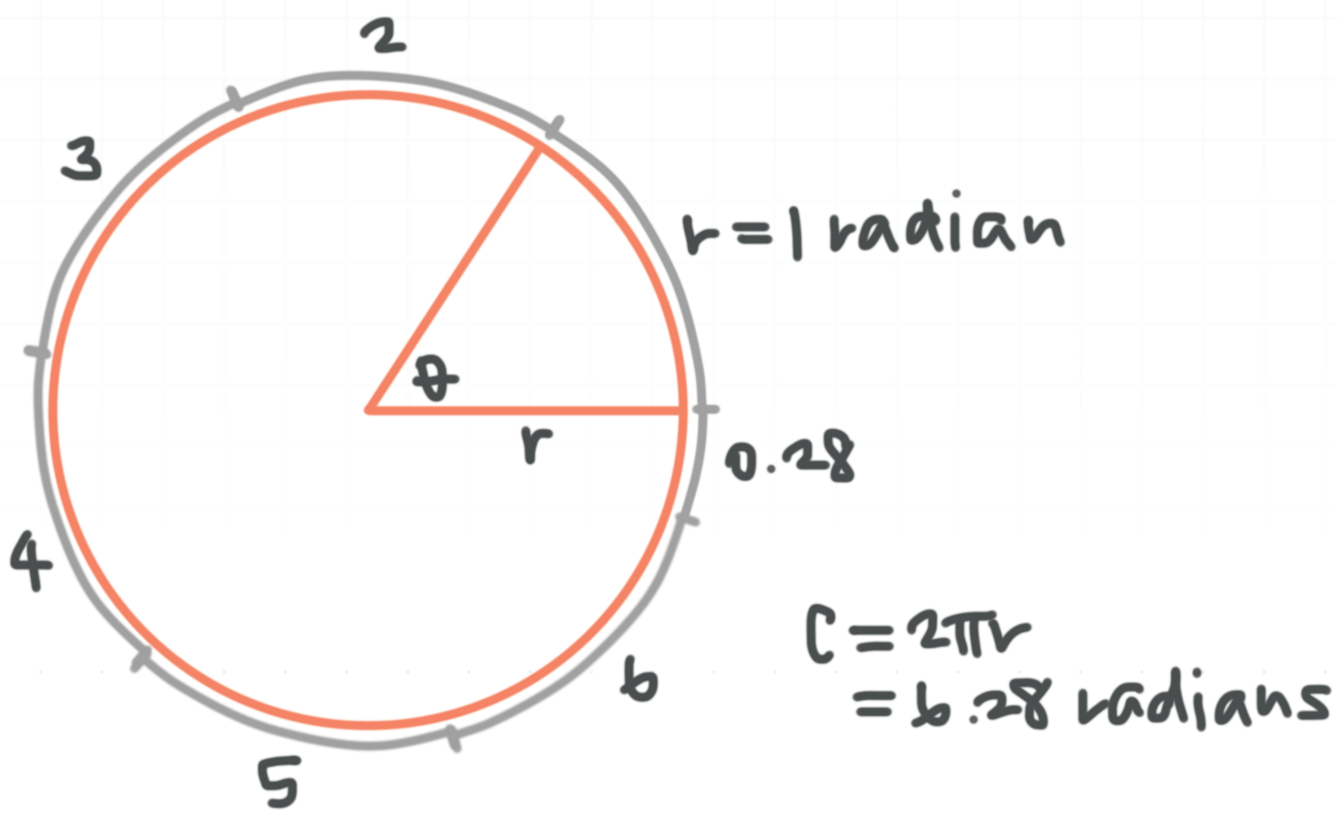
In other words, if you take the radius of a circle and lay the radius out along the edge of the circle, the angle that defines that arc has a measure of 1 radian.

Because we also know that it takes 2π radians to complete one full circle, and $\pi \approx 3.14$, we know there are approximately $2\pi = 2(3.14) = 6.28$ radians in any circle. Put a different way, regardless of the size of the circle, wrapping the radius around the circle about 6.28 times will trace out the full circle.



Of course, this should make sense to us when we think about the equation for the circumference of a circle, $C = 2\pi r$. As we know, the circumference of a circle is the full distance one time around the circle, and the formula here is telling us that the distance one time around the circle is the same as about 6.28 times the radius, which is what we just learned.

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And if we know that it takes about 6.28 radians to get around one full circle, and we also know that one full circle is defined by 360°, then we know the degree measure of exactly 1 radian is approximately

$$\frac{360^\circ}{6.28} \approx 57.32^\circ \approx 57^\circ 19' 29''$$

We see that these measures look approximately correct from the image we drew above where we sketched out the 6.28 radians around the circle.

Arc length

This definition of one radian leads us to the idea of arc length. We just said that laying out the length of one radius around the edge of a circle gave us an angle that measures 1 radian.

But what we really did there was create an **arc**, which is just two points connected by a curved line. A **circular arc**, specifically, is when the curve of the arc follows the perimeter of a circle.

Arc length is always given by

$$s = r\theta$$

where s is the length of the arc, r is the radius of the circle, and θ is the central angle that carves out that particular arc.

Keep in mind that whenever we use the formula for arc length, θ has to be in radians; we can't plug in a value for θ in degrees or DMS.

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Let's do an example with the arc length formula.

Example

Find the length of an arc carved out by a central angle of 60° in a circle of radius $r = 2$.

We can only use an angle defined in radians in the arc length formula, so we'll need to convert 60° to radians.

$$60^\circ \left(\frac{\pi \text{ radians}}{180^\circ}\right) = \frac{\pi}{3} \text{ radians}$$

Now we'll plug what we know into the arc length formula.

$$s = r\theta$$

$$s = 2\left(\frac{\pi}{3}\right)$$

$$s = \frac{2\pi}{3}$$

Let's do another example where we know the circumference of the circle.

Example

If the circumference of a circle is 9π, find the length of an arc that lies on the circle and subtends a central angle of 20°.

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If we solve both the arc length formula $s = r\theta$ for r, and the circumference formula $C = 2\pi r$ for r, we get $r = s/\theta$ and $r = C/2\pi$. Then we can set the equations equal to one another.

$$\frac{s}{\theta} = \frac{C}{2\pi}$$

$$\frac{s}{20^\circ} = \frac{9\pi}{2\pi}$$

Solve for arc length.

$$s = \frac{9}{2}(20^\circ)$$

$$s = 90^\circ$$

$$s = \frac{\pi}{2}$$

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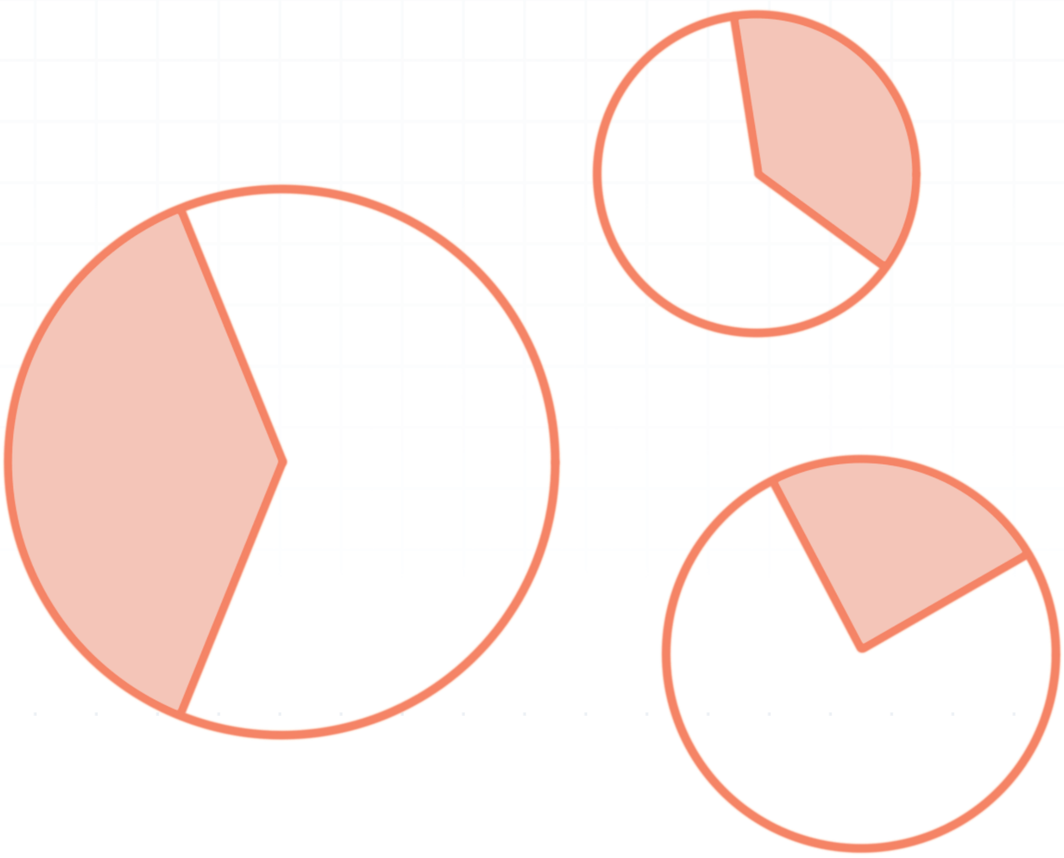
Area of a circular sector

Think of a **circular sector** as a wedge in a circle, like a piece in a pie. Whenever we have one sector in a circle, keep in mind that the rest of the circle also forms another circular sector.

In this figure, each of the red sections are the sectors we've carved out, but then by definition each of the white sections is also a sector.



Every circular sector will have a corresponding arc along the perimeter of the circle. When the two sectors in a circle have equal length arcs, it means each arc is a **semicircle**, or exactly half of the circle. Otherwise, if the arcs aren't the same length, the shorter arc is the **minor arc**, and the longer one is the **major arc**.

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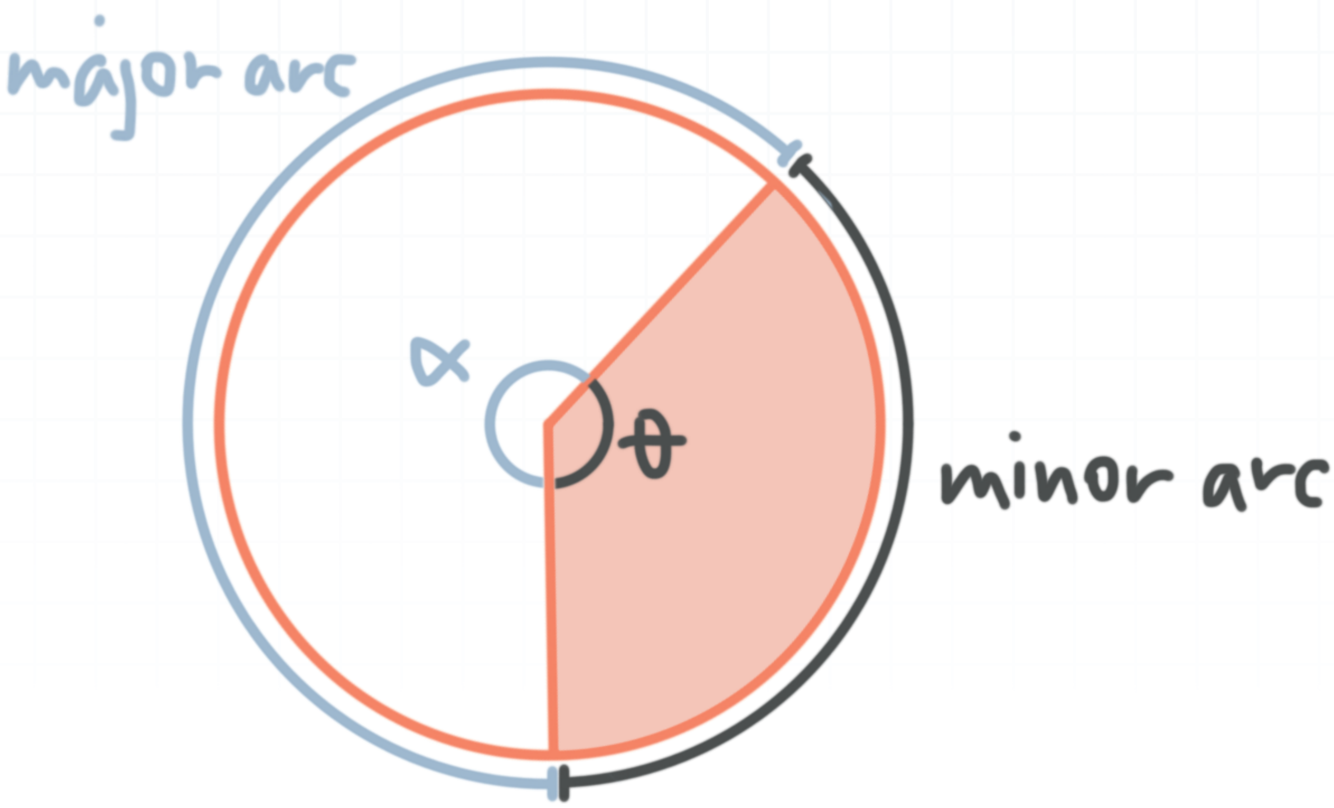
The angle at the center of the circle that defines the circular sector is the **central angle**. We say that the arc around the circular sector “subtends” the central angle, or that the central angle “is subtended” by the arc.

So in the figure, θ is the central angle subtended by the **minor arc** (the arc less than half the circle), and α is the central angle subtended by the **major arc** (the arc greater than half the circle).



Area of a circular sector

The area of a circular sector is always proportional to the size of the central angle θ. When θ is defined in radians, the area of a circular sector is

$$A = \frac{1}{2}r^2\theta$$

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If the angle we're given is measured in degrees, we can either convert it to radians, or we can use the formula for the area with a degree angle:

$$A = \left(\frac{\pi}{360}\right) r^2\theta$$

Let's use the formula to find the area of a circular sector, given the central angle and the radius of the circle.

Example

Find the area A in square inches of the circular sector with a central angle of $\pi/4$ radians, if the circle has a radius of 9 inches.

Since θ is in radians, the area of this circular sector is

$$A = \frac{1}{2}r^2\theta$$

$$A = \frac{1}{2}(9)^2\left(\frac{\pi}{4}\right)$$

$$A = \frac{81\pi}{8}$$

Let's do another example, but this time with an angle measured in degrees.

Example

In a circle of radius 10 centimeters, calculate the area A in square centimeters of a circular sector with a central angle of 80°.

Since θ is in degrees,

$$A = \left(\frac{\pi}{360}\right) r^2\theta$$

$$A = \left(\frac{\pi}{360}\right)(10)^2(80)$$

$$A = \frac{8{,}000\pi}{360}$$

$$A = \frac{200\pi}{9}$$

Area of the full circle

We may also run into circular sector problems where we're asked to solve for the area of the the entire circle, not just the sector, or where we're given information about the entire circle and asked to solve for the area of a sector of it.

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Usually we'll be given the coordinates for the center of the circle and the coordinates of one point on the perimeter of the circle.

The standard form of the equation of a circle is

$$(x-h)^2+(y-k)^2=r^2$$

where (h,k) is the center of the circle, r is its radius, and (x,y) represents the coordinates of any point on the circle.

Let's do an example problem where we use information about the circle to find the area of a sector within it.

Example

A circle passes through the point $(9,-7)$ and has its center at $(4,5)$. Find the area A of a sector of this circle if the sector is defined by the central angle $\theta=\pi/15$ radians.

Since the center of the circle is at $(4,5)$, notice that every point on this circle satisfies the equation

$$(x-4)^2+(y-5)^2=r^2$$

where r is the radius. Moreover, this circle passes through the point $(9,-7)$, so by letting $(x,y)=(9,-7)$, we can find the radius.

$$(9-4)^2+(-7-5)^2=r^2$$

$$5^2+(-12)^2=r^2$$

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$$25 + 144 = r^2$$

$$169 = r^2$$

$$r = \pm\sqrt{169}$$

$$r = \pm 13$$

The radius of a circle is a length, so it can't be negative, which means the circle's radius is $r = 13$.

We know the central angle of the circular sector is $\theta = \pi/15$ radians, so we'll plug everything into the formula for the area of a circular sector (with an angle in radians).

$$A = \frac{1}{2}r^2\theta$$

$$A = \frac{1}{2}(13)^2\left(\frac{\pi}{15}\right)$$

$$A = \frac{169\pi}{30}$$

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Trig functions of real numbers

Up to now, we've been evaluating trig functions at angles. In other words, we've treated the domain of the six trig functions as a set of angles, meaning that we've only plugged angle values in the trig functions.

But sometimes we want to evaluate trig functions at real numbers, not angles. When we use the six trig functions with real-number arguments, we often call them the six **circular functions**. And we usually indicate the real-number argument with a variable like s or t.

So when we see $\sin\theta$, it indicates a trig function being evaluated at an angle. But when we see $\sin t$ or $\sin s$, it indicates a trig function being evaluated at a real number.

Radian interpretation

When we see a trig function with a real-number argument, like $\sin 2$, we should always interpret it as a real number or as an angle in radians, such that $\sin 2$ means "sine of the real number 2" or "sine of 2 radians." We should never interpret $\sin 2$ to mean "sine of 2 degrees." If we'd wanted to specify the argument as an angle in degrees, we'd use the degree symbol to write $\sin 2^\circ$.

Therefore, because we're always interpreting the real-number argument in radians, if we're using a calculator to find values, we have to make sure the calculator is set to radian mode, not degree mode.

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Let's do an example.

Example

Evaluate the six circular functions at $t = 1.732$.

We'll use a calculator to evaluate the circular functions at 1.732, making sure the calculator is set to radian mode.

$$\sin 1.732 \approx 0.9870 \qquad \csc 1.732 \approx 1.0131$$

$$\cos 1.732 \approx -0.1605 \qquad \sec 1.732 \approx -6.2303$$

$$\tan 1.732 \approx -6.1495 \qquad \cot 1.732 \approx -0.1626$$

In the same way we evaluated trig functions at angles, realize here that, alternatively, we could have used the calculator to find sine and cosine at 1.732, and then used the quotient identity to find tangent, and then the reciprocal identities to find cosecant, secant, and cotangent.

Let's do one more example with a negative real number.

Example

Find the values of the six circular functions at $s = -0.6428$.

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We'll use a calculator to evaluate the circular functions at -0.6428, making sure the calculator is set to radian mode.

$$\sin(-0.6428) \approx -0.5994 \qquad \csc(-0.6428) \approx -1.6682$$

$$\cos(-0.6428) \approx 0.8004 \qquad \sec(-0.6428) \approx 1.2493$$

$$\tan(-0.6428) \approx -0.7489 \qquad \cot(-0.6428) \approx -1.3353$$

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Linear and angular velocity

Linear and angular velocity are great real-world applications of what we've already learned about angles in circles. But before we jump into talking about these velocities, we need to understand the difference between speed and velocity, and their relationship to one another.

Speed and velocity

Velocity is defined by two factors, magnitude and direction. So a large positive velocity tells us that we're moving forward at high speed, whereas a small negative velocity tells us that we're moving backward at low speed.

In contrast, **speed** is only the magnitude portion of velocity. Speed tells us how fast we're moving, but doesn't tell us the direction of movement. This makes sense, too, because speed is the absolute value of velocity, which means speed is always a positive number. A small positive value for speed tells us we're moving slowly, whereas a large positive value for speed tells us we're moving quickly.

Linear and angular velocity

With this in mind, we'll define linear and angular velocity, which are both related to the arcs and circles that we've been learning about.

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Linear velocity tells us how fast the length of an arc is changing. Imagine the arc of a circle. If the angle that creates the arc is growing, such that the length of the arc is increasing, then linear velocity will be positive, because the length of the arc is increasing over time. We use the formula

$$v = \frac{s}{t}$$

where v is linear velocity, s is arc length, and t is time.

On the other hand, while linear velocity gives the rate of change of the arc length, **angular velocity** tells us the rate of change of the interior angle (the rate at which the central angle is swept out as we move around the circle). The formula we use for angular velocity is

$$\omega = \frac{\theta}{t}$$

where ω (omega) is angular velocity, and θ is the radian measure of the interior angle at time t. Since θ is an angle measure and t is time, ω will be an angle measure per unit time, like radians per second, degrees per minute, etc.

Angular speed and angular velocity use the same formula; the difference between the two is that angular speed is a scalar quantity, while angular velocity is a vector quantity.

Let's do some examples, starting with one where we solve for angular velocity.

Example

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What is the angular velocity, in radians per second, of a disc that rotates at a constant rate and sweeps out an angle of 36.4π radians in 8.39 seconds?

To find the angular velocity ω, we'll divide the total angle swept out θ by the total time t.

$$\omega = \frac{\theta}{t}$$

$$\omega = \frac{36.4\pi \text{ radians}}{8.39 \text{ seconds}}$$

$$\omega = \frac{36.4\pi}{8.39} \text{ radians per second}$$

$$\omega \approx 4.34\pi \text{ radians per second}$$

We always want to be careful about units, and make sure that we have matching units of time. If we don't, we'll need to do some conversions.

Example

What is the angular velocity, in radians per second, of a wheel that rotates at a constant rate and sweeps out an angle of 72.7π radians in 3.2 minutes?

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Since we're asked to find angular velocity in radians per second, but we're given the rotation in minutes, we'll need to first convert the minutes into seconds.

$$t = (3.2 \text{ minutes})\left(\frac{60 \text{ seconds}}{1 \text{ minute}}\right)$$

$$t = 3.2(60) \text{ seconds}$$

$$t = 192 \text{ seconds}$$

Now we can find angular velocity.

$$\omega = \frac{\theta}{t}$$

$$\omega = \frac{72.7\pi \text{ radians}}{192 \text{ second}}$$

$$\omega = \frac{72.7\pi}{192} \text{ radians per second}$$

$$\omega \approx 0.379\pi \text{ radians per second}$$

Let's do an example where we need to convert between radians and degrees.

Example

Find the angular velocity, in radians per second, of an object that rotates at a constant rate and sweeps out an angle of $1{,}043^\circ$ in 5.9 seconds.

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We'll first convert the angle $1{,}043^\circ$ to radians.

$$\theta = 1{,}043^\circ\left(\frac{\pi \text{ radians}}{180^\circ}\right)$$

$$\theta = \frac{1{,}043\pi}{180}\text{radians}$$

Now we'll find angular velocity.

$$\omega = \frac{\theta}{t}$$

$$\omega = \frac{\frac{1{,}043\pi}{180}\text{ radians}}{5.9\text{ seconds}}$$

$$\omega = \frac{1{,}043\pi}{180(5.9)}\text{ radians per second}$$

$$\omega \approx 0.982\pi\text{ radians per second}$$

We also often express angular velocity in *revolutions* per unit of time. When we want to convert from one set of units to another, we'll need to remember that there's 1 revolution per 2π radians.

Example

Express an angular velocity of 31 radians per second in units of revolutions per minute.

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We know the angular velocity ω, and we just need to convert it to different units.

$$\omega = 31\ \frac{\text{radians}}{\text{second}}$$

$$\omega = \left(31\ \frac{\text{rad}}{\text{sec}}\right)\left(\frac{1\ \text{rev}}{2\pi\ \text{rad}}\right)\left(\frac{60\ \text{sec}}{1\ \text{min}}\right)$$

$$\omega = \frac{31(60)}{2\pi}\ \text{revolutions per minute}$$

If we say $\pi \approx 3.14$, we get

$$\omega \approx 296\ \text{revolutions per minute}$$

Here's an example of a conversion in the opposite direction.

Example

Express angular velocity of 86.3 revolutions per minute in units of radians per second.

We already know angular velocity, we just need to convert the units.

$$\omega = 86.3\ \frac{\text{revolutions}}{\text{minute}}$$

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$$\omega = \left(86.3\ \frac{\text{rev}}{\text{min}}\right)\left(\frac{2\pi \text{ rad}}{1 \text{ rev}}\right)\left(\frac{1 \text{ min}}{60 \text{ sec}}\right)$$

$$\omega = \frac{86.3(2)\pi}{60} \text{ radians per second}$$

$$\omega \approx 2.88\pi \text{ radians per second}$$

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Relating linear and angular velocity

Now we're ready to discuss linear velocity in connection with angular velocity. To relate them to each other, we can use the arc length formula that we learned before, $s = r\theta$. Because $s = r\theta$, we'll substitute $r\theta$ into the linear velocity formula for s.

$$v = \frac{s}{t} = r\frac{\theta}{t}$$

Then, because we know that angular velocity is $\omega = \theta/t$, we can replace the θ/t in this linear velocity formula with ω.

$$v = \frac{s}{t} = r\frac{\theta}{t} = r\omega$$

So now we have a formula relating linear velocity directly to angular velocity, which tells us that linear velocity is equivalent to the product of the length of the radius and angular velocity,

$$v = r\omega$$

or that angular velocity is the quotient of linear velocity and the radius of the circle.

$$\omega = \frac{v}{r}$$

Let's do an example where we calculate linear velocities when we know the angular velocity.

Example

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What are the linear velocities, in inches per second, of points on a rotating disc that has an angular velocity of 9.4 radians per second if those points are located 1.3 and 2.6 inches, respectively, from the center of the disc?

For the point that's 1.3 inches from the center of the disc, we'll say $r = 1.3$ and $\omega = 9.4$.

$$v = r\omega$$

$$v = (1.3 \text{ inches})\left(\frac{9.4}{\text{second}}\right)$$

$$v = 1.3(9.4) \text{ inches per second}$$

$$v = 12.22 \text{ inches per second}$$

For the point that's 2.6 inches from the center of the disc, $r = 2.6$ but ω is still 9.4.

$$v = r\omega$$

$$v = (2.6 \text{ inches})\left(\frac{9.4}{\text{second}}\right)$$

$$v = 2.6(9.4) \text{ inches per second}$$

$$v = 24.44 \text{ inches per second}$$

Notice that the linear velocity of the second point is double the linear velocity of the first point. That's because the arc traced out by the second point is twice as long as the arc traced out by the first point.

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Sometimes we'll have to convert units before we can solve the problem.

Example

What is the linear velocity, in inches per second, of a point that's 5.6 centimeters from the center of the disc if the disc is rotating at 42.8 revolutions per minute?

We'll need to convert the lengths from centimeters to inches, and we'll need to convert the angular velocity from revolutions per minute to radians per second.

For the first conversion, we'll use the fact that 1 inch is about 2.54 centimeters.

$$r = 5.6 \text{ cm}$$

$$r \approx (5.6 \text{ cm})\left(\frac{1 \text{ in}}{2.54 \text{ cm}}\right)$$

$$r \approx \frac{5.6}{2.54} \text{ in}$$

$$r \approx 2.20 \text{ in}$$

To express the angular velocity in radians per second, we'll get

$$\omega = 42.8 \frac{\text{rev}}{\text{min}}$$

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$$\omega = \left(42.8\ \frac{\text{rev}}{\text{min}}\right)\left(\frac{2\pi\ \text{rad}}{1\ \text{rev}}\right)\left(\frac{1\ \text{min}}{60\ \text{sec}}\right)$$

$$\omega = \frac{42.8(2)\pi}{60} \text{ radians per second}$$

$$\omega \approx 1.43\pi \text{ radians per second}$$

Now we'll combine these two results to get the linear speed.

$$v = r\omega$$

$$v \approx (2.20\text{in})\left(\frac{1.43\pi}{\text{sec}}\right)$$

$$v \approx 2.20(1.43\pi) \text{ inches per second}$$

$$v \approx 9.88 \text{ inches per second}$$

Let's do an example where we asked to solve for an unknown central angle.

Example

An object moves at a constant linear speed of 15 m/sec around a circle of radius 5 m. How large of a central angle does it sweep out in 2.5 seconds?

The formula for linearly velocity is $v = r\omega$, and we know that $\omega = \theta/t$, so

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$$v = \frac{r\theta}{t}$$

Solve this equation for θ, then substitute the values from the question.

$$\theta = \frac{vt}{r} = \frac{15 \cdot 2.5}{5} = 7.5 \text{ radians}$$

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Sketching sine and cosine

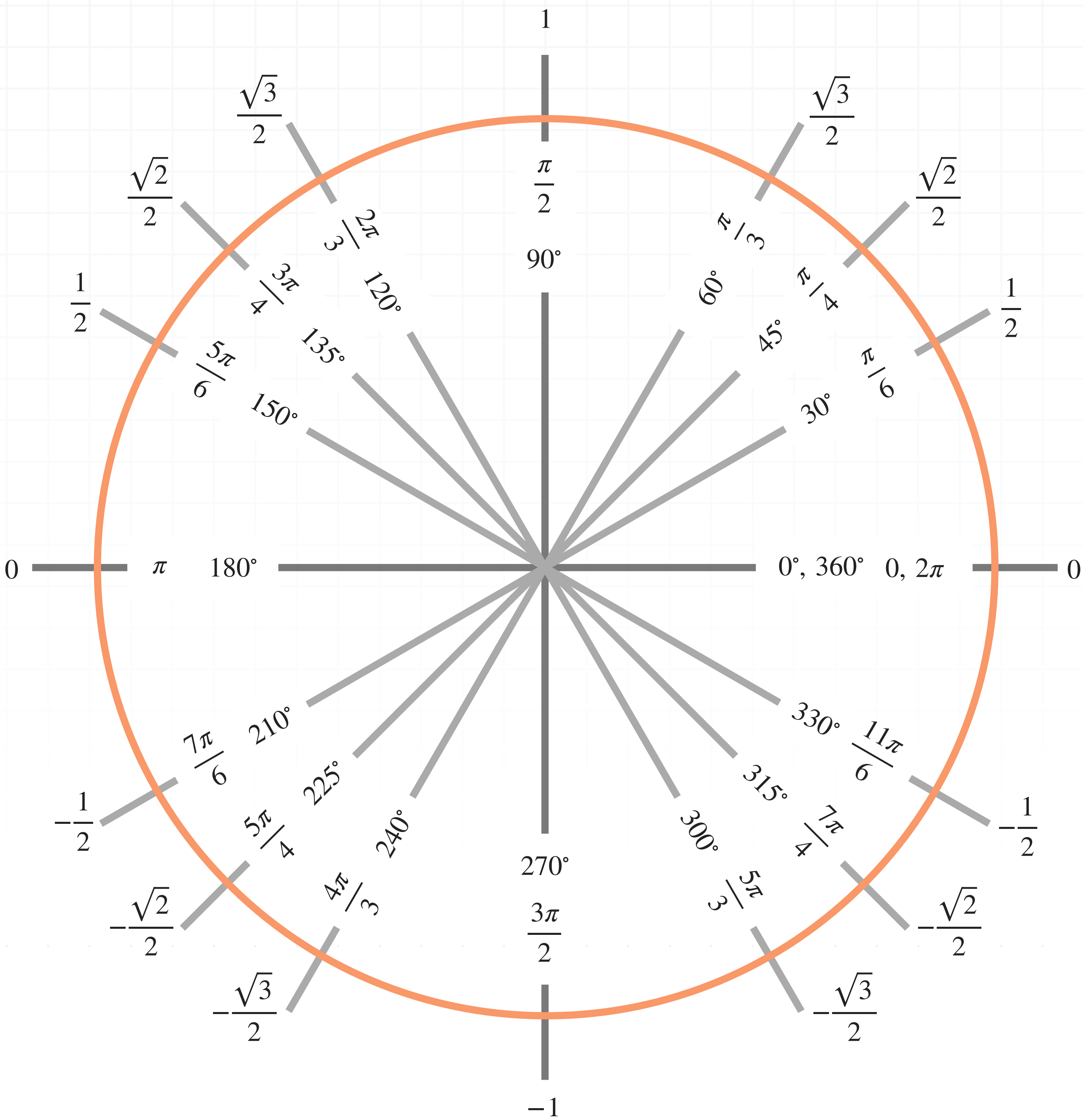
At this point we've worked extensively with the six trig functions, and now we want to turn our attention toward sketching their graphs. We'll start with the graphs of sine and cosine, and then later we'll look at the graphs of the other four trig functions.

The sine and cosine graphs

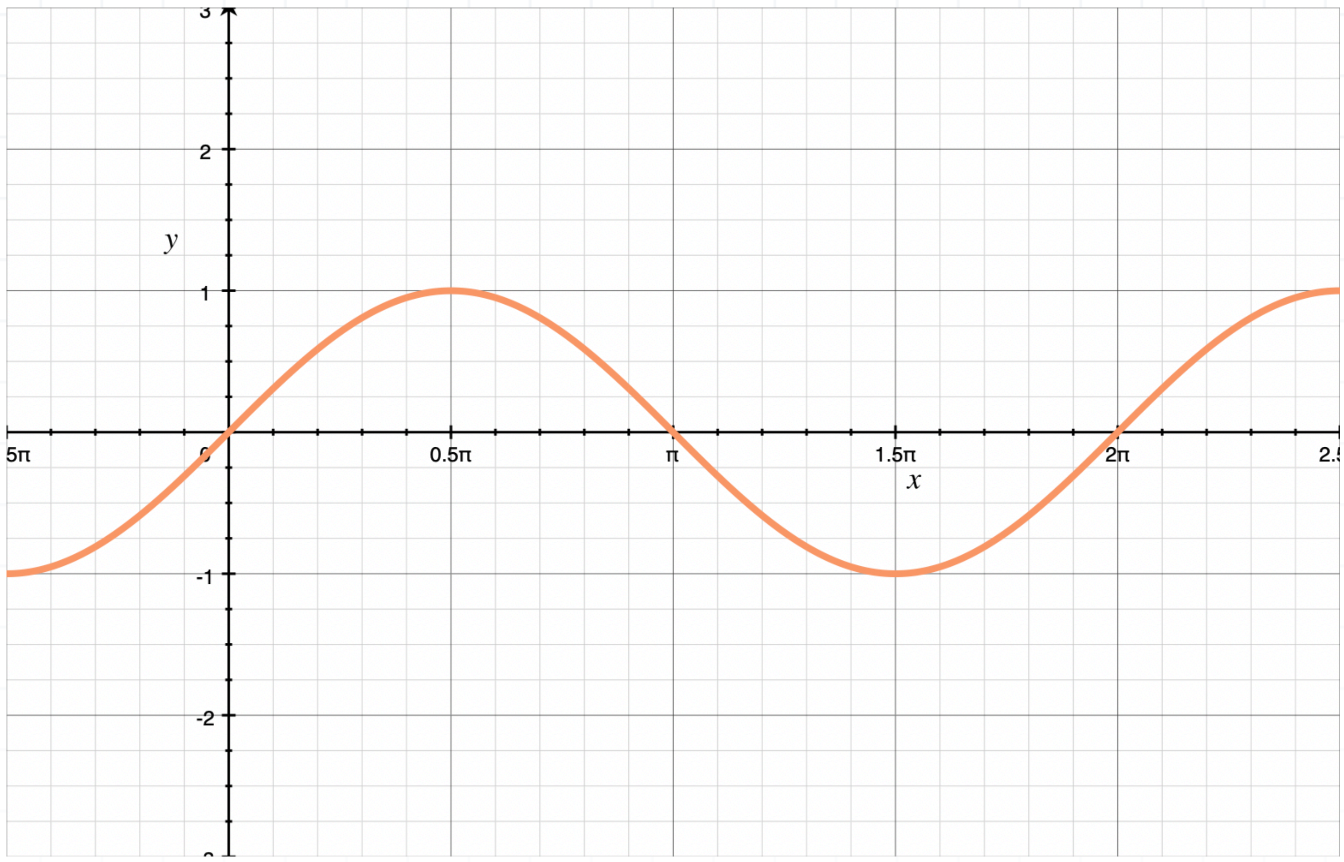
We already have some idea of the values of the sine and cosine function, based on what we know they do in the unit circle.

For instance, in the unit circle below, which shows only the sine values along the perimeter, we see that the value of sine is 0 when $\theta = 0$, then the value of sine increases as we head up toward $\theta = \pi/2$, eventually reaching its maximum value of 1.

Its value starts to decrease as we head back down toward π until it reaches 0, then the value of sine eventually reaches its minimum value of -1 at $\theta = 3\pi/2$, and finally it returns to 0 when we get back around to $\theta = 2\pi$.



If we keep wrapping around the circle again and again, the value of sine continues to follow the same pattern. And if we plot those sine values in an xy-plane, we get the curve for $y = \sin x$:



On the other hand, we can see from the unit circle below, which shows only the cosine values along the perimeter, the value of cosine is 1 when $\theta = 0$, then the value of cosine decreases as we head up toward $\theta = \pi/2$, eventually reaching 0. Its value continues to decrease as we head back down toward π until it reaches its minimum value of -1, then the value of cosine gets back to 0 at $\theta = 3\pi/2$, and finally it returns to 1 when we get back around to $\theta = 2\pi$.

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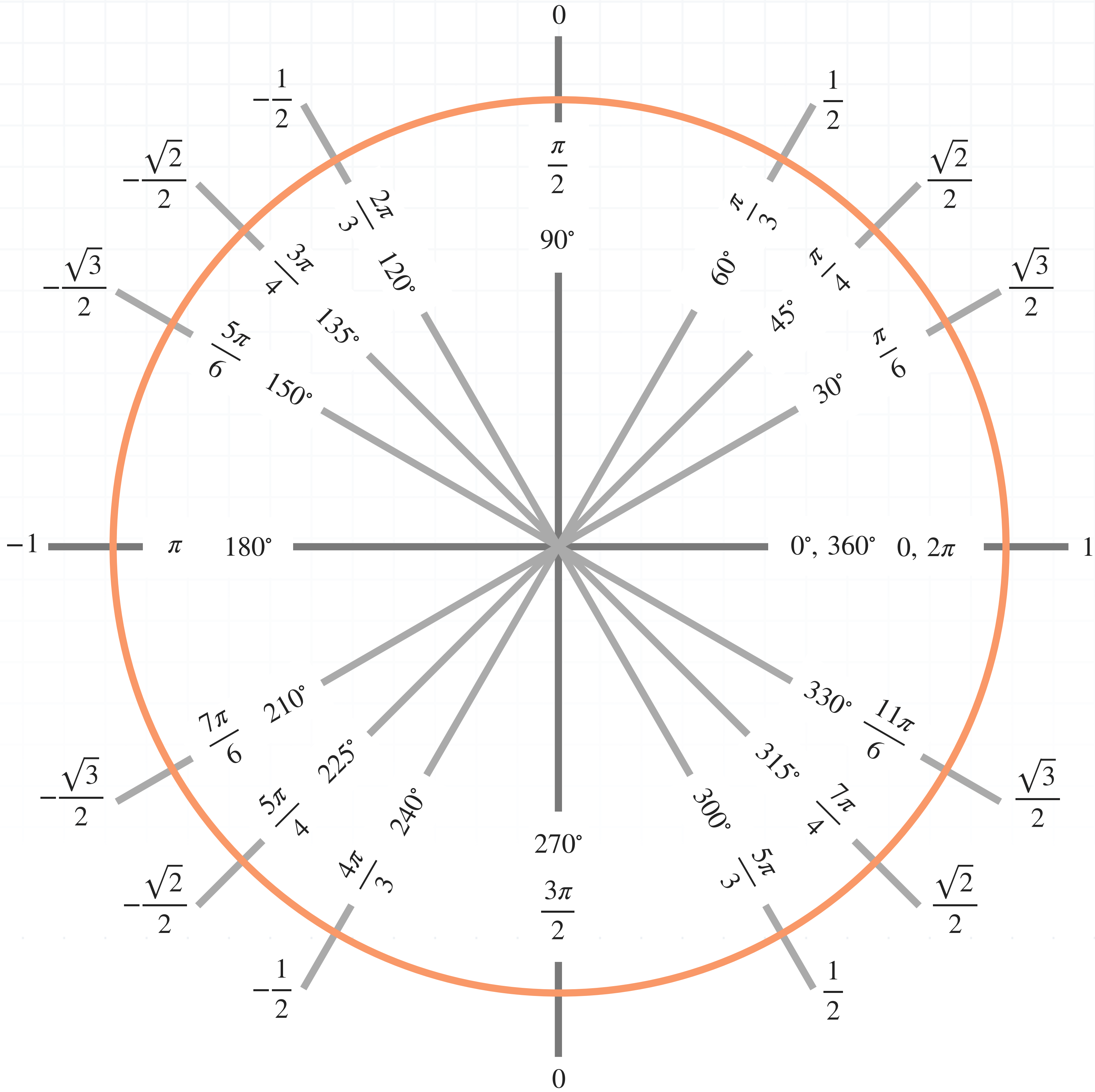
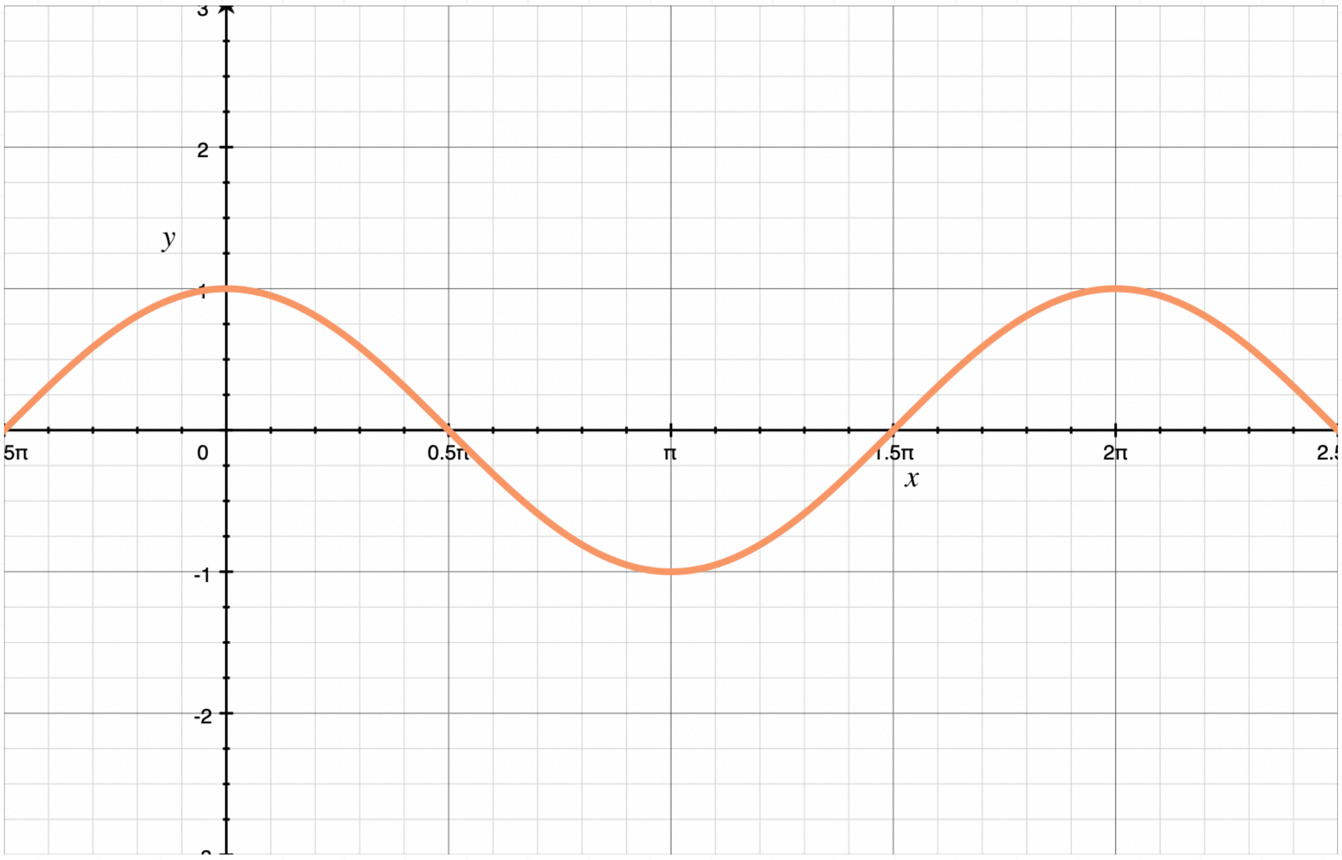


If we keep wrapping around the circle again and again, the value of cosine continues to follow the same pattern. And if we plot those cosine values in an xy-plane, we get the curve for $y = \cos x$:



These two basic sine and cosine curves model the equations $y = \sin x$ and $y = \cos x$. But we actually want to format every sine and cosine equation as

$$y = a\sin(b(x + c)) + d$$

$$y = a\cos(b(x + c)) + d$$

where a, b, c, and d are real numbers. The $y = \sin x$ and $y = \cos x$ equations actually follow this same format, just with $a = 1$, $b = 1$, $c = 0$, and $d = 0$.

Changing the value of a

When we have a sine or cosine equation with $b = 1$, $c = 0$, and $d = 0$, the format simplifies to

$$y = a\sin x$$

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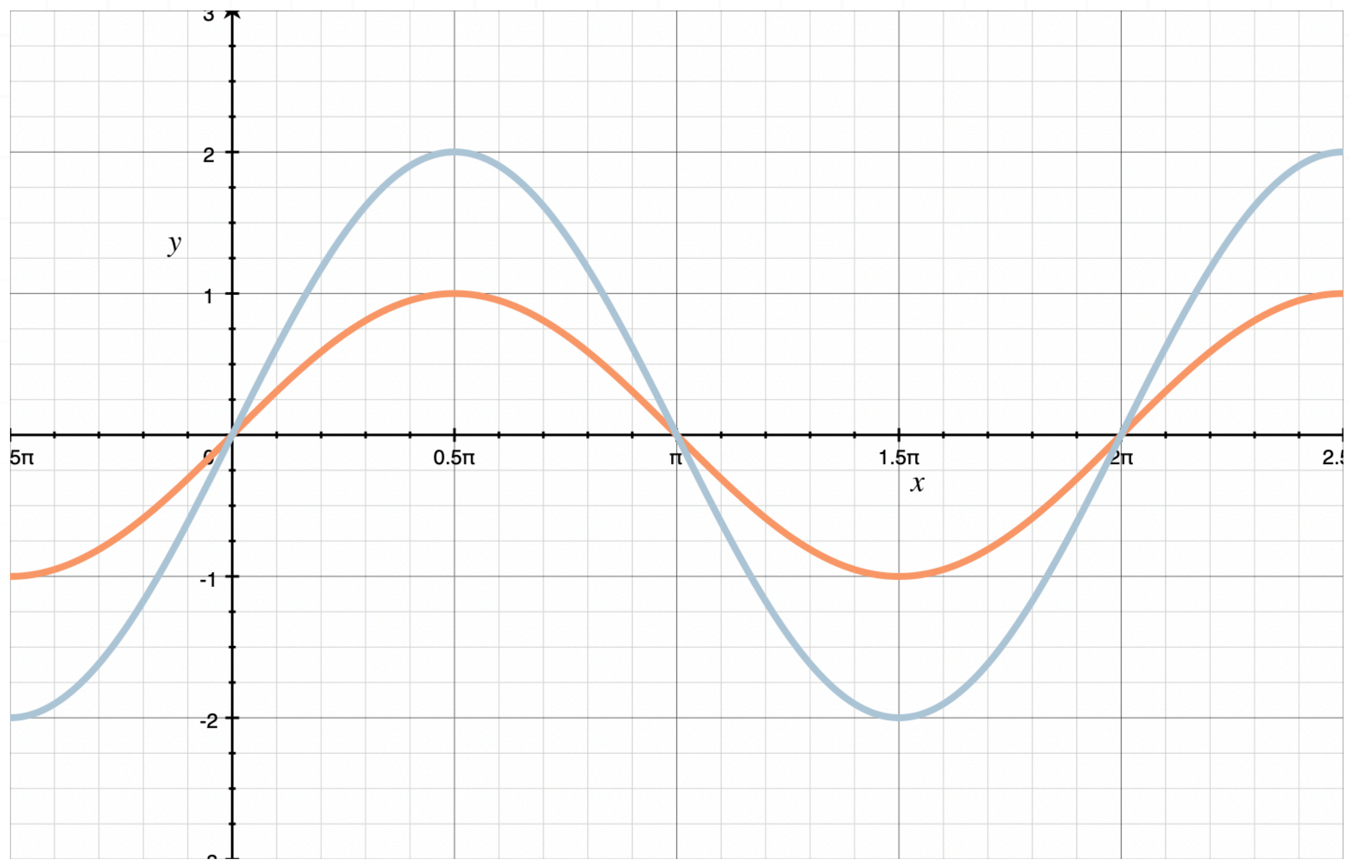
$$y = a \cos x$$

We've already seen what the sine and cosine graphs look like with $a = 1$. When a is some number other than 1 (and not 0), we'll stretch or compress vertically by the value of a. In other words, if $a = 2$, the graph will be stretched vertically by a factor of 2, meaning that it'll be twice as tall. Here are the graphs of the original sine function $y = \sin x$ in red and $y = 2 \sin x$ in blue,



and the original cosine function $y = \cos x$ in red with $y = 2 \cos x$ in blue.

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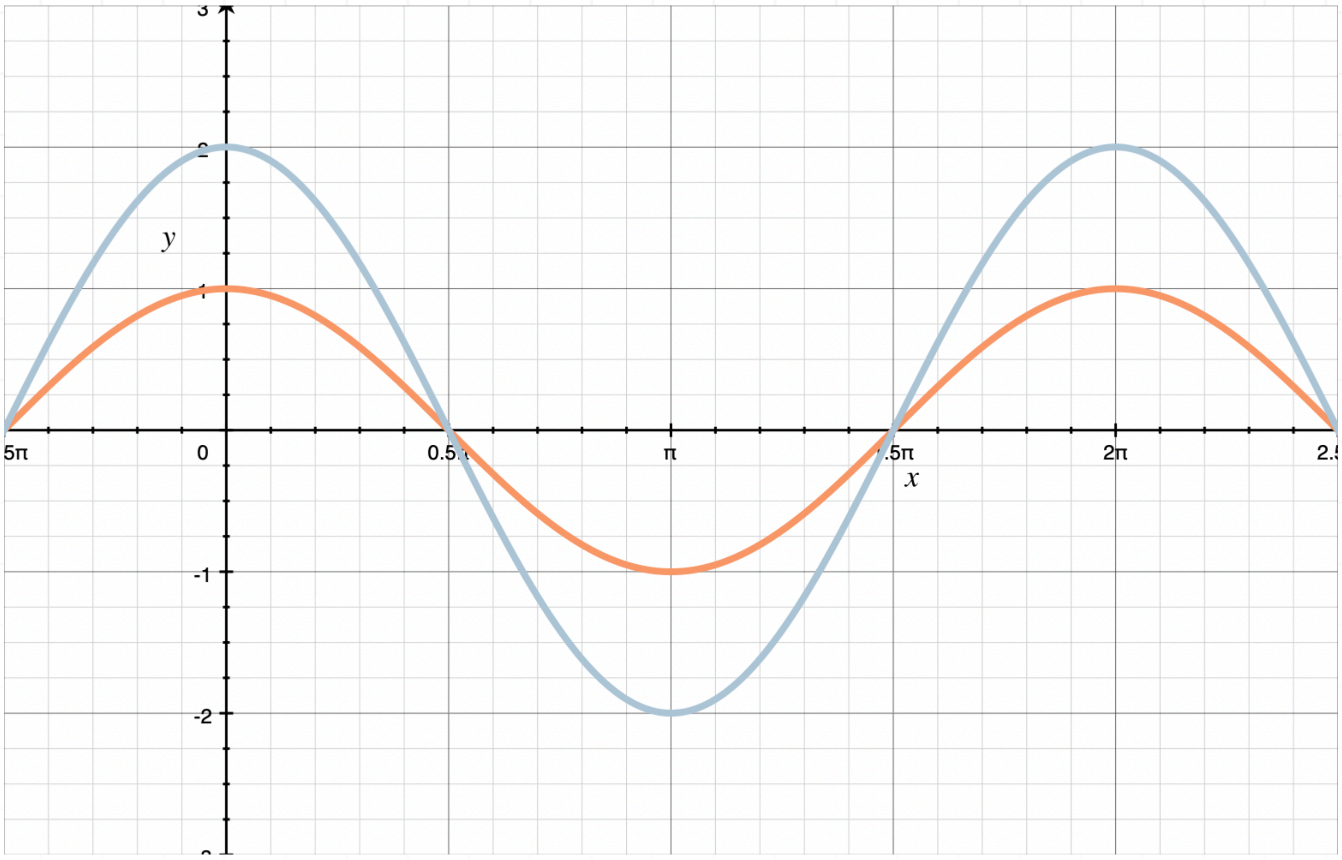
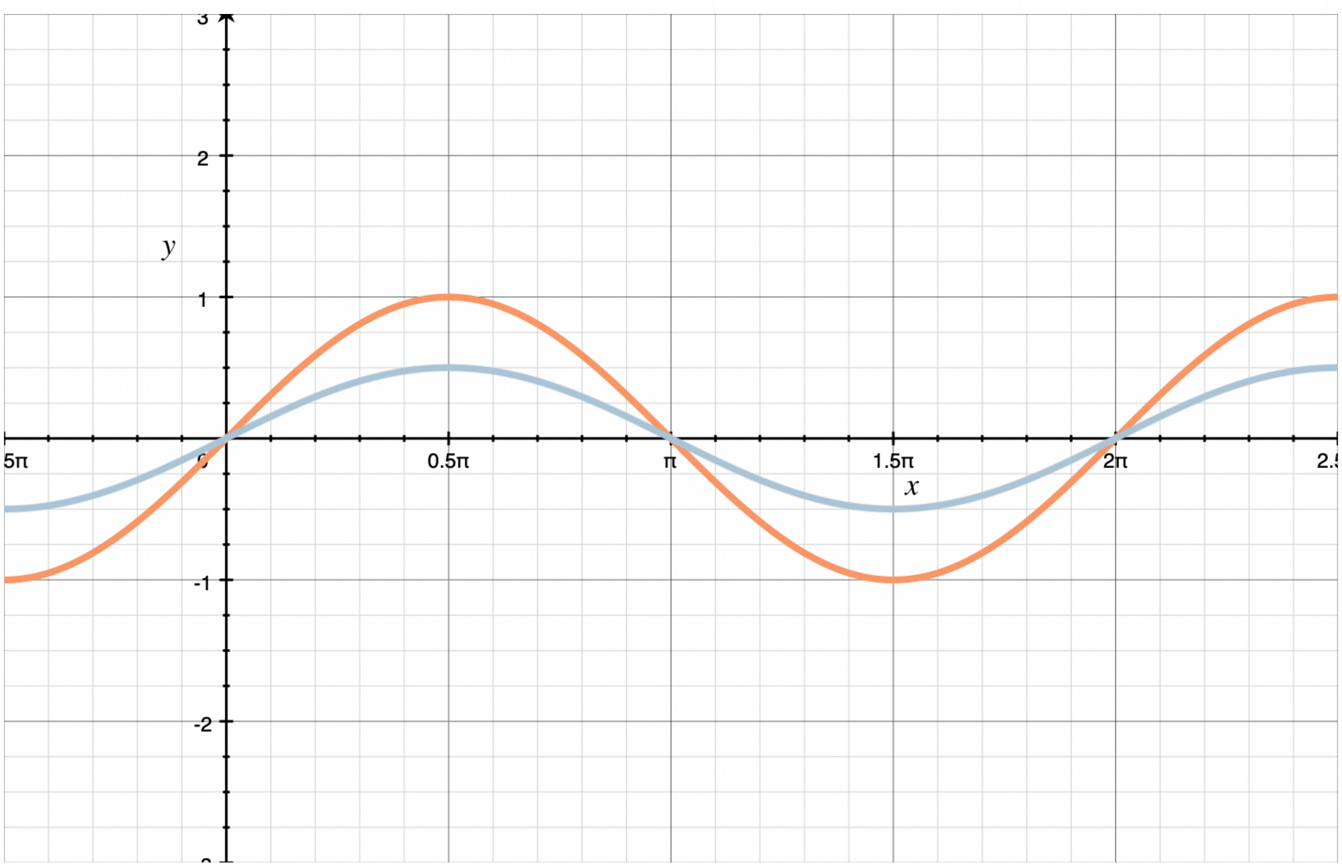


But if $a = 1/2$, the graph will be compressed vertically by a factor of $1/2$, meaning that it'll be half as tall. Here are the graphs of the original sine function $y = \sin x$ in red and $y = (1/2)\sin x$ in blue,



and the original cosine function $y = \cos x$ in red with $y = (1/2)\cos x$ in blue.

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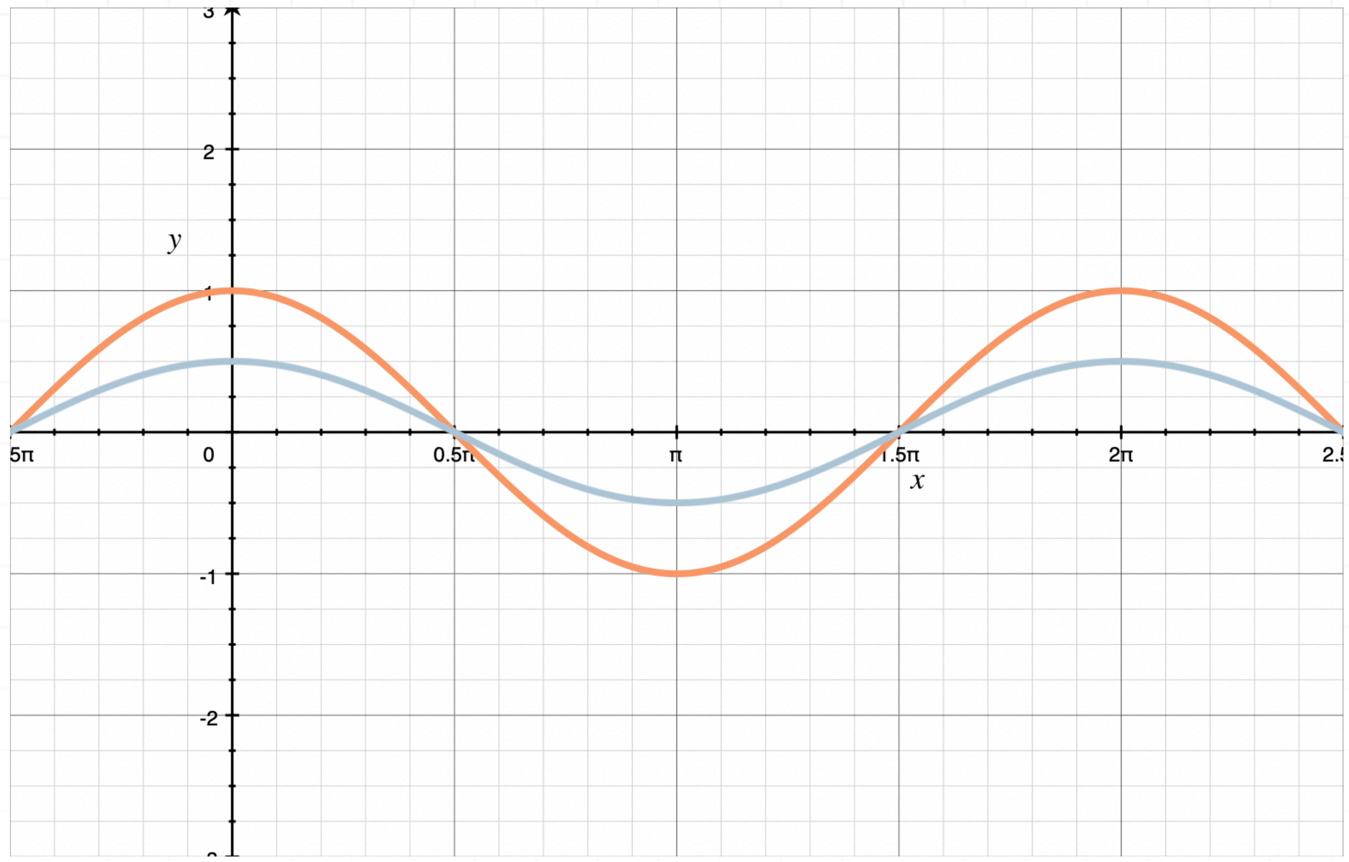
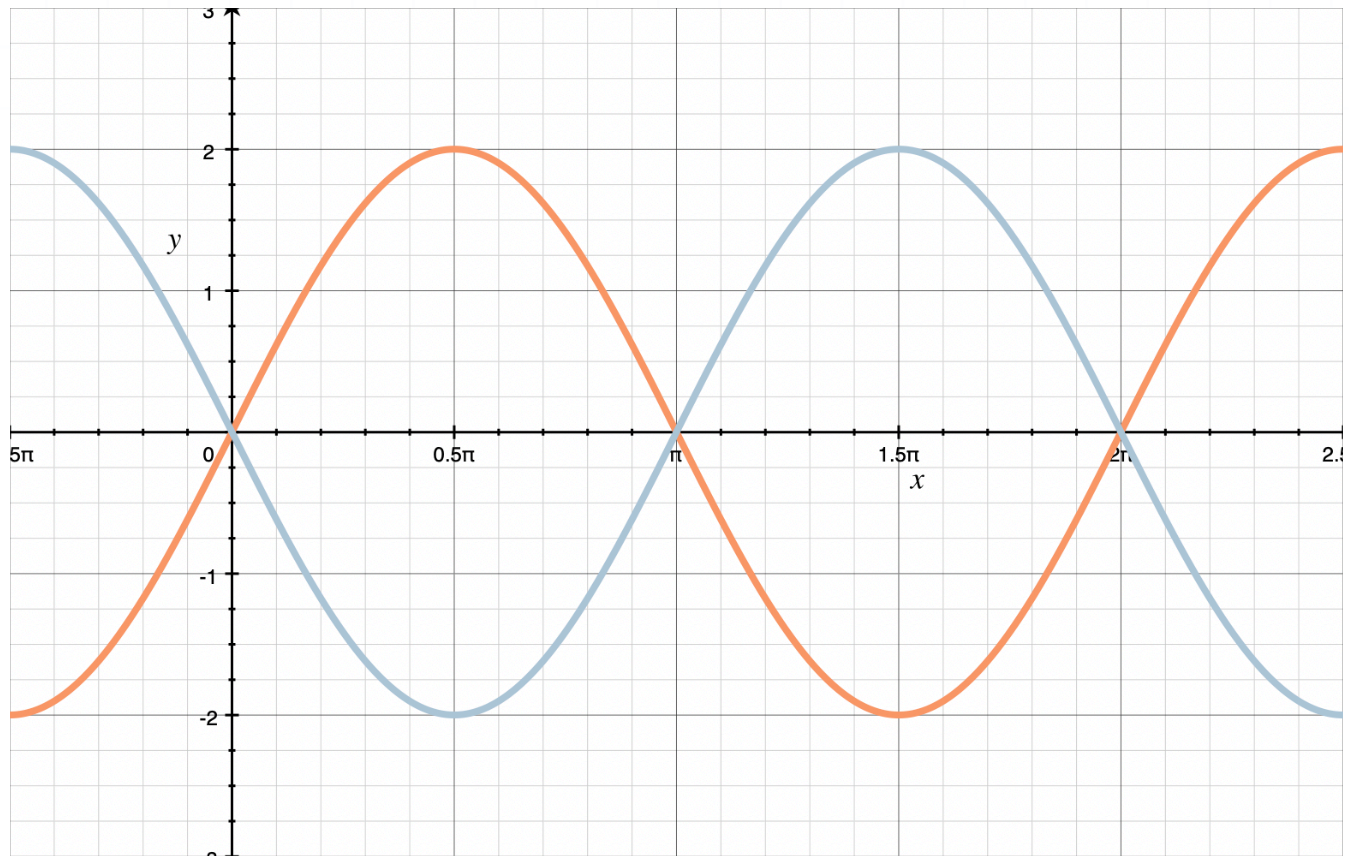


When the value of a is negative, the function is just reflected over the x-axis. As an example, $y = 2\sin x$ is graphed in red with $y = -2\sin x$ in blue:



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Changing the value of b

We just set $b = 1$, $c = 0$, and $d = 0$ so that we could focus on what happens to the function as we change the value of a. Now let's set $a = 1$, $c = 0$, and $d = 0$, such that the sine and cosine equation format simplifies to

$$y = \sin(bx)$$

$$y = \cos(bx)$$

and we can focus on what happens to the function as we change the value of b.

We've already seen what the graphs of sine and cosine look like when $b = 1$. Similar to the way that changing the value of a would stretch or compress the function vertically, changing the value of b will stretch or compress the function horizontally. We can almost think about holding both ends of a spring and pulling it apart or pushing it back together.

For instance, this graph shows $y = \sin x$ in red and $y = \sin(2x)$ in blue graphed together:

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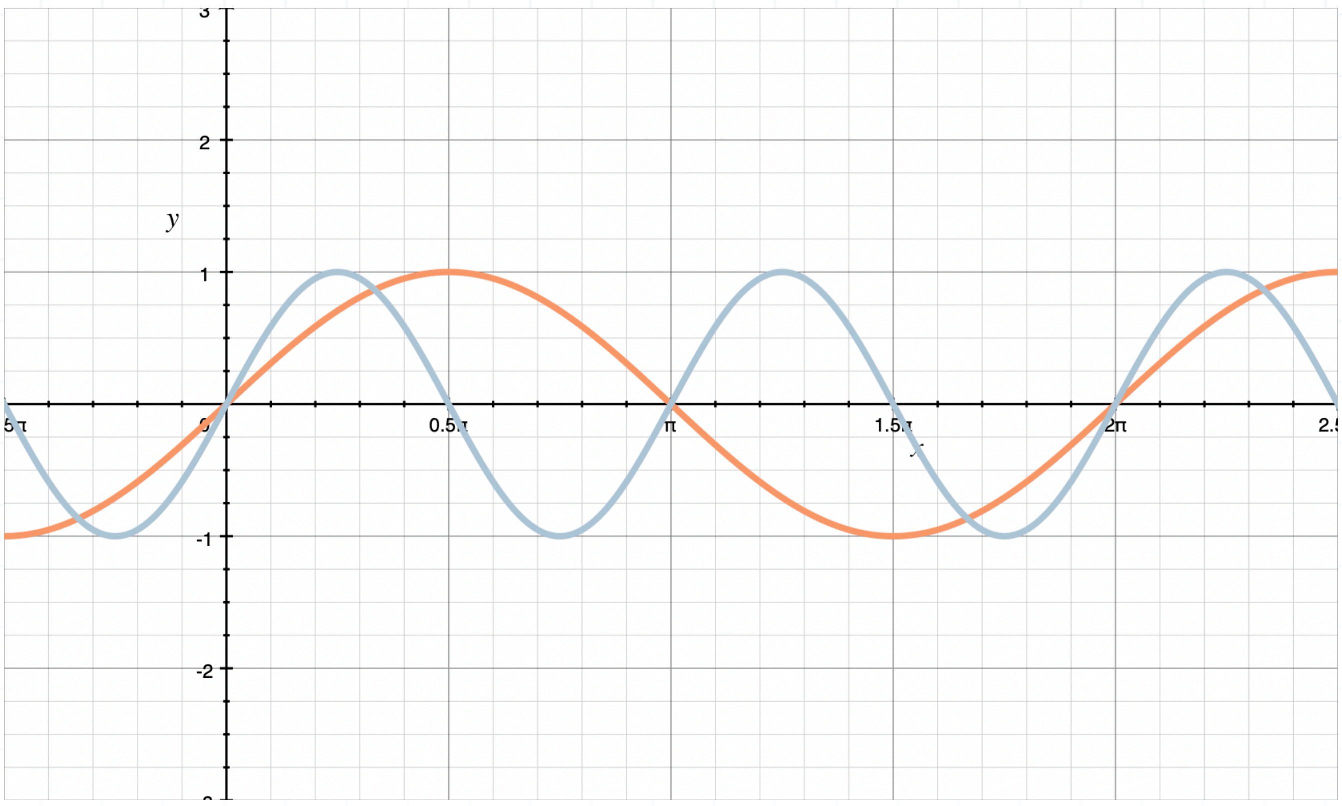
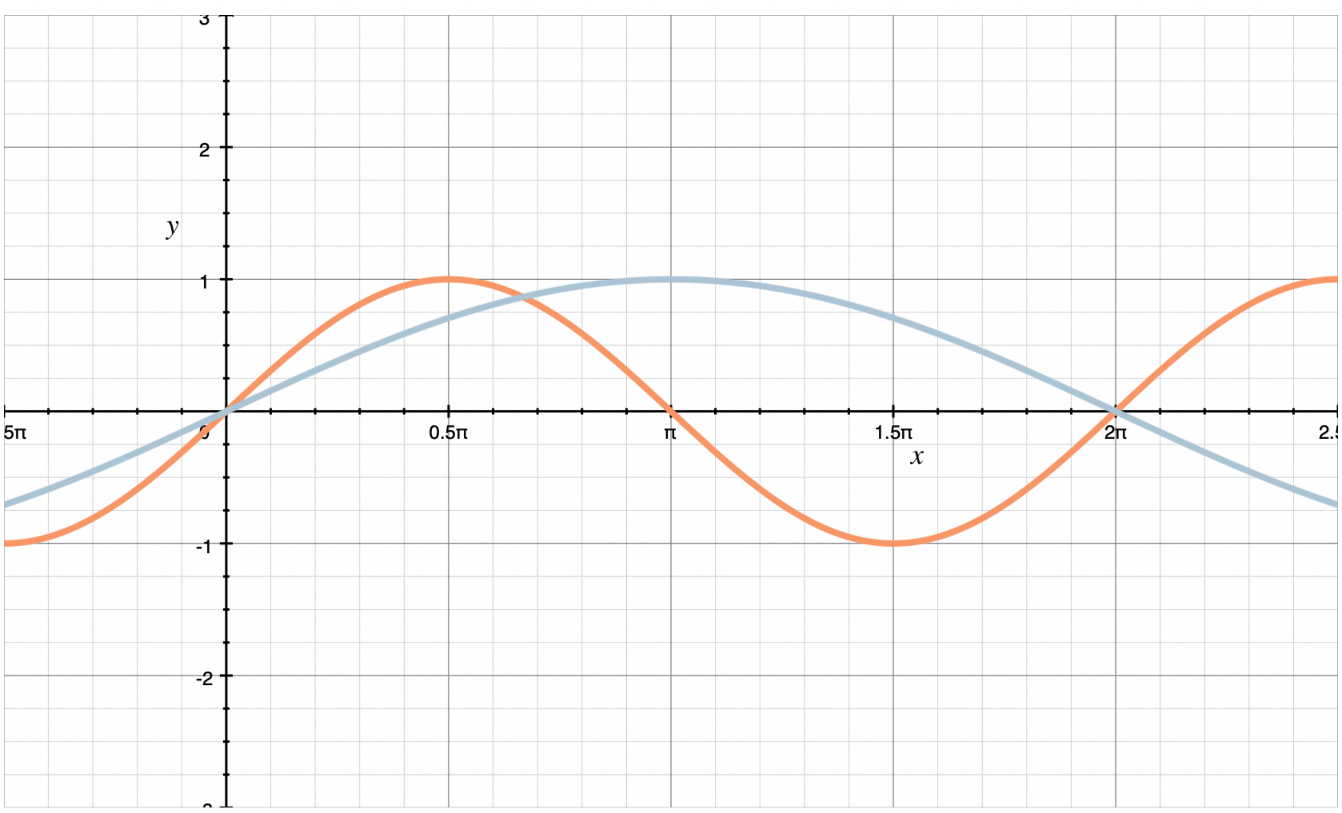


Notice how *increasing* the value of b horizontally *compresses* the function. As you might suspect now, *decreasing* the value of b will horizontally *stretch* the function. We can see that in the graph of $y = \sin x$ in red, together with $y = \sin((1/2)x)$ in blue.



The same is true for the cosine function. The graph of $y = \cos(2x)$ in blue is horizontally compressed compared to the graph of $y = \cos x$ in red,

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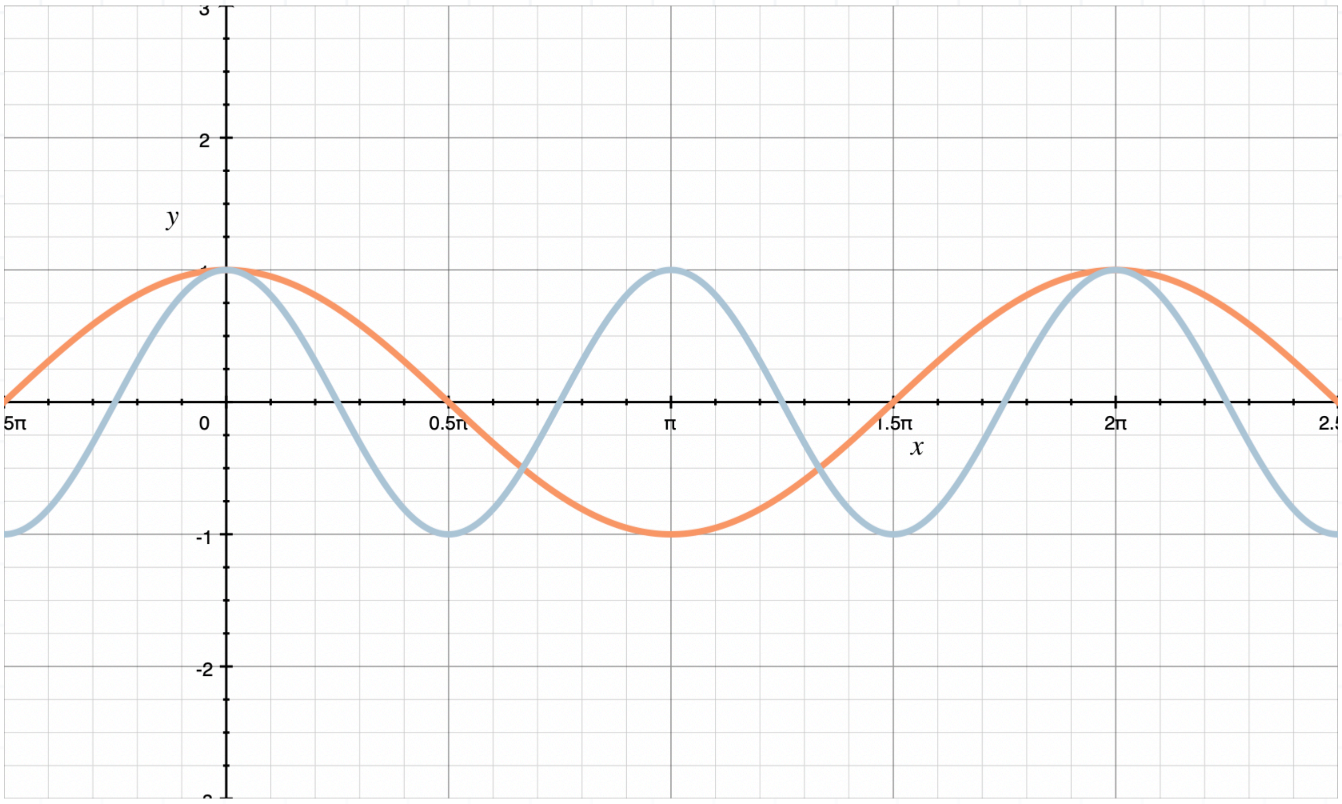
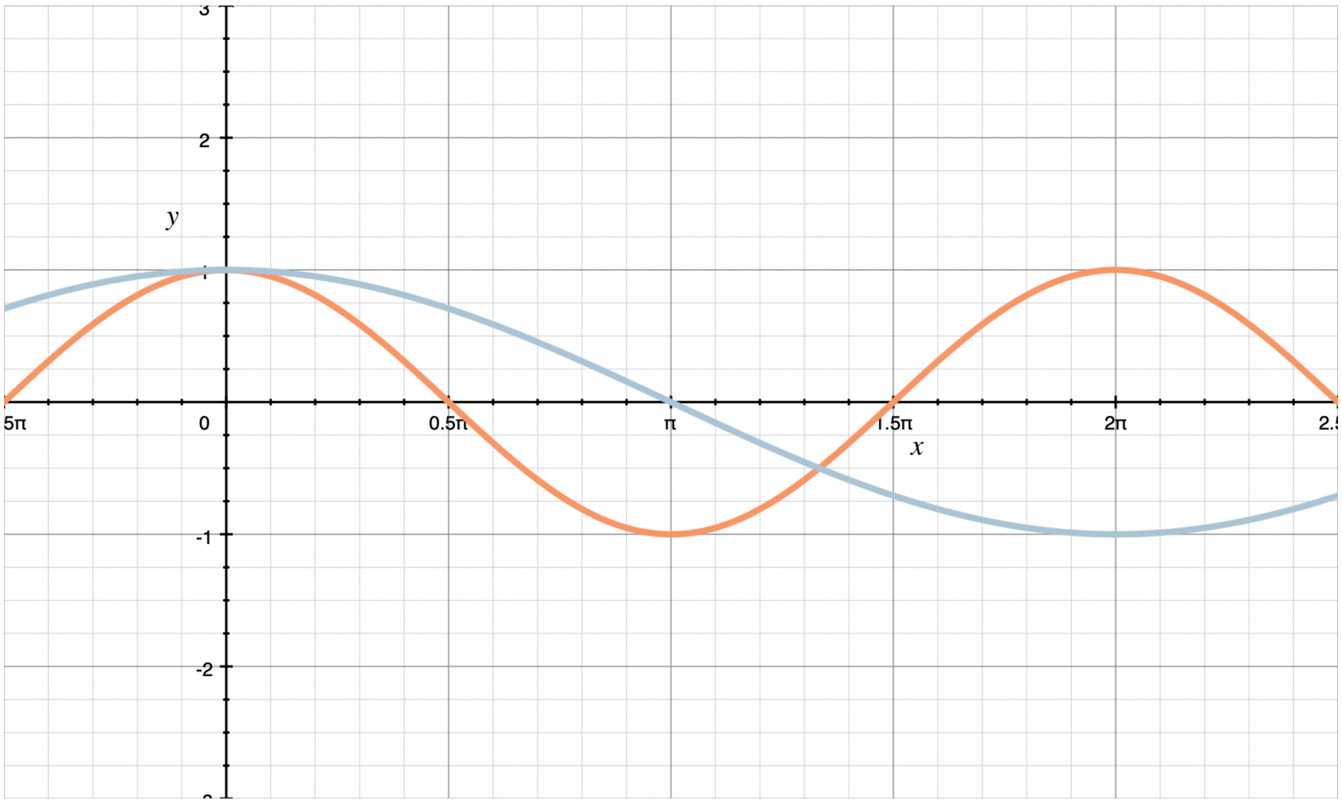


and the graph of $y = \cos((1/2)x)$ in blue is horizontally stretched compared to the graph of $y = \cos x$ in red:



When the value of b is negative, the function is just reflected over the y-axis. For example, we can see the graph of $y = \sin x$ in red alongside the graph of $y = \sin(-x)$ in blue.

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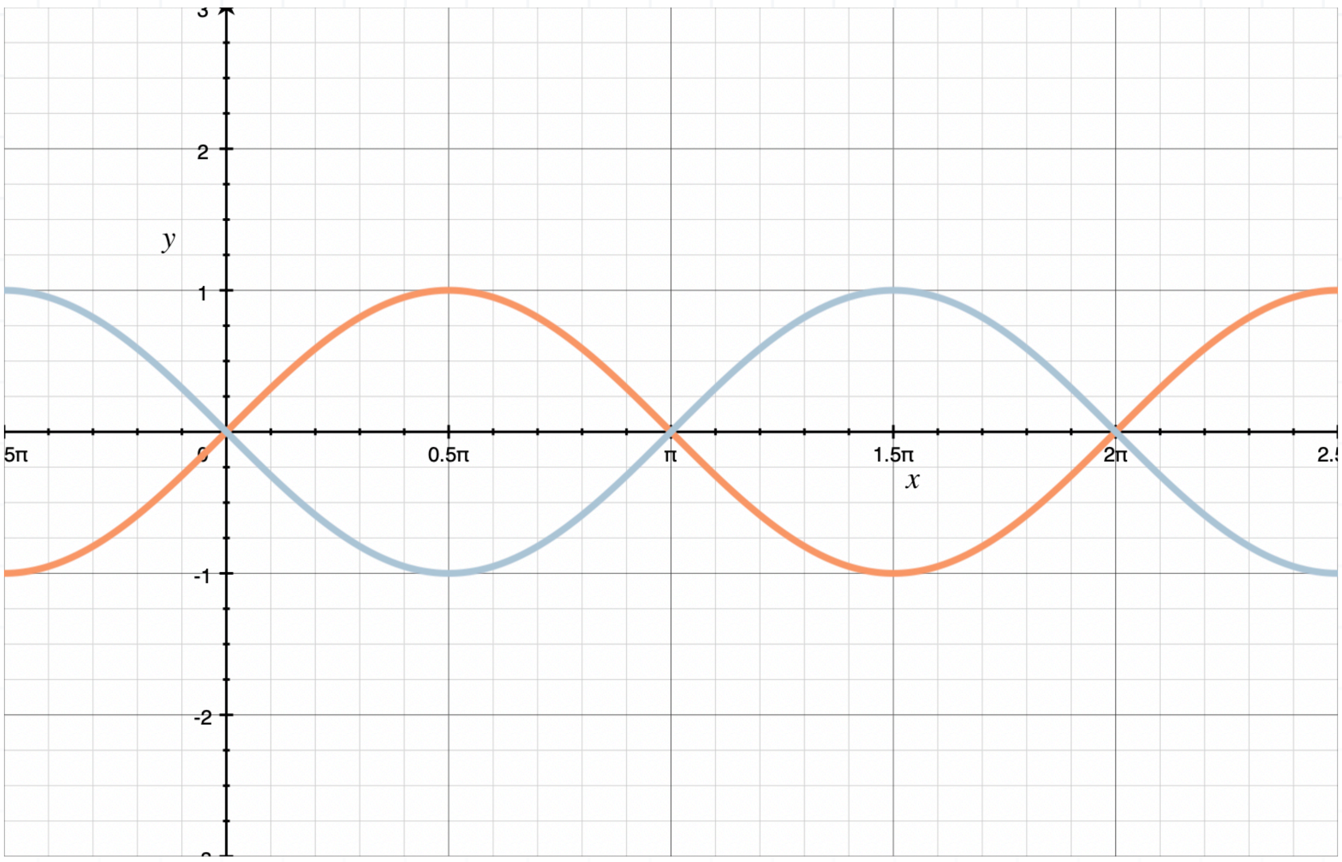

In a later lesson we'll look more at the effect of a and b on the graphs of trig functions, and we'll also work through what it means to add in non-zero values for c and d.

But for now, let's do some examples where we sketch the graphs of sine and cosine for different values of a and b.

Example

Sketch the graph of $y = 3\sin(2\theta)$.

In this case $a = 3$ and $b = 2$. Setting $b = 2$ means we'll compress $y = \sin\theta$ horizontally by a factor of 2. So if $y = \sin\theta$ includes the points

$$y = \sin\theta: \qquad (0,0) \qquad \left(\frac{\pi}{2},1\right) \qquad (\pi,0) \qquad \left(\frac{3\pi}{2},-1\right) \qquad (2\pi,0)$$

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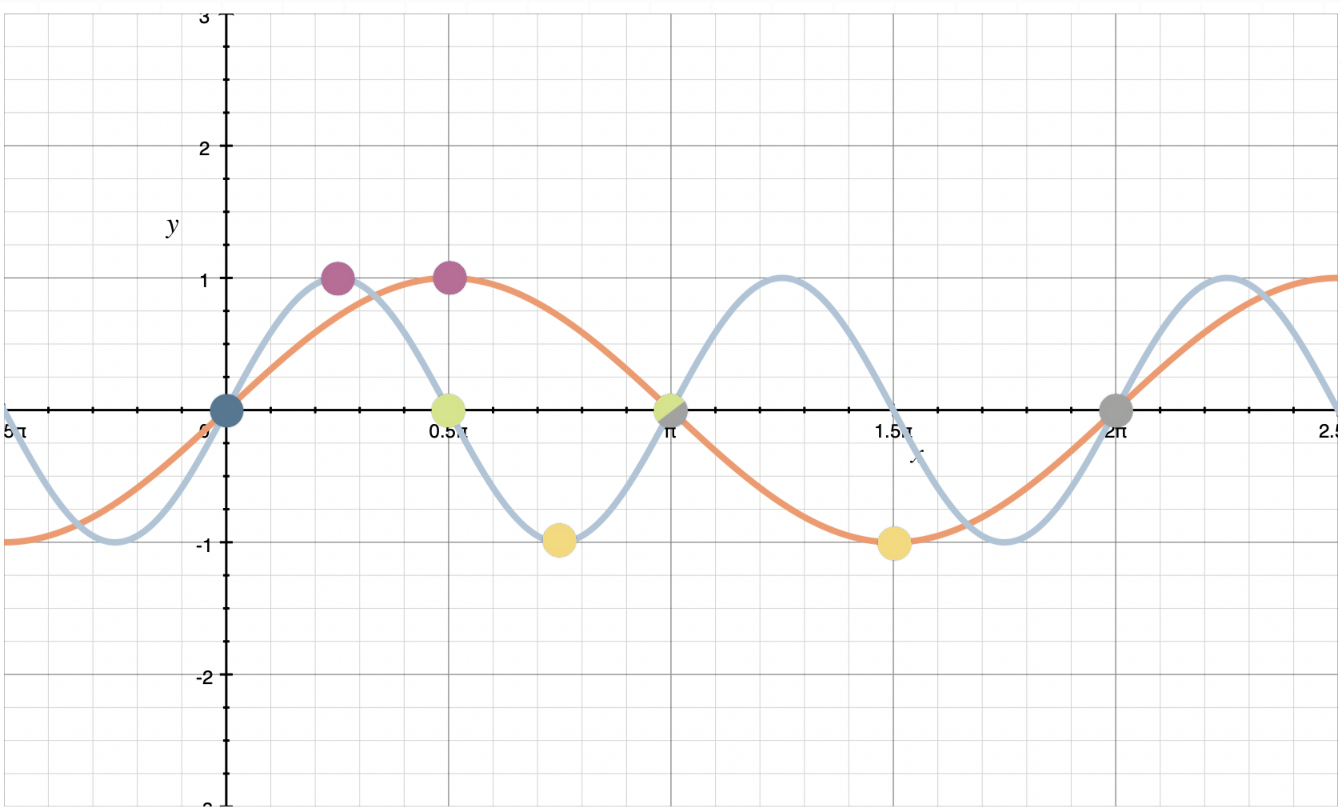
then $y = \sin(2\theta)$ will halve the x-values in these coordinates, while keeping the y-values the same.

$$y = \sin\theta: \quad (0,0) \quad \left(\frac{\pi}{2}, 1\right) \quad (\pi, 0) \quad \left(\frac{3\pi}{2}, -1\right) \quad (2\pi, 0)$$

$$y = \sin(2\theta): \quad (0,0) \quad \left(\frac{\pi}{4}, 1\right) \quad \left(\frac{\pi}{2}, 0\right) \quad \left(\frac{3\pi}{4}, -1\right) \quad (\pi, 0)$$

We see how the points shift and the graph compresses horizontally if we sketch $y = \sin\theta$ in red and $y = \sin(2\theta)$ in blue:



Notice how the purple point on the red curve moves to the left to become the corresponding purple point on the blue curve. In the same way, the green point on the red curve moves left to become the corresponding green point on the blue curve. The yellow point also moves left from the red curve to the blue curve, and the gray point moves left from the red curve to the blue curve.

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Now that we've dealt with $b = 2$ by compressing the graph horizontally, we can deal with the fact that $a = 3$.

The function $y = \sin(2\theta)$ oscillates vertically back and forth between $y = [-1,1]$. Because $a = 3$ in the function we're graphing, that means we'll stretch the graph of $y = \sin(2\theta)$ vertically by a factor of 3, such that $y = 3\sin(2\theta)$ will oscillate vertically back and forth between $y = [-3,3]$.

So if $y = \sin(2\theta)$ includes the points

$$y = \sin(2\theta): \quad (0,0) \qquad \left(\frac{\pi}{4},1\right) \qquad \left(\frac{\pi}{2},0\right) \qquad \left(\frac{3\pi}{4},-1\right) \qquad (\pi,0)$$

then $y = 3\sin(2\theta)$ will triple the y-values in these coordinates, while keeping the x-values the same.

$$y = \sin(2\theta): \quad (0,0) \qquad \left(\frac{\pi}{4},1\right) \qquad \left(\frac{\pi}{2},0\right) \qquad \left(\frac{3\pi}{4},-1\right) \qquad (\pi,0)$$

$$y = 3\sin(2\theta): (0,0) \qquad \left(\frac{\pi}{4},3\right) \qquad \left(\frac{\pi}{2},0\right) \qquad \left(\frac{3\pi}{4},-3\right) \qquad (\pi,0)$$

We see how this change stretches the graph vertically when we sketch $y = \sin(2\theta)$ in blue and $y = 3\sin(2\theta)$ in purple:

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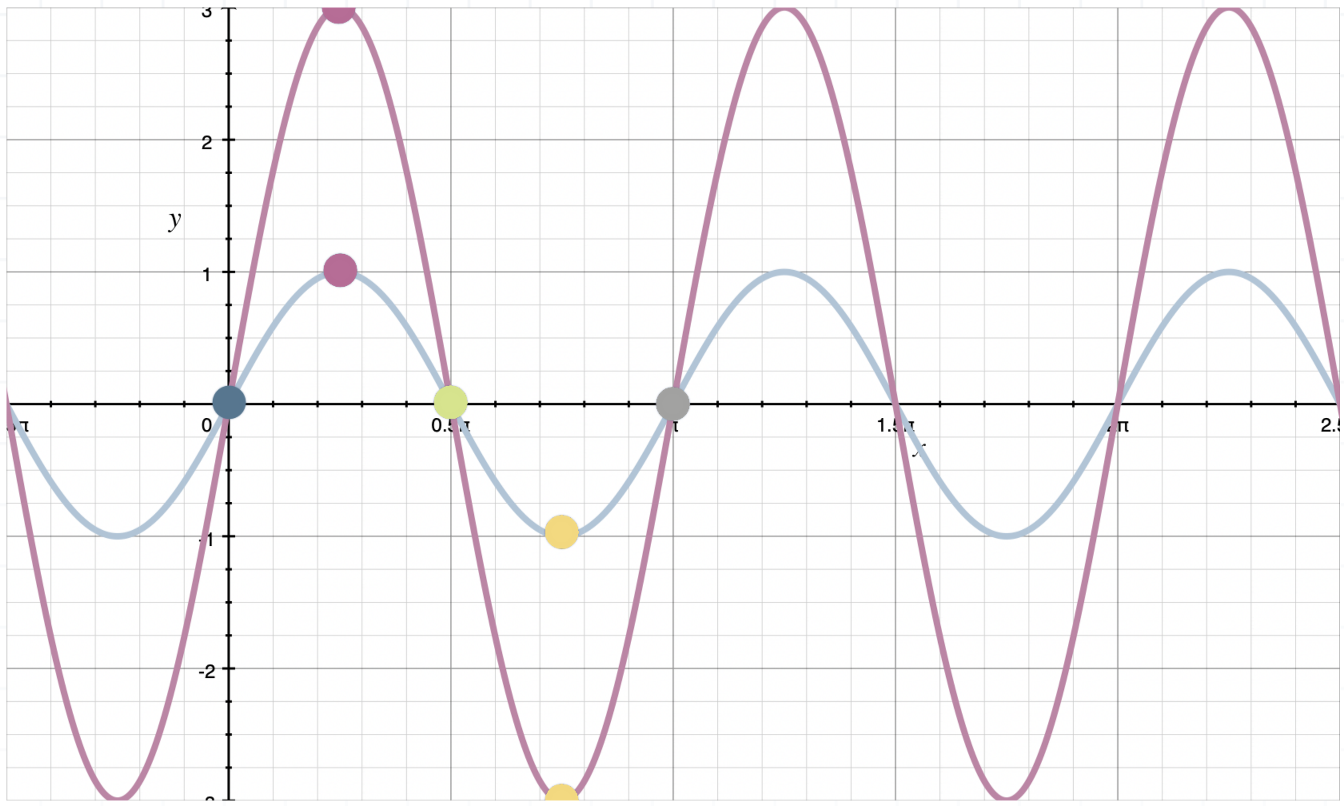
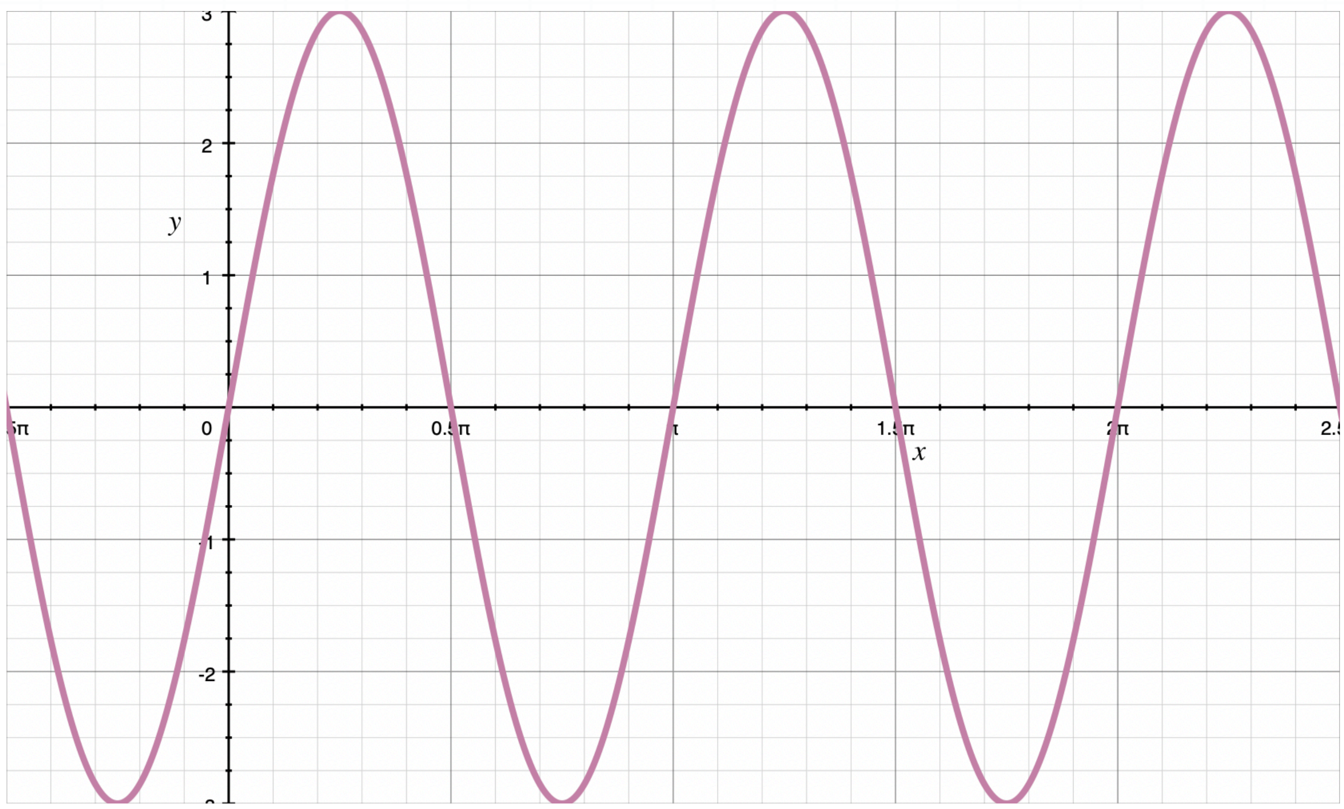


Therefore, a sketch of the final graph of $y = 3\sin(2\theta)$, by itself, is



Let's do another example with a negative value of a.

Example

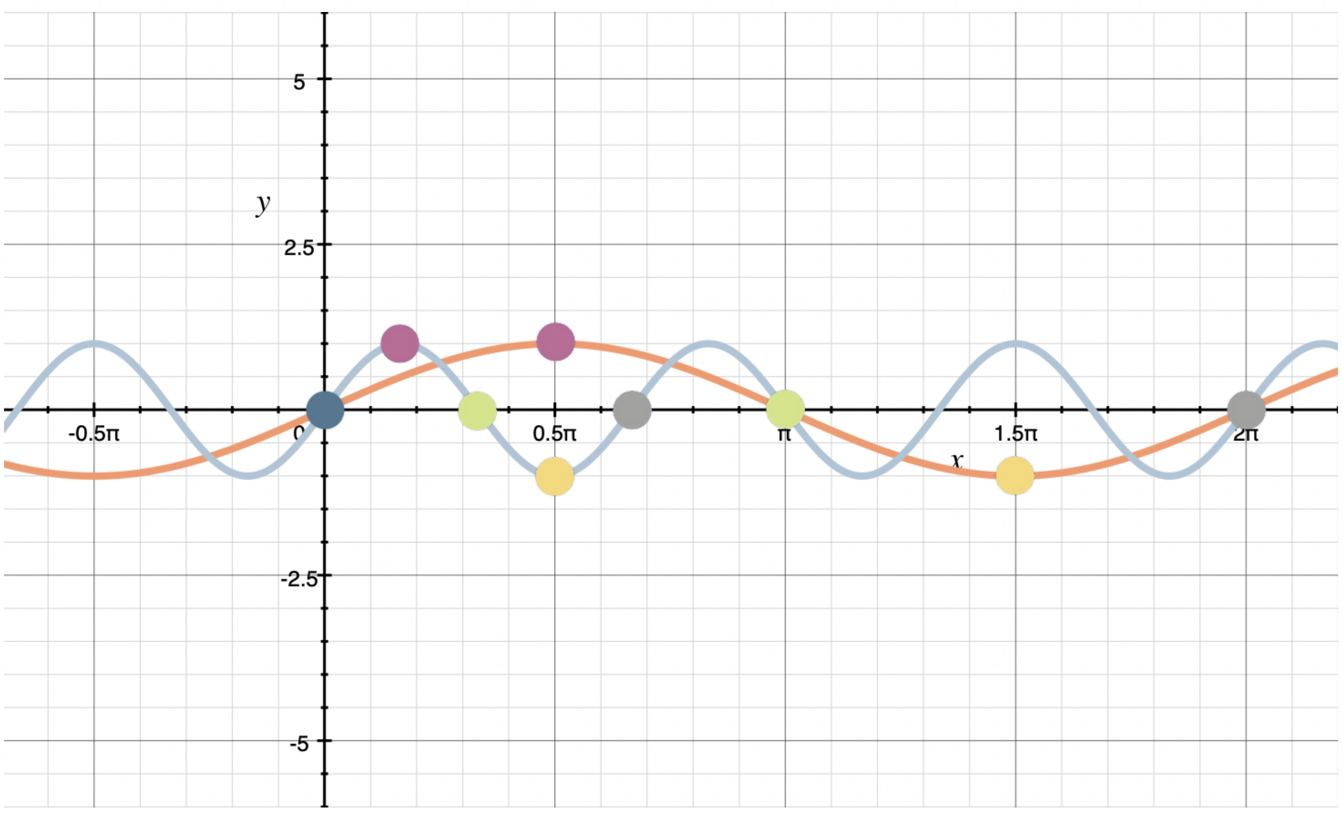
Sketch the graph of $y = -5\sin(3\theta)$.

Similar to the previous example, if we want to change the graph of $y = \sin\theta$ into the graph of $y = \sin(3\theta)$, we know the graph of $y = \sin(3\theta)$ will be horizontally compressed by a factor of 3, which means we'll take the points along $y = \sin\theta$ and divide the x-values by 3 in order to get points on $y = \sin(3\theta)$ while keeping the y-values the same.

$$y = \sin\theta: \qquad (0,0) \qquad \left(\frac{\pi}{2},1\right) \qquad (\pi,0) \qquad \left(\frac{3\pi}{2},-1\right) \qquad (2\pi,0)$$

$$y = \sin(3\theta): \qquad (0,0) \qquad \left(\frac{\pi}{6},1\right) \qquad \left(\frac{\pi}{3},0\right) \qquad \left(\frac{\pi}{2},-1\right) \qquad \left(\frac{2\pi}{3},0\right)$$

Then we can graph $y = \sin\theta$ in red, and $y = \sin(3\theta)$ in blue:



Then if we want to change the graph of $y = \sin(3\theta)$ into the graph of $y = -5\sin(3\theta)$, we know the graph of $y = 5\sin(3\theta)$ will be vertically stretched

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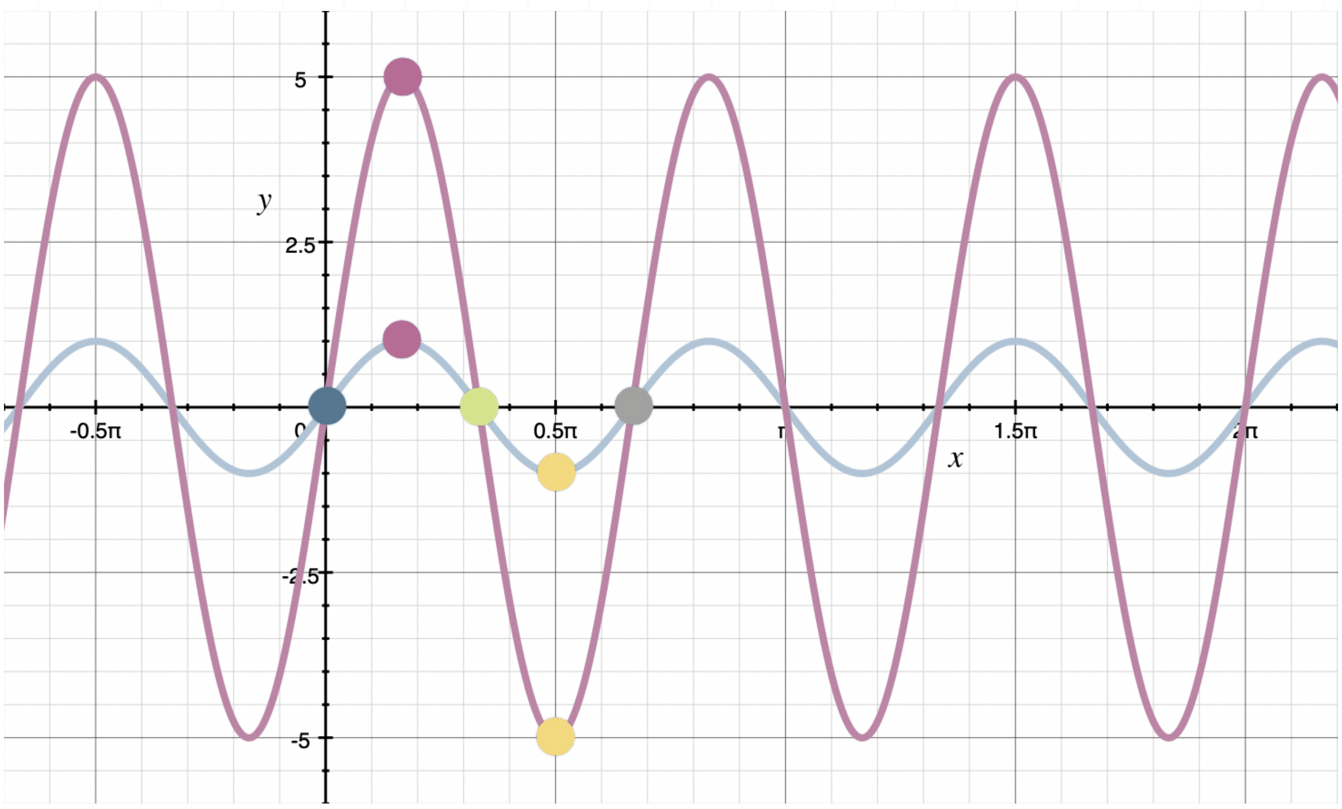
by a factor of 5, which means we'll take the points along $y = \sin(3\theta)$ and multiply the y-values by 5 in order to get points on $y = 5\sin(3\theta)$, while keeping the x-values the same.

$$y = \sin(3\theta): \quad (0,0) \quad \left(\frac{\pi}{6},1\right) \quad \left(\frac{\pi}{3},0\right) \quad \left(\frac{\pi}{2},-1\right) \quad \left(\frac{2\pi}{3},0\right)$$

$$y = 5\sin(3\theta): \quad (0,0) \quad \left(\frac{\pi}{6},5\right) \quad \left(\frac{\pi}{3},0\right) \quad \left(\frac{\pi}{2},-5\right) \quad \left(\frac{2\pi}{3},0\right)$$

Then we can graph $y = \sin(3\theta)$ in blue and $y = 5\sin(3\theta)$ in purple.



Then if we want to change the graph of $y = 5\sin(3\theta)$ into the graph of $y = -5\sin(3\theta)$, we need to reflect $y = 5\sin(3\theta)$ over the x-axis to get $y = -5\sin(3\theta)$, which means we need to change the sign on every y-value while keeping the x-values the same.

$$y = 5\sin(3\theta): \quad (0,0) \quad \left(\frac{\pi}{6},5\right) \quad \left(\frac{\pi}{3},0\right) \quad \left(\frac{\pi}{2},-5\right) \quad \left(\frac{2\pi}{3},0\right)$$

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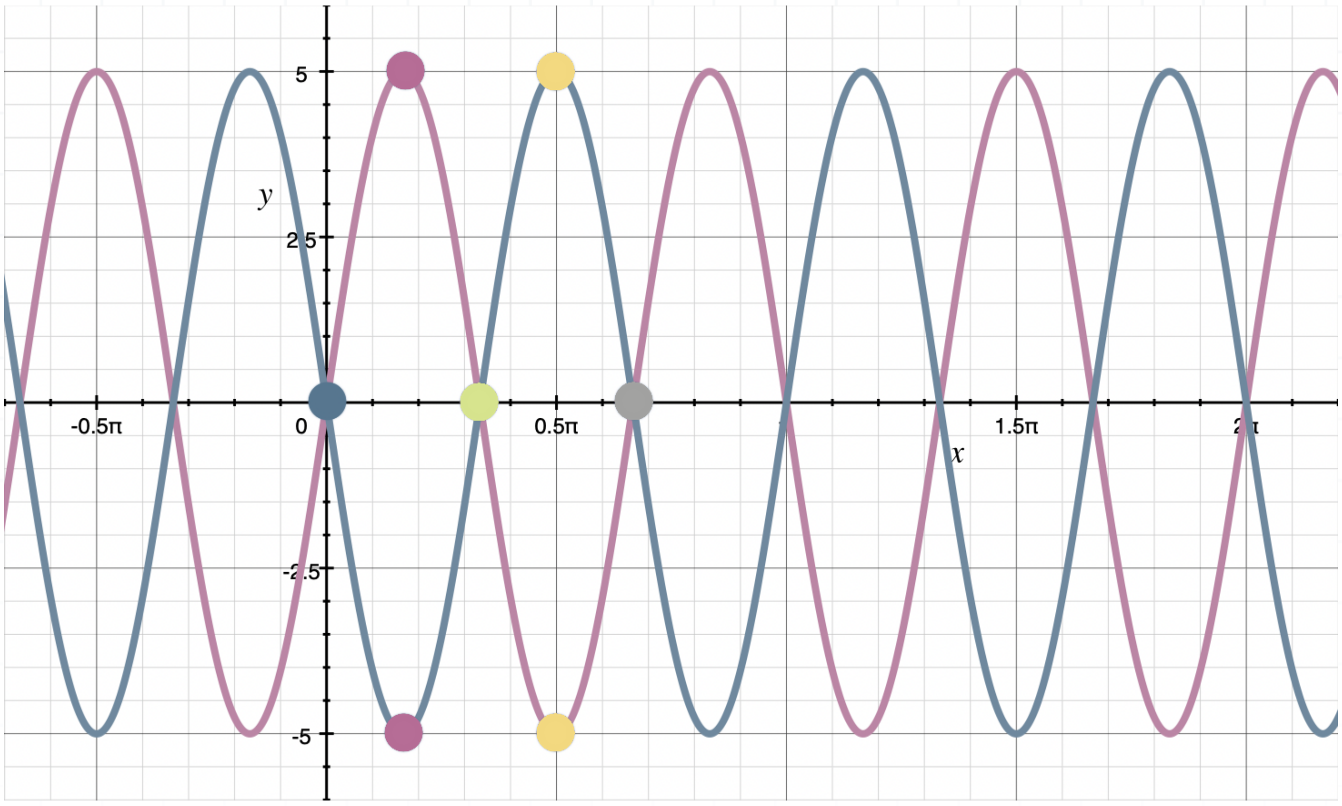
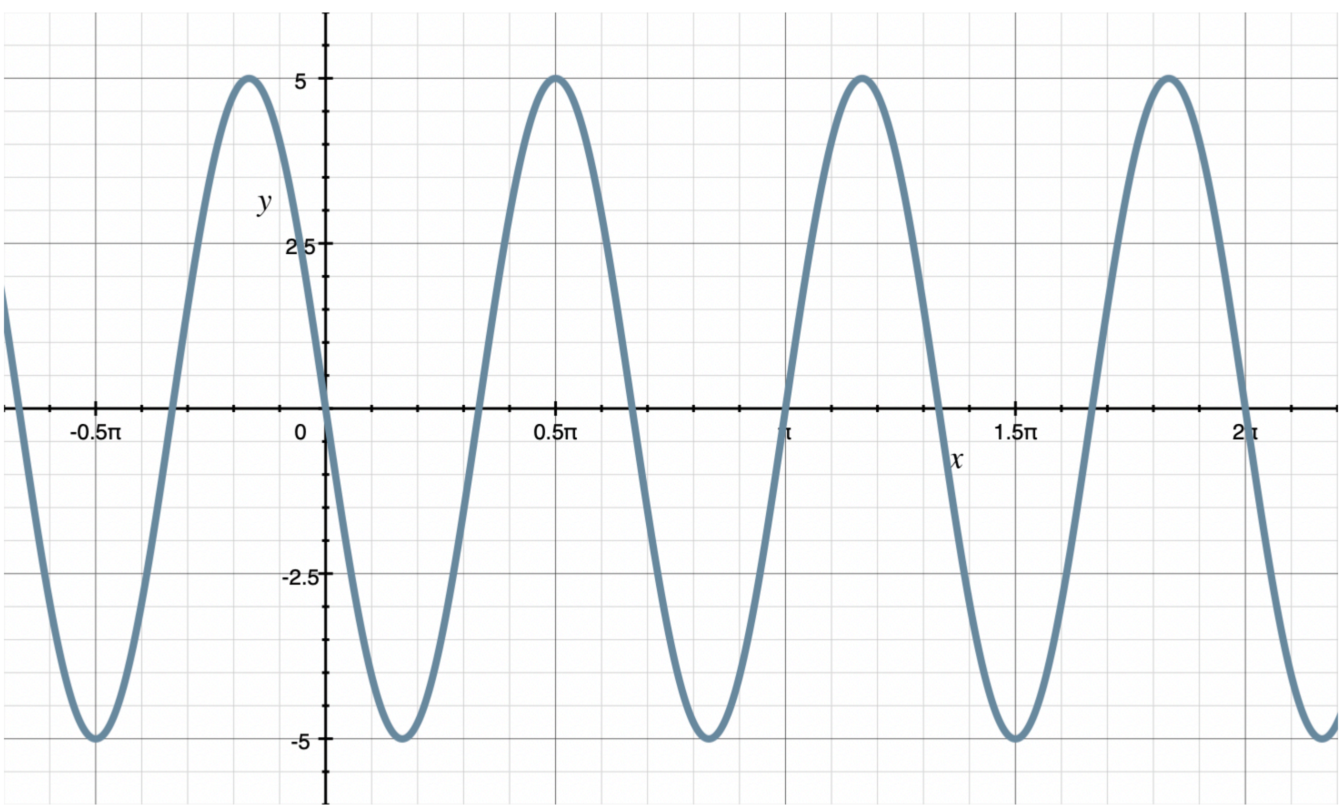
$$y = -5\sin(3\theta):\ (0,0) \qquad \left(\frac{\pi}{6}, -5\right)\ \left(\frac{\pi}{3}, 0\right) \qquad \left(\frac{\pi}{2}, 5\right) \qquad \left(\frac{2\pi}{3}, 0\right)$$

Then we can graph $y = 5\sin(3\theta)$ in purple and $y = -5\sin(3\theta)$ in dark blue.



Therefore, a sketch of the final graph of $y = -5\sin(3\theta)$, by itself, is



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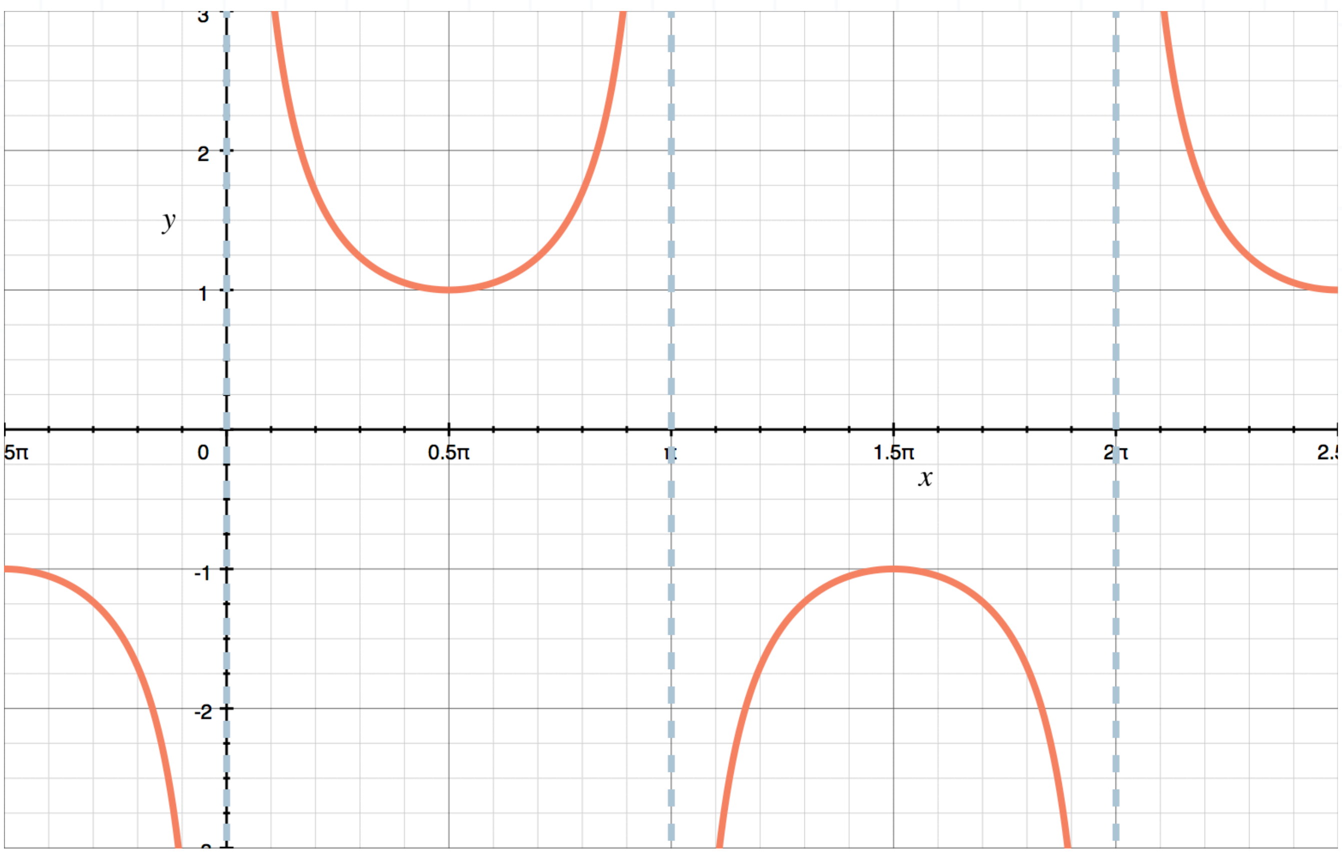
Sketching cosecant and secant

Once we know how to sketch the graphs of the sine and cosine functions, sketching cosecant and secant comes relatively easily.

Let's start our discussion by laying out the graphs of these two functions. The graph of $y = \csc x$ is



and the graph of $y = \sec x$ is

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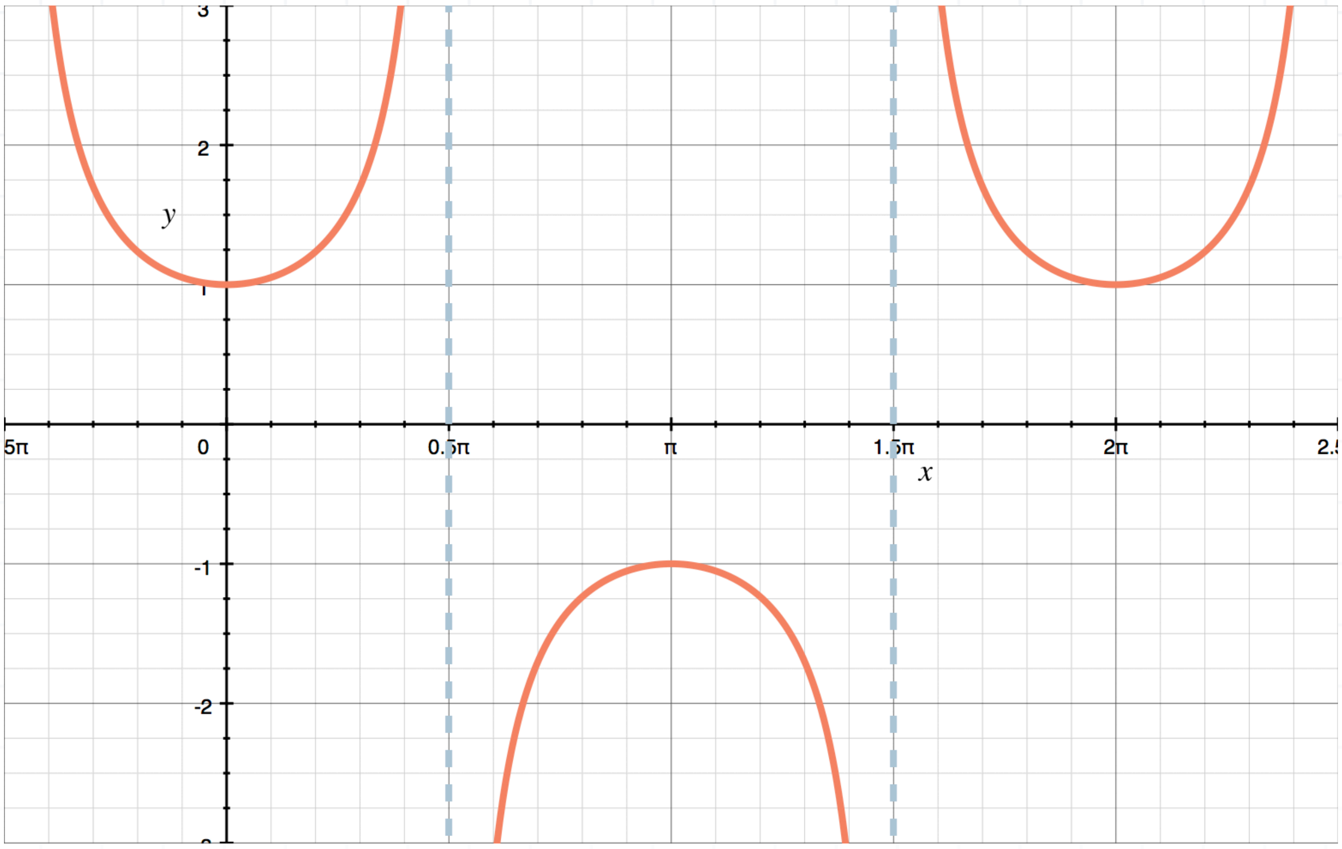

This gives us a picture of what the standard forms of these functions looks like, but now we want to dive deeper into the process we'll use for actually sketching these curves.

Sketching $\csc x$ and $\sec x$

We notice something interesting if we sketch the reciprocal functions $\sin x$ in red and $\csc x$ in blue on the same graph, we get:

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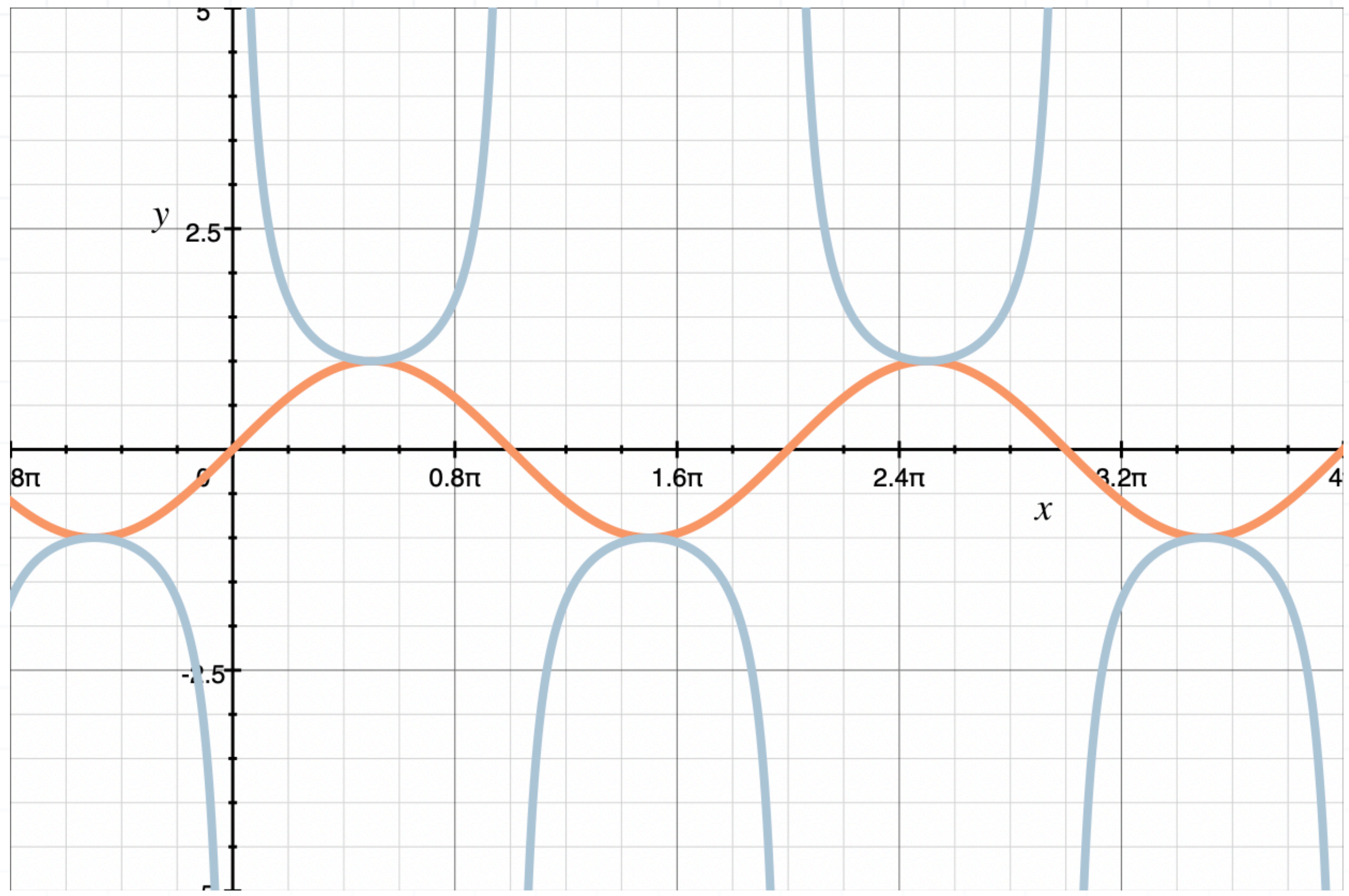
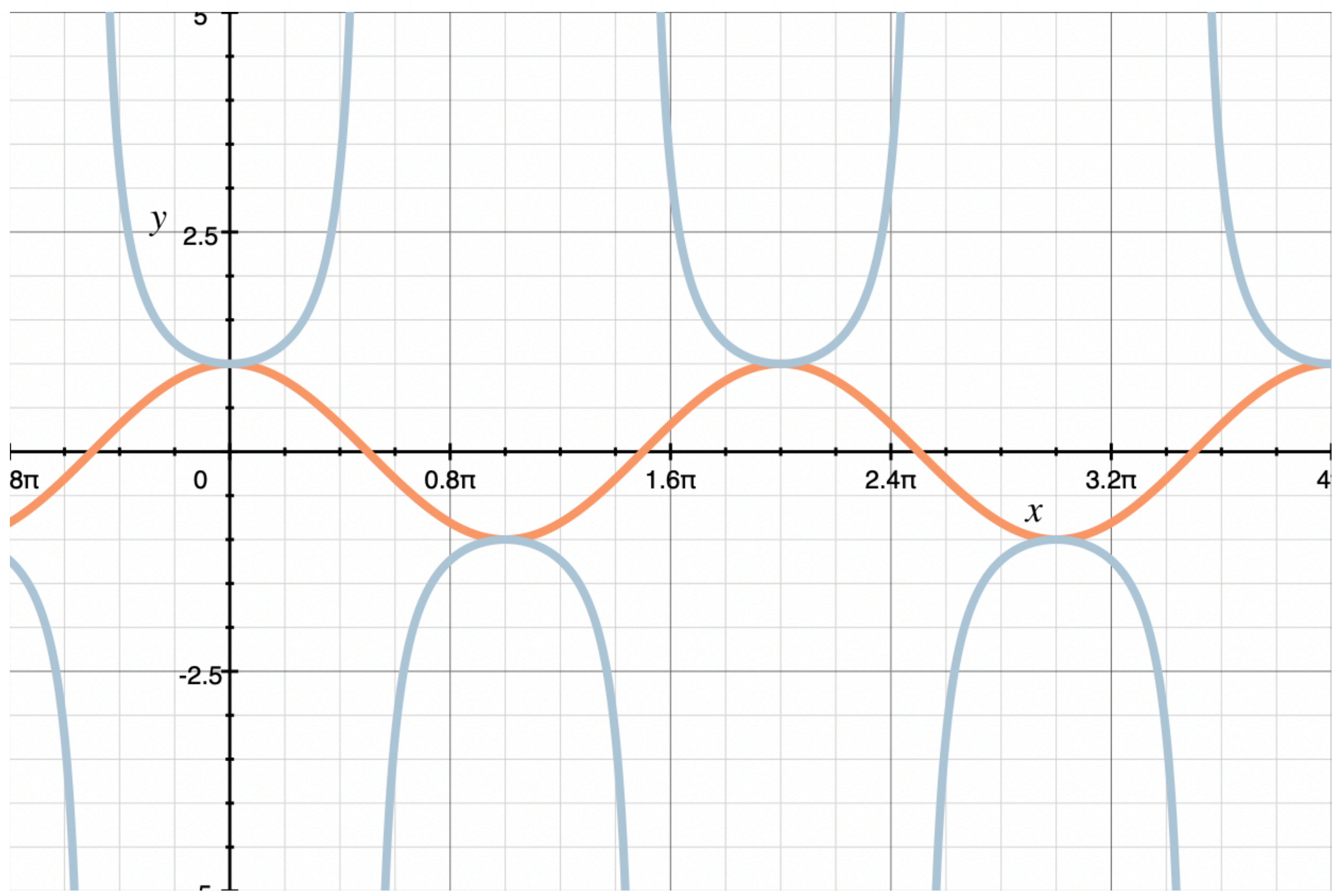


Similarly, if we sketch the reciprocal functions $\cos x$ in red and $\sec x$ in blue on the same graph, we get:



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Notice how the cosecant and secant functions in blue are these "U-shaped" curves that touch the sine and cosine functions in red at their maximum and minimum points.

Therefore, when it comes to graphing these cosecant and secant functions, one way to do it is to graph the corresponding sine or cosine function first, and then fill in the U-shapes. In other words, we'll follow these steps:

1. Sketch the corresponding reciprocal function. For cosecant this will be sine, for secant it'll be cosine.

2. Sketch in vertical asymptotes where the sine or cosine curve crosses its own midline, since these are the points at which the secant or cosecant functions are undefined.

3. Sketch the U-shapes of the cosecant or secant curve at the maximum and minimum points of the sine or cosine graph, and in between the vertical asymptotes.

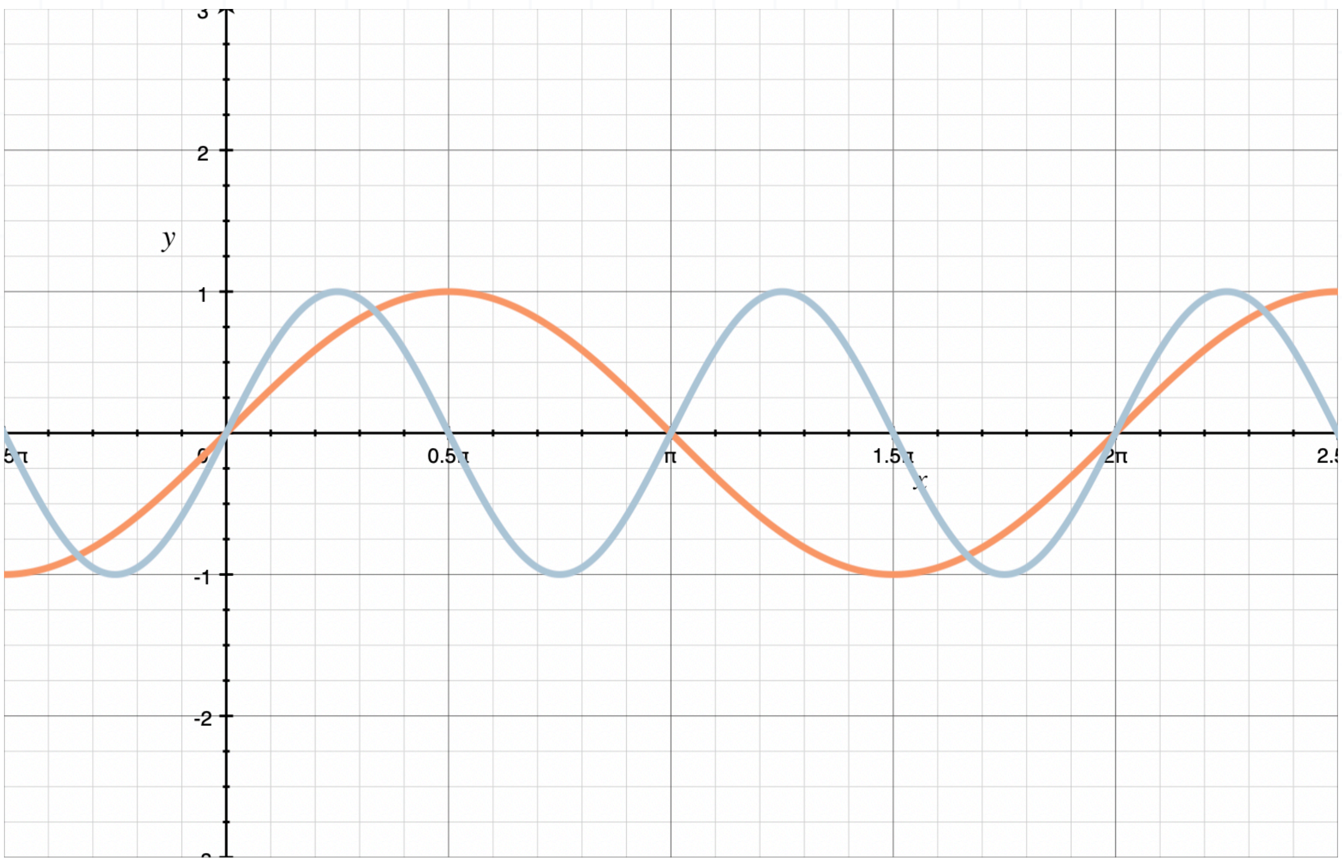
Let's do an example where we use the graph of sine to help us sketch the graph of cosecant.

Example

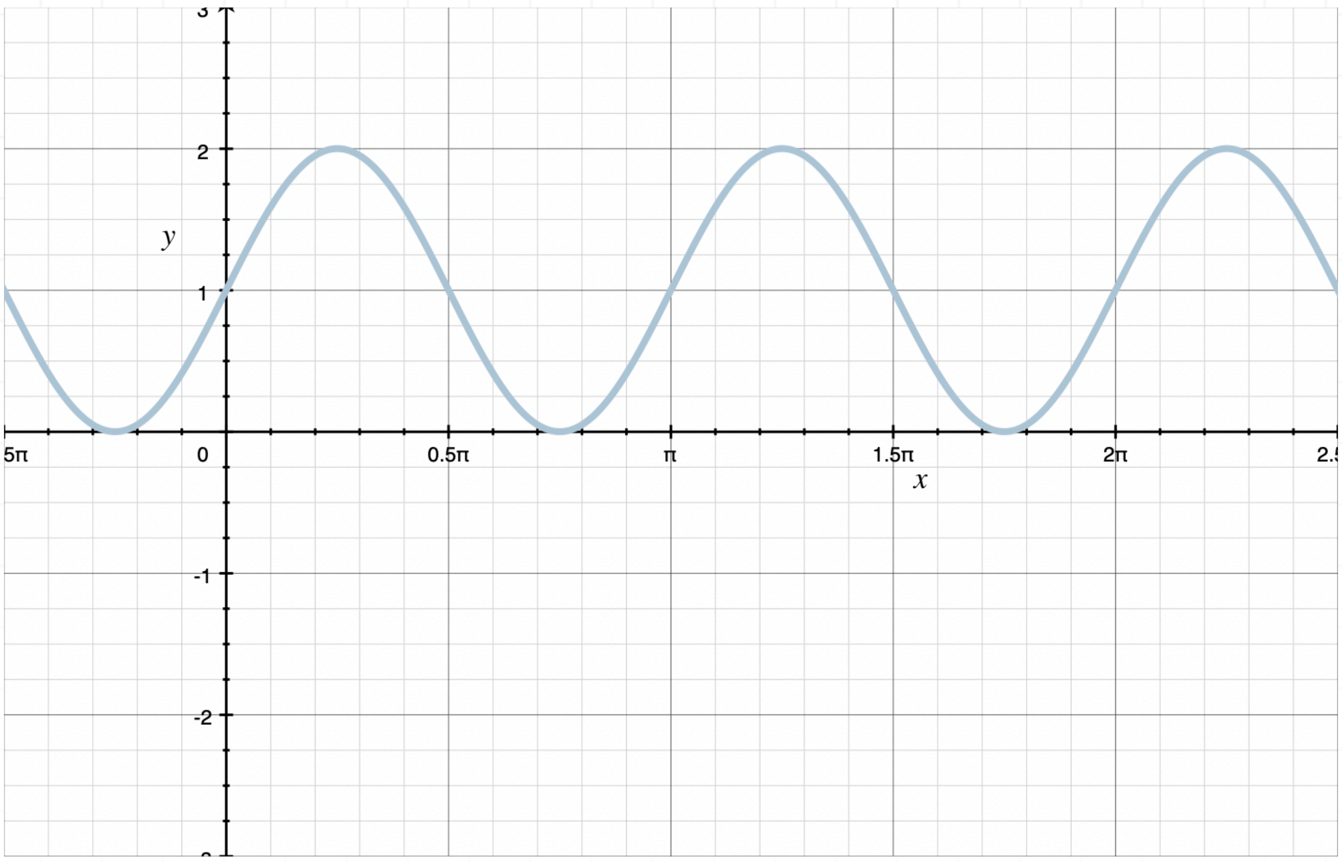
Sketch the graph of $y = \csc(2x) + 1$. Hint: adding 1 to $y = \csc(2x)$ shifts the graph up vertically by 1 unit.

Because the reciprocal function of cosecant is sine, we want to start by replacing cosecant with sine in the function we've been given. In other words, the corresponding function is $y = \sin(2x) + 1$.

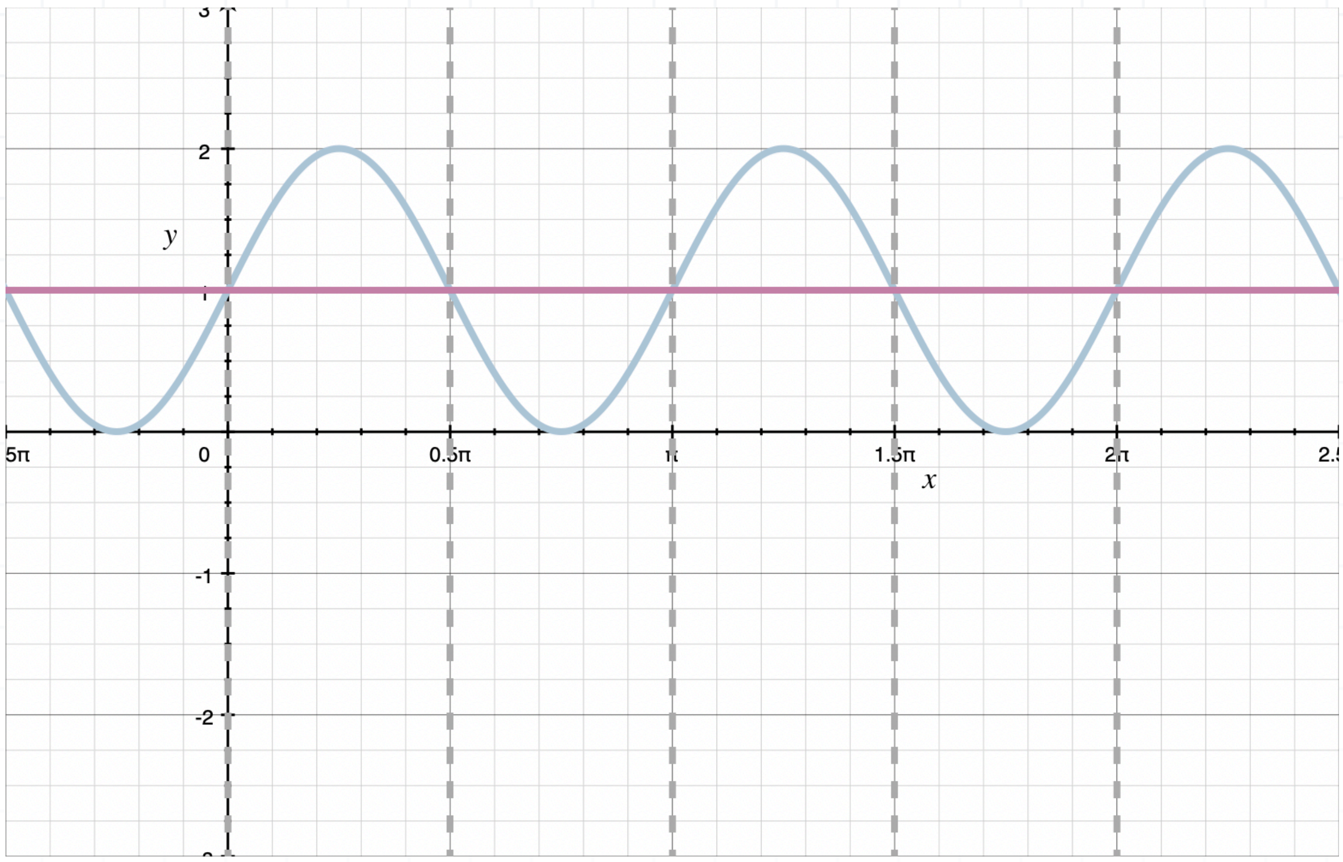
To graph $y = \sin(2x) + 1$, we see that $b = 2$, so we'll horizontally compress the sine curve by halving all the x-values. So if we sketch $y = \sin x$ in red and $y = \sin(2x)$ in blue, we get



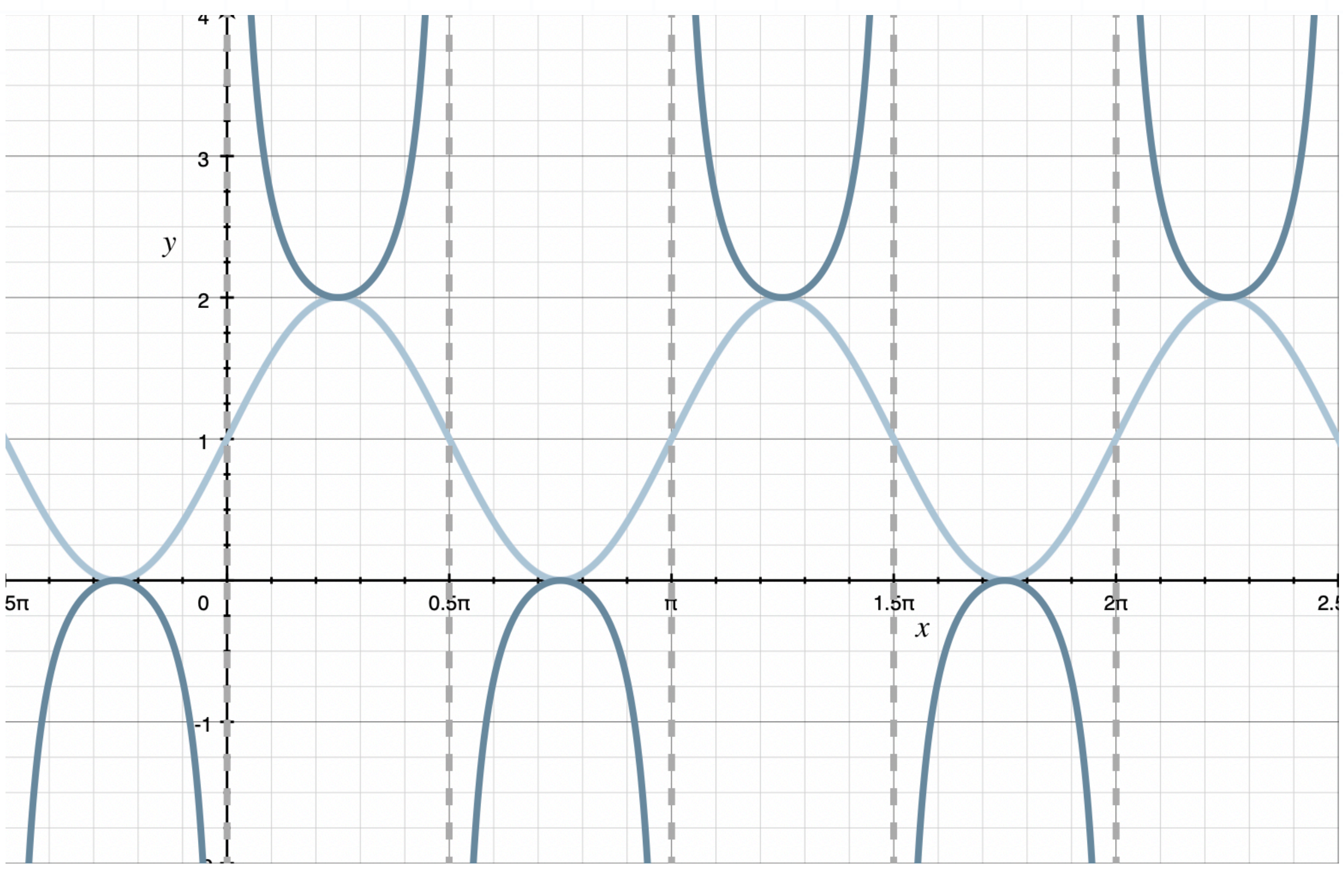
Then to get $y = \sin(2x) + 1$, we shift $y = \sin(2x)$ upward vertically by 1 unit (we'll learn about vertical shifts in more detail in a future lesson).



Then we'll sketch in vertical asymptotes at the midline points of $y = \sin(2x) + 1$. The midline of a sine or cosine function is the horizontal center line about which the functions oscillates above and below. The midline is parallel to the x-axis and is halfway between the curve's maximum and minimum values. So if we sketch in the midline of $y = \sin(2x) + 1$ and the vertical asymptotes where $y = \sin(2x) + 1$ crosses its own midline, we get



We'll sketch in the U-shapes for $y = \csc(2x) + 1$.



Finally, we'll take away the sine curve and the vertical asymptotes away to get the final sketch of $y = \csc(2x) + 1$.

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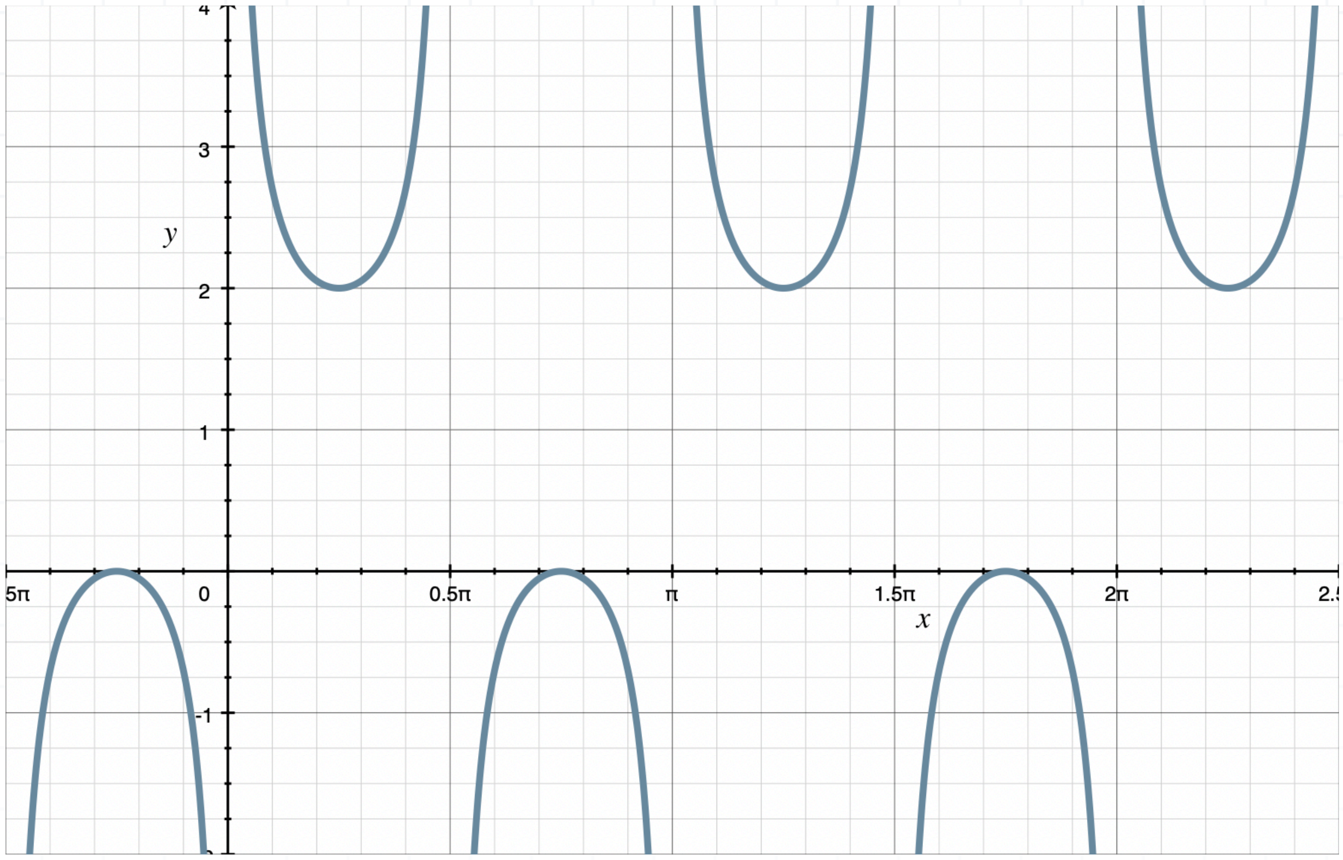

Let's do another example where we use the graph of cosine to help us sketch the graph of secant.

Example

Sketch the graph of $y = 2\sec x - 2$. Hint: subtracting 2 from $y = 2\sec x$ shifts the graph down vertically by 2 units.

Because the reciprocal function of secant is cosine, we want to start by replacing secant with cosine in the function we've been given. In other words, the corresponding function is $y = 2\cos x - 2$.

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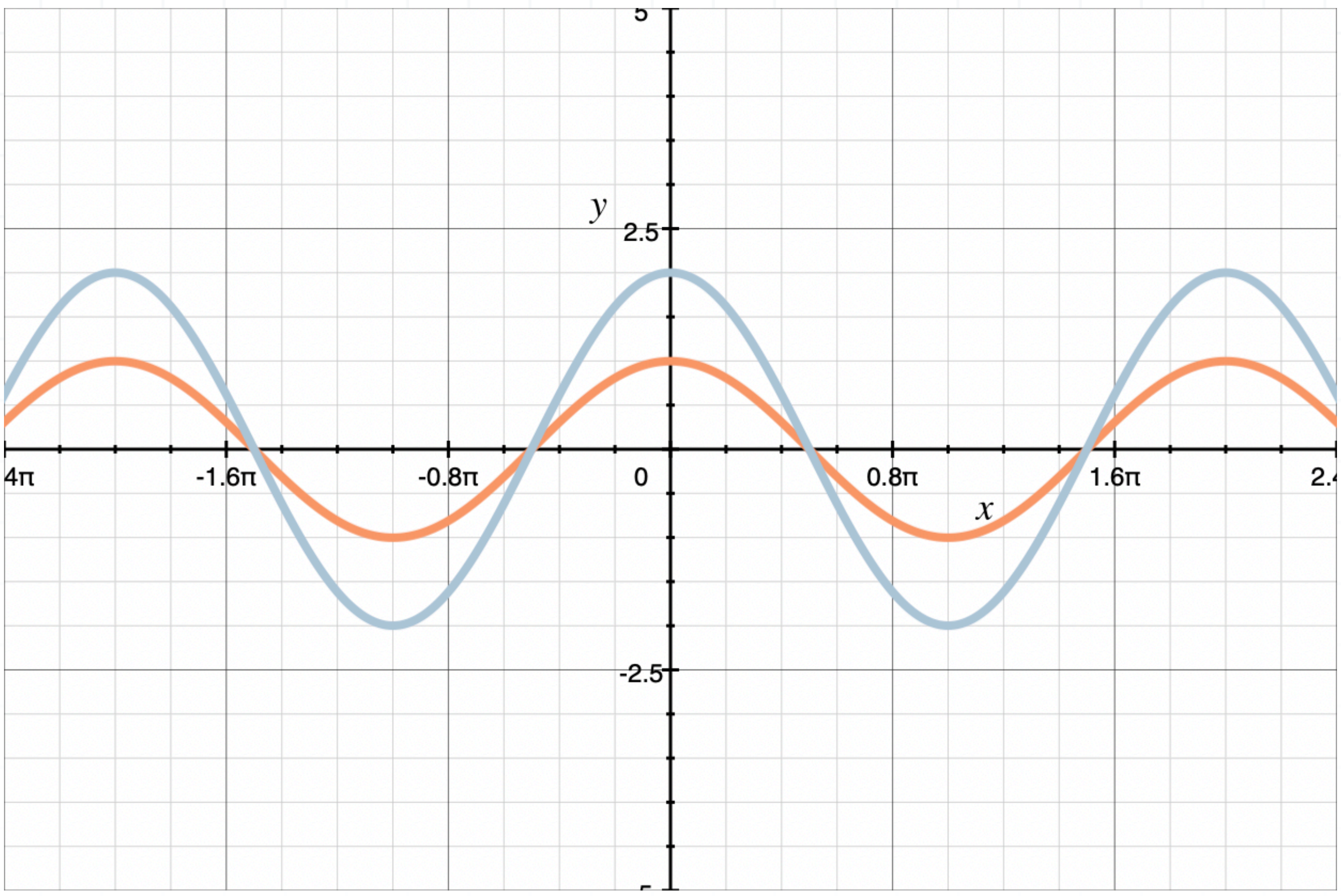
To graph $y = 2\cos x - 2$, we see that $a = 2$, so we'll vertically stretch the cosine curve by doubling all the y-values. So if we sketch $y = \cos x$ in red and $y = 2\cos x$ in blue, we get



Then to get $y = 2\cos x - 2$, we shift $y = 2\cos x$ downward vertically by 2 units.

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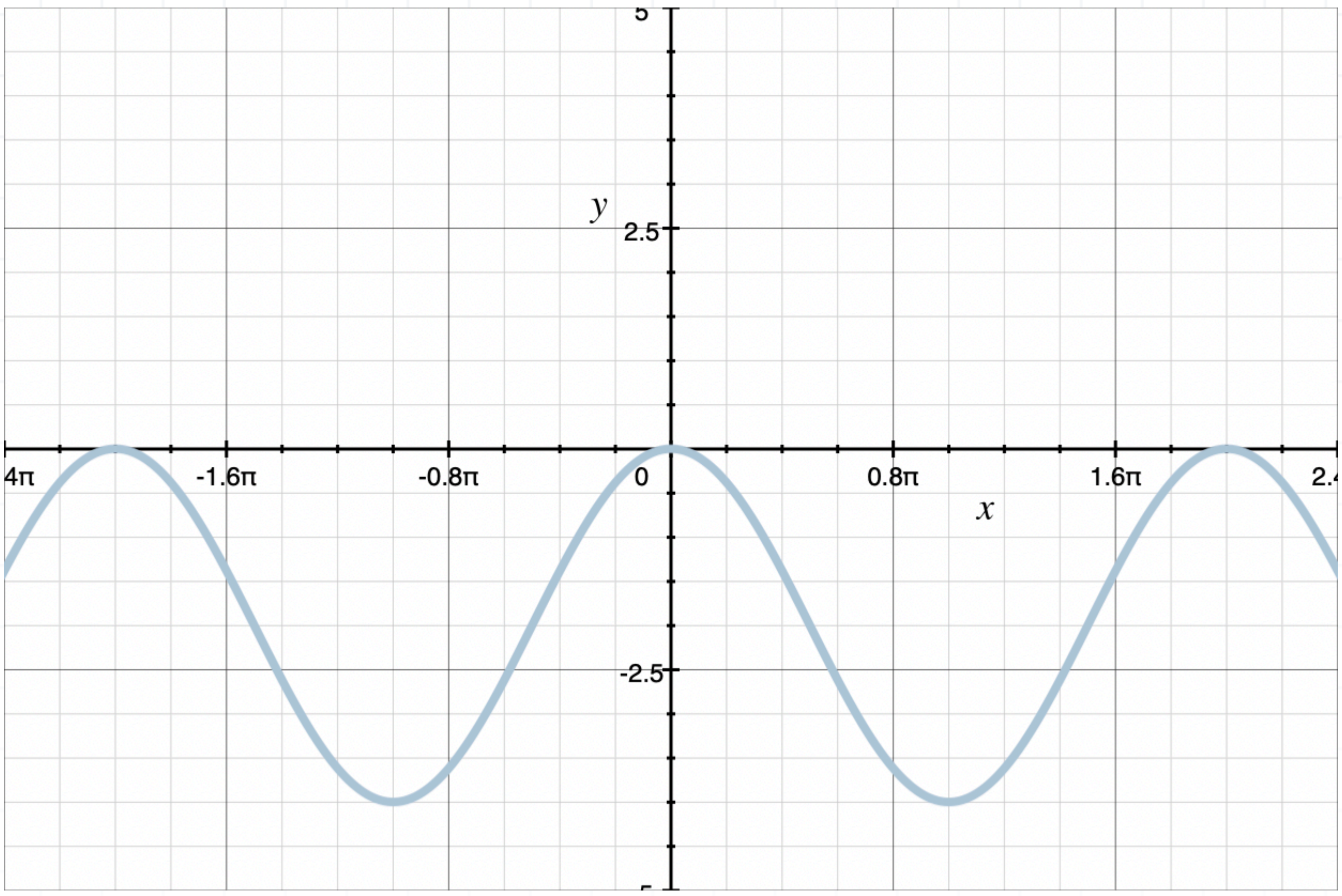
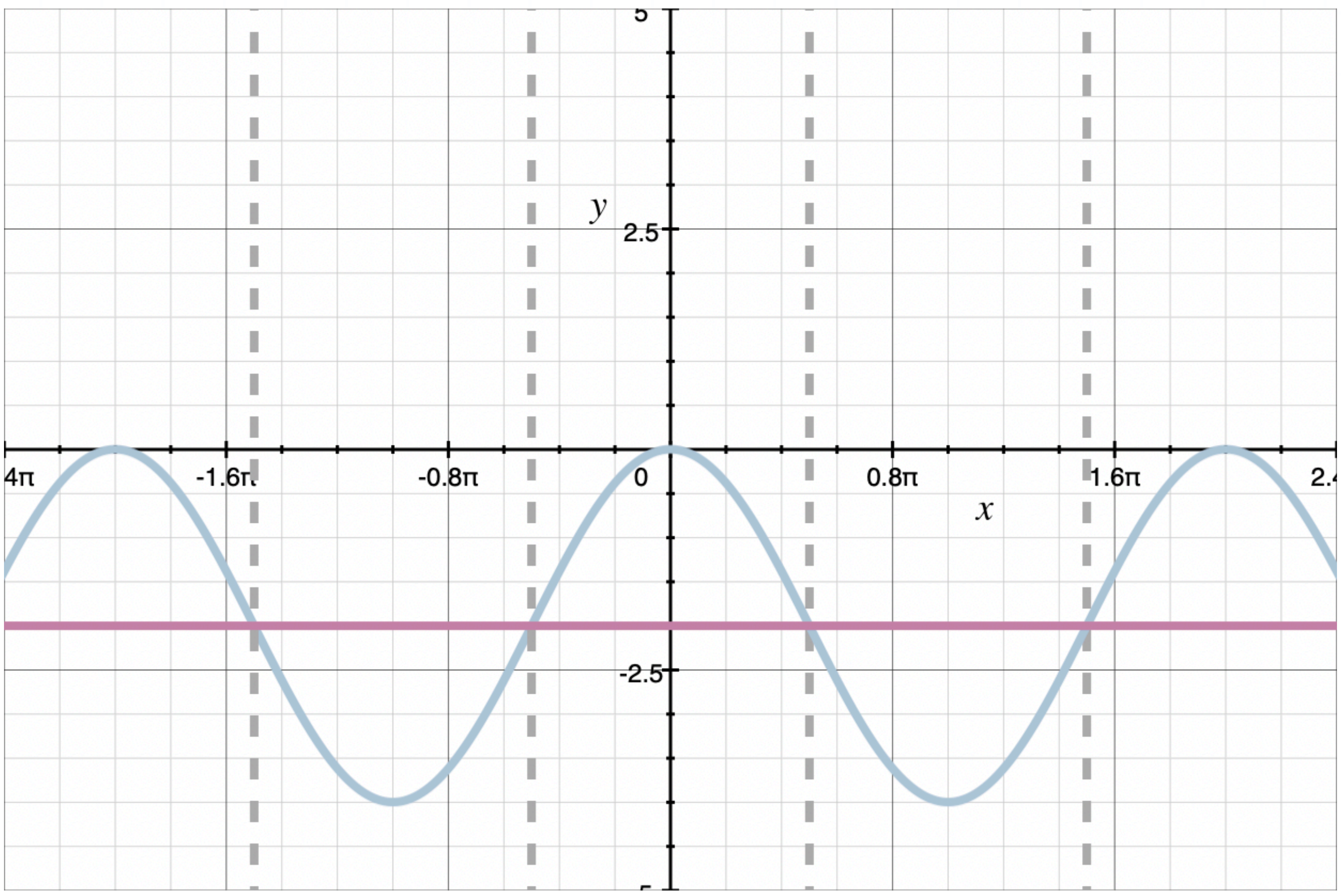


Then we'll sketch in vertical asymptotes at the midline points of $y = 2\cos x - 2$.



We'll sketch in the U-shapes for $y = 2\sec x - 2$.

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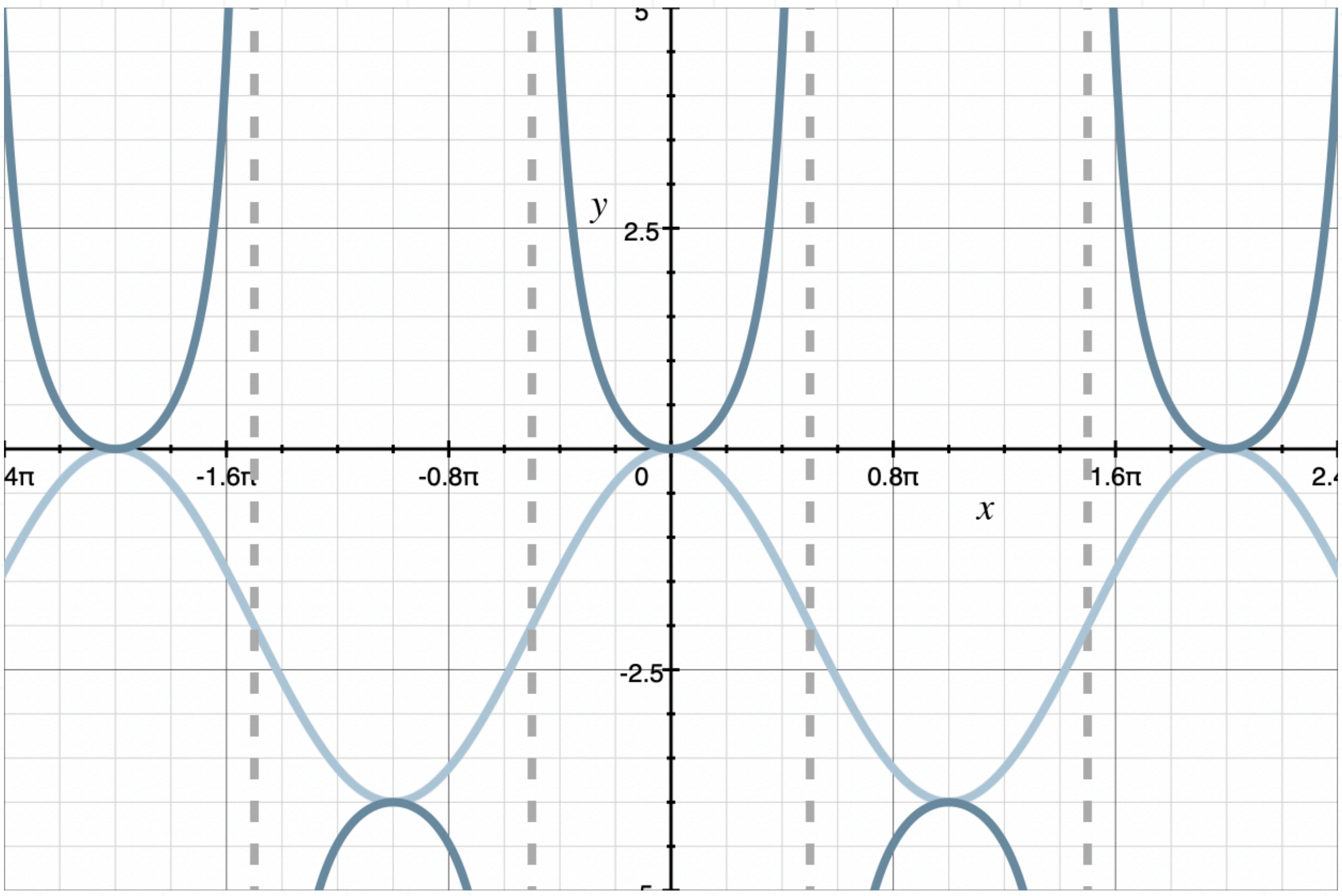
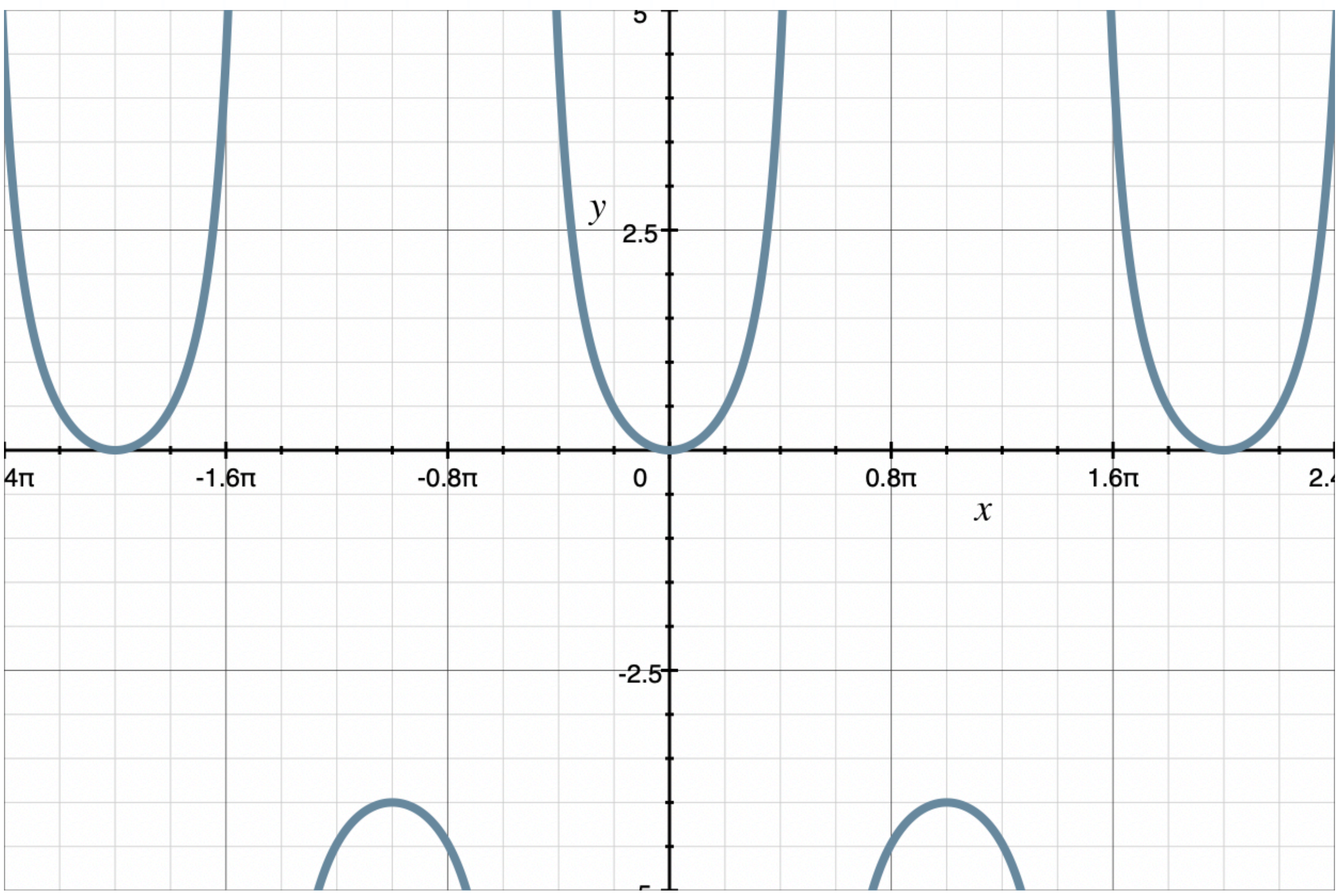


Finally, we'll take away the sine curve and the vertical asymptotes away to get the final sketch of $y = 2\sec x - 2$.



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Period and amplitude

Previously we looked at the graphs of sine and cosine, when the equations of those trig functions are given in the form

$$y = a\sin(b(x+c)) + d$$

$$y = a\cos(b(x+c)) + d$$

where $c = 0$ and $d = 0$. We're still not going to look at what happens to the graph when we make c or d non-zero; we'll save that for another lesson.

Instead, we'll dive deeper into the values of a and b, and also talk about their effect on the other four trig functions, not just sine and cosine.

Amplitude

We already know that the value of a determines how much we vertically stretch or compress the function.

Technically, we call $|a|$ the **amplitude** of the function, and the amplitude is the distance from the "midline" of the function's curve to the very top of the curve or the very bottom of the curve. Alternatively, we can measure the height from the highest to lowest point, and then divide the result by 2.

$$|a| = \frac{\text{max} - \text{min}}{2}$$

For instance, the function $y = \sin x$ has its midline at $y = 0$, right along the x-axis:

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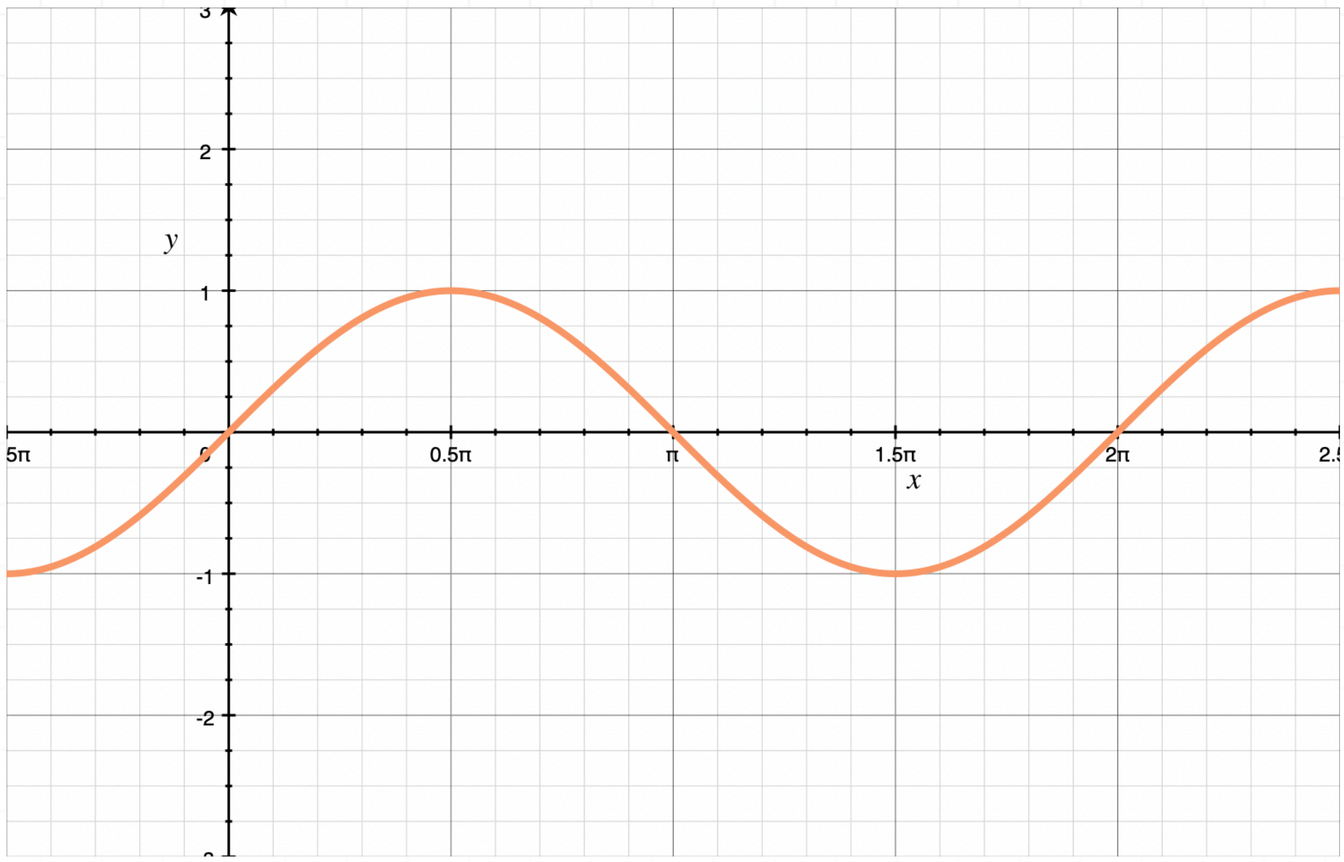

The distance from that midline to the very top of the function or the very bottom of the function is 1 in both directions, so the amplitude of the function is $|a| = 1$.

Notice how we use the absolute value of a to indicate amplitude. The absolute value guarantees that, regardless of whether a is positive or negative in the function, we'll always get a positive value for amplitude. That's what we want, because amplitude is supposed to indicate distance (the distance from the midline to the top or bottom of the graph), and it doesn't make logical sense to indicate a negative distance.

Lastly, before we move on from talking about a to talking about b, let's look at amplitude for the other four trig functions. We've already seen the graphs of the cosecant and secant functions, and we'll look at the graphs of tangent and cotangent in the next lesson, but for now, here are the graphs of the other four trig functions, $y = \csc x$,

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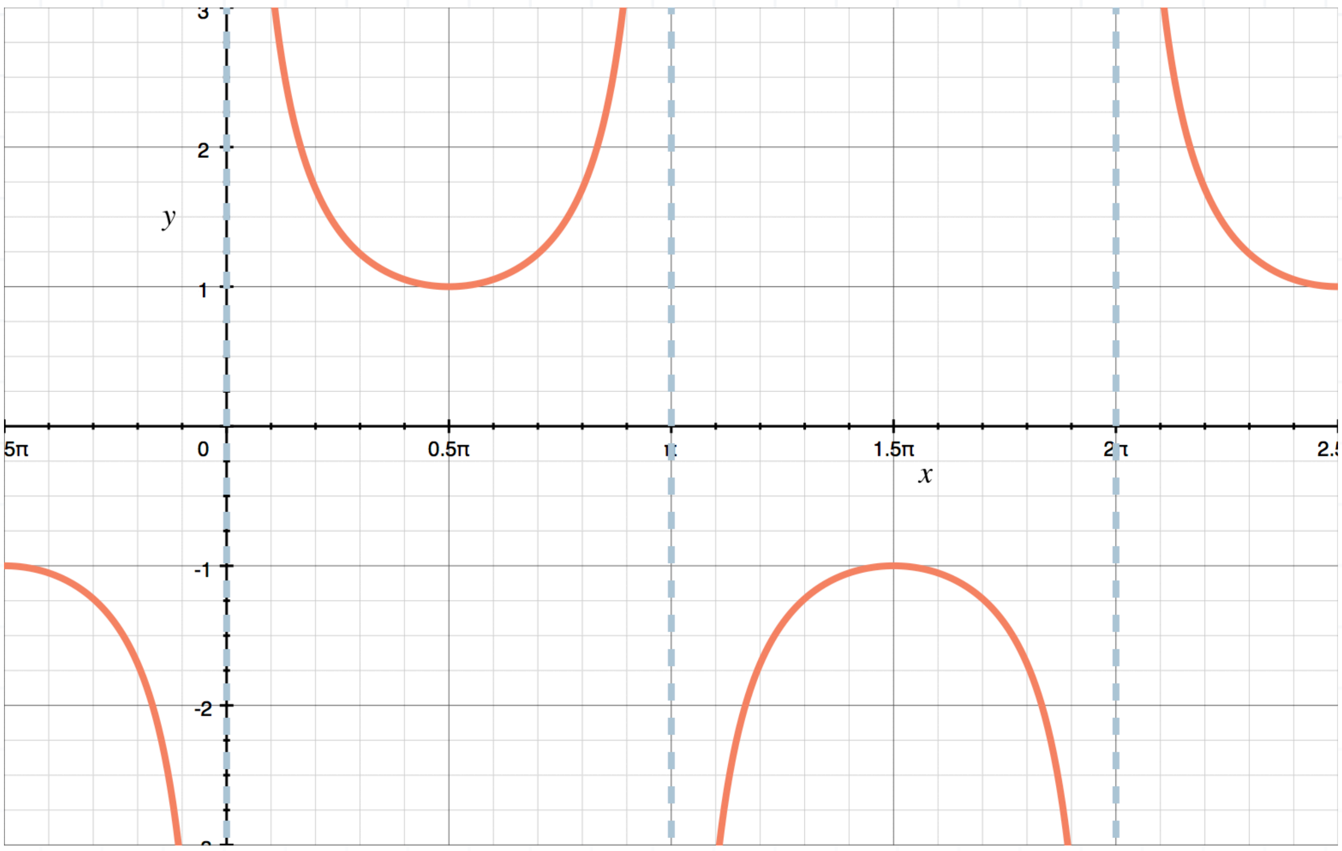
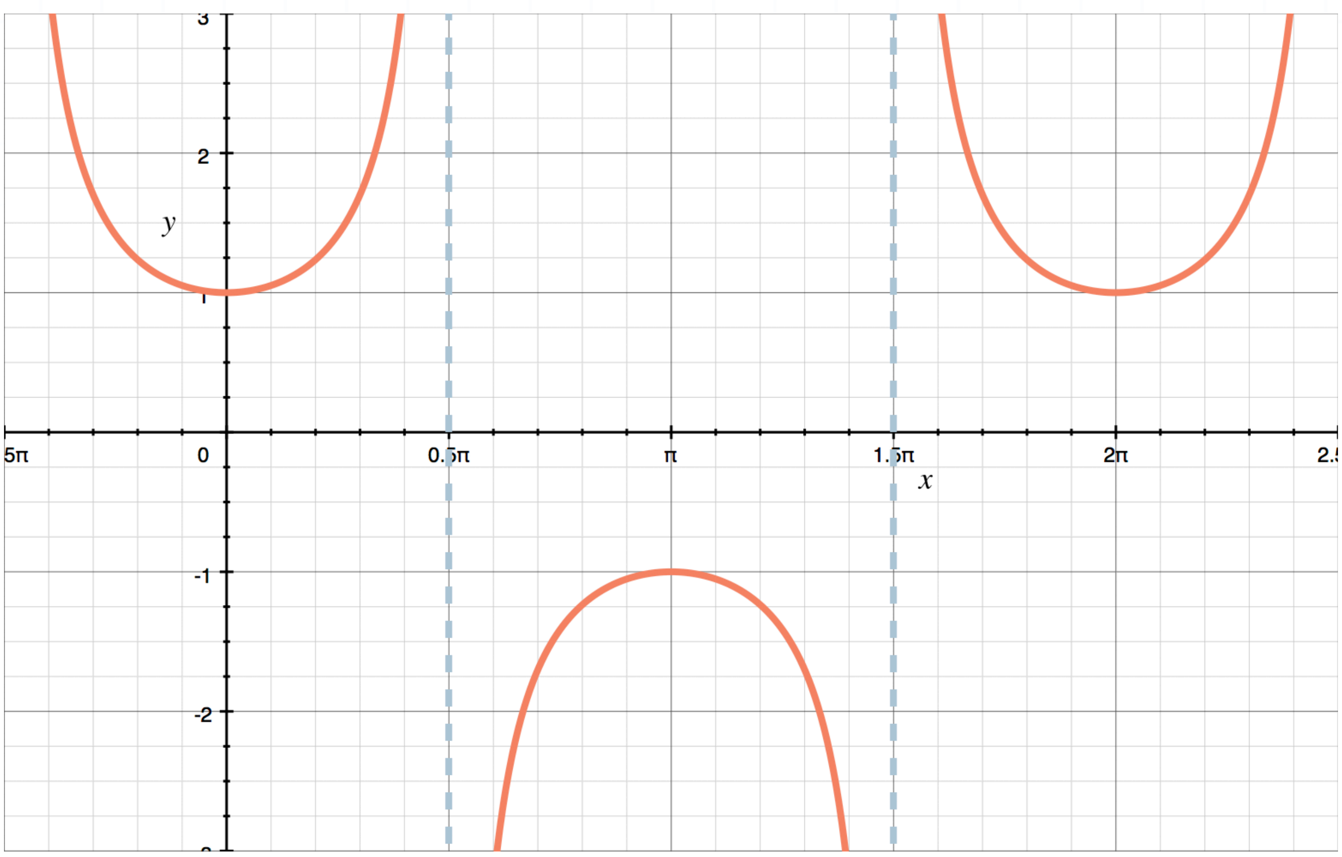


$y = \sec x,$



$y = \tan x$

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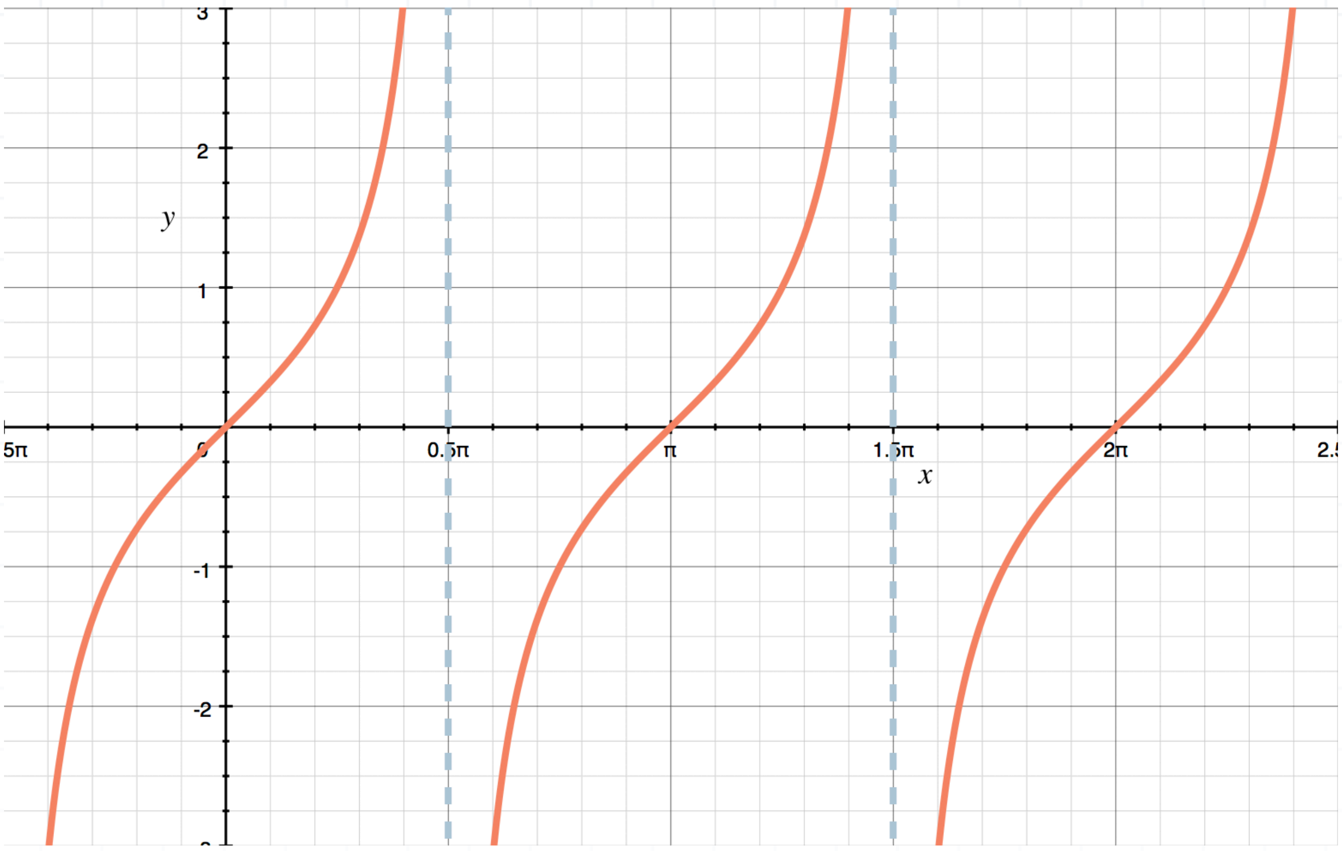
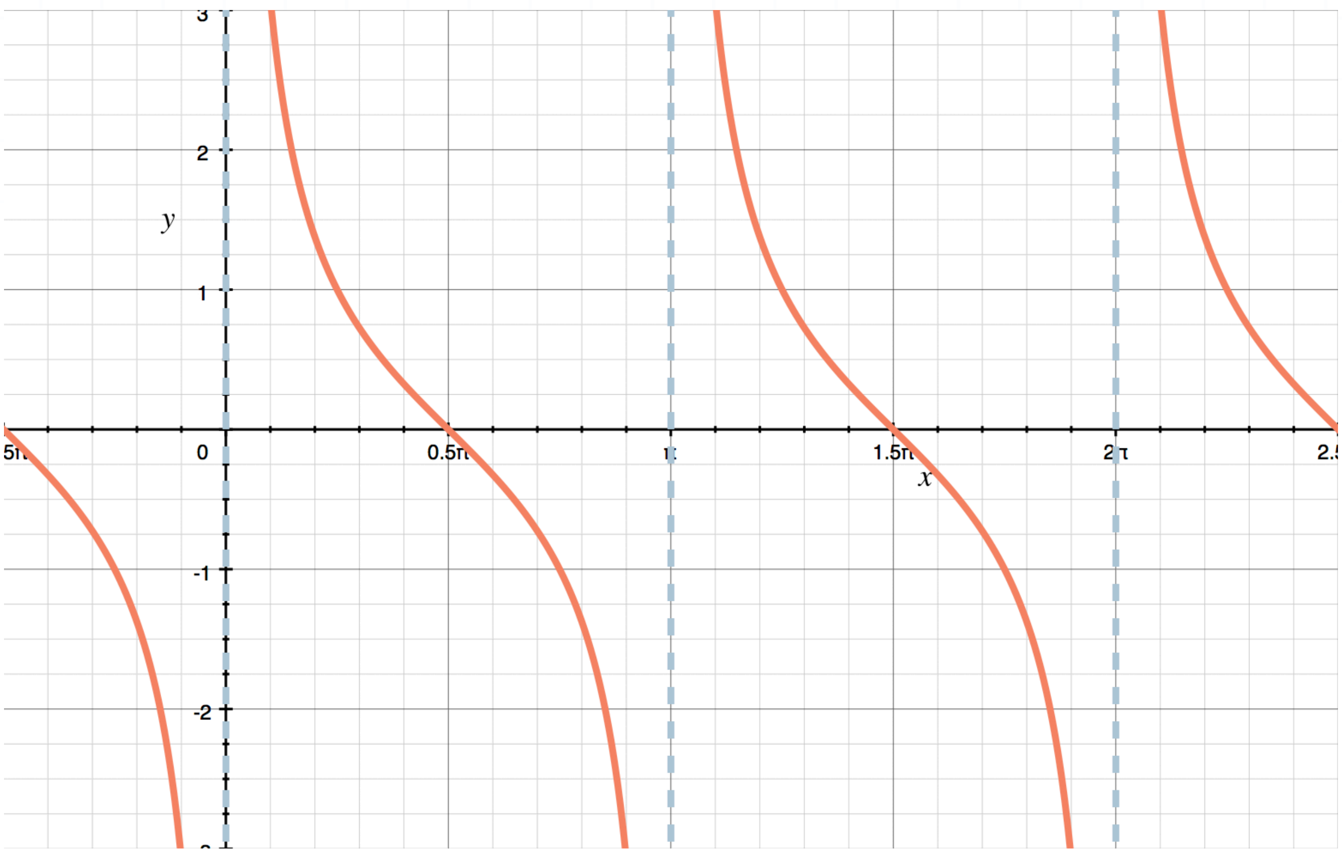


and $y = \cot x$.



Notice how the graphs of each of these four trig functions stretches infinitely toward $y = -\infty$ and $y = \infty$ around the vertical asymptotes. Because they have no maximum or minimum values, there's really no

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defined "height" of these four functions, since the height is infinite, we don't really define any amplitude for tangent, cosecant, secant, or cotangent. Which means we only give a value for amplitude when we're sketching sine and cosine functions.

Let's do a quick example with amplitude.

Example

Find the amplitude of $3\sin\theta$, $3\cos\theta$, $-5\sin\theta$, and $-5\cos\theta$.

For the functions $3\sin\theta$ and $3\cos\theta$, the value of a is 3, so the amplitude is $|a|=|3|=3$.

For the functions $-5\sin\theta$ and $-5\cos\theta$, we have $a=-5$, so the amplitude is $|a|=|-5|=5$.

Let's do an example where we find the amplitude from the graph of the function, instead of from the function's equation.

Example

Find the amplitude of the function shown in the graph below.

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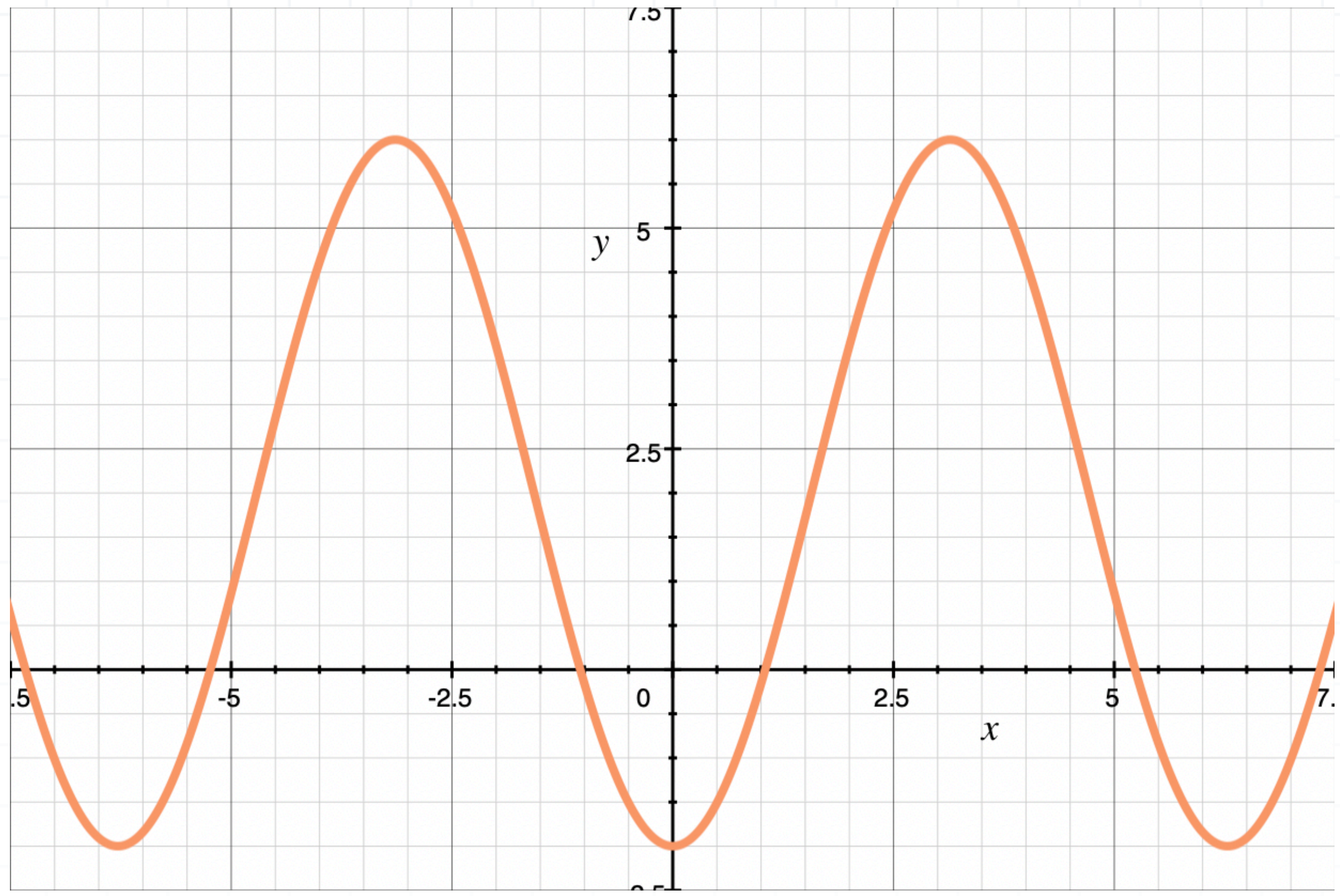
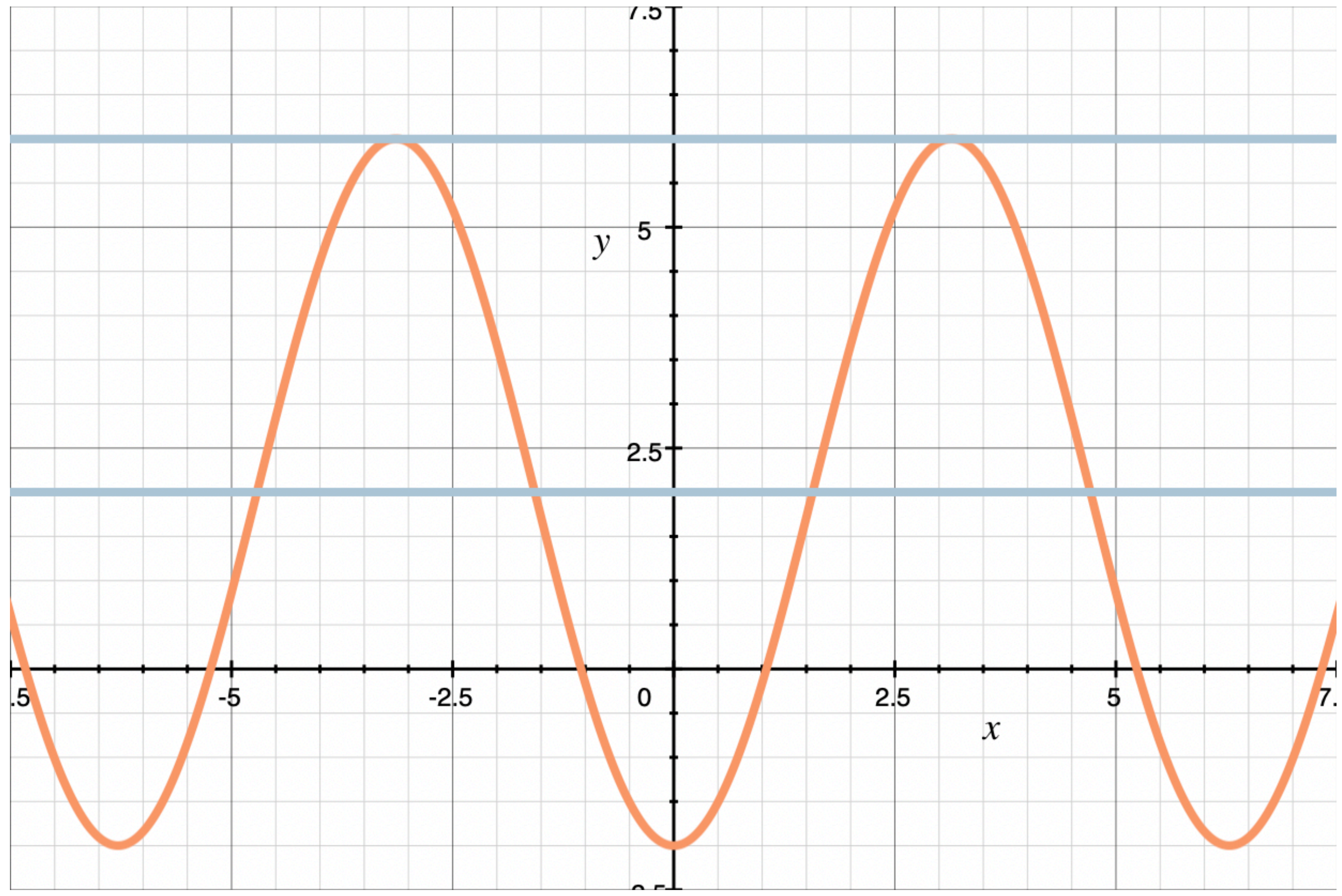


Since the midline runs through $y = 2$, and the top of the curve reaches $y = 6$,



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the distance from the midline of the curve to the very top of the curve is $6 - 2 = 4$. Therefore, the amplitude of the function is 4.

Or, alternatively, we can use the amplitude formula.

$$|a| = \frac{\text{max} - \text{min}}{2} = \frac{6 - (-2)}{2} = \frac{6 + 2}{2} = \frac{8}{2} = 4$$

Period

Now we'll dive deeper into the value of b. Remember how we learned in the last lesson that changing the value of b in either a sine or cosine function,

$$y = a\sin(b(x + c)) + d$$

$$y = a\cos(b(x + c)) + d$$

would horizontally stretch and compress the function. Formally, we say that the **period** of the function is

$2\pi/|b|$ for sine, cosine, secant, and cosecant

$\pi/|b|$ for tangent and cotangent

Think about the period as one full rotation of the function, or how long it takes for the function to get back to its "starting point." For instance, let's look back at the graph of $y = \sin x$ from earlier:

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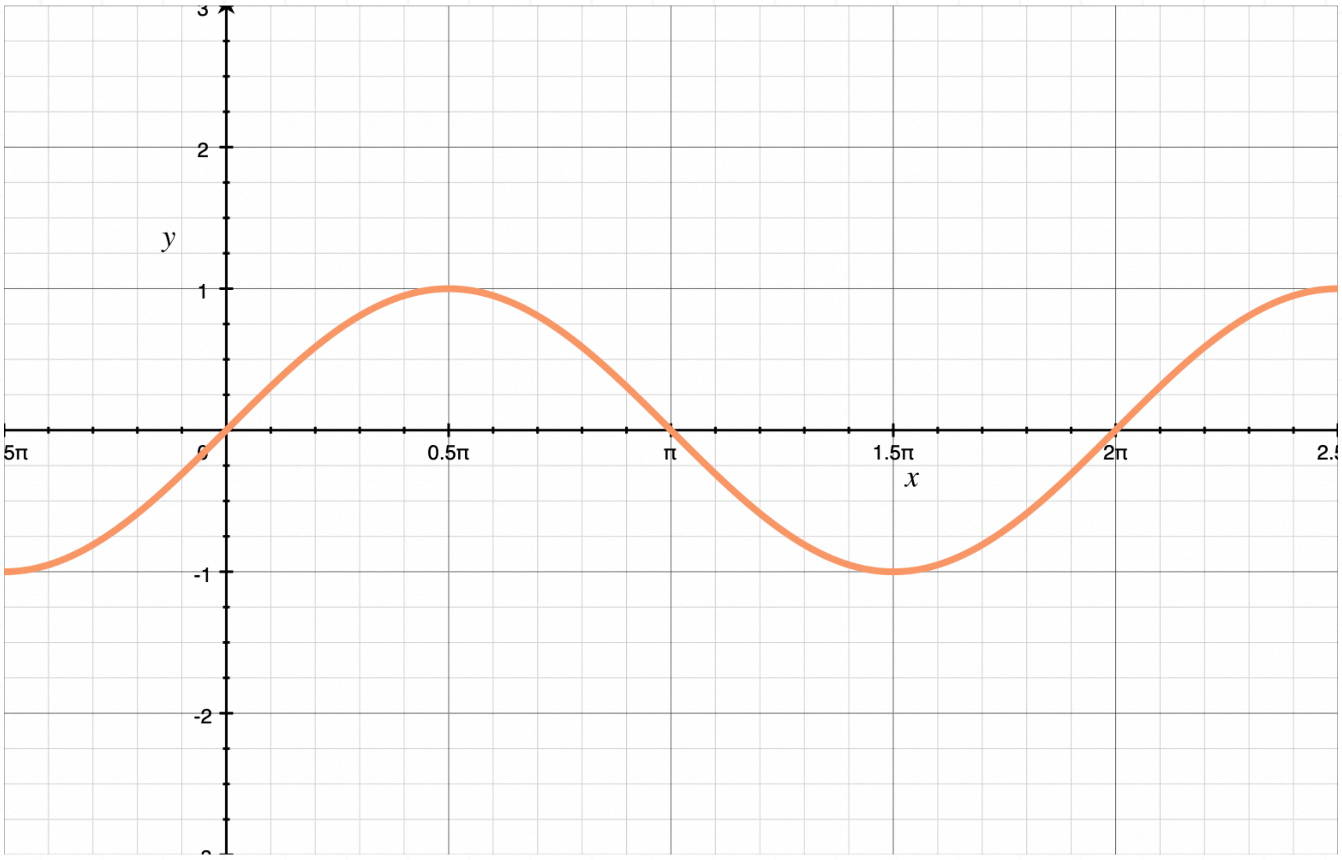
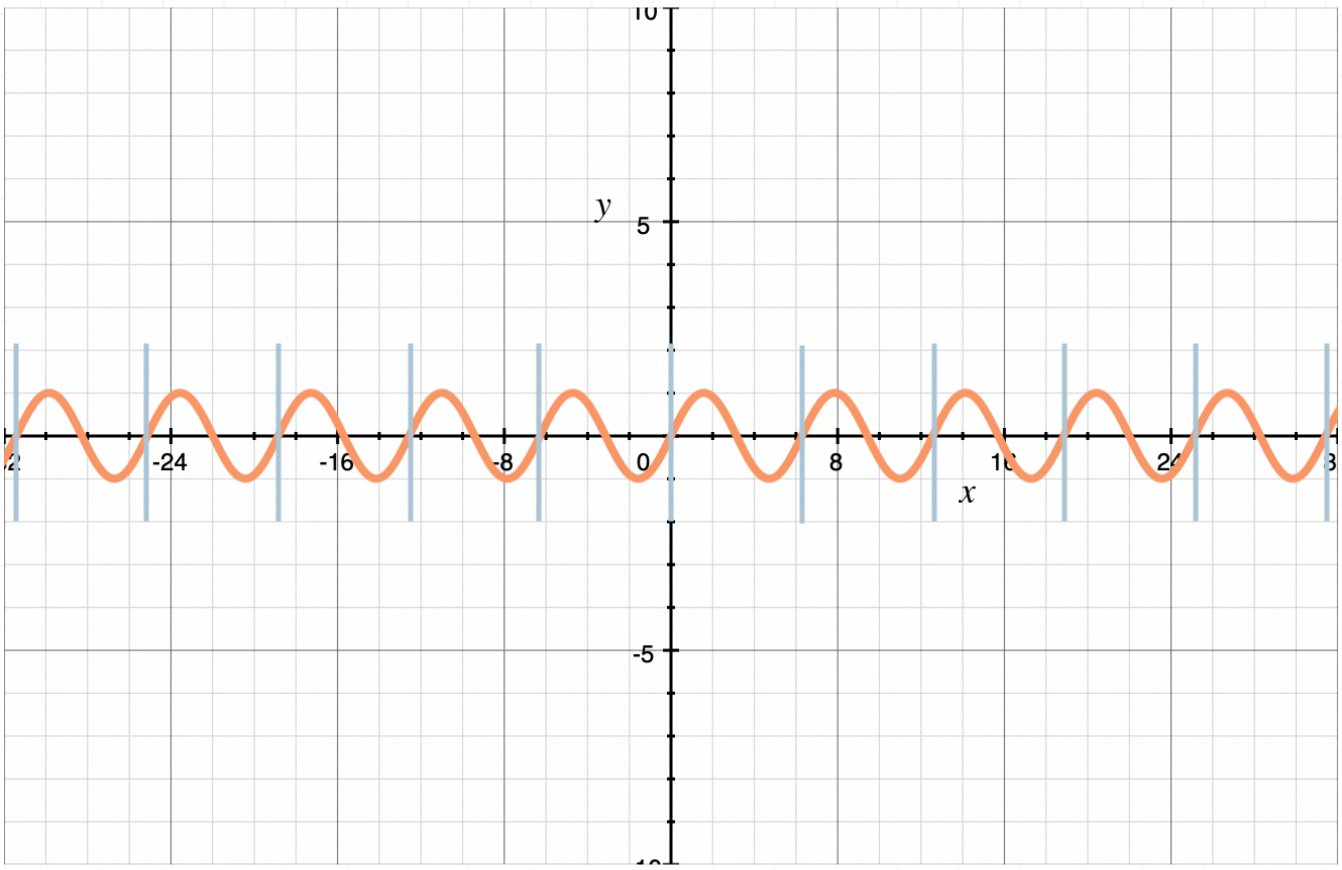


Let's say that $(0,0)$ is the "starting point." At that point, the function's value is 0. As we travel along the graph to the right, we see that the curve has a pattern to it. There's a unique "S" shape from $(0,0)$ to $(2\pi,0)$. But after that, the function sort of starts over, repeating this same "S" curve over and over again, infinitely, as we continue moving to the right. The same pattern also repeats itself as we move out to the left. If we zoom out on the graph of $y = \sin x$, we can see many periods at once:



The formula $2\pi/|b|$ tells us how long each period will be, or how long it takes before the function starts to repeat itself again. Of course, when $b = 1$, the period of both the sine and cosine functions are $2\pi/|1| = 2\pi$.

Think about for a second how this idea of repeating values should make sense to us, given what we know about the unit circle. In the unit circle, the values of sine and cosine vary from the angle $\theta = 0$ until we get back around to our “starting point” at $\theta = 2\pi$. But once we’ve rotated all the way around the circle, and we start our second rotation, the values of sine and cosine will start to repeat, following the same pattern they did through the first rotation. All we’re doing here is stretching out that pattern of repeated values in the xy-plane.

Let’s do an example with a sine function.

Example

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Find the amplitude and period of $4\sin(6\theta)$.

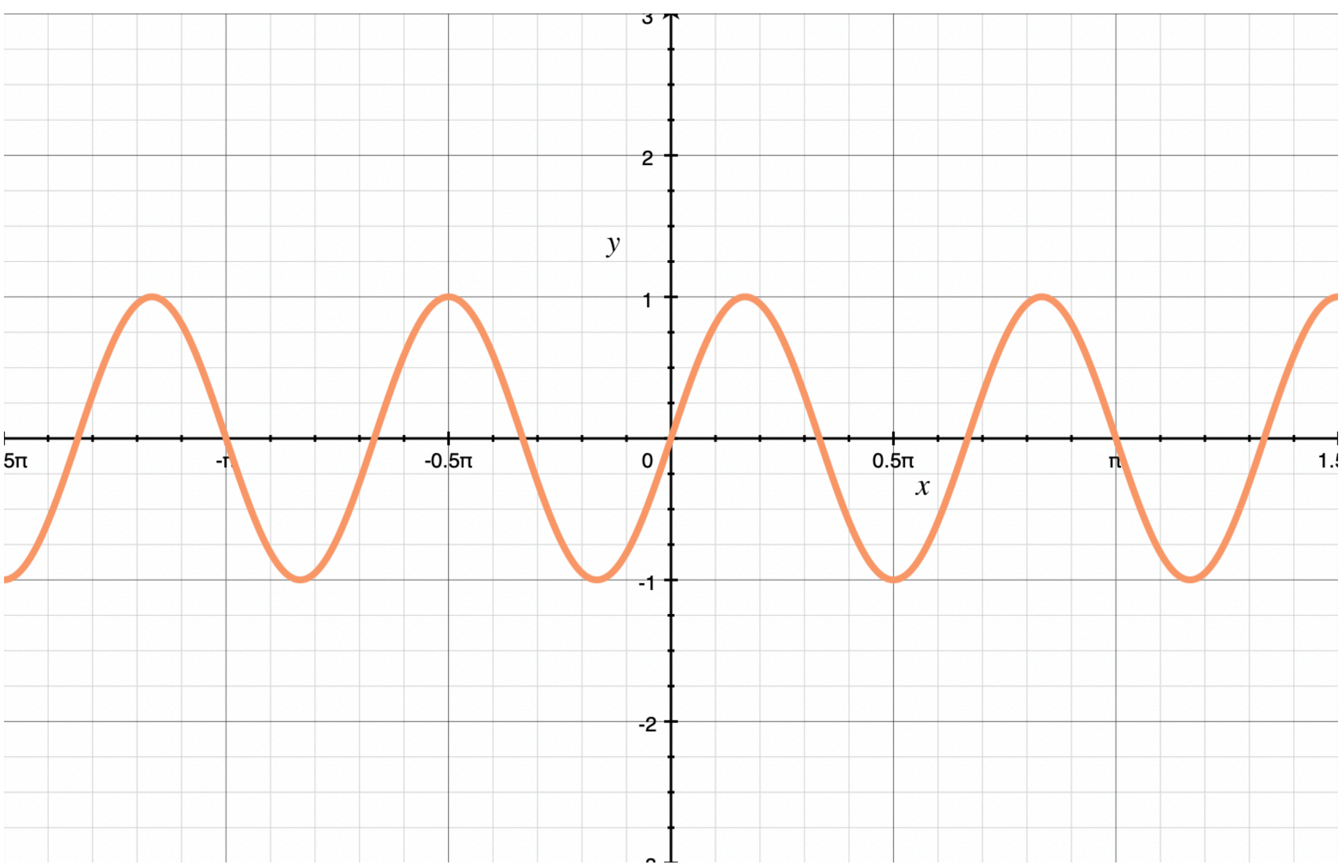
For this sine function, $a=4$ and $b=6$, so the amplitude is $|a|=|4|=4$ and the period is

$$\frac{2\pi}{|b|}=\frac{2\pi}{|6|}=\frac{2\pi}{6}=\frac{\pi}{3}$$

Now we'll look at an example where we find the period from the graph of the function.

Example

Find the period of the function shown in the graph below.



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From the graph, we see a unique "S" shape from $(0,0)$ to $(2\pi/3,0)$, after which the pattern of the curve starts to repeat itself. Therefore, the period of the function is $2\pi/3$.

If we look at the graphs of the other four trig functions, we can see that the period of tangent is π, the period of cosecant is 2π, the period of secant is 2π, and the period of cotangent is π. Let's summarize those in a table.

Function	**Period**
$y = \sin x$	2π
$y = \cos x$	2π
$y = \tan x$	π
$y = \csc x$	2π
$y = \sec x$	2π
$y = \cot x$	π

Of course, like we saw in the last lesson, if we increase the value of b for any of these functions from 1 to some value greater than 1, then the period shrinks, and the graph gets horizontally compressed.

But if we decrease the value of b from 1 to some value less than 1, then the period grows, and the graph gets horizontally stretched.

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To figure out what happens when b is negative, we have to think about the even-odd identities that we learned about before:

$$\sin(-\theta) = -\sin\theta \qquad \csc(-\theta) = -\csc\theta$$

$$\cos(-\theta) = \cos\theta \qquad \sec(-\theta) = \sec\theta$$

$$\tan(-\theta) = -\tan\theta \qquad \cot(-\theta) = -\cot\theta$$

They show us that for sine, tangent, cosecant, and cotangent, the negative sign on b can be moved to the front of the function and attached to a. Which means that the period will remain the same, but the graph will get reflected over the x-axis.

Let's do an example where we use the even-odd identities.

Example

Express the functions $7\sin(-3\theta)$ and $7\cos(-3\theta)$ in the form $a\sin(b\theta)$ and $a\cos(b\theta)$ where a and b are real numbers and b is positive, and then state their amplitudes and periods.

Since $\sin(-3\theta) = -\sin(3\theta)$ and $\cos(-3\theta) = \cos(3\theta)$, we can express the functions $7\sin(-3\theta)$ and $7\cos(-3\theta)$ as $-7\sin(3\theta)$ and $7\cos(3\theta)$.

For $-7\sin(3\theta)$, we see $a = -7$ and $b = 3$, so the amplitude is $|a| = |-7| = 7$ and the period is

$$\frac{2\pi}{|b|} = \frac{2\pi}{|3|} = \frac{2\pi}{3}$$

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For $7\cos(3\theta)$, we see $a = 7$ and $b = 3$, so the amplitude is $|a| = |7| = 7$ and the period is

$$\frac{2\pi}{|b|} = \frac{2\pi}{|3|} = \frac{2\pi}{3}$$

Let's do one more example here.

Example

Express the function $y = 2\cos(-3x + 7)$ in the form $y = a\cos(bx + c)$, where a and b are real numbers and where b is positive. Then state the function's amplitude and period.

Since $\cos(-x) = \cos x$, we can rewrite the function as

$$y = 2\cos(-3x + 7)$$

$$y = 2\cos(-(3x - 7))$$

$$y = 2\cos(3x - 7)$$

We see $a = 2$ and $b = 3$. So the amplitude is $|a| = |2| = 2$ and the period is

$$\frac{2\pi}{|b|} = \frac{2\pi}{|3|} = \frac{2\pi}{3}$$

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Here's an example with cosecant and secant.

Example

Find the periods of $-9\sec(3\theta/2)$ and $2.5\csc(\theta/6)$.

For $-9\sec(3\theta/2)$, we see $b = 3/2$, so the period is

$$\frac{2\pi}{|b|} = \frac{2\pi}{\left|\frac{3}{2}\right|} = 2\pi\left(\frac{2}{3}\right) = \frac{4\pi}{3}$$

For $2.5\csc(\theta/6)$, we see $b = 1/6$, so the period is

$$\frac{2\pi}{|b|} = \frac{2\pi}{\left|\frac{1}{6}\right|} = 2\pi\left(\frac{6}{1}\right) = 12\pi$$

The first four trig functions have a period of 2π, so we can find their period using $2\pi/|b|$, but tangent and cotangent have a period of π, which means we'll find their periods using $\pi/|b|$.

Example

Find the periods of $25\tan(\theta/4)$ and $-(1/9)\cot(3\theta)$.

For $25\tan(\theta/4)$, the value of b is $b = 1/4$, so the period is

$$\frac{\pi}{|b|} = \frac{\pi}{\left|\frac{1}{4}\right|} = \pi\left(\frac{4}{1}\right) = 4\pi$$

For $-(1/9)\cot(3\theta)$, the value of b is $b = 3$, so the period is

$$\frac{\pi}{|b|} = \frac{\pi}{|3|} = \frac{\pi}{3}$$

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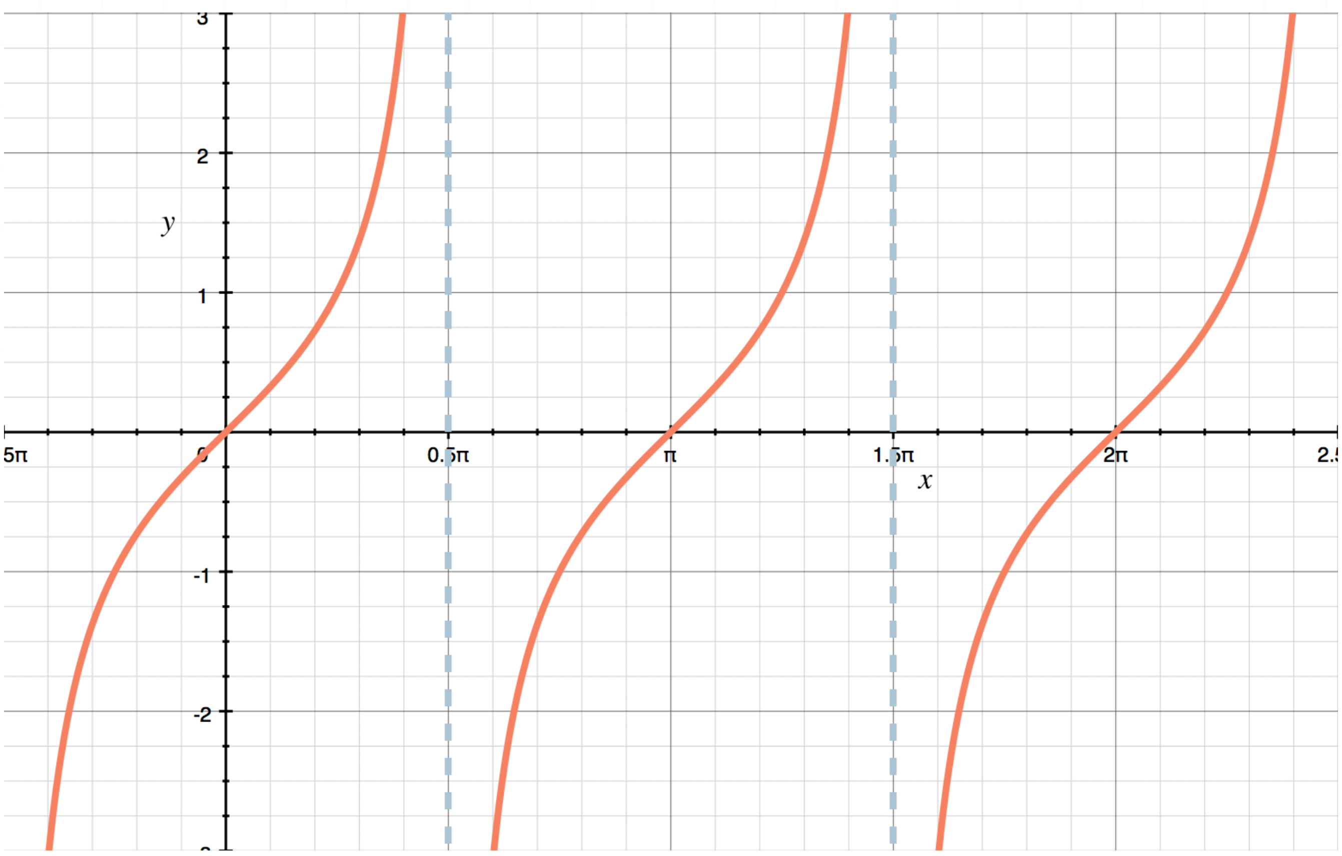
Sketching tangent and cotangent

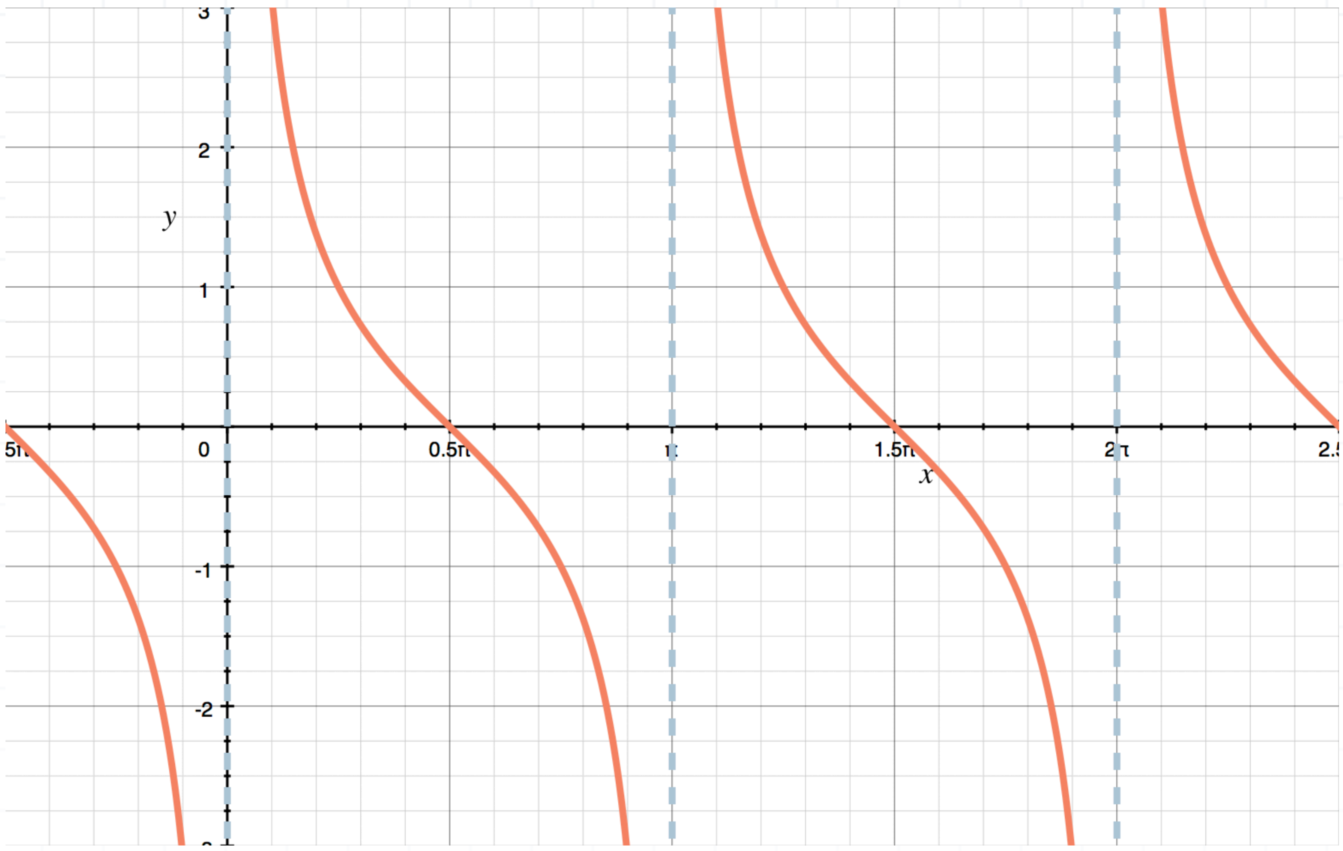
We started this section by looking at the graphs of sine and cosine and how to sketch them. Then we saw that the cosecant and secant functions can be easily sketched from their reciprocal sine and cosine functions.

In this lesson, we want to finish off the six trig functions by learning how to sketch the tangent and cotangent functions.

We looked at both of these in the last lesson, but let's start our discussion here with a reminder that the graph of $y = \tan x$ is



and the graph of $y = \cot x$ is



Sketching $\tan x$ and $\cot x$

To sketch tangent and cotangent functions in the form

$$y = a\tan(b(x+c)) + d$$

$$y = a\cot(b(x+c)) + d$$

where $c = 0$ and $d = 0$, we want to start by locating two adjacent vertical asymptotes. We'll follow these steps:

1. For $y = a\tan(bx)$, solve the equations $bx = -\pi/2$ and $bx = \pi/2$ to get adjacent vertical asymptotes, and for $y = a\cot(bx)$, solve $bx = 0$ and $bx = \pi$ to get adjacent vertical asymptotes.

2. Sketch this pair of vertical asymptotes, then divide the interval in between the pair into four equal parts.

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3. Evaluate the function at each quarter line and the midline of the interval.

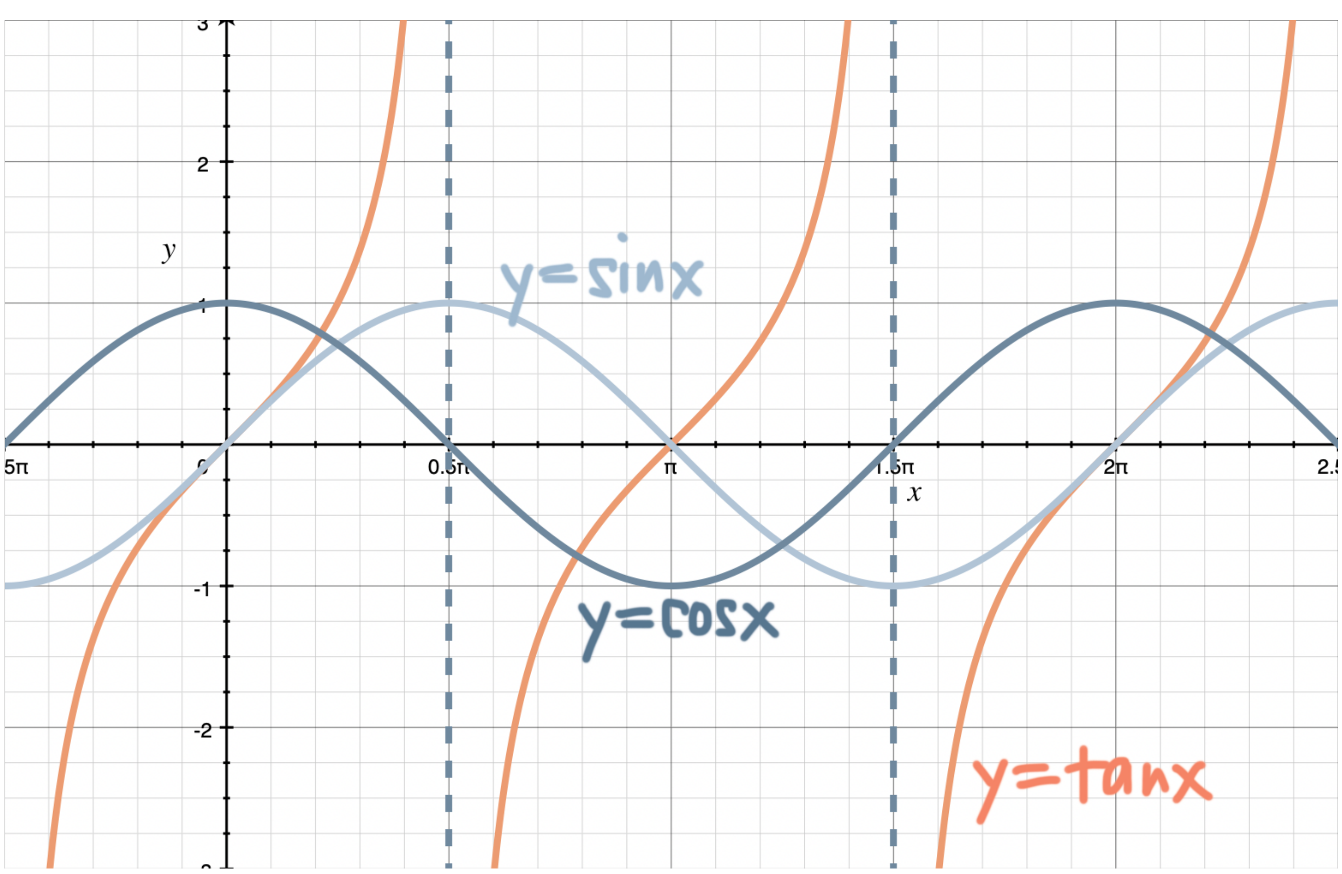
4. Join those points with a smooth curve, letting the curve approach the vertical asymptotes, then once the function is sketched in one interval, duplicate that curve in other intervals to the left and right.

Remember also the quotient identity for tangent:

$$\tan x = \frac{\sin x}{\cos x}$$

With this in mind, we can say that tangent is undefined when $\cos x = 0$, so the graph of tangent will have vertical asymptotes where the cosine function crosses the x-axis.

Similarly, tangent is 0 when $\sin x = 0$, so the graph of tangent will cross the x-axis where the sine function crosses the x-axis. If we sketch tangent in red, sine in blue, and cosine in dark blue, we see these relationships:



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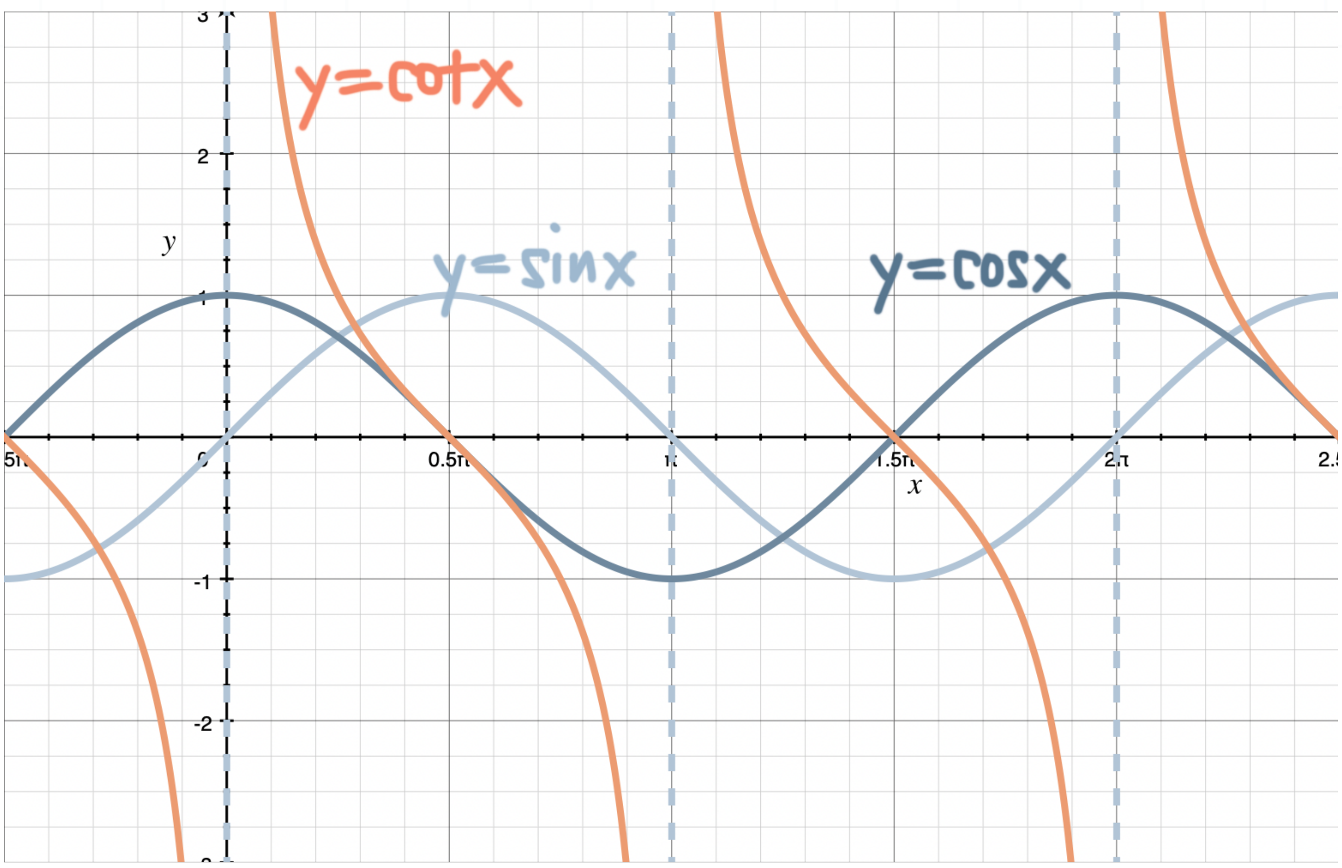
Now think about the quotient identity for cotangent:

$$\cot x = \frac{\cos x}{\sin x}$$

From this quotient identity, we know that cotangent is undefined when $\sin x = 0$, so the graph of cotangent will haver vertical asymptotes where the sine function crosses the x-axis.

Similarly, cotangent is 0 when $\cos x = 0$, so the graph of cotangent will cross the x-axis where the cosine function crosses the x-axis. If we sketch cotangent in red, sine in blue, and cosine in dark blue, we see these relationships:



The easiest way to understand these steps is to work through an example, so let's do one with a tangent function.

Example

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Sketch the graph of $y = -2\tan(2x) + 1$. Hint: adding 1 to $y = -2\tan(2x)$ shifts the graph up vertically by 1 unit.

We'll find two adjacent vertical asymptotes by solving $bx = -\pi/2$ and $bx = \pi/2$ for x. With $b = 2$, we get

$$2x = -\frac{\pi}{2}$$

$$x = -\frac{\pi}{4}$$

and

$$2x = \frac{\pi}{2}$$

$$x = \frac{\pi}{4}$$

We'll sketch in the vertical asymptotes $x = -\pi/4$ and $x = \pi/4$,

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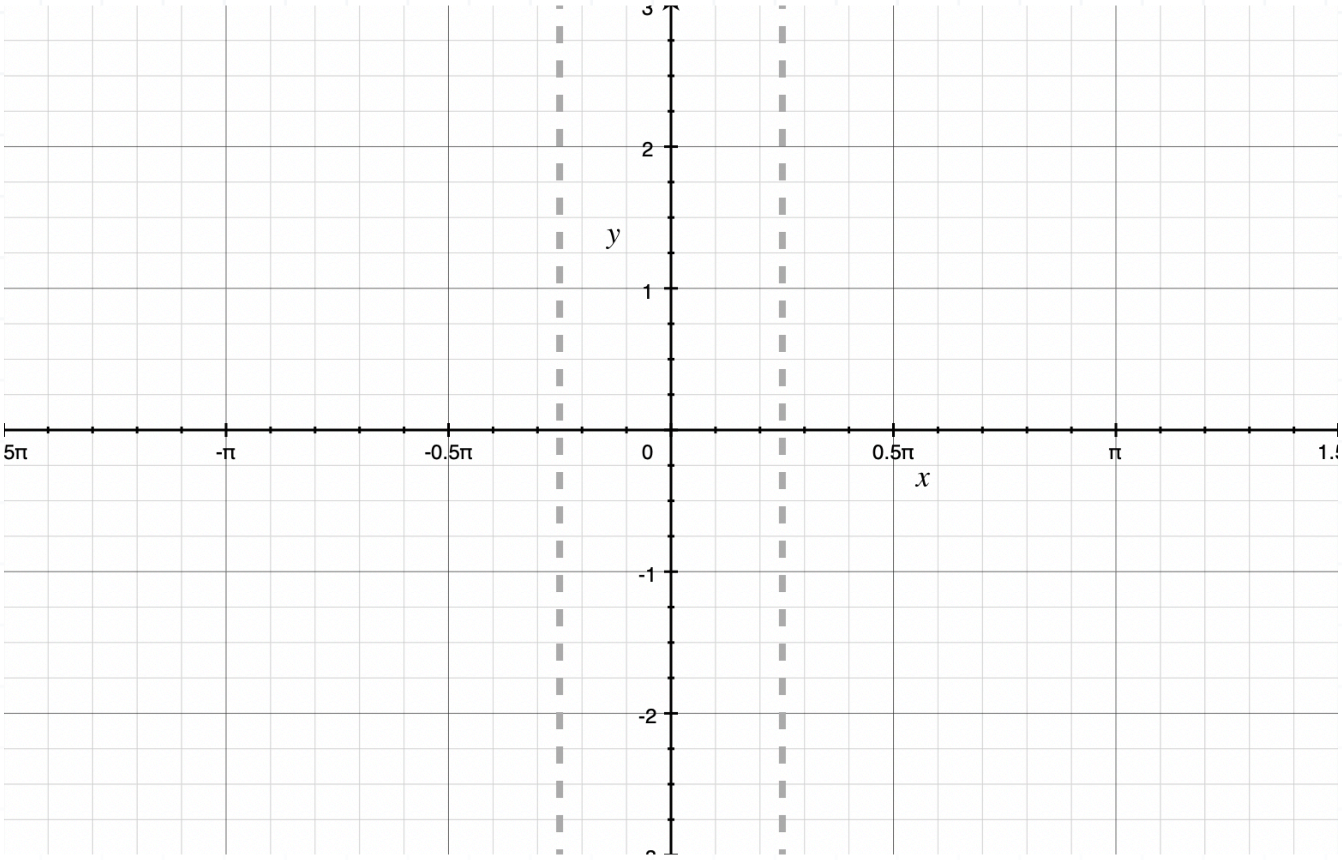
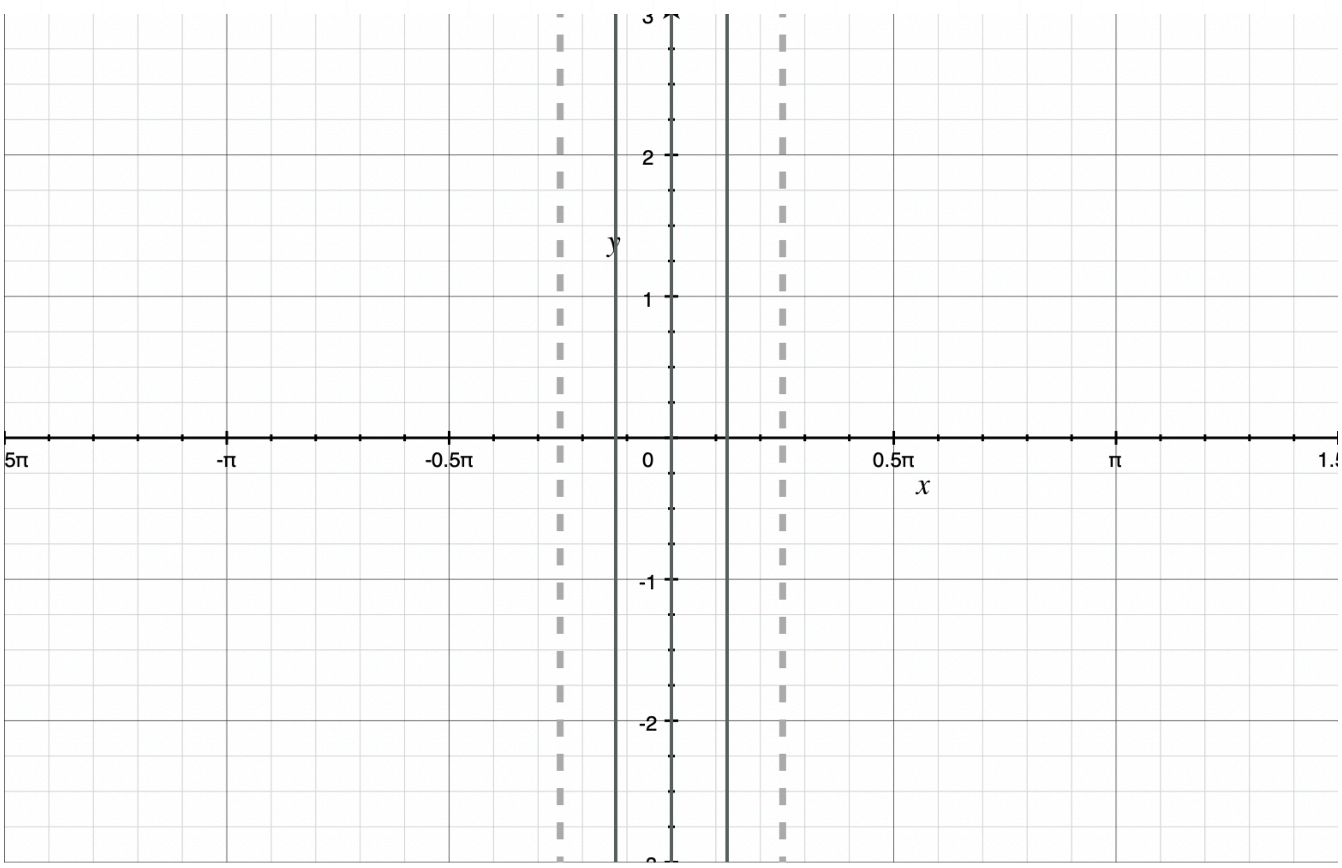


and then divide the interval between $x = -\pi/4$ and $x = \pi/4$ into four equal parts:



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The dividing lines of each of these four sub-intervals are $x = -\pi/8$, $x = 0$, and $x = \pi/8$, so we'll evaluate $y = -2\tan(2x) + 1$ at those three values. We'll get

$$y = -2\tan\left(2 \cdot -\frac{\pi}{8}\right) + 1$$

$$y = -2\tan\left(-\frac{\pi}{4}\right) + 1$$

$$y = -2(-1) + 1$$

$$y = 2 + 1$$

$$y = 3$$

and

$$y = -2\tan(2 \cdot 0) + 1$$

$$y = -2\tan(0) + 1$$

$$y = -2(0) + 1$$

$$y = 0 + 1$$

$$y = 1$$

and

$$y = -2\tan\left(2 \cdot \frac{\pi}{8}\right) + 1$$

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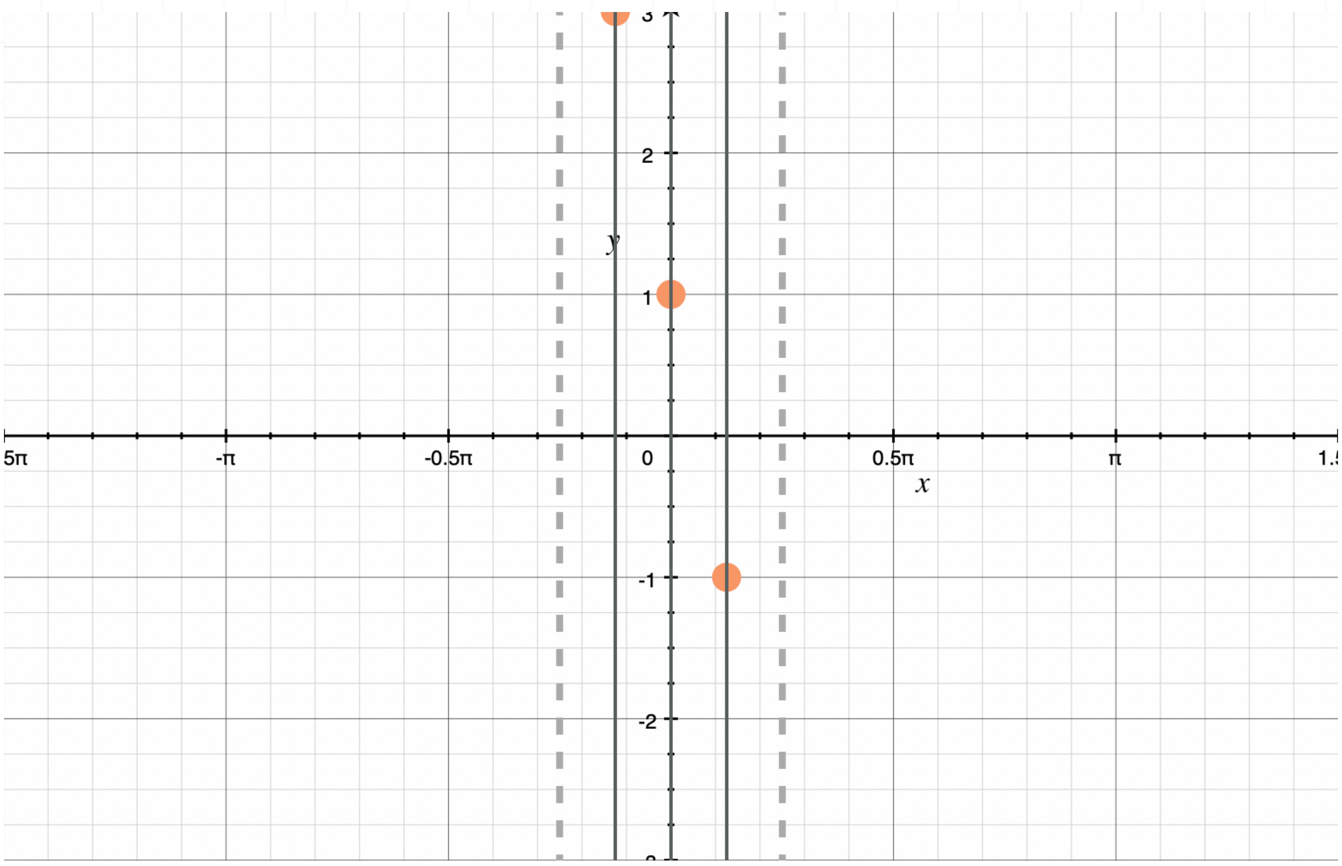
$$y = -2\tan\left(\frac{\pi}{4}\right) + 1$$

$$y = -2(1) + 1$$

$$y = -2 + 1$$

$$y = -1$$

We’ll plot those points,



and then connect them with a smooth curve, respecting the asymptotes.

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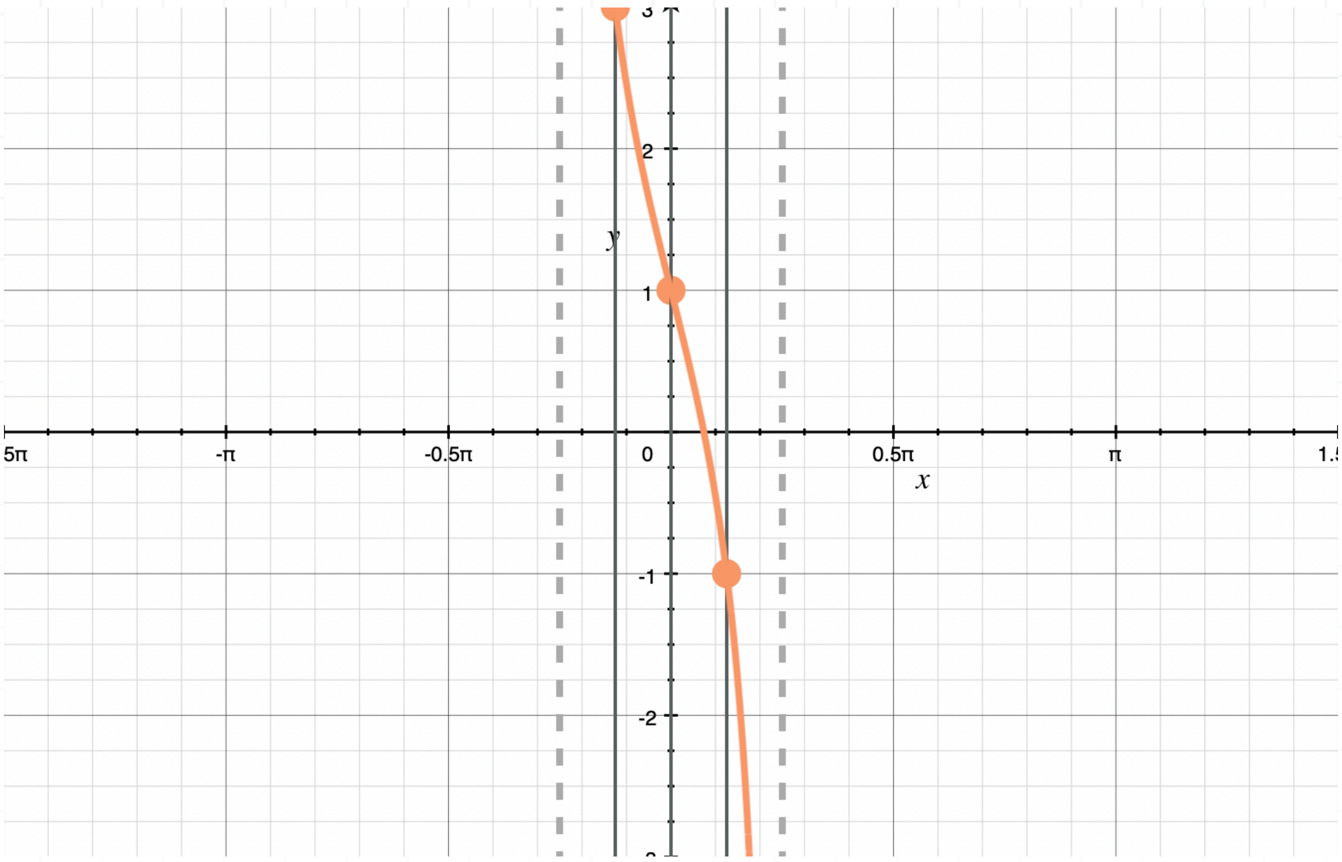

Because the period of this tangent function is $\pi/|b| = \pi/2$, we can sketch more vertical asymptotes at

$$\frac{\pi}{4} + \frac{\pi}{2} = \frac{\pi}{4} + \frac{2\pi}{4} = \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}, \ldots$$

$$-\frac{\pi}{4} - \frac{\pi}{2} = -\frac{\pi}{4} - \frac{2\pi}{4} = -\frac{3\pi}{4}, \ -\frac{5\pi}{4}, \ -\frac{7\pi}{4}, \ldots$$

sketching repeated periods of the tangent function in between them. After repeating this pattern both to the left and right, and taking away the asymptotes and other guiding lines that we sketched, and we'll get the final graph of $y = -2\tan(2x) + 1$.

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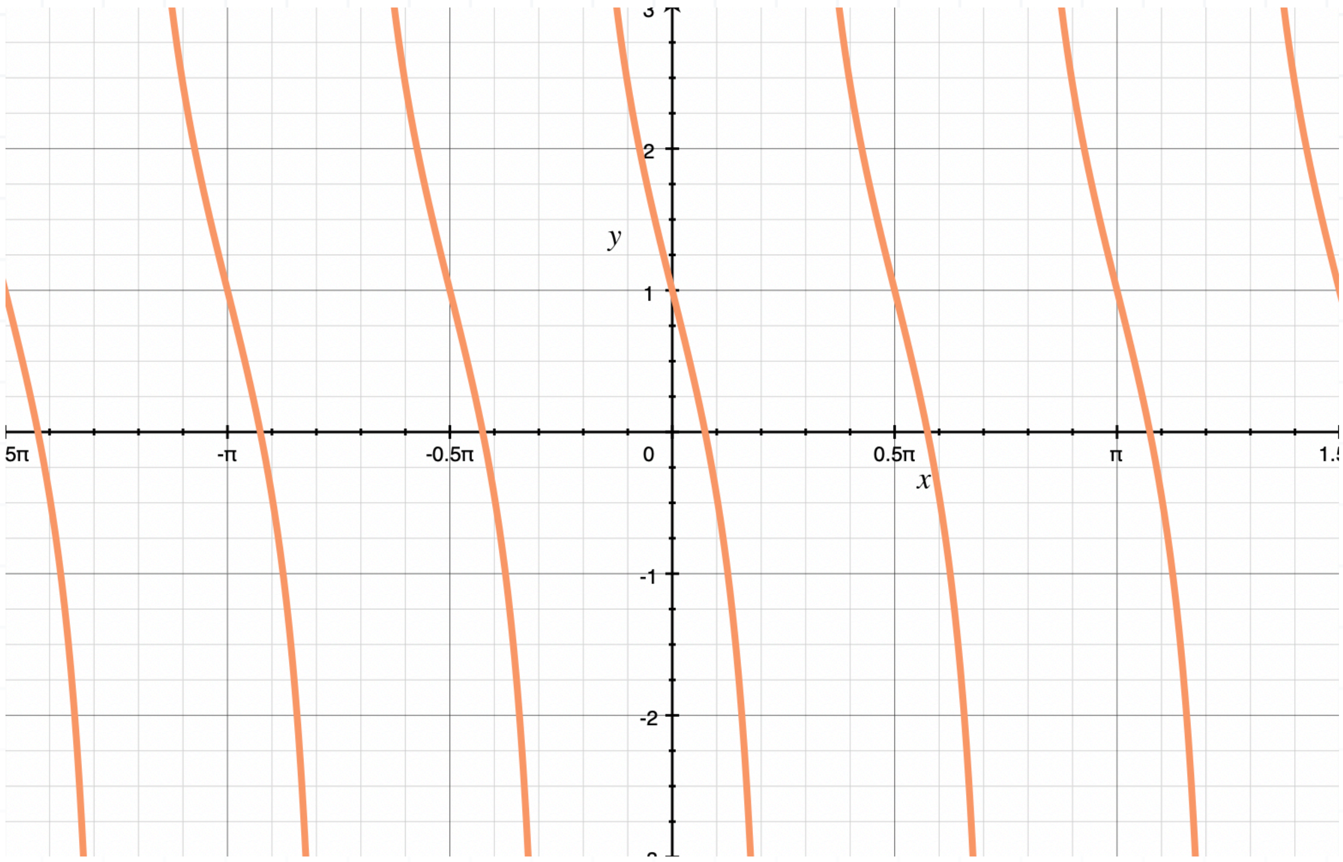


Now let's do an example with a cotangent function.

Example

Sketch the graph of $y = \cot x - 2$. Hint: subtracting 2 from $y = \cot x$ shifts the graph down vertically by 2 units.

We'll find two adjacent vertical asymptotes by solving $bx = 0$ and $bx = \pi$ for x. With $b = 1$, we get

$$1x = 0$$

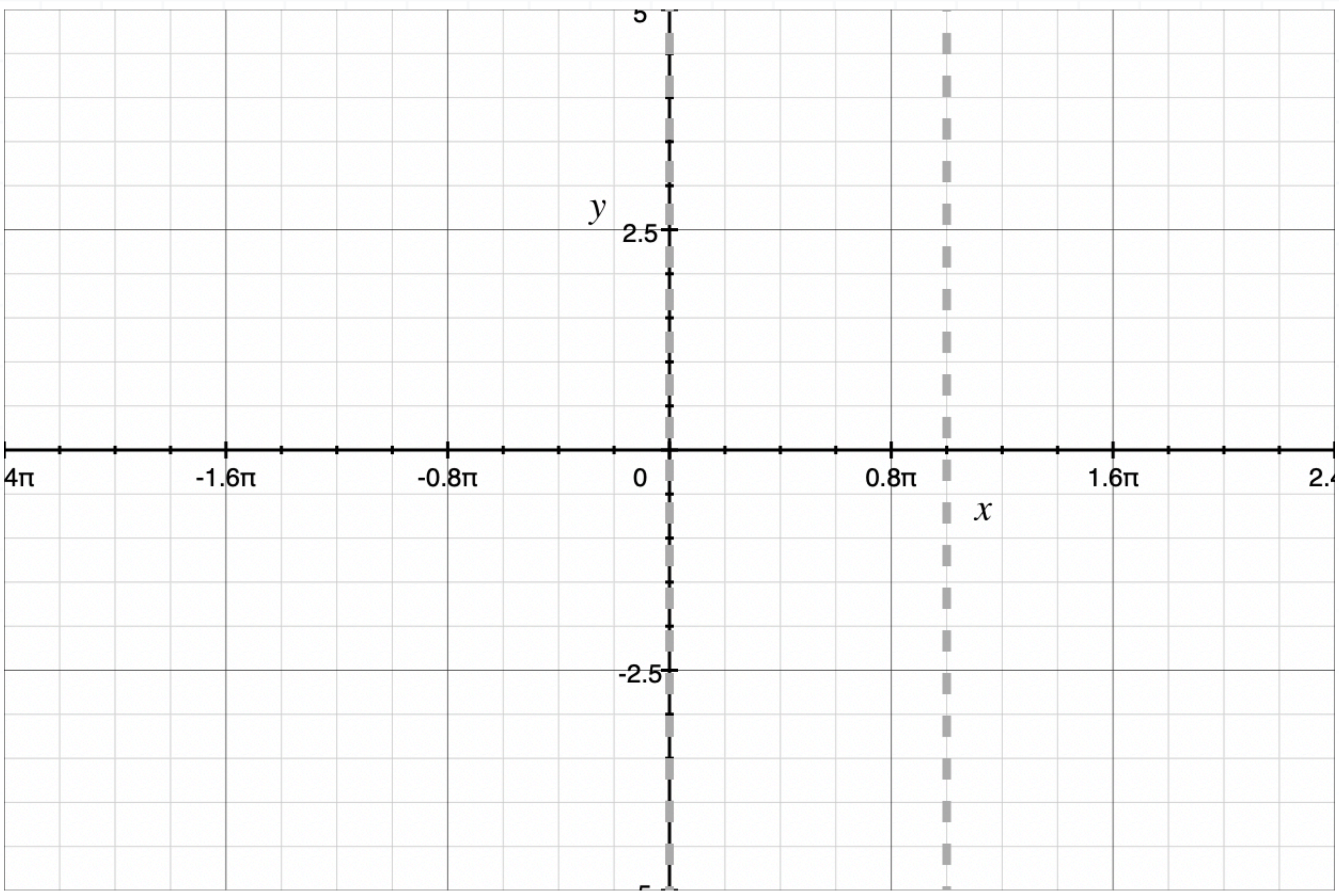
$$x = 0$$

and

$$1x = \pi$$

$$x = \pi$$

We'll sketch in the vertical asymptotes $x = 0$ and $x = \pi$,



and then divide the interval between $x = 0$ and $x = \pi$ into four equal parts:

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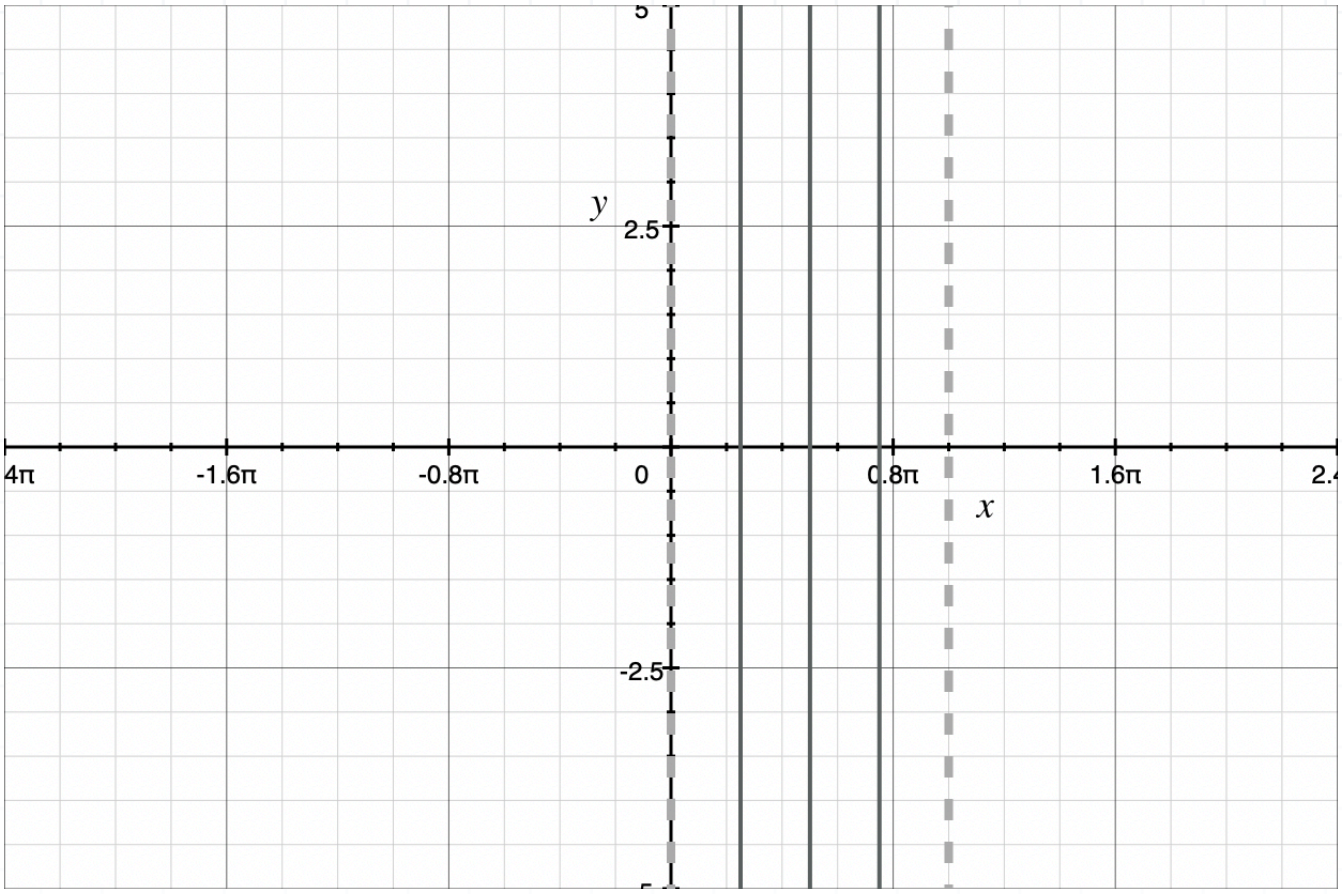

The dividing lines of each of these four sub-intervals are $x = \pi/4$, $x = \pi/2$, and $x = 3\pi/4$, so we'll evaluate $y = \cot x - 2$ at those three values. We'll get

$$y = \cot \frac{\pi}{4} - 2$$

$$y = 1 - 2$$

$$y = -1$$

and

$$y = \cot \frac{\pi}{2} - 2$$

$$y = 0 - 2$$

$$y = -2$$

and

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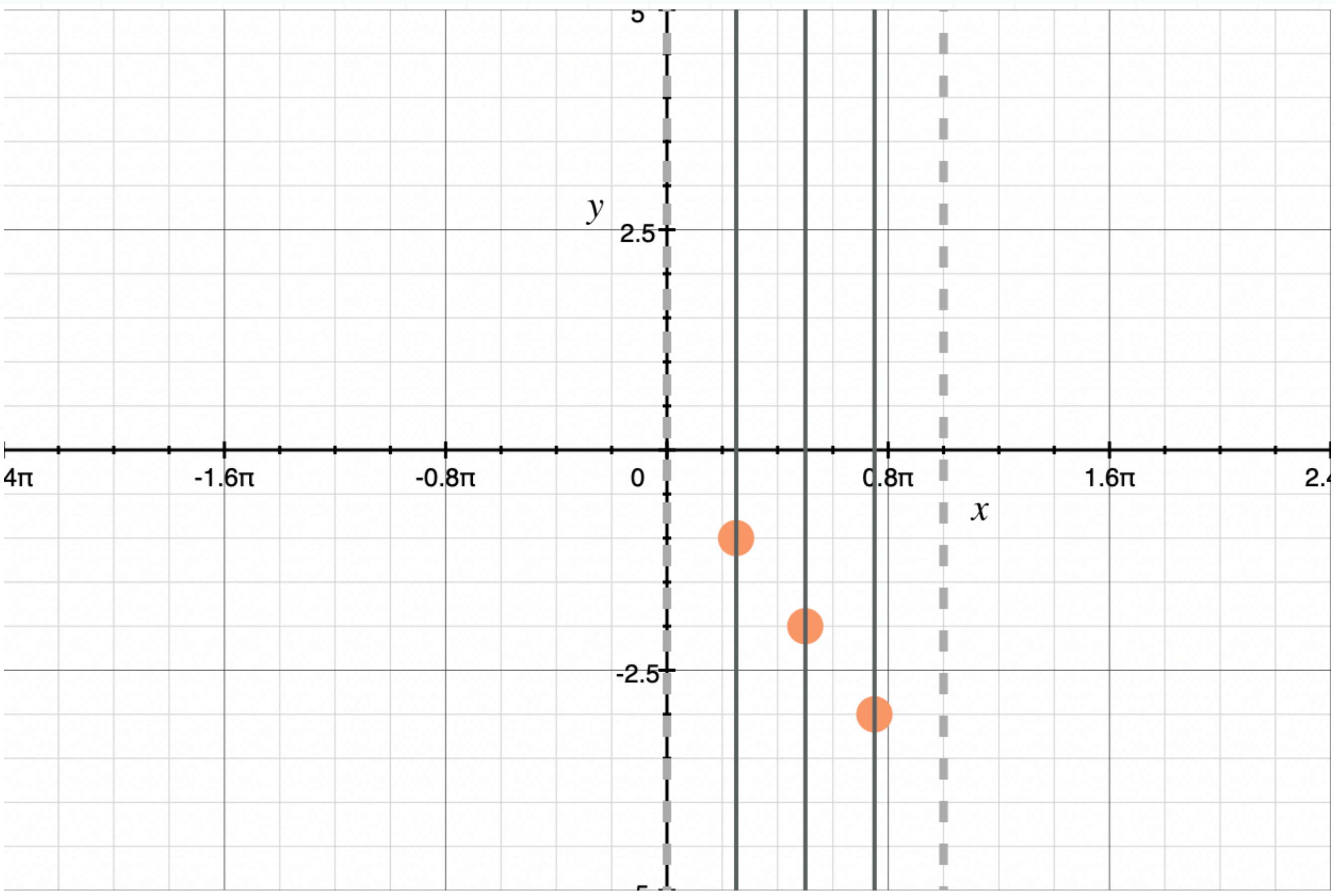
$$y = \cot \frac{3\pi}{4} - 2$$

$$y = -1 - 2$$

$$y = -3$$

We'll plot those points,



and then connect them with a smooth curve, respecting the asymptotes.

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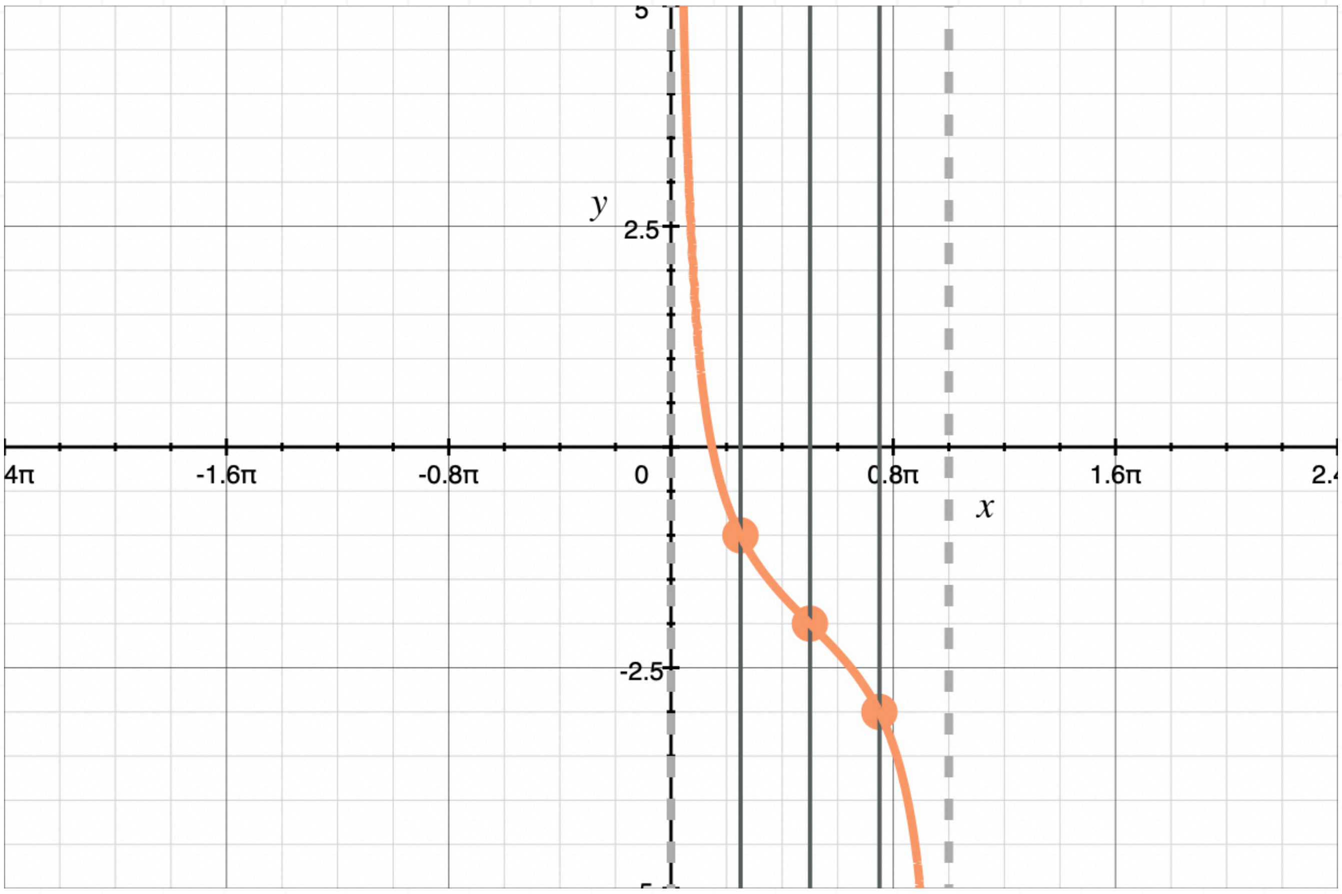
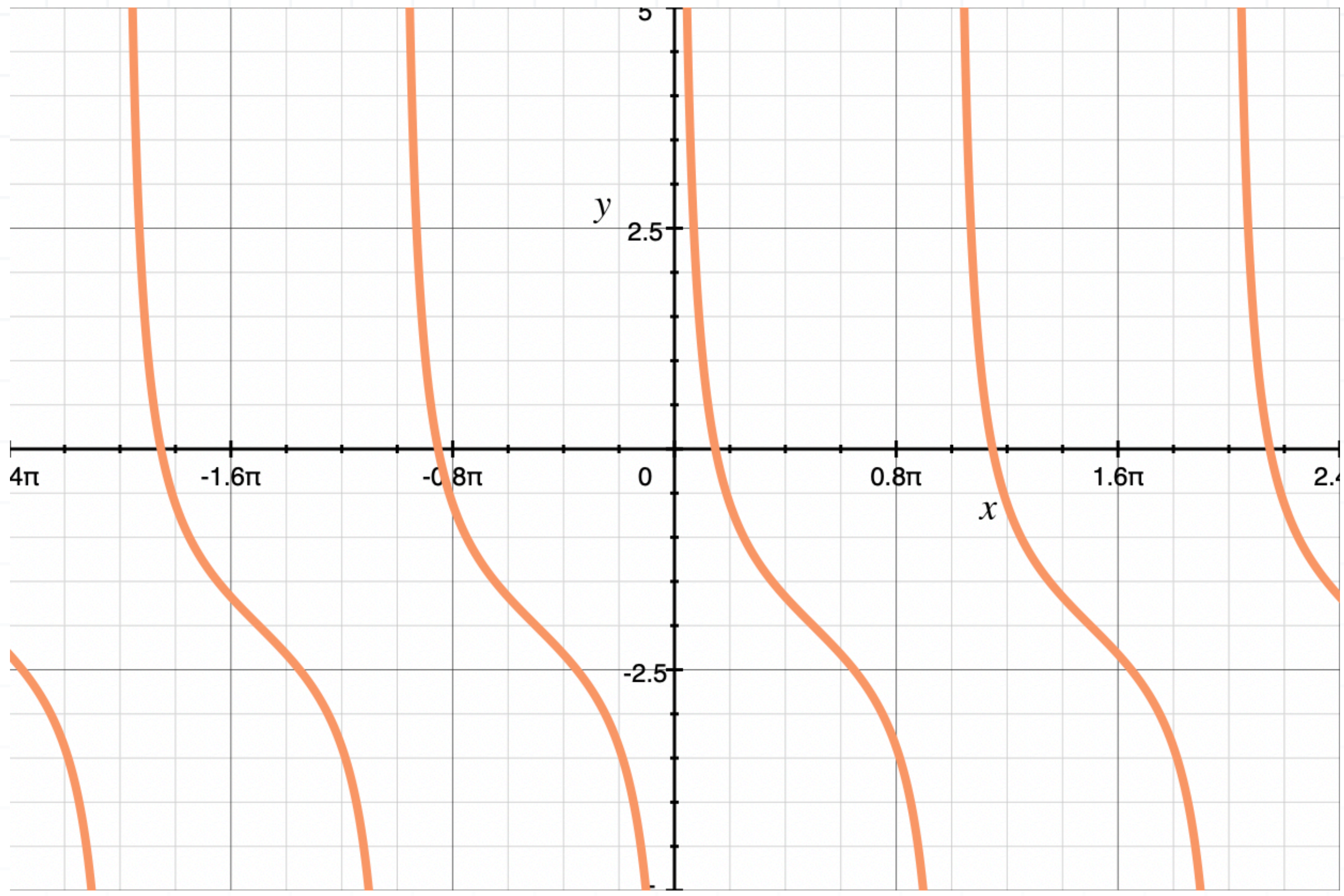


Because the period of this cotangent function is $\pi/|b| = \pi/1 = \pi$, we can sketch more vertical asymptotes at

$$\pi + \pi = 2\pi,\ 3\pi,\ 4\pi,\ \ldots$$

$$0 - \pi = -\pi,\ \ -2\pi,\ \ -3\pi,\ \ldots$$

sketching repeated periods of the cotangent function in between them. After repeating this pattern both to the left and right, and taking away the asymptotes and other guiding lines that we sketched, and we'll get the final graph of $y = \cot x - 2$.


5
y
2.5
-2.5
4π
-1.6π
-0.8π
0
0.8π
1.6π
x

Horizontal and vertical shifts

So far, while we've been looking at the graphs of the six trig functions, we've talked about how the amplitude is dictated by $|a|$, and how the period is dictated by $2\pi/|b|$ for sine, cosine, cosecant, and secant, and by $\pi/|b|$ for tangent and cotangent. So in functions like

$$y = a\sin(b(x+c)) + d$$

$$y = a\cos(b(x+c)) + d$$

we know what a and b are doing. But now, finally, we want to start talking about c and d.

Horizontal shifts

To start, for the above sine and cosine functions, if we make c any non-zero value, then the graph will be shifted horizontally either to the right or to the left. When we shift a graph to the left or right, we call it a **phase shift**.

If c is positive, then the sine and cosine functions both get shifted c units to the left horizontally, which means we'll subtract c from each x-value. But if c is negative, then the sine and cosine functions both get shifted c units to the right horizontally, which means we'll add c to each x-value.

Let's do an example with a sine function and a horizontal shift to the left.

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Example

Sketch the graph of $\sin(\theta + (\pi/4))$.

We know that a set of points along $y = \sin\theta$ is

$$y = \sin\theta: \qquad (0,0) \qquad \left(\frac{\pi}{2},1\right) \qquad (\pi,0) \qquad \left(\frac{3\pi}{2},-1\right) \qquad (2\pi,0)$$

Because $c = \pi/4$, which means c is positive, the graph of $y = \sin(\theta + (\pi/4))$ is shifted to the left, which means we'll subtract $\pi/4$ from each x-value.

$$y = \sin\theta: \qquad (0,0) \qquad \left(\frac{\pi}{2},1\right) \qquad (\pi,0) \qquad \left(\frac{3\pi}{2},-1\right) \qquad (2\pi,0)$$

$$y = \sin(\theta + (\pi/4)): \quad \left(-\frac{\pi}{4},0\right) \quad \left(\frac{\pi}{4},1\right) \quad \left(\frac{3\pi}{4},0\right) \quad \left(\frac{5\pi}{4},-1\right) \quad \left(\frac{7\pi}{4},0\right)$$

Then if we plot the points on $y = \sin(\theta + (\pi/4))$, we can see the graph.

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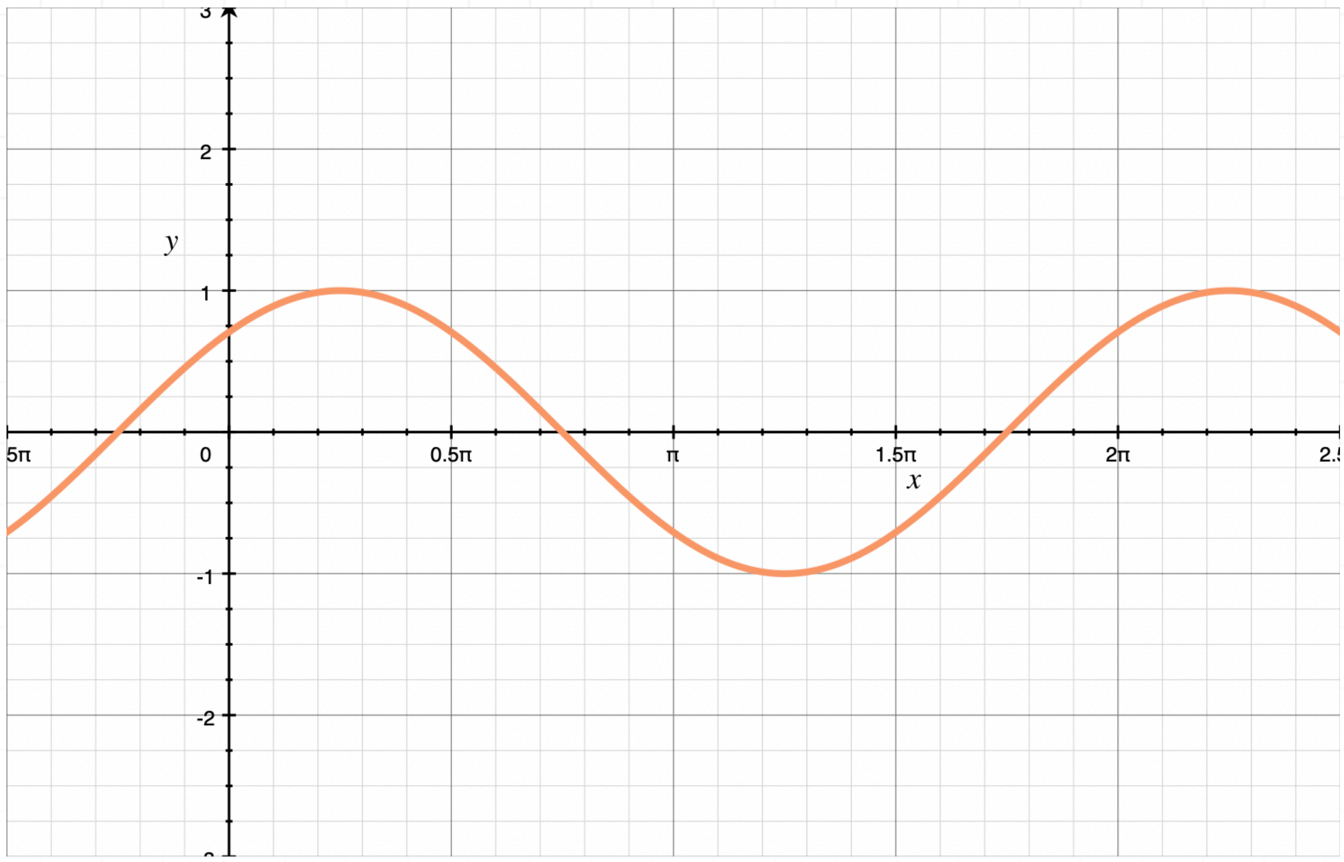


Next let's look at a cosine function with a negative c-value that will shift the function to the right.

Example

Sketch the graph of $\cos(\theta - (\pi/3))$.

We know that a set of points along $y = \cos\theta$ is

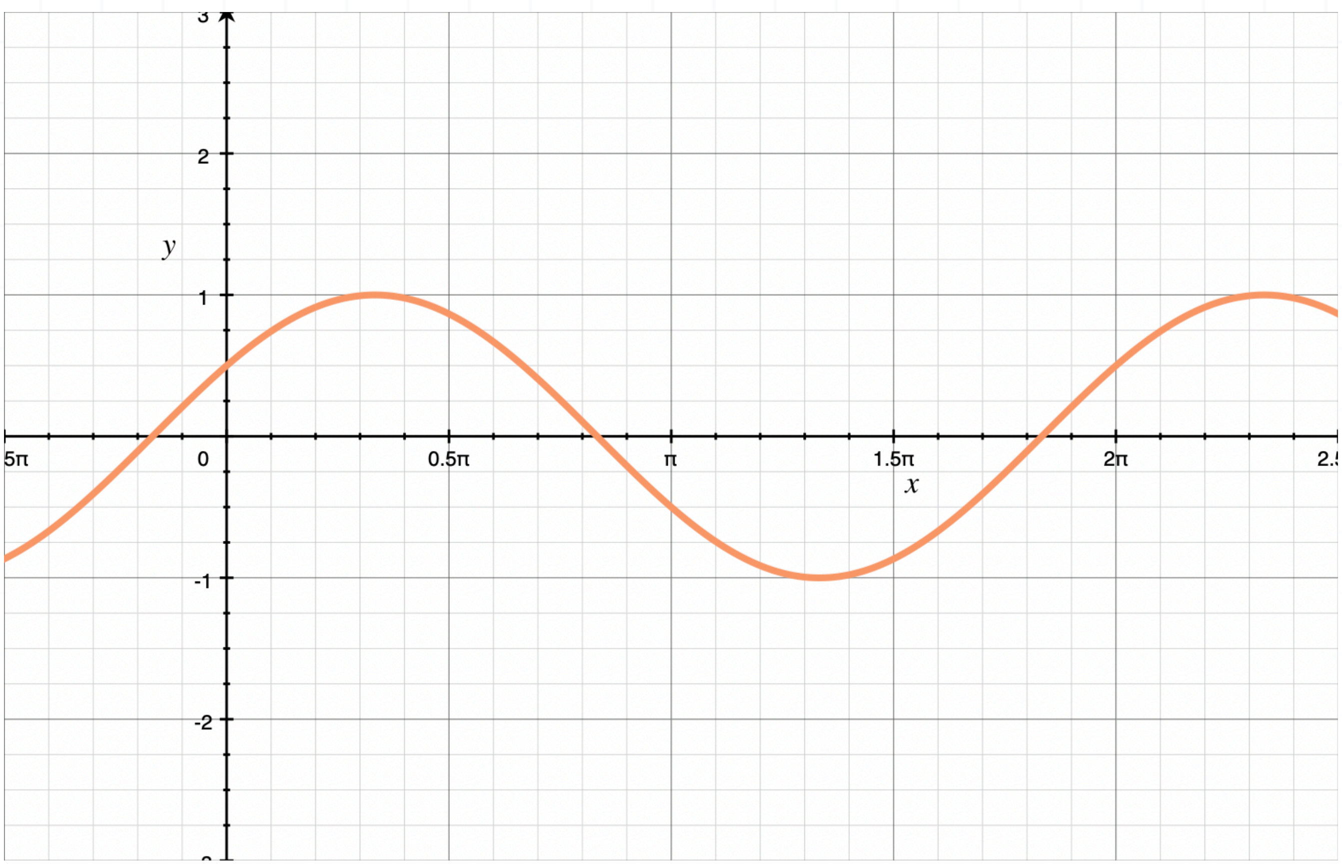
$$y = \cos\theta: \qquad (0,1) \qquad \left(\frac{\pi}{2},0\right) \qquad (\pi,-1) \qquad \left(\frac{3\pi}{2},0\right) \qquad (2\pi,1)$$

Because $c = -\pi/3$, which means c is negative, the graph of $y = \cos(\theta - (\pi/3))$ is shifted to the right, which means we'll add $\pi/3$ to each x-value.

$$y = \cos\theta: \quad (0,1) \quad \left(\frac{\pi}{2},0\right) \quad (\pi,-1) \quad \left(\frac{3\pi}{2},0\right) \quad (2\pi,1)$$

$$y = \cos(\theta - (\pi/3)): \quad \left(\frac{\pi}{3},1\right) \quad \left(\frac{5\pi}{6},0\right) \quad \left(\frac{4\pi}{3},-1\right) \quad \left(\frac{11\pi}{6},0\right) \quad \left(\frac{7\pi}{3},1\right)$$

Then if we plot the points on $y = \cos(\theta - (\pi/3))$, we can see the graph.



These kinds of horizontal shifts to the left and right will apply to cosecant, secant, tangent, and cotangent in the same way that they apply to sine and cosine.

Let's look at an example where we have a horizontal shift, but $b \neq 1$.

Example

Sketch the graph of the sine function.

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$$y = \sin\left(2\theta - \frac{\pi}{2}\right)$$

First, we need to rewrite the function in the form $y = a\sin(b(x + c)) + d$. We'll factor out a 2 from the argument.

$$2\theta - \frac{\pi}{2} = 2\left(\theta - \frac{\frac{\pi}{2}}{2}\right) = 2\left(\theta - \frac{\pi}{4}\right)$$

We know that a set of points along $y = \sin\theta$ is

$$(0,0), \left(\frac{\pi}{2},1\right), (\pi,0), \left(\frac{3\pi}{2}, -1\right), \text{ and } (2\pi,0)$$

Since $b = 2$, this means we'll horizontally compress $y = \sin\theta$ by a factor of 2. So to find points along $y = \sin(2\theta)$, we'll halve the x-values in the coordinates above, keeping the y-values the same.

And because $c = -\pi/4$, c is negative and therefore the graph of $y = \sin(2(\theta - (\pi/4)))$ is shifted to the right by $\pi/4$. So we'll add $\pi/4$ to each x-value.

Putting these two transformations together, we'll change each x-value to $(1/2)x + (\pi/4)$.

$$y = \sin\theta: \quad (0,0), \left(\frac{\pi}{2},1\right), (\pi,0), \left(\frac{3\pi}{2}, -1\right), \text{ and } (2\pi,0)$$

$$y = \sin\left(2\left(\theta - \frac{\pi}{4}\right)\right): \quad \left(\frac{\pi}{4},0\right), \left(\frac{\pi}{2},1\right), \left(\frac{3\pi}{4},0\right), (\pi, -1), \left(\frac{5\pi}{4},0\right)$$

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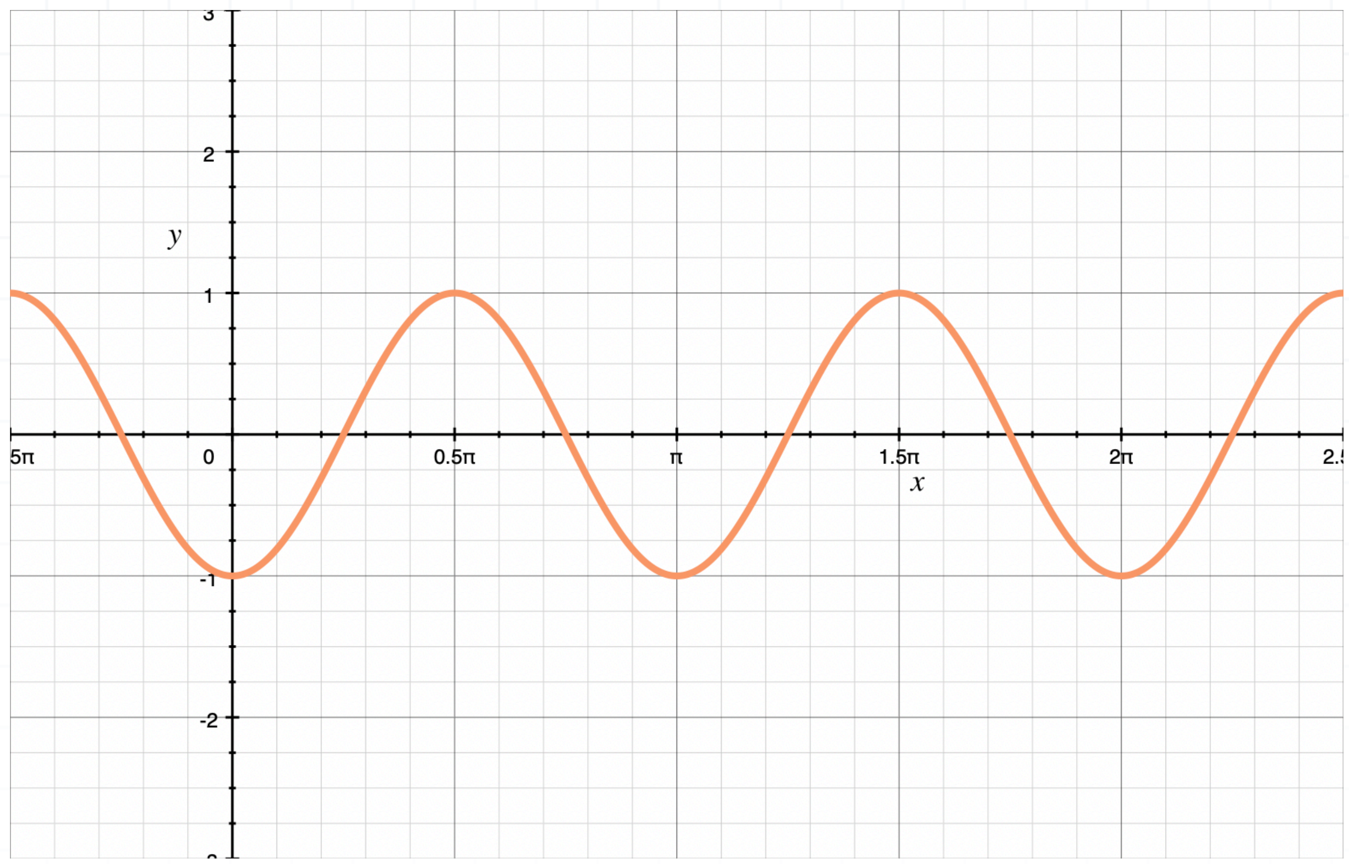
If we plot these new points, we see the graph of the transformed sine function.



Vertical shifts

So we've looked at a, b, and c, and now we want to start talking about d. In the same way that c indicated a horizontal shift in the graph to the left or right, d indicates a vertical shift in the graph up or down.

$$y = a\sin(b(x+c)) + d$$

$$y = a\cos(b(x+c)) + d$$

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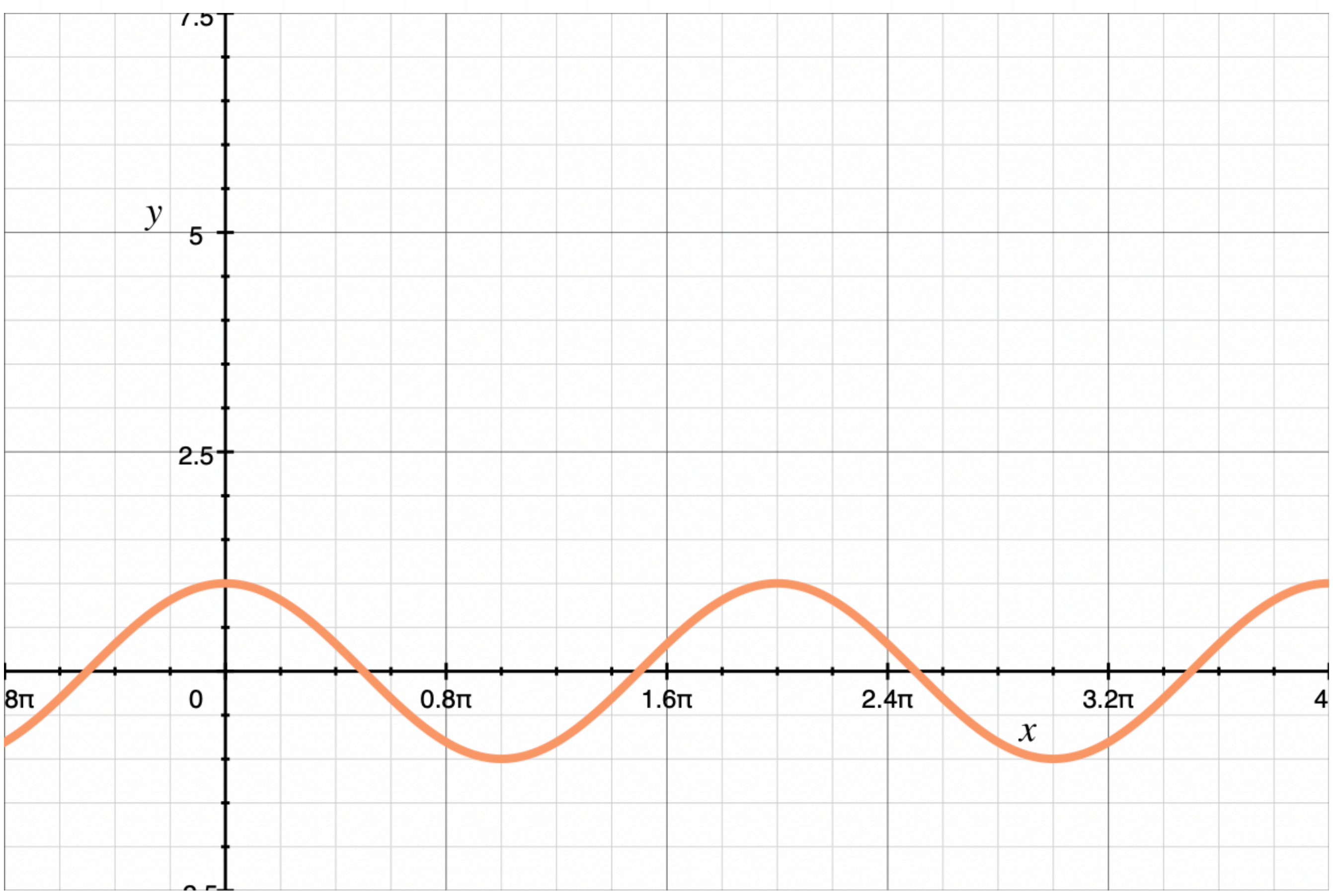
If d is positive, then the curve is shifted d units upward, which means we'll add d to each y-value. But if d is negative, then the curve is shifted d units downward, which means we'll subtract d from each y-value.

Vertical shifts apply to all six trigonometric functions in the same way, but let's work through an example with a cosine function.

Example

Sketch the graph of $y = \cos\theta + 4$.

We can start by sketching the graph of $y = \cos\theta$.



Then the graph of $y = \cos\theta + 4$ will be identical, but just shifted 4 units upward. We could also find points on $y = \cos\theta + 4$ algebraically by adding 4 to the y-values in coordinate points on $y = \cos\theta$.

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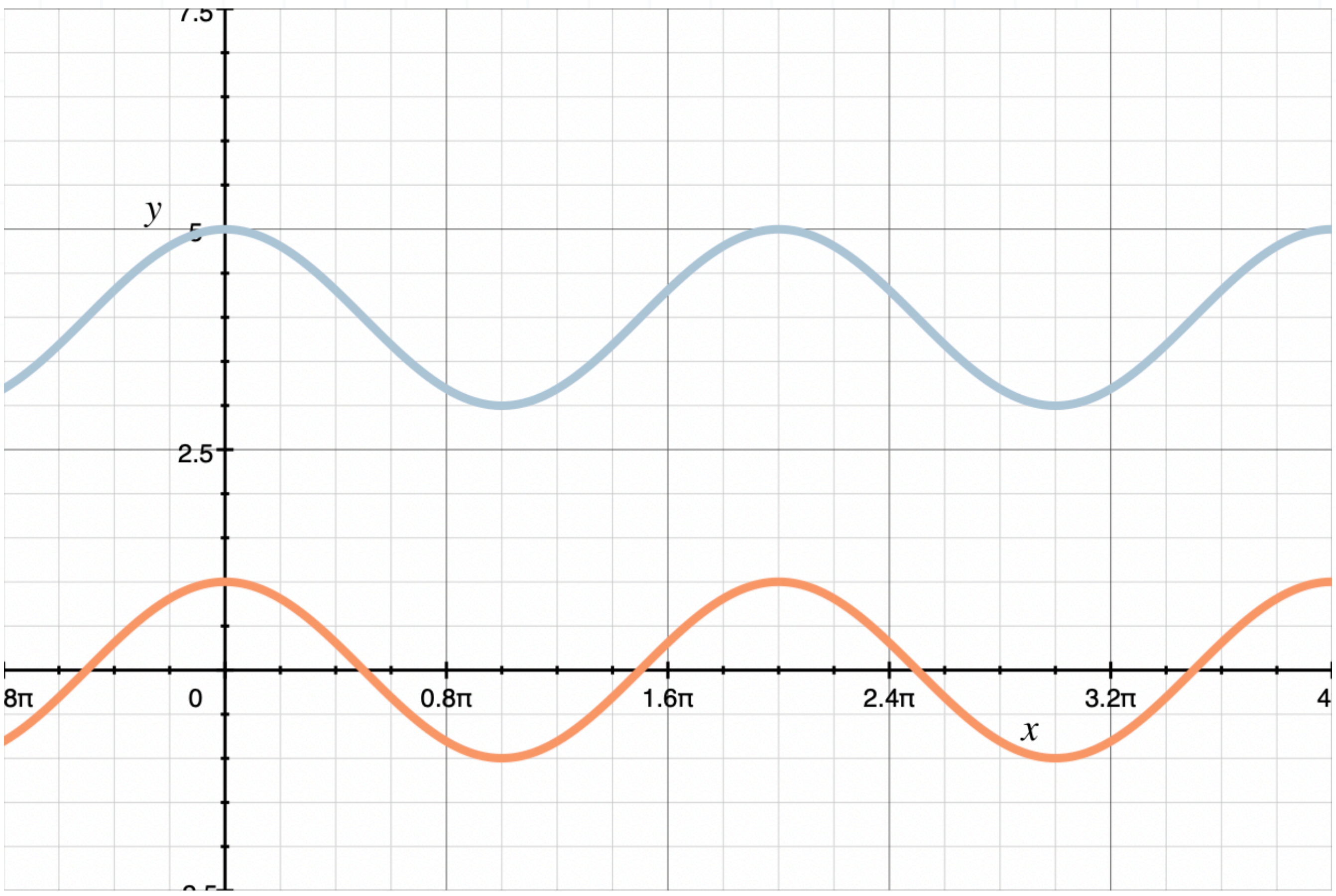
$y=\cos\theta$:	$(0,1)$	$\left(\frac{\pi}{2},0\right)$	$(\pi,-1)$	$\left(\frac{3\pi}{2},0\right)$	$(2\pi,1)$
$y=\cos\theta+4$:	$(0,5)$	$\left(\frac{\pi}{2},4\right)$	$(\pi,3)$	$\left(\frac{3\pi}{2},4\right)$	$(2\pi,5)$

If we keep the graph of $y=\cos\theta$ in red and add the graph of $y=\cos\theta+4$ in blue, we get



Let's do another cosine example where $d<1$.

Example

Sketch the graph of $y=\cos\theta+0.75$.

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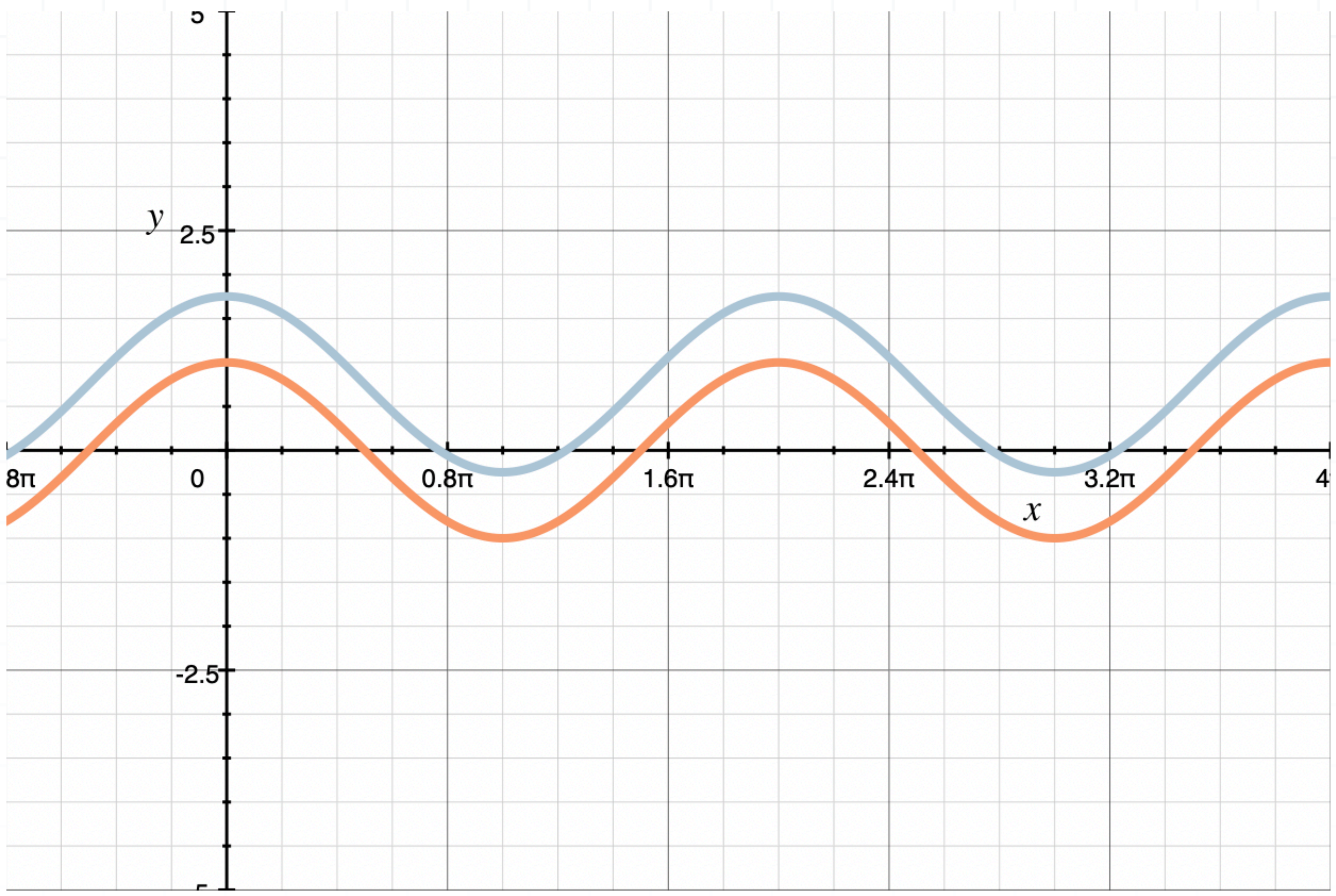
The addition of 0.75 to the value of $\cos\theta$ shifts the entire graph upward by 0.75 units. If we graph $y = \cos\theta$ in red, then we get the graph of $y = \cos\theta + 0.75$ in blue.

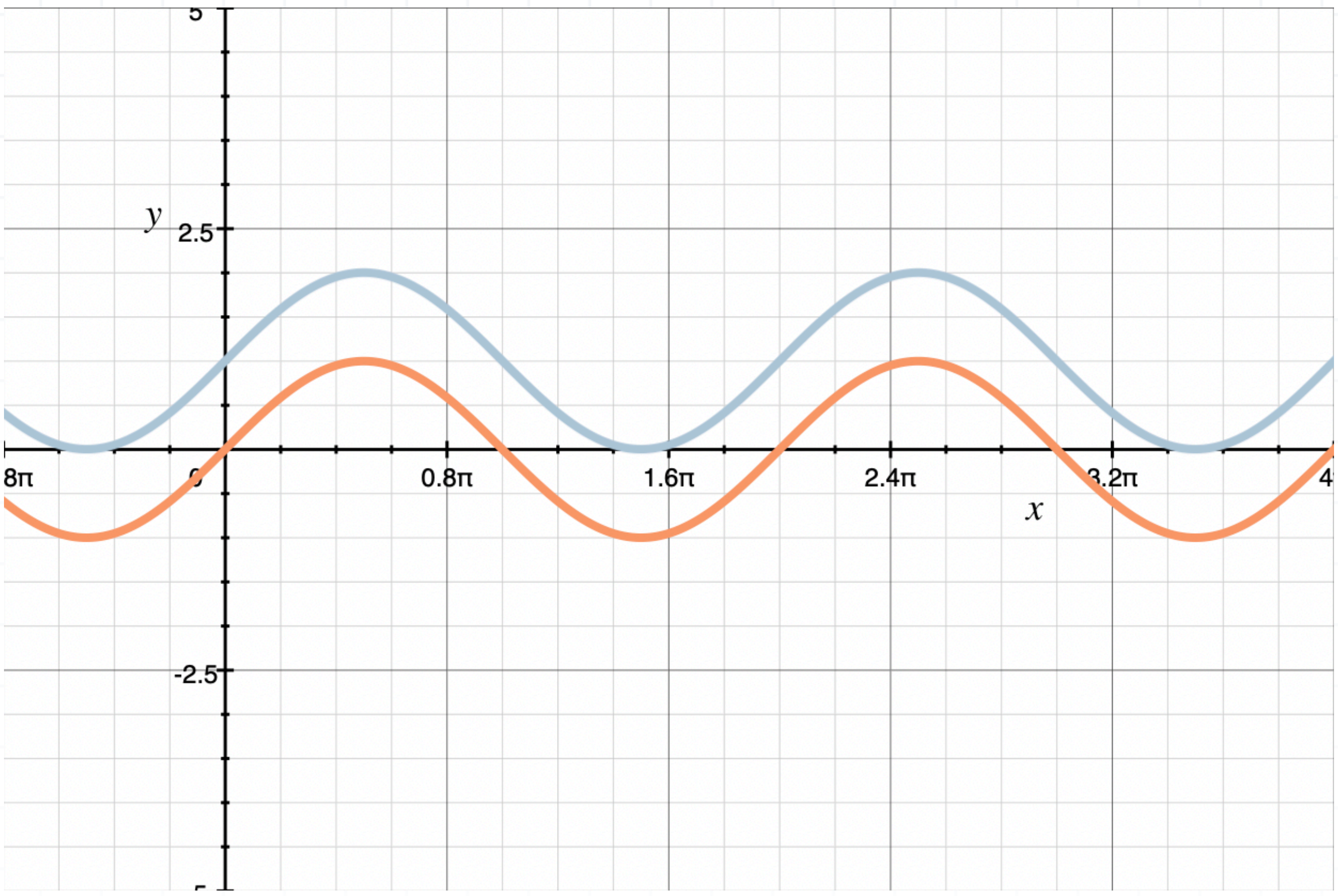


Let's look at vertical shifts with the sine function.

Example

Sketch the graph of $y = \sin\theta + 1$.

The addition of 1 shifts the graph of $y = \sin\theta$ upward by 1 unit. If we graph $y = \sin\theta$ in red, then we get the graph of $y = \sin\theta + 1$ in blue.



We'll do one last example of a sine function with a negative vertical shift.

Example

Sketch the graph of $y = \sin\theta - 3$.

By subtracting 3, we shift the graph of $y = \sin\theta$ downward by 3 units. If we graph $y = \sin\theta$ in red, then we get the graph of $y = \sin\theta - 3$ in blue.

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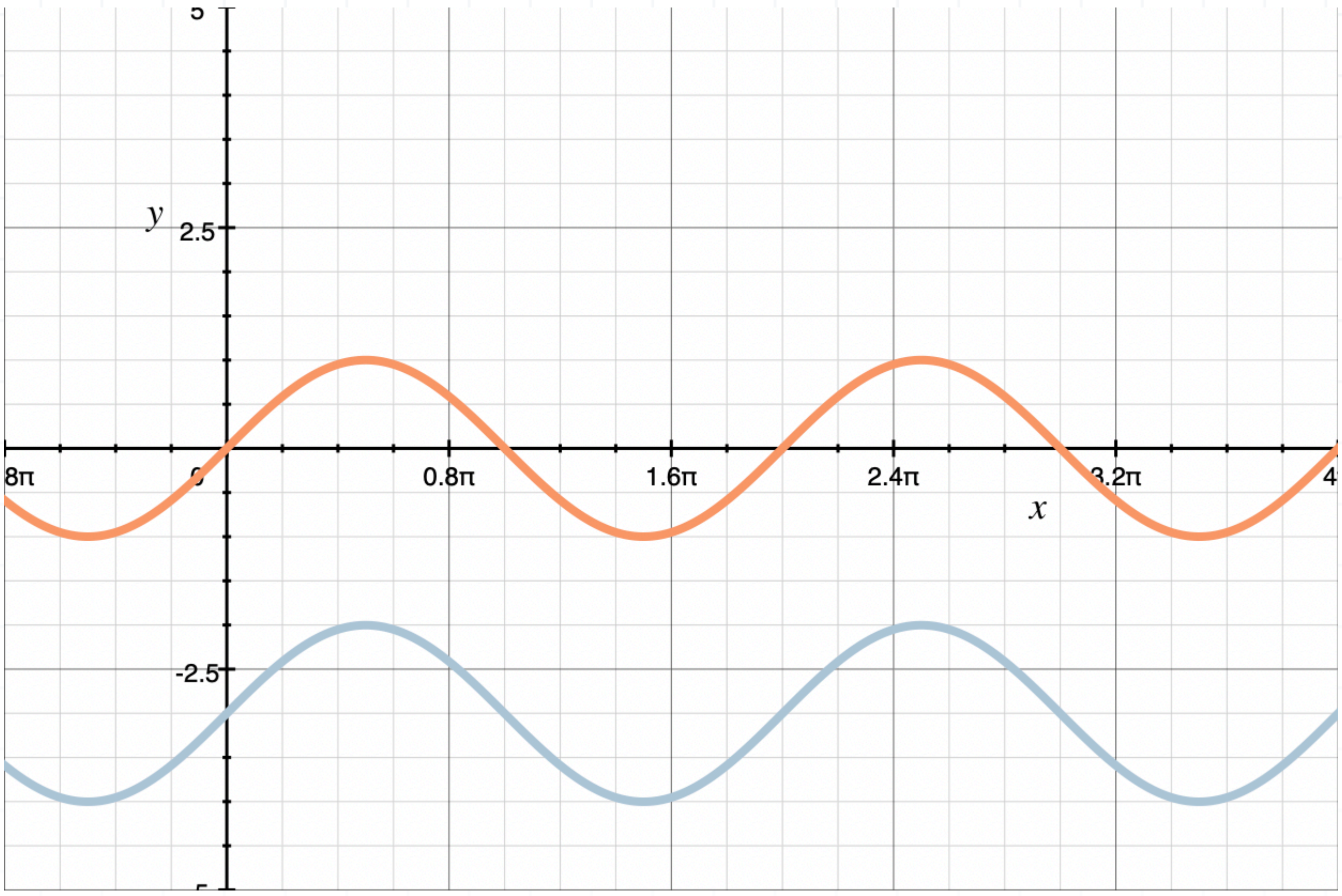

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Graphing transformations

The horizontal and vertical stretches, compressions, shifts, and reflections that we've learned about so far, which are determined by the values of a, b, c, and d in functions like

$$a\sin(b(\theta+c))+d$$

$$a\cos(b(\theta+c))+d$$

are all just different types of **transformations**. When we graph functions that include these kinds of transformations, we'll use a systematic process and start with the graph of the basic sine or cosine function, and then gradually work our way through each transformation.

Keep in mind that we always want to start at the "inside" and work our way toward the "outside." In functions given in the format above, we'll want to

1. **horizontally stretch** or compress the function by a factor of b
2. **horizontally shift** to the left or right by c
3. **reflect** over the y-axis if b is negative
4. **vertically stretch** or compress the function by a factor of a
5. **reflect** over the x-axis if a is negative
6. **vertically shift** the function upward or downward by d

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We've already been seeing some of these transformations in action, but let's work through some more examples where a, b, c, and d all come into play.

Example

What transformations are applied to transform $y = \cos\theta$ into the given function?

$$y = \cos\left(\frac{3\theta}{2}\right) - 4$$

We want to make the given cosine function match the exact format of $a\cos(b(\theta + c)) + d$, so we'll start by rewriting the argument of the cosine as

$$\frac{3\theta}{2} = \left(\frac{3}{2}\right)\theta = \frac{3}{2}(\theta - 0)$$

Then the given function can be rewritten as

$$y = 1\cos\left(\frac{3}{2}(\theta - 0)\right) - 4$$

and we have $a = 1$, $b = 3/2$, $c = 0$, and $d = -4$. Because $c = 0$, there's no horizontal shift to the left or right. Then because $b = 3/2$, the first transformation that brings about an actual change in the graph is a horizontal compression by a factor of $b = 3/2$. Therefore, we can shift points along $y = \cos\theta$ by dividing the x-values by $b = 3/2$ while keeping the y-values the same.

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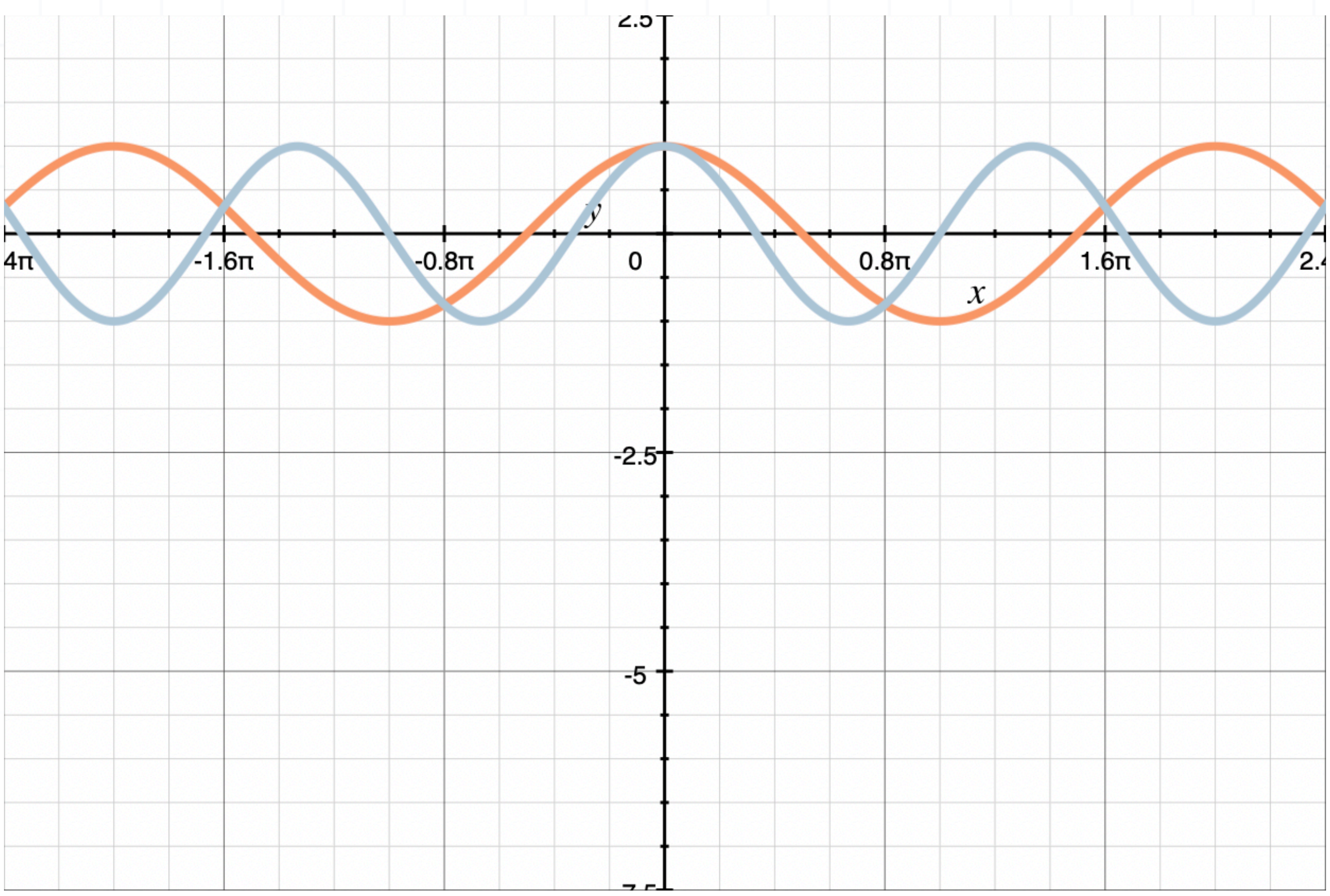
$$y = \cos\theta: \qquad (0,1) \qquad \left(\frac{\pi}{2},0\right) \qquad (\pi,-1) \qquad \left(\frac{3\pi}{2},0\right) \qquad (2\pi,1)$$

$$y = \cos\left(\frac{3}{2}\theta\right): \qquad (0,1) \qquad \left(\frac{\pi}{3},0\right) \qquad \left(\frac{2\pi}{3},-1\right) \qquad (\pi,0) \qquad \left(\frac{4\pi}{3},1\right)$$

Then we'll graph $y = \cos\theta$ in red and $y = \cos((3/2)\theta)$ in blue:



Since b is not negative, there won't be a reflection over the y-axis, since $a = 1$ there won't be any vertical stretch or compression, and since a is not negative there won't be a reflection over the x-axis.

But then since $d = -4$, the next transformation is a vertical shift downward by 4. If we sketch the graph of $y = \cos((3/2)\theta)$ in blue and $y = \cos((3/2)\theta) - 4$ in dark blue, we get

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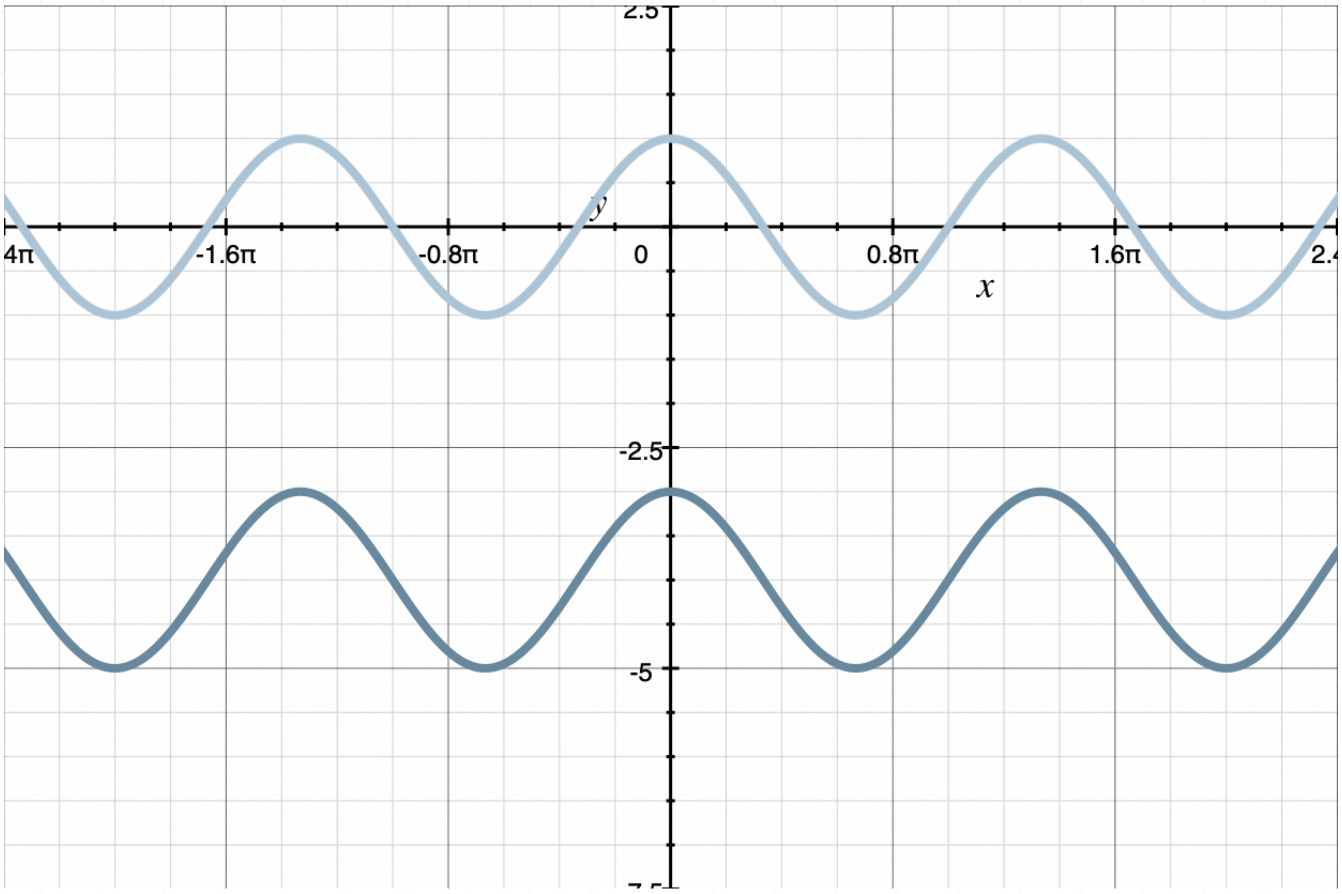
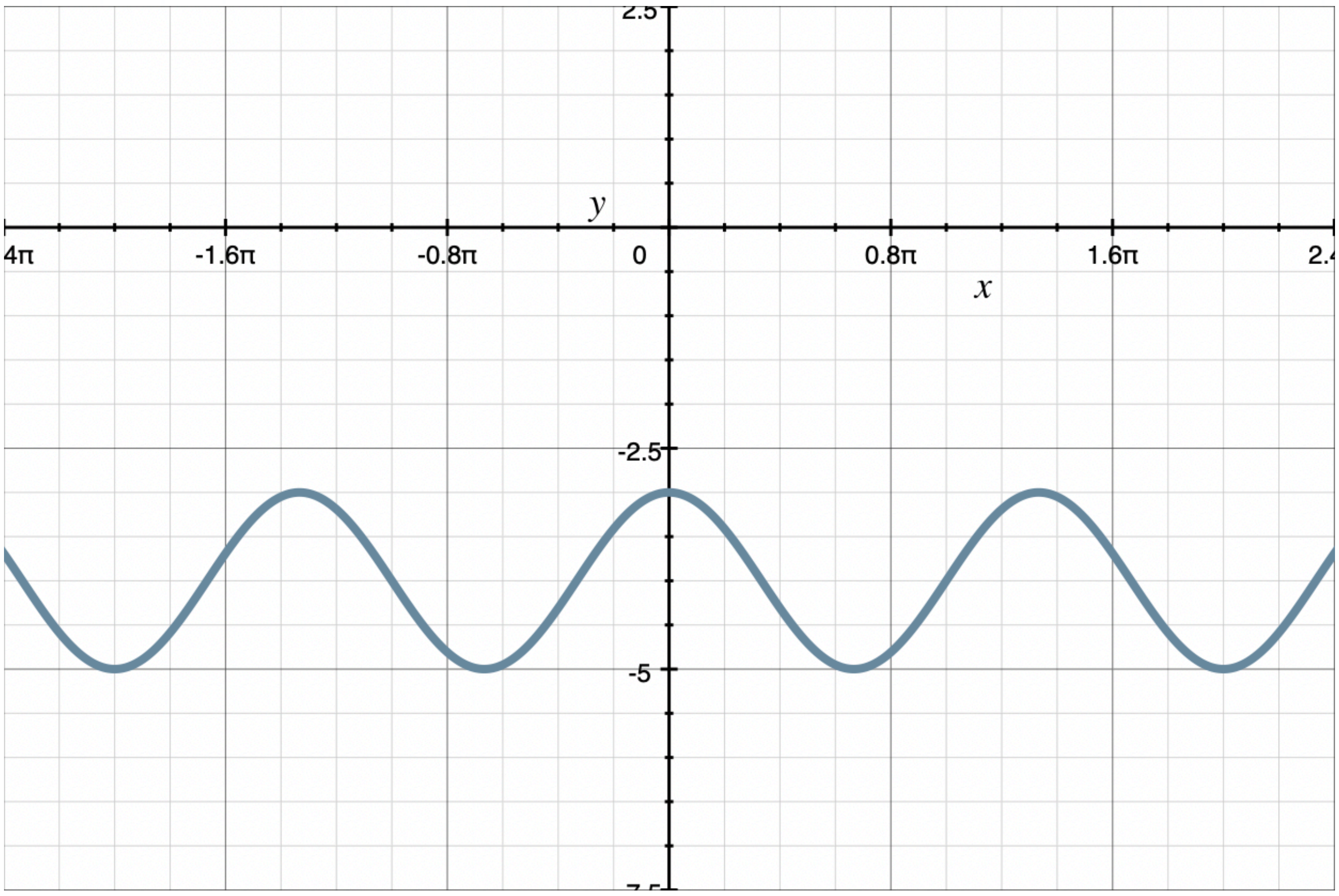


Taking the previous graph away, we get the final graph of $y = \cos((3/2)\theta) - 4$.



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The last example only had two transformations being applied to the cosine function, so let's do an example where we use three transformations on the sine function.

Example

What transformations are applied to transform $y = \sin\theta$ into the given function?

$$y = 1.7\sin\left(2\left(\theta - \frac{\pi}{6}\right)\right)$$

The given sine function is already written as $a\sin(b(\theta + c)) + d$, so we can see right away that we have $a = 1.7$, $b = 2$, $c = -\pi/6$, and $d = 0$.

Since $b = 2$, the function gets horizontally compressed by a factor of 2, which means all the x-values in the coordinate points get halved, while the y-values stay the same. If we graph $y = \sin\theta$ in red and $y = \sin(2\theta)$ in blue, we get

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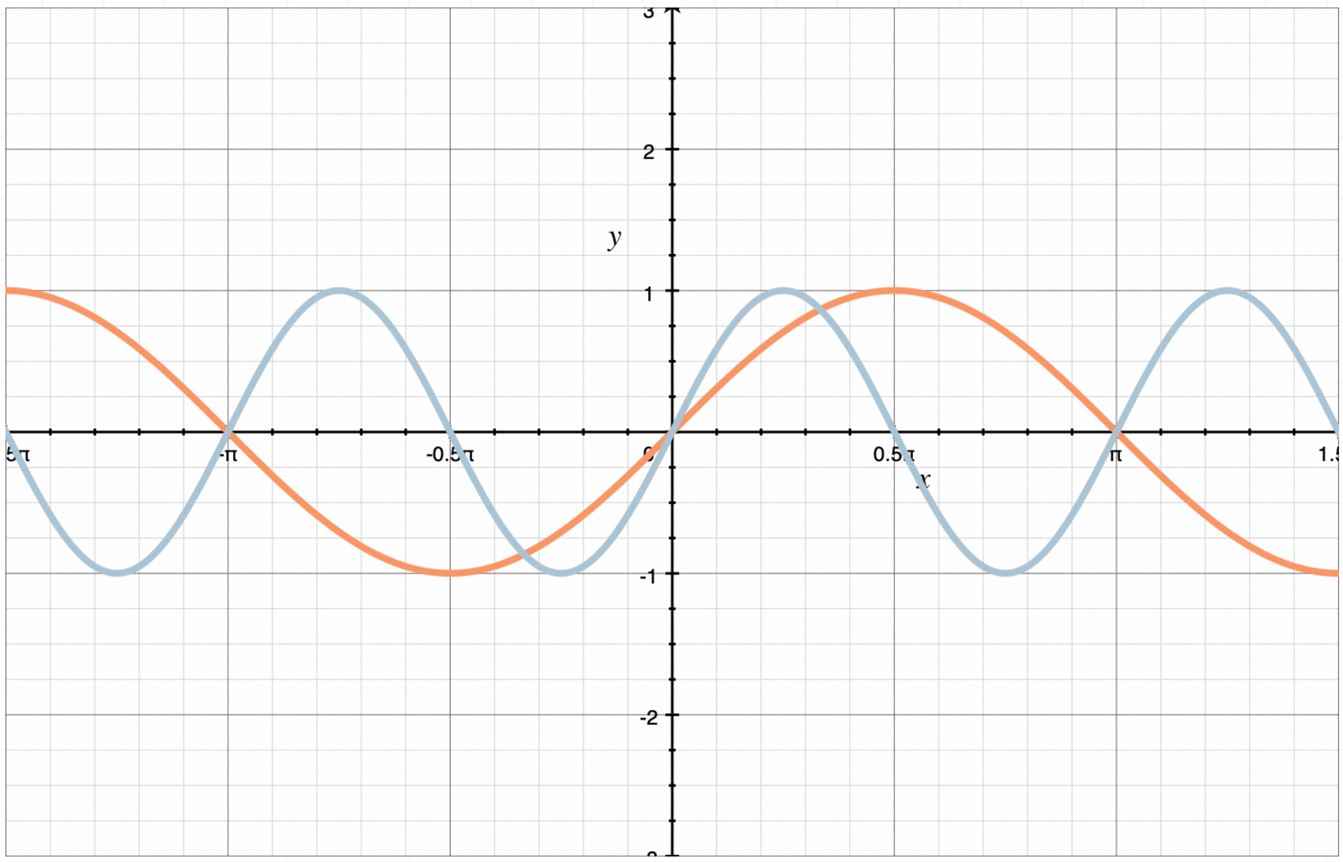

Then we apply the horizontal shift given by $c = -\pi/6$. Compared with the graph of $y = \sin(2\theta)$, the graph of $y = \sin(2(\theta - (\pi/6)))$ will be shifted $\pi/6$ units to the right, which means we'll add $\pi/6$ to the x-value of coordinate points on $y = \sin\theta$, while keeping the y-values the same.

If we graph $y = \sin(2\theta)$ in blue and $y = \sin(2(\theta - (\pi/6)))$ in dark blue, we get

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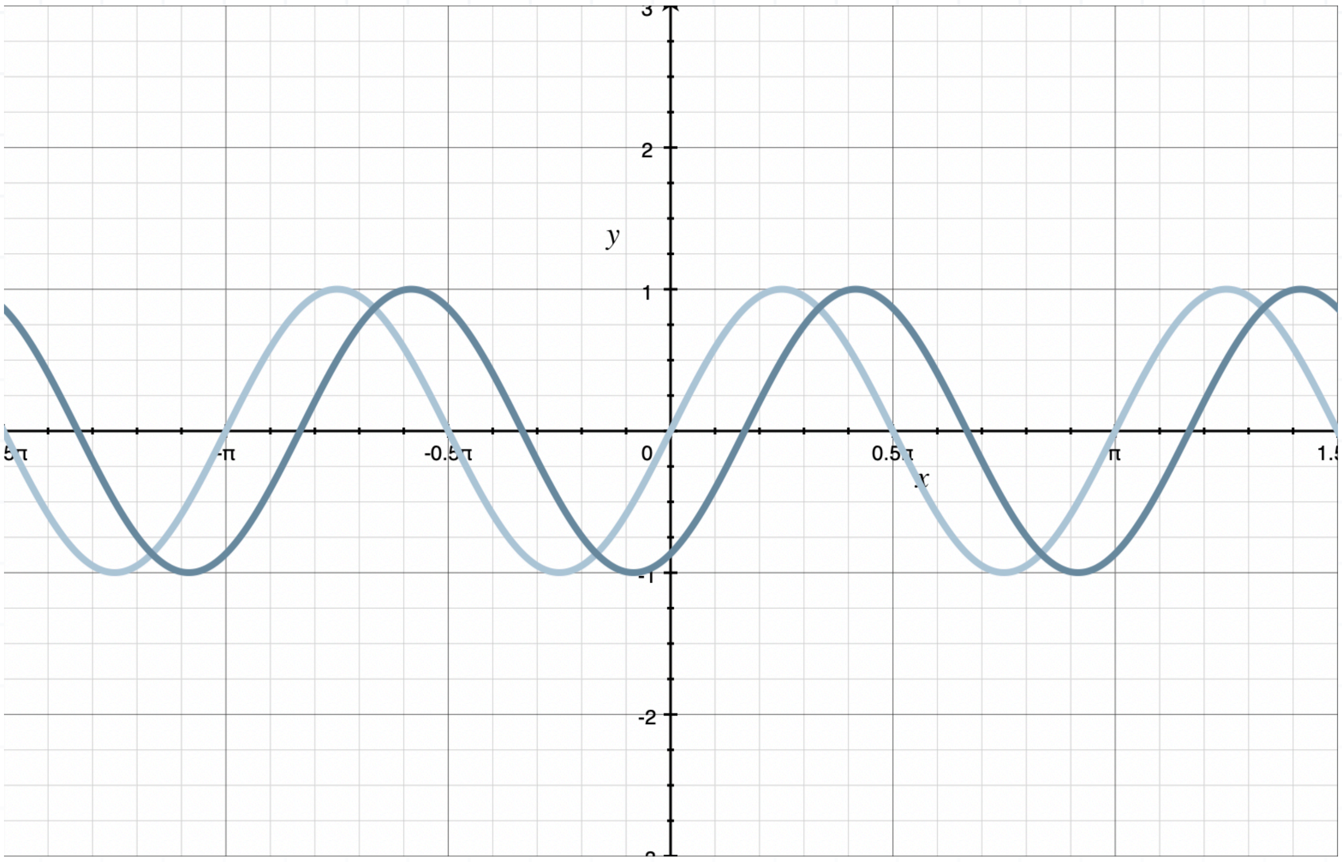
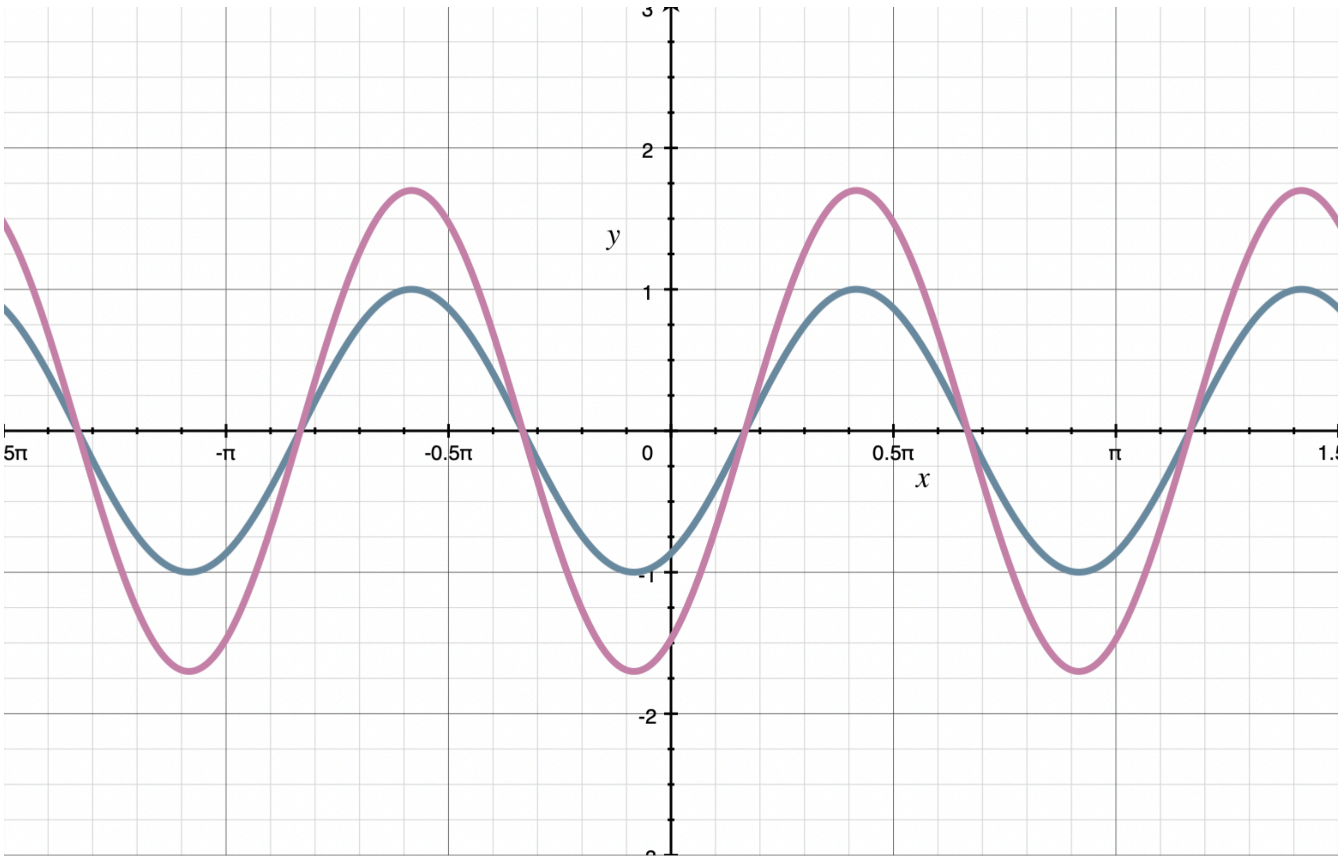
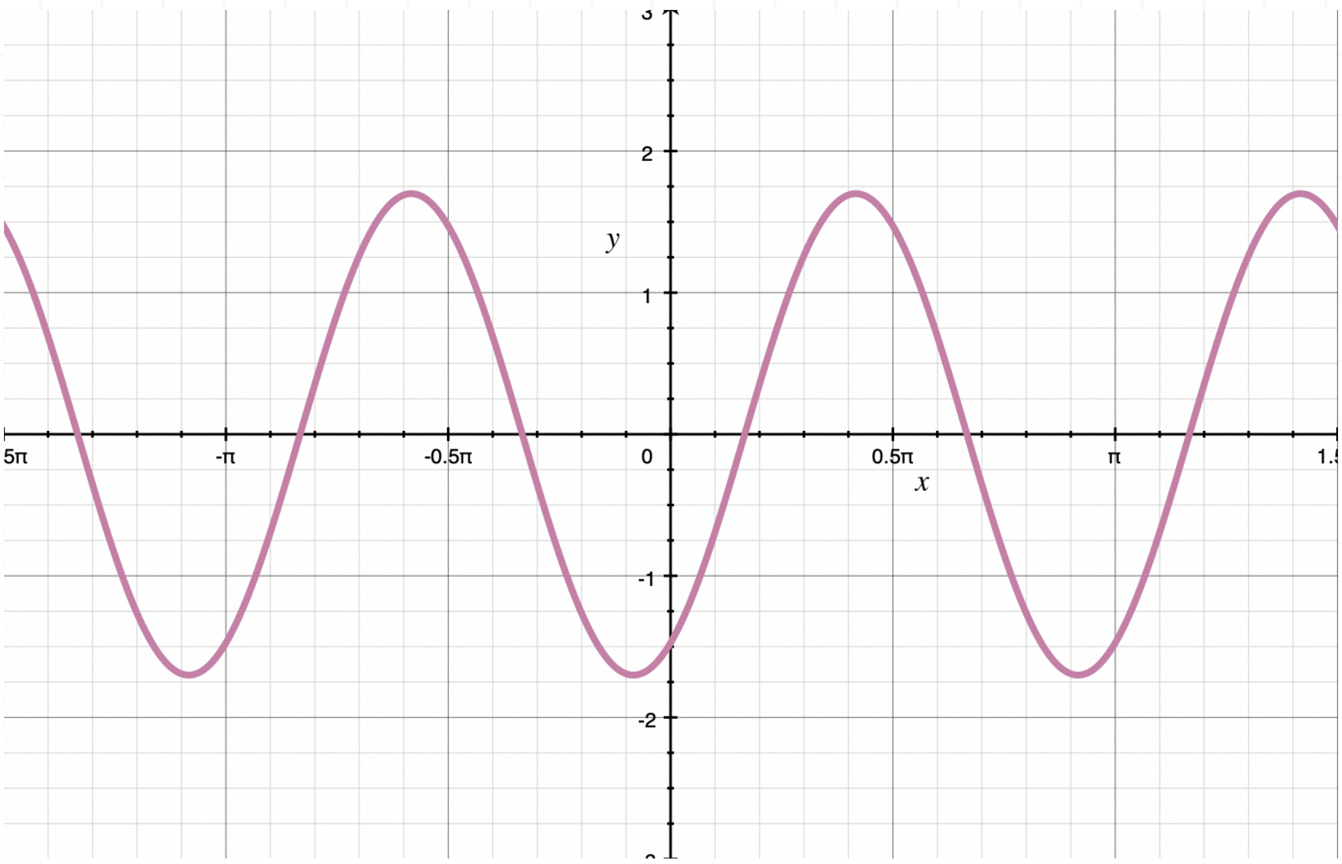


Since $a = 1.7$ and $d = 0$, the final transformation is a vertical stretch by a factor of 1.7, which means all the y-values in the coordinate points get multiplied by 1.7. If we sketch $y = \sin(2(\theta - (\pi/6)))$ in dark blue and $y = 1.7\sin(2(\theta - (\pi/6)))$ in purple, we get



Taking the previous graph away, we get the final graph of $y = 1.7\sin(2(\theta - (\pi/6)))$.



Let's do one more example with a cosine function to which five transformation types are applied.

Example

What transformations are applied to transform $y = \cos\theta$ into the given function?

$$y = -1.5\cos\left(\frac{3\theta}{4} + \frac{\pi}{4}\right) + 2$$

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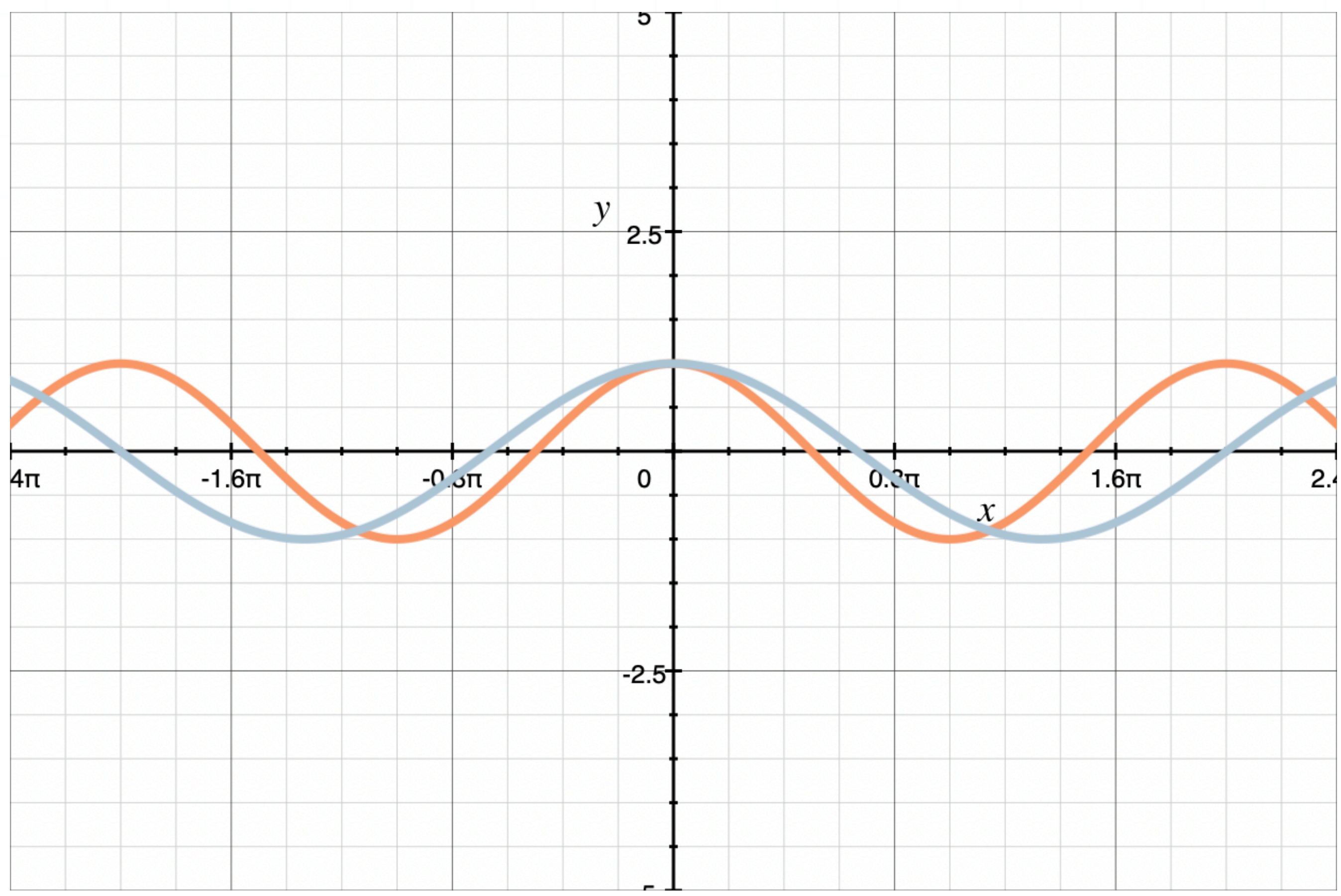
To put the given cosine function into the form $a\cos(b(\theta + c)) + d$, we need to factor $3/4$ out of the argument of the cosine function.

$$y = -1.5\cos\left(\frac{3}{4}\left(\theta + \frac{\pi}{3}\right)\right) + 2$$

With the function now in the form $a\cos(b(\theta + c)) + d$, we can see that we have $a = -1.5$, $b = 3/4$, $c = \pi/3$, and $d = 2$.

Since $b = 3/4$, the function gets horizontally stretched by a factor of $3/4$, which means all the x-values in the coordinate points get divided by $3/4$, while the y-values stay the same. If we graph $y = \cos\theta$ in red and $y = \cos((3/4)(\theta))$ in blue, we get



The horizontal shift given by $c = \pi/3$. Compared with the graph of $y = \cos((3/4)(\theta))$, the graph of $y = \cos((3/4)(\theta + (\pi/3)))$ will be shifted $\pi/3$ units to the left, which means we'll subtract $\pi/3$ from the x-value of coordinate points on $y = \cos((3/4)(\theta))$, while keeping the y-values the same.

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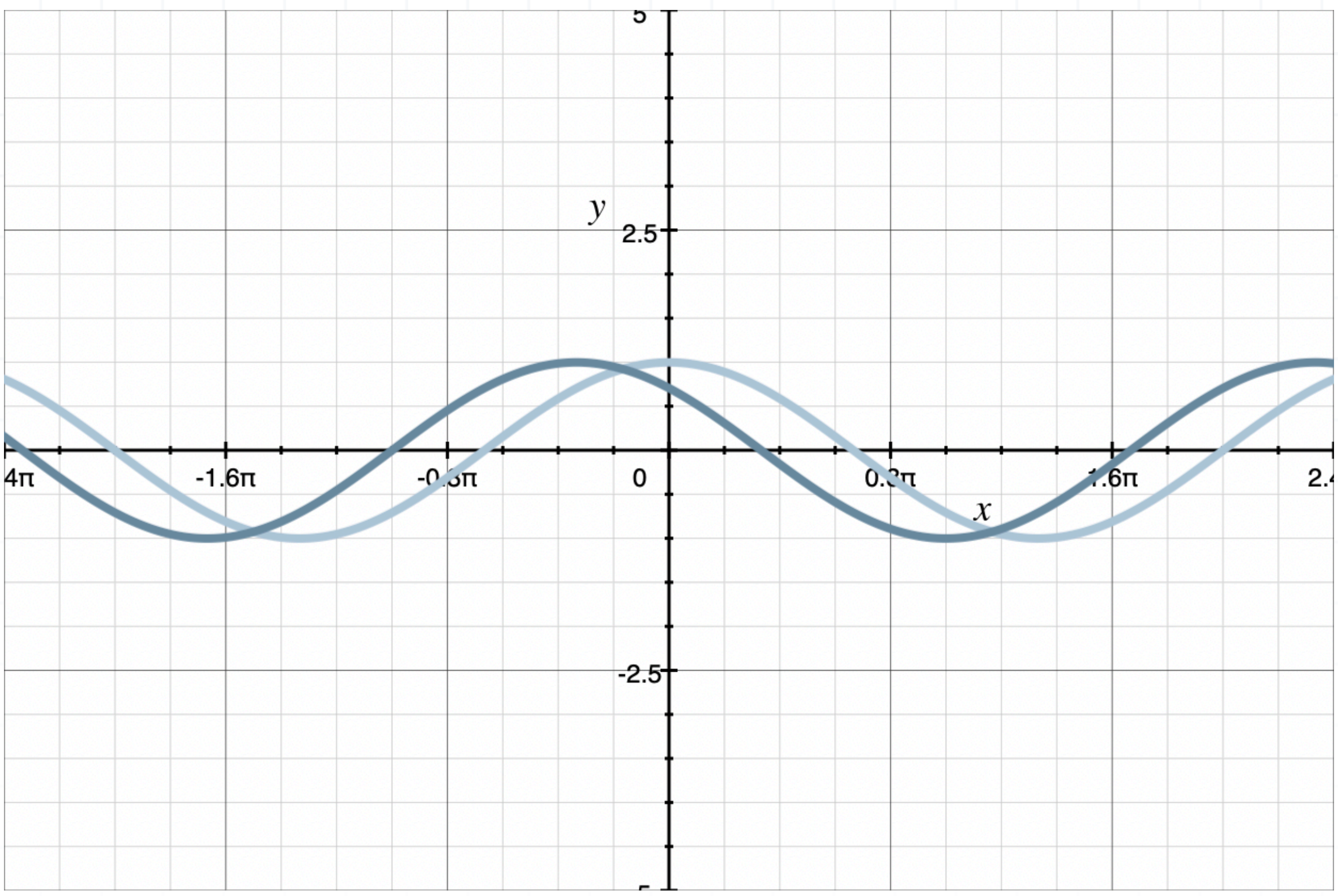
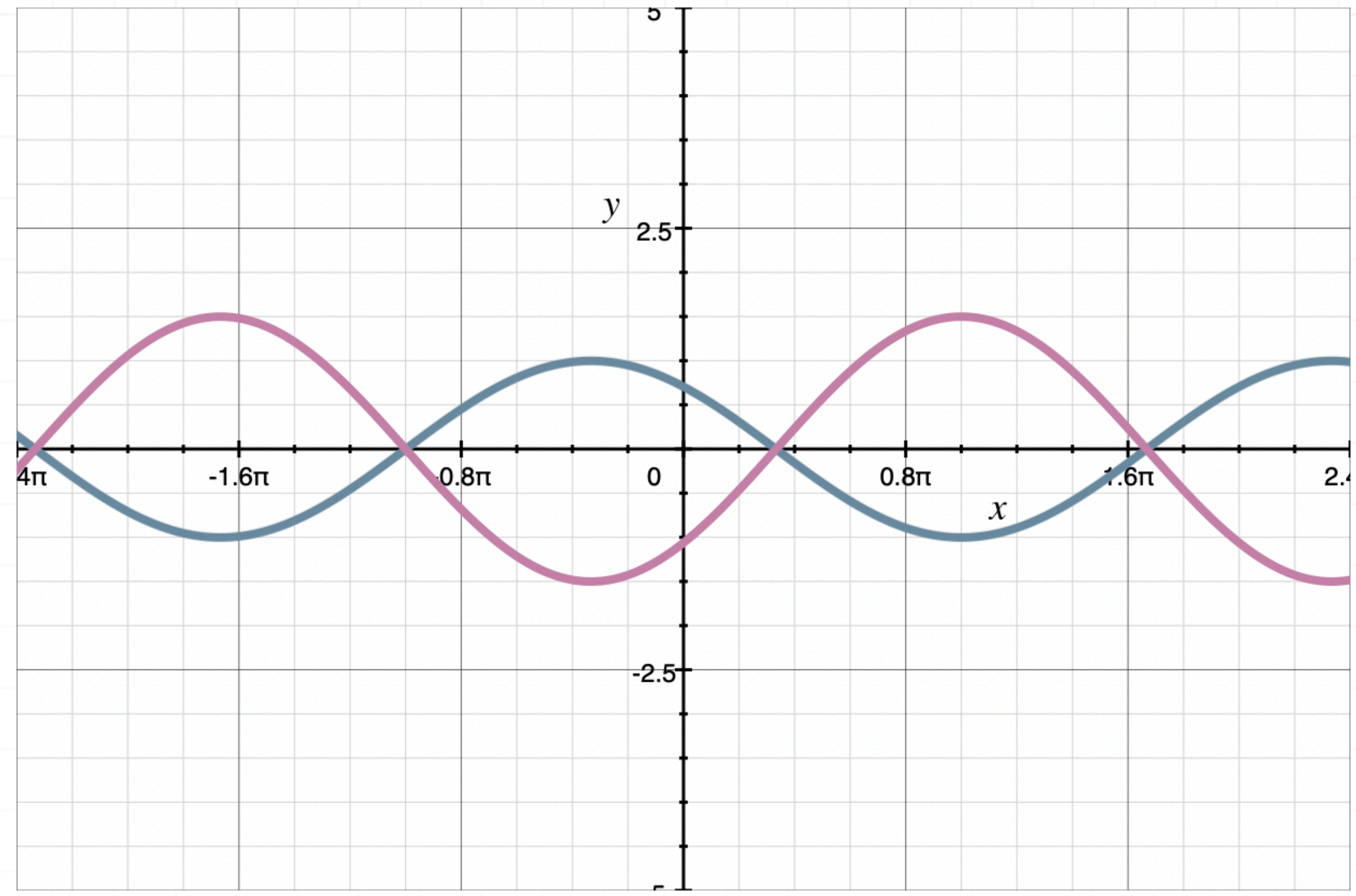
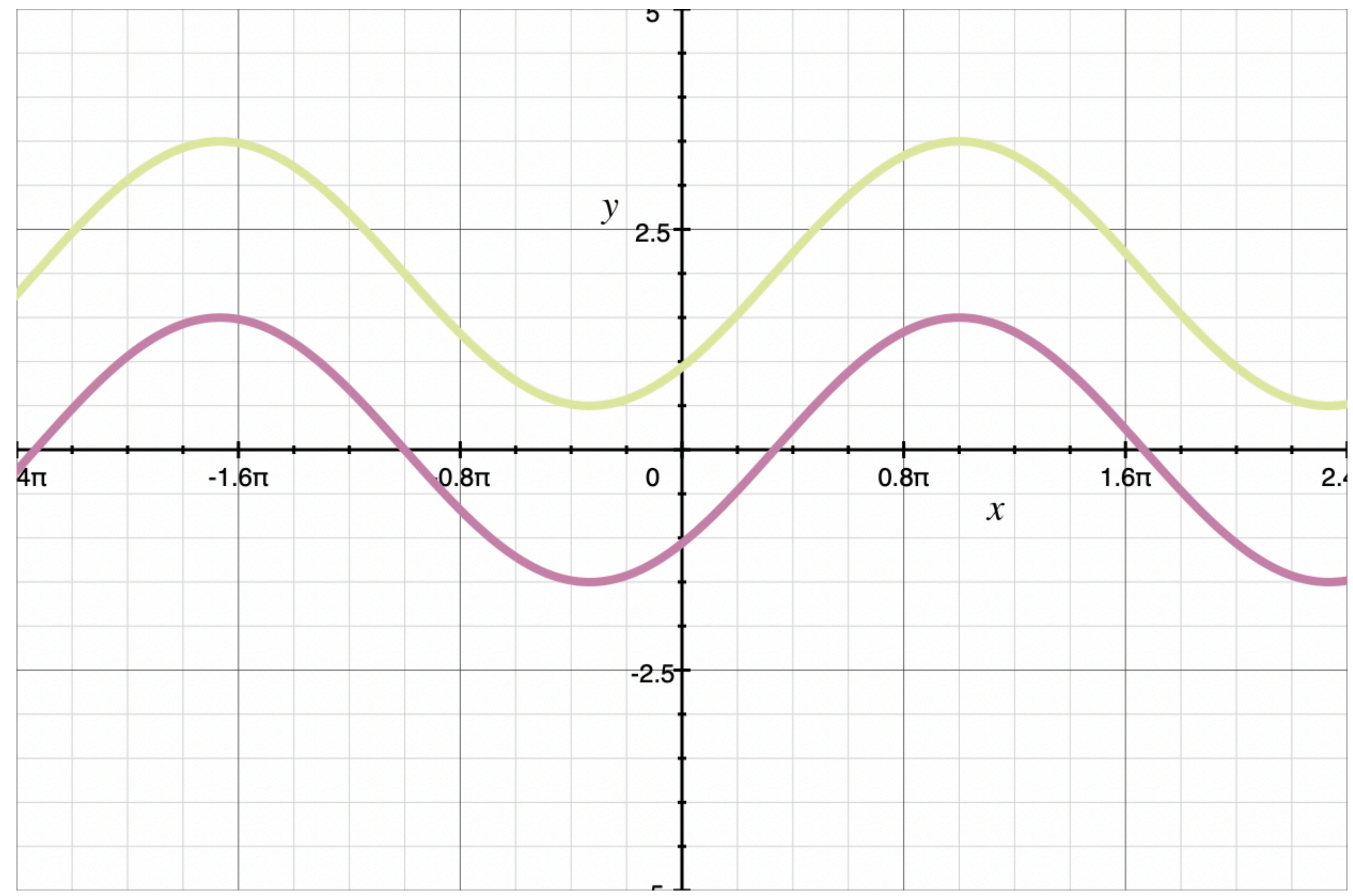
If we graph $y = \cos((3/4)(\theta))$ in blue and $y = \cos((3/4)(\theta + (\pi/3)))$ in dark blue, we get



Since $a = -1.5$, the function gets vertically stretched by a factor of 1.5, which means the y-value of every point gets multiplied by 1.5, but then because a is negative, the graph gets reflected over the x-axis. If we graph $y = \cos((3/4)(\theta + (\pi/3)))$ in dark blue and $y = -1.5\cos((3/4)(\theta + (\pi/3)))$ in purple, we get



Finally, since $d = 2$, there's a vertical shift upwards by 2, which means we'll add 2 to the y-value in each coordinate point, while keeping the x-values the same. If we graph $y = -1.5\cos((3/4)(\theta + (\pi/3)))$ in purple and $y = -1.5\cos((3/4)(\theta + (\pi/3))) + 2$ in green, we get



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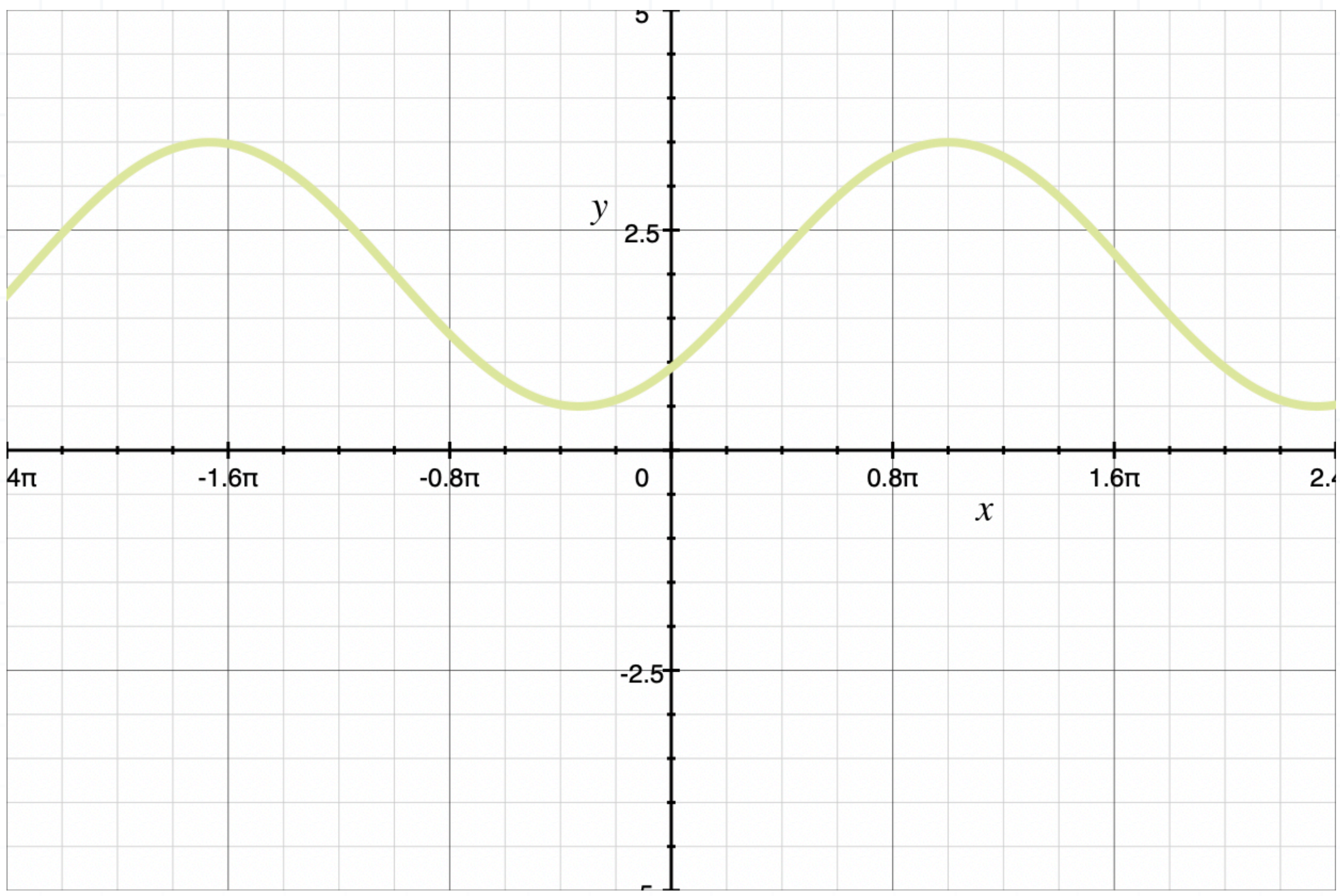
Taking the previous graph away, we get the final graph of
$y = -1.5\cos((3/4)(\theta + (\pi/3))) + 2$.



Let's do one more example.

Example

What transformations are applied to transform $y = \sin\theta$ into the given function?

$$y = 2\sin\left(-2\theta + \frac{2\pi}{3}\right)$$

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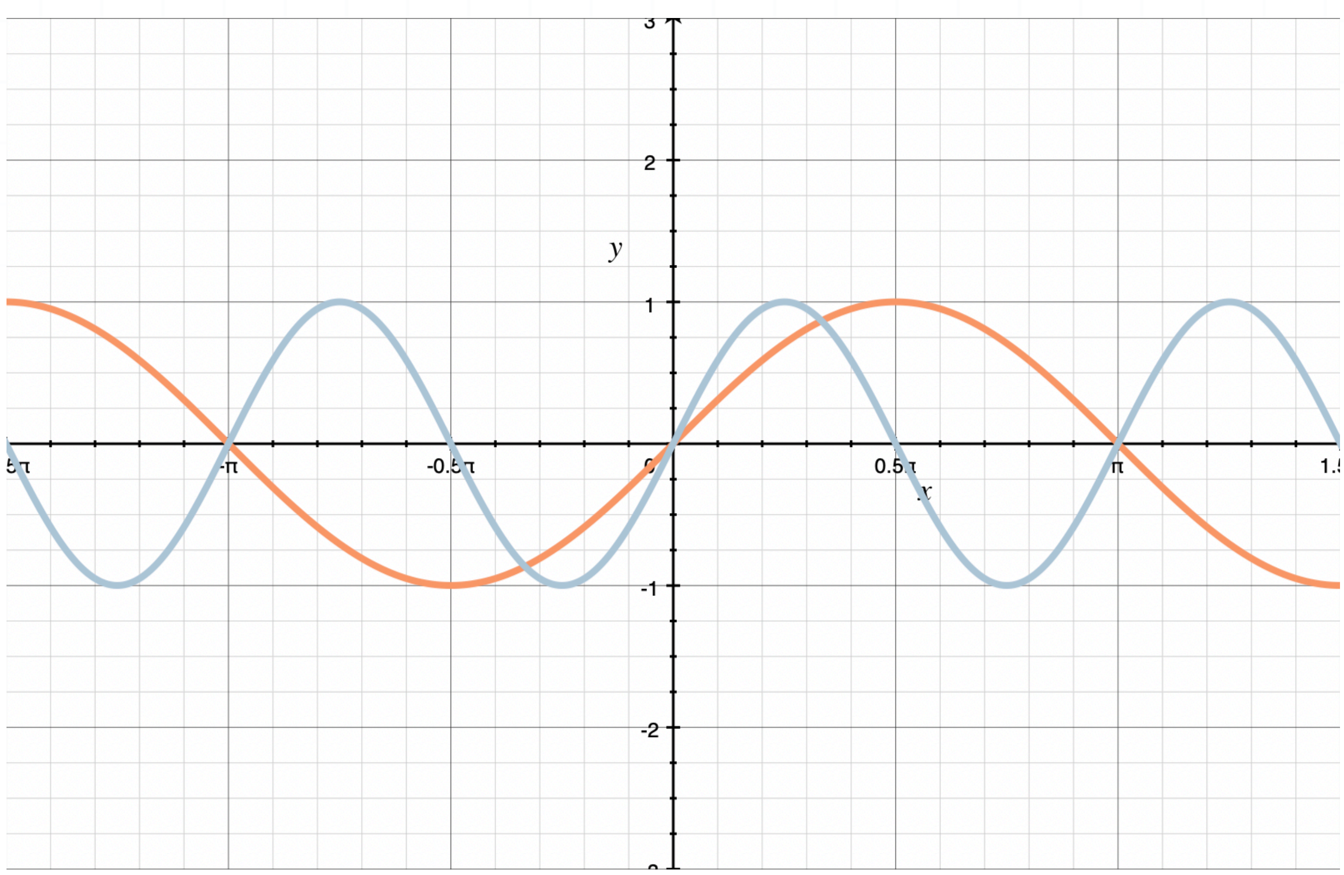
To put the given sine function into the form $a\sin(b(\theta+c))+d$, we need to factor -2 out of the argument of the sine function.

$$y = 2\sin\left(-2\left(\theta - \frac{\pi}{3}\right)\right)$$

With the function now in the form $a\sin(b(\theta+c))+d$, we can see that we have $a=2$, $b=-2$, $c=-\pi/3$, and $d=0$.

Since $b=-2$, the function gets horizontally compressed by a factor of 2, which means all the x-values in the coordinate points get divided by 2, while the y-values stay the same. If we graph $y=\sin\theta$ in red and $y=\sin(2\theta)$ in blue, we get



But then because b is negative, the graph gets reflected over the y-axis. If we graph $y=\sin(2\theta)$ in blue and $y=\sin(-2\theta)$ in dark blue, we get

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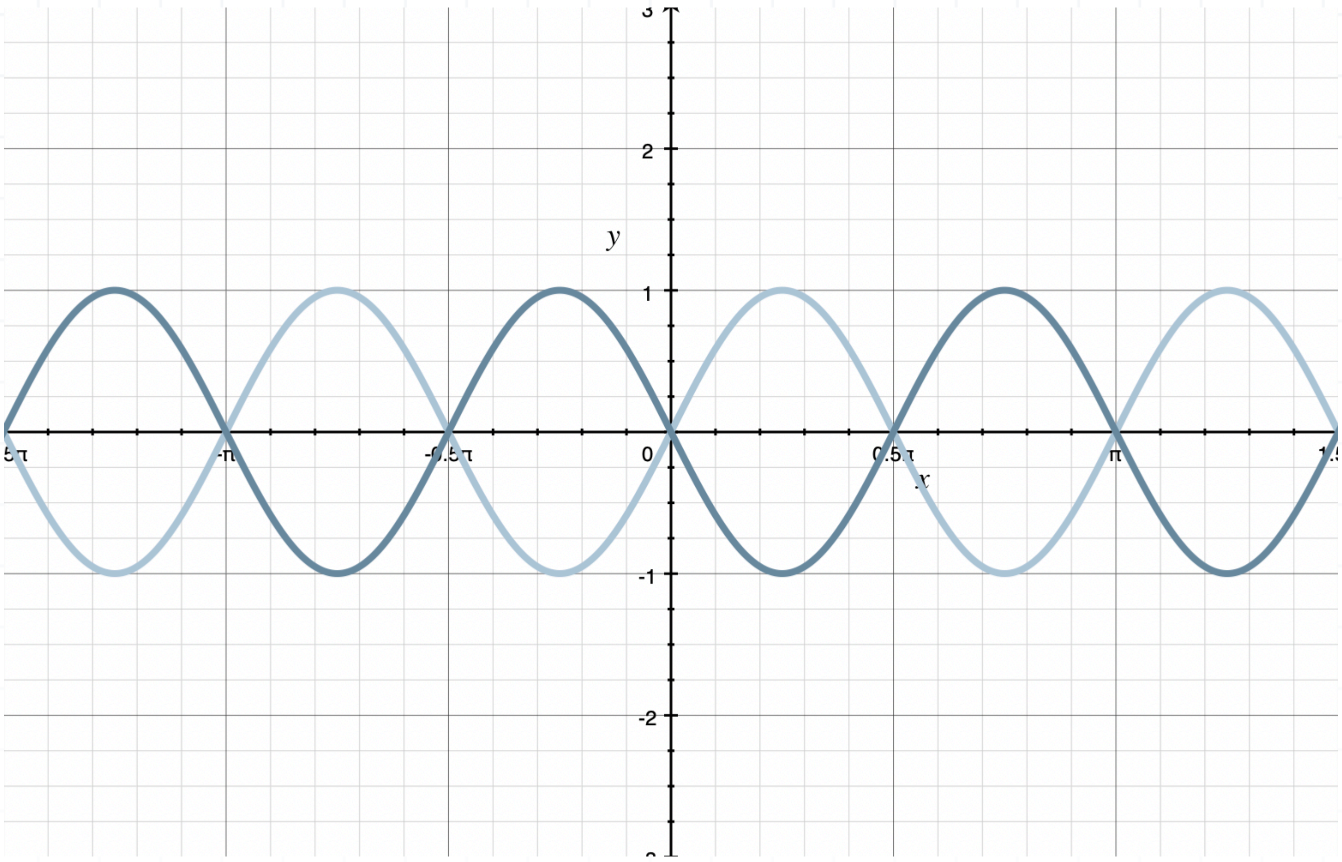

The horizontal shift is given by $c = -\pi/3$. Compared with the graph of $y = \sin(-2\theta)$, the graph of $y = \sin(-2(\theta - \pi/3))$ will be shifted $\pi/3$ units to the right, which means we'll add $\pi/3$ to the x-values of coordinate points on $y = \sin(-2\theta)$, while keeping the y-values the same.

If we graph $y = \sin(-2\theta)$ in dark blue and $y = \sin(-2(\theta - \pi/3))$ in purple, we get

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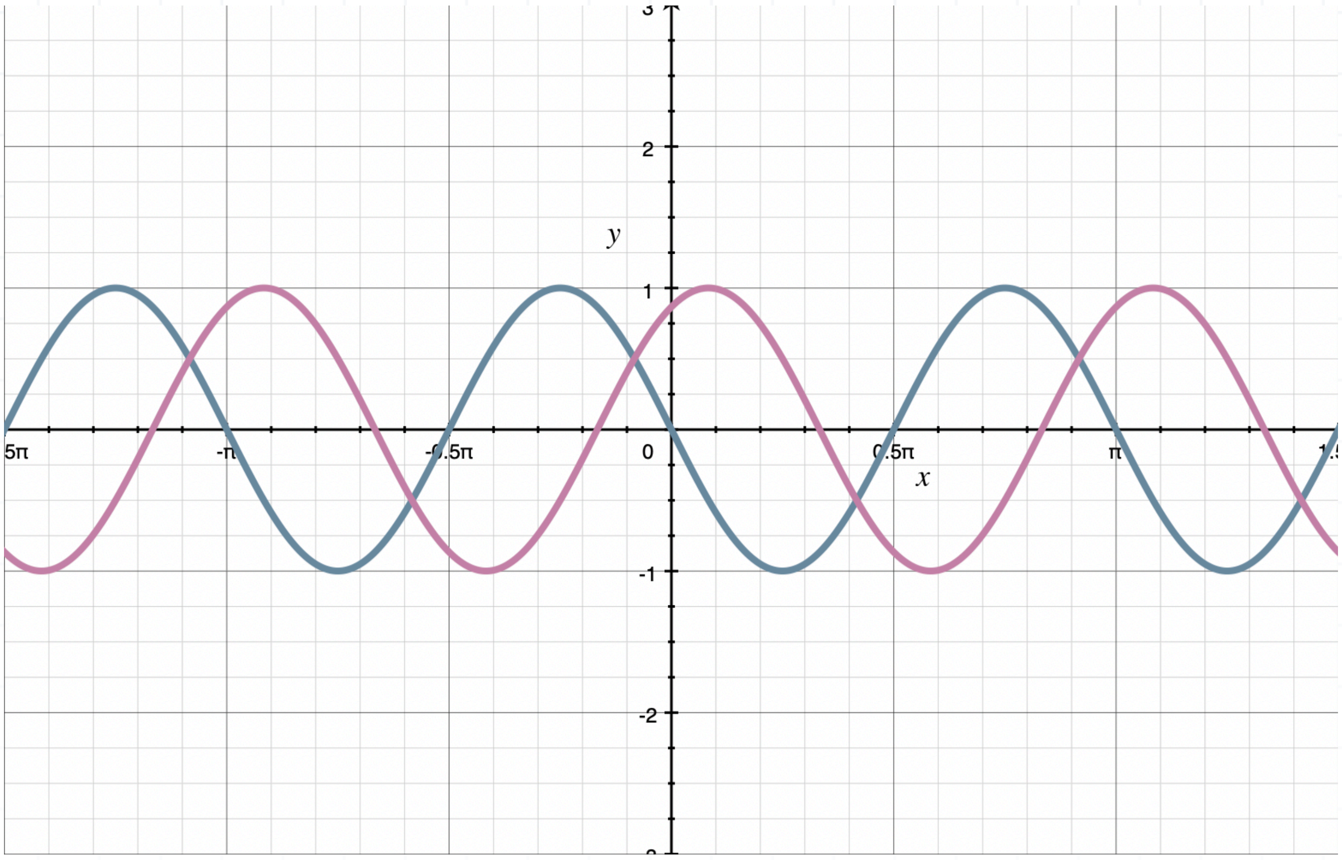
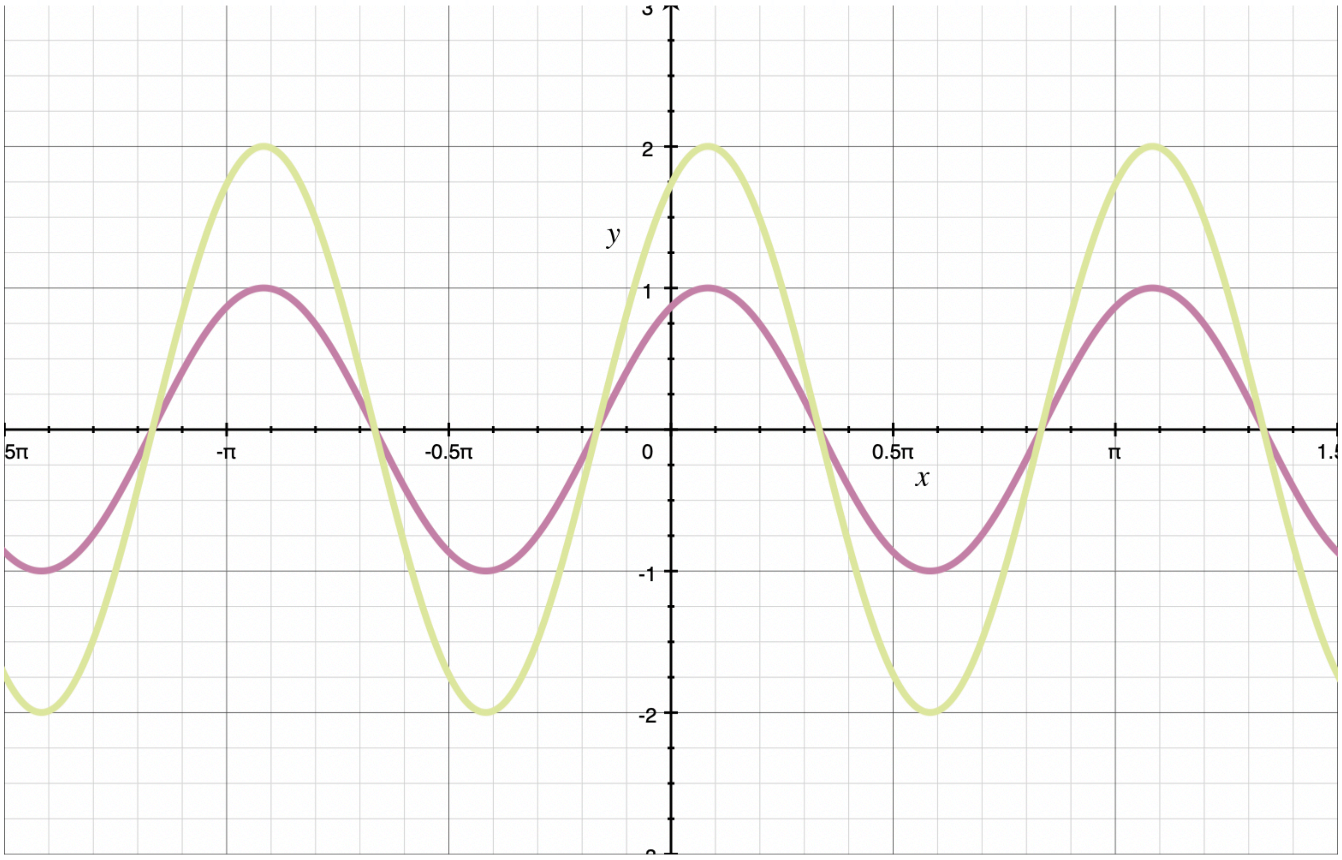
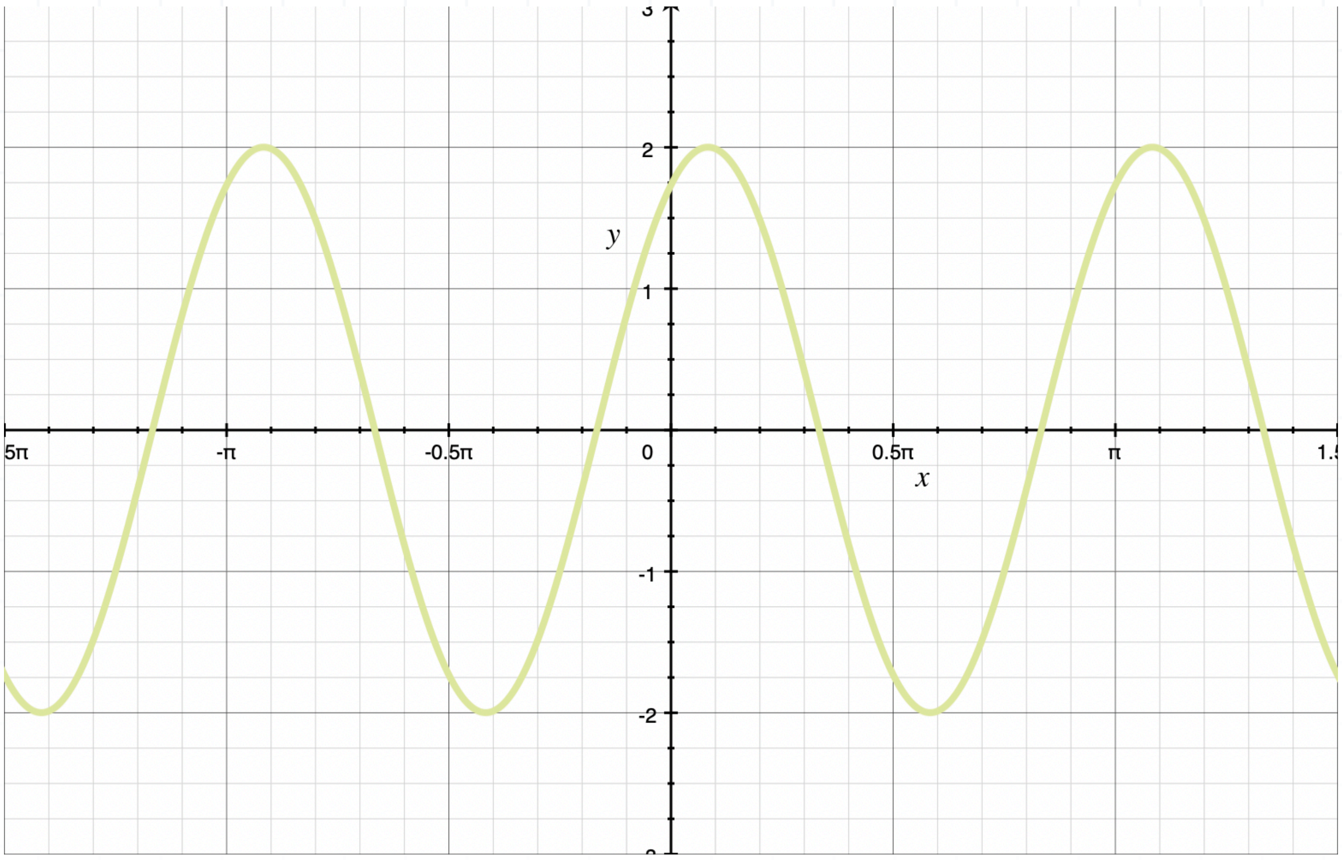


Since $a = 2$, the next transformation is a vertical stretch by a factor of 2, which means the y-value of every point gets multiplied by 2. If we graph $y = \sin(-2(\theta - \pi/3))$ in purple and $y = 2\sin(-2(\theta - \pi/3))$ in green, we get



Taking the previous graph away, we get the final graph of
$y = 2\sin(-2(\theta - \pi/3))$.



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Graphing combinations

Now that we've seen how to graph all six trig functions, including how to deal with the transformations caused by a, b, c, and d, we want to turn toward combinations of these functions.

A **combination** of two functions is the result of adding, subtracting, multiplying, or dividing those functions. What we want to see is how to graph these combinations when we're given the pair of original functions.

Usually, it'll be really helpful to graph each of the original functions individually, and then use those two graphs together to figure out what the graph of the combination will look like.

Let's do an example where the combination is the sum of sine and cosine functions.

Example

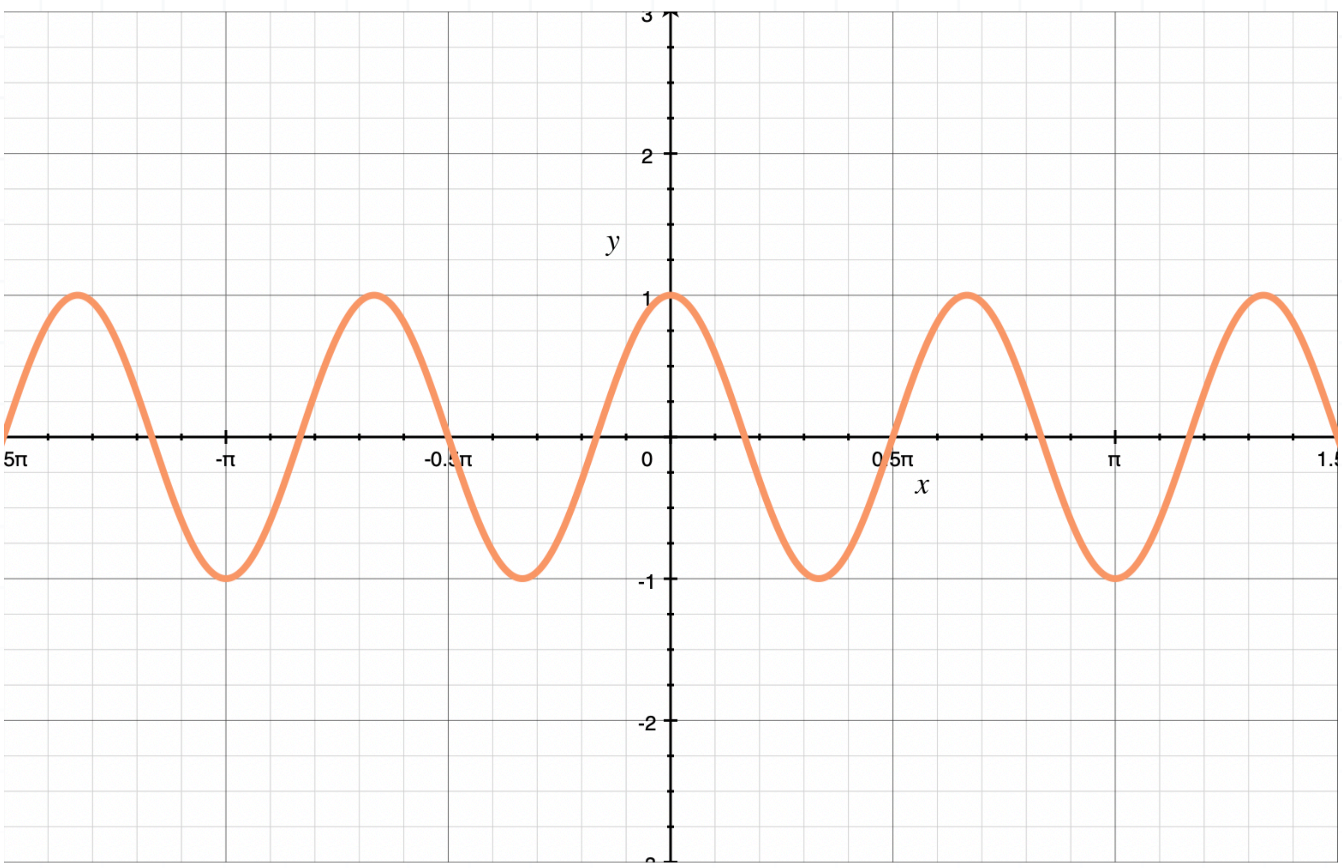
Graph the combination function $\cos(3\theta) + \sin(\theta - \pi)$.

We want to recognize that $\cos(3\theta) + \sin(\theta - \pi)$ is really just the combination of $\cos(3\theta)$ and $\sin(\theta - \pi)$. We've added the two functions together to get the combination. So let's look at each of these two functions individually.

If we write $\cos(3\theta)$ in the form $a\cos(b(\theta + c)) + d$, we'll get

$$\cos(3\theta) = 1\cos(3(\theta + 0)) + 0$$

Which means that $a = 1$ (amplitude is 1), $b = 3$ (period is $2\pi/3$), $c = 0$ (there's no horizontal shift), and $d = 0$ (there's no vertical shift). And therefore the graph of $\cos(3\theta)$ is



If we write $\sin(\theta - \pi)$ in the form $a\sin(b(\theta + c)) + d$, we'll get

$$\sin(\theta - \pi) = 1\sin(1(\theta - \pi)) + 0$$

so $a = 1$ (amplitude is 1), $b = 1$ (period is 2π), $c = -\pi$ (there's a horizontal shift of π units to the right), and $d = 0$ (there's no vertical shift). And therefore the graph of $\sin(\theta - \pi)$ is

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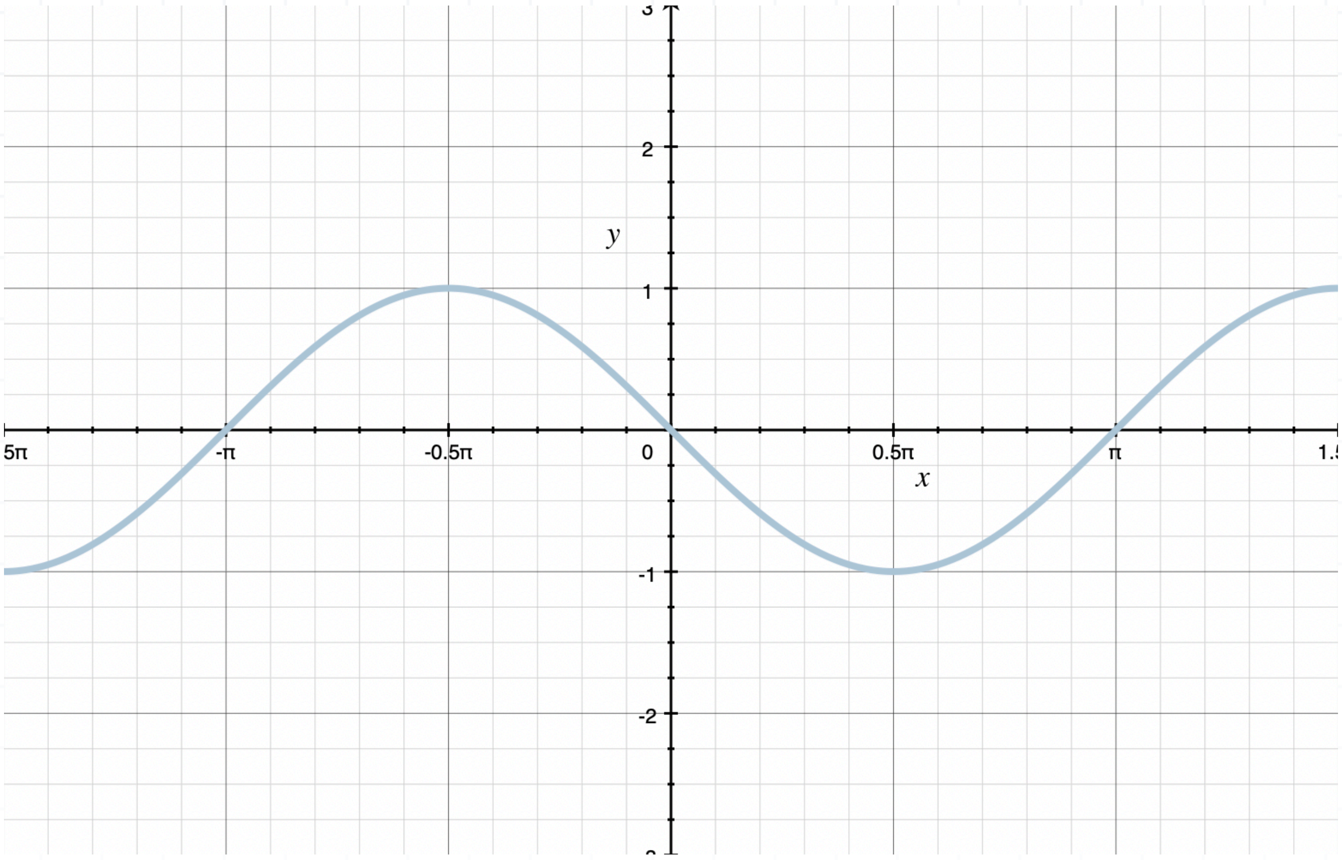

The period of $\sin(\theta - \pi)$ is 2π, and the period of $\cos(3\theta)$ is $2\pi/3$, so the period of $\sin(\theta - \pi)$ is three times the period of $\cos(3\theta)$. When the combination function is a sum or difference like it is in this problem, we take the least common multiple of the two periods of the original functions as the period of the combination. So since the least common multiple of $2\pi/3$ and 2π is 2π, the period of the combination is 2π.

We can sketch the graph of the combination simply by adding together the values of the other two functions at each point. For example, at $\theta = 0$, $\cos(3\theta)$ has a value of 1, and $\sin(\theta - \pi)$ has a value of 0, which means $\cos(3\theta) + \sin(\theta - \pi)$ will have a value of $1 + 0 = 1$, and we can plot that point. We'll continue on like this until we have a rough sketch of the graph over.

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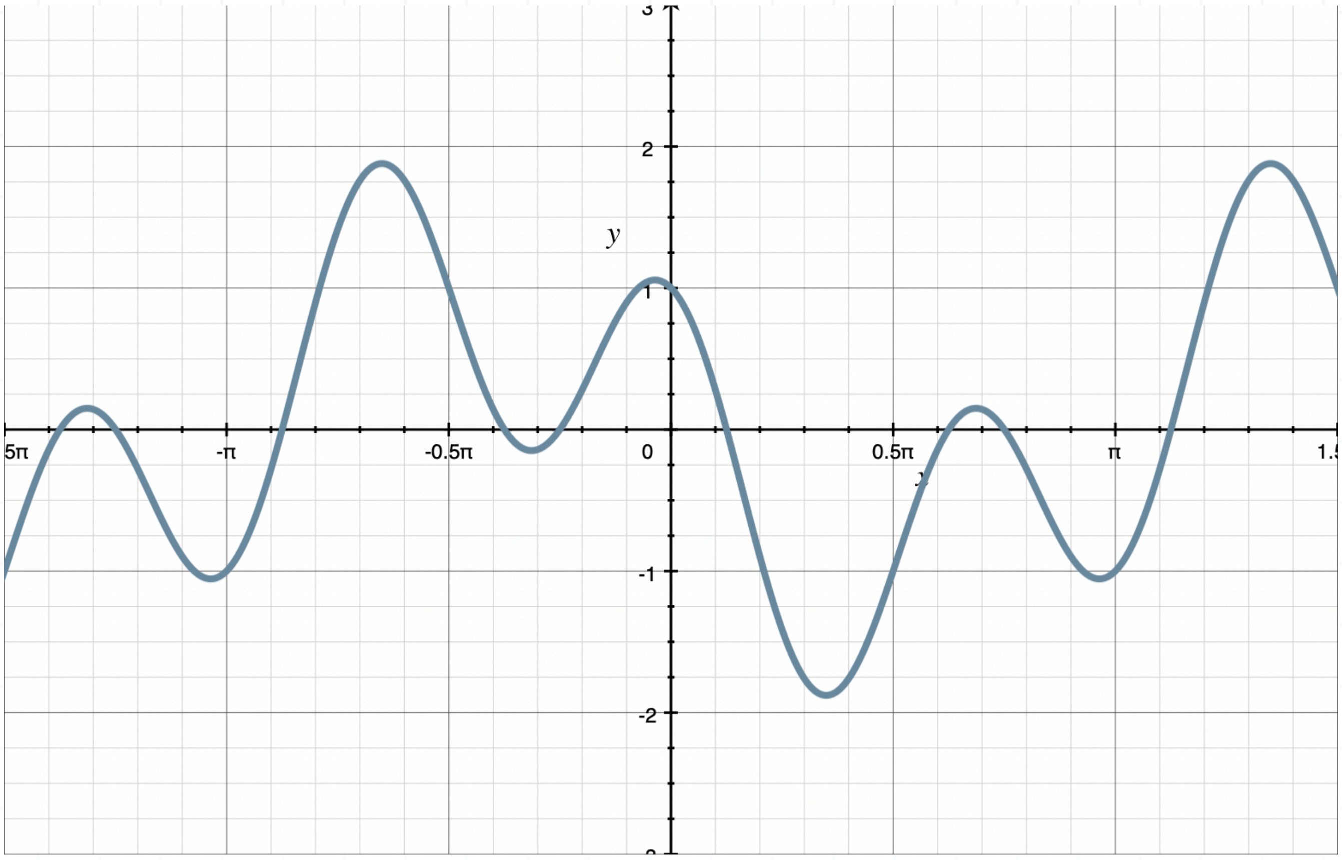


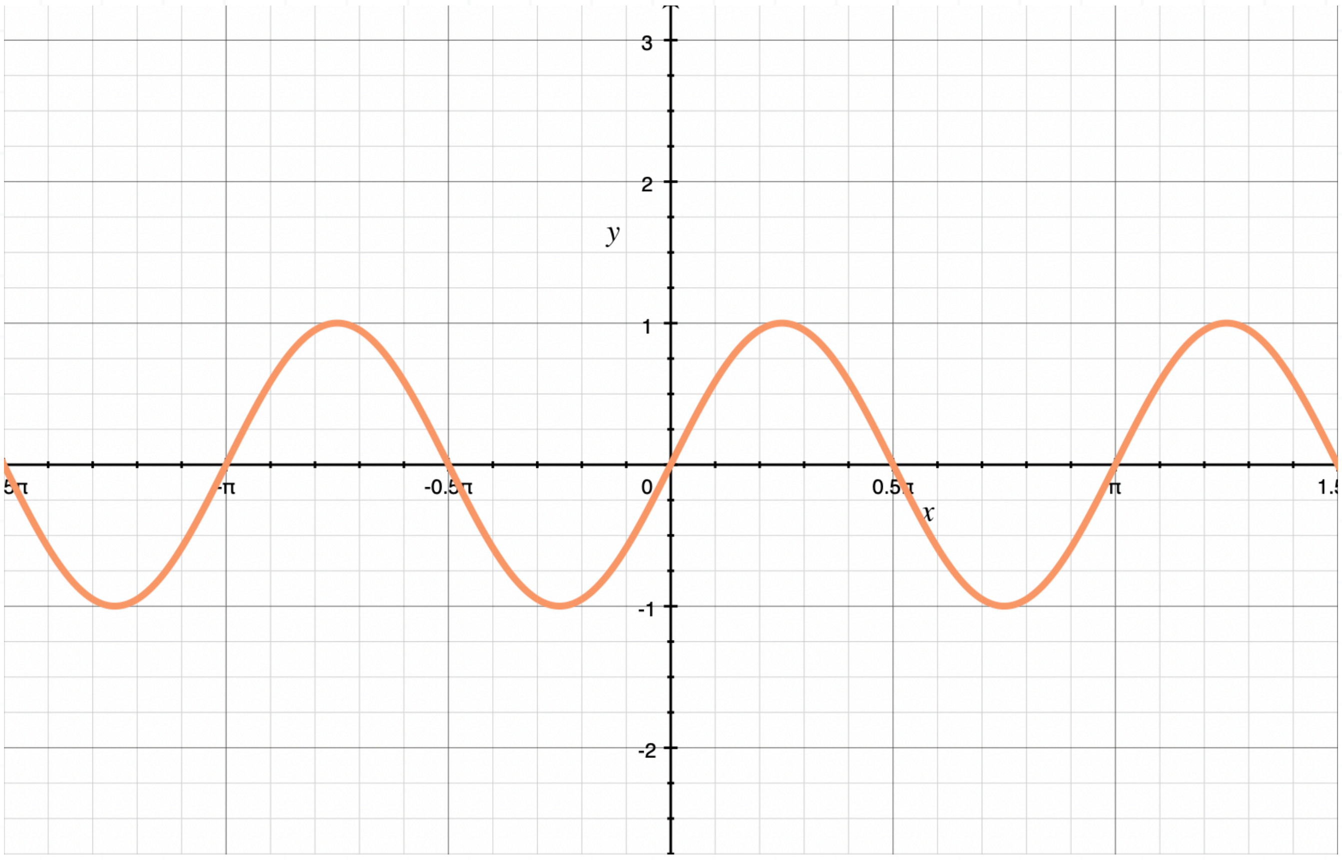
The difference of two functions, $f - g$, is just the sum of f and $-g$, so let's take a look at the product of sine and cosine functions.

Example

Sketch the graph of $(\sin(2\theta))(2\cos\theta + 1)$.

We want to recognize that $(\sin(2\theta))(2\cos\theta + 1)$ is really just the combination of $\sin(2\theta)$ and $2\cos\theta + 1$. We've taken the product of the two functions together to get the combination. So let's look at each of these two functions individually.

We can write $\sin(2\theta)$ as $1\sin(2(\theta+0))$, so $a=1$ (amplitude is 1), $b=2$ (period is $2\pi/2=\pi$), $c=0$ (there's no horizontal shift), and $d=0$ (there's no vertical shift). Therefore, the graph is



We can write $2\cos\theta+1$ as $2\cos(1(\theta+0))+1$, so this cosine function has $a=2$ (amplitude is 2), $b=1$ (period is 2π), $c=0$ (there's no horizontal shift), and $d=1$ (there's a vertical shift upward by 1 unit). Therefore, the graph is

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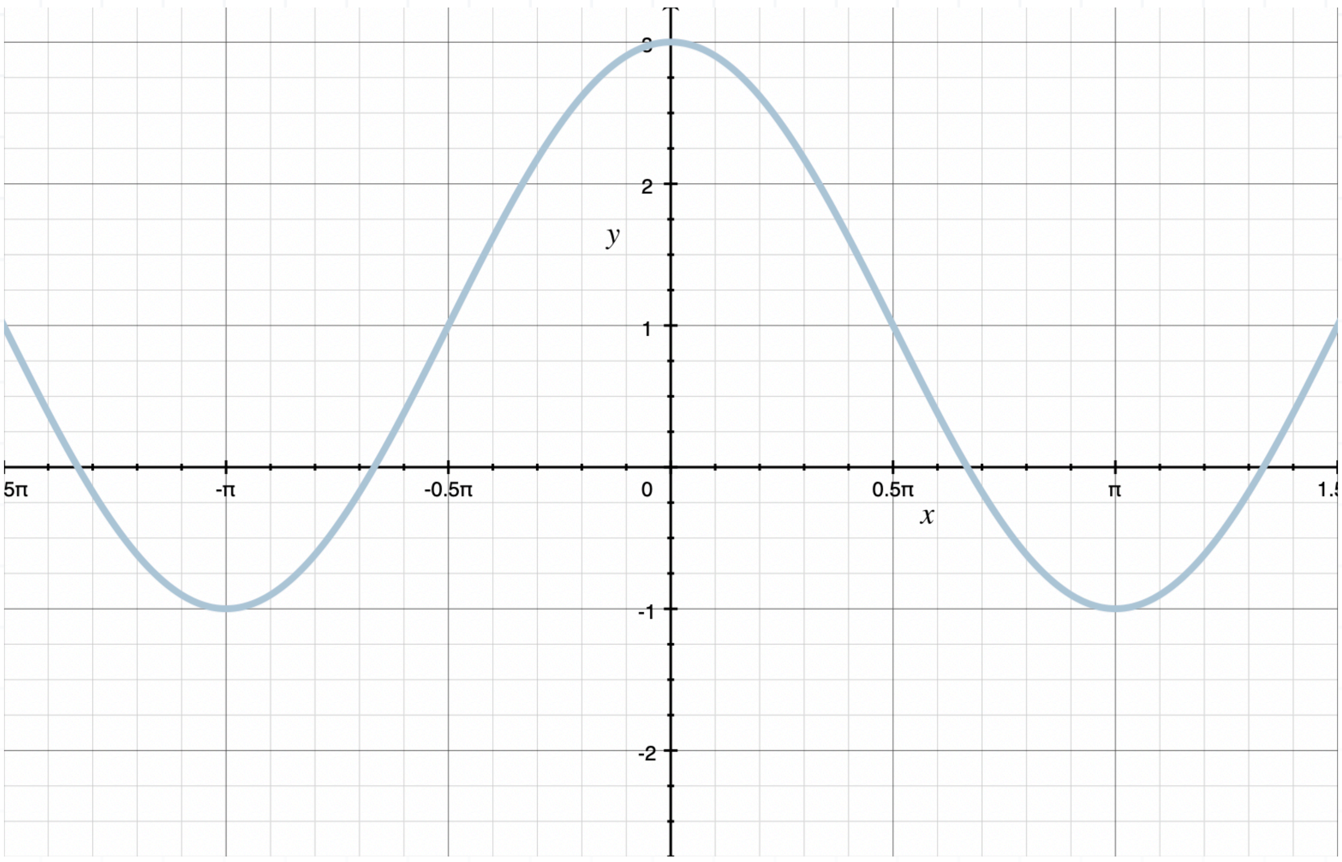
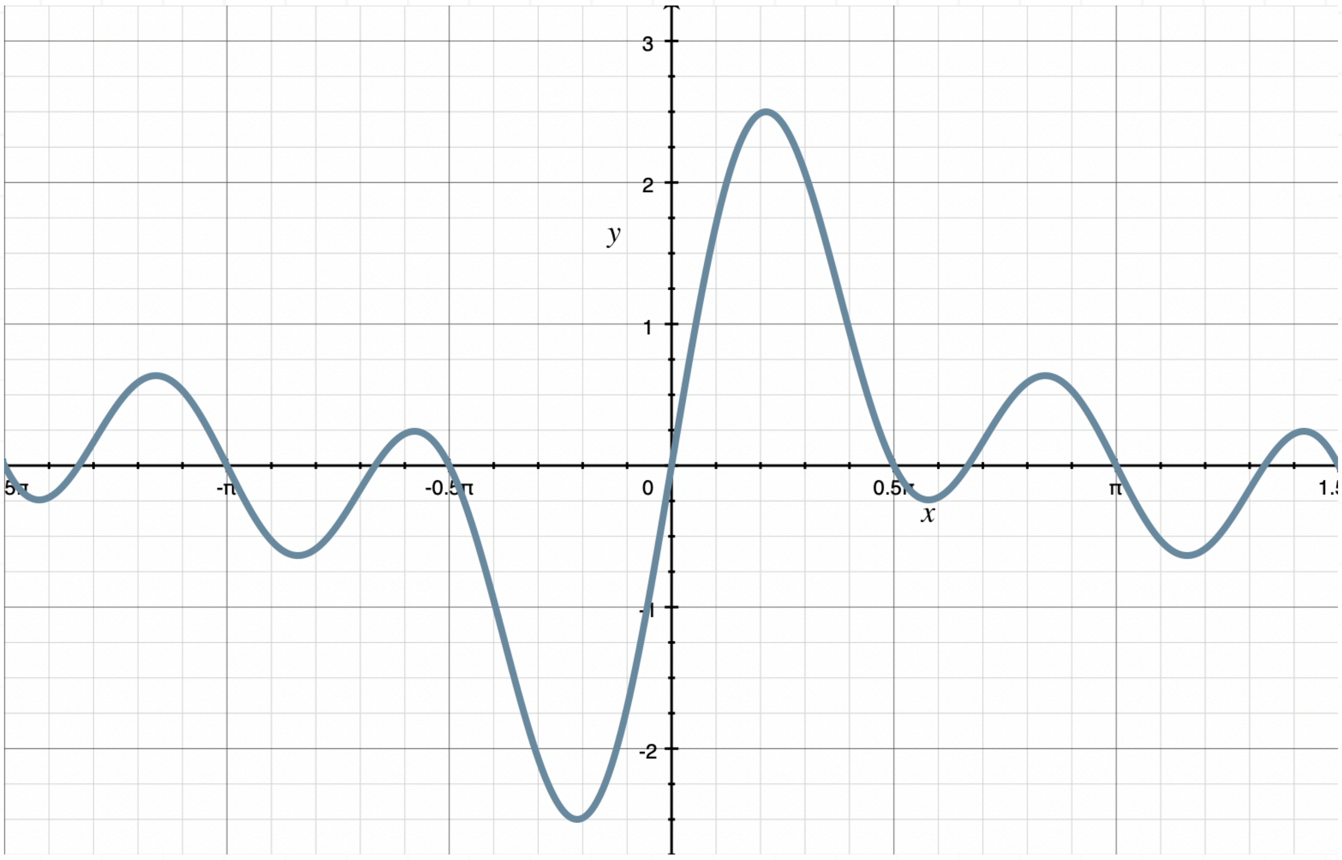


The period of $2\cos\theta + 1$ is 2π, and the period of $\sin(2\theta)$ is π, so the period of $2\cos\theta + 1$ is twice the period of $\sin(2\theta)$. When the combination function is a product or quotient like it is in this problem, we take the least common multiple of the original functions as the period of the combination. So since the least common multiple of 2π and π is 2π, the period of the combination is 2π.

We can sketch the graph of the combination simply by multiplying together the values of the other two functions at each point. For example, at $\theta = 0$, $2\cos\theta + 1$ has a value of 3, and $\sin(2\theta)$ has a value of 0, which means $(\sin(2\theta))(2\cos\theta + 1)$ will have a value of $3(0) = 0$, and we can plot that point. We'll continue on like this until we have a rough sketch of the graph.



Things get more complicated when we consider a quotient of trig functions, because the quotient is undefined at every angle θ that causes the denominator to be 0. Let's see an example of that.

Example

Sketch the graph of the quotient combination.

$$\frac{\cos\left(\theta+\frac{\pi}{4}\right)}{\sin\left(3\theta-\frac{\pi}{2}\right)}$$

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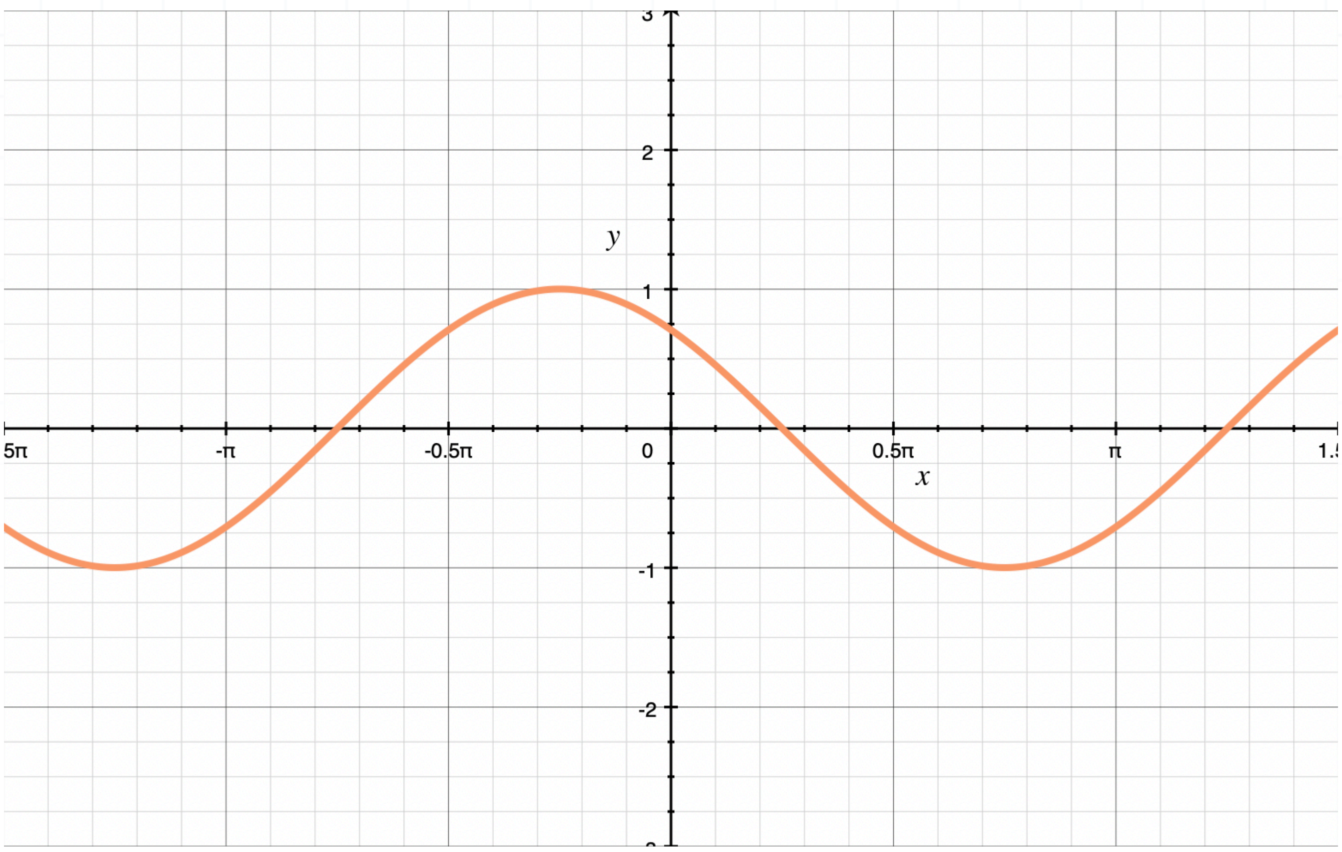
We want to recognize that the given quotient is really just the combination of $\cos(\theta + (\pi/4))$ and $\sin(3\theta - (\pi/2))$. We've divided the two functions to get the combination. So let's look at each of these two functions individually.

The function in the numerator, $\cos(\theta + (\pi/4))$, can be expressed in the form $a\cos(b(\theta + c)) + d$ with $a = 1$ (amplitude is 1), $b = 1$ (period is 2π), $c = \pi/4$ (there's a horizontal shift $\pi/4$ units to the left), and $d = 0$ (there's no vertical shift). So the graph is



The function in the denominator, $\sin(3\theta - (\pi/2))$, can be rewritten as $\sin(3(\theta - (\pi/6)))$, so $a = 1$ (amplitude is 1), $b = 3$ (period is $2\pi/3$), $c = -\pi/6$ (there's a horizontal shift $\pi/6$ units to the right), and $d = 0$ (there's no vertical shift). So the graph is

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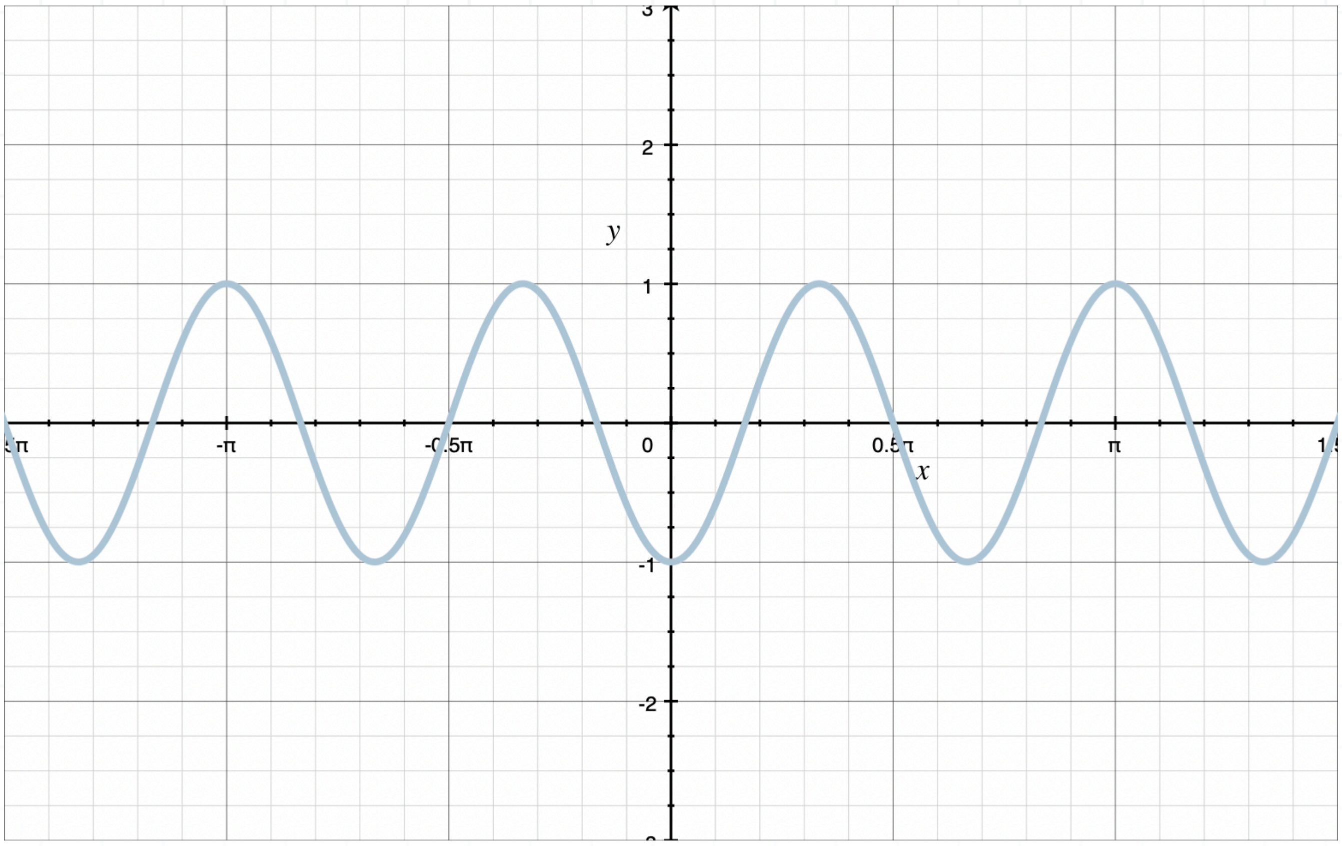

We expect that the behavior of the quotient combination will be a little bizarre near any values where $\sin(3\theta - (\pi/2))$ gets close to 0, since $\sin(3\theta - (\pi/2))$ is the denominator of the quotient combination, and a function is undefined whenever it's denominator is 0.

Let's look at where $\sin(3\theta - (\pi/2))$ will get close to 0. We'll set $\sin(3\theta - (\pi/2))$ equal to 0 and solve for θ.

$$\sin\left(3\theta - \frac{\pi}{2}\right) = 0$$

The sine function is equal to 0 at angles of $0, \pi, 2\pi, 3\pi, 4\pi, \ldots,$ in other words, at multiples of π, or any value $n\pi$ where n is an integer. Therefore we can solve an equation for θ:

$$3\theta - \frac{\pi}{2} = n\pi$$

$$6\theta - \pi = 2n\pi$$

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$$6\theta = 2n\pi + \pi$$

$$6\theta = (2n + 1)\pi$$

$$\theta = \frac{(2n + 1)\pi}{6}$$

Let's plug in some integers for n, to see which values we get for θ.

$n = 0$ $\quad\theta = [2(0) + 1]\left(\frac{\pi}{6}\right) = \frac{\pi}{6}$

$n = 1$ $\quad\theta = [2(1) + 1]\left(\frac{\pi}{6}\right) = \frac{3\pi}{6} = \frac{\pi}{2}$

$n = 2$ $\quad\theta = [2(2) + 1]\left(\frac{\pi}{6}\right) = \frac{5\pi}{6}$

$n = 3$ $\quad\theta = [2(3) + 1]\left(\frac{\pi}{6}\right) = \frac{7\pi}{6}$

$n = 4$ $\quad\theta = [2(4) + 1]\left(\frac{\pi}{6}\right) = \frac{9\pi}{6} = \frac{3\pi}{2}$

$n = 5$ $\quad\theta = [2(5) + 1]\left(\frac{\pi}{6}\right) = \frac{11\pi}{6}$

These will all be vertical asymptotes of the quotient combination, and of course we could continue finding more evenly-spaced asymptotes to the left and right.

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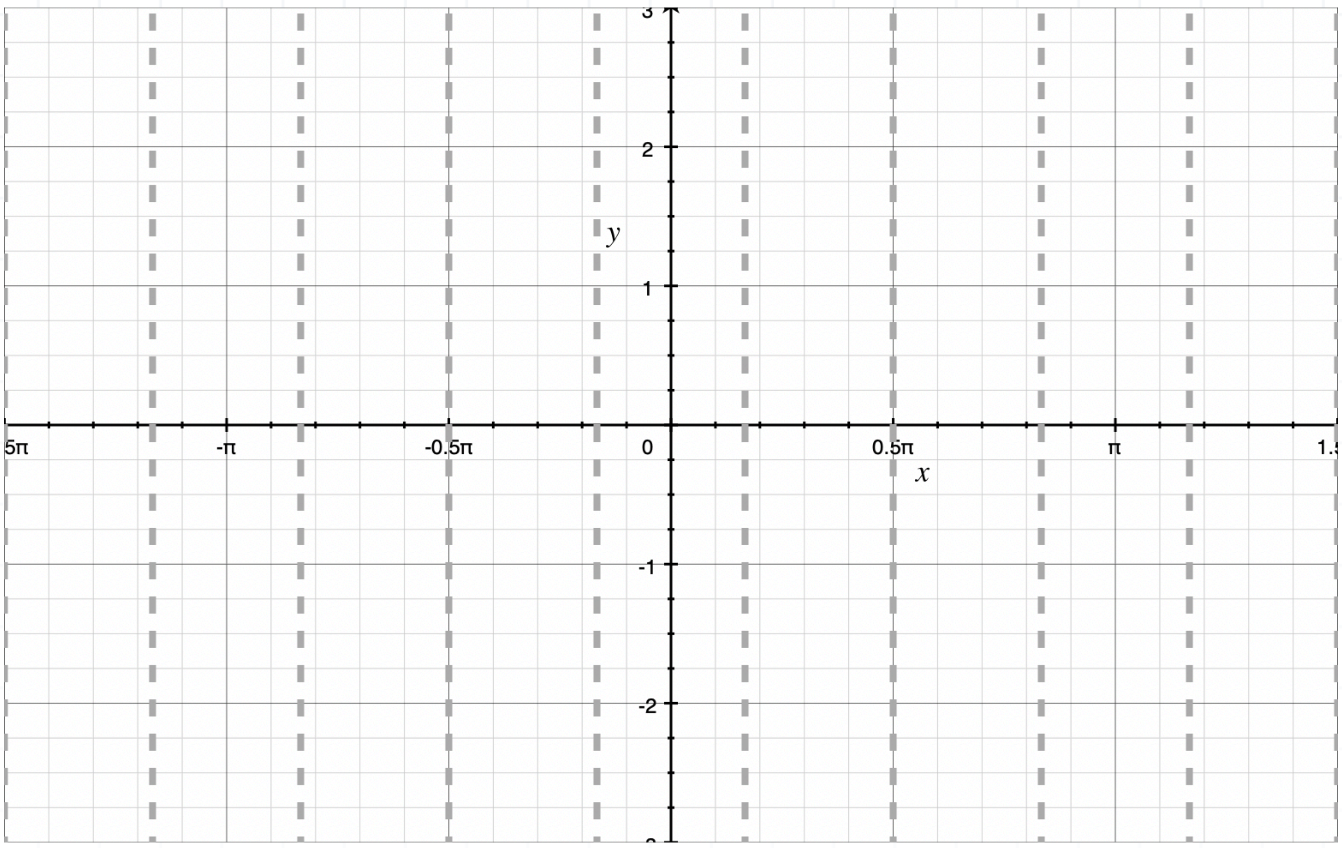

The period of $\cos(\theta + (\pi/4))$ is 2π, and the period of $\sin(3\theta - (\pi/2))$ is $2\pi/3$, so the period of $\cos(\theta + (\pi/4))$ is three times the period of $\sin(3\theta - (\pi/2))$. When the combination function is a product or quotient like it is in this problem, we take the least common multiple of the original functions as the period of the combination. So since the least common multiple of 2π and $2\pi/3$ is 2π, the period of the combination is 2π. However, when we look at the final graph, we'll see that 2π will actually include two full repeated periods. One single period will be defined in an interval of just π, so we'll reduce the period from 2π to π.

We can sketch the graph of the combination simply by dividing the values of the other two functions at each point. For example, at $\theta = 0$, $\cos(\theta + (\pi/4))$ has a value of $\sqrt{2}/2$, and $\sin(3\theta - (\pi/2))$ has a value of -1, which means the combination quotient will have a value of

$$\frac{\sqrt{2}}{2}/(-1) = -\frac{\sqrt{2}}{2}$$

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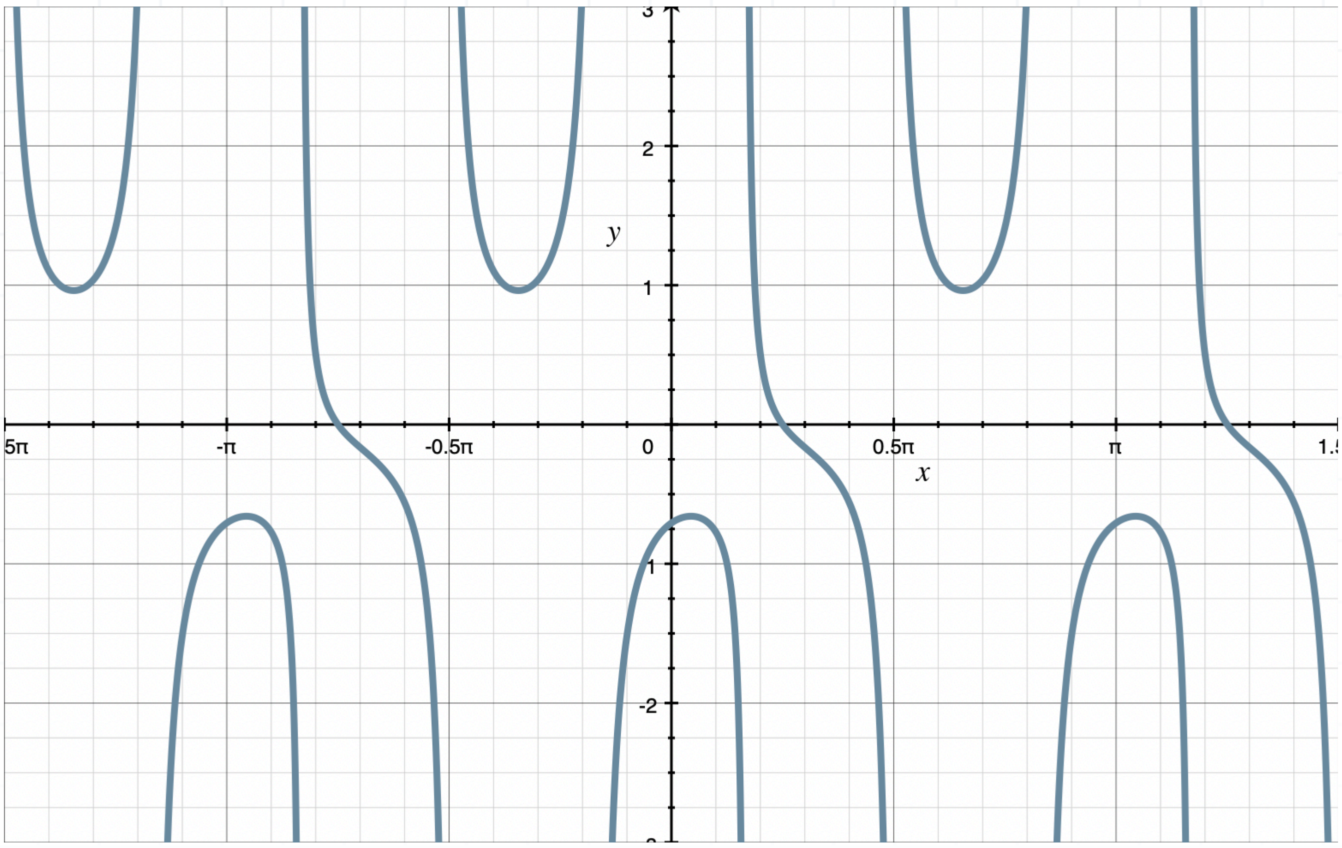
and we can plot that point. We'll continue on like this until we have a rough sketch of the graph, making sure to respect the asymptotes as we go.



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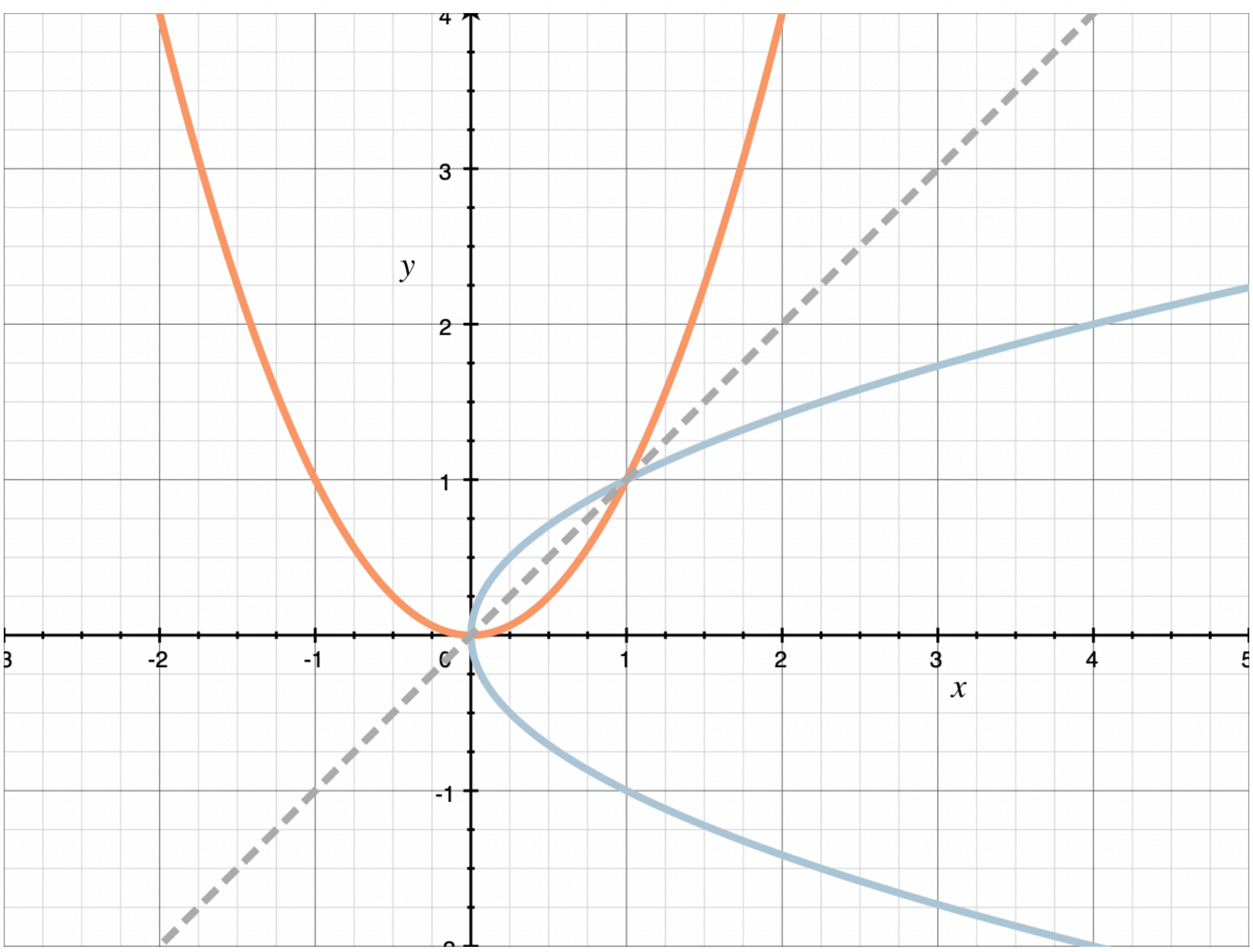
Inverse trig relations

In Algebra, we learned that equations are inverses of one another when they have their x- and y-values swapped. For instance, the inverse of $y = x^2$ is $x = y^2$, because we've changed the places of the x and y variables.

Inverses as reflections

When we learned about inverses, we also learned that inverse curves will be reflections of each other over the line $y = x$. So if we sketch the graphs of $y = x^2$ in red and $x = y^2$ in blue, along with the dotted line $y = x$ in gray, we should see that $y = x^2$ and $x = y^2$ are perfect mirror images of one another across the line.



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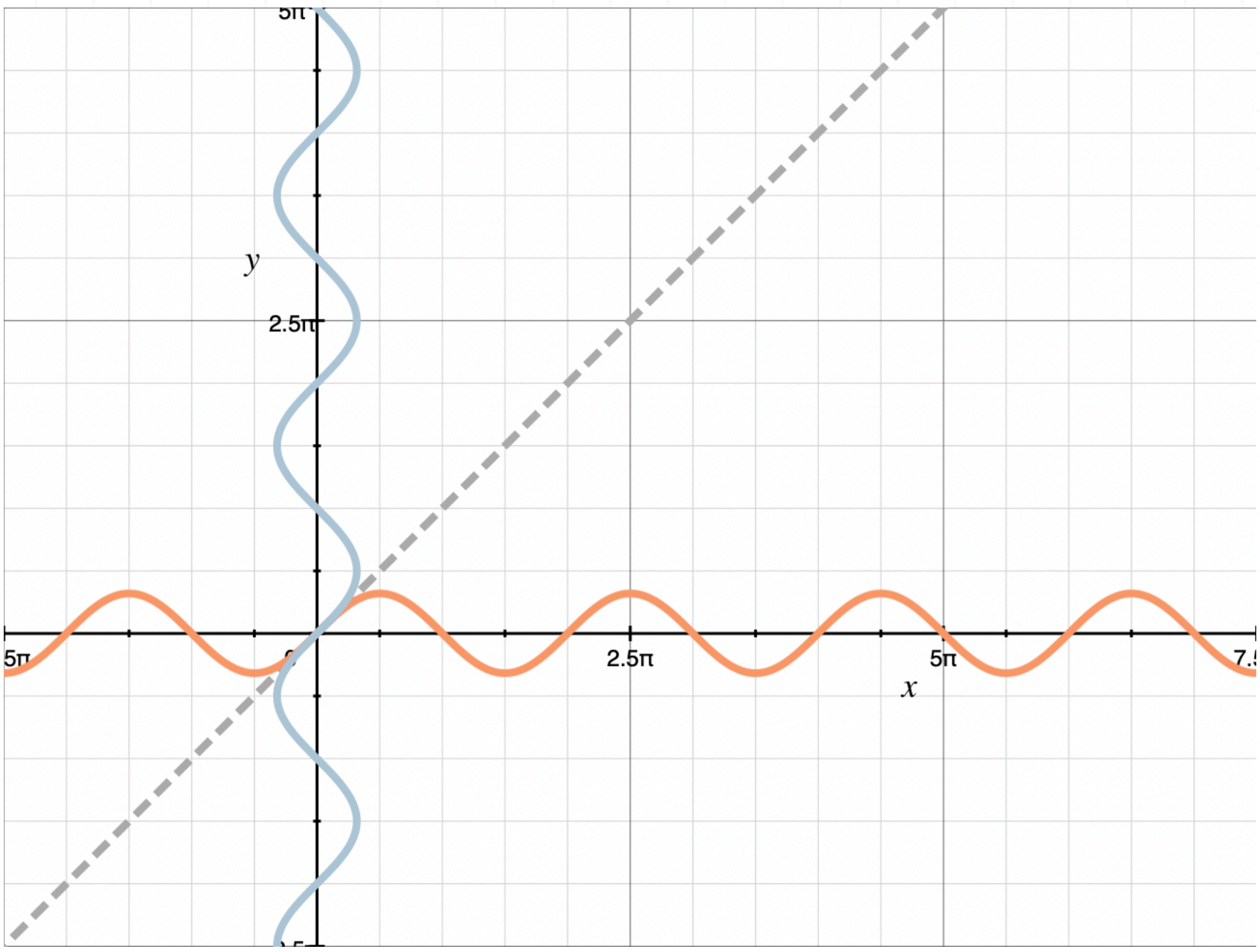
Inverse trig functions are just like these inverse functions from Algebra. To find a pair of inverse trig relations, we just swap the x- and y-values. So for example, the inverse of $y = \sin x$ is $x = \sin y$. We can see that these equations are inverses of each other if we sketch their graphs and the line $y = x$.



Notice how the blue curve $x = \sin y$ can't be a function. We learned about functions in Algebra, but as a review, a **function** will always pass the **Vertical Line Test**, which means that no perfectly vertical line will intersect the curve at more than one point.

Because we could draw many vertical lines near $x = 0$ that would cross the blue curve at more than one point (in fact, at infinitely many points), the blue curve $x = \sin y$ is not a function. For this reason, we call these **inverse trig relations**, instead of inverse trig functions.

Working backwards to solve inverse equations

So we understand the general idea of an inverse, and what inverse trig equations look like, but now we want to talk about the meaning behind trig functions and their inverses.

We know that a trig function $y = \sin\theta$ is an equation that lets us input a particular angle θ, and get back the corresponding value y. This is the kind of relationship we're used to dealing with so far: we choose an angle θ, and then find the value of a trig function at that angle. So the question we've been answering is "What value y do we get for a given angle θ?"

If we take the inverse of $y = \sin\theta$, we're really switching y and θ, and we get $\theta = \sin y$. With this equation, we're answering the opposite question: "What angle θ do we get for a given y-value?"

So if we think back to the unit circle, we can see these opposite questions in action. Previously, we'd take something like $\sin(\pi/2)$, and say that the value of $\sin(\pi/2)$ is 1. When we have the inverse instead, we're looking at the equation

$$\sin y = 1$$

To solve this equation, we have to think about where the sine function is equal to 1. In the unit circle, sine of $\pi/2$ gives 1, and so does sine of all angles coterminal with $\pi/2$. So if we know that the result of the sine function is 1, then the solution to the equation above is

$$y = \frac{\pi}{2} + 2n\pi$$

where n is any integer.

Let's do a few examples in which we've already been given the value from the coordinate point on the unit circle, and we need to find the angle that's giving us that value. For reference in the next two examples, here is the unit circle again:

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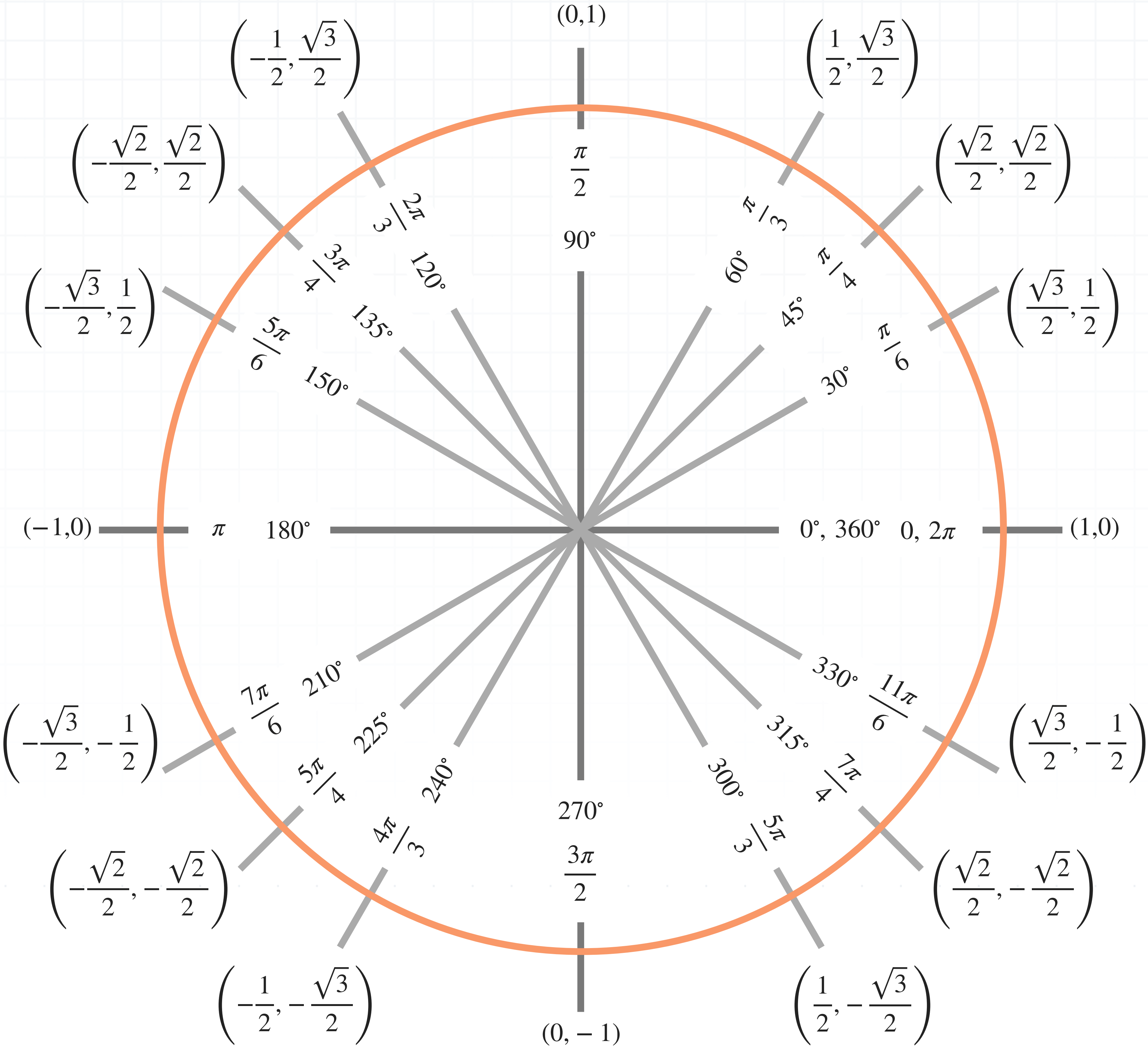

Example

In both radians and degrees, use the unit circle to find the set of angles whose sine is $1/2$.

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On the unit circle, we know that the y-value in the coordinate point is the value that gives us the sine of the angle. Therefore, because we're told that sine of the angle is $1/2$, we need to find the angles in the unit circle where the corresponding coordinate point has a y-value equal to $1/2$.

Those angles are $\pi/6$ and $5\pi/6$. To give the full set of radian angles, we have to give all of the angles that are coterminal with these two:

$$\theta = \frac{\pi}{6} + 2n\pi \text{ and } \theta = \frac{5\pi}{6} + 2n\pi$$

We know that $\pi/6 = 30^\circ$ and $5\pi/6 = 150^\circ$, so the set of angles in degrees will be:

$$\theta = 30^\circ + n(360^\circ) \text{ and } \theta = 150^\circ + n(360^\circ)$$

Let's do an example with cosine.

Example

In both radians and degrees, use the unit circle to find the set of angles whose cosine is -1.

On the unit circle, we know that the x-value in the coordinate point is the value that gives us the cosine of the angle. Therefore, because we're told that cosine of the angle is -1, we need to find the angles in the unit circle where the corresponding coordinate point has an x-value equal to -1.

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The only angle that does this is the angle π. To give the full set of radian angles, we have to give all of the angles that are coterminal π:

$$\theta = \pi + 2n\pi$$

We know that $\pi = 180^\circ$, so the set of angles in degrees will be:

$$\theta = 180^\circ + n(360^\circ)$$

How we express inverse relations

We know that $x = \sin y$ is the inverse sine relation, but we usually like to express equations for y in terms of x, which means we want to be able to solve $x = \sin y$ for y.

The way we do that is by applying the inverse sine to both sides of the equation. We indicate inverse sine as either $\sin^{-1}$ or as $\arcsin$. These mean the same thing, they're just written different way. When we take inverse sine of both sides, we get

$$\arcsin(x) = \arcsin(\sin y)$$

On the right side, the $\arcsin$ will cancel out the $\sin$, leaving us with just y.

$$\arcsin(x) = y$$

$$y = \arcsin x$$

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This equation can also be written as $y=\sin^{-1}x$. The statement we're making with both equation is "y is the angle whose sine is x."

We also have to be careful when we use the notation $\sin^{-1}$. In Algebra, we would make a negative exponent positive by moving the term to the denominator. For instance, x^{-2} could be rewritten as $1/(x^2)$. But the -1 in $\sin^{-1}$ isn't a negative exponent, it's simply notation to indicate "inverse sine." So

$$\sin^{-1}x \neq \frac{1}{\sin x}$$

If we extend this to the other trig functions, we get the following table:

The inverse of	**is written as**	**and is equivalent to**
$y=\sin x$	$y=\sin^{-1}x$ or $y=\arcsin x$	$x=\sin y$
$y=\cos x$	$y=\cos^{-1}x$ or $y=\arccos x$	$x=\cos y$
$y=\tan x$	$y=\tan^{-1}x$ or $y=\arctan x$	$x=\tan y$
$y=\csc x$	$y=\csc^{-1}x$ or $y=$ arccsc$x$	$x=\csc y$
$y=\sec x$	$y=\sec^{-1}x$ or $y=$ arcsec$x$	$x=\sec y$
$y=\cot x$	$y=\cot^{-1}x$ or $y=$ arccot$x$	$x=\cot y$

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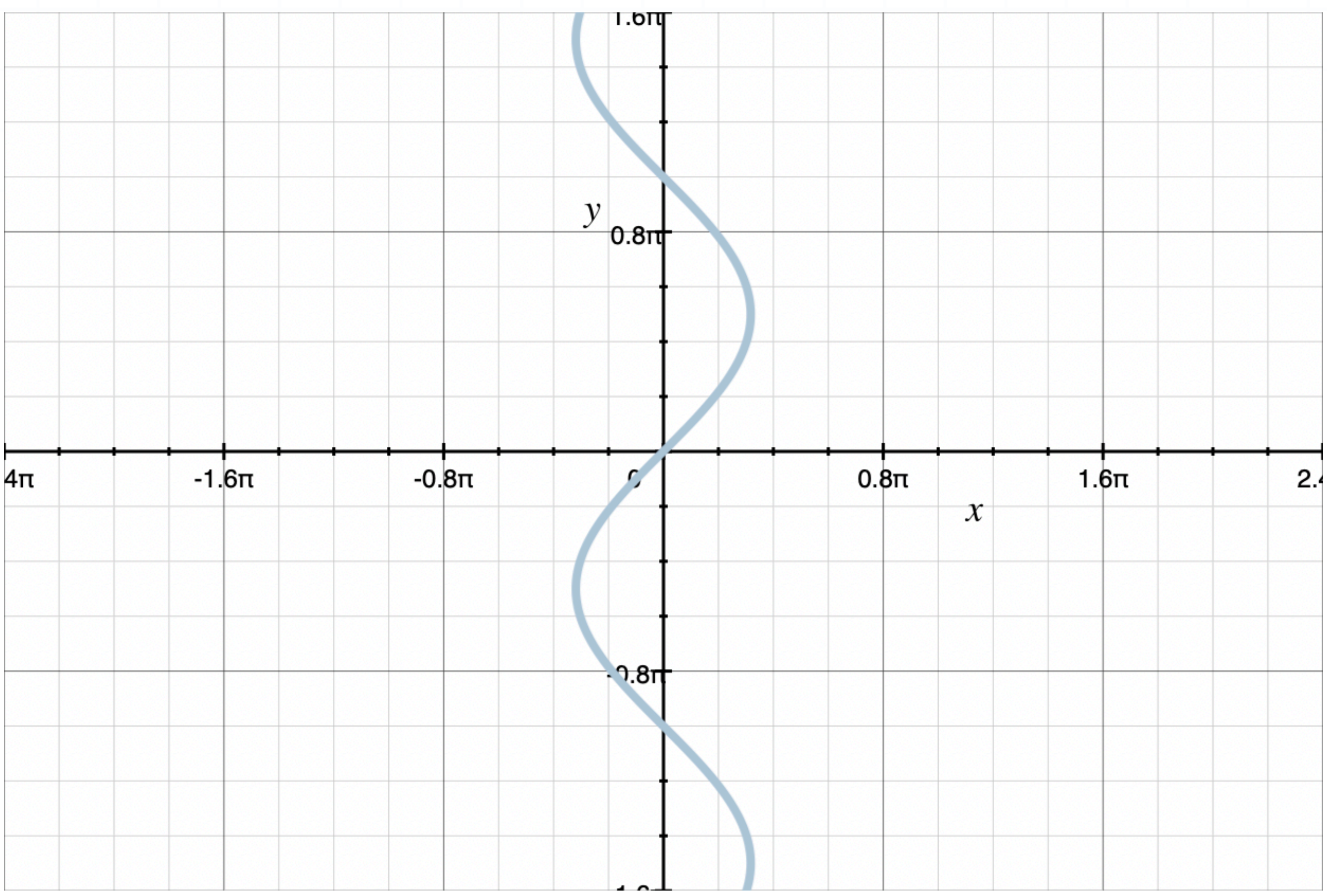
Inverse trig functions

In the last lesson we looked at inverse trig relations, which were the equations we got when we flipped the variables in a standard trig function. In other words, starting with the standard trig function $y = \sin x$, we got its inverse trig relation by swapping the variables to get $x = \sin y$.

Remember though that we called $x = \sin y$ a "relation" because it wasn't a function, because it didn't pass the Vertical Line Test. The way that we turn the inverse trig relation into an inverse trig function is by limiting the range of the relation to just one period.

So in the last lesson we showed that the graph of $y = \sin^{-1} x$ was



But if we want to turn this relation into a function, then we limit the range to $-\pi/2 \leq y \leq \pi/2$. When we restrict the range that way, the graph gets cut down to just one portion:

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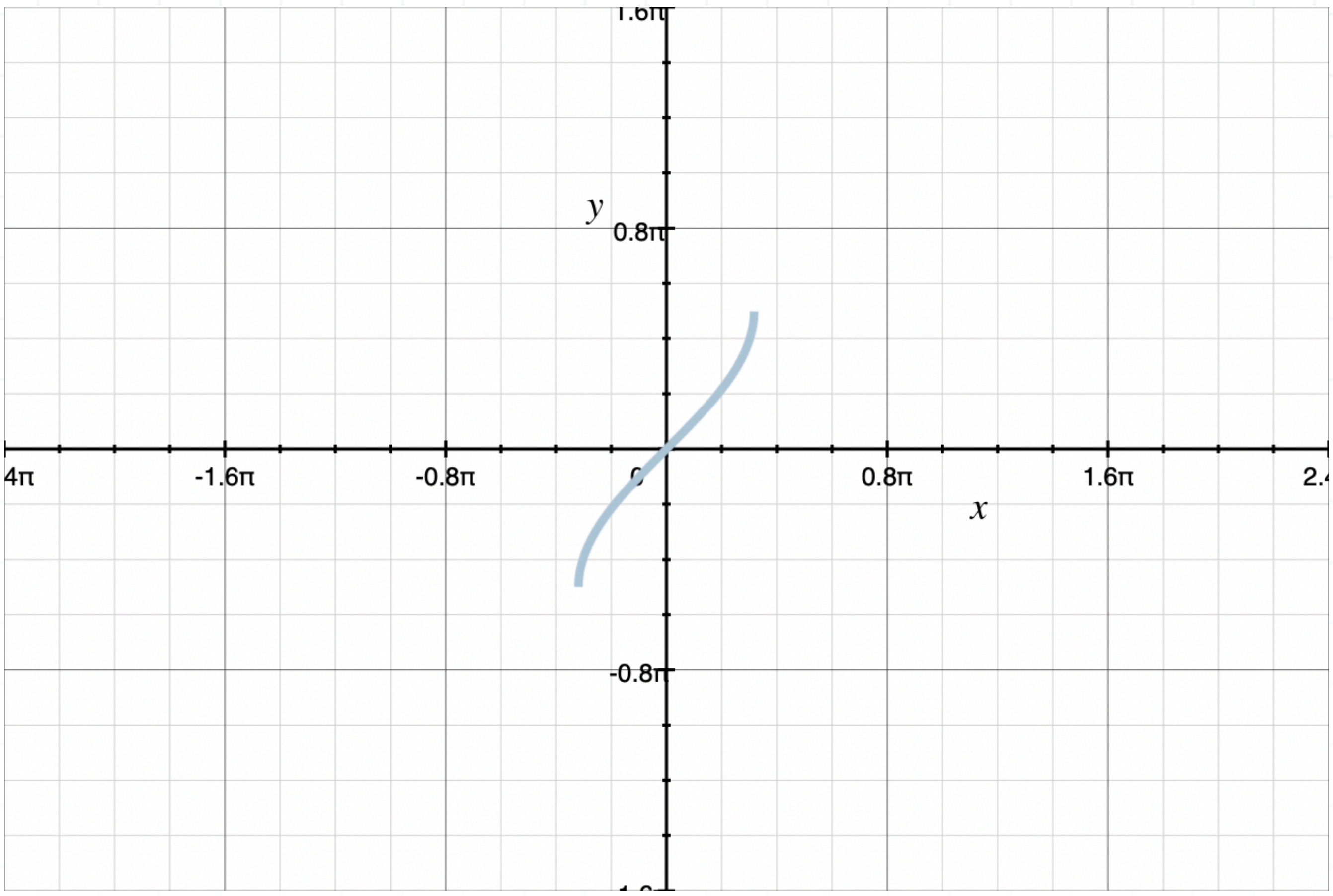

Notice how this curve now represents a function, since it could pass the Vertical Line Test. Again, that means we could draw any perfectly vertical line anywhere along this curve, and it would only intersect the curve at one point. Therefore, if we name the curve this way,

$$y = \sin^{-1} x \text{ for } -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$$

then we can call it an **inverse trig function**, because it's now a function that will pass the Vertical Line Test. Sometimes we indicate an inverse trig function, as opposed to an inverse trig relation, with a capital letter. So if you see

$$y = \text{Sin}^{-1} x \text{ or } y = \text{Arcsin} x$$

it indicates an inverse trig function where the range is restricted. But if you see just

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$$y = \sin^{-1} x \text{ or } y = \arcsin x$$

without any inequality restricting the range, it means we're dealing with the inverse relation. Of course, if we add an inequality to restrict the range, even if we use a lower case letter, then we're still defining the inverse trig function.

The other inverse trig functions

We've already defined the inverse sine function as

$$y = \sin^{-1} x \text{ for } -\frac{\pi}{2} \le y \le \frac{\pi}{2}$$

Now we want to define the inverse cosine and tangent functions. The inverse cosine function and its graph are

$$y = \cos^{-1} x \text{ for } 0 \le y \le \pi$$

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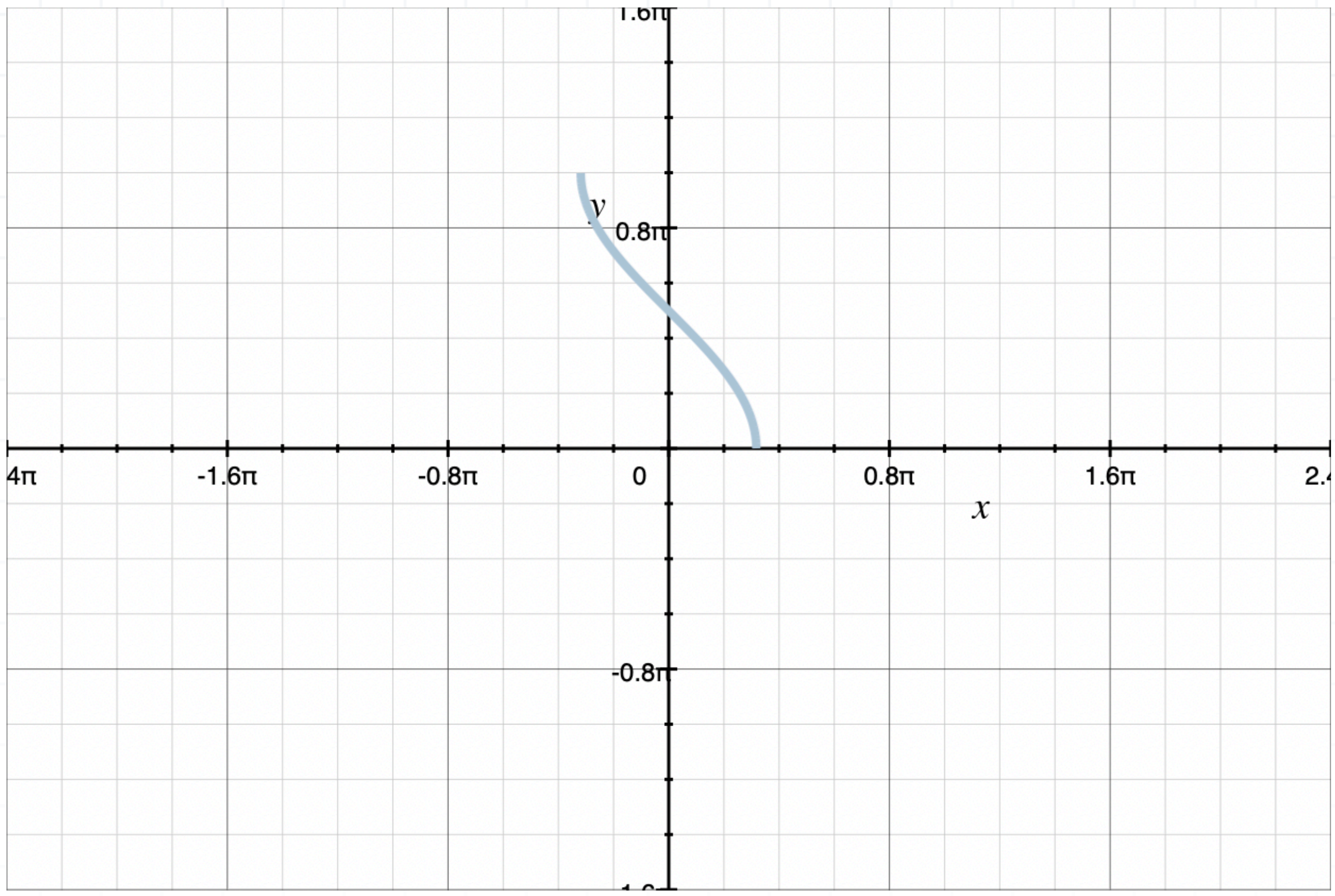
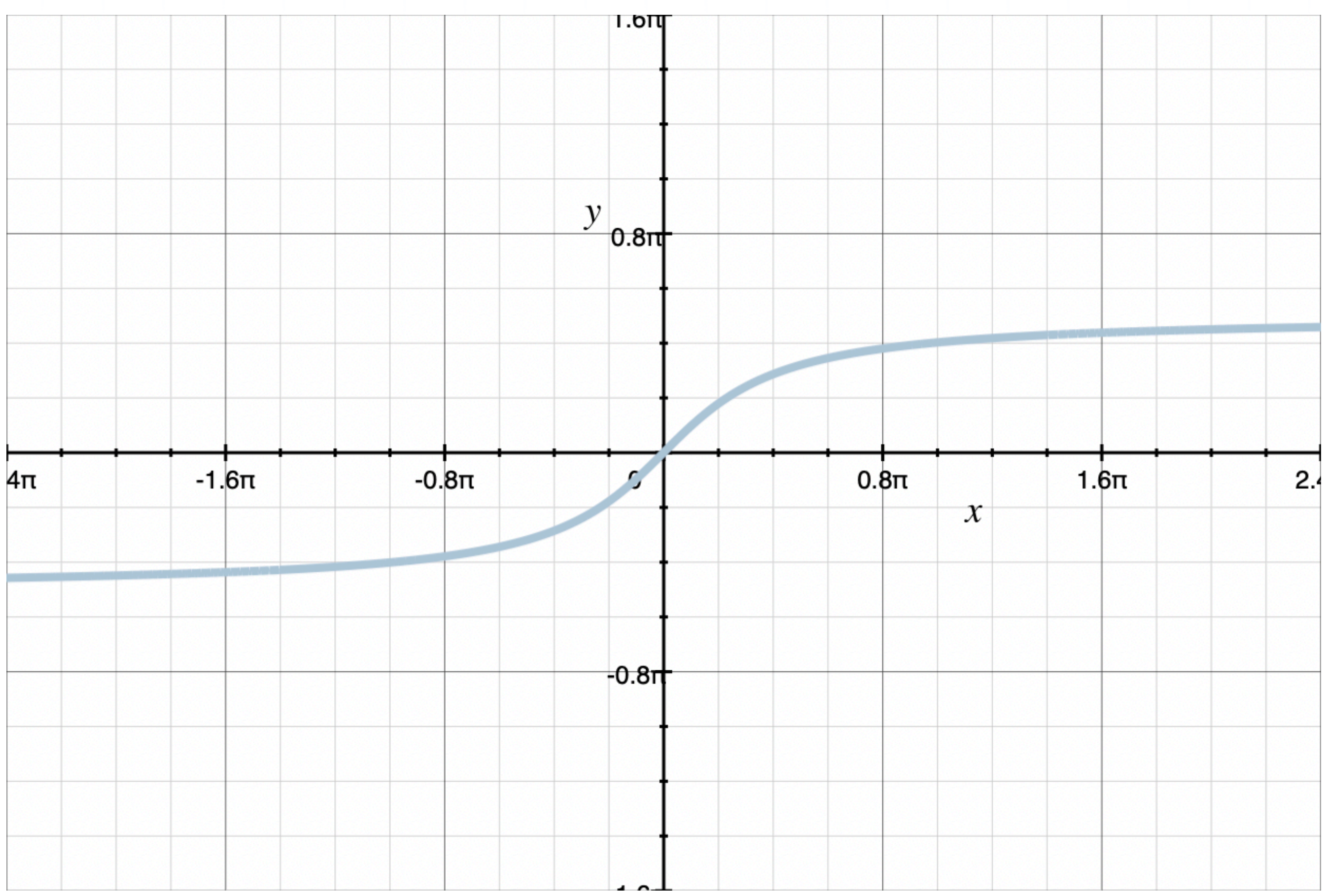


and the inverse tangent function and its graph are

$$y = \tan^{-1} x \text{ for } -\frac{\pi}{2} < y < \frac{\pi}{2}$$



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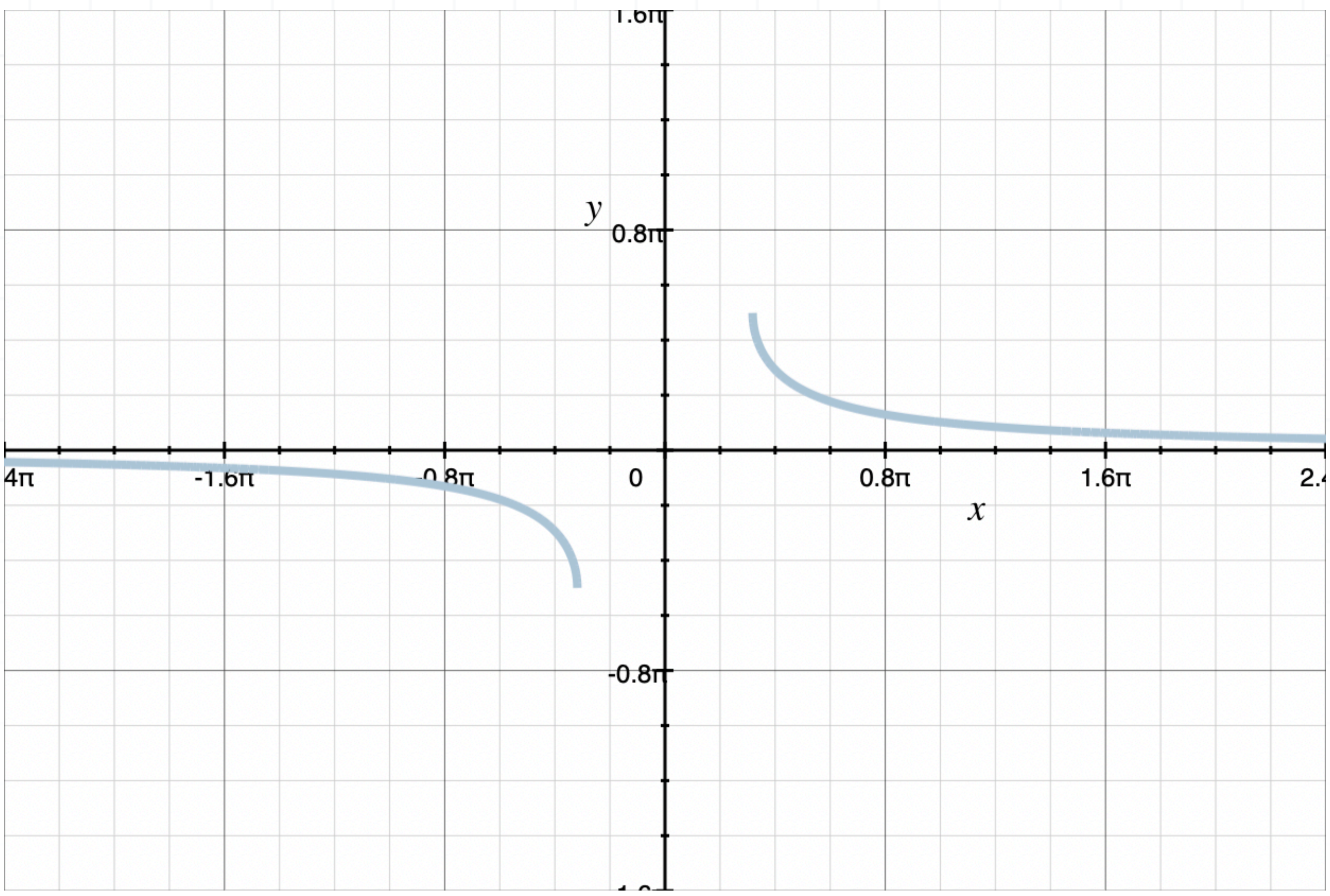
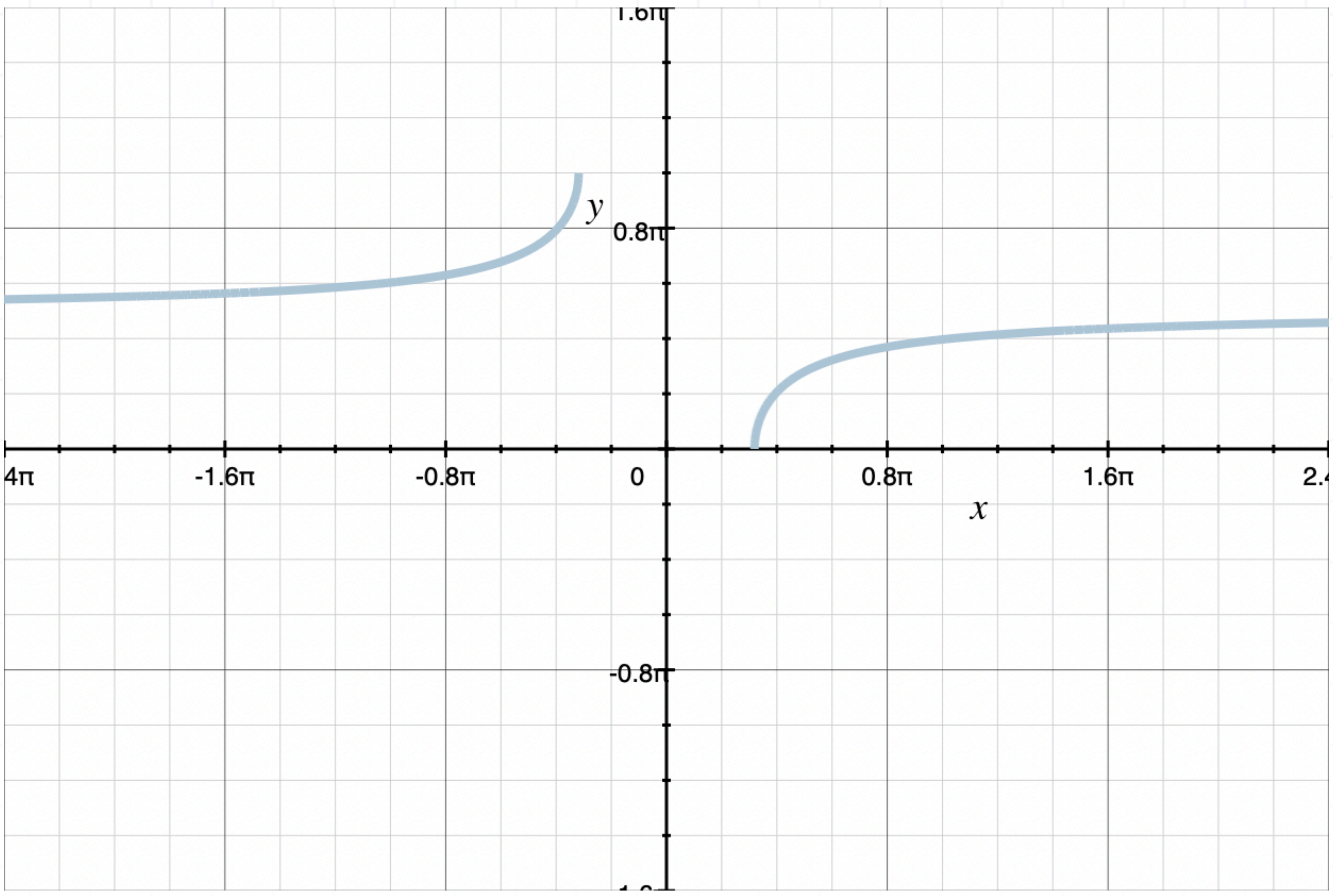
We can evaluate the other three trig functions as the reciprocals of the inverse sine, cosine, and tangent functions.

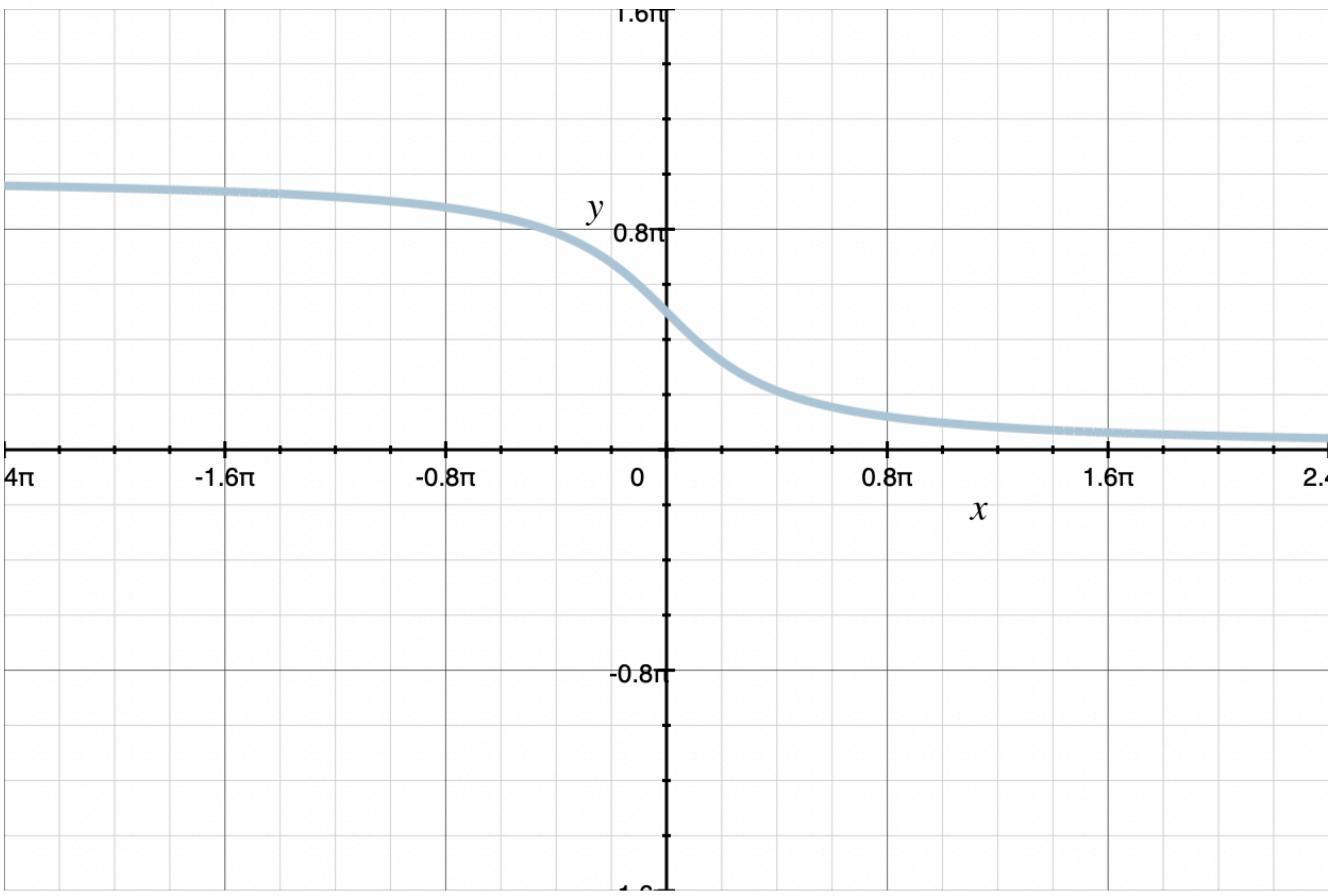
$$y = \csc^{-1} x = \frac{1}{\sin^{-1} x} \text{ for } -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}, \text{ except at } y = 0$$



$$y = \sec^{-1} x = \frac{1}{\cos^{-1} x} \text{ for } 0 \leq y \leq \pi, \text{ except at } y = \frac{\pi}{2}$$



$$y = \cot^{-1} x = \frac{1}{\tan^{-1} x} \text{ for } 0 < y < \pi$$



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Here's a summary of the domain and range of all six inverse trig functions.

Inverse function	**Domain**	**Range**
$y=\sin^{-1}x$	$x=[-1,1]$	$y=\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$
$y=\cos^{-1}x$	$x=[-1,1]$	$y=[0,\pi]$
$y=\tan^{-1}x$	$x=(-\infty,\infty)$	$y=\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$
$y=\csc^{-1}x$	$x=(-\infty,-1]\cup[1,\infty)$	$y=\left[-\frac{\pi}{2},0\right)\cup\left(0,\frac{\pi}{2}\right]$
$y=\sec^{-1}x$	$x=(-\infty,-1]\cup[1,\infty)$	$y=\left[0,\frac{\pi}{2}\right)\cup\left(\frac{\pi}{2},\pi\right]$
$y=\cot^{-1}x$	$x=(-\infty,\infty)$	$y=(0,\pi)$

Remember that to evaluate inverse trig functions, we need to "think backwards," meaning that we'll start from the value of the trig function, and then figure out the angle to which it corresponds.

Let's do an example with inverse sine.

Example

Find the value of the inverse sine function.

$$\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$$

If we look at the unit circle, we can see that the sine function is $-\sqrt{2}/2$ when $\theta = 5\pi/4$ and when $\theta = 7\pi/4$. But because we're dealing with the inverse sine function, we only want an angle in the interval $[-\pi/2,\pi/2]$.

Both $\theta = 5\pi/4$ and $\theta = 7\pi/4$ fall outside the interval $[-\pi/2,\pi/2]$, which means we'll need to find an angle coterminal with either $\theta = 5\pi/4$ or $\theta = 7\pi/4$ that falls within $[-\pi/2,\pi/2]$.

Remember that the interval $[-\pi/2,\pi/2]$ defines the fourth and first quadrant. The angle $\theta = 5\pi/4$ is in the third quadrant, and the angle $\theta = 7\pi/4$ is in the fourth quadrant. Which means the angle we need is one that's coterminal with $\theta = 7\pi/4$, but in the interval $[-\pi/2,\pi/2]$. That angle that works is $\theta = -\pi/4$, so we'll say

$$\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right) = -\frac{\pi}{4}$$

Let's do another example, this time with the inverse cosine function.

Example

Find the value of the inverse cosine function.

$$\cos^{-1}\left(\frac{1}{2}\right)$$

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If we look at the unit circle, we can see that the cosine function is $1/2$ when $\theta = \pi/3$ and when $\theta = 5\pi/3$. But because we're dealing with the inverse cosine function, we only want an angle in the interval $[0,\pi]$.

The angle $\theta = \pi/3$ is the only angle in $[0,\pi]$, so

$$\cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$$

To get the inverses for the reciprocal functions, you do the same thing, but we'll take the reciprocal of what's in the parentheses and then use the "normal" trig functions. For example, to get $\sec^{-1}(2)$, we have to look for $\cos^{-1}(1/2)$, which is $\pi/3$ and $5\pi/3$. But because we're dealing with the inverse cosine function, we only want an angle in the interval $[0,\pi]$. Therefore, $\sec^{-1}(2) = \cos^{-1}(1/2) = \pi/3$, or 60°.

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Trig functions of inverse trig functions

Previously in an Algebra course, you may have studied different kinds of inverse relationships.

For example, exponents and radicals are inverses. So taking the square root of something is inverse to raising it to the power of 2, and vice versa. Those operations undo each other.

$$\sqrt{x^2} = (\sqrt{x})^2 = x$$

Similarly, exponential and log functions undo each other, because they're inverses. So taking the natural log of something is inverse to raising something to the base e, or vice versa.

$$e^{\ln x} = \ln(e^x) = x$$

Inverse operations with trig functions

Trig functions and inverse trig functions are the same way. So taking the sine of something is inverse to taking the inverse sine of something, which means those operations undo each other.

$\sin^{-1}(\sin x) = x$ for $x = [-\pi/2,\pi/2]$

$\sin(\sin^{-1} x) = x$ for $x = [-1,1]$

The same is true for all six of the trig functions.

$\cos^{-1}(\cos x) = x$ for $x = [0,\pi]$

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$\cos(\cos^{-1} x) = x$	for $x = [-1,1]$
$\tan^{-1}(\tan x) = x$	for $x = (-\pi/2,\pi/2)$
$\tan(\tan^{-1} x) = x$	for $x = (-\infty, \infty)$
$\csc^{-1}(\csc x) = x$	for $x = [-\pi/2,0)$ or $x = (0,\pi/2]$
$\csc(\csc^{-1} x) = x$	for $x = (-\infty, -1]$ or $x = [1,\infty)$
$\sec^{-1}(\sec x) = x$	for $x = [0,\pi/2)$ or $x = (\pi/2,\pi]$
$\sec(\sec^{-1} x) = x$	for $x = (-\infty, -1]$ or $x = [1,\infty)$
$\cot^{-1}(\cot x) = x$	for $x = (0,\pi)$
$\cot(\cot^{-1} x) = x$	for $x = (-\infty, \infty)$

But we can also pair together trig functions and inverse trig functions that don't "match." For instance, we can calculate $\cos(\sin^{-1} x)$, $\cot(\cos^{-1} x)$, or $\csc(\tan^{-1} x)$, etc.

Below is a table of formulas showing how we calculate every trig function of every inverse trig function. Keep in mind that it only gives values for the trig function of an inverse trig function, not the other way around. In other words, the second row third column shows $\cos(\tan^{-1} x)$. There is no cell in the table that gives $\tan^{-1}(\cos x)$.

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	$\sin^{-1}x$	$\cos^{-1}x$	$\tan^{-1}x$	$\csc^{-1}x$	$\sec^{-1}x$	$\cot^{-1}x$
sin of	x	$\sqrt{1-x^2}$	$\frac{x}{\sqrt{x^2+1}}$	$\frac{1}{x}$	$\sqrt{1-\frac{1}{x^2}}$	$-\frac{1}{x\sqrt{\frac{1}{x^2}+1}}$
cos of	$\sqrt{1-x^2}$	x	$\frac{1}{\sqrt{x^2+1}}$	$\sqrt{1-\frac{1}{x^2}}$	$\frac{1}{x}$	$\frac{1}{\sqrt{\frac{1}{x^2}+1}}$
tan of	$\frac{x}{\sqrt{1-x^2}}$	$\frac{\sqrt{1-x^2}}{x}$	x	$\frac{1}{x\sqrt{1-\frac{1}{x^2}}}$	$x\sqrt{1-\frac{1}{x^2}}$	$\frac{1}{x}$
csc of	$\frac{1}{x}$	$\frac{1}{\sqrt{1-x^2}}$	$\frac{\sqrt{x^2+1}}{x}$	x	$\frac{1}{\sqrt{1-\frac{1}{x^2}}}$	$x\sqrt{\frac{1}{x^2}+1}$
sec of	$\frac{1}{\sqrt{1-x^2}}$	$\frac{1}{x}$	$\sqrt{x^2+1}$	$\frac{1}{\sqrt{1-\frac{1}{x^2}}}$	x	$\sqrt{\frac{1}{x^2}+1}$
cot of	$\frac{\sqrt{1-x^2}}{x}$	$\frac{x}{\sqrt{1-x^2}}$	$\frac{1}{x}$	$x\sqrt{1-\frac{1}{x^2}}$	$\frac{1}{x\sqrt{1-\frac{1}{x^2}}}$	x

It would be really difficult to remember all these formulas, but they're fairly easy to build from what we know about right triangles.

For instance, consider $\sin(\cos^{-1}x)$, which we see in the first row and second column of the table. We can rewrite about $\cos^{-1}x$ as

$$\cos^{-1}\left(\frac{x}{1}\right)$$

Because this is the inverse cosine function, the input $x/1$ represents adjacent/hypotenuse, and the result of $\cos^{-1}(x/1)$ will be the angle within the triangle.

So if the adjacent side of the triangle is given by x, and the hypotenuse of the triangle is given by 1, then the opposite side of the triangle, by the Pythagorean Theorem, must be

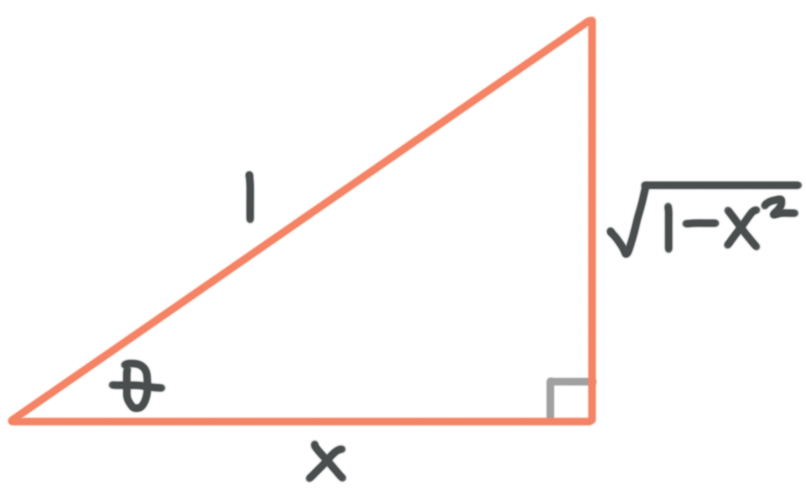
$$a^2 + b^2 = c^2$$

$$x^2 + b^2 = 1^2$$

$$b^2 = 1 - x^2$$

$$b = \sqrt{1 - x^2}$$

Now we know that the triangle we're describing has adjacent side x, opposite side $\sqrt{1 - x^2}$, and hypotenuse 1,



we can find the sine of the interior angle of that triangle. Because sine is equivalent to opposite/hypotenuse, we get

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$$\frac{\text{opposite}}{\text{hypotenuse}} = \frac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2}$$

Therefore, $\sin(\cos^{-1} x) = \sqrt{1-x^2}$.

Let's do another example where we build another of these formulas.

Example

Find the value of $\sec(\cot^{-1} x)$.

Set $\theta = \cot^{-1} x$. Then we can say

$$\theta = \cot^{-1}\left(\frac{x}{1}\right)$$

$$\theta = \cot^{-1}\left(\frac{x = \text{adjacent}}{1 = \text{opposite}}\right)$$

Given a triangle with adjacent leg x and opposite leg 1, the hypotenuse must be

$$a^2 + b^2 = c^2$$

$$x^2 + 1^2 = c^2$$

$$c^2 = 1 + x^2$$

$$c = \sqrt{1+x^2}$$

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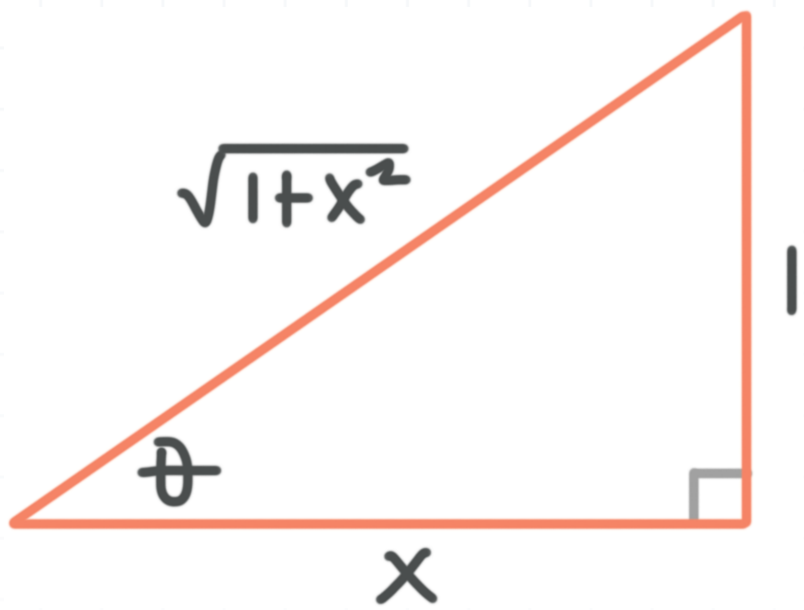
Now we know that the triangle we're describing has adjacent leg x, opposite leg 1, and hypotenuse $\sqrt{1+x^2}$,



we can find the secant of the interior angle of that triangle. Because secant is equivalent to hypotenuse/adjacent, we get

$$\frac{\text{hypotenuse}}{\text{adjacent}} = \frac{\sqrt{1+x^2}}{x}$$

This expression represents $\sec(\cot^{-1} x)$, and now we just need to simplify it by putting the entire fraction under one square root. To do that, we can rewrite x as $\sqrt{x^2}$.

$$\sec(\cot^{-1} x) = \frac{\sqrt{1+x^2}}{\sqrt{x^2}}$$

When both the numerator and denominator of a fraction are under a root, we can simplify by taking the root of the whole fraction.

$$\sec(\cot^{-1} x) = \sqrt{\frac{1+x^2}{x^2}}$$

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$$\sec(\cot^{-1} x) = \sqrt{\frac{1}{x^2} + \frac{x^2}{x^2}}$$

$$\sec(\cot^{-1} x) = \sqrt{\frac{1}{x^2} + 1}$$

Now let's work through an example where we evaluate one of these values at a particular point.

Example

Find the value of the expression.

$$\tan\left(\cos^{-1}\left(-\frac{3}{5}\right)\right)$$

Let θ represent the angle in $[0,\pi]$ (because this is the range of the inverse cosine function) whose cosine is $-3/5$. Then we can say

$$\theta = \cos^{-1}\left(-\frac{3}{5}\right)$$

$$\cos\theta = -\frac{3}{5}$$

If θ is in $[0,\pi]$, it must be in the first or second quadrant. The value of $\cos\theta$ is positive in the first quadrant, and negative in the second quadrant, so because $\cos\theta$ is negative, θ can only be in the second quadrant.

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If we imagine our right triangle in the second quadrant, because cosine is always equal to adjacent/hypotenuse, the adjacent side is -3 and the hypotenuse is 5. We can use the Pythagorean Theorem to find the length of the opposite side.

$$a^2 + b^2 = c^2$$

$$(-3)^2 + b^2 = 5^2$$

$$b^2 = 25 - 9$$

$$b^2 = 16$$

$$b = 4$$

Because tangent is equivalent to opposite/adjacent, tangent must be

$$\frac{\text{opposite}}{\text{adjacent}} = \frac{4}{-3} = -\frac{4}{3}$$

so

$$\tan\left(\cos^{-1}\left(-\frac{3}{5}\right)\right) = \tan\theta = -\frac{4}{3}$$

This value also matches what we would have found using the formula from the table for $\tan(\cos^{-1} x)$,

$$\frac{\sqrt{1-x^2}}{x}$$

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Let's do one more example.

Example

Find the value of the expression.

$$\sin^{-1}\left(\sin\left(\frac{4\pi}{3}\right)\right)$$

The inverse property $\sin^{-1}(\sin x) = x$ applies for every x in $[-\pi/2,\pi/2]$. The value $x = 4\pi/3$ does not lie in $[-\pi/2,\pi/2]$. To evaluate this expression, we first need to find $\sin(4\pi/3)$.

$$\sin\left(\frac{4\pi}{3}\right) = -\frac{\sqrt{3}}{2}$$

The angle in $[-\pi/2,\pi/2]$ whose sine is $-\sqrt{3}/2$ is $-\pi/3$.

Therefore,

$$\sin^{-1}\left(\sin\left(\frac{4\pi}{3}\right)\right) = \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) = -\frac{\pi}{3}$$

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Sum-difference identities for sine and cosine

Previously, we introduced the reciprocal identities, quotient identities, Pythagorean identities, and the even-odd identities (which are sometimes called the negative-angle identities). Together, all of these make up the **fundamental identities**.

Now we want to look at some of the other trig identities that we'll use to solve trig equations, starting in this lesson with the sum-difference identities (also called the addition and subtraction identities) for sine and cosine.

The sum-difference identities for sine and cosine

When we want to find the sine or cosine of the sum or difference of two angles, we can use the sum-difference identities. For two angles θ and α, the sum-difference identities for sine are

$$\sin(\theta + \alpha) = \sin\theta\cos\alpha + \cos\theta\sin\alpha$$

$$\sin(\theta - \alpha) = \sin\theta\cos\alpha - \cos\theta\sin\alpha$$

and the sum-difference identities for cosine are

$$\cos(\theta + \alpha) = \cos\theta\cos\alpha - \sin\theta\sin\alpha$$

$$\cos(\theta - \alpha) = \cos\theta\cos\alpha + \sin\theta\sin\alpha$$

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These identities are especially helpful for finding sine and cosine of angles that aren't directly represented on the unit circle.

Remember that the unit circle only shows angles that are multiples of $\pi/6$, like $\pi/6$, $\pi/3$, $\pi/2$, $2\pi/3$, etc., and angles that are multiples of $\pi/4$, like $\pi/4$, $\pi/2$, $3\pi/4$, π, etc.

Because the unit circle only gives us this finite set of angles, up to now we haven't had a way to calculate sine and cosine of an angle like $\pi/12$. But if we realize that $\pi/12$ is equivalent to

$$\frac{\pi}{3}-\frac{\pi}{4}$$

$$\frac{\pi}{3}\left(\frac{4}{4}\right)-\frac{\pi}{4}\left(\frac{3}{3}\right)$$

$$\frac{4\pi}{12}-\frac{3\pi}{12}$$

$$\frac{\pi}{12}$$

And the unit circle *does* give us the values of $\pi/3$ and $\pi/4$. So even though we can't get $\pi/12$ from the unit circle directly, we can express it as the difference $(\pi/3)-(\pi/4)$. Then the sine and cosine of $\pi/12$ can be calculated as

$$\sin\frac{\pi}{12}=\sin\left(\frac{\pi}{3}-\frac{\pi}{4}\right)$$

$$\sin\frac{\pi}{12}=\left(\sin\frac{\pi}{3}\right)\left(\cos\frac{\pi}{4}\right)-\left(\cos\frac{\pi}{3}\right)\left(\sin\frac{\pi}{4}\right)$$

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$$\sin\frac{\pi}{12}=\left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)-\left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right)$$

$$\sin\frac{\pi}{12}=\frac{\sqrt{6}}{4}-\frac{\sqrt{2}}{4}$$

$$\sin\frac{\pi}{12}=\frac{\sqrt{6}-\sqrt{2}}{4}$$

and

$$\cos\frac{\pi}{12}=\cos\left(\frac{\pi}{3}-\frac{\pi}{4}\right)$$

$$\cos\frac{\pi}{12}=\left(\cos\frac{\pi}{3}\right)\left(\cos\frac{\pi}{4}\right)+\left(\sin\frac{\pi}{3}\right)\left(\sin\frac{\pi}{4}\right)$$

$$\cos\frac{\pi}{12}=\left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right)+\left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)$$

$$\cos\frac{\pi}{12}=\frac{\sqrt{2}}{4}+\frac{\sqrt{6}}{4}$$

$$\cos\frac{\pi}{12}=\frac{\sqrt{2}+\sqrt{6}}{4}$$

Let’s do another example with some more sine and cosine values.

Example

Find the exact values of $\cos(7\pi/12)$ and $\sin(7\pi/12)$.

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From just the unit circle, we wouldn't know the values of sine and cosine at $7\pi/12$, but we can rewrite $7\pi/12$ as

$$\frac{7\pi}{12}=\frac{(4+3)\pi}{12}=\frac{4\pi}{12}+\frac{3\pi}{12}=\frac{\pi}{3}+\frac{\pi}{4}$$

Therefore, by the sum identity for the cosine function,

$$\cos\frac{7\pi}{12}=\cos\left(\frac{\pi}{3}+\frac{\pi}{4}\right)$$

$$\cos\frac{7\pi}{12}=\left(\cos\frac{\pi}{3}\right)\left(\cos\frac{\pi}{4}\right)-\left(\sin\frac{\pi}{3}\right)\left(\sin\frac{\pi}{4}\right)$$

$$\cos\frac{7\pi}{12}=\left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right)-\left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)$$

$$\cos\frac{7\pi}{12}=\frac{\sqrt{2}}{4}-\frac{\sqrt{6}}{4}$$

$$\cos\frac{7\pi}{12}=\frac{\sqrt{2}-\sqrt{6}}{4}$$

By the sum identity for the sine function,

$$\sin\frac{7\pi}{12}=\sin\left(\frac{\pi}{3}+\frac{\pi}{4}\right)$$

$$\sin\frac{7\pi}{12}=\left(\sin\frac{\pi}{3}\right)\left(\cos\frac{\pi}{4}\right)+\left(\cos\frac{\pi}{3}\right)\left(\sin\frac{\pi}{4}\right)$$

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$$\sin\frac{7\pi}{12} = \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) + \left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right)$$

$$\sin\frac{7\pi}{12} = \frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4}$$

$$\sin\frac{7\pi}{12} = \frac{\sqrt{6}+\sqrt{2}}{4}$$

Sometimes we won't have both the sine and cosine of the two angles we're using in the sum difference identities. Let's do an example where we only know the sine of one angle, the cosine of the angle, and the quadrants of both angles.

Example

Suppose θ is an angle in the fourth quadrant whose cosine is $2\sqrt{5}/5$, and α is an angle in the second quadrant whose sine is $1/5$. Find the exact values of $\sin\theta$, $\cos\alpha$, $\cos(\theta-\alpha)$, and $\sin(\theta+\alpha)$.

We're starting with two angles, θ and α. We have the cosine for θ and we have the sine for α. So we need to start by finding the sine for θ and the cosine for α.

Remember that we can always find cosine of an angle when we know the sine and the quadrant, and we can always find the sine of an angle when we know the cosine and the quadrant.

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To do that, we'll rewrite the Pythagorean identity for sine and cosine as

$$\sin^2\theta + \cos^2\theta = 1$$

$$\sin^2\theta = 1 - \cos^2\theta$$

Let's start by working on sine of the angle θ. From the information we were given, we'll substitute $\cos\theta = 2\sqrt{5}/5$ to get

$$\sin^2\theta = 1 - \left(\frac{2\sqrt{5}}{5}\right)^2$$

$$\sin^2\theta = 1 - \frac{4(5)}{25}$$

$$\sin^2\theta = 1 - \frac{4}{5}$$

$$\sin^2\theta = \frac{1}{5}$$

$$\sin\theta = \pm\sqrt{\frac{1}{5}}$$

Since θ is in the fourth quadrant, we know that $\sin\theta$ is negative. So we can ignore the positive value and say

$$\sin\theta = -\sqrt{\frac{1}{5}} = -\frac{\sqrt{1}}{\sqrt{5}} = -\frac{1}{\sqrt{5}} = -\frac{1}{\sqrt{5}}\left(\frac{\sqrt{5}}{\sqrt{5}}\right) = -\frac{\sqrt{5}}{5}$$

Now we'll work on cosine of the angle α. Again by the Pythagorean identity,

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$$\sin^2\alpha + \cos^2\alpha = 1$$

$$\cos^2\alpha = 1 - \sin^2\alpha$$

Substituting $\sin\alpha = 1/5$, we get

$$\cos^2\alpha = 1 - \left(\frac{1}{5}\right)^2$$

$$\cos^2\alpha = 1 - \frac{1}{25}$$

$$\cos^2\alpha = \frac{24}{25}$$

$$\cos\alpha = \pm\sqrt{\frac{24}{25}}$$

Since α is in the second quadrant, we know that $\cos\alpha$ is negative. So we can ignore the positive value and say that

$$\cos\alpha = -\sqrt{\frac{24}{25}} = -\frac{\sqrt{24}}{\sqrt{25}} = -\frac{2\sqrt{6}}{5}$$

Now we have both sine and cosine for both θ and α:

$$\sin\theta = -\frac{\sqrt{5}}{5}$$

$$\cos\theta = \frac{2\sqrt{5}}{5}$$

$$\sin\alpha = \frac{1}{5}$$

$$\cos\alpha = -\frac{2\sqrt{6}}{5}$$

Which means of the values we were asked to find, $\sin\theta$, $\cos\alpha$, $\cos(\theta-\alpha)$, and $\sin(\theta+\alpha)$, we've already found $\sin\theta$ and $\cos\alpha$. Now we just need to find $\cos(\theta-\alpha)$ and $\sin(\theta+\alpha)$, which we can get from the sum-difference identities.

By the difference identity for the cosine function,

$$\cos(\theta-\alpha) = \cos\theta\cos\alpha + \sin\theta\sin\alpha$$

$$\cos(\theta-\alpha) = \left(\frac{2\sqrt{5}}{5}\right)\left(-\frac{2\sqrt{6}}{5}\right) + \left(-\frac{\sqrt{5}}{5}\right)\left(\frac{1}{5}\right)$$

$$\cos(\theta-\alpha) = -\frac{4\sqrt{30}}{25} - \frac{\sqrt{5}}{25}$$

$$\cos(\theta-\alpha) = -\frac{4\sqrt{30}+\sqrt{5}}{25}$$

By the sum identity for the sine function,

$$\sin(\theta+\alpha) = \sin\theta\cos\alpha + \cos\theta\sin\alpha$$

$$\sin(\theta+\alpha) = \left(-\frac{\sqrt{5}}{5}\right)\left(-\frac{2\sqrt{6}}{5}\right) + \left(\frac{2\sqrt{5}}{5}\right)\left(\frac{1}{5}\right)$$

$$\sin(\theta+\alpha) = \frac{2\sqrt{30}}{25} + \frac{2\sqrt{5}}{25}$$

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$$\sin(\theta + \alpha) = \frac{2\sqrt{30} + 2\sqrt{5}}{25}$$

Cofunction identities

Next, we'll look at the cofunction identities, which give us a relationship between sine and cosine, a relationship between secant and cosecant, and a relationship between tangent and cotangent.

If we take the graph of $y = \sin x$ and shift it to the left by $\pi/2$ units, it looks exactly like the graph of $y = \cos x$. The same is true for tangent and cotangent, as well as for secant and cosecant. That's the basic premise of co-function identities — the sine and cosine functions take on the same values, but those values are shifted slightly on the coordinate plane when we look at one function compared to the other.

In degrees, the cofunction identities are

$$\sin\theta = \cos(90^\circ - \theta) \qquad \csc\theta = \sec(90^\circ - \theta)$$

$$\cos\theta = \sin(90^\circ - \theta) \qquad \sec\theta = \csc(90^\circ - \theta)$$

$$\tan\theta = \cot(90^\circ - \theta) \qquad \cot\theta = \tan(90^\circ - \theta)$$

and of course, in radians these identities are

$$\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right) \qquad \csc\theta = \sec\left(\frac{\pi}{2} - \theta\right)$$

$$\cos\theta = \sin\left(\frac{\pi}{2} - \theta\right) \qquad \sec\theta = \csc\left(\frac{\pi}{2} - \theta\right)$$

$$\tan\theta = \cot\left(\frac{\pi}{2} - \theta\right) \qquad \cot\theta = \tan\left(\frac{\pi}{2} - \theta\right)$$

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Notice how sine and co(sine) are cofunctions, tangent and co(tangent) are cofunctions, and secant and co(secant) are cofunctions. The fact that the functions are named the way they are makes it really easy to remember which pairs of trig functions are cofunctions.

The value of a trigonometric function of an angle is equal to the value of the cofunction of the angle's complement. Remember from geometry that complementary angles are angles that sum to 90°.

These identities are useful when we know sine of an angle and want to find cosine, or vice versa, when we know tangent of an angle and want to find cotangent, or vice versa, or when we know secant of an angle and want to find cosecant, or vice versa.

Let's do an example where we use a cofunction identity to find the value of a trig function from the value of another trig function at the same angle.

Example

Find an angle θ that satisfies the equation.

$$\sec\frac{3\pi}{4} = \csc\theta$$

The equation we're given tells us that the cosecant of some angle is equivalent to secant of $3\pi/4$. Secant and cosecant are cofunctions, which means we can plug into the cofunction identity for secant that relates them.

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$$\sec\theta = \csc\left(\frac{\pi}{2} - \theta\right)$$

$$\sec\frac{3\pi}{4} = \csc\left(\frac{\pi}{2} - \frac{3\pi}{4}\right)$$

Find a common denominator.

$$\sec\frac{3\pi}{4} = \csc\left(\frac{\pi}{2}\left(\frac{2}{2}\right) - \frac{3\pi}{4}\right)$$

$$\sec\frac{3\pi}{4} = \csc\left(\frac{2\pi}{4} - \frac{3\pi}{4}\right)$$

$$\sec\frac{3\pi}{4} = \csc\left(-\frac{\pi}{4}\right)$$

So the angle θ that satisfies the equation is $\theta = -\pi/4$. And this result tells us that secant of the angle $3\pi/4$ has the same value as cosecant of the angle $-\pi/4$.

Let's do one more example where we use a cofunction identity for sine and cosine.

Example

Find an angle θ that satisfies the equation.

$$\sin\left(-\frac{\pi}{6}\right) = \cos\theta$$

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The equation we're given tells us that the cosine of some angle is equivalent to sine of $-\pi/6$. Sine and cosine are cofunctions, which means we can plug into the cofunction identity for sine that relates them.

$$\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$$

$$\sin\left(-\frac{\pi}{6}\right) = \cos\left(\frac{\pi}{2} - \left(-\frac{\pi}{6}\right)\right)$$

$$\sin\left(-\frac{\pi}{6}\right) = \cos\left(\frac{\pi}{2} + \frac{\pi}{6}\right)$$

Find a common denominator.

$$\sin\left(-\frac{\pi}{6}\right) = \cos\left(\frac{\pi}{2}\left(\frac{3}{3}\right) + \frac{\pi}{6}\right)$$

$$\sin\left(-\frac{\pi}{6}\right) = \cos\left(\frac{3\pi}{6} + \frac{\pi}{6}\right)$$

$$\sin\left(-\frac{\pi}{6}\right) = \cos\frac{4\pi}{6}$$

$$\sin\left(-\frac{\pi}{6}\right) = \cos\frac{2\pi}{3}$$

So the angle θ that satisfies the equation is $\theta = 2\pi/3$. And this result tells us that sine of the angle $-\pi/6$ has the same value as cosine of the angle $2\pi/3$.

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Sum-difference identities for tangent

In the same way that we had sum-difference identities for the sine and cosine functions, we also have a sum identity and difference identity for the tangent function:

$$\tan(\theta+\alpha)=\frac{\tan\theta+\tan\alpha}{1-\tan\theta\tan\alpha}$$

$$\tan(\theta-\alpha)=\frac{\tan\theta-\tan\alpha}{1+\tan\theta\tan\alpha}$$

Because $\tan\theta=\sin\theta/\cos\theta$, we can actually build these sum-difference identities for tangent by dividing the sine identity by the cosine identity. Here's how we build the sum identity for tangent,

$$\tan(\theta+\alpha)=\frac{\sin(\theta+\alpha)}{\cos(\theta+\alpha)}$$

$$\tan(\theta+\alpha)=\frac{\sin\theta\cos\alpha+\cos\theta\sin\alpha}{\cos\theta\cos\alpha-\sin\theta\sin\alpha}$$

$$\tan(\theta+\alpha)=\frac{\sin\theta\cos\alpha+\cos\theta\sin\alpha}{\cos\theta\cos\alpha-\sin\theta\sin\alpha}\cdot\frac{\frac{1}{\cos\theta\cos\alpha}}{\frac{1}{\cos\theta\cos\alpha}}$$

$$\tan(\theta+\alpha)=\frac{\frac{\sin\theta\cos\alpha+\cos\theta\sin\alpha}{\cos\theta\cos\alpha}}{\frac{\cos\theta\cos\alpha-\sin\theta\sin\alpha}{\cos\theta\cos\alpha}}$$

$$\tan(\theta+\alpha)=\frac{\frac{\sin\theta\cos\alpha}{\cos\theta\cos\alpha}+\frac{\cos\theta\sin\alpha}{\cos\theta\cos\alpha}}{\frac{\cos\theta\cos\alpha}{\cos\theta\cos\alpha}-\frac{\sin\theta\sin\alpha}{\cos\theta\cos\alpha}}$$

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$$\tan(\theta+\alpha)=\frac{\frac{\sin\theta}{\cos\theta}+\frac{\sin\alpha}{\cos\alpha}}{1-\frac{\sin\theta\sin\alpha}{\cos\theta\cos\alpha}}$$

$$\tan(\theta+\alpha)=\frac{\tan\theta+\tan\alpha}{1-\tan\theta\tan\alpha}$$

and here's how we build the difference identity for tangent.

$$\tan(\theta-\alpha)=\frac{\sin(\theta-\alpha)}{\cos(\theta-\alpha)}$$

$$\tan(\theta-\alpha)=\frac{\sin\theta\cos\alpha-\cos\theta\sin\alpha}{\cos\theta\cos\alpha+\sin\theta\sin\alpha}$$

$$\tan(\theta-\alpha)=\frac{\sin\theta\cos\alpha-\cos\theta\sin\alpha}{\cos\theta\cos\alpha+\sin\theta\sin\alpha}\cdot\frac{\frac{1}{\cos\theta\cos\alpha}}{\frac{1}{\cos\theta\cos\alpha}}$$

$$\tan(\theta-\alpha)=\frac{\frac{\sin\theta\cos\alpha-\cos\theta\sin\alpha}{\cos\theta\cos\alpha}}{\frac{\cos\theta\cos\alpha+\sin\theta\sin\alpha}{\cos\theta\cos\alpha}}$$

$$\tan(\theta-\alpha)=\frac{\frac{\sin\theta\cos\alpha}{\cos\theta\cos\alpha}-\frac{\cos\theta\sin\alpha}{\cos\theta\cos\alpha}}{\frac{\cos\theta\cos\alpha}{\cos\theta\cos\alpha}+\frac{\sin\theta\sin\alpha}{\cos\theta\cos\alpha}}$$

$$\tan(\theta-\alpha)=\frac{\frac{\sin\theta}{\cos\theta}-\frac{\sin\alpha}{\cos\alpha}}{1+\frac{\sin\theta\sin\alpha}{\cos\theta\cos\alpha}}$$

$$\tan(\theta-\alpha)=\frac{\tan\theta-\tan\alpha}{1+\tan\theta\tan\alpha}$$

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Alternatively, to determine the difference identity for tangent, we can use the even-odd identity $\tan(-\theta) = -\tan(\theta)$. Substitute $-\alpha$ for α into the sum identity for tangent.

$$\tan(\theta + \alpha) = \frac{\tan\theta + \tan\alpha}{1 - \tan\theta\tan\alpha}$$

$$\tan(\theta + (-\alpha)) = \frac{\tan\theta + \tan(-\alpha)}{1 - \tan\theta\tan(-\alpha)}$$

$$\tan(\theta - \alpha) = \frac{\tan\theta - \tan\alpha}{1 - \tan\theta(-\tan\alpha)}$$

$$\tan(\theta - \alpha) = \frac{\tan\theta - \tan\alpha}{1 + \tan\theta\tan\alpha}$$

Let's do an example where we know the cosines of two different angles, and the quadrants of each angle, and need to use that information to find the tangent of both the sum and difference.

Example

Find the exact values of $\tan(\theta + \alpha)$ and $\tan(\theta - \alpha)$ if θ is an angle in the third quadrant whose cosine is $-2\sqrt{7}/7$ and α is an angle in the first quadrant whose cosine is $4/7$.

Before we can find the values of $\tan(\theta + \alpha)$ and $\tan(\theta - \alpha)$, we need to find the values of $\sin\theta$ and $\sin\alpha$, because right now we only have the cosines. To find the sines from the cosines, we'll rewrite the Pythagorean identity with sine and cosine,

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$$\sin^2\theta + \cos^2\theta = 1$$

$$\sin^2\theta = 1 - \cos^2\theta$$

and then substitute $\cos\theta = -2\sqrt{7}/7$.

$$\sin^2\theta = 1 - \left(-\frac{2\sqrt{7}}{7}\right)^2$$

$$\sin^2\theta = 1 - \frac{4(7)}{49}$$

$$\sin^2\theta = 1 - \frac{4}{7}$$

$$\sin^2\theta = \frac{3}{7}$$

$$\sin\theta = \pm\sqrt{\frac{3}{7}}$$

Since θ is in the third quadrant, we know that $\sin\theta$ is negative. So we can ignore the positive value and say

$$\sin\theta = -\sqrt{\frac{3}{7}} = -\frac{\sqrt{3}}{\sqrt{7}} = -\frac{\sqrt{3}\sqrt{7}}{7} = -\frac{\sqrt{21}}{7}$$

Now we need to find the value of $\sin\alpha$. Again by the Pythagorean identity,

$$\sin^2\alpha + \cos^2\alpha = 1$$

$$\sin^2\alpha = 1 - \cos^2\alpha$$

Substituting $\cos\alpha = 4/7$, we get

$$\sin^2\alpha = 1 - \left(\frac{4}{7}\right)^2$$

$$\sin^2\alpha = 1 - \frac{16}{49}$$

$$\sin^2\alpha = \frac{33}{49}$$

$$\sin\alpha = \pm\sqrt{\frac{33}{49}}$$

Since α is in the first quadrant, we know that $\sin\alpha$ is positive. So we can ignore the negative value and say

$$\sin\alpha = \sqrt{\frac{33}{49}} = \frac{\sqrt{33}}{\sqrt{49}} = \frac{\sqrt{33}}{7}$$

Now we can find the values of $\tan\theta$ and $\tan\alpha$. Using the definition of the tangent function, we get

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{-\frac{\sqrt{21}}{7}}{-\frac{2\sqrt{7}}{7}} = \frac{\sqrt{21}}{7}\cdot\frac{7}{2\sqrt{7}} = \frac{\sqrt{21}}{2\sqrt{7}} = \frac{\sqrt{7}\sqrt{3}}{2\sqrt{7}} = \frac{\sqrt{3}}{2}$$

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \frac{\frac{\sqrt{33}}{7}}{\frac{4}{7}} = \frac{\sqrt{33}}{7}\cdot\frac{7}{4} = \frac{\sqrt{33}}{4}$$

By the sum identity for the tangent function,

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$$\tan(\theta+\alpha)=\frac{\tan\theta+\tan\alpha}{1-\tan\theta\tan\alpha}$$

$$\tan(\theta+\alpha)=\frac{\frac{\sqrt{3}}{2}+\frac{\sqrt{33}}{4}}{1-\left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{33}}{4}\right)}$$

$$\tan(\theta+\alpha)=\frac{\frac{2\sqrt{3}+\sqrt{33}}{4}}{1-\frac{\sqrt{99}}{8}}$$

$$\tan(\theta+\alpha)=\frac{\frac{2\sqrt{3}+\sqrt{3}\sqrt{11}}{4}}{1-\frac{3\sqrt{11}}{8}}$$

$$\tan(\theta+\alpha)=\frac{\frac{\sqrt{3}(2+\sqrt{11})}{4}}{\frac{8-3\sqrt{11}}{8}}$$

$$\tan(\theta+\alpha)=\frac{\sqrt{3}(2+\sqrt{11})}{4}\cdot\frac{8}{8-3\sqrt{11}}$$

$$\tan(\theta+\alpha)=\frac{8\sqrt{3}(2+\sqrt{11})}{4(8-3\sqrt{11})}$$

$$\tan(\theta+\alpha)=\frac{2\sqrt{3}(2+\sqrt{11})}{8-3\sqrt{11}}$$

By the difference identity for the tangent function,

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$$\tan(\theta - \alpha) = \frac{\tan\theta - \tan\alpha}{1 + \tan\theta\tan\alpha}$$

$$\tan(\theta - \alpha) = \frac{\frac{\sqrt{3}}{2} - \frac{\sqrt{33}}{4}}{1 + \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{33}}{4}\right)}$$

$$\tan(\theta - \alpha) = \frac{\frac{2\sqrt{3} - \sqrt{33}}{4}}{1 + \frac{\sqrt{99}}{8}}$$

$$\tan(\theta - \alpha) = \frac{\frac{2\sqrt{3} - \sqrt{3}\sqrt{11}}{4}}{1 + \frac{3\sqrt{11}}{8}}$$

$$\tan(\theta - \alpha) = \frac{\frac{\sqrt{3}(2 - \sqrt{11})}{4}}{\frac{8 + 3\sqrt{11}}{8}}$$

$$\tan(\theta - \alpha) = \frac{\sqrt{3}(2 - \sqrt{11})}{4} \cdot \frac{8}{8 + 3\sqrt{11}}$$

$$\tan(\theta - \alpha) = \frac{8\sqrt{3}(2 - \sqrt{11})}{4(8 + 3\sqrt{11})}$$

$$\tan(\theta - \alpha) = \frac{2\sqrt{3}(2 - \sqrt{11})}{8 + 3\sqrt{11}}$$

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Double-angle identities

Very often we'll handle trig functions with an angle of 2θ, instead of just θ, like $\sin 2\theta$. We'll usually want to convert these to get the 2 out of the angle.

But we can't simply say $\sin 2\theta = 2\sin\theta$; that's just not true. 2θ is the argument of the sine function; we're evaluating the sine function at the angle 2θ, so the $\sin$ and 2θ aren't simply multiplied together such that we can pull the 2 out in front of the sine function.

The way we'll get the 2 out of the argument is by using the double-angle identities, which are actually derived from the sum identities we learned about earlier.

Double-angle identities from the sum identities

To build the double-angle identity for sine, we start with the sum identity for sine, $\sin(\theta+\alpha) = \sin\theta\cos\alpha + \cos\theta\sin\alpha$, and replace α with θ.

$$\sin(\theta+\alpha) = \sin\theta\cos\alpha + \cos\theta\sin\alpha$$

$$\sin(\theta+\theta) = \sin\theta\cos\theta + \cos\theta\sin\theta$$

$$\sin(2\theta) = \sin\theta\cos\theta + \sin\theta\cos\theta$$

$$\sin(2\theta) = 2\sin\theta\cos\theta$$

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Similarly, we build the double-angle identity for cosine by starting with the sum identity for cosine, $\cos(\theta+\alpha)=\cos\theta\cos\alpha-\sin\theta\sin\alpha$, and replacing α with θ.

$$\cos(\theta+\alpha)=\cos\theta\cos\alpha-\sin\theta\sin\alpha$$

$$\cos(\theta+\theta)=\cos\theta\cos\theta-\sin\theta\sin\theta$$

$$\cos 2\theta=\cos^2\theta-\sin^2\theta$$

Keep in mind that we'll also often see this double-angle cosine identity rewritten using the Pythagorean identity with sine and cosine. From the Pythagorean identity, we can rewrite the double-angle cosine identity by substituting $\cos^2\theta=1-\sin^2\theta$,

$$\cos 2\theta=(1-\sin^2\theta)-\sin^2\theta$$

$$\cos 2\theta=1-\sin^2\theta-\sin^2\theta$$

$$\cos 2\theta=1-2\sin^2\theta$$

or by substituting $\sin^2\theta=1-\cos^2\theta$.

$$\cos 2\theta=\cos^2\theta-(1-\cos^2\theta)$$

$$\cos 2\theta=\cos^2\theta-1+\cos^2\theta$$

$$\cos 2\theta=2\cos^2\theta-1$$

And to build the double-angle identity for tangent, we'll again start with the sum identity for tangent and replace α with θ.

$$\tan(\theta+\alpha)=\frac{\tan\theta+\tan\alpha}{1-\tan\theta\tan\alpha}$$

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$$\tan(\theta + \theta) = \frac{\tan\theta + \tan\theta}{1 - \tan\theta\tan\theta}$$

$$\tan(2\theta) = \frac{2\tan\theta}{1 - \tan^2\theta}$$

To summarize, the **double-angle identities**, including the alternate forms of the double-angle identity for cosine, are

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$$

$$\cos 2\theta = 1 - 2\sin^2\theta$$

$$\cos 2\theta = 2\cos^2\theta - 1$$

All of these give us a way to change the angle 2θ into the angle θ, which we'll be something we'll want to do all the time in Trigonometry and beyond.

Let's do an example with the double-angle cosine identity.

Example

If θ is an angle in the second quadrant with $\cos\theta = -\sqrt{5}/6$, find $\cos 2\theta$ and $\sin 2\theta$.

By the double-angle identity for cosine,

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

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In order to use the double-angle identity, we first need to find $\sin^2\theta$. By the basic Pythagorean identity,

$$\sin^2\theta + \cos^2\theta = 1$$

$$\sin^2\theta = 1 - \cos^2\theta$$

Since $\cos\theta = -\sqrt{5}/6$, we get

$$\sin^2\theta = 1 - \left(-\frac{\sqrt{5}}{6}\right)^2$$

$$\sin^2\theta = 1 - \frac{5}{36}$$

$$\sin^2\theta = \frac{31}{36}$$

Plus, since we know that $\cos\theta = -\sqrt{5}/6$, we can say

$$\cos^2\theta = \left(-\frac{\sqrt{5}}{6}\right)^2$$

$$\cos^2\theta = \frac{5}{36}$$

Now to find $\cos 2\theta$, we'll substitute into the double-angle identity for cosine.

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$\cos 2\theta = \frac{5}{36} - \frac{31}{36}$$

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$$\cos 2\theta = -\frac{26}{36}$$

$$\cos 2\theta = -\frac{13}{18}$$

To find $\sin 2\theta$ for this same angle θ, we'll use the double-angle identity for sine.

$$\sin 2\theta = 2\sin\theta\cos\theta$$

We've already found

$$\sin^2\theta = \frac{31}{36}$$

but we don't yet know the value of $\sin\theta$. Since θ is in the second quadrant, $\sin\theta$ is positive. Therefore,

$$\sin\theta = \sqrt{\frac{31}{36}} = \frac{\sqrt{31}}{\sqrt{36}} = \frac{\sqrt{31}}{6}$$

Now we're ready to apply the double-angle identity for sine.

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$\sin 2\theta = 2\left(\frac{\sqrt{31}}{6}\right)\left(-\frac{\sqrt{5}}{6}\right)$$

$$\sin 2\theta = -\frac{2\sqrt{31}\sqrt{5}}{36}$$

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$$\sin 2\theta = -\frac{\sqrt{155}}{18}$$

Let's look at an example where we only know the value of $\sin \theta$ and the quadrant of the angle, and we need to find $\sin 2\theta$ and $\cos 2\theta$.

Example

Find $\sin 2\theta$ and $\cos 2\theta$ for an angle θ which lies in the third quadrant and has $\sin \theta = -3/7$.

By the double-angle identity for sine,

$$\sin 2\theta = 2 \sin \theta \cos \theta$$

We already know that $\sin \theta = -3/7$, but we need to find $\cos \theta$. We'll plug into the Pythagorean identity with sine and cosine.

$$\sin^2 \theta + \cos^2 \theta = 1$$

$$\cos^2 \theta = 1 - \sin^2 \theta$$

$$\cos^2 \theta = 1 - \left(-\frac{3}{7}\right)^2$$

$$\cos^2 \theta = 1 - \frac{9}{49}$$

$$\cos^2\theta = \frac{40}{49}$$

$$\cos\theta = \pm\sqrt{\frac{40}{49}}$$

$$\cos\theta = \pm\frac{\sqrt{40}}{7}$$

Since θ is in the third quadrant, we know that $\cos\theta$ is negative, so we can ignore the positive value and say

$$\cos\theta = -\frac{\sqrt{40}}{7}$$

Now, substituting $\cos\theta = -\sqrt{40}/7$ and $\sin\theta = -3/7$ into the double-angle identity for sine, we get

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$\sin 2\theta = 2\left(-\frac{3}{7}\right)\left(-\frac{\sqrt{40}}{7}\right)$$

$$\sin 2\theta = \frac{6\sqrt{40}}{49}$$

$$\sin 2\theta = \frac{12\sqrt{10}}{49}$$

By the double-angle identity for cosine, we get

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

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$$\cos 2\theta = \left(-\frac{\sqrt{40}}{7}\right)^2 - \left(-\frac{3}{7}\right)^2$$

$$\cos 2\theta = \frac{40}{49} - \frac{9}{49}$$

$$\cos 2\theta = \frac{31}{49}$$

If we're given just the value of $\tan\theta$ for some angle θ, we can compute the value of $\tan 2\theta$ (even if we don't know the quadrant in which θ lies) by using the double-angle identity for tangent.

Example

Find $\tan 2\theta$ if $\tan\theta = \sqrt{23}$.

By the double-angle identity for tangent,

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$$

$$\tan 2\theta = \frac{2\sqrt{23}}{1-(\sqrt{23})^2}$$

$$\tan 2\theta = \frac{2\sqrt{23}}{1-23}$$

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$$\tan 2\theta = \frac{2\sqrt{23}}{-22}$$

$$\tan 2\theta = -\frac{\sqrt{23}}{11}$$

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Half-angle identities

In this lesson we're looking at the set of half-angle identities. They allow us to rewrite a trig function when the argument is $\theta/2$. They transform the argument from $\theta/2$ to just θ.

The good news is that we can build these identities directly from the double-angle identities we just learned.

Half-angle identities from the double-angle identities

To find the half-angle identity for sine, we start with the double-angle identity for cosine, $\cos 2\theta = 1 - 2\sin^2\theta$, and solve it for $\sin\theta$.

$$\cos 2\theta = 1 - 2\sin^2\theta$$

$$2\sin^2\theta + \cos 2\theta = 1$$

$$2\sin^2\theta = 1 - \cos 2\theta$$

$$\sin^2\theta = \frac{1 - \cos 2\theta}{2}$$

$$\sin\theta = \pm\sqrt{\frac{1 - \cos 2\theta}{2}}$$

Now we'll make the substitution $2\theta = \alpha$. If we solve $2\theta = \alpha$ for θ, we also get $\theta = \alpha/2$. So we'll replace θ with $\alpha/2$, and replace 2θ with α, and we get

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$$\sin\frac{\alpha}{2}=\pm\sqrt{\frac{1-\cos\alpha}{2}}$$

We build the half-angle identity for cosine in the same way, but we start with the $\cos 2\theta = 2\cos^2\theta - 1$ double-angle identity instead, solving it for $\cos\theta$.

$$\cos 2\theta = 2\cos^2\theta - 1$$

$$2\cos^2\theta = 1 + \cos 2\theta$$

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}$$

$$\cos\theta = \pm\sqrt{\frac{1+\cos 2\theta}{2}}$$

Now we'll make the substitution $2\theta = \alpha$. If we solve $2\theta = \alpha$ for θ, we also get $\theta = \alpha/2$. So we'll replace θ with $\alpha/2$, and replace 2θ with α, and we get

$$\cos\frac{\alpha}{2}=\pm\sqrt{\frac{1+\cos\alpha}{2}}$$

Then to get the half-angle identity for tangent, we'll use the half-angles we just got for sine and cosine.

$$\tan\frac{\alpha}{2}=\frac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}}$$

$$\tan\frac{\alpha}{2}=\frac{\pm\sqrt{\frac{1-\cos\alpha}{2}}}{\pm\sqrt{\frac{1+\cos\alpha}{2}}}$$

$$\tan\frac{\alpha}{2}=\frac{\pm\frac{\sqrt{1-\cos\alpha}}{\sqrt{2}}}{\pm\frac{\sqrt{1+\cos\alpha}}{\sqrt{2}}}$$

$$\tan\frac{\alpha}{2}=\pm\frac{\sqrt{1-\cos\alpha}}{\sqrt{2}}\cdot\pm\frac{\sqrt{2}}{\sqrt{1+\cos\alpha}}$$

$$\tan\frac{\alpha}{2}=\pm\frac{\sqrt{1-\cos\alpha}}{\sqrt{1+\cos\alpha}}$$

There are also two other alternate forms of this half-angle tangent identity. So if we pull together all three of the half-angle identities we've built so far, along with the two alternate forms for the tangent identity, we can summarize the **half-angle identities** as

$$\cos\frac{\theta}{2}=\pm\sqrt{\frac{1+\cos\theta}{2}}$$

$$\sin\frac{\theta}{2}=\pm\sqrt{\frac{1-\cos\theta}{2}}$$

$$\tan\frac{\theta}{2}=\pm\frac{\sqrt{1-\cos\theta}}{\sqrt{1+\cos\theta}}$$

$$\tan\frac{\theta}{2}=\frac{\sin\theta}{1+\cos\theta}$$

$$\tan\frac{\theta}{2}=\frac{1-\cos\theta}{\sin\theta}$$

Notice here that three of these half-angle identities include a ± sign. The sign we choose will depend on the quadrant of the angle. For instance, using the half-angle cosine identity, cosine is positive in the first and fourth quadrants and negative in the second and third quadrants. So if the angle

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is in the first or fourth quadrant, we'll choose the positive value of the root on the right side of that cosine identity.

$$\cos\frac{\theta}{2}=\sqrt{\frac{1+\cos\theta}{2}}$$

But if the angle is in the second or third quadrant, we'll choose the negative value of the root on the right side of that cosine identity.

$$\cos\frac{\theta}{2}=-\sqrt{\frac{1+\cos\theta}{2}}$$

Let's do an example where we find the half-angle values of sine and cosine for an angle in the third quadrant.

Example

If $\pi<\theta<3\pi/2$ and $\cos\theta=-\sqrt{6}/7$, find $\cos(\theta/2)$ and $\sin(\theta/2)$.

If we substitute $\cos\theta=-\sqrt{6}/7$ into the half-angle identity for cosine, we get

$$\cos\frac{\theta}{2}=\pm\sqrt{\frac{1+\cos\theta}{2}}$$

$$\cos\frac{\theta}{2}=\pm\sqrt{\frac{1+\left(-\frac{\sqrt{6}}{7}\right)}{2}}$$

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$$\cos\frac{\theta}{2}=\pm\sqrt{\frac{\frac{7-\sqrt{6}}{7}}{2}}$$

$$\cos\frac{\theta}{2}=\pm\sqrt{\frac{7-\sqrt{6}}{14}}$$

If we also substitute $\cos\theta=-\sqrt{6}/7$ into the half-angle identity for sine, we get

$$\sin\frac{\theta}{2}=\pm\sqrt{\frac{1-\cos\theta}{2}}$$

$$\sin\frac{\theta}{2}=\pm\sqrt{\frac{1-\left(-\frac{\sqrt{6}}{7}\right)}{2}}$$

$$\sin\frac{\theta}{2}=\pm\sqrt{\frac{\frac{7+\sqrt{6}}{7}}{2}}$$

$$\sin\frac{\theta}{2}=\pm\sqrt{\frac{7+\sqrt{6}}{14}}$$

To figure out the quadrant of $\theta/2$, we'll start with the fact that we were told θ is in the third quadrant, $\pi<\theta<3\pi/2$. We'll divide through the inequality by 2 to change θ into $\theta/2$.

$$\pi<\theta<\frac{3\pi}{2}$$

$$\frac{\pi}{2} < \frac{\theta}{2} < \frac{3\pi}{4}$$

This inequality tells us that the angle $\theta/2$ falls between $\pi/2$ (which is along the positive direction of the y-axis) and $3\pi/4$ (which is halfway through the second quadrant). Therefore, the bounds $\theta/2 = [\pi/2, 3\pi/4]$ define space only in the second quadrant, so $\theta/2$ must be in the second quadrant.

For any angle in the second quadrant, cosine is negative and sine is positive.

$$\cos\frac{\theta}{2} = -\sqrt{\frac{7-\sqrt{6}}{14}}$$

$$\sin\frac{\theta}{2} = \sqrt{\frac{7+\sqrt{6}}{14}}$$

Let's do an example with an angle that's outside the interval $[0,2\pi)$.

Example

If $-11\pi/2 < \theta < -5\pi$ and $\sin\theta = 1/3$, find $\cos(\theta/2)$ and $\sin(\theta/2)$.

We'll start by using $\sin\theta = 1/3$ and the Pythagorean identity with sine and cosine to find the corresponding value of $\cos\theta$.

$$\sin^2\theta + \cos^2\theta = 1$$

$$\cos^2\theta = 1 - \sin^2\theta$$

$$\cos^2\theta = 1 - \left(\frac{1}{3}\right)^2$$

$$\cos^2\theta = 1 - \frac{1}{9}$$

$$\cos^2\theta = \frac{8}{9}$$

$$\cos\theta = \pm\sqrt{\frac{8}{9}}$$

We were told that $-11\pi/2 < \theta < -5\pi$. To figure out the quadrant in which θ lies, we'll find coterminal angles for both $-11\pi/2$ and -5π by adding 6π to both angles.

$$-\frac{11\pi}{2} + 6\pi < \theta < -5\pi + 6\pi$$

$$-\frac{11\pi}{2} + \frac{12\pi}{2} < \theta < \pi$$

$$\frac{\pi}{2} < \theta < \pi$$

So θ is in the second quadrant. The cosine of every angle in the second quadrant is negative, so

$$\cos\theta = -\sqrt{\frac{8}{9}} = -\frac{\sqrt{8}}{\sqrt{9}} = -\frac{2\sqrt{2}}{3}$$

Then, by the half-angle identity for cosine, we get

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$$\cos\frac{\theta}{2}=\pm\sqrt{\frac{1+\cos\theta}{2}}$$

$$\cos\frac{\theta}{2}=\pm\sqrt{\frac{1+\left(-\frac{2\sqrt{2}}{3}\right)}{2}}$$

$$\cos\frac{\theta}{2}=\pm\sqrt{\frac{\frac{3-2\sqrt{2}}{3}}{2}}$$

$$\cos\frac{\theta}{2}=\pm\sqrt{\frac{3-2\sqrt{2}}{6}}$$

By the half-angle identity for sine, we get

$$\sin\frac{\theta}{2}=\pm\sqrt{\frac{1-\cos\theta}{2}}$$

$$\sin\frac{\theta}{2}=\pm\sqrt{\frac{1-\left(-\frac{2\sqrt{2}}{3}\right)}{2}}$$

$$\sin\frac{\theta}{2}=\pm\sqrt{\frac{\frac{3+2\sqrt{2}}{3}}{2}}$$

$$\sin\frac{\theta}{2}=\pm\sqrt{\frac{3+2\sqrt{2}}{6}}$$

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To find the quadrant of the angle $\theta/2$, we'll divide through the inequality we were given by 2.

$$-\frac{11\pi}{2} < \theta < -5\pi$$

$$-\frac{11\pi}{4} < \frac{\theta}{2} < -\frac{5\pi}{2}$$

To see where the angles $-11\pi/4$ and $-5\pi/4$ lie, we'll add 4π to both angles to find coterminal angles.

$$-\frac{11\pi}{4} + 4\pi < \frac{\theta}{2} < -\frac{5\pi}{2} + 4\pi$$

$$-\frac{11\pi}{4} + \frac{16\pi}{4} < \frac{\theta}{2} < -\frac{5\pi}{2} + \frac{8\pi}{2}$$

$$\frac{5\pi}{4} < \frac{\theta}{2} < \frac{3\pi}{2}$$

The angle $5\pi/4$ is halfway through the third quadrant, and the angle $3\pi/2$ is along the negative side of the y-axis, so $\theta/2$ has to be in the third quadrant. Both the cosine function and the sine function are negative for all angles in the third quadrant, so

$$\cos\frac{\theta}{2} = -\sqrt{\frac{3-2\sqrt{2}}{6}}$$

$$\sin\frac{\theta}{2} = -\sqrt{\frac{3+2\sqrt{2}}{6}}$$

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Even if we're given the value of $\tan\theta$ for some angle θ and the interval in which it lies, we can find $\cos(\theta/2)$ and $\sin(\theta/2)$.

Example

Find $\sin(\theta/2)$ and $\cos(\theta/2)$ for the angle θ such that $\tan\theta = 21$ and θ lies in the interval $(10\pi, 10\pi + (\pi/2))$.

We'll start with the Pythagorean identity with secant and tangent.

$$\sec^2\theta = 1 + \tan^2\theta$$

$$\sec^2\theta = 1 + (21^2)$$

$$\sec^2\theta = 1 + 441$$

$$\sec^2\theta = 442$$

Using the reciprocal identity for cosine, we get

$$\cos^2\theta = \frac{1}{\sec^2\theta} = \frac{1}{442}$$

We've been told that

$$10\pi < \theta < 10\pi + \frac{\pi}{2}$$

Since 10π is an integer multiple of 2π, that angle lies along the positive side of the x-axis. Which means that $10\pi + (\pi/2)$ must be along the positive side

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of the y-axis. These two angles therefore bound the entire first quadrant, which means θ is in the first quadrant, so $\cos\theta$ is positive.

$$\cos\theta = \sqrt{\frac{1}{442}} = \frac{\sqrt{1}}{\sqrt{442}} = \frac{1}{\sqrt{442}} = \frac{\sqrt{442}}{\sqrt{442}\sqrt{442}} = \frac{\sqrt{442}}{442}$$

Then by the half-angle identity for cosine,

$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}}$$

$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\frac{\sqrt{442}}{442}}{2}}$$

$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{\frac{442+\sqrt{442}}{442}}{2}}$$

$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{442+\sqrt{442}}{884}}$$

And by the half-angle identity for sine,

$$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}}$$

$$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\frac{\sqrt{442}}{442}}{2}}$$

$$\sin\frac{\theta}{2} = \pm\sqrt{\frac{\frac{442-\sqrt{442}}{442}}{2}}$$

$$\sin\frac{\theta}{2} = \pm\sqrt{\frac{442-\sqrt{442}}{884}}$$

If we divide through the inequality we were given by 2, we get

$$10\pi < \theta < 10\pi + \frac{\pi}{2}$$

$$5\pi < \frac{\theta}{2} < 5\pi + \frac{\pi}{4}$$

The angle 5π is an integer multiple of π, so it lies on the negative x-axis, which means the angle $5\pi + (\pi/4)$ lies halfway through the third quadrant, and therefore that $\theta/2$ must lie in the third quadrant. So both $\cos\theta$ and $\sin\theta$ are negative.

$$\cos\frac{\theta}{2} = -\sqrt{\frac{442+\sqrt{442}}{884}}$$

$$\sin\frac{\theta}{2} = -\sqrt{\frac{442-\sqrt{442}}{884}}$$

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Product-to-sum identities

Sometimes we'll have the product of trig functions and we'll want to break up the product into a sum or difference. For instance, given the product $\sin\theta\cos\alpha$, we'll want to break up the product and be able to write this as

$$\frac{1}{2}\left[\sin(\theta+\alpha)+\sin(\theta-\alpha)\right]$$

which is the sum of two different sine functions. This is what the product-to-sum identities do; they change a product into a sum (or a difference).

Product-to-sum identities from the sum-difference identities

These product-to-sum identities can be built directly from the sum-difference identities we learned about earlier. As a reminder, here are the sum-difference identities:

$$\sin(\theta+\alpha)=\sin\theta\cos\alpha+\cos\theta\sin\alpha$$

$$\sin(\theta-\alpha)=\sin\theta\cos\alpha-\cos\theta\sin\alpha$$

$$\cos(\theta+\alpha)=\cos\theta\cos\alpha-\sin\theta\sin\alpha$$

$$\cos(\theta-\alpha)=\cos\theta\cos\alpha+\sin\theta\sin\alpha$$

If we add the two sine identities together, we'll add the left sides of the equations, $\sin(\theta+\alpha)$ and $\sin(\theta-\alpha)$, and the right sides of the equations, $\sin\theta\cos\alpha+\cos\theta\sin\alpha$ and $\sin\theta\cos\alpha-\cos\theta\sin\alpha$.

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$$\left[\sin(\theta+\alpha)=\sin\theta\cos\alpha+\cos\theta\sin\alpha\right]+\left[\sin(\theta-\alpha)=\sin\theta\cos\alpha-\cos\theta\sin\alpha\right]$$

$$\sin(\theta+\alpha)+\sin(\theta-\alpha)=\sin\theta\cos\alpha+\cos\theta\sin\alpha+\sin\theta\cos\alpha-\cos\theta\sin\alpha$$

$$\sin(\theta+\alpha)+\sin(\theta-\alpha)=\sin\theta\cos\alpha+\sin\theta\cos\alpha$$

$$\sin(\theta+\alpha)+\sin(\theta-\alpha)=2\sin\theta\cos\alpha$$

$$\sin\theta\cos\alpha=\frac{1}{2}\left[\sin(\theta+\alpha)+\sin(\theta-\alpha)\right]$$

This is the first product-to-sum identity. We can find another one for sine by subtracting the identity for $\sin(\theta-\alpha)$ from the identity for $\sin(\theta+\alpha)$.

$$\left[\sin(\theta+\alpha)=\sin\theta\cos\alpha+\cos\theta\sin\alpha\right]-\left[\sin(\theta-\alpha)=\sin\theta\cos\alpha-\cos\theta\sin\alpha\right]$$

$$\sin(\theta+\alpha)-\sin(\theta-\alpha)=\sin\theta\cos\alpha+\cos\theta\sin\alpha-(\sin\theta\cos\alpha-\cos\theta\sin\alpha)$$

$$\sin(\theta+\alpha)-\sin(\theta-\alpha)=\sin\theta\cos\alpha+\cos\theta\sin\alpha-\sin\theta\cos\alpha+\cos\theta\sin\alpha$$

$$\sin(\theta+\alpha)-\sin(\theta-\alpha)=\cos\theta\sin\alpha+\cos\theta\sin\alpha$$

$$\sin(\theta+\alpha)-\sin(\theta-\alpha)=2\cos\theta\sin\alpha$$

$$\cos\theta\sin\alpha=\frac{1}{2}\left[\sin(\theta+\alpha)-\sin(\theta-\alpha)\right]$$

In the same way we just found these first two product-to-sum identities, we can find two more product-to-sum identities by adding and subtracting the sum-difference identities for cosine. When we add the identities, we get

$$\left[\cos(\theta+\alpha)=\cos\theta\cos\alpha-\sin\theta\sin\alpha\right]+\left[\cos(\theta-\alpha)=\cos\theta\cos\alpha+\sin\theta\sin\alpha\right]$$

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$$\cos(\theta+\alpha)+\cos(\theta-\alpha)=\cos\theta\cos\alpha-\sin\theta\sin\alpha+\cos\theta\cos\alpha+\sin\theta\sin\alpha$$

$$\cos(\theta+\alpha)+\cos(\theta-\alpha)=\cos\theta\cos\alpha+\cos\theta\cos\alpha$$

$$\cos(\theta+\alpha)+\cos(\theta-\alpha)=2\cos\theta\cos\alpha$$

$$\cos\theta\cos\alpha=\frac{1}{2}\left[\cos(\theta+\alpha)+\cos(\theta-\alpha)\right]$$

and when we subtract the identities, we get

$$\left[\cos(\theta+\alpha)=\cos\theta\cos\alpha-\sin\theta\sin\alpha\right]-\left[\cos(\theta-\alpha)=\cos\theta\cos\alpha+\sin\theta\sin\alpha\right]$$

$$\cos(\theta+\alpha)-\cos(\theta-\alpha)=\cos\theta\cos\alpha-\sin\theta\sin\alpha-(\cos\theta\cos\alpha+\sin\theta\sin\alpha)$$

$$\cos(\theta+\alpha)-\cos(\theta-\alpha)=\cos\theta\cos\alpha-\sin\theta\sin\alpha-\cos\theta\cos\alpha-\sin\theta\sin\alpha$$

$$\cos(\theta+\alpha)-\cos(\theta-\alpha)=-\sin\theta\sin\alpha-\sin\theta\sin\alpha$$

$$\cos(\theta+\alpha)-\cos(\theta-\alpha)=-2\sin\theta\sin\alpha$$

$$\sin\theta\sin\alpha=-\frac{1}{2}\left[\cos(\theta+\alpha)-\cos(\theta-\alpha)\right]$$

$$\sin\theta\sin\alpha=\frac{1}{2}\left[-\cos(\theta+\alpha)+\cos(\theta-\alpha)\right]$$

$$\sin\theta\sin\alpha=\frac{1}{2}\left[\cos(\theta-\alpha)-\cos(\theta+\alpha)\right]$$

If we summarize what we've built, we get the four **product-to-sum identities**.

$$\sin\theta\cos\alpha=\frac{1}{2}\left[\sin(\theta+\alpha)+\sin(\theta-\alpha)\right]$$

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$$\cos\theta\sin\alpha = \frac{1}{2}\left[\sin(\theta+\alpha) - \sin(\theta-\alpha)\right]$$

$$\cos\theta\cos\alpha = \frac{1}{2}\left[\cos(\theta+\alpha) + \cos(\theta-\alpha)\right]$$

$$\sin\theta\sin\alpha = \frac{1}{2}\left[\cos(\theta-\alpha) - \cos(\theta+\alpha)\right]$$

Notice also that identities do something for us other than break up products into sums and differences.

The first identity takes the product of a sine and cosine function and changes it into the sum of two sine functions, thereby eliminating cosine from the expression completely.

The second identity takes the product of a sine and cosine function and changes it into the difference of two sine functions, again eliminating cosine completely.

And the fourth identity takes the product of two sines functions and changes it into the difference of two cosine functions, completely eliminating sine from the expression.

So these product-to-sum identities can also help us eliminate sine or cosine from the expression, or put the expression completely in terms of sine only or cosine only. And sometimes that'll be valuable to us.

Let's do an example where we use a product-to-sum identity to break up the product of two sine functions.

Example

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Express $\sin(7\theta)\sin(11\theta)$ as the sum or difference of trig functions.

Because we have the product of sine functions, we'll use the product-to-sum identity for the product of sine functions.

$$\sin\theta\sin\alpha = \frac{1}{2}\left[\cos(\theta-\alpha)-\cos(\theta+\alpha)\right]$$

$$\sin(7\theta)\sin(11\theta) = \frac{1}{2}\left[\cos(7\theta-11\theta)-\cos(7\theta+11\theta)\right]$$

$$\sin(7\theta)\sin(11\theta) = \frac{1}{2}\left[\cos(-4\theta)-\cos(18\theta)\right]$$

We could leave the expression this way. But we can also further simplify the first cosine expression using the even identity for cosine, which tells us that

$$\cos(-4\theta) = \cos(4\theta)$$

Therefore, we'll simplify the equation to

$$\sin(7\theta)\sin(11\theta) = \frac{1}{2}\left[\cos(4\theta)-\cos(18\theta)\right]$$

Let's look at another example where we use the product-to-sum identities to calculate the values of a set of expressions.

Example

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Find the exact values of $(\sin \pi/8)(\cos \pi/8)$, $\sin^2(\pi/8)$, and $\cos^2(\pi/8)$.

To compute $(\sin \pi/8)(\cos \pi/8)$, we can use the product-to-sum identity for the product of a sine and cosine function.

$$\sin\theta\cos\alpha = \frac{1}{2}\left[\sin(\theta+\alpha)+\sin(\theta-\alpha)\right]$$

$$\left(\sin\frac{\pi}{8}\right)\left(\cos\frac{\pi}{8}\right) = \frac{1}{2}\left[\sin\left(\frac{\pi}{8}+\frac{\pi}{8}\right)+\sin\left(\frac{\pi}{8}-\frac{\pi}{8}\right)\right]$$

$$\left(\sin\frac{\pi}{8}\right)\left(\cos\frac{\pi}{8}\right) = \frac{1}{2}\left(\sin\frac{\pi}{4}+\sin 0\right)$$

Pulling the values of sine on the right side from the unit circle, we get

$$\left(\sin\frac{\pi}{8}\right)\left(\cos\frac{\pi}{8}\right) = \frac{1}{2}\left(\frac{\sqrt{2}}{2}+0\right)$$

$$\left(\sin\frac{\pi}{8}\right)\left(\cos\frac{\pi}{8}\right) = \frac{\sqrt{2}}{4}$$

To find $\sin^2(\pi/8)$, we'll use the product-to-sum identity for the product of two sine functions.

$$\sin\theta\sin\alpha = \frac{1}{2}\left[\cos(\theta-\alpha)-\cos(\theta+\alpha)\right]$$

$$\left(\sin\frac{\pi}{8}\right)\left(\sin\frac{\pi}{8}\right) = \frac{1}{2}\left[\cos\left(\frac{\pi}{8}-\frac{\pi}{8}\right)-\cos\left(\frac{\pi}{8}+\frac{\pi}{8}\right)\right]$$

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$$\sin^2 \frac{\pi}{8} = \frac{1}{2}\left(\cos 0 - \cos \frac{\pi}{4}\right)$$

Pulling the values of cosine on the right side from the unit circle, we get

$$\sin^2 \frac{\pi}{8} = \frac{1}{2}\left(1 - \frac{\sqrt{2}}{2}\right)$$

$$\sin^2 \frac{\pi}{8} = \frac{1}{2} - \frac{\sqrt{2}}{4}$$

$$\sin^2 \frac{\pi}{8} = \frac{2}{4} - \frac{\sqrt{2}}{4}$$

$$\sin^2 \frac{\pi}{8} = \frac{2 - \sqrt{2}}{4}$$

To get the value of $\cos^2(\pi/8)$, we'll use the product-to-sum identity for the product of two cosine functions.

$$\cos\theta\cos\alpha = \frac{1}{2}\left[\cos(\theta + \alpha) + \cos(\theta - \alpha)\right]$$

$$\left(\cos\frac{\pi}{8}\right)\left(\cos\frac{\pi}{8}\right) = \frac{1}{2}\left[\cos\left(\frac{\pi}{8} + \frac{\pi}{8}\right) + \cos\left(\frac{\pi}{8} - \frac{\pi}{8}\right)\right]$$

$$\cos^2 \frac{\pi}{8} = \frac{1}{2}\left(\cos \frac{\pi}{4} + \cos 0\right)$$

Pulling the values of cosine on the right side from the unit circle, we get

$$\cos^2 \frac{\pi}{8} = \frac{1}{2}\left(\frac{\sqrt{2}}{2} + 1\right)$$

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$$\cos^2\frac{\pi}{8} = \frac{\sqrt{2}}{4} + \frac{1}{2}$$

$$\cos^2\frac{\pi}{8} = \frac{\sqrt{2}}{4} + \frac{2}{4}$$

$$\cos^2\frac{\pi}{8} = \frac{2+\sqrt{2}}{4}$$

Let's do one more example where we use the product-to-sum identity to break up the product of a cosine and sine function.

Example

Find the exact value of $\cos(17\pi/12)\sin(\pi/12)$.

We'll use the product-to-sum identity for the product of sine and cosine.

$$\cos\theta\sin\alpha = \frac{1}{2}\left[\sin(\theta+\alpha) - \sin(\theta-\alpha)\right]$$

$$\left(\cos\frac{17\pi}{12}\right)\left(\sin\frac{\pi}{12}\right) = \frac{1}{2}\left[\sin\left(\frac{17\pi}{12}+\frac{\pi}{12}\right) - \sin\left(\frac{17\pi}{12}-\frac{\pi}{12}\right)\right]$$

$$\left(\cos\frac{17\pi}{12}\right)\left(\sin\frac{\pi}{12}\right) = \frac{1}{2}\left[\sin\left(\frac{18\pi}{12}\right) - \sin\left(\frac{16\pi}{12}\right)\right]$$

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$$\left(\cos\frac{17\pi}{12}\right)\left(\sin\frac{\pi}{12}\right)=\frac{1}{2}\left[\sin\left(\frac{3\pi}{2}\right)-\sin\left(\frac{4\pi}{3}\right)\right]$$

Pulling the values of sine on the right side from the unit circle, we get

$$\left(\cos\frac{17\pi}{12}\right)\left(\sin\frac{\pi}{12}\right)=\frac{1}{2}\left[-1-\left(-\frac{\sqrt{3}}{2}\right)\right]$$

$$\left(\cos\frac{17\pi}{12}\right)\left(\sin\frac{\pi}{12}\right)=\frac{1}{2}\left(-1+\frac{\sqrt{3}}{2}\right)$$

$$\left(\cos\frac{17\pi}{12}\right)\left(\sin\frac{\pi}{12}\right)=-\frac{1}{2}+\frac{\sqrt{3}}{4}$$

$$\left(\cos\frac{17\pi}{12}\right)\left(\sin\frac{\pi}{12}\right)=-\frac{2}{4}+\frac{\sqrt{3}}{4}$$

$$\left(\cos\frac{17\pi}{12}\right)\left(\sin\frac{\pi}{12}\right)=\frac{-2+\sqrt{3}}{4}$$

Sum-to-product identities

In the last lesson, we looked at the product-to-sum identities in order to to break down the product of trig functions into a sum or difference of trig functions.

Now we want to go the other direction, converting the sum or difference of trig functions into the product of trig functions.

To convert in this opposite direction, we'll use the sum-to-product identities, which we can build from the product-to-sum identities we just introduced.

Sum-to-product identities from the product-to-sum identities

As a reminder, the product-to-sum identities are

$$\sin\theta\cos\alpha = \frac{1}{2}\left[\sin(\theta+\alpha)+\sin(\theta-\alpha)\right]$$

$$\cos\theta\sin\alpha = \frac{1}{2}\left[\sin(\theta+\alpha)-\sin(\theta-\alpha)\right]$$

$$\cos\theta\cos\alpha = \frac{1}{2}\left[\cos(\theta+\alpha)+\cos(\theta-\alpha)\right]$$

$$\sin\theta\sin\alpha = \frac{1}{2}\left[\cos(\theta-\alpha)-\cos(\theta+\alpha)\right]$$

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With these identities in mind, let $x = \theta + \alpha$ and let $y = \theta - \alpha$. Then add these equations,

$$x + y = \theta + \alpha + \theta - \alpha$$

$$x + y = \theta + \theta$$

$$x + y = 2\theta$$

$$\theta = \frac{x + y}{2}$$

and subtract these equations.

$$x - y = \theta + \alpha - (\theta - \alpha)$$

$$x - y = \theta + \alpha - \theta + \alpha$$

$$x - y = \alpha + \alpha$$

$$x - y = 2\alpha$$

$$\alpha = \frac{x - y}{2}$$

Now with these different values for θ and α, we can substitute into the product-to-sum identities. The first identity becomes

$$\sin\theta\cos\alpha = \frac{1}{2}\left[\sin(\theta + \alpha) + \sin(\theta - \alpha)\right]$$

$$\sin\left(\frac{x + y}{2}\right)\cos\left(\frac{x - y}{2}\right) = \frac{1}{2}(\sin x + \sin y)$$

$$\sin x + \sin y = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

the second becomes,

$$\cos\theta\sin\alpha = \frac{1}{2}\left[\sin(\theta+\alpha) - \sin(\theta-\alpha)\right]$$

$$\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right) = \frac{1}{2}(\sin x - \sin y)$$

$$\sin x - \sin y = 2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$

the third becomes,

$$\cos\theta\cos\alpha = \frac{1}{2}\left[\cos(\theta+\alpha) + \cos(\theta-\alpha)\right]$$

$$\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right) = \frac{1}{2}(\cos x + \cos y)$$

$$\cos x + \cos y = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

and the fourth becomes

$$\sin\theta\sin\alpha = \frac{1}{2}\left[\cos(\theta-\alpha) - \cos(\theta+\alpha)\right]$$

$$\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right) = \frac{1}{2}(\cos y - \cos x)$$

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$$\cos y - \cos x = 2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$

or

$$\cos x - \cos y = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$

If we summarize what we've just built, we get the four **sum-to-product identities**.

$$\sin\theta + \sin\alpha = 2\sin\left(\frac{\theta+\alpha}{2}\right)\cos\left(\frac{\theta-\alpha}{2}\right)$$

$$\sin\theta - \sin\alpha = 2\cos\left(\frac{\theta+\alpha}{2}\right)\sin\left(\frac{\theta-\alpha}{2}\right)$$

$$\cos\theta + \cos\alpha = 2\cos\left(\frac{\theta+\alpha}{2}\right)\cos\left(\frac{\theta-\alpha}{2}\right)$$

$$\cos\theta - \cos\alpha = -2\sin\left(\frac{\theta+\alpha}{2}\right)\sin\left(\frac{\theta-\alpha}{2}\right)$$

Let's do an example where we use the fourth sum-to-product identity to turn the difference of cosine functions into the product of sine functions.

Example

Express $\cos(6\theta) - \cos(15\theta)$ as a product of trig functions.

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Because we have the difference of cosine functions, we'll plug into the fourth sum-to-product identity.

$$\cos\theta - \cos\alpha = -2\sin\left(\frac{\theta+\alpha}{2}\right)\sin\left(\frac{\theta-\alpha}{2}\right)$$

$$\cos(6\theta) - \cos(15\theta) = -2\sin\left(\frac{6\theta+15\theta}{2}\right)\sin\left(\frac{6\theta-15\theta}{2}\right)$$

$$\cos(6\theta) - \cos(15\theta) = -2\sin\left(\frac{21\theta}{2}\right)\sin\left(-\frac{9\theta}{2}\right)$$

We could certainly leave the equation this way. But we can further simplify the right side by using the odd identity for the sine function.

$$\cos(6\theta) - \cos(15\theta) = -2\sin\left(\frac{21\theta}{2}\right)\left[-\sin\left(\frac{9\theta}{2}\right)\right]$$

$$\cos(6\theta) - \cos(15\theta) = 2\sin\left(\frac{21\theta}{2}\right)\sin\left(\frac{9\theta}{2}\right)$$

Let's do a different kind of example, where we're asked to use the sum-to-product identities to prove a trig equation. To prove that a trig equation is true, we need to show that both sides are identical, by changing one or both sides until they're identical.

Example

Use a sum-to-product identity to prove the trig equation.

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$$\sin\theta = \sin(\pi - \theta)$$

If we subtract $\sin(\pi - \theta)$ from both sides to rewrite the equation as

$$\sin\theta - \sin(\pi - \theta) = 0$$

then we can use the sum-to-product identity for the difference of sine functions,

$$\sin\theta - \sin\alpha = 2\cos\left(\frac{\theta + \alpha}{2}\right)\sin\left(\frac{\theta - \alpha}{2}\right)$$

with $\alpha = \pi - \theta$.

$$\sin\theta - \sin(\pi - \theta) = 2\cos\left(\frac{\theta + (\pi - \theta)}{2}\right)\sin\left(\frac{\theta - (\pi - \theta)}{2}\right)$$

$$\sin\theta - \sin(\pi - \theta) = 2\cos\left(\frac{\theta + \pi - \theta}{2}\right)\sin\left(\frac{\theta - \pi + \theta}{2}\right)$$

$$\sin\theta - \sin(\pi - \theta) = 2\cos\left(\frac{\pi}{2}\right)\sin\left(\frac{2\theta - \pi}{2}\right)$$

$$\sin\theta - \sin(\pi - \theta) = 2(0)\sin\left(\frac{2\theta - \pi}{2}\right)$$

$$\sin\theta - \sin(\pi - \theta) = 0$$

Let's do another example.

Example

Use a sum-to-product identity to find the value of $\cos(14\pi/3) + \cos(13\pi/3)$.

We'll use the sum-to-product identity for the sum of cosine functions, with $\theta = 14\pi/3$ and $\alpha = 13\pi/3$.

$$\cos\theta + \cos\alpha = 2\cos\left(\frac{\theta+\alpha}{2}\right)\cos\left(\frac{\theta-\alpha}{2}\right)$$

$$\cos\left(\frac{14\pi}{3}\right) + \cos\left(\frac{13\pi}{3}\right) = 2\cos\left(\frac{\frac{14\pi}{3}+\frac{13\pi}{3}}{2}\right)\cos\left(\frac{\frac{14\pi}{3}-\frac{13\pi}{3}}{2}\right)$$

$$\cos\left(\frac{14\pi}{3}\right) + \cos\left(\frac{13\pi}{3}\right) = 2\cos\left(\frac{\frac{27\pi}{3}}{2}\right)\cos\left(\frac{\frac{\pi}{3}}{2}\right)$$

$$\cos\left(\frac{14\pi}{3}\right) + \cos\left(\frac{13\pi}{3}\right) = 2\cos\left(\frac{27\pi}{6}\right)\cos\left(\frac{\pi}{6}\right)$$

$$\cos\left(\frac{14\pi}{3}\right) + \cos\left(\frac{13\pi}{3}\right) = 2\cos\left(\frac{9\pi}{2}\right)\cos\left(\frac{\pi}{6}\right)$$

$$\cos\left(\frac{14\pi}{3}\right) + \cos\left(\frac{13\pi}{3}\right) = 2(0)\cos\left(\frac{\pi}{6}\right)$$

$$\cos\left(\frac{14\pi}{3}\right) + \cos\left(\frac{13\pi}{3}\right) = 0$$

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Proving the trig equation

Now that we've introduced so many trig identities, we want to talk about one way we'll use them, which is to prove trig equations. As a reminder, here's a full list of the identities we've looked at so far:

Reciprocal identities	Cofunction identities
Quotient identities	Double-angle identities
Pythagorean identities	Half-angle identities
Even-odd identities	Product-to-sum identities
Sum-difference identities	Sum-to-product identities

When we use trig identities to prove a trig equation, we often start with one side of the equation and manipulate it using algebra and one or more trig identities in order to get the expression on the other side of the equation.

Mostly, these kinds of problems just take lots of practice, so that's what we'll focus on in this lesson. But there are some general tips that we want to keep in mind when we're trying to figure out the best way to approach the problem.

1. Try to express every trig function in the equation in terms of sine and cosine. For cosecant and secant, we'll do this with the reciprocal identities, and for tangent and cotangent we'll do this with the quotient identities.

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2. Make sure all the angles are the same. For example, when we have $\sin(2\alpha)$ on one side of the equation, and $\sin\alpha$ on the other side, it's difficult to prove the equation. The same applies for addition and subtraction: don't try working with $\sin(\alpha+\beta)$ and $\sin\alpha$.

3. Try rewriting the more complicated side of the equation in order to match the simpler side.

4. If we need to add more powers or remove them, we can use the Pythagorean identities like $\cos^2 x + \sin^2 x = 1$. We can always multiply by 1 without changing the meaning, so we can always multiply by $\cos^2 x + \sin^2 x$.

5. Look for fractions that can be combined or pulled apart, and consider whether or not the equation might be factorable.

6. Look for a trig function that links the trig functions in the equation. For instance, if we have sine and cotangent in an equation, we know that tangent is a linking function, because tangent=sine/cosine, and tangent=1/cotangent.

7. If we ever have a value which is the sum or difference of a constant and a trig function, like $1+\cos\theta$, consider multiplying by the conjugate. The conjugate is the same two terms, but with the opposite sign between them. So the conjugate of $1+\cos\theta$ is $1-\cos\theta$.

These tips aren't the only things we can try, but they *are* common operations we'll use. Above anything, don't be afraid to try different things! Some people feel paralyzed because they don't know where to

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start. But there's no harm in starting somewhere, even if we don't know where it'll lead us.

So when we tackle these kinds of problems, we just need to start *somewhere*, and try *something*. It we feel at any point like we're running into a dead-end, we can just back up and try something else. We might have to work through a couple different approaches before we find the one that works.

With all this in mind, let's get some practice by working through a few examples.

Example

Prove the trig equation.

$$\frac{\cot\theta}{\csc\theta} = \cos\theta$$

We can't simplify the right side at all, so we'll try to rewrite the left side. Our goal would be to rewrite the left side to eventually show that it's equivalent to $\cos\theta$.

Let's start with our first tip in the list, which is to put everything in terms of sine and cosine. Since $\cot\theta = \cos\theta/\sin\theta$, and $\csc\theta = 1/\sin\theta$, we'll rewrite the equation as

$$\frac{\frac{\cos\theta}{\sin\theta}}{\frac{1}{\sin\theta}} = \cos\theta$$

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$$\frac{\cos\theta}{\sin\theta}\left(\frac{\sin\theta}{1}\right)=\cos\theta$$

The $\sin\theta$ will cancel from the numerator and denominator, leaving just

$$\cos\theta=\cos\theta$$

Let's try one where we use quotient and Pythagorean identities.

Example

Prove the trig equation.

$$\frac{\tan\theta-\cot\theta}{\tan\theta+\cot\theta}=1-2\cos^2\theta$$

We'll start with the left side, and try to show that it's equal to the right side. We can use the quotient identities

$$\tan\theta=\frac{\sin\theta}{\cos\theta} \text{ and } \cot\theta=\frac{\cos\theta}{\sin\theta}$$

to put everything on the left in terms of sine and cosine.

$$\frac{\tan\theta-\cot\theta}{\tan\theta+\cot\theta}=1-2\cos^2\theta$$

$$\frac{\frac{\sin\theta}{\cos\theta}-\frac{\cos\theta}{\sin\theta}}{\frac{\sin\theta}{\cos\theta}+\frac{\cos\theta}{\sin\theta}}=1-2\cos^2\theta$$

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Find common denominators in both the numerator and denominator, then simplify them.

$$\frac{\frac{\sin\theta}{\cos\theta}\left(\frac{\sin\theta}{\sin\theta}\right)-\frac{\cos\theta}{\sin\theta}\left(\frac{\cos\theta}{\cos\theta}\right)}{\frac{\sin\theta}{\cos\theta}\left(\frac{\sin\theta}{\sin\theta}\right)+\frac{\cos\theta}{\sin\theta}\left(\frac{\cos\theta}{\cos\theta}\right)}=1-2\cos^2\theta$$

$$\frac{\frac{\sin^2\theta}{\sin\theta\cos\theta}-\frac{\cos^2\theta}{\sin\theta\cos\theta}}{\frac{\sin^2\theta}{\sin\theta\cos\theta}+\frac{\cos^2\theta}{\sin\theta\cos\theta}}=1-2\cos^2\theta$$

$$\frac{\frac{\sin^2\theta-\cos^2\theta}{\sin\theta\cos\theta}}{\frac{\sin^2\theta+\cos^2\theta}{\sin\theta\cos\theta}}=1-2\cos^2\theta$$

$$\frac{\sin^2\theta-\cos^2\theta}{\sin\theta\cos\theta}\left(\frac{\sin\theta\cos\theta}{\sin^2\theta+\cos^2\theta}\right)=1-2\cos^2\theta$$

The $\sin\theta\cos\theta$ will cancel from the numerator and denominator, leaving just

$$\frac{\sin^2\theta-\cos^2\theta}{\sin^2\theta+\cos^2\theta}=1-2\cos^2\theta$$

With the Pythagorean identity $\sin^2\theta+\cos^2\theta=1$, we get

$$\frac{\sin^2\theta-\cos^2\theta}{1}=1-2\cos^2\theta$$

$$\sin^2\theta-\cos^2\theta=1-2\cos^2\theta$$

Using one of the other forms of the basic Pythagorean identity, $\sin^2\theta=1-\cos^2\theta$, we'll make a substitution for $\sin^2\theta$.

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$$(1-\cos^2\theta)-\cos^2\theta=1-2\cos^2\theta$$

$$1-\cos^2\theta-\cos^2\theta=1-2\cos^2\theta$$

$$1-2\cos^2\theta=1-2\cos^2\theta$$

Sometimes it's easier or more convenient to work on the expressions on both sides of the equation separately (one after the other), and prove that they're both equal to some other third expression.

Example

Prove the trig equation.

$$\csc\theta+\cot\theta=\frac{\sin\theta}{1-\cos\theta}$$

We'll start by working on the left side. Using the reciprocal and quotient identities

$$\csc\theta=\frac{1}{\sin\theta} \text{ and } \cot\theta=\frac{\cos\theta}{\sin\theta}$$

we'll rewrite the left side as

$$\csc\theta+\cot\theta$$

$$\frac{1}{\sin\theta}+\frac{\cos\theta}{\sin\theta}$$

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$$\frac{1+\cos\theta}{\sin\theta}$$

With the left side in this form, the full equation is currently

$$\frac{1+\cos\theta}{\sin\theta}=\frac{\sin\theta}{1-\cos\theta}$$

Now we'll work on the right side. We'll multiply both the numerator and denominator by the conjugate of the denominator, $1+\cos\theta$. (Alternatively, we could continue working on the left side by multiplying both the numerator and denominator by the conjugate of the numerator, $1-\cos\theta$.)

$$\frac{1+\cos\theta}{\sin\theta}=\frac{\sin\theta}{1-\cos\theta}\left(\frac{1+\cos\theta}{1+\cos\theta}\right)$$

$$\frac{1+\cos\theta}{\sin\theta}=\frac{\sin\theta(1+\cos\theta)}{1+\cos\theta-\cos\theta-\cos^2\theta}$$

$$\frac{1+\cos\theta}{\sin\theta}=\frac{\sin\theta(1+\cos\theta)}{1-\cos^2\theta}$$

Use the rewritten form $1-\cos^2\theta=\sin^2\theta$ of the Pythagorean identity with sine and cosine to substitute for the denominator on the right side.

$$\frac{1+\cos\theta}{\sin\theta}=\frac{\sin\theta(1+\cos\theta)}{\sin^2\theta}$$

On the right side, we can cancel one factor of $\sin\theta$ from both the numerator and denominator.

$$\frac{1+\cos\theta}{\sin\theta}=\frac{1+\cos\theta}{\sin\theta}$$

Let's do an example using the double- and half-angle identities.

Example

Prove the trig equation.

$$\cot\left(\frac{\theta}{2}\right)=\frac{\sin\theta}{1-\cos\theta}$$

We'll start with the expression on the left side. Using the quotient identity, we'll rewrite the cotangent function in terms of sine and cosine.

$$\frac{\cos\left(\frac{\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)}=\frac{\sin\theta}{1-\cos\theta}$$

Multiplying both the numerator and denominator by $2\sin(\theta/2)$.

$$\frac{2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)}{2\sin\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)}=\frac{\sin\theta}{1-\cos\theta}$$

By the double-angle identity for sine, we know that the numerator of the left side is equivalent to

$$2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)=\sin\left(2\cdot\frac{\theta}{2}\right)$$

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So we'll replace the numerator on the left side.

$$\frac{\sin\left(2\cdot\frac{\theta}{2}\right)}{2\sin\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)}=\frac{\sin\theta}{1-\cos\theta}$$

$$\frac{\sin\theta}{2\sin\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)}=\frac{\sin\theta}{1-\cos\theta}$$

$$\frac{\sin\theta}{2\sin^2\left(\frac{\theta}{2}\right)}=\frac{\sin\theta}{1-\cos\theta}$$

With the half-angle identity for sine,

$$\sin^2\left(\frac{\theta}{2}\right)=\frac{1-\cos\theta}{2}$$

the left side of the equation can be rewritten.

$$\frac{\sin\theta}{2\left(\frac{1-\cos\theta}{2}\right)}=\frac{\sin\theta}{1-\cos\theta}$$

$$\frac{\sin\theta}{1-\cos\theta}=\frac{\sin\theta}{1-\cos\theta}$$

In the next example, we'll apply sum, double-angle, and Pythagorean identities.

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Example

Prove the trig equation.

$$\cos(3\theta) = \cos\theta(1 - 4\sin^2\theta)$$

If we think of 3θ as $2\theta + \theta$, then we can apply the sum identity for cosine to the left side.

$$\cos(3\theta) = \cos\theta(1 - 4\sin^2\theta)$$

$$\cos(2\theta + \theta) = \cos\theta(1 - 4\sin^2\theta)$$

$$\cos(2\theta)(\cos\theta) - \sin(2\theta)(\sin\theta) = \cos\theta(1 - 4\sin^2\theta)$$

The double-angle identity for cosine,

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta$$

can be applied to the left side to get

$$(\cos^2\theta - \sin^2\theta)(\cos\theta) - \sin(2\theta)(\sin\theta) = \cos\theta(1 - 4\sin^2\theta)$$

Then the double-angle identity for sine,

$$\sin(2\theta) = 2\sin\theta\cos\theta$$

can be applied to the left side to get

$$(\cos^2\theta - \sin^2\theta)(\cos\theta) - (2\sin\theta\cos\theta)(\sin\theta) = \cos\theta(1 - 4\sin^2\theta)$$

$$\cos^3\theta - \sin^2\theta\cos\theta - 2\sin^2\theta\cos\theta = \cos\theta(1 - 4\sin^2\theta)$$

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We can see that $\cos\theta$ is a common factor, so we'll factor it out.

$$\cos\theta(\cos^2\theta - \sin^2\theta - 2\sin^2\theta) = \cos\theta(1 - 4\sin^2\theta)$$

$$\cos\theta(\cos^2\theta - 3\sin^2\theta) = \cos\theta(1 - 4\sin^2\theta)$$

If we express the Pythagorean identity with sine and cosine as $\cos^2\theta = 1 - \sin^2\theta$, the left side becomes

$$\cos\theta(1 - \sin^2\theta - 3\sin^2\theta) = \cos\theta(1 - 4\sin^2\theta)$$

$$\cos\theta(1 - 4\sin^2\theta) = \cos\theta(1 - 4\sin^2\theta)$$

Let's do one more example, this one with quotient, difference, and cofunction identities.

Example

Prove the trig equation.

$$\tan\left(\frac{\pi}{2} - \theta\right) = \cot\theta$$

We'll start by working on the left side. By the quotient identity for tangent,

$$\frac{\sin\left(\frac{\pi}{2} - \theta\right)}{\cos\left(\frac{\pi}{2} - \theta\right)} = \cot\theta$$

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The difference identity for sine tells us that

$$\sin\left(\frac{\pi}{2}-\theta\right)=\sin\left(\frac{\pi}{2}\right)(\cos\theta)-\cos\left(\frac{\pi}{2}\right)(\sin\theta)$$

so we'll replace the numerator of the left side of our equation.

$$\frac{\sin\left(\frac{\pi}{2}\right)(\cos\theta)-\cos\left(\frac{\pi}{2}\right)(\sin\theta)}{\cos\left(\frac{\pi}{2}-\theta\right)}=\cot\theta$$

And the difference identity for cosine tells us that

$$\cos\left(\frac{\pi}{2}-\theta\right)=\cos\left(\frac{\pi}{2}\right)(\cos\theta)+\sin\left(\frac{\pi}{2}\right)(\sin\theta)$$

so we'll use this value to replace the denominator of the left side.

$$\frac{\sin\left(\frac{\pi}{2}\right)(\cos\theta)-\cos\left(\frac{\pi}{2}\right)(\sin\theta)}{\cos\left(\frac{\pi}{2}\right)(\cos\theta)+\sin\left(\frac{\pi}{2}\right)(\sin\theta)}=\cot\theta$$

$$\frac{(1)(\cos\theta)-(0)(\sin\theta)}{(0)(\cos\theta)+(1)(\sin\theta)}=\cot\theta$$

$$\frac{\cos\theta}{\sin\theta}=\cot\theta$$

By the quotient identity for cotangent, the left side becomes

$$\cot\theta=\cot\theta$$

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Complete solution set of the equation

Much earlier on in Trigonometry, when we introduced coterminal angles, we showed how, if one angle satisfied a trig equation, then every angle coterminal with the original angle would also satisfy the equation.

For example, given the equation $\cos\theta = 1$, we know from the unit circle that the cosine function is 1 when $\theta = 0$, so $\theta = 0$ is a solution. But we also know that every angle coterminal with $\theta = 0$ is a solution, so we gave the complete solution set of $\cos\theta = 1$ as

$$\theta = 0 + 2n\pi$$

$$\theta = 2n\pi$$

where n is any integer. Which means that the equation $\cos\theta = 1$ is satisfied by an infinitely large set of angles:

$$\ldots - 8\pi,\ -6\pi,\ -4\pi,\ -2\pi,\ 0,\ 2\pi,\ 4\pi,\ 6\pi,\ 8\pi\ldots$$

In this lesson, we want to build on this same idea, but we'll be dealing with trig equations that are more complex than equations like $\cos\theta = 1$. The equations in this lesson may require us to apply trig identities first to simplify the equation, and then find the full solution set of coterminal angles.

Let's work through some examples, starting with a sine equation.

Example

Find all the values of θ that satisfy $\sin(2\theta) = \sin\theta$.

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We'll start by using the double-angle identity $\sin(2\theta) = 2\sin\theta\cos\theta$ to rewrite the left side of the equation.

$$\sin(2\theta) = \sin\theta$$

$$2\sin\theta\cos\theta = \sin\theta$$

Subtract $\sin\theta$ from both sides, then factor out a $\sin\theta$.

$$2\sin\theta\cos\theta - \sin\theta = 0$$

$$\sin\theta(2\cos\theta - 1) = 0$$

The only way the left side of the equation is 0 is if $\sin\theta = 0$, $2\cos\theta - 1 = 0$, or both. So we need to solve these equations individually to find the values of θ that satisfy the equation. We get

$$\sin\theta = 0$$

and

$$2\cos\theta - 1 = 0$$

$$2\cos\theta = 1$$

$$\cos\theta = \frac{1}{2}$$

The equation $\sin\theta = 0$ is true when $\theta = 0, \pi, 2\pi, 3\pi, 4\pi, \ldots$, which is just all multiples of π. So the solution set of $\sin\theta = 0$ is $\theta = n\pi$, where n is the set of all integers.

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The equation $\cos\theta = 1/2$ is true at $\theta = \pi/3$ and $\theta = 5\pi/3$. But the set of all angles coterminal with these two angles is

$$\theta = \frac{\pi}{3} + 2n\pi \text{ and } \theta = \frac{5\pi}{3} + 2n\pi$$

Putting all these sets together, we can say that the complete solution set of $\sin(2\theta) = \sin\theta$ includes all of these, where n is any integer:

$$\theta = n\pi$$

$$\theta = \frac{\pi}{3} + 2n\pi$$

$$\theta = \frac{5\pi}{3} + 2n\pi$$

In some cases, we may need to find the roots of a polynomial to get the solutions of a trig equation. Let's look at an example like this in which we're asked to limit the solutions to the interval $[0,2\pi)$.

Example

Find the set of angles in the interval $[0,2\pi)$ which satisfies the trig equation.

$$\tan^2\theta + 2\tan\theta + 1 = 0$$

Let's substitute $u = \tan\theta$ to rewrite the equation.

$$u^2 + 2u + 1 = 0$$

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In this form, we can see that the equation will factor as $(u+1)^2$. Which means the original equation will factor as

$$(\tan\theta + 1)^2 = 0$$

$$\tan\theta + 1 = 0$$

$$\tan\theta = -1$$

Tangent of an angle will be -1 when the sine and cosine values are equal, but with opposite signs. This happens in the second and fourth quadrants at $\theta = 3\pi/4$ and $\theta = 7\pi/4$.

$$\tan\left(\frac{3\pi}{4}\right) = -1$$

$$\tan\left(\frac{7\pi}{4}\right) = -1$$

And of course, every angle that's coterminal with these two also has a tangent of -1, which means they are solutions as well. So θ is a solution of $\tan^2\theta + 2\tan\theta + 1 = 0$ when n is any integer for all of these:

$$\theta = \frac{3\pi}{4} + 2n\pi$$

$$\theta = \frac{7\pi}{4} + 2n\pi$$

Or since the period of tangent is π, we see that $3\pi/4$ and $7\pi/4$ differ by π, which means that these two expressions can be combined to just

$$\theta = \frac{3\pi}{4} + n\pi$$

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While this is the complete solution set, we were only asked for the solutions in $[0,2\pi)$. Of the full solution set we found, the only angles that lie in the interval are the original angles, $\theta = 3\pi/4$ and $\theta = 7\pi/4$.

Let's do one more example in which we need to first apply a trig identity.

Example

Find all the solutions of the trig equation, then list only the solutions that lie in the interval $[0,2\pi)$.

$$\sin^2\theta + \cos(2\theta) = 1$$

If we use the double-angle identity for cosine, $\cos 2\theta = \cos^2\theta - \sin^2\theta$, we can rewrite the equation as

$$\sin^2\theta + \cos^2\theta - \sin^2\theta = 1$$

$$\cos^2\theta = 1$$

Take the square root of both sides.

$$\cos\theta = \pm\sqrt{1}$$

$$\cos\theta = \pm 1$$

From the unit circle, we know that $\cos\theta = 1$ at

$$\theta = 0,\ 2\pi,\ 4\pi,\ 6\pi,\ \ldots$$

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and that $\cos\theta = -1$ at

$$\theta = \pi,\ 3\pi,\ 5\pi,\ 7\pi,\ \ldots$$

Putting these sets of solutions together, we get

$$\theta = 0,\ \pi,\ 2\pi,\ 3\pi,\ 4\pi,\ 5\pi,\ 6\pi,\ \ldots$$

The only angles in this set that fall in the interval $[0,2\pi)$ are the angles $\theta = 0$ and $\theta = \pi$. The interval notation $[0,2\pi)$ tells us that $\theta = 0$ is included in the interval (because there's a square bracket around the 0), but that $\theta = 2\pi$ is excluded from the interval (because there's a parenthesis around the 2π). That's why we say $\theta = 0$ falls in the interval, while $\theta = 2\pi$ does not.

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Law of sines

Earlier we learned how to solve right triangles by finding all three side lengths and all three interior angle measures.

But we'd also like to be able to solve **oblique triangles**, which are triangles that aren't right (they don't include a right angle). Of course, every triangle has six values we want to find: three side lengths and three interior angles. We'll be able to solve any oblique triangle whenever we know three of these six pieces of information, as long as one of the things we know is one side length.

In other words, the only time we won't be able to solve the triangle is when we only know the three interior angle measures (AAA). That's because knowing three angles gives us the shape of the triangle, but tells us nothing about the size of the triangle. There will be an infinite number of **similar triangles** (triangles with the same shape but different size) that match the three-angle set, but we'll have no way of finding a single set of side lengths.

But for any set of information other than AAA, we'll be able to solve the triangle. We want to address each possible combination of information, but let's go ahead and summarize all of them here:

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Known information	How to solve
SAA or ASA One side and two angles	1. Use A+B+C=180° to find the remaining angle 2. Use law of sines to find the remaining sides
SAS Two sides and the included angle	1. Use law of cosines to find the third side 2. Use law of sines to find another angle 3. Use A+B+C=180° to find the remaining angle
SSS Three sides	1. Use law of cosines to find the largest angle 2. Use law of sines to find either remaining angle 3. Use A+B+C=180° to find the remaining angle
SSA Two sides and a non-included angle	**The ambiguous case**. If two triangles exist, use this same set of steps to find both triangles. 1. Use law of sines to find an angle 2. Use A+B+C=180° to find the remaining angle 3. Use law of sines to find the remaining side

If we know one side and two angles (SAA or ASA), or if we know two sides and the included angle (SAS), or if we know all three sides (SSS), then there's exactly one triangle that can be a solution.

But if we know two sides and a non-included angle (SSA), we call this the ambiguous case because it's possible that there are 0, 1, or 2 triangles that satisfy the given conditions. In the ambiguous case, we'll need to first determine how many triangles we have. Then, if we have one or two triangles that satisfy the information, then we'll need to solve for each triangle.

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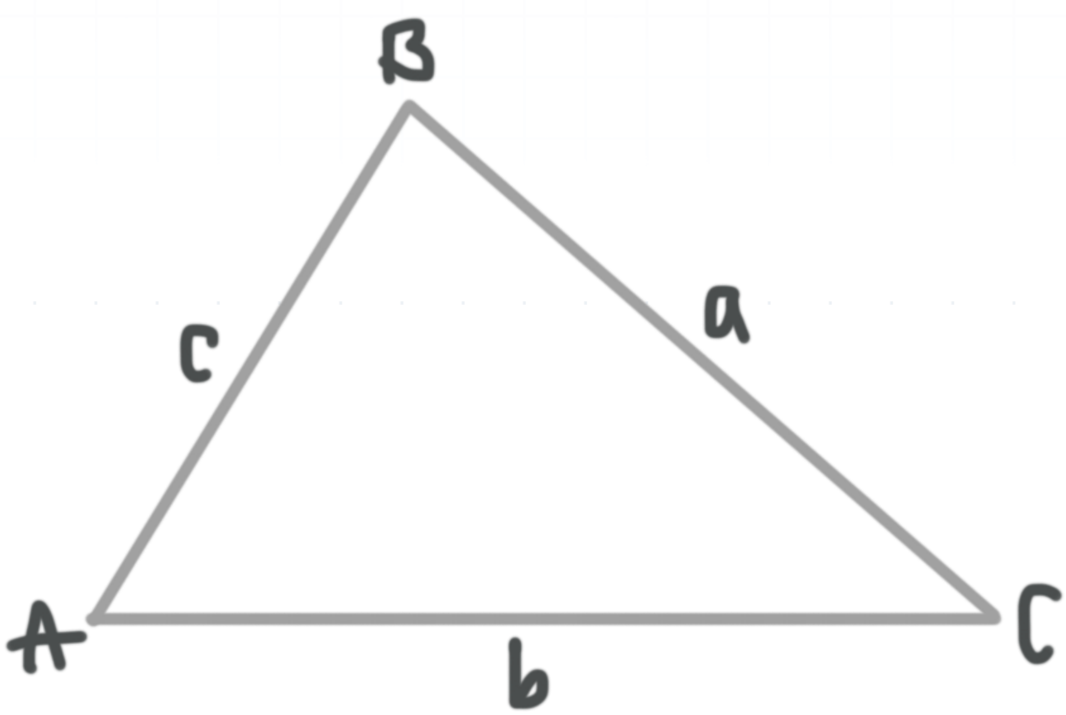
In this lesson, we'll tackle the SAA or ASA case, which will require us to use the law of sines. In the next lesson, we'll look at the SSA ambiguous case since it also uses the law of sines.

Then, later on, we'll look at the law of cosines and the SAS and SSS cases that require that law.

For now, let's introduce the law of sines.

The law of sines

For any triangle with vertices A, B, and C, where side a is opposite angle A, side b is opposite angle B, and side c is opposite angle C,



the **law of sines** says

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

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It’s equivalent to write this equation as

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

Which of these two equations we use depends on the value we’re trying to solve for. It’s usually easiest to set up the equation such that the value we’re trying to find is in the numerator. Which means that we’ll prefer the first equation if we’re trying to solve for a side length, and we’ll prefer the second equation if we’re trying to solve for an angle.

But which equation we choose really doesn’t matter, because we’ll always be able to use algebra to rewrite the equation and solve for the value we need, regardless of whether we start with the unknown value in the numerator or denominator.

The idea of these law of sines equations is that we’ll have the same ratio between the angles and sides of the triangle.

And even though the law of sines is a three-part equation, we can always pull apart the equation and include only two pieces of it. So each of these equations is also valid:

$$\frac{a}{\sin A} = \frac{b}{\sin B} \qquad \frac{\sin A}{a} = \frac{\sin B}{b}$$

$$\frac{a}{\sin A} = \frac{c}{\sin C} \qquad \frac{\sin A}{a} = \frac{\sin C}{c}$$

$$\frac{b}{\sin B} = \frac{c}{\sin C} \qquad \frac{\sin B}{b} = \frac{\sin C}{c}$$

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SAA or ASA

Now let's tackle the first case in our table, which is when we know the measures of any two interior angles and the length of one of the sides.

In the SAA case, we know two adjacent angles and a side opposite one of those angles. In the ASA case, we know two adjacent angles and the side between them.

Either way, to solve these kinds of triangles, we'll start by using $A+B+C=180^\circ$ to find the measure of the remaining unknown angle. Once we know all three angles, we'll use the law of sines to find the length of the two remaining unknown sides.

Let's do an example with an ASA triangle.

Example

Solve the triangle that has angles 38° and 64°, where the length of the side opposite the third angle is 55.

We'll let angle $A=38^\circ$ and angle $B=64^\circ$, then we'll find the measure of the third angle.

$$A+B+C=180^\circ$$

$$38^\circ+64^\circ+C=180^\circ$$

$$C=180^\circ-38^\circ-64^\circ$$

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$$C = 78^\circ$$

The known side is opposite this third angle C, so we'll say $c = 55$. Plugging this side length and all three angle measures into the law of sines gives

$$\frac{a}{\sin 38^\circ} = \frac{b}{\sin 64^\circ} = \frac{55}{\sin 78^\circ}$$

We'll use just the first and third parts of the three-part equation in order to solve for a.

$$\frac{a}{\sin 38^\circ} = \frac{55}{\sin 78^\circ}$$

$$a = \frac{55 \sin 38^\circ}{\sin 78^\circ}$$

$$a \approx 34.6$$

To solve for b, we'll use just the second and third parts of the three-part equation.

$$\frac{b}{\sin 64^\circ} = \frac{55}{\sin 78^\circ}$$

$$b = \frac{55 \sin 64^\circ}{\sin 78^\circ}$$

$$b \approx 50.6$$

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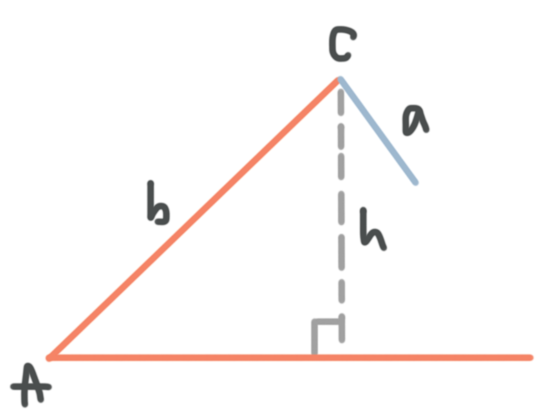
The ambiguous case of the law of sines

In the last lesson, we mentioned the ambiguous case of the law of sines, which we said occurred in an SSA triangle, for which we know the length of two sides of the triangle, and a non-included angle (one of the angles that's *not* between the two known side lengths).

In this SSA case, it's possible that exactly zero triangles are possible, that exactly one triangle is possible, or that exactly two triangles are possible. How many triangles we can get from the given information depends on the lengths of the two known sides and the measure of the one known angle.

Because we don't know initially how many triangles we'll have, this is called the **ambiguous case** of the law of sines. Below is a table summarizing all possible ways that we can get 0, 1, or 2 triangles. In every case in the table, we're given two sides a and b and the non-included angle A. By the definition of sine, the altitude is $h = b \sin A$.

Case	# of triangles	Sketch	Conditions
A is acute			
I	0		$a < h,\ h = b \sin A$

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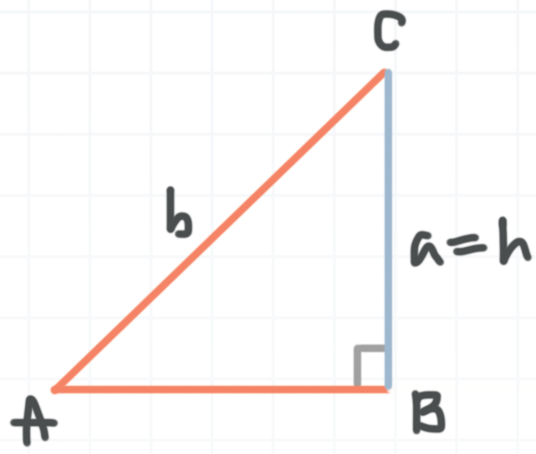
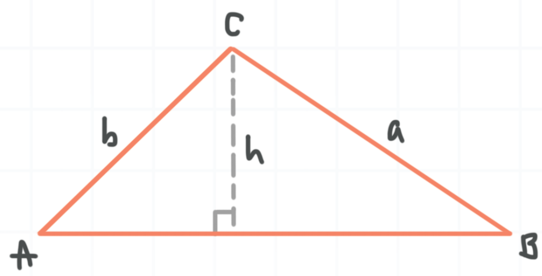
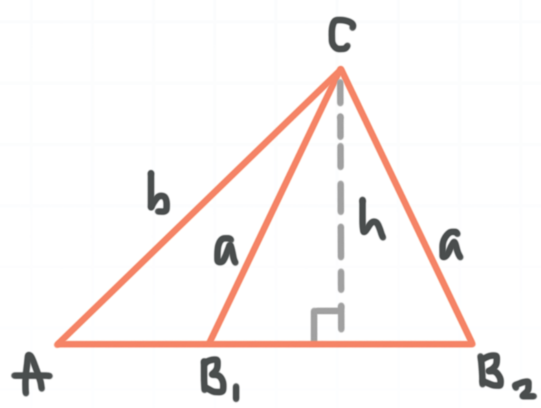
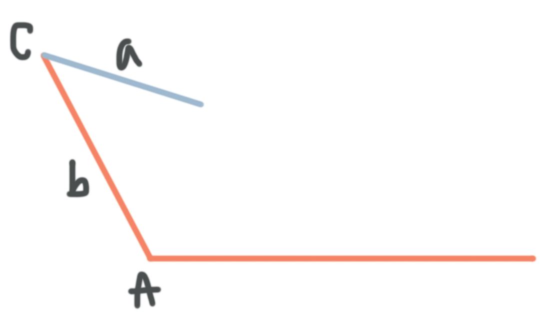
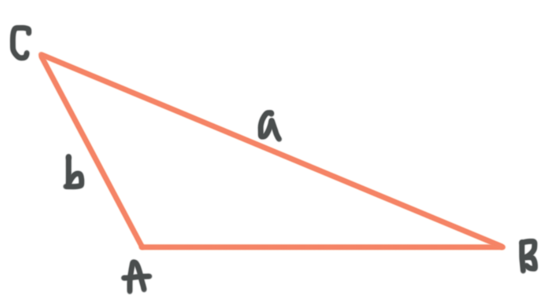
II	1		$a = h$
III	1		$a > b$
IV	2		$b > a > h$
	A is obtuse		
V	0		$a \leq b$
VI	1		$a > b$

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Notice in this table how Case II is a right triangle. As we've already seen, right triangles will always have exactly one solution, so it makes sense that there's exactly 1 triangle possible in a Case II situation.

To solve the ambiguous case by figuring out the number of possible triangles and then solving for every angle and side length for any possible triangle(s), here's the row from the table in the previous lesson showing the problem solving process:

Known information	**How to solve**
SSA Two sides and a non-included angle	**The ambiguous case**. If two triangles exist, use this same set of steps to find both triangles. 1. Use law of sines to find an angle 2. Use A+B+C=180º to find the remaining angle 3. Use law of sines to find the remaining side

Let's do an example of case VI, which is the case where there's exactly one triangle possible, and the triangle includes an obtuse angle.

Example

A triangle has one side with length 3 and another with length 5. The angle opposite the side with length 5 is 40°. Complete the triangle.

Let $a=3$ and $b=5$, and let $B=40^\circ$ be the angle opposite b. Substituting these values into the law of sines gives

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\frac{3}{\sin A} = \frac{5}{\sin 40^\circ} = \frac{c}{\sin C}$$

Use just the first two parts of this three-part equation in order to solve for A.

$$\frac{3}{\sin A} = \frac{5}{\sin 40^\circ}$$

$$3 = \frac{5}{\sin 40^\circ}(\sin A)$$

$$3 \sin 40^\circ = 5 \sin A$$

$$\sin A = \frac{3 \sin 40^\circ}{5}$$

$$\sin A \approx 0.386$$

Apply the inverse sine function to both sides to cancel the sine on the left and solve for A.

$$A \approx \arcsin(0.386)$$

$$A \approx 22.7^\circ$$

Since the sum of the interior angles of any triangle is 180°, the measure of the third interior angle C is approximately

$$C \approx 180^\circ - 40^\circ - 22.7^\circ$$

$$C \approx 117.3^\circ$$

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Use the second and third parts of the three-part equation to solve for c.

$$\frac{5}{\sin 40^\circ} = \frac{c}{\sin C}$$

$$\frac{5}{\sin 40^\circ} \approx \frac{c}{\sin 117.3^\circ}$$

$$c \approx \frac{5}{\sin 40^\circ}(\sin 117.3^\circ)$$

$$c \approx 6.91$$

The side lengths of the triangle are $a = 3$, $b = 5$, and $c \approx 6.91$, and the angle measures are $A \approx 22.7^\circ$, $B = 40^\circ$, and $C = 117.3^\circ$.

Let's do an example of Case V, where no triangle is possible.

Example

Solve the triangle with side lengths 0.68 and 0.92, if angle $C = 118^\circ$ is opposite the side with length 0.68.

Because the side with length 0.68 is opposite the angle $C = 118^\circ$, we'll name that side as $c = 0.68$. We'll also name $b = 0.92$, then we'll plug everything we know into the law of sines.

$$\frac{a}{\sin A} = \frac{0.92}{\sin B} = \frac{0.68}{\sin 118^\circ}$$

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Use the second and third parts of this three-part equation to solve for B.

$$\frac{0.92}{\sin B} = \frac{0.68}{\sin 118^\circ}$$

$$0.92 = \frac{0.68}{\sin 118^\circ}(\sin B)$$

$$0.92 \sin 118^\circ = 0.68 \sin B$$

$$\sin B = \frac{0.92 \sin 118^\circ}{0.68}$$

$$\sin B \approx 1.195$$

Remember that the sine of any angle must have a value on the interval $[-1,1]$. And when we're talking about a triangle, sine of an angle in the triangle needs to be a positive value on the interval $(0,1]$.

Because we're getting $\sin B \approx 1.195$, which is a value for sine greater than 1, a triangle with the given measurements is impossible.

Finally, let's do an example of Case IV, where two triangles are possible.

Example

Solve the triangle with side lengths 41 and 54, where the angle opposite $a = 41$ is 38°.

The problem names side a as $a=41$. Since 38° is opposite that side, we'll say $A=38^\circ$. And we'll name $b=54$. Then we can substitute into the law of sines.

$$\frac{\sin 38^\circ}{41}=\frac{\sin B}{54}=\frac{\sin C}{c}$$

We'll use the first two parts of this three-part equation to solve for the angle B.

$$\frac{\sin 38^\circ}{41}=\frac{\sin B}{54}$$

$$\sin B=\frac{54\sin 38^\circ}{41}$$

$$\sin B\approx 0.811$$

There are two possibilities here. We could find $\sin B\approx 0.811$ in both the first and second quadrants, since those are the quadrants where sine is positive. To find the value of angle B in the first quadrant, we'll apply the inverse sine function to both sides to cancel the sine on the left and solve for B.

$$B\approx \arcsin(0.811)$$

$$B\approx 54^\circ$$

Then to find the other possible value of B in the second quadrant, we'll subtract $B\approx 54^\circ$ from 180°.

$$B'\approx 180^\circ-54^\circ$$

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$$B' \approx 126^\circ$$

Then the remaining angle is either

$$C \approx 180^\circ - 38^\circ - 54^\circ$$

$$C \approx 88^\circ$$

or

$$C' \approx 180^\circ - 38^\circ - 126^\circ$$

$$C' \approx 16^\circ$$

Finally, we can find the measure of the remaining side. If we plug everything we know so far into the law of sines, we get either

$$\frac{\sin 38^\circ}{41} \approx \frac{\sin 54^\circ}{54} \approx \frac{\sin 88^\circ}{c}$$

$$0.0150 \approx 0.0150 \approx \frac{\sin 88^\circ}{c}$$

$$c \approx \frac{\sin 88^\circ}{0.0150}$$

$$c \approx 67$$

or

$$\frac{\sin 38^\circ}{41} \approx \frac{\sin 126^\circ}{54} \approx \frac{\sin 16^\circ}{c'}$$

$$0.0150 \approx 0.0150 \approx \frac{\sin 16^\circ}{c'}$$

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$$c' = \frac{\sin 16^\circ}{0.0150}$$

$$c' \approx 18$$

If we summarize what we've found, we get the values for both triangles:

Triangle #1: $A = 38^\circ$, $B \approx 54^\circ$, $C \approx 88^\circ$, $a = 41$, $b = 54$, and $c \approx 67$

Triangle #2: $A = 38^\circ$, $B' \approx 126^\circ$, $C' \approx 16^\circ$, $a = 41$, $b = 54$, and $c \approx 18$

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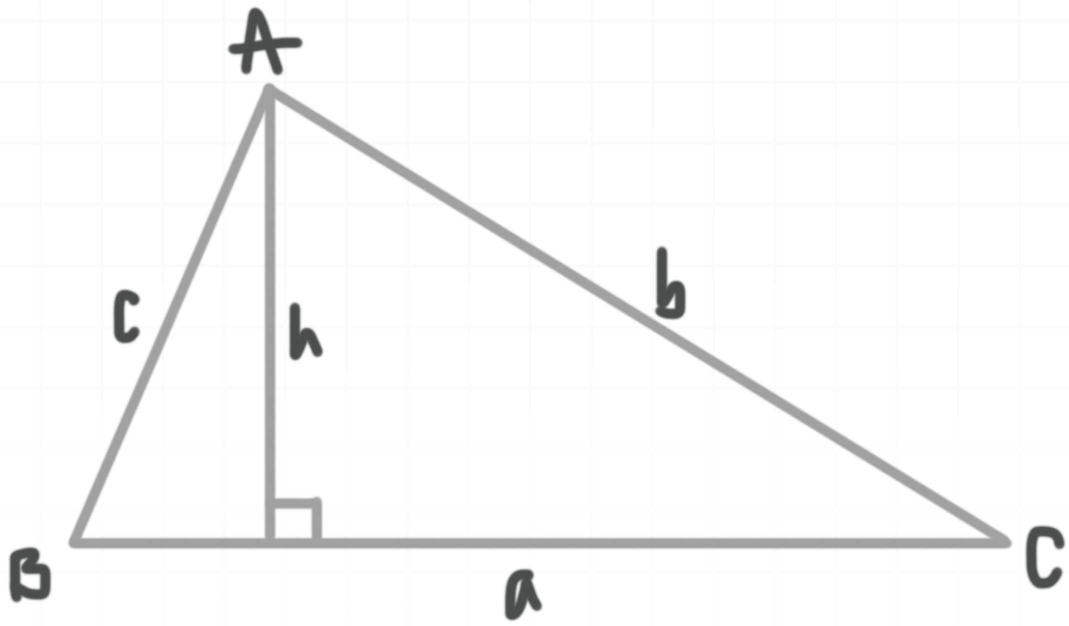
Area from the law of sines

From the law of sines, we can find formulas that give the area of an oblique triangle. Given any oblique triangle with angles A, B, and C, and side lengths a, b, and c,



The area of the triangle will be $A = (1/2)ah$, where a is the base and h is height (when the height is drawn perpendicular to a). By the definition of sine, the height is $h = b \sin C$. Therefore, we can rewrite the area formula as $A = (1/2)ab \sin C$. If we choose either of the other two sides as the triangle's "base," then we can write similar formulas for the area of the triangle in terms of $\sin A$ and $\sin B$:

$$\text{Area} = \frac{1}{2}ab \sin C \qquad \text{Area} = \frac{1}{2}ac \sin B \qquad \text{Area} = \frac{1}{2}bc \sin A$$

Notice that it doesn't matter which sides or angle we use. What matters is that we use two sides, and then the angle we choose should be the one opposite the side we didn't use.

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That happens to always be the angle that's included between the two sides we chose. In other words, we always need to use two sides and their included angle.

These area formulas are sometimes called the **law of sines for the area of a triangle**, and we can apply it anytime we know the lengths of two sides of a triangle and the measure of the included angle.

Let's do an example where we apply the law of sines for the area of a triangle.

Example

Find the area of the triangle in which two of the sides have lengths 23 and 5 and the measure of the included angle is 38°.

Let $a = 23$ and $b = 5$, and let angle $C = 38^\circ$ be the included angle. Then we'll use the area formula that includes sides a and b and the angle C.

$$\text{Area} = \frac{1}{2}ab\sin C$$

$$\text{Area} = \frac{1}{2}(23)(5)(\sin 38^\circ)$$

$$\text{Area} = \frac{115\sin 38^\circ}{2}$$

$$\text{Area} \approx 35.4$$

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Realize here that we could have used any of the area formulas and the result would have come out the same. For instance, we could have used the second formula, setting $a = 23$, $b = 5$, and $B = 38^\circ$.

Notice also what happens to these area formulas when we use an angle of 90° as the included angle. Let's plug $C = 90^\circ$ into the first area formula, naming a and b as the side lengths adjacent to C.

$$\text{Area} = \frac{1}{2}ab\sin C$$

$$\text{Area} = \frac{1}{2}ab\sin 90^\circ$$

$$\text{Area} = \frac{1}{2}ab(1)$$

$$\text{Area} = \frac{1}{2}ab$$

Using $C = 90^\circ$ means we're dealing with a right triangle. When that's the case, the area formula simplifies to $(1/2)ab$.

Because a and b are the sides around the right angle, we can think of b as the base of the triangle and a as the height of the triangle. So we could rewrite the area formula as

$$\text{Area} = \frac{1}{2}bh$$

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And we recognize that this is the standard formula for the area of a right triangle. So it makes sense that this would be the result when we use $C = 90^\circ$ as the included angle.

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Law of cosines

When we first learned about solving oblique triangles, we listed out in the table below the full set of scenarios in which we'd need to use either the law of sines or law of cosines.

Known information	How to solve
SAA or ASA One side and two angles	1. Use $A+B+C=180^\circ$ to find the remaining angle 2. Use law of sines to find the remaining sides
SAS Two sides and the included angle	1. Use law of cosines to find the third side 2. Use law of sines to find another angle 3. Use $A+B+C=180^\circ$ to find the remaining angle
SSS Three sides	1. Use law of cosines to find the largest angle 2. Use law of sines to find either remaining angle 3. Use $A+B+C=180^\circ$ to find the remaining angle
SSA Two sides and a non-included angle	**The ambiguous case**. If two triangles exist, use this same set of steps to find both triangles. 1. Use law of sines to find an angle 2. Use $A+B+C=180^\circ$ to find the remaining angle 3. Use law of sines to find the remaining side

We've already talked about the SAA or ASA case and the SSA ambiguous case, which both require the law of sines. In this lesson, we'll talk about the SAS and SSS cases, both of which requires the law of cosines.

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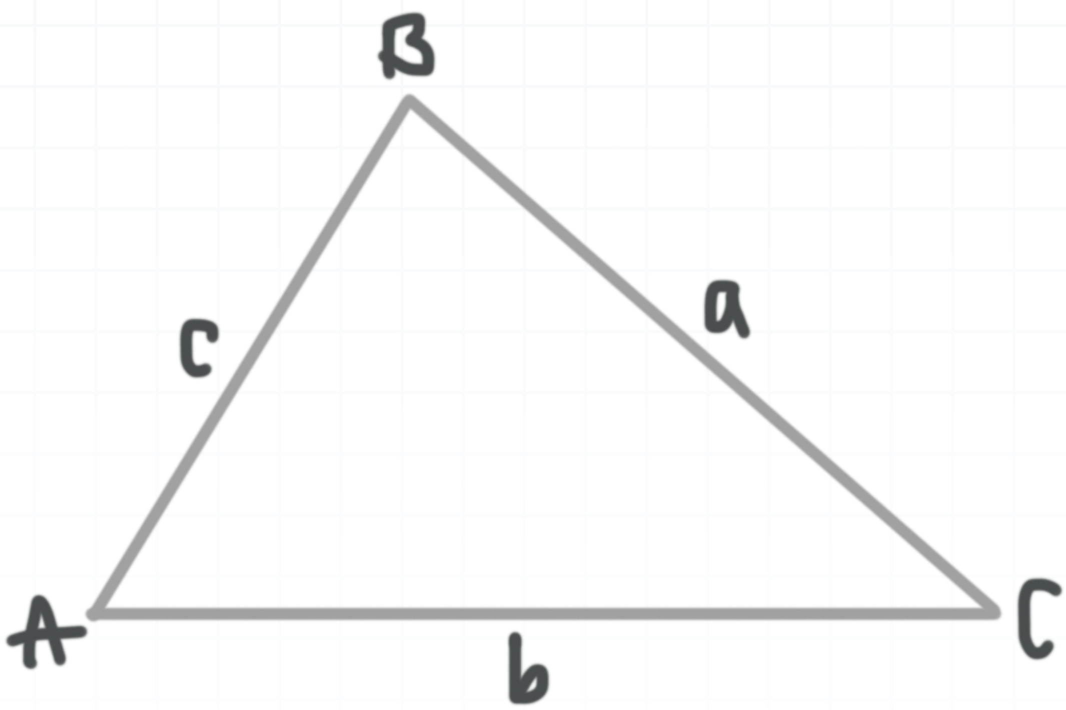
The law of cosines

For any triangle with vertices A, B, and C, where side a is opposite angle A, side b is opposite angle B, and side c is opposite angle C,



the **law of cosines** comes in three parts:

$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$b^2 = a^2 + c^2 - 2ac\cos B$$

$$a^2 = b^2 + c^2 - 2bc\cos A$$

There are several things we want to say about these formulas. First, realize that they all follow the same pattern.

Look at the first formula, which is the one that includes the angle C. For the angle in the formula, we always use its corresponding side on the other side of the formula. So the formula with angle C is set equal to c^2. The other two sides, a and b, are only the right side as $a^2 + b^2 - 2ab$. For the formula that includes the angle B, we have the matching b^2 on the left side,

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and then the other two sides a and c follow the same $a^2 + c^2 - 2ac$ pattern on the right.

In other words, these three formulas are actually all the same and can be used interchangeably. All that matters is that we get the correct relationships between the angles and sides.

Second, we want to notice what happens when we use a 90° angle. We'll plug into the first formula, and we get

$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$c^2 = a^2 + b^2 - 2ab\cos 90^\circ$$

$$c^2 = a^2 + b^2 - 2ab(0)$$

$$c^2 = a^2 + b^2$$

$$a^2 + b^2 = c^2$$

We've arrived at the Pythagorean theorem, which makes sense. Because $C = 90^\circ$, we have a right triangle, and the Pythagorean theorem gives us the relationship between the side lengths in a right triangle.

Lastly, these law of cosines formulas can be solved for the cosine functions as

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

$$\cos B = \frac{a^2 - b^2 + c^2}{2ac}$$

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$$\cos A = \frac{-a^2 + b^2 + c^2}{2bc}$$

Sometimes it'll be more convenient to apply the formulas in this form, which is why it's nice to know them. To remember the formulas this way, realize that the negative sign attaches to the side length in the numerator on the right side that corresponds to the angle from the left side.

Let's look at an SSS example where we use the law of cosines to complete a triangle when we know all three side lengths.

Example

Solve the triangle that has side lengths 11, 6, and 9.

We'll plug what we know into the law of cosines in order to find one angle in the triangle. If we let $a = 11$, $b = 6$, and $c = 9$, then for angle A we'll get

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$11^2 = 6^2 + 9^2 - 2(6)(9)\cos A$$

$$121 = 36 + 81 - 108 \cos A$$

$$4 = -108 \cos A$$

$$\cos A = -\frac{1}{27}$$

$$\cos A \approx -0.0370$$

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Solve for A by applying the inverse cosine function to both sides.

$$A \approx \arccos(-0.0370)$$

$$A \approx 92^\circ$$

The easiest way to get the next angle will be to plug into the law of sines.

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

$$\frac{\sin 92^\circ}{11} = \frac{\sin B}{6} = \frac{\sin C}{9}$$

Use the first and second parts of this three-part equation to solve for the angle B.

$$\frac{\sin 92^\circ}{11} = \frac{\sin B}{6}$$

$$\sin B = \frac{6 \sin 92^\circ}{11}$$

$$\sin B \approx 0.5451$$

Solve for B by applying the inverse sine function to both sides.

$$B \approx \arcsin(0.5451)$$

$$B \approx 33^\circ$$

Remember that once we have the first two angles, we can simply find the third angle by subtracting the first two from 180°

$$C = 180^\circ - A - B$$

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$$C \approx 180^\circ - 92^\circ - 33^\circ$$

$$C \approx 55^\circ$$

To summarize, the given triangle is defined by $a = 11$, $b = 6$, $c = 9$, $A \approx 92^\circ$, $B \approx 33^\circ$, and $C \approx 55^\circ$.

Let's look at another example where only the three side lengths are known. This time though, the triangle won't be defined.

Example

Solve the triangle with side lengths 18, 15, and 2.

Let $a = 18$, $b = 15$, and $c = 2$, plugging into the law of cosines formula that's solved for $\cos A$. We choose the $\cos A$ formula because a is the longest side, so A will be the largest angle.

$$\cos A = \frac{-a^2 + b^2 + c^2}{2bc}$$

$$\cos A = \frac{-18^2 + 15^2 + 2^2}{2(15)(2)}$$

$$\cos A = \frac{-324 + 225 + 4}{30}$$

$$\cos A = -\frac{95}{30}$$

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$$\cos A \approx -3.17$$

But remember, the cosine function has a range of $[-1,1]$. So it's impossible to find $\cos A \approx -3.17$, which means that the triangle with the given side lengths actually can't exist.

Finally, let's do an SAS example where we know the length of two sides and the measure of their "included angle," which is the angle between those two sides.

Example

Solve the triangle where two of the sides are 25 and 21 and the measure of their included angle is 70°.

We'll start by using the law of cosines to the find length of the third side. Let $a = 25$ and $b = 21$. The included angle will then be $C = 70^\circ$.

$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$c^2 = 25^2 + 21^2 - 2(25)(21)\cos 70^\circ$$

$$c^2 = 625 + 441 - 1{,}050\cos 70^\circ$$

$$c^2 = 1{,}066 - 1{,}050\cos 70^\circ$$

$$c^2 \approx 1{,}066 - 1{,}050(0.342)$$

$$c^2 \approx 1{,}066 - 359$$

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$$c^2 \approx 707$$

$$c \approx \sqrt{707}$$

$$c \approx 26.6$$

Now that we have this one side length, we'll use the law of sines to find another angle.

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

$$\frac{\sin A}{25} = \frac{\sin B}{21} = \frac{\sin 70^\circ}{26.6}$$

Use the first and third parts of this three-part equation to find the measure of angle A.

$$\frac{\sin A}{25} = \frac{\sin 70^\circ}{26.6}$$

$$\sin A = \frac{25 \sin 70^\circ}{26.6}$$

$$\sin A \approx 0.8832$$

$$A \approx \arcsin(0.8832)$$

$$A \approx 62^\circ$$

Now find the measure of the third angle.

$$B = 180^\circ - A - C$$

$$B = 180^\circ - 62^\circ - 70^\circ$$

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$$B = 48^\circ$$

To summarize, the given triangle is defined by $a = 25$, $b = 21$, $c \approx 26.6$, $A \approx 62^\circ$, $B = 48^\circ$, and $C = 70^\circ$.

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Heron's formula

We've already looked at how to use the law of sines for finding the area of a triangle, but that's not the only formula we can use to find area.

Heron's formula for area of a triangle

When we know the lengths of all three sides of a triangle, we can find its area using Heron's formula. And this area formula works for all oblique triangles, not just right triangles. **Heron's formula** is

$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)}$$

where a, b, and c are the lengths of the sides of the triangle and

$$s = \frac{1}{2}(a+b+c)$$

which is half the perimeter of the triangle.

Let's look at an example where we use the three side lengths in Heron's formula to calculate area.

Example

Apply Heron's formula to find the area of the triangle that has side lengths 16, 19, and 7.

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Let $a=16$, $b=19$, and $c=7$, then calculate s.

$$s=\frac{1}{2}(a+b+c)$$

$$s=\frac{1}{2}(16+19+7)$$

$$s=\frac{1}{2}(35+7)$$

$$s=\frac{1}{2}(42)$$

$$s=21$$

Now we'll plug into Heron's formula.

$$\text{Area}=\sqrt{s(s-a)(s-b)(s-c)}$$

$$\text{Area}=\sqrt{21(21-16)(21-19)(21-7)}$$

$$\text{Area}=\sqrt{21(5)(2)(14)}$$

$$\text{Area}=\sqrt{21(140)}$$

$$\text{Area}=\sqrt{2{,}940}$$

$$\text{Area}\approx 54.2$$

What we've shown is that, if we know the lengths of two sides of a triangle and the measure of the included angle, we can use the law of sines for the

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area of a triangle to compute the area. And if we know the lengths of all three sides of a triangle, we can use Heron's formula to compute area.

But how do we get the area of a triangle if we don't have either of these particular information sets?

In that's the case, we might need to apply the law of sines or cosines first to find whatever information we're missing, and then go on to find area once we have everything we need.

Example

Find the area of the triangle that has side lengths 10 and 5 and where the angle opposite the side with length 10 is 40°.

Let $a=10$ and $b=5$. Then the angle opposite the side of length 10 is $A=40^\circ$, the angle opposite the side of length 5 is B, and the included angle is C.

We want to be able to eventually apply the law of sines for the area of a triangle, so we'll first find the measure of B, and then use that to get the measure of C.

By the law of sines, we get

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$$

$$\frac{10}{\sin 40^\circ}=\frac{5}{\sin B}=\frac{c}{\sin C}$$

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Use the first and second parts of this three-part equation to solve for B.

$$\frac{10}{\sin 40^\circ} = \frac{5}{\sin B}$$

$$\frac{10}{\sin 40^\circ}(\sin B) = 5$$

$$10 \sin B = 5 \sin 40^\circ$$

$$\sin B = \frac{5 \sin 40^\circ}{10}$$

$$\sin B \approx 0.322$$

$$B \approx \arcsin(0.322)$$

$$B \approx 19^\circ$$

Then the measure of angle C is

$$C \approx 180^\circ - 40^\circ - 19^\circ$$

$$C \approx 121^\circ$$

Now we're ready to compute the area of the triangle. Plugging what we know into the law of sines for the area of a triangle, we get

$$\text{Area} = \frac{1}{2}ab \sin C$$

$$\text{Area} \approx \frac{1}{2}(10)(5)\sin 121^\circ$$

$$\text{Area} \approx 25(0.857)$$

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Area ≈ 21.4

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