Graphing limacons

In the previous lesson, we looked at how to graph cardioids, which are a specific category of limaçon (limaçon is the French word for "snail," which is a nod to the shape of the curve). Limaçons are polar equations in one of these forms,

$$r = a + b\cos\theta \qquad\qquad r = a + b\sin\theta$$

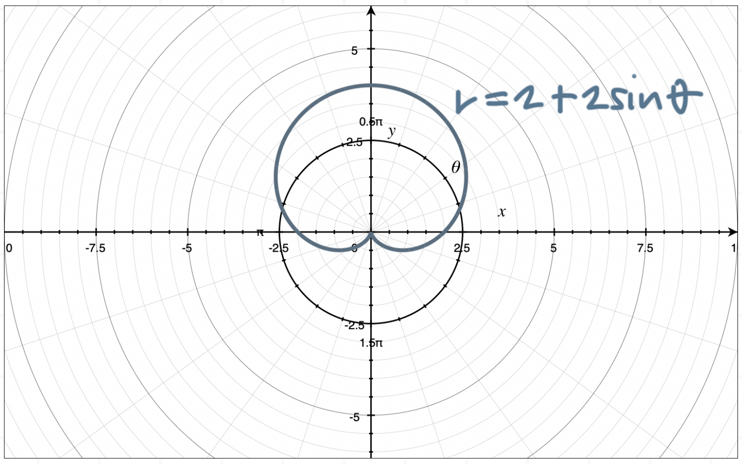
$$r = a - b\cos\theta \qquad\qquad r = a - b\sin\theta$$

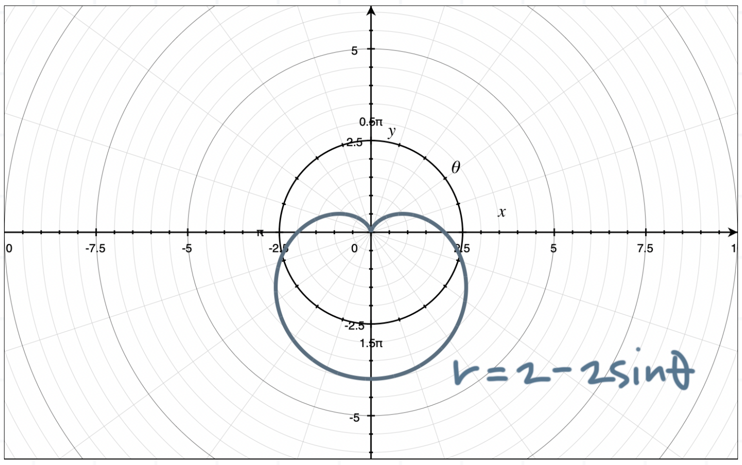
where a and b are positive numbers and where $a \neq b$. A cardioid is just a category of limaçon where $a = b$, so in this lesson we'll focus just on limaçons where $a \neq b$.

Properties of limaçons

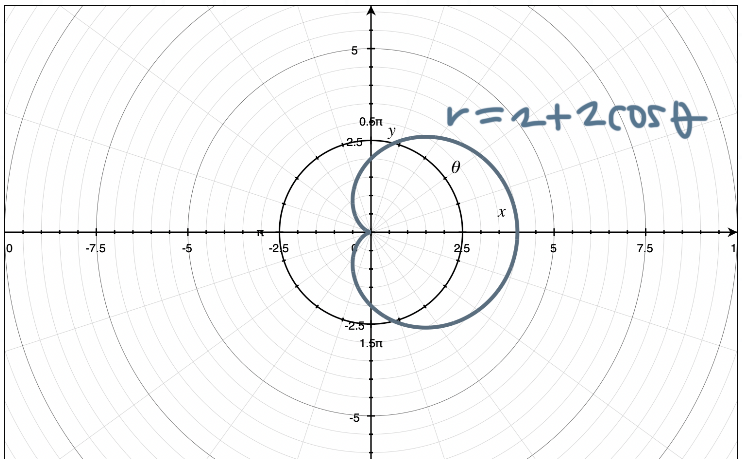
Just like for cardioids specifically, the location of the graphs of limaçons depends on whether the curve is a sine limaçon or a cosine limaçon, and whether the sign between the terms in the equation is a positive or negative sign.

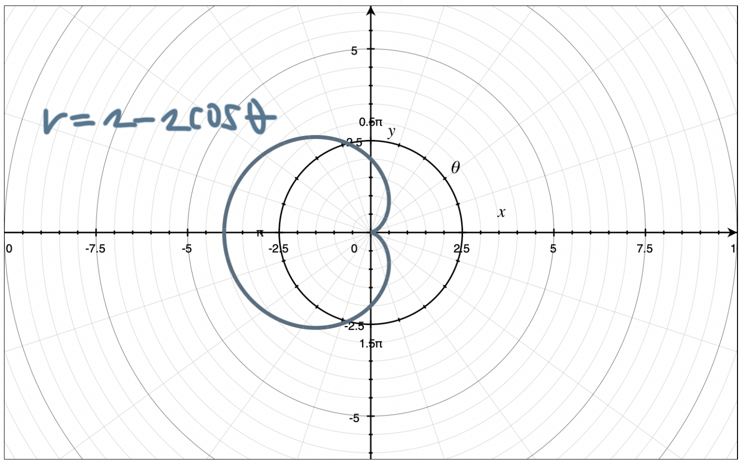
Sine limaçons are symmetric about the vertical axis, and sine limaçon equations with a positive sign will sit mostly above the horizontal axis, while sine limaçon equations with a negative sign will sit mostly below the horizontal axis.





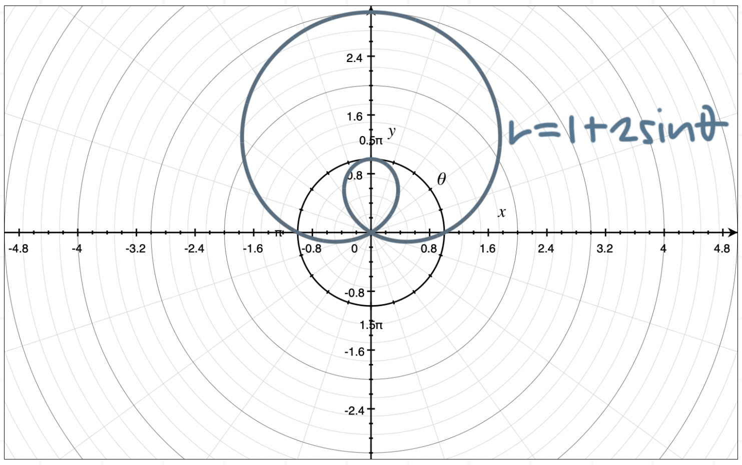
Cosine limaçons are symmetric above the horizontal axis, and cosine limaçon equations with a positive sign will sit mostly to the right of the vertical axis, while cosine limaçon equations with a negative sign will sit mostly to the left of the vertical axis.

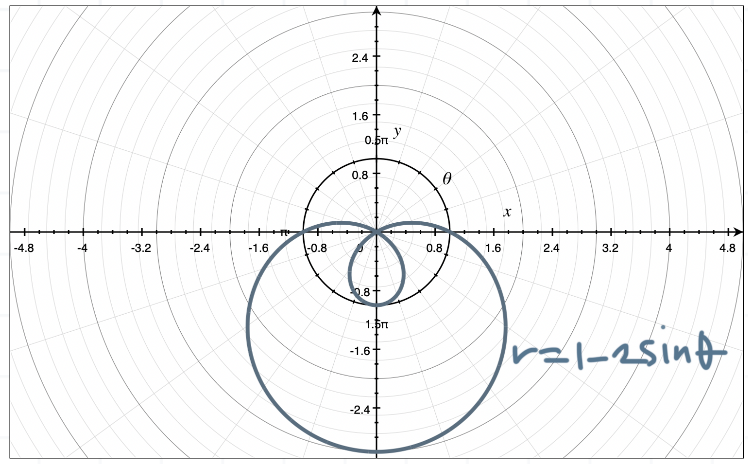


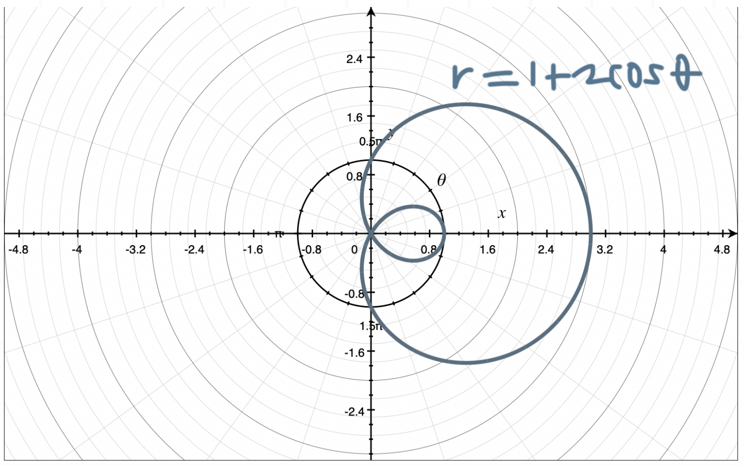


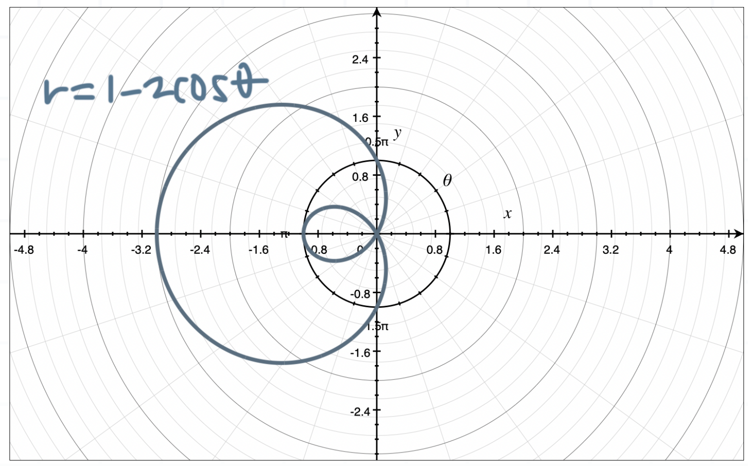
The relationship between a and b will determine the exact shape of the limaçon. We've already seen that the limaçon will be a cardioid when $a = b$, or put another way, when $a/b = 1$.

When $a/b < 1$, the graph of the limaçon will include a small loop.

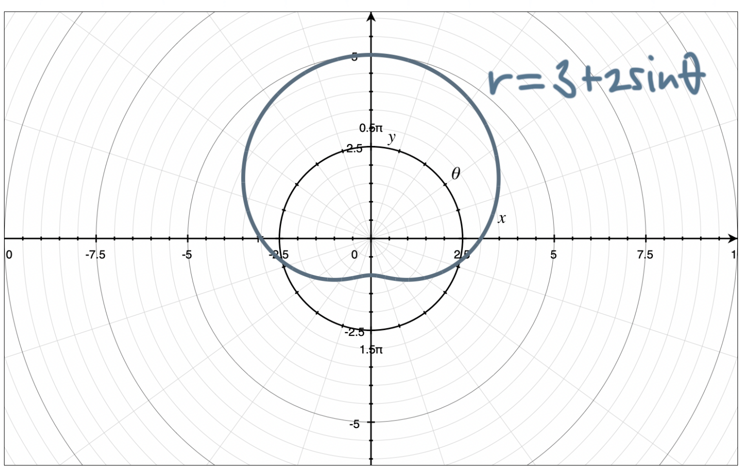


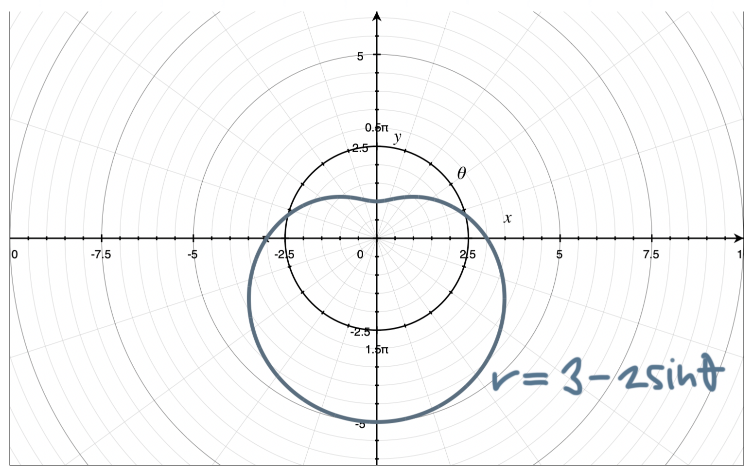


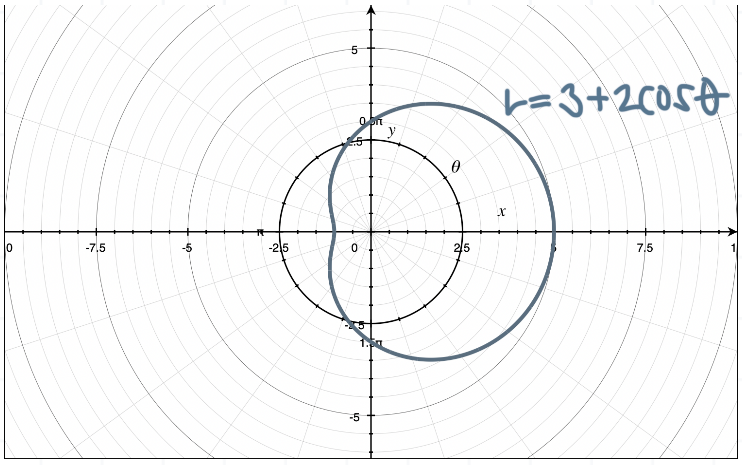


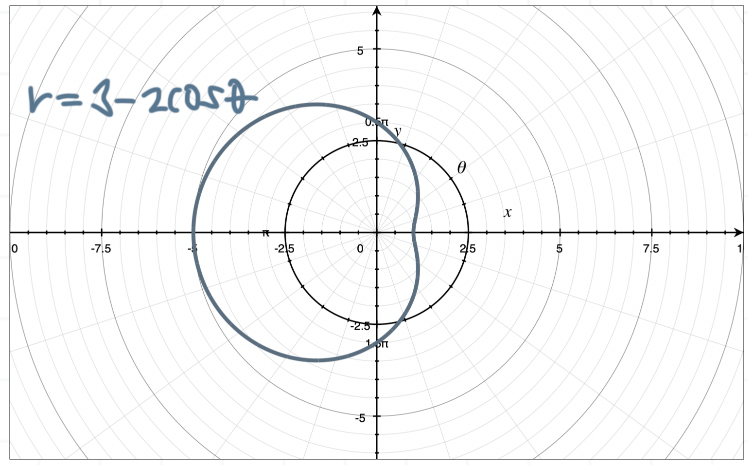


When $1 < a/b < 2$, the graph of the limaçon will include a small dip, like the cardioids we looked at earlier, but less severe.

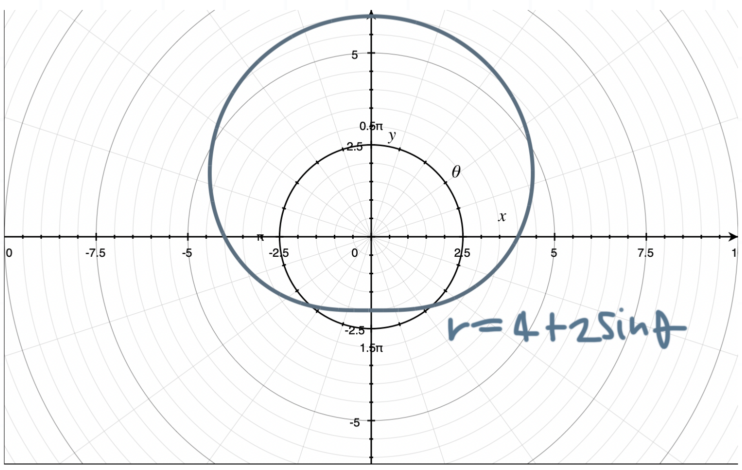


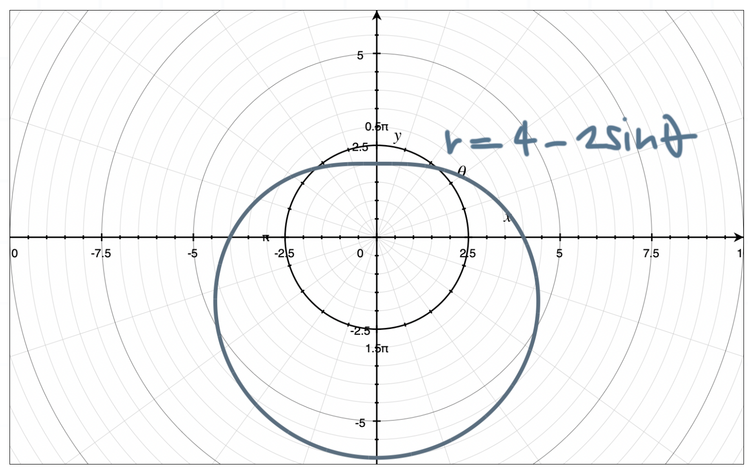


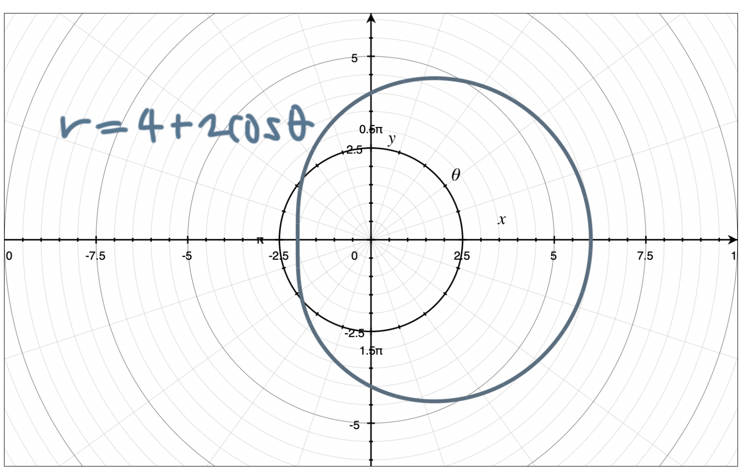


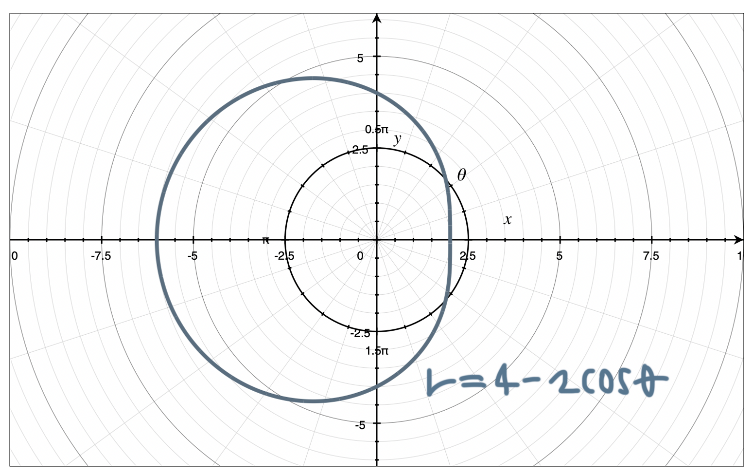


And when $a/b \geq 2$, the graph of the limaçon will include the smallest dip yet, even less severe than the $1 < a/b < 2$ case. The shape of these curves is close to, but not quite, a perfect circle.









How to sketch limaçons

We'll use the same approach to sketch limaçons that we've used previously to sketch circles, roses, and cardioids:

1. Set the argument of the trigonometric function equal to $\pi/2$, and then solve the equation for θ.

2. Evaluate the polar curve at multiples of the θ-value we solved for in Step 1, starting with $\theta = 0$, and plot the resulting points on the polar graph.

3. Connect the points on the polar graph with a smooth curve.

Let's do an example where we sketch the graph of a cosine limaçon.

Example

Sketch the graph of $r = 3 + 4\cos\theta$.

With $a = 3$ and $b = 4$, we get $a/b = 3/4 < 1$, which means the graph of this limaçon will include a small loop. And because this is a cosine limaçon with a positive sign separating the terms, it'll be symmetric about the horizontal axis, and sit mostly to the right of the vertical axis.

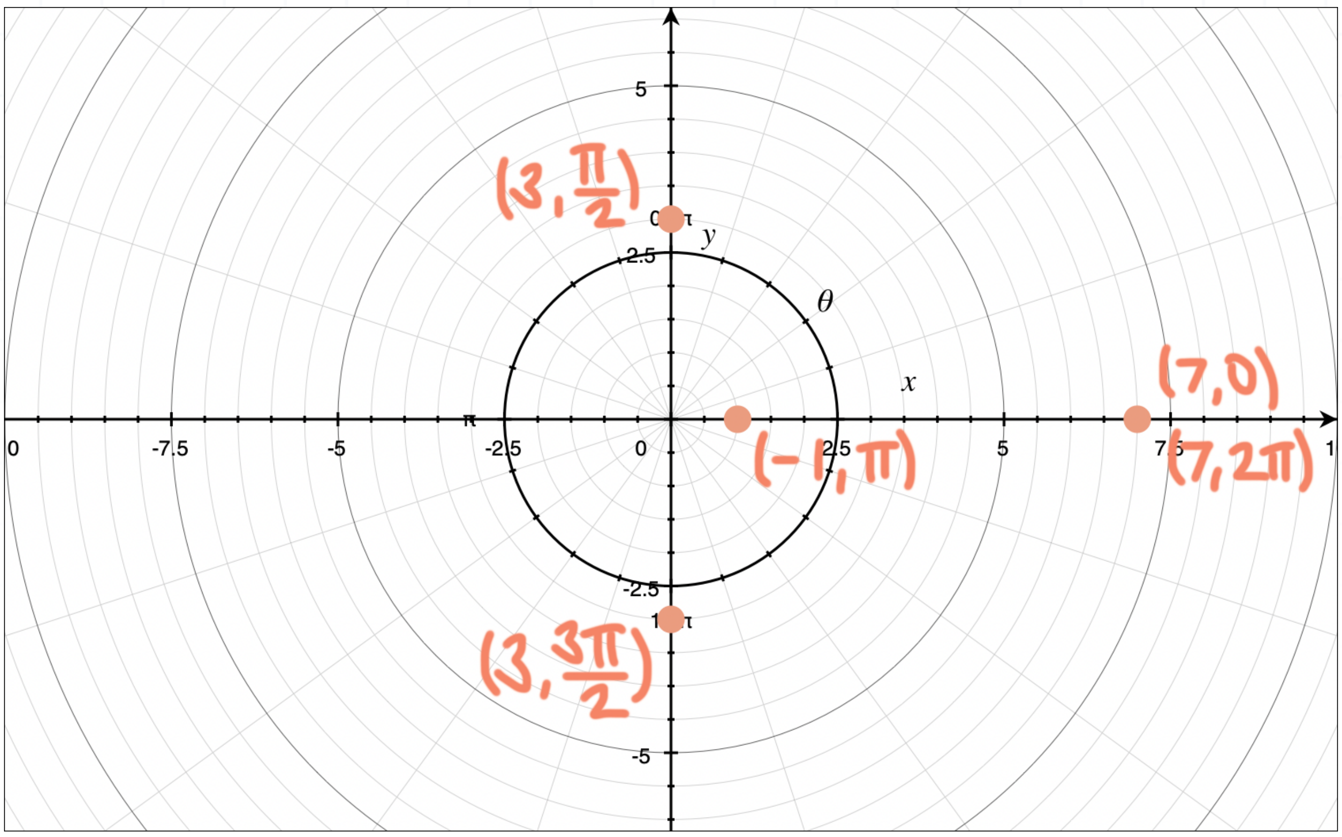
The trigonometric function in this polar equation is $\cos\theta$, and its argument (the angle at which cosine is evaluated) is θ. So we'll set

$$\theta = \frac{\pi}{2}$$

Now we'll make a table with multiples of $\pi/2$, like $\theta = 0$, $\pi/2$, π, $3\pi/2$, 2π, etc., and include the values of r that correspond to each of these θ-values.

θ	0	$\pi/2$	π	$3\pi/2$	2π
r	7	3	-1	3	7

Plotting these points on the polar graph gives



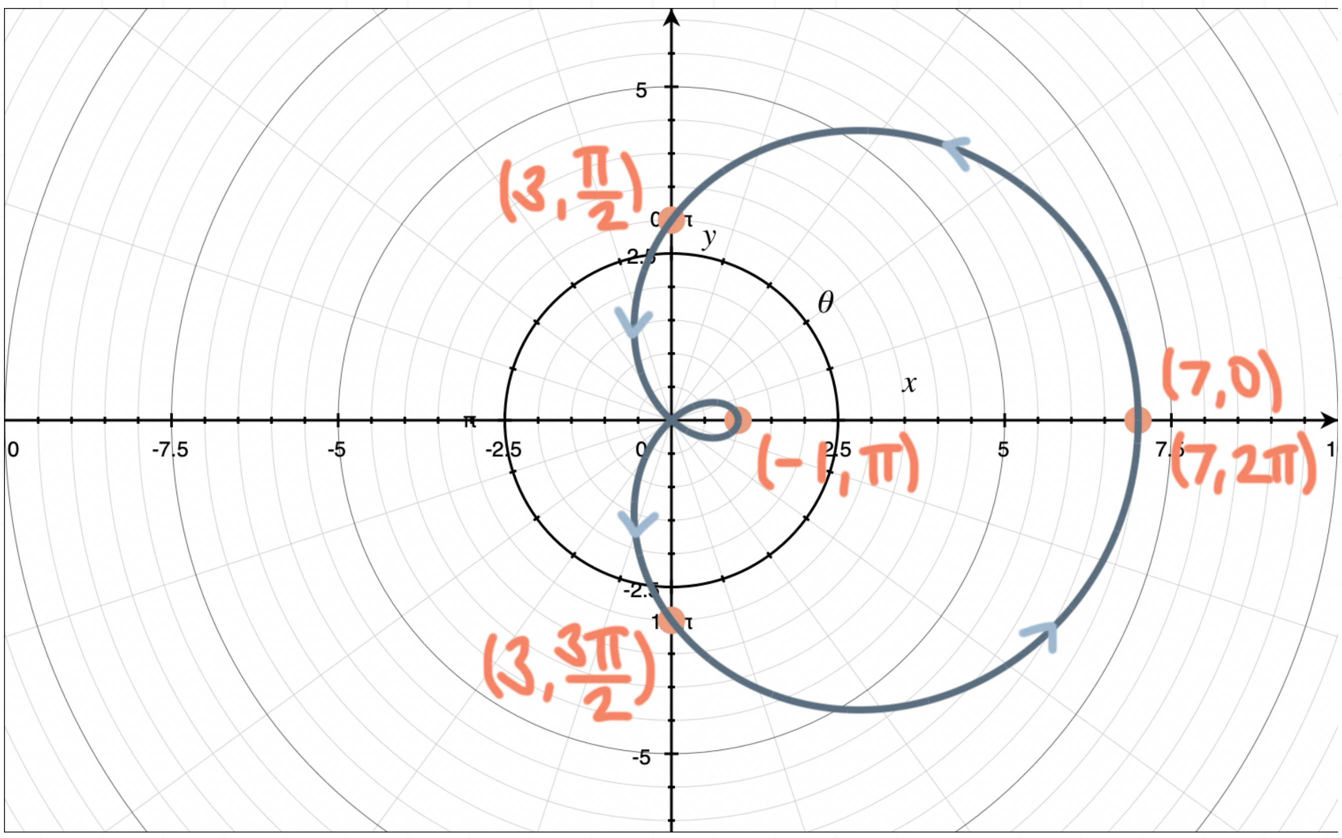
And if we connect these points with a smooth curve, in order, we see the graph of the limaçon. We start at $(7,0)$, and then

loop to $(3,\pi/2)$,

loop to $(-1,\pi)$,

loop to $(3,3\pi/2)$,

then finally loop back to $(7,2\pi)$, which is actually the same point as $(7,0)$. From there on, we're retracing the same pieces of the limaçon over and over.



Let's do another example, this time with a sine limaçon and a different a/b ratio.

Example

Sketch the graph of $r = 3 - 2\sin\theta$.

With $a = 3$ and $b = 2$, we get $a/b = 3/2$, and $1 < 3/2 < 2$, which means the graph of this limaçon will include a small dip. And because this is a sine limaçon with a negative sign separating the terms, it'll be symmetric about the vertical axis, and sit mostly below the vertical axis.

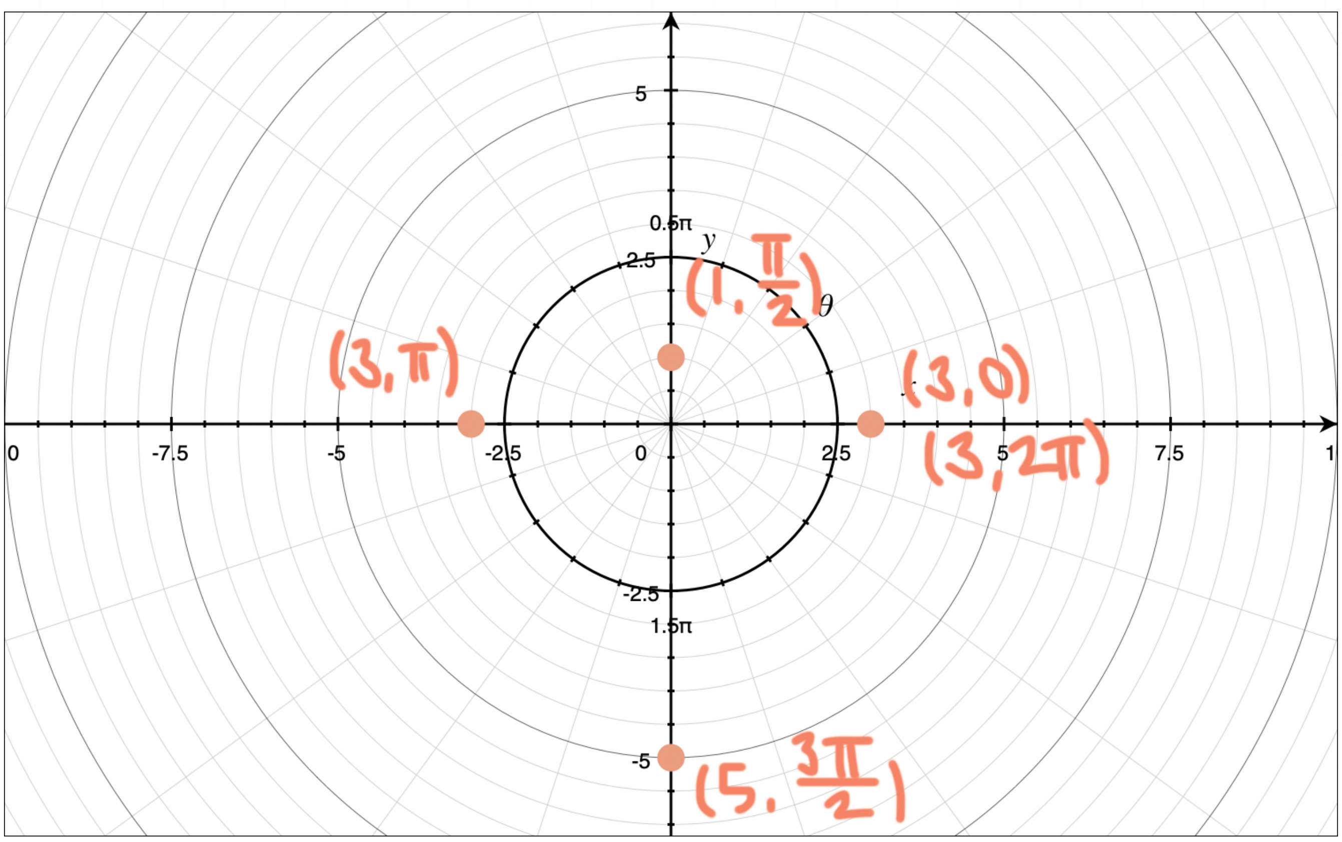
The trigonometric function in this polar equation is $\sin\theta$, and its argument (the angle at which sine is evaluated) is θ. So we'll set

$$\theta = \frac{\pi}{2}$$

Now we'll make a table with multiples of $\pi/2$, like $\theta = 0,\ \pi/2,\ \pi,\ 3\pi/2,\ 2\pi$, etc., and include the values of r that correspond to each of these θ-values.

θ	0	$\pi/2$	π	$3\pi/2$	2π
r	3	1	3	5	3

Plotting these points on the polar graph gives



And if we connect these points with a smooth curve, in order, we see the graph of the limaçon. We start at $(3,0)$, and then

loop to $(1,\pi/2)$,

loop to $(3,\pi)$,

loop to $(5,3\pi/2)$,

then finally loop back to $(3,2\pi)$, which is actually the same point as $(3,0)$. From there on, we're retracing the same pieces of the limaçon over and over.