Multiple ways to express polar points

In Trigonometry we learned about coterminal angles, which are angles that differ by one full 2π rotation. For instance, the angles $\pi/4$ and $9\pi/4$ are coterminal angles because they differ by 2π.

Given an angle θ, the expression

$\alpha = \theta + n(2\pi)$, where n is any integer

represents the complete set of angles (all the angles α) that are coterminal with θ.

Finding equivalent points by changing θ

So given a polar point like $(2,\pi)$, we know that we can find equivalent points just by changing the θ value from the point by an integer-multiple of 2π. So all of these angles are coterminal to π, the angle in the polar point $(2,\pi)$:

For $n = -3$ $\quad \alpha = \pi - 3(2\pi) = \pi - 6\pi = -5\pi$

For $n = -2$ $\quad \alpha = \pi - 2(2\pi) = \pi - 4\pi = -3\pi$

For $n = -1$ $\quad \alpha = \pi - 1(2\pi) = \pi - 2\pi = -\pi$

For $n = 1$ $\quad \alpha = \pi + 1(2\pi) = \pi + 2\pi = 3\pi$

For $n = 2$ $\quad \alpha = \pi + 2(2\pi) = \pi + 4\pi = 5\pi$

For $n = 3$ $\quad \alpha = \pi + 3(2\pi) = \pi + 6\pi = 7\pi$

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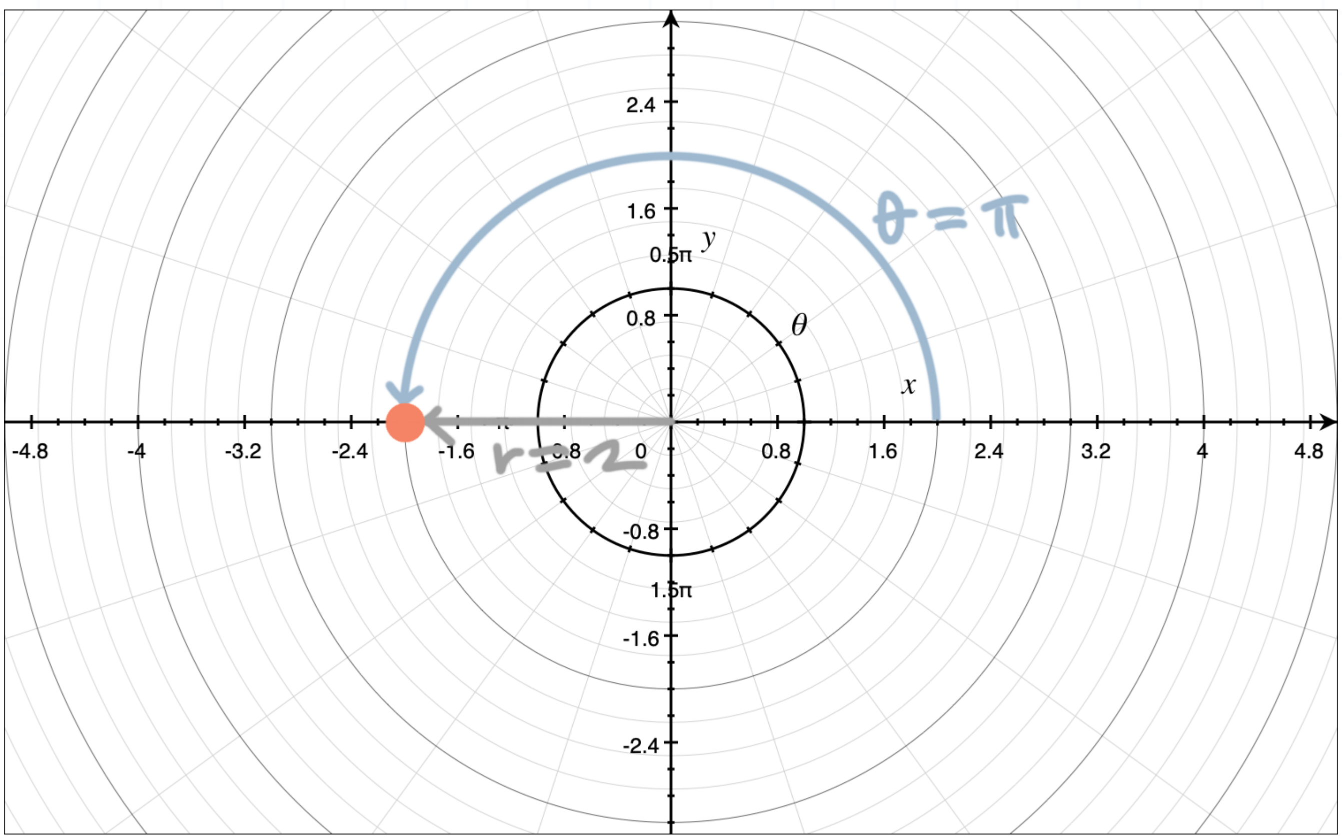
Therefore, all of these points are equivalent to $(2,\pi)$:

$(2, -5\pi)$ $\quad$ $(2,3\pi)$

$(2, -3\pi)$ $\quad$ $(2,5\pi)$

$(2, -\pi)$ $\quad$ $(2,7\pi)$

Notice how all of these points have the same r-value as our original point $(2,\pi)$, but we've just changed the θ-value by some multiple of 2π. All of these points are plotted in the same spot,



so we've now shown that we can express the same polar point in infinitely many ways, just by continuing to add or take away an extra 2π rotation from the angle θ.

Finding equivalent points by changing r

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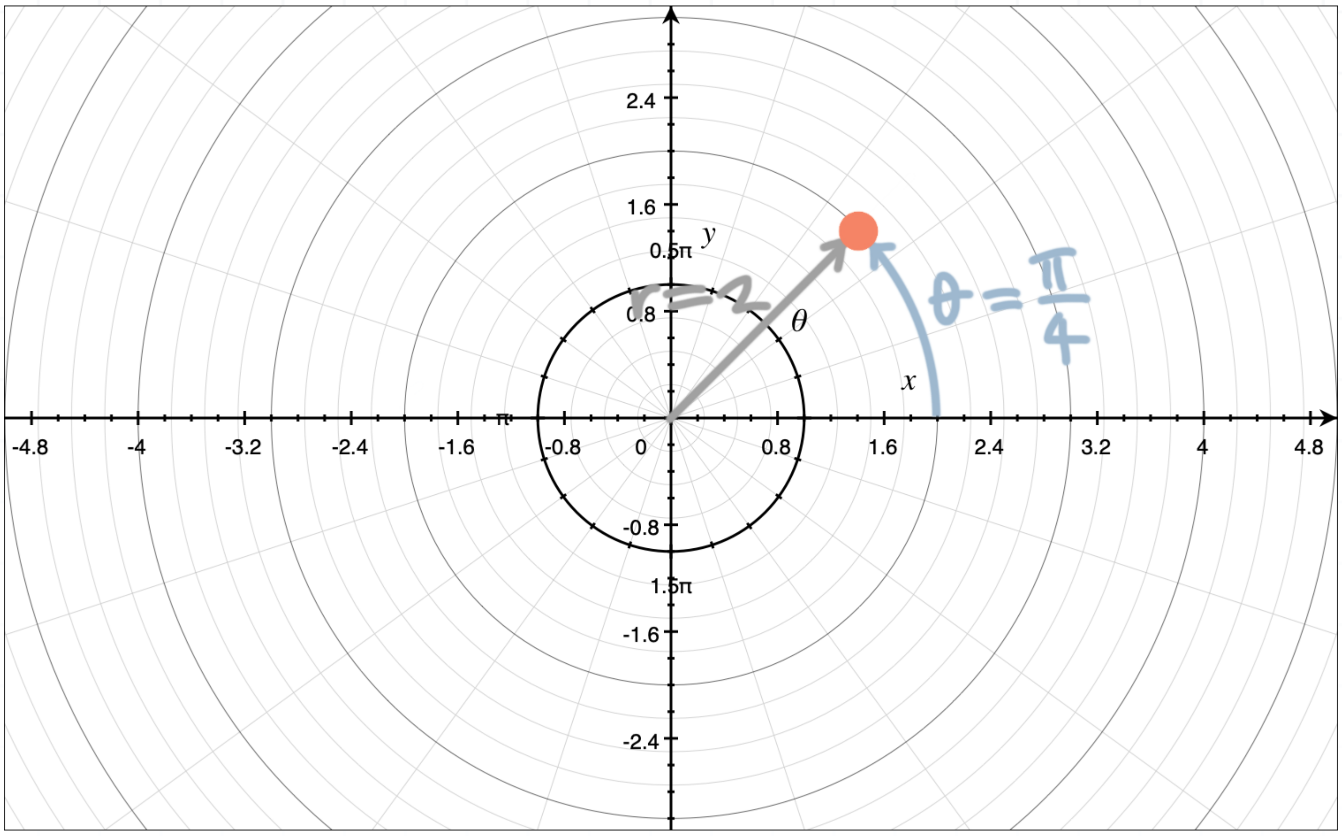
When we see a polar point with a positive r-value, it tells us to walk forward toward the angle θ. For instance, given the point $(2,\pi/4)$, we rotate toward the angle $\pi/4$ in the first quadrant, and then walk forward into the first quadrant, to arrive at $(2,\pi/4)$.



But when we see a polar point with a negative r-value, it tells us to walk backward away from the angle θ. For instance, given the point $(-2,\pi/4)$, we rotate toward the angle $\pi/4$ in the first quadrant, and then walk straight backward into the third quadrant, to arrive at $(-2,\pi/4)$.

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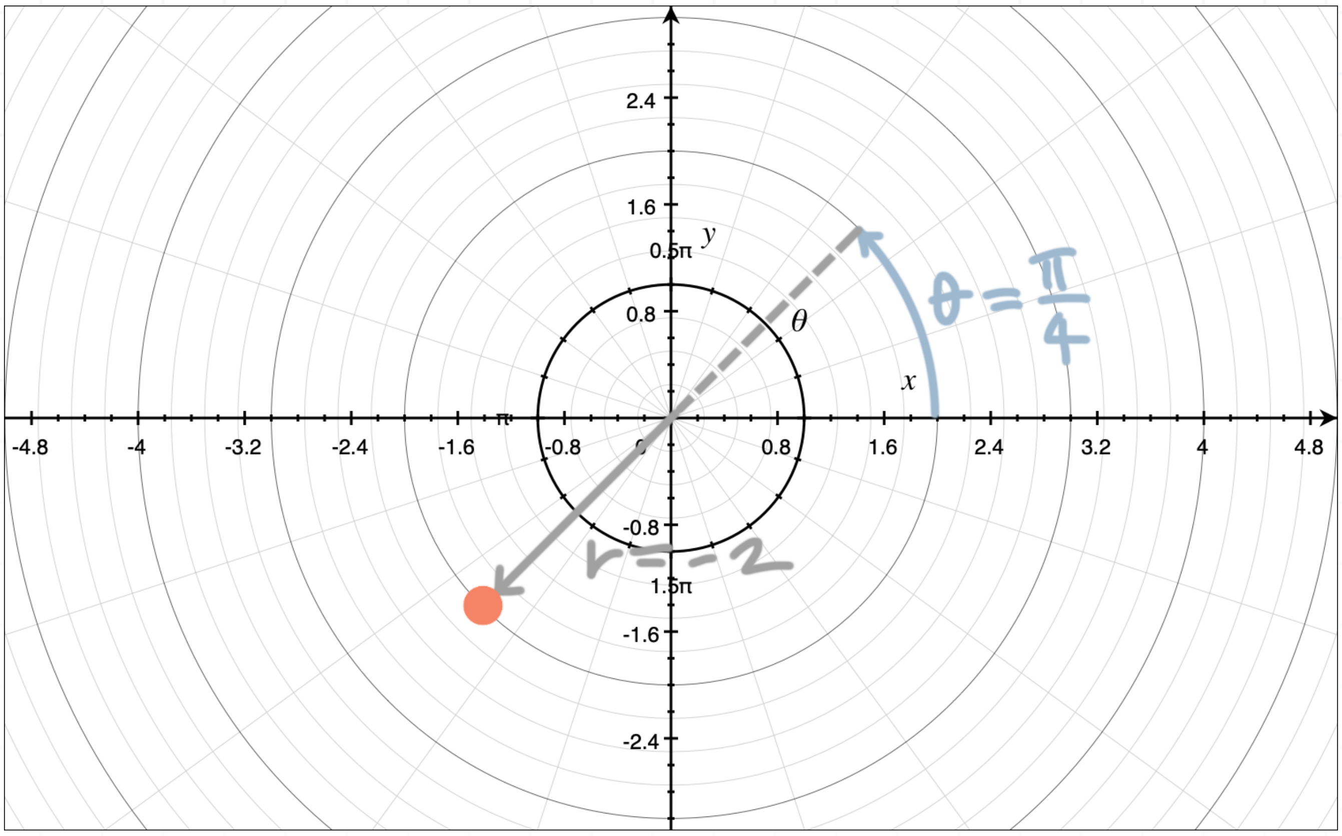
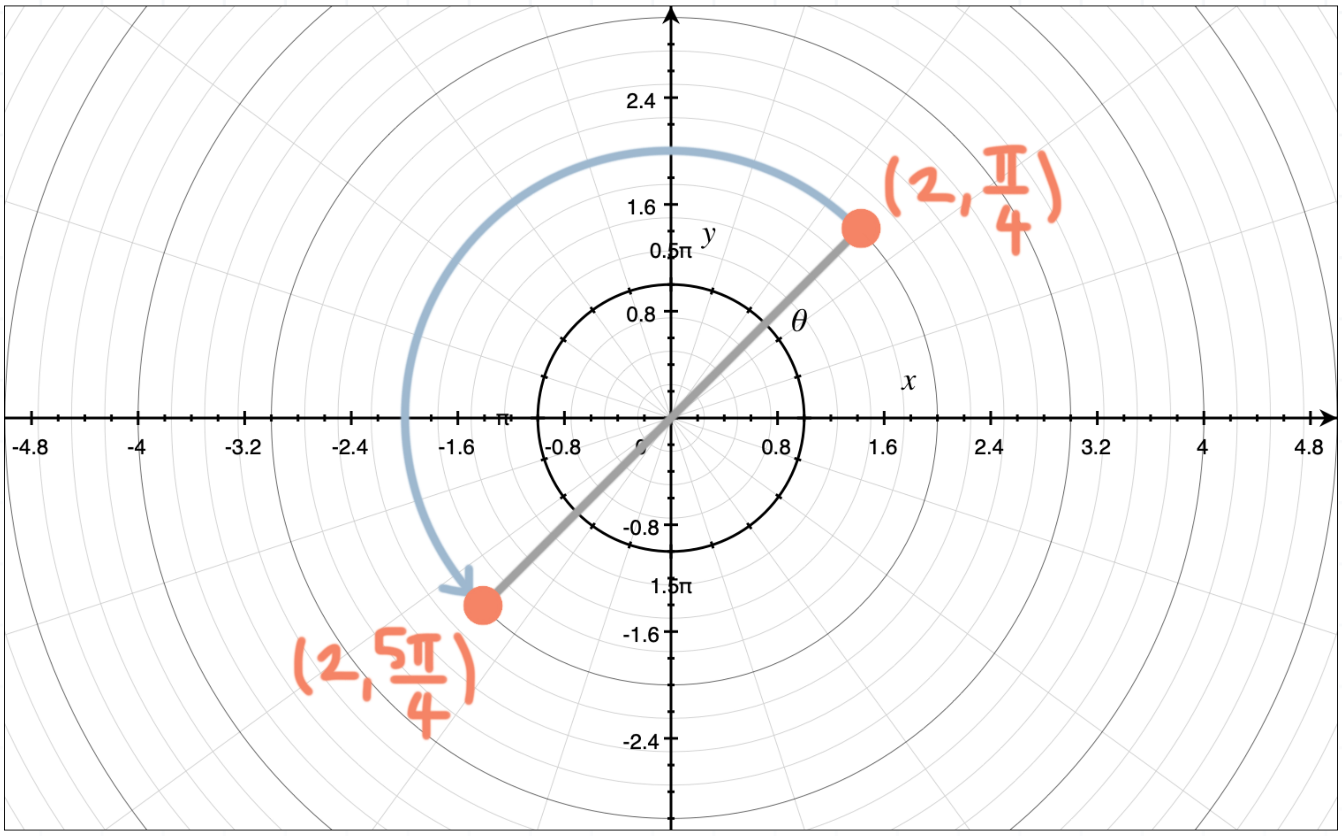


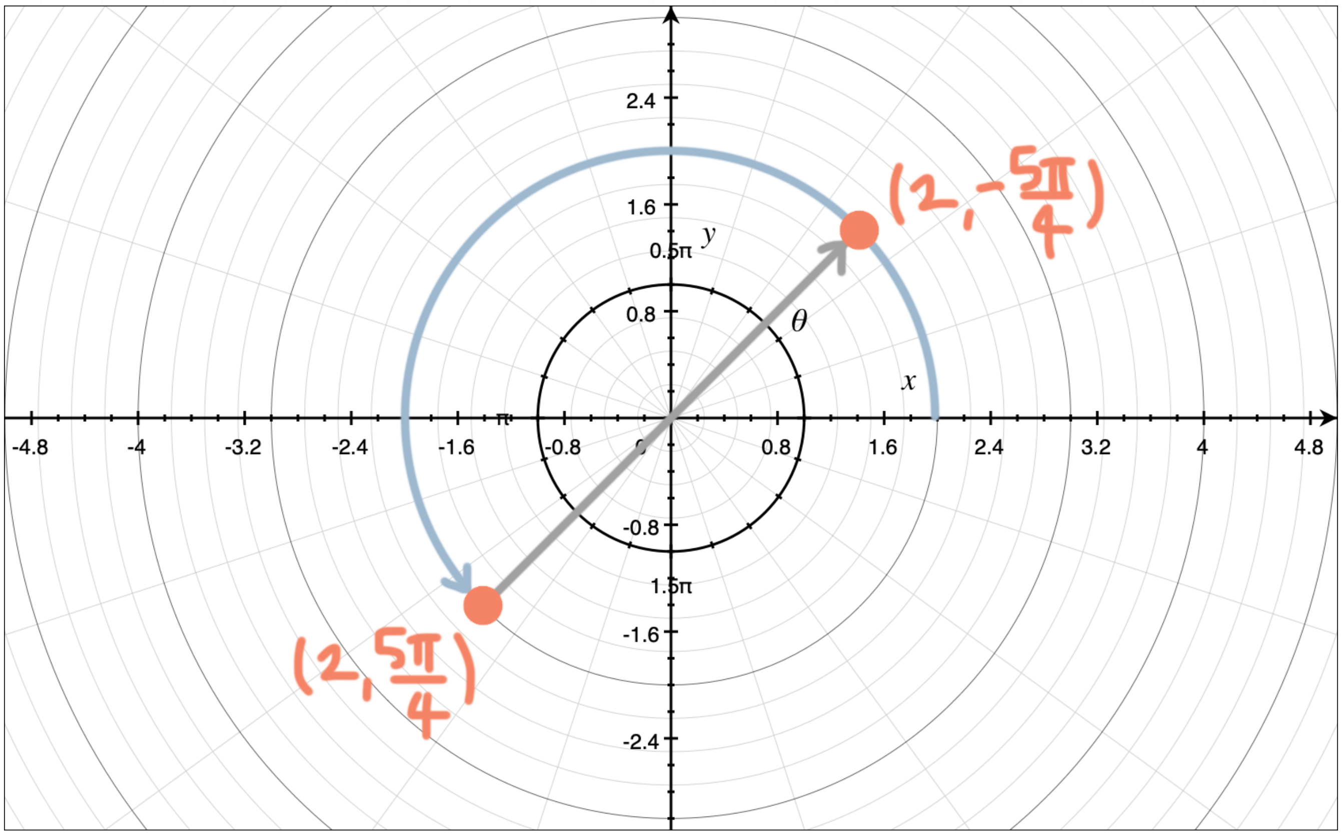
We want to notice from this example that the point $(-2,\pi/4)$ is a half-circle rotation away (a π rotation away) from $(2,\pi/4)$. Which means we could also express $(-2,\pi/4)$ as

$$\left(2,\frac{\pi}{4}+\pi\right)=\left(2,\frac{\pi}{4}+\frac{4\pi}{4}\right)=\left(2,\frac{5\pi}{4}\right)$$

So the point $(-2,\pi/4)$ is equivalent to the point $(2,5\pi/4)$. But remember, we're looking for points that are equivalent to $(2,\pi/4)$. What we've learned is that we can rotate a half circle from $(2,\pi/4)$ to get to $(2,5\pi/4)$,



and then change the value of r from 2 to -2 in order to get back to the original point.



So $(-2,5\pi/4)$ is the same point as $(2,\pi/4)$. And because we know that we can change θ by any multiple of 2π to find an equivalent point, all of these points are equivalent to $(-2,5\pi/4)$, and therefore also equivalent to $(2,\pi/4)$:

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For $n = -3$ $\quad \left(-2, \frac{5\pi}{4} - 3(2\pi)\right) = \left(-2, -\frac{19\pi}{4}\right)$

For $n = -2$ $\quad \left(-2, \frac{5\pi}{4} - 2(2\pi)\right) = \left(-2, -\frac{11\pi}{4}\right)$

For $n = -1$ $\quad \left(-2, \frac{5\pi}{4} - 1(2\pi)\right) = \left(-2, -\frac{3\pi}{4}\right)$

For $n = 1$ $\quad \left(-2, \frac{5\pi}{4} + 1(2\pi)\right) = \left(-2, \frac{13\pi}{4}\right)$

For $n = 2$ $\quad \left(-2, \frac{5\pi}{4} + 2(2\pi)\right) = \left(-2, \frac{21\pi}{4}\right)$

For $n = 3$ $\quad \left(-2, \frac{5\pi}{4} + 3(2\pi)\right) = \left(-2, \frac{29\pi}{4}\right)$

To summarize, we can say that there are an infinite number of ways to express the same point in space in polar coordinates. We can

1. Keep the value of r the same but add or subtract any multiple of 2π from θ.

2. Change the value of r to $-r$ while we add or subtract any odd multiple of π from θ.

Both of these options produce an infinite number of equivalent points for our original polar point. Let's do an example where we find a different way to express the same polar coordinate point.

Example

Find a point in polar coordinates which is equivalent to $(14,31\pi/7)$.

We can find equivalent points by adding or subtracting multiples of 2π from the angle θ in the point, so all of these are examples of some equivalent polar points:

For $n=-3$ $\quad\left(14,\frac{31\pi}{7}-3(2\pi)\right)=\left(14,-\frac{11\pi}{7}\right)$

For $n=-2$ $\quad\left(14,\frac{31\pi}{7}-2(2\pi)\right)=\left(14,\frac{3\pi}{7}\right)$

For $n=-1$ $\quad\left(14,\frac{31\pi}{7}-1(2\pi)\right)=\left(14,\frac{17\pi}{7}\right)$

For $n=1$ $\quad\left(14,\frac{31\pi}{7}+1(2\pi)\right)=\left(14,\frac{45\pi}{7}\right)$

For $n=2$ $\quad\left(14,\frac{31\pi}{7}+2(2\pi)\right)=\left(14,\frac{59\pi}{7}\right)$

For $n=3$ $\quad\left(14,\frac{31\pi}{7}+3(2\pi)\right)=\left(14,\frac{73\pi}{7}\right)$

We could also find an equivalent point by simultaneously changing $r=14$ to $r=-14$, and adding π to $31\pi/7$ to get

$$\left(-14,\frac{31\pi}{7}+\pi\right)=\left(-14,\frac{38\pi}{7}\right)$$

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Then we can find more equivalent polar points with the $r = -14$ value by adding and subtracting multiples of 2π from the angle θ, so this is another set of points which are also equivalent to $(14,31\pi/7)$:

For $n = -3$ $$\left(-14,\frac{38\pi}{7}-3(2\pi)\right)=\left(-14,-\frac{4\pi}{7}\right)$$

For $n = -2$ $$\left(-14,\frac{38\pi}{7}-2(2\pi)\right)=\left(-14,\frac{10\pi}{7}\right)$$

For $n = -1$ $$\left(-14,\frac{38\pi}{7}-1(2\pi)\right)=\left(-14,\frac{24\pi}{7}\right)$$

For $n = 1$ $$\left(-14,\frac{38\pi}{7}+1(2\pi)\right)=\left(-14,\frac{52\pi}{7}\right)$$

For $n = 2$ $$\left(-14,\frac{38\pi}{7}+2(2\pi)\right)=\left(-14,\frac{66\pi}{7}\right)$$

For $n = 3$ $$\left(-14,\frac{38\pi}{7}+3(2\pi)\right)=\left(-14,\frac{80\pi}{7}\right)$$

These are all examples of points that are equivalent to $(14,31\pi/7)$, but we could list infinitely more.

Let's do one more, but this time we'll start with a negative value of r.

Example

Find some polar points that are equivalent to $(-20,-18\pi/11)$.

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We can find equivalent points by adding or subtracting multiples of 2π from the angle θ in the point, so all of these are examples of some equivalent polar points:

For $n=-3$ $$\left(-20,-\frac{18\pi}{11}-3(2\pi)\right)=\left(-20,-\frac{84\pi}{11}\right)$$

For $n=-2$ $$\left(-20,-\frac{18\pi}{11}-2(2\pi)\right)=\left(-20,-\frac{62\pi}{11}\right)$$

For $n=-1$ $$\left(-20,-\frac{18\pi}{11}-1(2\pi)\right)=\left(-20,-\frac{40\pi}{11}\right)$$

For $n=1$ $$\left(-20,-\frac{18\pi}{11}+1(2\pi)\right)=\left(-20,\frac{4\pi}{11}\right)$$

For $n=2$ $$\left(-20,-\frac{18\pi}{11}+2(2\pi)\right)=\left(-20,\frac{26\pi}{11}\right)$$

For $n=3$ $$\left(-20,-\frac{18\pi}{11}+3(2\pi)\right)=\left(-20,\frac{48\pi}{11}\right)$$

We could also find an equivalent point by simultaneously changing $r=-20$ to $r=20$, and adding π to $-18\pi/11$ to get

$$\left(20,-\frac{18\pi}{11}+\pi\right)=\left(20,-\frac{7\pi}{11}\right)$$

Then we can find more equivalent polar points with the $r=20$ value by adding and subtracting multiples of 2π from the angle θ, so this is another set of points which are also equivalent to $(-20,-18\pi/11)$:

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For $n = -3$ $\qquad \left(20, -\frac{7\pi}{11} - 3(2\pi)\right) = \left(20, -\frac{73\pi}{11}\right)$

For $n = -2$ $\qquad \left(20, -\frac{7\pi}{11} - 2(2\pi)\right) = \left(20, -\frac{51\pi}{11}\right)$

For $n = -1$ $\qquad \left(20, -\frac{7\pi}{11} - 1(2\pi)\right) = \left(20, -\frac{29\pi}{11}\right)$

For $n = 1$ $\qquad \left(20, -\frac{7\pi}{11} + 1(2\pi)\right) = \left(20, \frac{15\pi}{11}\right)$

For $n = 2$ $\qquad \left(20, -\frac{7\pi}{11} + 2(2\pi)\right) = \left(20, \frac{37\pi}{11}\right)$

For $n = 3$ $\qquad \left(20, -\frac{7\pi}{11} + 3(2\pi)\right) = \left(20, \frac{59\pi}{11}\right)$

These are all examples of points that are equivalent to $(-20, -18\pi/11)$, but we could list infinitely more.