Calculus 1 Workbook Solutions

Intermediate Value Theorem

krista king MATH

INTERMEDIATE VALUE THEOREM WITH AN INTERVAL

1. The value $c = -1$ satisfies the conditions of the Intermediate Value Theorem for the function on the interval $[-3,5]$ because $f(c)$ equals what value?

$$f(x) = \frac{1}{4}(2x+5)(x-3)^2$$

Solution:

The Intermediate Value Theorem (IVT) states that a function $y = f(x)$ is continuous on a closed interval $[a, b]$ and takes on every value between $f(a)$ and $f(b)$. In other words, if y_0 is between $f(a)$ and $f(b)$, then $y_0 = f(c)$ for some c in $[a, b]$. In this problem, $f(a) = f(-3) = -9$ and $f(b) = f(5) = 15$. Then,

$$f(c) = f(-1) = \frac{1}{4}(2(-1)+5)(-1-3)^2 = 12$$

The IVT requires that $f(a) \le f(c) \le f(b)$ and $-9 \le 12 \le 15$.

2. The value $c = 2$ does not satisfy the conditions of the Intermediate Value Theorem for $g(x) = 2x^2 - 11x + 4$ on the interval $[-2,4]$ because $g(c)$ equals what value?

Solution:

The Intermediate Value Theorem (IVT) states that a function $y = f(x)$ is continuous on a closed interval $[a, b]$ takes on every value between $f(a)$ and $f(b)$. In other words, if y_0 is between $f(a)$ and $f(b)$, then $y_0 = f(c)$ for some c in $[a, b]$. In this problem, $g(a) = g(-2) = 34$ and $g(b) = g(4) = -8$. However, $g(c) = g(2) = 2(2)^2 - 11(2) + 4 = -10$. The IVT requires that $g(a) \le g(c) \le g(b)$ or $g(b) \le g(c) \le g(a)$, but -10 is not between -8 and 34.

■ 3. What value of c is guaranteed by the Intermediate Value Theorem on the interval $[-3,3]$ if $h(x) = 3(x+1)^3$ and $h(c) = 24$?

Solution:

The Intermediate Value Theorem (IVT) states that a function $y = f(x)$ is continuous on a closed interval $[a, b]$ and takes on every value between $f(a)$ and $f(b)$. In other words, if y_0 is between $f(a)$ and $f(b)$, then $y_0 = f(c)$ for some c in $[a, b]$. In this problem, $h(a) = h(-3) = -24$ and $h(b) = h(3) = 192$. Thus, if $h(c) = 24$, the IVT requires that since $f(a) \le f(c) \le f(b)$, $a \le c \le b$. Thus, since $h(c) = 24$, we get

$$3(c+1)^3 = 24$$

$$(c+1)^3 = 8$$

$$c + 1 = 2$$

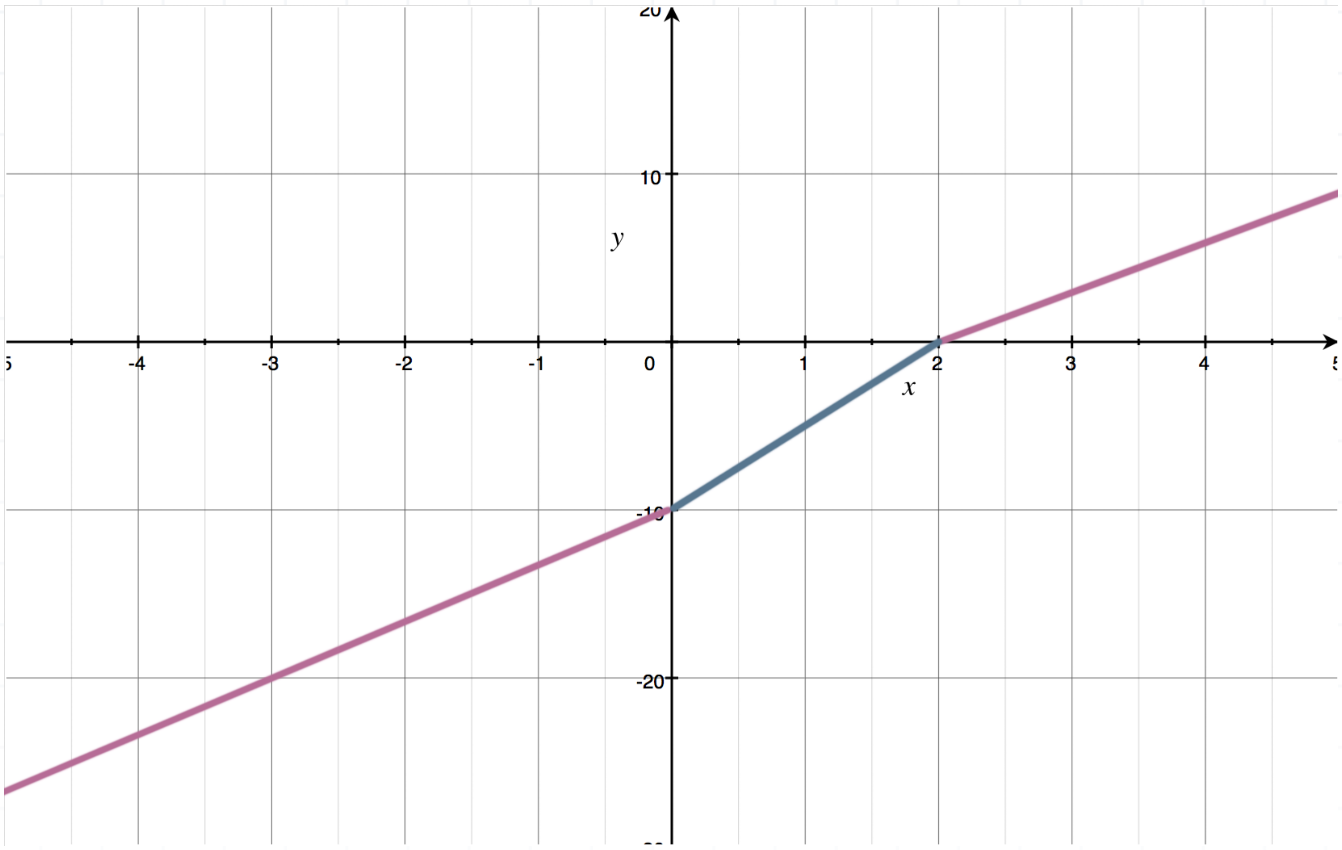
$$c = 1$$

■ 4. What value of c is guaranteed by the Intermediate Value Theorem on the interval $[-5,6]$ if $f(c) = -6$ and

$$f(x) = \begin{cases} 3x - 10 & \text{if } x \leq 0 \\ x^2 + 3x - 10 & \text{if } 0 < x < 2 \\ 3x - 6 & \text{if } x \geq 2 \end{cases}$$

Solution:

The Intermediate Value Theorem (IVT) states that a function $y = f(x)$ is continuous on a closed interval $[a, b]$ and takes on every value between $f(a)$ and $f(b)$. In other words, if y_0 is between $f(a)$ and $f(b)$, then $y_0 = f(c)$ for some c in $[a, b]$. In this problem, first confirm that the function $f(x)$ is continuous on the interval $[-5,6]$ by evaluating the function on both sides of $x = 0$ and $x = 2$. The function is continuous, as shown in the graph below.



$f(a) = f(-5) = -25$ and $f(b) = f(6) = 12$. Thus, if $f(c) = -6$, the IVT requires that since $f(a) \le f(c) \le f(b)$, $a \le c \le b$. Thus, since $f(c) = -6$,

$$c^2 + 3c - 10 = -6$$

$$c^2 + 3c - 4 = 0$$

$$(c + 4)(c - 1) = 0$$

$c = -4$ and $c = 1$

when using $x^2 + 3x - 10$, but we consider $x^2 + 3x - 10$ on the interval $(0,2)$, so $c = -4$ does not satisfy the conditions. Because $f(x)$ is defined piecewise, there can be other values of c that might satisfy the IVT, but there are no other values of c that satisfy the conditions.

5. Show that the function has a zero in the interval $[2,9]$ and find the solution.

$$g(x) = \frac{x^2 - 9}{x + 3}$$

Solution:

The IVT states that a function $y = f(x)$ is continuous on a closed interval $[a, b]$ and takes on every value between $f(a)$ and $f(b)$. In other words, if y_0 is between $f(a)$ and $f(b)$, then $y_0 = f(c)$ for some c in $[a, b]$. In this problem, $g(a) = g(2) = -5/6$ and $g(b) = g(9) = 6$.

Because the function is below the x-axis at the left edge of the interval, and above the x-axis at the right edge of the interval, we can say $g(a) < g(c) < g(b)$, or more specifically, $-5/6 < g(c) < 6$, where $g(c) = 0$.

Therefore, by the IVT, it must be true that the function has a root on the interval $[2,9]$. To find the root, which is the point where the graph of the function crosses the x-axis, we'll set the function equal to 0.

$$\frac{x^2 - 9}{x + 3} = 0$$

$$\frac{(x + 3)(x - 3)}{x + 3} = 0$$

$$x - 3 = 0$$

$$x = 3$$

Therefore, the root in the interval $[2,9]$ is at $x = 3$, or the point $(3,0)$.

■ 6. What value of c is guaranteed by the Intermediate Value Theorem on the interval $[3,6]$ if c is a root of $h(x)$.

$$h(x) = \frac{x^3 - 4x^2 - 11x + 30}{x^2 - 4}$$

Solution:

The IVT states that a function $y = f(x)$ is continuous on a closed interval $[a, b]$ and takes on every value between $f(a)$ and $f(b)$. In other words, if y_0 is between $f(a)$ and $f(b)$, then $y_0 = f(c)$ for some c in $[a, b]$.

In this problem, $h(a) = h(3) = -12/5$ and $h(b) = h(6) = 9/8$. Thus, if $h(c)$ is a root of $h(x)$, then $h(c) = 0$. The IVT requires that since $f(a) \le f(c) \le f(b)$, $a \le c \le b$. Thus, since $h(c) = 0$, then

$$\frac{c^3 - 4c^2 - 11c + 30}{c^2 - 4} = 0$$

Solving this equation gives $c = 5$ and $c = -3$, but -3 is not in the interval $[3,6]$. Note that although $h(x)$, as defined, contains discontinuities at $x = -2$ and $x = 2$, the function is continuous in the given interval, therefore satisfying the IVT.

INTERMEDIATE VALUE THEOREM WITHOUT AN INTERVAL

■ 1. Use the Intermediate Value Theorem to prove that the equation $2e^x = 3\cos x$ has at least one positive solution. In what interval is that solution?

Solution:

Let $f(x) = 2e^x - 3\cos x$. The root of $f(x)$ is a solution to the given equation. The Intermediate Value Theorem guarantees that the function $f(x)$ has a root in a certain closed interval if the function's value changes sign in that closed interval.

Consider the interval $[0,1]$. Then,

$$f(0) = 2e^0 - 3\cos 0 = 2 - 3 = -1$$

$$f(1) = 2e^1 - 3\cos 1 = 2e - 1.6209$$

which is approximately 3.8157. Since the function's value changes sign in the interval $[0,1]$, and since $f(x)$ is continuous in the interval, by the Intermediate Value Theorem the function has a zero in that interval.

■ 2. Use the Intermediate Value Theorem to prove that the equation $3\sin x + 7 = x^2 - 2x - 2$ has at least one positive solution. In what interval is that solution?

Solution:

Let $g(x) = 3\sin x + 7 - (x^2 - 2x - 2)$. The root of $g(x)$ is a solution to the given equation. The Intermediate Value Theorem guarantees that the function $g(x)$ has a root in a certain closed interval if the function's value changes sign in that closed interval.

Consider the interval $[3,4]$. Then,

$$g(3) = 3\sin(3) + 7 - (3^2 - 2(3) - 2) = 6.4234$$

$$g(4) = 3\sin(4) + 7 - (4^2 - 2(4) - 2) = -1.2704$$

Since the function's value changes sign in the interval $[3,4]$, and $g(x)$ is continuous on the interval, the function has a zero in that interval.

■ 3. Use the Intermediate Value Theorem to prove that the equation $x^6 - 9x^4 + 7 = x^5 - 8x^3 - 9$ has at least one positive solution. In what interval is that solution?

Solution:

Let $h(x) = (x^6 - 9x^4 + 7) - (x^5 - 8x^3 - 9)$. The root of $h(x)$ is a solution to the given equation. The Intermediate Value Theorem guarantees that the function $h(x)$ has a root in a certain closed interval if the function's value of changes sign in that closed interval.

Consider the interval $[1,2]$. Then,

$$h(1) = ((1)^6 - 9(1)^4 + 7) - ((1)^5 - 8(1)^3 - 9) = 15$$

$$h(2) = ((2)^6 - 9(2)^4 + 7) - ((2)^5 - 8(2)^3 - 9) = -32$$

Since the function's value changes sign in the interval $[1,2]$, and $h(x)$ is continuous on the interval, by the Intermediate Value Theorem the function has a zero in that interval.

■ 4. Use the Intermediate Value Theorem to prove that the equation $4e^{x-3} = 2(x^3 - 5x + 9)$ has at least one negative solution. In what interval is that solution?

Solution:

Let $f(x) = 4e^{x-3} - 2(x^3 - 5x + 9)$. The root of $f(x)$ is a solution to the given equation. The Intermediate Value Theorem guarantees that the function $f(x)$ has a root in a certain closed interval if the function's value changes sign in that closed interval.

Consider the interval $[-3,-2]$. Then,

$$f(-3) = 4e^{-3-3} - 2((-3)^3 - 5(-3) + 9) = 6.0099$$

$$f(-2) = 4e^{-2-3} - 2((-2)^3 - 5(-2) + 9) = -21.97$$

Since the function's value changes sign in the interval $[-3,-2]$, and $f(x)$ is continuous on the interval, by the Intermediate Value Theorem the function has a zero in that interval.

■ 5. Use the Intermediate Value Theorem to show that the equation has at least one positive solution. In what interval is that solution?

$$6e^{-x} = -\left(\frac{1}{5}x^2 - 4x + 9\right)$$

Solution:

Let

$$g(x) = 6e^{-x} + \frac{1}{5}x^2 - 4x + 9$$

The root of $g(x)$ is a solution to the given equation. The Intermediate Value Theorem guarantees that the function $g(x)$ has a root in a certain closed interval if the function's value changes sign in that closed interval.

Consider the interval $[2,3]$. Then,

$$g(2) = 6e^{-2} + \frac{1}{5}(2)^2 - 4(2) + 9 = 2.612$$

$$g(3) = 6e^{-3} + \frac{1}{5}(3)^2 - 4(3) + 9 = -0.9013$$

Since the function's value changes sign in the interval $[2,3]$, and $g(x)$ is continuous on the interval, by the Intermediate Value Theorem the function has a zero in that interval.

■ 6. Use the Intermediate Value Theorem to show that the equation $2\sin(4x-1)=\cos(2x-3)$ has at least one negative solution. In what interval is that solution?

Solution:

Let $h(x)=2\sin(4x-1)-\cos(2x-3)$. The root of $h(x)$ is a solution to the given equation. The Intermediate Value Theorem guarantees that the function $h(x)$ has a root in a certain closed interval if the function's value changes sign in that closed interval.

Consider the interval $[-2,-1]$. Then,

$$h(-2)=2\sin(4(-2)-1)-\cos(2(-2)-3)=-1.578$$

$$h(-1)=2\sin(4(-1)-1)-\cos(2(-1)-3)=1.634$$

Since the function's value changes sign in the interval $[-2,-1]$, and $h(x)$ is continuous on the interval, by the Intermediate Value Theorem the function has a zero in that interval.

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