Graphing cardioids

Our next category of polar curves are called "cardioids" because of the way that they resemble the shape of a heart, and their equations take the form

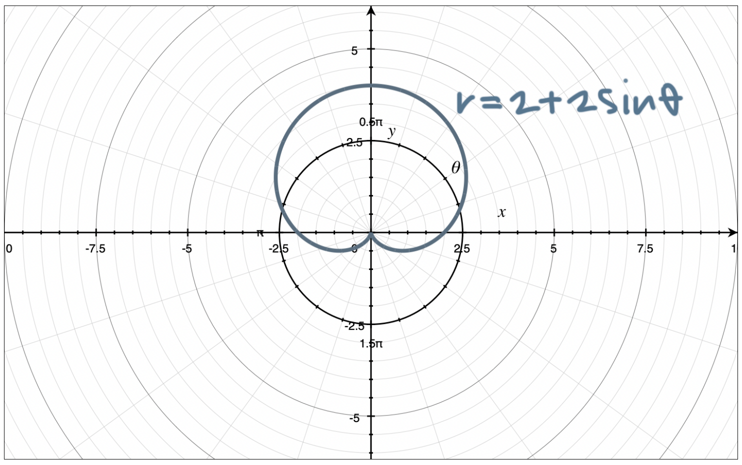
$$r = c + c\cos\theta \qquad r = c + c\sin\theta$$

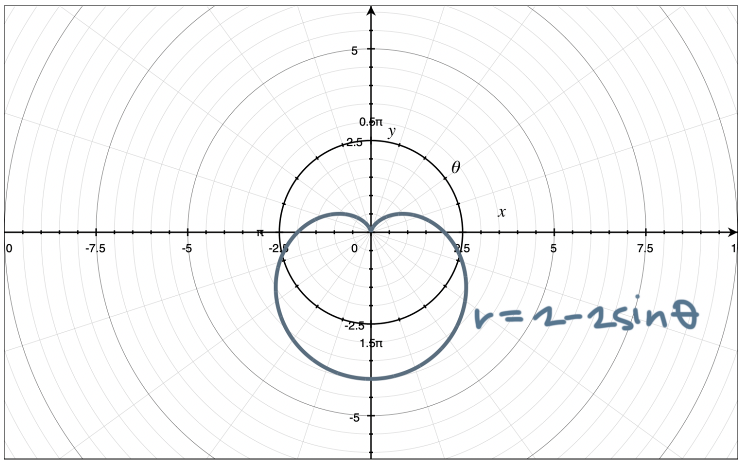
$$r = c - c\cos\theta \qquad r = c - c\sin\theta$$

where c is a positive constant. In the equations of cardioids, the coefficient on the argument θ is always 1.

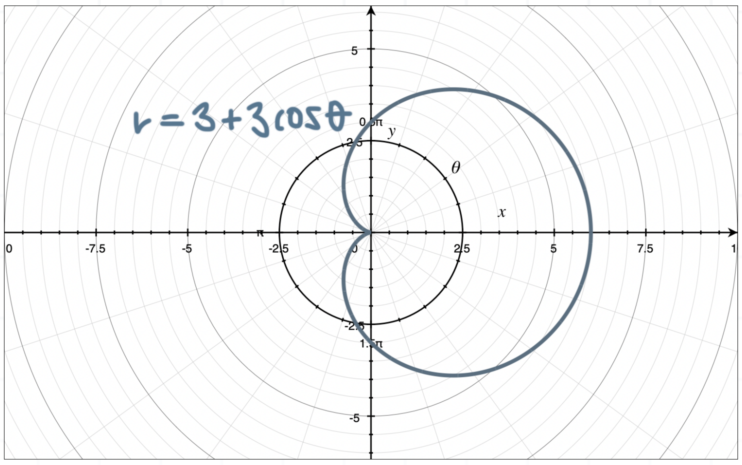
Properties of cardioids

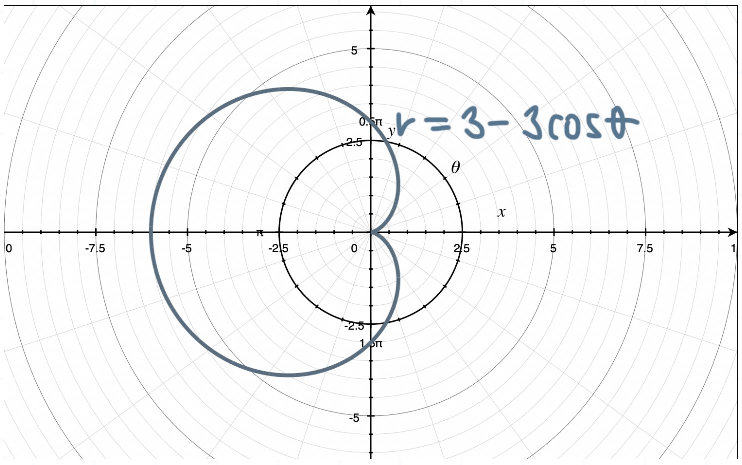
Sine cardioids will have symmetry around the vertical axis, and $r = c + c\sin\theta$ will sit mostly above the horizontal axis while $r = c - c\sin\theta$ will sit mostly below the horizontal axis. For example, the graphs of $r = 2 + 2\sin\theta$ and $r = 2 - 2\sin\theta$ are





Cosine cardioids will have symmetry around the horizontal axis, and $r = c + c\cos\theta$ will sit mostly to the right of the vertical axis while $r = c - c\cos\theta$ will sit mostly to the left of the vertical axis. For example, the graphs of $r = 3 + 3\cos\theta$ and $r = 3 - 3\cos\theta$ are





The cardioid's furthest distance from the origin will always be at a distance of $2c$, which is why we see the two sine graphs above, $r = 2 + 2\sin\theta$ and $r = 2 - 2\sin\theta$, extend out to $2c = 2(2) = 4$, while the two cosines graphs above, $r = 3 + 3\cos\theta$ and $r = 3 - 3\cos\theta$, extend out to $2c = 2(3) = 6$.

How to sketch cardioids

We'll use the same approach to sketch cardioids that we've used previously to sketch circles and roses:

1. Set the argument of the trigonometric function equal to $\pi/2$, and then solve the equation for θ.

2. Evaluate the polar curve at multiples of the θ-value we solved for in Step 1, starting with $\theta = 0$, and plot the resulting points on the polar graph.

3. Connect the points on the polar graph with a smooth curve.

Let's work through an example of how to sketch a cosine cardioid.

Example

Sketch the graph of $r = 4 + 4\cos\theta$.

Because $c = 4$, this cardioid will extend out to a distance of $2c = 2(4) = 8$ from the origin. Because it's a cosine curve where the sign between the terms is positive, the graph will sit mostly to the right of the vertical axis, with symmetry across the horizontal axis.

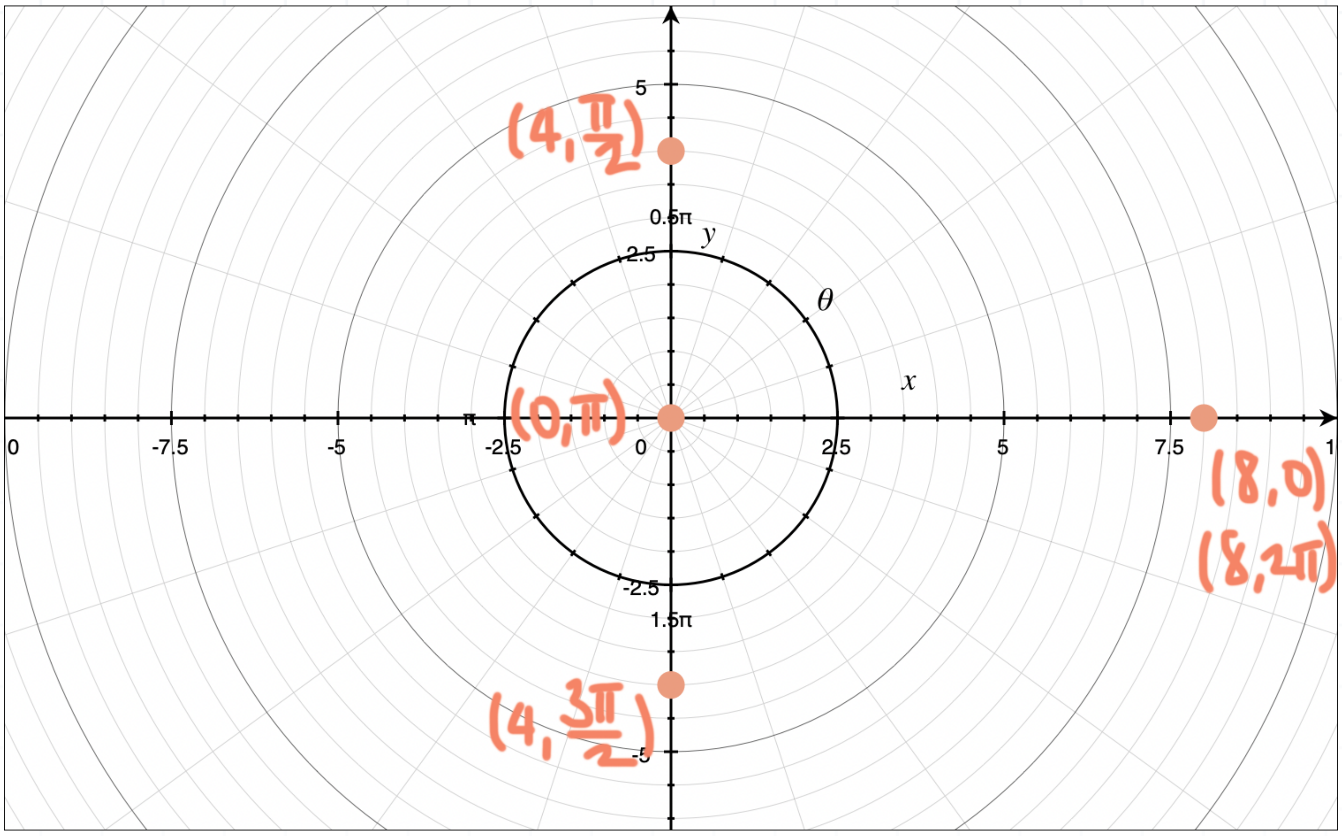
To sketch the graph, we recognize that the trigonometric function in this polar equation is $\cos\theta$, and its argument (the angle at which cosine is evaluated) is θ. So we'll set

$$\theta = \frac{\pi}{2}$$

Now we'll make a table with multiples of $\pi/2$, like $\theta = 0$, $\pi/2$, π, $3\pi/2$, 2π, $5\pi/2$, 3π, etc., and include the values of r that correspond to each of these θ-values.

θ	0	$\pi/2$	π	$3\pi/2$	2π
r	8	4	0	4	8

Plotting these points on the polar graph gives



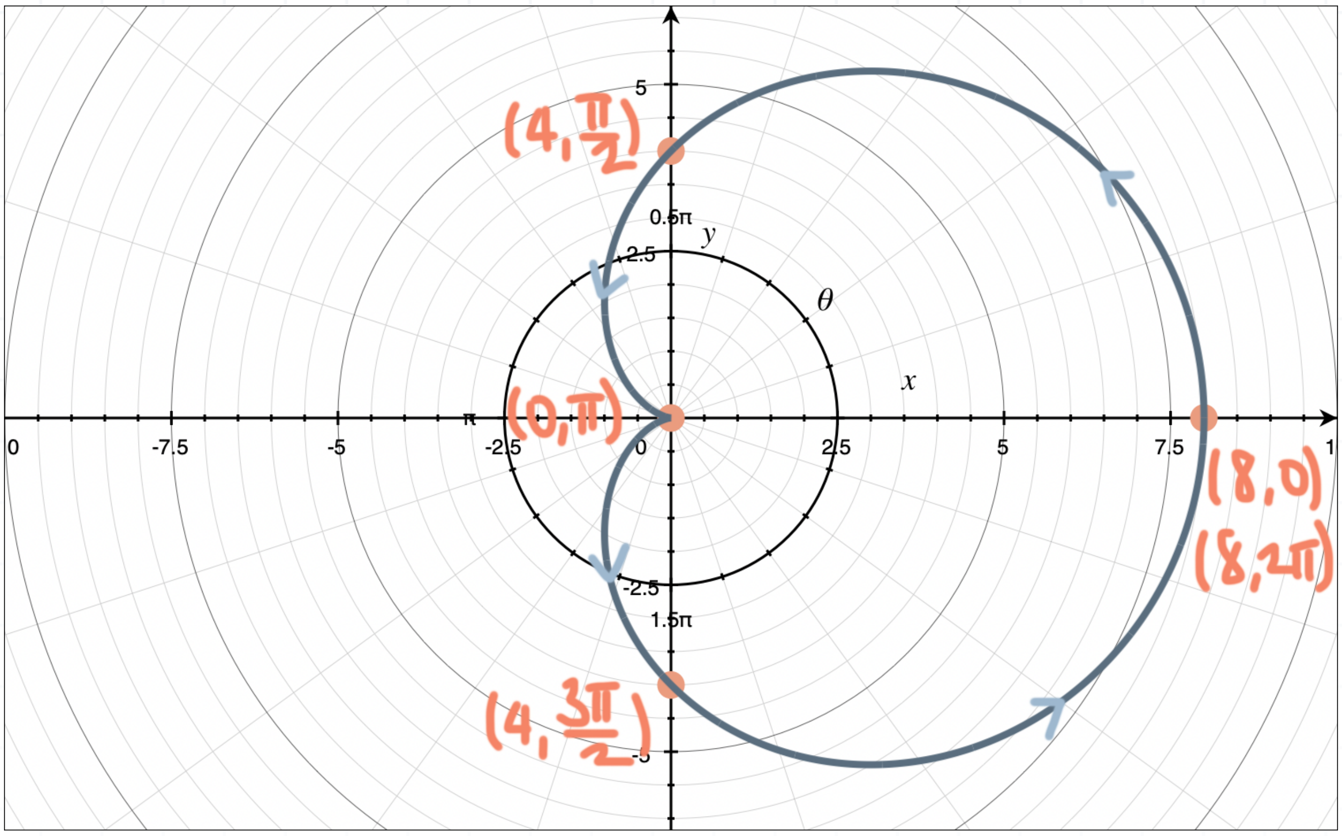
And if we connect these points with a smooth curve, in order, we see the graph of the cardioid. We start at $(8,0)$, and then

loop to $(4,\pi/2)$,

loop back to the origin at $(0,\pi)$,

loop to $(4,3\pi/2)$,

then finally loop back to $(8,2\pi)$, which is actually the same point as $(8,0)$. From there on, we're retracing the same pieces of the cardioid over and over.



Now let's do an example with a cardioid in the form $r = c - c\cos\theta$.

Example

Sketch the graph of $r = 4 - 4\cos\theta$.

Because $c = 4$, this cardioid will extend out to a distance of $2c = 2(4) = 8$ from the origin. Because it's a cosine curve where the sign between the terms is negative, the graph will sit mostly to the left of the vertical axis, with symmetry across the horizontal axis.

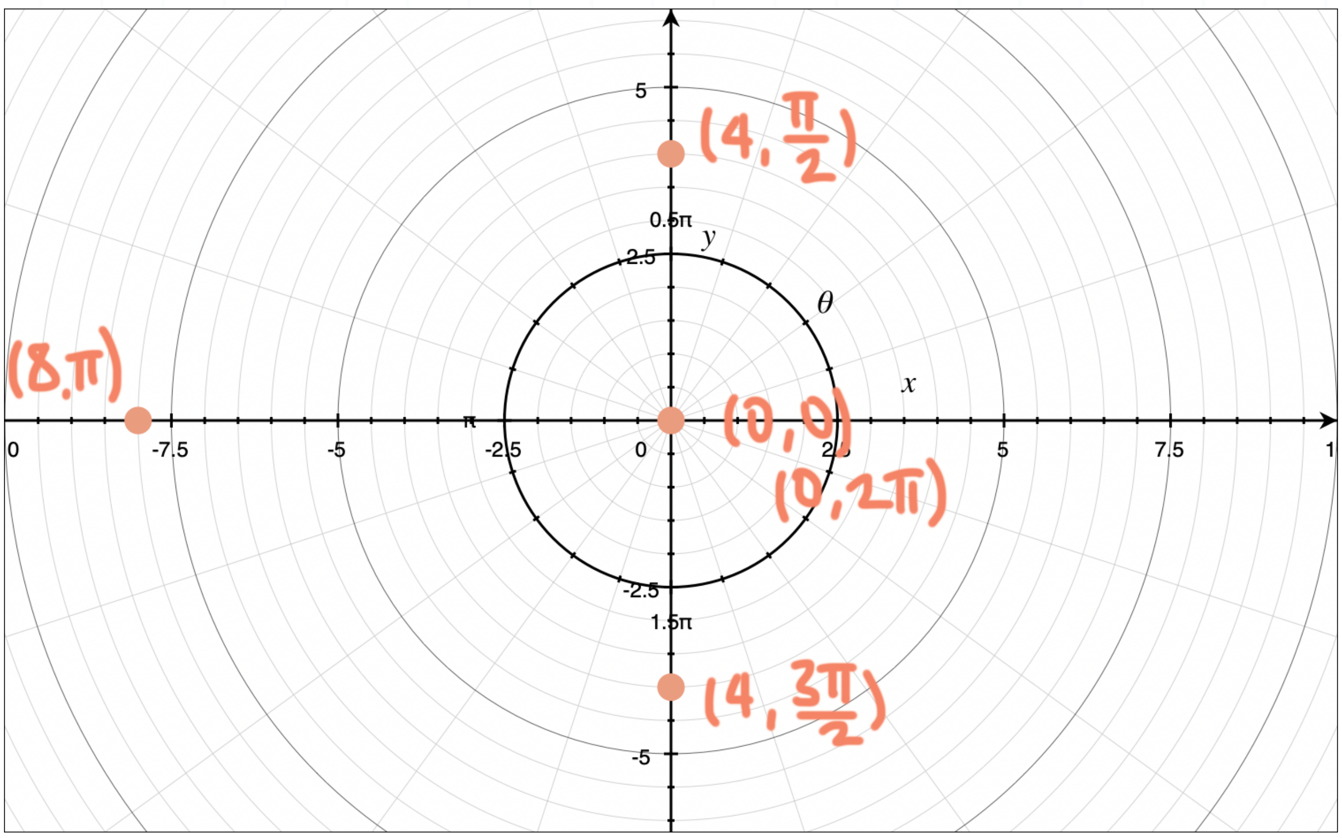
To sketch the graph, we recognize that the trigonometric function in this polar equation is $\cos\theta$, and its argument (the angle at which cosine is evaluated) is θ. So we'll set

$$\theta = \frac{\pi}{2}$$

Now we'll make a table with multiples of $\pi/2$, like $\theta = 0$, $\pi/2$, π, $3\pi/2$, 2π, $5\pi/2$, 3π, etc., and include the values of r that correspond to each of these θ -values.

θ	0	$\pi/2$	π	$3\pi/2$	2π
r	0	4	8	4	0

Plotting these points on the polar graph gives



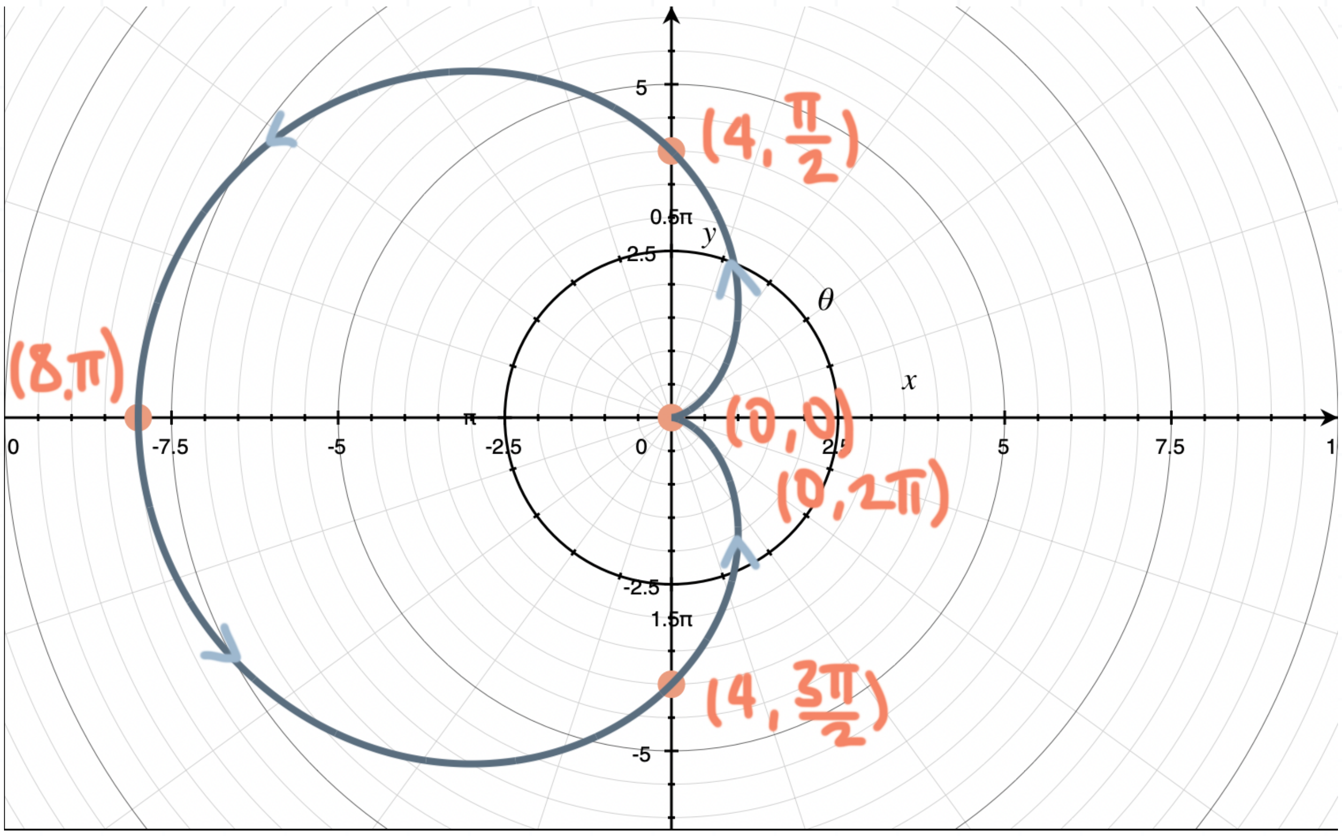
And if we connect these points with a smooth curve, in order, we see the graph of the cardioid. We start at $(0,0)$, and then

loop to $(4,\pi/2)$,

loop to $(8,\pi)$,

loop to $(4,3\pi/2)$,

then finally loop back to $(0,2\pi)$, which is actually the same point as $(0,0)$. From there on, we're retracing the same pieces of the cardioid over and over.



Now let's look at the graphs of sine cardioids.

Example

Sketch the graph of $r = 3 + 3\sin\theta$.

Because $c = 3$, this cardioid will extend out to a distance of $2c = 2(3) = 6$ from the origin. Because it's a sine curve where the sign between the terms if positive, the graph will sit mostly above the horizontal axis, with symmetry across the vertical axis.

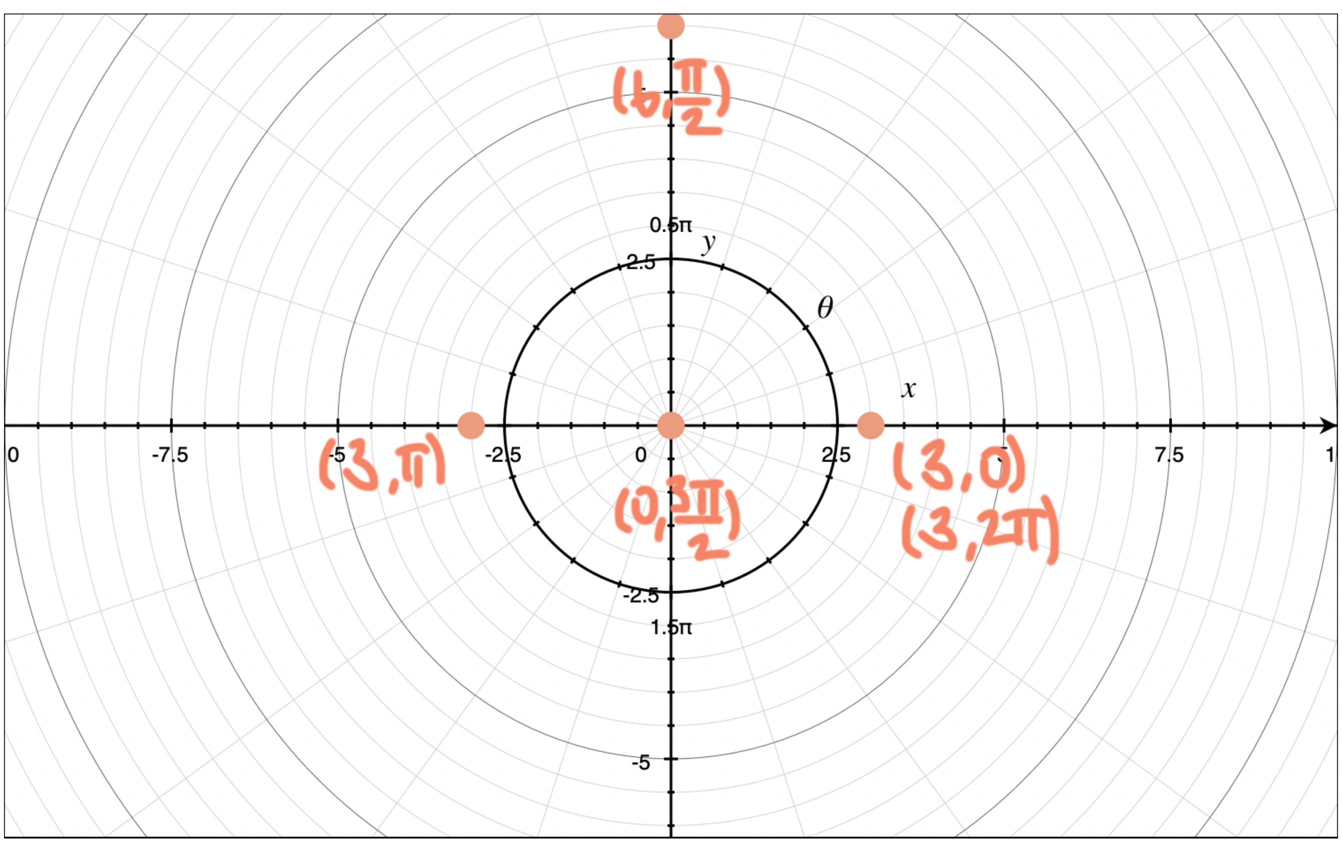
To sketch the graph, we recognize that the trigonometric function in this polar equation is $\sin\theta$, and its argument (the angle at which sine is evaluated) is θ. So we'll set

$$\theta = \frac{\pi}{2}$$

Now we'll make a table with multiples of $\pi/2$, like $\theta = 0$, $\pi/2$, π, $3\pi/2$, 2π, $5\pi/2$, 3π, etc., and include the values of r that correspond to each of these θ-values.

θ	0	$\pi/2$	π	$3\pi/2$	2π
r	3	6	3	0	3

Plotting these points on the polar graph gives



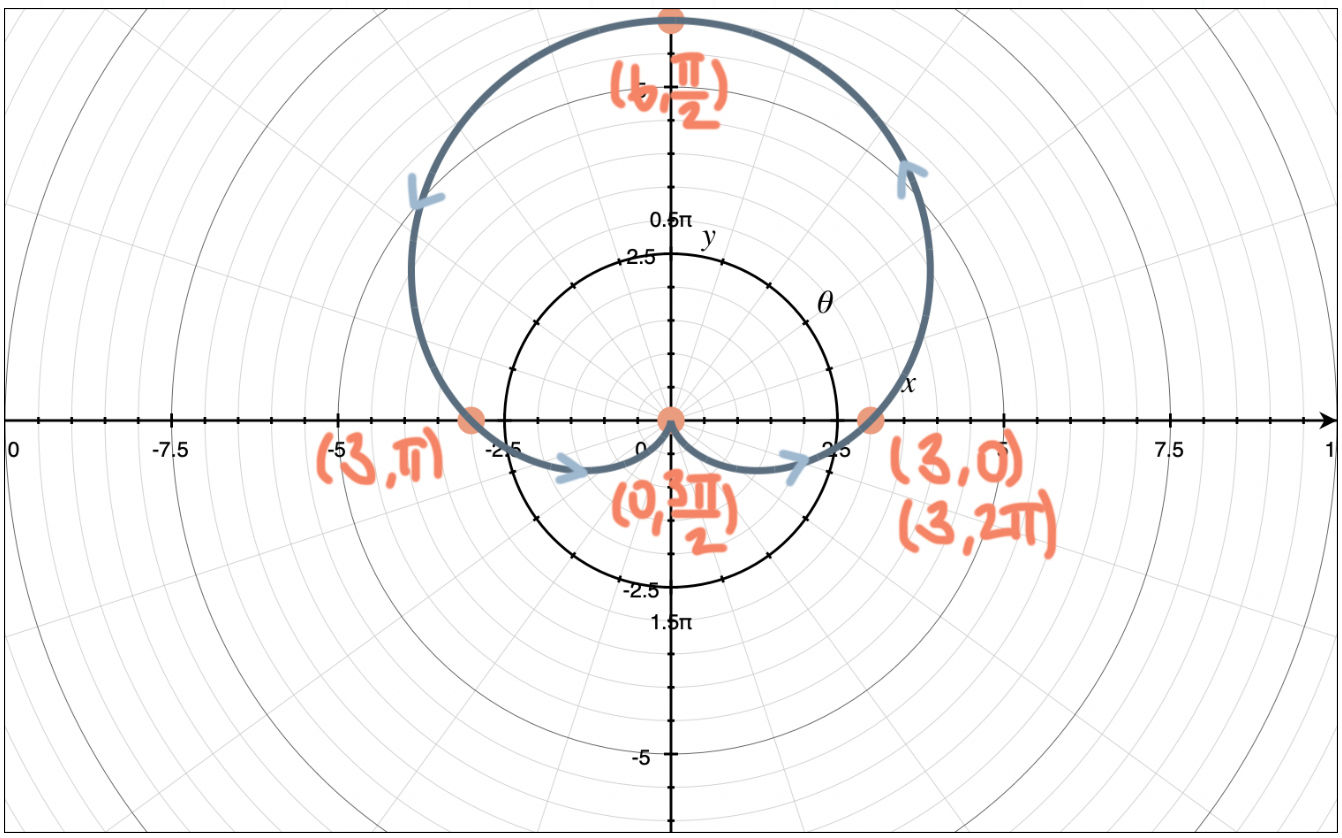
And if we connect these points with a smooth curve, in order, we see the graph of the cardioid. We start at $(3,0)$, and then

loop to $(6,\pi/2)$,

loop to $(3,\pi)$,

loop to $(0,3\pi/2)$,

then finally loop back to $(3,2\pi)$, which is actually the same point as $(3,0)$. From there on, we're retracing the same pieces of the cardioid over and over.



Let's do one more example to see what happens when $c<0$.

Example

Sketch the graph of $r = -4 - 4\cos\theta$.

Remember we said earlier that the value of c needs to be positive. Because this equation begins with $c = -4$, we need to factor out a negative sign.

$$r = -(4 + 4\cos\theta)$$

Now we have the cardioid equation $r = 4 + 4\cos\theta$, and we'll just need to apply the negative sign to each of our r-values.

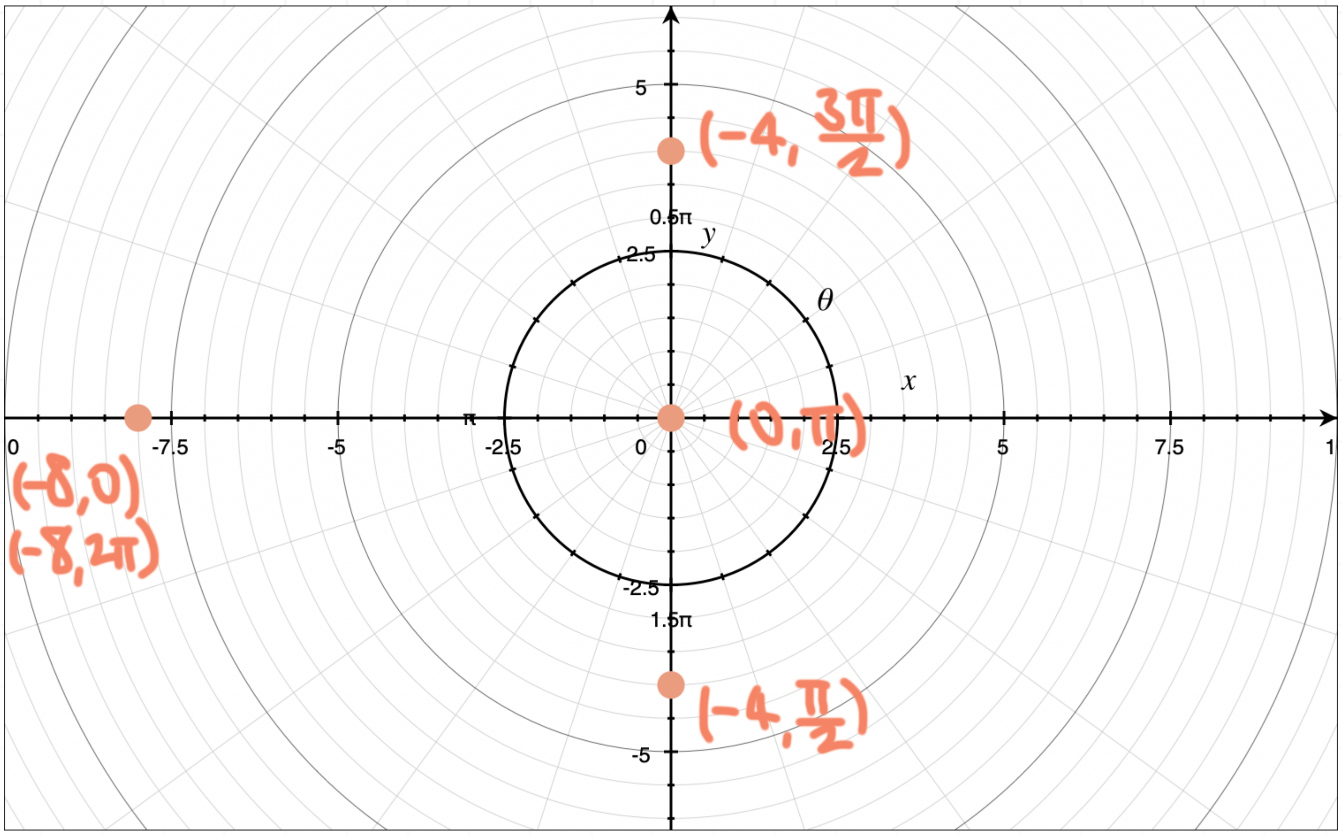
To sketch the graph, we recognize that the trigonometric function in $r = -(4 + 4\cos\theta)$ is $\cos\theta$, and its argument (the angle at which cosine is evaluated) is θ. So we'll set

$$\theta = \frac{\pi}{2}$$

Now we'll make a table with multiples of $\pi/2$, like $\theta = 0$, $\pi/2$, π, $3\pi/2$, 2π, $5\pi/2$, 3π, etc., and include the values of r that correspond to each of these θ-values.

θ	0	$\pi/2$	π	$3\pi/2$	2π
r	-8	-4	0	-4	-8

Plotting these points on the polar graph gives



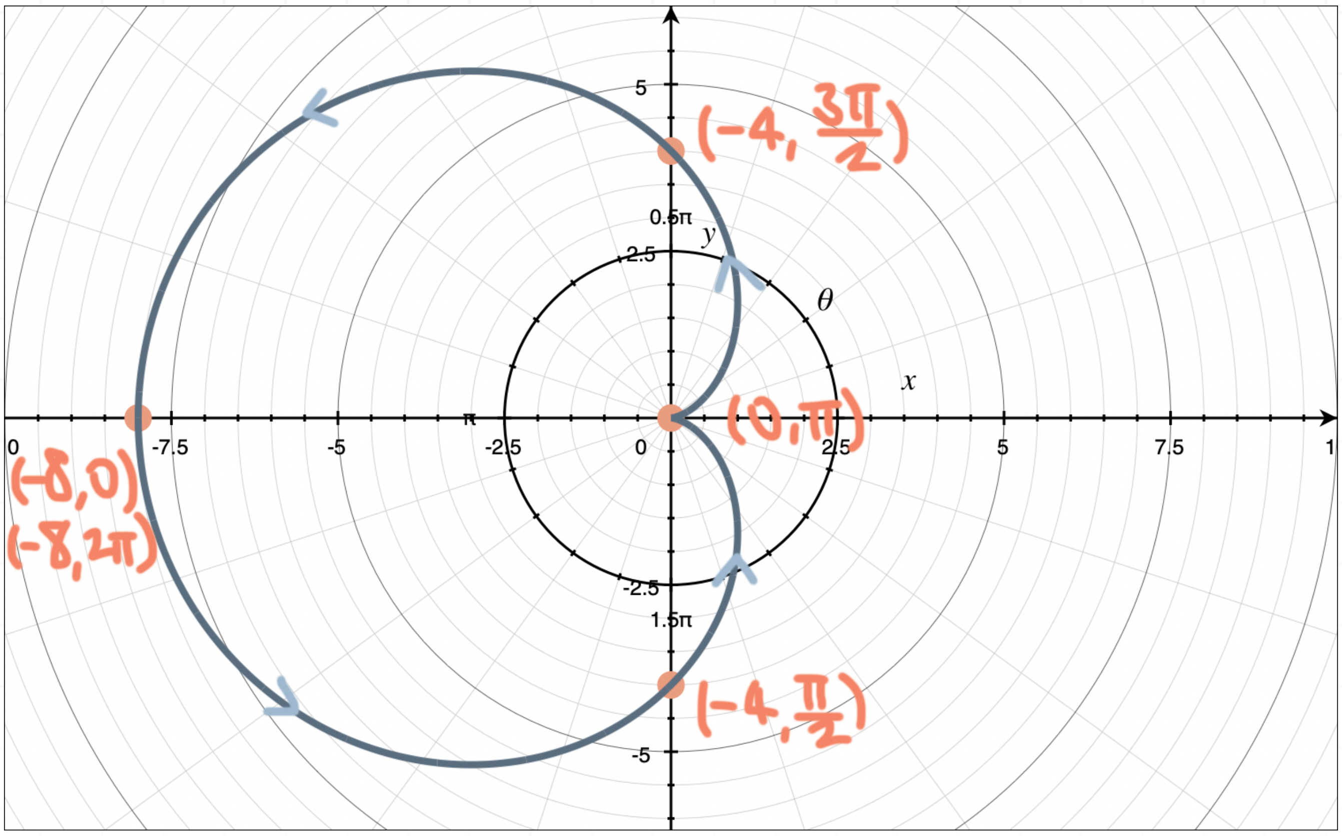
And if we connect these points with a smooth curve, in order, we see the graph of the cardioid. We start at $(-8,0)$, and then

loop to $(-4,\pi/2)$,

loop to $(0,\pi)$,

loop to $(-4,3\pi/2)$,

then finally loop back to $(-8,2\pi)$, which is actually the same point as $(-8,0)$. From there on, we're retracing the same pieces of the cardioid over and over.



In this last example, what we notice is that the graph of $r = -(4 + 4\cos\theta)$ is simply the graph of $r = 4 + 4\cos\theta$ reflected over the vertical axis. Which means $r = -(4 + 4\cos\theta)$ is actually equivalent to the cosine curve

$$r = 4 - 4\cos\theta$$

That's why we include the condition that c should be positive in these cardioid equations, because leading the cardioid equation with a negative value of c never actually gives us a new curve.

$r = -c + c\cos\theta$ turns out to be equivalent to $r = c + c\cos\theta$

$r = -c - c\cos\theta$ turns out to be equivalent to $r = c - c\cos\theta$

$r = -c + c\sin\theta$ turns out to be equivalent to $r = c + c\sin\theta$

$r = -c - c\sin\theta$ turns out to be equivalent to $r = c - c\sin\theta$