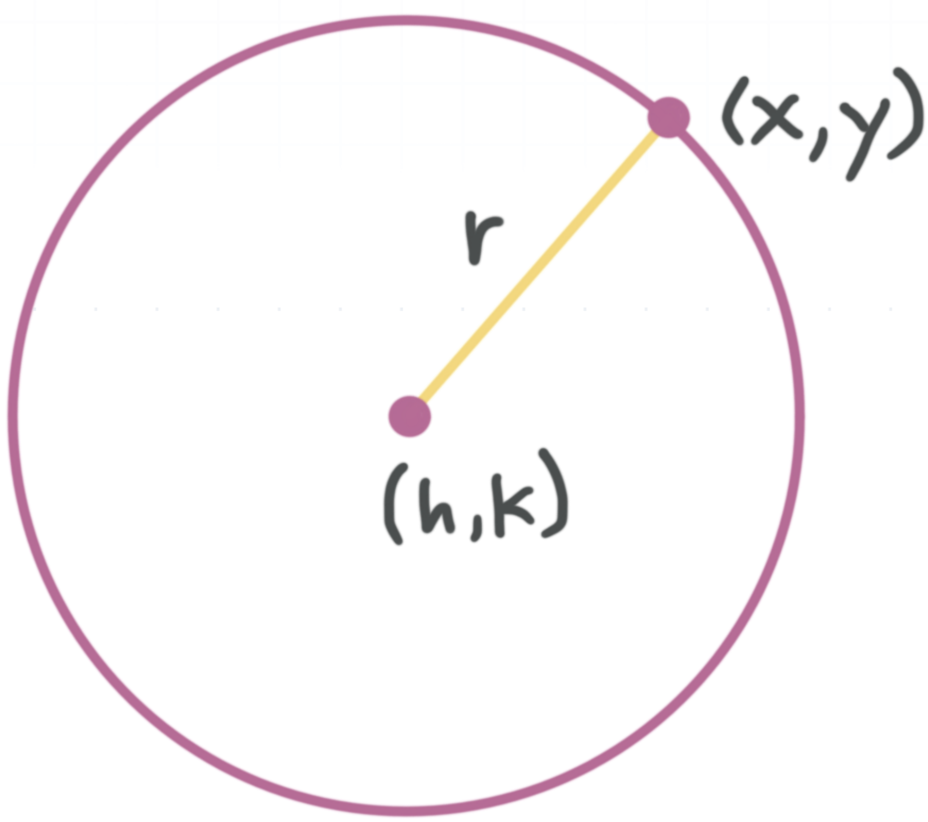
Equation of a circle

In this lesson we'll look at the equation of a circle, $(x-h)^2+(y-k)^2=r^2$, and how to use it to graph a circle, interpret points on a circle, and write the equation of a circle, given a graph or special features of the circle.

A circle can be defined by the point at its center and its radius. In the equation of a circle

$$(x-h)^2+(y-k)^2=r^2$$

the coordinates of the center are (h,k) and the radius is r. From the equation, you can see that the circle is the collection of all the points (x,y) that are a distance r from the center.

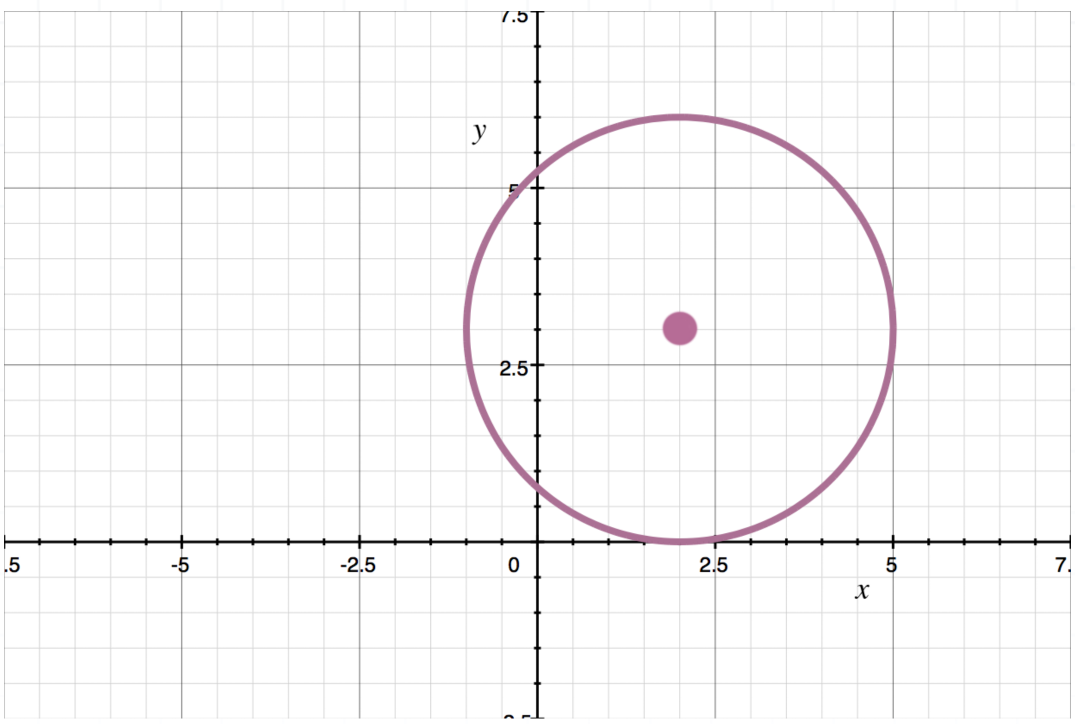


That collection of points includes only the points on the arc (curve). The center of the circle isn't actually a point of the circle.

Let's start by working through an example.

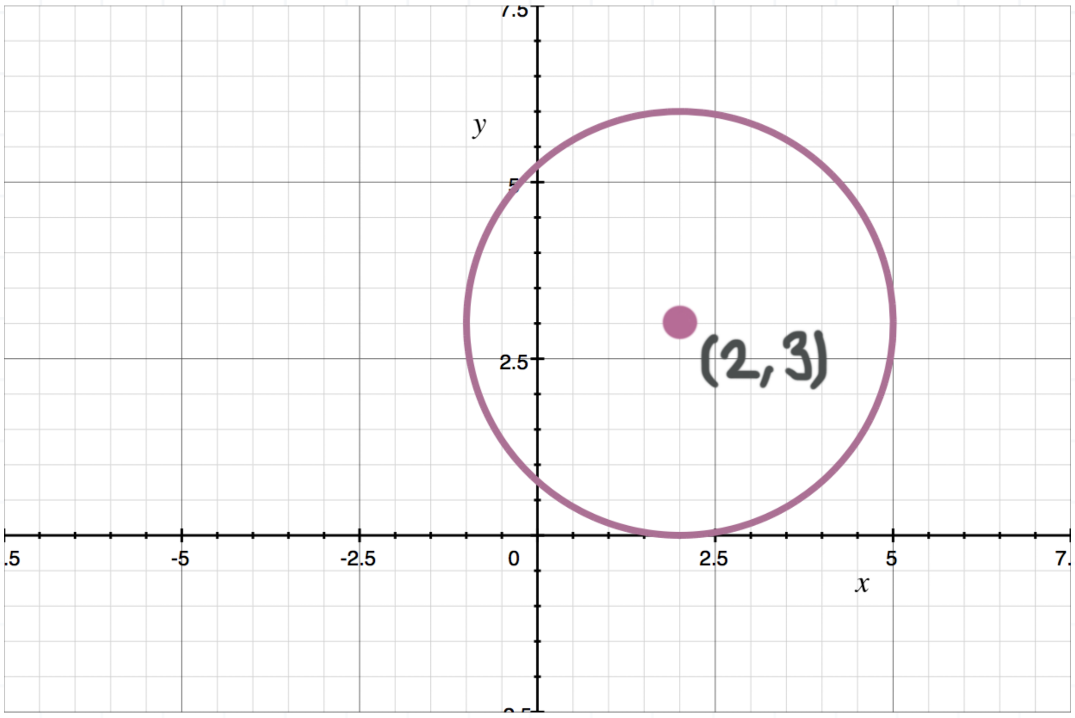
Example

What is the equation of the circle?

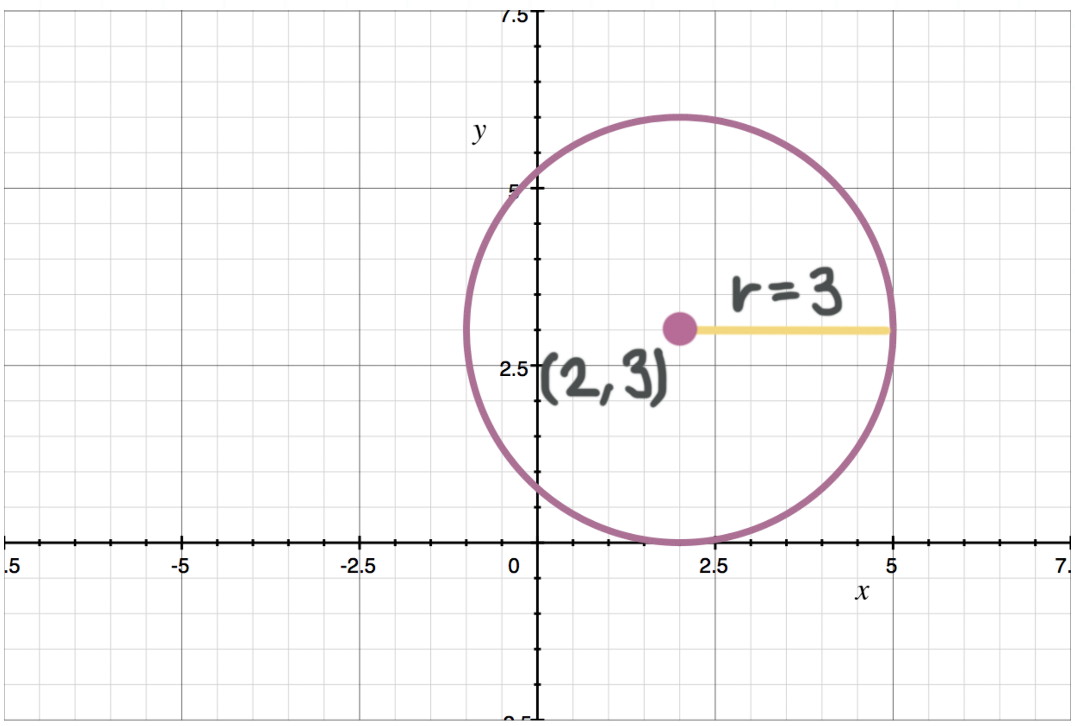


We need to find the equation of this circle in the form $(x-h)^2+(y-k)^2=r^2$, which means we need to find the coordinates of its center, (h,k), and its radius r.

The center is at $(2,3)$, so $h=2$ and $k=3$.



Now let's count from the center to a point on the circle to find the radius.



One point on the circle is $(5,3)$. The distance of that point from the center of the circle is 3, so $r = 3$. Now let's plug everything into the standard form of the equation of a circle.

$$(x-2)^2 + (y-3)^2 = 3^2$$

$$(x-2)^2 + (y-3)^2 = 9$$

Sometimes we want to know the center and radius of a circle given the equation of the circle.

Example

What are the center and radius of the circle?

$$x^2 + (y - 3)^2 = 27$$

We can rewrite the equation as

$$(x - 0)^2 + (y - 3)^2 = 27$$

Which lets us identify h and k as 0 and 3, respectively, so the center is at $(0,3)$. And the radius is $\sqrt{27}$, so

$$r = \sqrt{27}$$

$$r = \sqrt{9 \cdot 3}$$

$$r = \sqrt{9} \cdot \sqrt{3}$$

$$r = 3\sqrt{3}$$

Sometimes we want to know the x-intercepts of a circle.

Example

What are the x-intercepts of the circle?

$$(x-2)^2+(y+1)^2=16$$

The x-intercepts are the points at which $y=0$, so set y to 0 and solve for x.

$$(x-2)^2+(y+1)^2=16$$

$$(x-2)^2+(0+1)^2=16$$

$$(x-2)^2+1^2=16$$

$$x^2-4x+4+1=16$$

$$x^2-4x+5=16$$

$$x^2-4x-11=0$$

If you can't factor the equation to solve for x, you can use the quadratic formula. In this case, $a=1$, $b=-4$, and $c=-11$.

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$x=\frac{4\pm\sqrt{(-4)^2-4(1)(-11)}}{2(1)}$$

$$x=\frac{4\pm\sqrt{60}}{2}=\frac{4\pm\sqrt{4\cdot 15}}{2}=\frac{4\pm 2\sqrt{15}}{2}=2\pm\sqrt{15}$$

The x-intercepts are $(2+\sqrt{15},0)$ and $(2-\sqrt{15},0)$.

Sometimes to find out information about a circle, you'll need to know how to complete the square.

Example

Find the center and radius of the circle.

$$x^2+y^2+24x+10y+160=0$$

In order to find the center and radius, we need to convert the equation of the circle into standard form, $(x-h)^2+(y-k)^2=r^2$. In order to get the equation into standard form, we have to complete the square with respect to both variables.

Grouping x's and y's together and moving the constant to the right side, we get

$$(x^2+24x)+(y^2+10y)=-160$$

Completing the square with respect to any variable requires us to take the coefficient of the first-degree term in that variable, divide it by 2, and then square the result before adding it to both sides of the equation.

The coefficient of the first-degree term in x is 24, so

$$\frac{24}{2}=12 \quad \to \quad 12^2=144$$

The coefficient of the first-degree term in y is 10, so

$$\frac{10}{2} = 5 \quad \to \quad 5^2 = 25$$

So we add 144 inside the parentheses with the x terms, and 25 inside the parentheses with the y terms, and we also add 144 and 25 to the right side of the equation (to -160).

$$(x^2 + 24x + 144) + (y^2 + 10y + 25) = -160 + 144 + 25$$

$$(x + 12)^2 + (y + 5)^2 = 9$$

Therefore, the center of the circle is at $(h, k) = (-12, -5)$ and its radius is $r = \sqrt{9} = 3$.