Intersection of polar curves

We also want to be able to find the points at which two polar curves intersect one another. We did the same thing with linear equations back in Algebra.

In both cases, with linear equations in Algebra, and now here with polar equations in Precalculus, we'll set the curves equal to each other in order to solve for points of intersection.

"Hidden" points of intersection

But we have to be careful. The points that satisfy both polar equations aren't necessarily the only points of intersection. Because we can represent the same polar point multiple ways (as we learned earlier), it's possible for two polar curves to intersect one another at the same location in space, but at different values of r and θ.

For example, we know that $(3,0)$ and $(-3,\pi)$ represent the same point in polar space.

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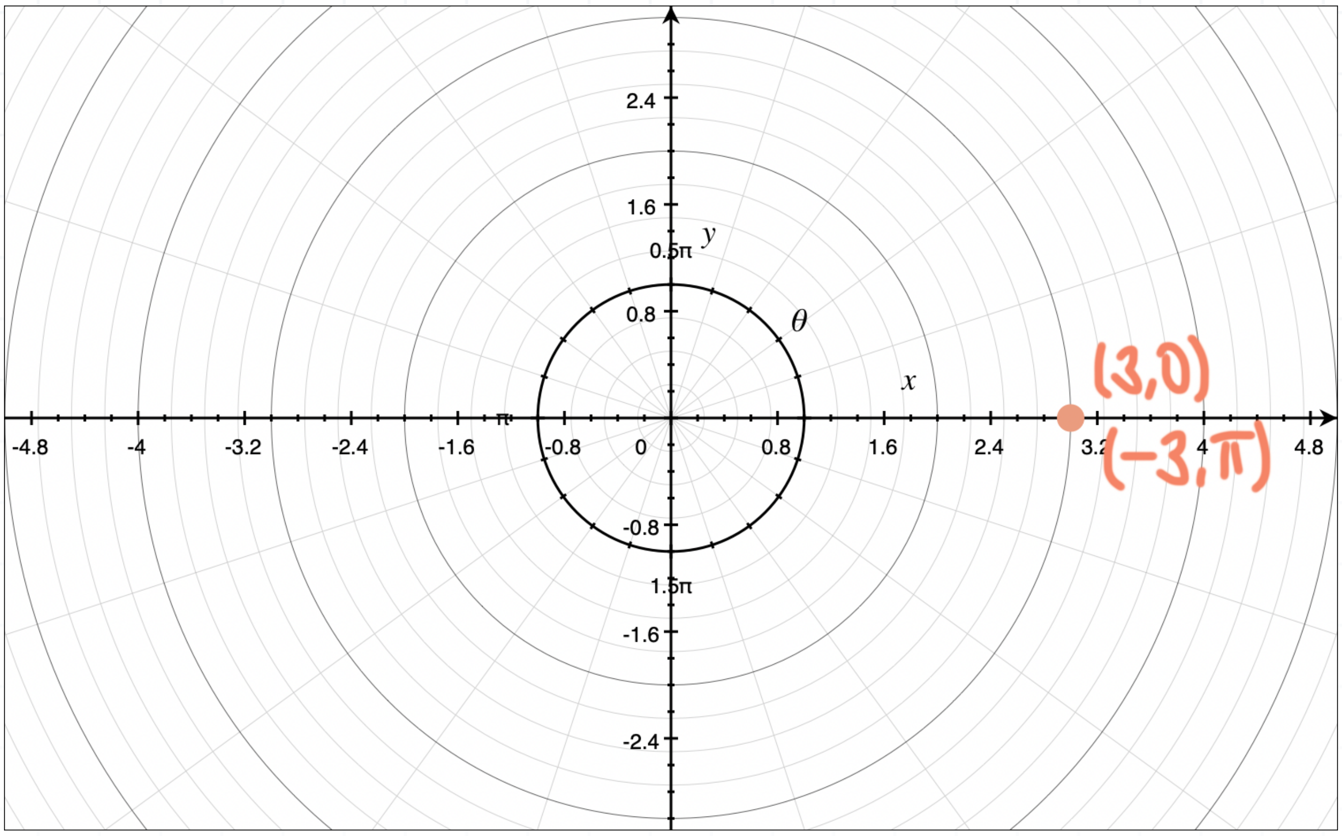

Let's say that we have one polar curve that intersects the point $(3,0)$, and another polar curve that intersects the point $(-3,\pi)$. When we set our two polar curves equal to one another, we won't find this point of intersection. That's because one curve satisfies $(3,0)$ but not $(-3,\pi)$, while the other curve satisfies $(-3,\pi)$ but not $(3,0)$. So even though both $(3,0)$ and $(-3,\pi)$ represent the same point in space, because they're represented differently, the intersection point won't show up when we set the curves equal to each other.

Therefore, our strategy for finding points of intersection for two polar curves will be to start by setting the curves equal to each other and solving for coordinate points that satisfy both equations.

But then we'll follow that up by sketching both curves to look for other "hidden" points of intersection that we might have missed. We'll pay special attention to the origin $(0,0)$, which is very common hidden point of intersection.

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If we find any hidden points of intersection, then we can use any representation of that point that we choose. In other words, in our example from earlier, we could choose either $(3,0)$ or $(-3,\pi)$ as the representation of the intersection. If possible, we try to choose a representation with an angle θ in the principal interval $[0,2\pi)$, and a positive value of r. So we'd prefer $(3,0)$ over $(-3,\pi)$, since r is positive in $(3,0)$.

Let's do an example where we find the intersection points of two circles.

Example

Find the points of intersection of $r=2$ and $r=-4\sin\theta$.

Because both equations are equal to r, we can set the right-hand sides equal to one another.

$$2=-4\sin\theta$$

$$\sin\theta=-\frac{1}{2}$$

From the unit circle from Trigonometry, we know that sine has a value of $-1/2$ at $\theta=7\pi/6$ and $\theta=11\pi/6$. If we plug these angles back into either of the polar equations, we get $r=2$ at both angles. Which means the points of intersection for these curves are

$$(r,\theta)=\left(2,\frac{7\pi}{6}\right)$$

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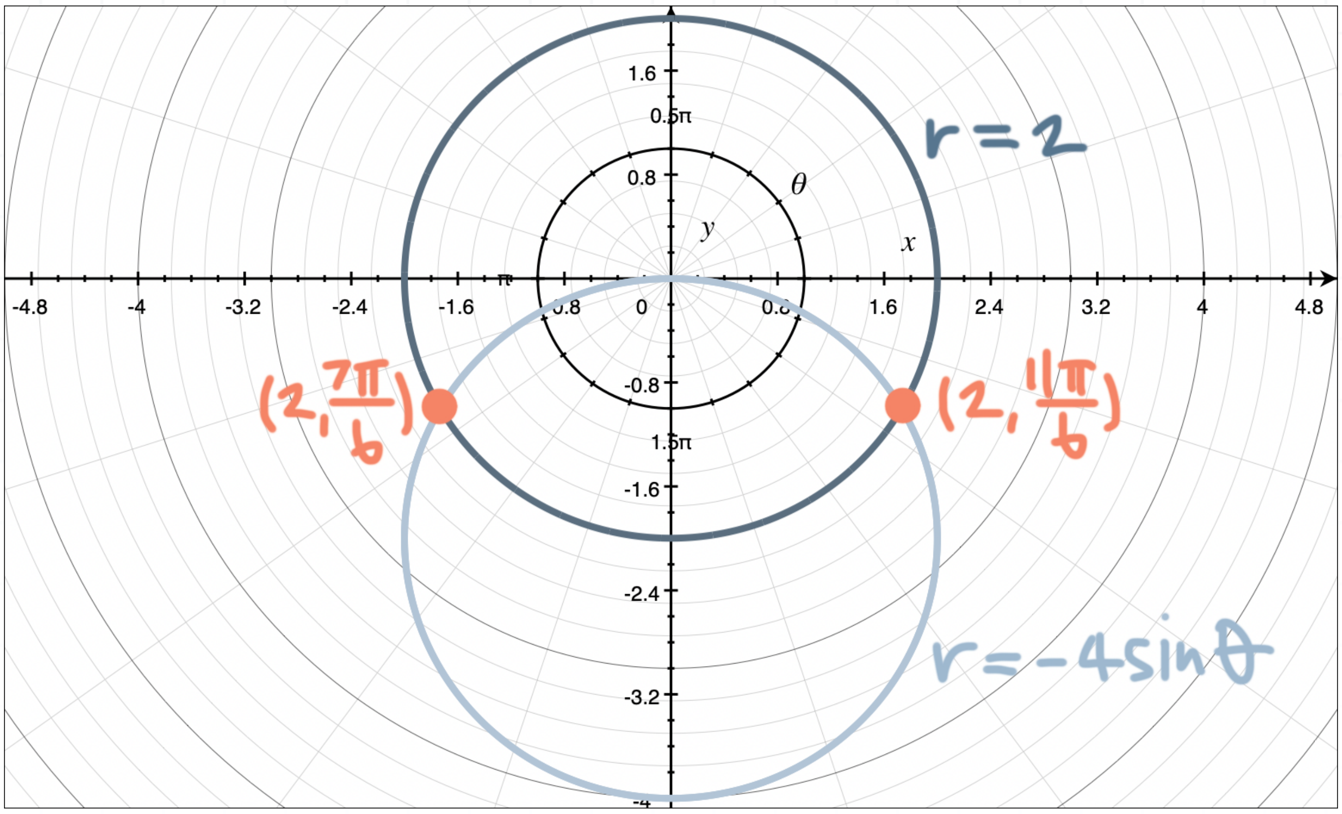
$$(r, \theta) = \left(2, \frac{11\pi}{6}\right)$$

If we sketch the graph of both curves on the same set of axes, we can see these points of intersection.



Let's do another example, this time with a circle and a limaçon.

Example

Find the points of intersection of $r = 2$ and $r = 2 - 4\cos\theta$.

In the limaçon equation, we have $a = 2$ and $b = 4$, which means $a/b < 1$, which tell us that the limaçon has a small loop. We'll start by setting the curves equal to one another.

$$2 = 2 - 4\cos\theta$$

$$0 = -4\cos\theta$$

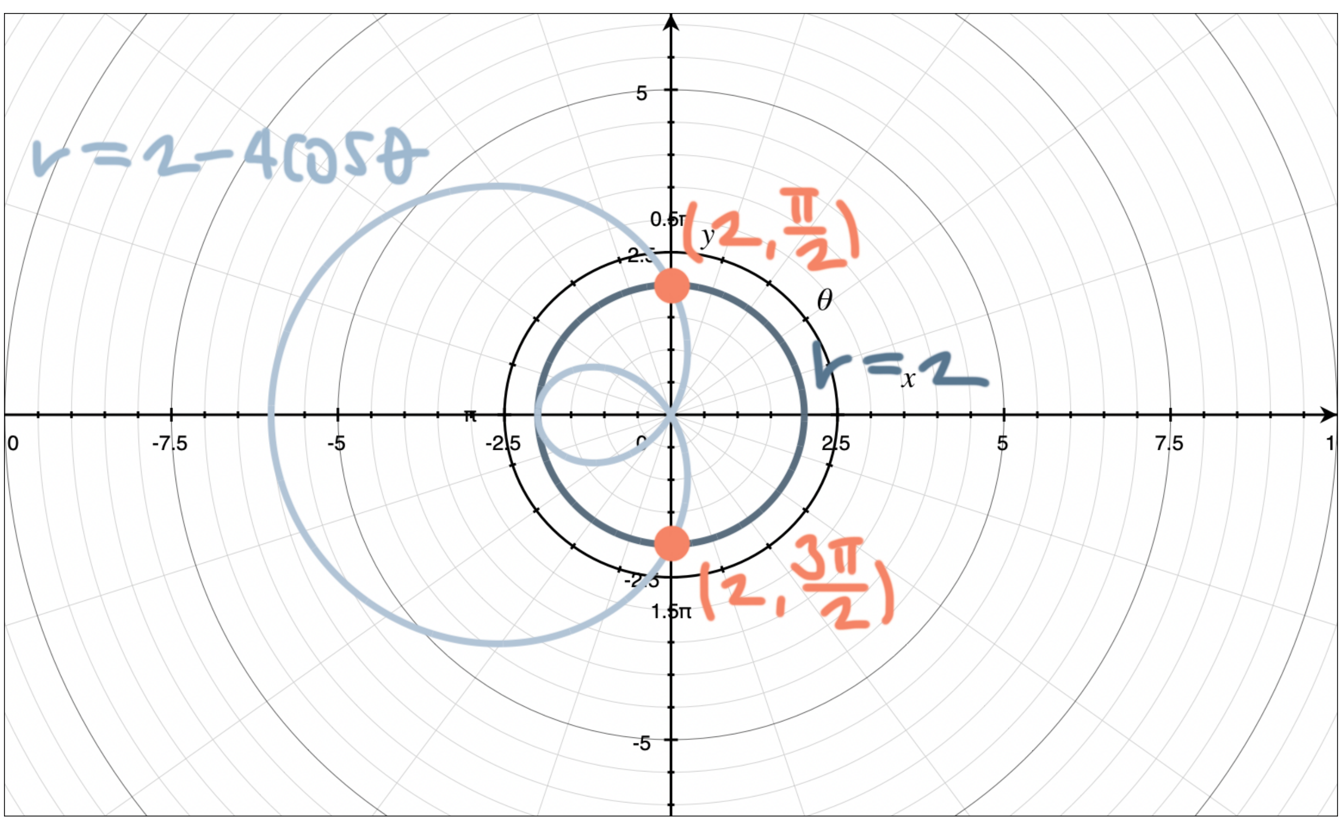
$$\cos\theta = 0$$

From the unit circle from Trigonometry, we know that cosine has a value of 0 at $\theta = \pi/2$ and $\theta = 3\pi/2$. If we plug these angles back into either of the polar equations, we get $r = 2$ at both angles. Which means the points of intersection for these curves are

$$(r, \theta) = \left(2, \frac{\pi}{2}\right)$$

$$(r, \theta) = \left(2, \frac{3\pi}{2}\right)$$

If we sketch the graph of both curves on the same set of axes, we can see these points of intersection.



What we notice when we sketch the curves is that we actually have a third "hidden" point of intersection at $(2,\pi)$ that we didn't uncover when we set the curves equal to each other and solved for θ.

That's because the circle reaches that third point at $(r,\theta)=(2,\pi)$, while the limaçon reaches that point at $(r,\theta)=(-2,0)$. We can pick either representation of this third point; we'll choose $(2,\pi)$ because it includes a positive value of r, so the points of intersection of the curves are

$$(r,\theta)=\left(2,\frac{\pi}{2}\right)$$

$$(r,\theta)=\left(2,\frac{3\pi}{2}\right)$$

$$(r,\theta)=(2,\pi)$$

Let's look at an example with a circle and a cardioid, where we again have "hidden" points.

Example

Find the points of intersection of $r=3\sin\theta$ and $r=1+\sin\theta$.

The cardioid is a sine cardioid with a positive sign between the terms, which means its graph is symmetric about the vertical axis and will sit mostly above the horizontal axis. We'll start by setting the curves equal to one another.

$$3\sin\theta = 1 + \sin\theta$$

$$2\sin\theta = 1$$

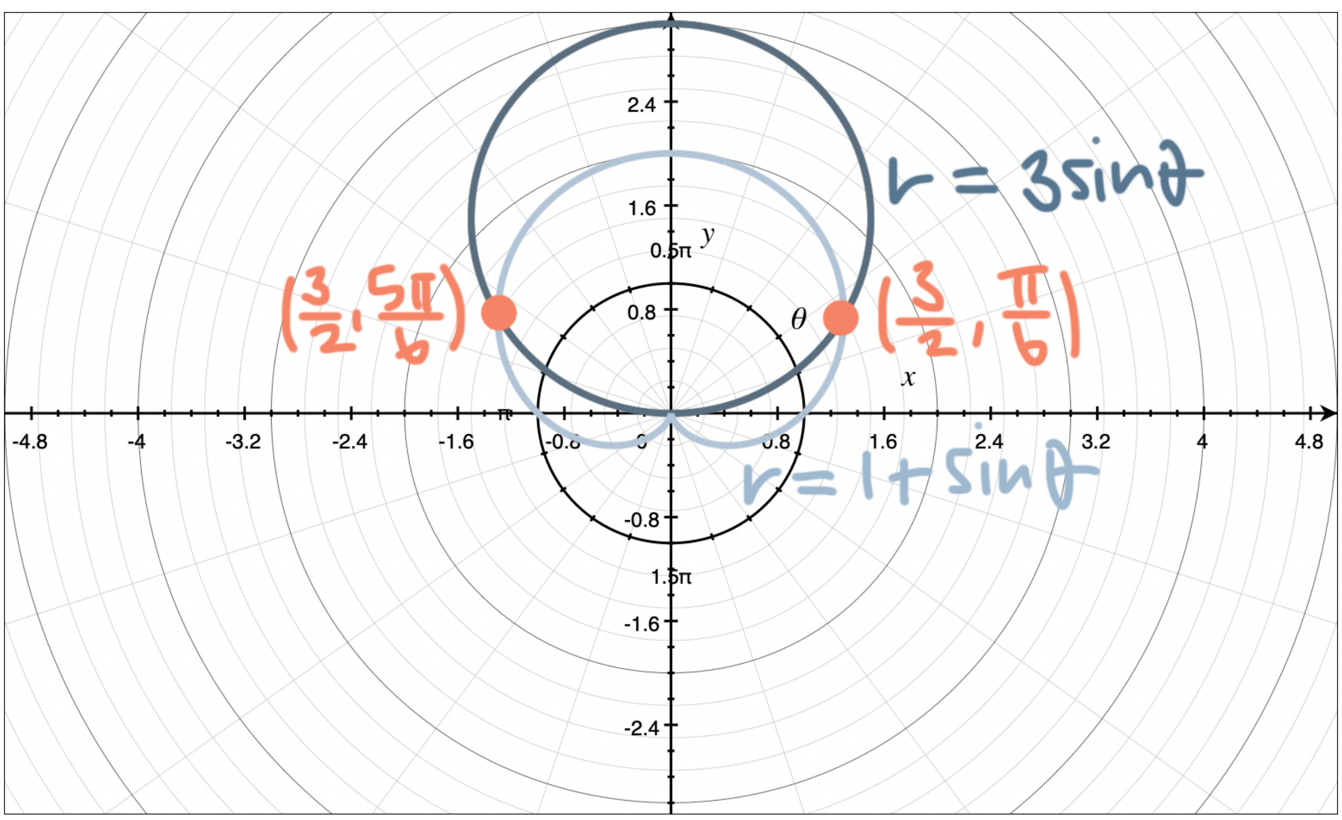
$$\sin\theta = \frac{1}{2}$$

From the unit circle from Trigonometry, we know that sine has a value of $1/2$ at $\theta = \pi/6$ and $\theta = 5\pi/6$. If we plug these angles back into either of the polar equations, we get $r = 3/2$ at both angles. Which means the points of intersection for these curves are

$$(r, \theta) = \left(\frac{3}{2}, \frac{\pi}{6}\right)$$

$$(r, \theta) = \left(\frac{3}{2}, \frac{5\pi}{6}\right)$$

If we sketch the graph of both curves on the same set of axes, we can see these points of intersection.



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What we notice when we sketch the curves is that we actually have a third "hidden" point of intersection at $(0,0)$ that we didn't uncover when we set the curves equal to each other and solved for θ.

That's because the circle reaches that third point at $(r,\theta)=(0,0)$, while the cardioid reaches that point at $(r,\theta)=(0,3\pi/2)$. We can pick either representation of this third point; we'll choose $(0,0)$, so the points of intersection of the curves are

$$(r,\theta)=\left(\frac{3}{2},\frac{\pi}{6}\right)$$

$$(r,\theta)=\left(\frac{3}{2},\frac{5\pi}{6}\right)$$

$$(r,\theta)=(0,0)$$

Let's look at another example with two cardioids.

Example

Find the points of intersection of $r=4-4\cos\theta$ and $r=4+4\sin\theta$.

The sine cardioid has a positive sign between the terms, which means its graph is symmetric about the vertical axis and will sit mostly above the horizontal axis, whereas the cosine cardioid has a negative sign between the terms, which means its graph is symmetric about the horizontal axis and will sit mostly to the left of the vertical axis.

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We'll start by setting the curves equal to one another.

$$4 - 4\cos\theta = 4 + 4\sin\theta$$

$$-4\cos\theta = 4\sin\theta$$

$$-\cos\theta = \sin\theta$$

From the unit circle from Trigonometry, we know that sine and cosine have equal values (except opposite signs) at $\theta = 3\pi/4$ and $\theta = 7\pi/4$. If we plug these angles back into either of the polar equations, we get $r = 4 + 2\sqrt{2}$ at $\theta = 3\pi/4$, and $r = 4 - 2\sqrt{2}$ at $\theta = 7\pi/4$. Which means the points of intersection for these curves are

$$(r, \theta) = \left(4 + 2\sqrt{2}, \frac{3\pi}{4}\right)$$

$$(r, \theta) = \left(4 - 2\sqrt{2}, \frac{7\pi}{4}\right)$$

If we sketch the graph of both curves on the same set of axes, we can see these points of intersection.

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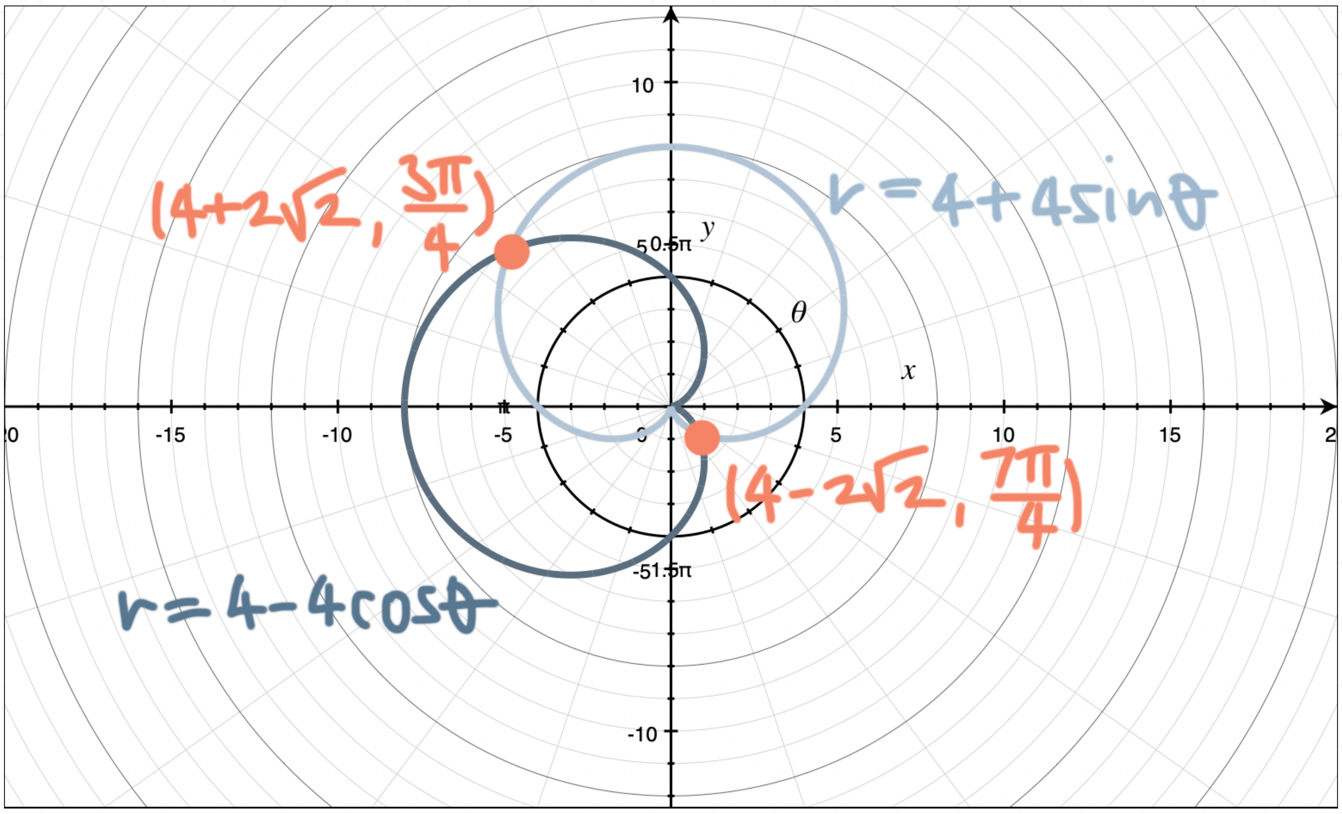

What we notice when we sketch the curves is that we actually have a third "hidden" point of intersection at $(0,0)$ that we didn't uncover when we set the curves equal to each other and solved for θ.

That's because the cosine cardioid reaches that third point at $(r,\theta)=(0,0)$, while the sine cardioid reaches that point at $(r,\theta)=(0,3\pi/2)$. We can pick either representation of this third point; we'll choose $(0,0)$, so the points of intersection of the curves are

$$(r,\theta)=\left(4+2\sqrt{2},\frac{3\pi}{4}\right)$$

$$(r,\theta)=\left(4-2\sqrt{2},\frac{7\pi}{4}\right)$$

$$(r,\theta)=(0,0)$$

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Finally, let's look at the intersection of a rose and a lemniscate.

Example

Find the points of intersection of $r = 3\cos(2\theta)$ and $r^2 = 9\cos(2\theta)$.

The rose has $|2n| = |2(2)| = 4$ petals that extend out to a distance of $c = 3$ from the origin. Because the equation of the rose is given for r, while the equation of the lemniscate is given for r^2, we'll square the rose equation,

$$r = 3\cos(2\theta)$$

$$r^2 = (3\cos(2\theta))^2$$

$$r^2 = 9\cos^2(2\theta)$$

then set the curves equal to one another.

$$9\cos^2(2\theta) = 9\cos(2\theta)$$

$$\cos^2(2\theta) = \cos(2\theta)$$

$$\cos^2(2\theta) - \cos(2\theta) = 0$$

$$\cos(2\theta)(\cos(2\theta) - 1) = 0$$

Apply the Zero Theorem from Algebra to create two equations that we can solve individually. We get

$$\cos(2\theta) = 0$$

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$$2\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \ldots$$

$$\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}, \ldots$$

and

$$\cos(2\theta) - 1 = 0$$

$$\cos(2\theta) = 1$$

$$2\theta = 0,\ 2\pi,\ 4\pi,\ 6\pi,\ \ldots$$

$$\theta = 0,\ \pi,\ 2\pi,\ 3\pi,\ \ldots$$

Combining these angle sets into one gives the complete set of angles that satisfy both equations.

$$\theta = 0, \frac{\pi}{4}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{7\pi}{4}, 2\pi, \ldots$$

If we plug these angles back into either of the polar equations, we get

	$r = 3\cos(2\theta)$	$r^2 = 9\cos(2\theta)$
$\theta = 0$	$r = 3$	$r = \pm 3$
$\theta = \pi/4$	$r = 0$	$r = 0$
$\theta = 3\pi/4$	$r = 0$	$r = 0$
$\theta = \pi$	$r = 3$	$r = \pm 3$
$\theta = 5\pi/4$	$r = 0$	$r = 0$

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$\theta = 7\pi/4$	$r = 0$	$r = 0$
$\theta = 2\pi$	$r = 3$	$r = \pm 3$

We have points of intersection only where we have matching values of r. We get $r = 3$ from both curves at $\theta = 0$, so $(r, \theta) = (3,0)$ is a point of intersection. We get $r = 0$ from both curves at $\theta = \pi/4$, so $(r, \theta) = (0,\pi/4)$ is a second point of intersection. We get $r = 3$ from both curves at $\theta = \pi$, so $(r, \theta) = (3,\pi)$ is a third point of intersection.

The table appears to include many more points of intersection, but any other matching values we found will just give us duplicates of the same three points, $(3,0)$, $(0,\pi/4)$, and $(3,\pi)$.

If we sketch the graph of both curves on the same set of axes, we can see these points of intersection.