Converting polar equations to rectangular

We know now how to convert polar coordinate points (r, θ) into rectangular coordinate points (x, y). We've used the conversion formulas

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$r^2 = x^2 + y^2$$

$$\tan\theta = \frac{y}{x}$$

In the same way that we used these conversion formulas to convert coordinate points, we can also use them to convert equations from polar coordinates into rectangular coordinates.

It's common to see polar equations like $r = 8\cos\theta$, where the equation is defined for r in terms of θ, in the same way that it's common to see rectangular equations like $y = x + 3$, where the equation is defined for y in terms of x.

But, just like we can have rectangular equations with only one variable, like $y = 2$ and $x = 4$, we can have polar equations with only one variable, like $r = 2$ and $\theta = \pi$. While $y = 2$ represents a perfectly horizontal line and $x = 4$ represents a perfectly vertical line, $r = 2$ represents a perfect circle around the origin and $\theta = \pi$ represents a line from the origin out toward the angle $\theta = \pi$.

We can convert all of these kinds of polar equations into rectangular equations. Let's do an example with an equation like $r = 2$.

Example

Convert the polar equation $r = 7$ to rectangular coordinates.

To convert a polar equation in this form, we'll plug $r = 7$ into the conversion equation $r^2 = x^2 + y^2$.

$$7^2 = x^2 + y^2$$

$$x^2 + y^2 = 49$$

We've converted the polar equation into a rectangular equation. So both $r = 7$ and $x^2 + y^2 = 49$ represent the identical curve.

Now let's look at an example with an equation like $\theta = \pi$.

Example

Convert the polar equation $\theta = \pi/3$ to rectangular coordinates.

To convert a polar equation in this form, take the tangent of both sides of the equation.

$$\tan\theta = \tan\left(\frac{\pi}{3}\right)$$

Now we can use the conversion equation $\tan\theta = y/x$ to get

$$\frac{y}{x} = \tan\left(\frac{\pi}{3}\right)$$

To simplify the right side, we'll use the quotient identity for tangent to rewrite tangent as sine over cosine.

$$\tan\left(\frac{\pi}{3}\right) = \frac{\sin\left(\frac{\pi}{3}\right)}{\cos\left(\frac{\pi}{3}\right)} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3}$$

Substituting this result into the equation from earlier, we get

$$\frac{y}{x} = \sqrt{3}$$

$$y = \sqrt{3}x$$

Let's do one last example, this time with the equation $r = 8\cos\theta$, so that we can see what happens when both r and θ are present in the equation.

Example

Convert the polar equation $r = 8\cos\theta$ to rectangular coordinates.

First, we'll rewrite the conversion equation $x = r\cos\theta$ as

$$\cos\theta = \frac{x}{r}$$

Now we can replace $\cos\theta$ in $r = 8\cos\theta$ with x/r.

$$r = 8\left(\frac{x}{r}\right)$$

$$r^2 = 8x$$

Then with the conversion equation $r^2 = x^2 + y^2$, we can substitute $x^2 + y^2$ for r^2.

$$x^2 + y^2 = 8x$$

$$x^2 - 8x + y^2 = 0$$