PH3203 Term Paper: Fermi's golden rule and Bardeen's tunneling theory

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Abstract

In this article, we study and analyze the derivation of Bardeen's tunneling theory [2] using Fermi's Golden Rule [4] for the tunneling of electrons across a vacuum gap between two metal electrodes, treating the appearance of the second electrode as a perturbation. We elucidate Fermi's Golden Rule by deriving it for a constant perturbation. This is followed by an analysis of an exactly solvable example of a rectangular barrier [4].

Introduction

Quantum tunneling is a phenomenon in which an object such as an electron or atom passes through a potential energy barrier that should not be passable (according to classical mechanics) due to the object not having sufficient energy to pass or surmount the barrier. Tunneling is due to the wavelike behaviour of the object, whose **wavefunction** is governed by the Schrödinger equation.

In this paper we explore the topics of Fermi's Golden Rule and Bardeen's Tunnelling Theory. We begin with Fermi's Golden Rule, which is a formulation to predict the probability of a quantum system transitioning from one state to another due to a perturbation in its Hamiltonian or total energy explicitly for weak interactions. We derive the rule for a time-dependent constant perturbation. This is relevant because, in the tunneling of electrons across the electrode-vacuum barrier, the second electrode is brought within a finite distance (in our case, $2d$) of the first electrode at some time t, which can be regarded as a time-dependent constant perturbation. Following this, we study Bardeen's Tunnelling Theory which is an alternative method using the Transfer Hamiltonian Approach. This approach involves dividing the system into distinct regions separated by potential barriers. Within each region, the behaviour of the wavefunction of the particle can be described by the Schrödinger equation. The transfer Hamiltonian describes the coupling between these regions, allowing for the calculation of tunnelling probabilities. We follow the approach taken by Reittu [4] in the tunneling (low rate) of electrons from one electrode to another across a vacuum gap, such that the appearance of the second electrode can be regarded as a perturbation. Bardeen's theory is then illustrated by an analysis of the exactly solvable example of a rectangular barrier.

Fermi's Golden Rule

Fermi's Golden Rule describes the transition rate of a quantum system from one of its eigenstates in a discrete spectrum to a state in a continuum, under the effect of a time-dependent perturbation. The transition rate (probability of a transition per unit time) is independent of time (given that the perturbation strength is independent of time). It is proportional to the square of the absolute value of matrix element of the perturbative Hamiltonian, and the density of states of the final continuum spectrum.

Setting up the System

We take a system in an initial eigenstate $|i\rangle$ of an unperturbed Hamiltonian H^0. The system is then subject to a perturbative Hamiltonian δH. The new Hamiltonian is of the form

$$H = H^0 + \delta H \tag{1}$$

As a result of the application of the first order perturbation, the probability of the system being found in a set of final states $|f\rangle$ to first order is given by:

$$P^{(1)}_{f\leftarrow i}(t) = \frac{2\pi}{\hbar} \left|\langle f|\delta H|i\rangle\right|^2 \tag{2}$$

The above expression is the mathematical expression of Fermi's Golden Rule.

Derivation

We derive Fermi's Golden Rule for a time-dependent constant perturbation i.e., $\delta H(t)$.

Constant Perturbation

We consider a constant perturbation with strength V, which switches on at time $t = 0$. The Hamiltonian for the system has the following form:

$$H = \begin{cases} H^{(0)}, & t \leq 0 \\ H^{(0)} + V, & t > 0 \end{cases} \tag{3}$$

Here, $\delta H(t) = V$ for $t \geq 0$. Using the expression for the first order probability of a transition from $|i\rangle$ to $|f\rangle$ at some time $t_0 (> 0)$, we get:

$$P^{(1)}_{f\leftarrow i}(t_0) = \left|\frac{1}{i\hbar}\int_0^{t_0} e^{i\omega_{fi}t}\delta H_{fi}\right|^2 = \left|\frac{1}{i\hbar}\int_0^{t_0} e^{i\omega_{fi}t}V_{fi}\right|^2 \tag{4}$$

where $\delta H_{fi} = V_{fi} = \langle f|V|i\rangle$ is the matrix element of the perturbation calculated using the initial and final states. We obtain

$$P^{(1)}_{f\leftarrow i}(t_0) = \frac{|V_{fi}|^2}{\hbar^2}\left|\frac{e^{i\omega_{fi}t_0} - 1}{i\omega_{fi}}\right|^2 = \frac{|V_{fi}|^2}{\hbar^2\omega_{fi}^2}\left|e^{\frac{i\omega_{fi}t_0}{2}} 2i\sin\left(\frac{\omega_{fi}t_0}{2}\right)\right|^2 \tag{5}$$

The final expression for the transition probability from $|i\rangle$ at $t = 0$ to $|f\rangle$ at $t = t_0$ is

$$\boxed{P^{(1)}_{f\leftarrow i}(t_0) = 4|V_{fi}|^2 \frac{\sin^2\left(\frac{\omega_{fi}t_0}{2}\right)}{(E_f - E_i)^2}} \tag{6}$$

where $E_f - E_i = \hbar\omega_{fi}$
This result is valid at time t_0 only if we have $P^{(1)}_{f\leftarrow i}(t_0) \ll 1$. If the transition probability at first order is high, then we must investigate higher orders.

There are two possibilities:

- $\mathbf{E_f \neq Ei}$: Energy non-conserving transition
- $\mathbf{E_f \rightarrow Ei}$: Energy conserving transition

Case 1: $\mathbf{E_f \neq Ei}$

For this scenario, the probability of transition is oscillatory in nature (with frequency ω_{fi}). The expression is accurate for all values of t_0 if the amplitude of oscillation is much smaller than 1. We also note that as the energy difference $|E_f - Ei|$ increases, the probability of transitioning becomes smaller. This is because a perturbation that switches on at some point of time and remains constant is not an efficient supply of energy *(Zwiebach, MIT OCW 8.06 lecture notes)*.

$$\frac{4\left|V_{fi}\right|^2}{\left(E_f - E_i\right)^2} \ll 1 \tag{7}$$

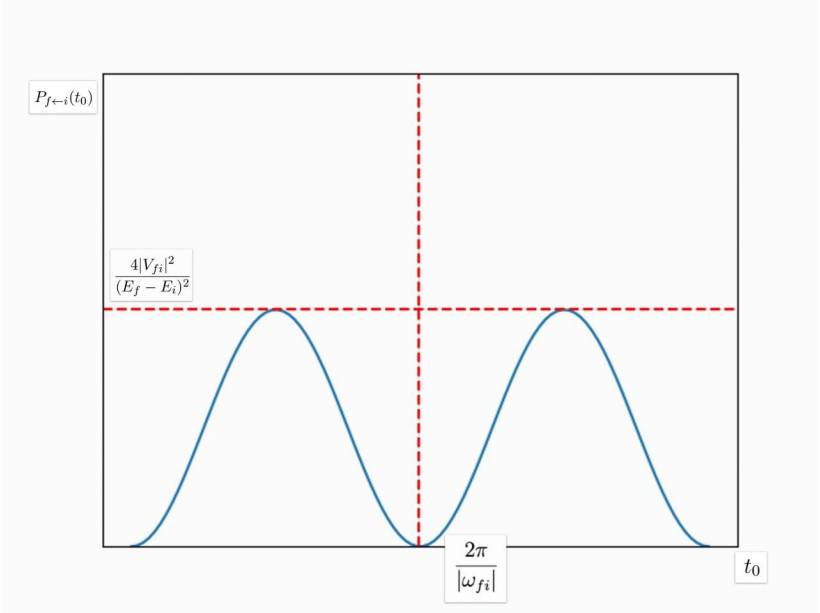

Figure 1: **Probability of transition as a function of t_0**

Case 2: $\mathbf{E_f \to Ei}$

In this situation, we have $\omega_{fi} \to 0$, and therefore

$$\lim_{E_f \to E_i} P^{(1)}_{f \leftarrow i}(t_0) = 4 \left|V_{fi}\right|^2 \frac{\sin^2\left(\frac{\omega_{fi} t_0}{2}\right)}{\hbar^2 \left(\frac{\omega_{fi} t_0}{2}\right)^2 \frac{4}{t_0^2}} = \frac{\left|V_{fi}\right|^2}{\hbar^2} t_0^2 \tag{8}$$

Since the result is an expression that varies quadratically with time, it can be assumed valid only for small t_0, ensuring that $P^{(1)}_{f \leftarrow i}(t_0) \ll 1$.
Now, we have to sum over the probabilities of transition for all the states $|f\rangle$ in the continuum of final states. The discrete sum over states can be approximated by an integral by utilizing the density of states $\rho(E_f)$ of the continuum.

$$\sum_f P^{(1)}_{f \leftarrow i}(t_0) = \int P^{(1)}_{f \leftarrow i}(t_0) \rho(E_f) dE_f = 4 \int \left|V_{fi}\right|^2 \frac{\sin^2\left(\frac{\omega_{fi} t_0}{2}\right)}{\left(E_f - E_i\right)^2} \rho(E_f) dE_f \tag{9}$$

It can be seen that with growing $(E_f - E_i)$, the integrand becomes smaller. Therefore, only the states (with energy E_f) lying within a narrow range of energies ΔE_f near E_i contribute **significantly** to the integral.

Since only a narrow range of states contribute to the integral, we can make the following assumption: $\rho(E_f)$ and $|V_{fi}|^2$ are almost constant over the range ΔE. This permits us to pull them out of the integral.

$$\sum_f P^{(1)}_{f \leftarrow i}(t_0) = 4 \left|V_{fi}\right|^2 \rho(E_f \approx E_i) I \tag{10}$$

where

$$I = \int \frac{\sin^2\left(\frac{\omega_{fi} t_0}{2}\right)}{\left(E_f - E_i\right)^2} dE_f \tag{11}$$

Writing I in terms of ω_{fi}, we obtain:

$$I = \frac{1}{\hbar} \int \frac{\sin^2\left(\frac{\omega_{fi} t_0}{2}\right)}{\omega_{fi}^2} d\omega_{fi} \tag{12}$$

We make the substitution

$$u = \frac{\omega_{fi} t_0}{2} \implies du = \frac{t_0}{2} d\omega_{fi}$$

We obtain

$$I = \frac{1}{\hbar}\int_{-\infty}^{\infty}\left(\frac{\sin^2 u}{4u^2}\right)t_0^2\frac{2}{t_0}du = \frac{t_0}{2\hbar}\int_{-\infty}^{\infty}\frac{\sin^2 u}{u^2} = \frac{\pi t_0}{2\hbar} \tag{13}$$

From the plot of the function $f(x) = \dfrac{\sin^2 x}{x}$, we note that the bulk of the contribution to the integral comes from the main lobe. Therefore, we can put the integration limits as $-\infty$ to ∞ without affecting the result (since the remaining lobes' contributions are negligible compared to that of the main lobe).

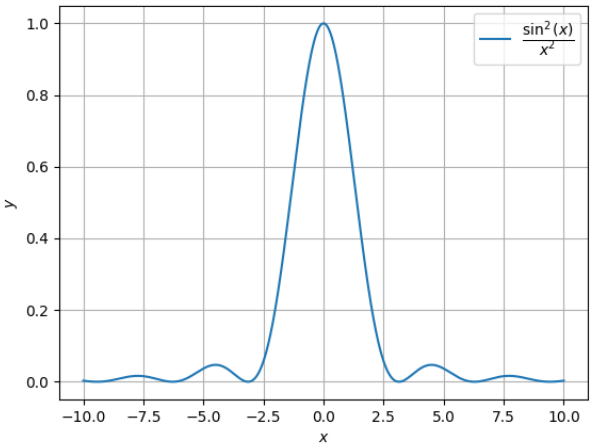

Figure 2: **Plot of** $\frac{\sin^2 x}{x^2}$

Replacing t_0 by t in the expression for I and plugging it into (10), we obtain

$$\sum_f P^{(1)}_{f\leftarrow i}(t_0) = \frac{2\pi}{\hbar}\left|V_{fi}\right|^2\rho(E_f)t \tag{14}$$

This expression tells us that the probability of transitions is linear in time. However, t cannot grow without bound as it is constrained by $P^{(1)}_{f\leftarrow i}(t_0) \ll 1$. In addition to this, we assumed that the the energy range of contributing states (ΔE_f) is quite narrow.

$$-\frac{2\pi}{t_0} < \omega_{fi} < \frac{2\pi}{t_0} \implies E_i - \frac{2\pi\hbar}{t_0} < E_f < E_i + \frac{2\pi\hbar}{t_0} \tag{15}$$

This implies that t_0 also has to be large enough to ensure that the band of contributing energy states is narrow enough to make the calculation of the transition probability valid.

Therefore, t_0 must lie in a sweet spot (not too large, and not too small) in order for this calculation to work.

Since the probability of transition is linearly dependent on time, we can define

$$w = \frac{1}{t}\sum_f P^{(1)}_{f\leftarrow i}(t) \tag{16}$$

This gives us **Fermi's Golden Rule** for a constant perturbation:

$$\boxed{w = \frac{2\pi}{\hbar}\left|V_{fi}\right|^2\rho(E_f)} \tag{17}$$

where $E_f = E_i$.

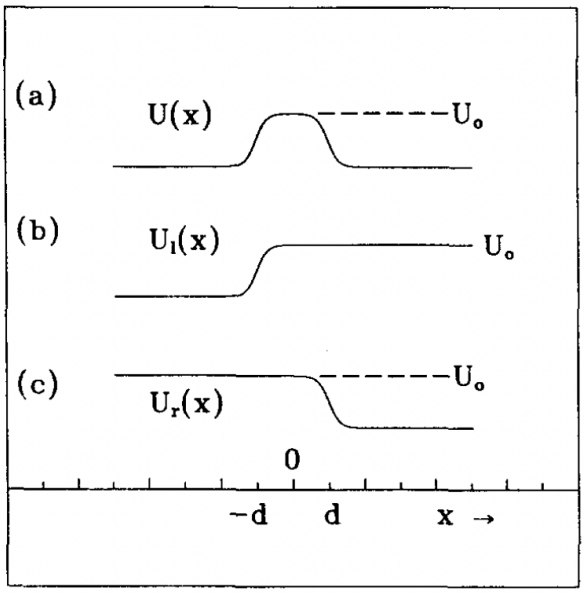

Figure 3: **(a) Plot of the potential barrier between left** $(x < -d)$ **and right** $(x > d)$ **electrodes. Plots (b) and (c) depict the steps of the potentials** $U_l(x)$ **and** $U_r(x)$ **(in our case,** $V_l(x)$ **and** $V_r(x)$ **respectively) at the surfaces of the left (b) and right (c) electrodes, when they are separated by an infinite distance.** U_0 **(**V_0 **in our case) is the constant value assumed by the potential in the vacuum region. Figure from:** [4]

Derivation of Bardeen's Tunnelling Theory

We are considering the tunnelling of electrons with effective mass m from left to right through a potential barrier in 1-dimension. The description of the potential is given in the figure below. We select an arbitrary reference point x_1 lying in the plateau region, assuming that the two potential barriers are separated by a very large distance. The potential corresponding to the left and right electrodes are V_l and V_r respectively. These potentials have steps in the vicinity of $x = -d$ (for V_l) and $x = d$ (for V_r). When the distance $2d$ is very large the left and right regions have stationary wavefunctions ψ_l and ψ_r.

Assumptions:

The assumptions [3] underlying Bardeen's tunnelling theory are:

1. this theory assumes the tunnelling to be weak enough such that the first order approximation is valid (since we calculate the transition probabilities to first order).

2. ψ_l and ψ_r are nearly orthogonal.

3. the interaction between electrons is ignored.

4. occupation probabilities for ψ_l and ψ_r are independent of each other, and do not change, despite the tunnelling.

Defining the Hamiltonians before perturbation

For the left region ($V_l(x) = V_0$, $x \geq x_0$),

$$H_l = \frac{p^2}{2m} + V_l \tag{18}$$

the Eigenvalue equation is,

$$H_l\psi_l = E_l\psi_l \tag{19}$$

For the right region ($V_r(x) = V_0$, $x \leq x_0$),

$$H_r = \frac{p^2}{2m} + V_r \tag{20}$$

The Eigenvalue equation is,

$$H_r\psi_r = E_r\psi_r \tag{21}$$

here, $p = -i\hbar\dfrac{d}{dx}$ and E_l, E_r are the energy of the electrons within the left and right electrodes respectively ($E_l, E_r < V_0$).

Hamiltonian of the system after applying Perturbation

Considering an electron initially in the ψ_l with energy E_l, i.e., $H_l\psi_l = E_l\psi_l$, its wavefunction evolves into $\psi_l(t)$ at time t from ψ_l at $t = 0$. If the evolution were governed solely by the sample Hamiltonian, then $\psi_l(t)$ would equal $e^{-\frac{itE_l}{\hbar}}\psi_l$. But here our Hamiltonian of the system is,

$$H = \frac{p^2}{2m} + V(x) \tag{22}$$

The final potential is given by

$$V(x) = \begin{cases} V_l(x) & \text{if } x < x_1, \\ V_r(x) & \text{if } x \geq x_1. \end{cases} \tag{23}$$

Here, x_1 is the same arbitrary point inside the barrier region.

Now, for $t \geq 0$, using equations (18), (20) and (22),

$$H = H_l - \Theta(x - x_1)[V_0 - V_r(x)] \tag{24}$$

where, $\Theta(x)$ is Heaviside's Step Function. Since at $t < 0$, $H = H_l$, hence we can define

$$H = H_l + \Theta(t)H' \tag{25}$$

$$H' = -\Theta(x - x_1)[V_0 - V_r(x)] \tag{26}$$

At $t = 0$, the states of the left electrode starts to evolve with time as the $\psi_l(x)$ are no longer the eigenstates of the Hamiltonian due to the presence of the Transfer Hamiltonian. Equation (26) is called the Transfer Hamiltonian. If the barrier is thick, the Transfer Hamiltonian H' can be considered as a perturbation causing transitions between the unperturbed states of ψ_l and ψ_r. Because H_r and H' shares same eigenbasis and if the tunnelling is weak, we anticipate that $\psi_l(t)$ will be close to $e^{-\frac{itE}{\hbar}}\psi$ when t is small enough. So we write:, the total wavefunction is,

$$\psi(x,t) = a(t)e^{-i\omega_l t}\psi_l(x) + \sum_r b_r(t)e^{-i\omega_r t}\psi_r(x) \tag{27}$$

Here, $\omega_{l,r} = \frac{E_{l,r}}{\hbar}$, $a(0) = 1$, and $b_r(0) = 0$ as the constant perturbation is turned on after $t \geq 0$.
Our goal is to approximate $b_r(t)$, and luckily, our first assumption about weak tunnelling in constant perturbation enables us to use Fermi's Golden Rule.
The transition probability per unit time from the initial state ψ_l to a continuum of final states ψ_r is given by Fermi's Golden Rule,

$$w_{lr} = \frac{2\pi}{\hbar}|H'_{lr}|^2\rho_r(E) \tag{28}$$

here,

$$H'_{lr} = \int_{-\infty}^{\infty} \psi_r^* H'\psi_l dx \tag{29}$$

where $E_l \approx E_r = E$ and

$$\rho_r(E) = \frac{g_r(E)}{N^2}, \quad N = \int_{-\infty}^{\infty} |\psi_r|^2 dx \tag{30}$$

here g_r is the density of states at the right electrode. We divided g_r by N^2 because at first wave functions are not normalized and instead of just $|H'_{lr}|$, we should use $\frac{|H'_{lr}|^2}{N^2}$. That N^2 is adjusted to density function expression.
Now, to verify the equivalence of Fermi's Golden Rule with Bardeen's Tunnelling Theory, we evaluate the transition matrix element through the corresponding matrix element (J_{lr}) current density operator at the barrier.

Substituting (26) into (29), we get,

$$H'_{lr} = -\int_{x_1}^{\infty} \psi_r^*[V_0 - V_r(x)]\psi_l dx \tag{31}$$

For the region $x \geq x_1$, we have $V_0 = H_l - \frac{p^2}{2m}$ and (31) becomes,

$$H'_{lr} = -\int_{x_1}^{\infty} \psi_r^*[E_l + \frac{\hbar^2}{2m}\frac{d^2}{dx^2} - V_r(x)]\psi_l dx \tag{32}$$

Integrating (32) twice using integration by parts, we obtain

$$H'_{lr} = J_{lr}(x_1) - E_l \int_{x_1}^{\infty} \psi_r^* \psi_l dx + \int_{x_1}^{\infty} \psi_l [V_r(x) - \frac{\hbar^2}{2m}\frac{d^2}{dx^2}]\psi_r^* dx \tag{33}$$

where,

$$J_{lr} = \frac{\hbar^2}{2m}(\psi_r^* \frac{d}{dx}\psi_l - \psi_l \frac{d}{dx}\psi_r^*)\Big|_{x=x_1} \tag{34}$$

Our wavefunctions can be normalized with our assumption criterions, so we can assume ψ_l and ψ_r are vanishing at the infinities. The last integral in (33) becomes the following when we substitute (20) and (21),

$$H'_{lr} = J_{lr}(x_1) - (E_l - E_r) \int_{x_1}^{\infty} \psi_r^* \psi_l dx \tag{35}$$

Since $E_l \approx E_r$, the second term in (35) vanishes and we get

$$H'_{lr} = J_{lr}(x_1) \tag{36}$$

(28), (34) and (36) are the basic formulations of Bardeen's Tunnelling Theory.

Points to be noted

1. This derivation is valid within the limits of Fermi's Golden Rule, as we have taken $E_l \approx E_r$, which indicates that only a narrow range of energies contribute to the transition probability.

2. We have assumed x_1 to be an arbitrary point, but in order for ψ_l and ψ_r to correspond to the wavefunctions of the independent electrodes, x_1 must lie within the plateau region where $V(x)$ has its vacuum value V_0.

Exactly Solvable Tunneling Problem

Rectangular Barrier

Time-Independent Way: Consider a rectangular potential barrier of height V_0 and width $2d$. To find the tunnelling probability, we need to find solutions of the Schrodinger equation separably in three regions and then find the coefficients by applying boundary conditions to them. For $V_0 > E$,

$$\begin{aligned} \Psi_{\mathrm{I}}(x) &= Ae^{ik_2 x} + Be^{-ik_2 x} && \text{for } x < -d \\ \Psi_{\mathrm{II}}(x) &= Ce^{ik_1 x} + De^{-ik_1 x} && \text{for } -d \le x \le d \\ \Psi_{\mathrm{III}}(x) &= Fe^{ik_2 x} && \text{for } x > d \end{aligned} \tag{37}$$

Here:

$$k_1 = \sqrt{\frac{2m(V_0 - E)}{\hbar^2}} \quad \text{(imaginary wave number within the barrier)}$$

$$k_2 = \sqrt{\frac{2mE}{\hbar^2}} \quad \text{(wave number in free space)}$$

We have assumed no wave is coming to the potential barrier from right to left. Now, the coefficients A, B, C, D, F, and G are determined by applying boundary conditions and continuity of the wave function and its derivative at $x = -d$ and $x = d$. Solving by Mathematica Software,

$$\begin{aligned} T = \frac{|F|^2}{|A|^2} =& \frac{1}{\cosh^2(2k_1 d) + \dfrac{(-k_1^2 + k_2^2)}{4k_1^2 k_2^2}\sinh^2(2k_1 d)} \\ \therefore T =& \frac{1}{1 + \dfrac{(k_1^2 + k_2^2)}{4k_1^2 k_2^2}\sinh^2(2k_1 d)} \end{aligned} \tag{38}$$

There is transmission, even though the energy is less than the barrier height – this tunnelling phenomenon is exhibited by quantum mechanical particles. To locate the particle inside the barrier, we must localize the particle well within the barrier,

$$\Delta x \ll 2d \Rightarrow \Delta p \gg \frac{\hbar}{2d}$$

This corresponds to the transfer of energy,

$$\therefore \Delta E \gg \frac{\hbar^2}{8md^2} \tag{39}$$

But the transfer of energy should not be large enough to change the sign of $V_0 - E$ and therefore,

$$\Delta E \ll V_0 - E = \frac{\hbar^2 k_1^2}{2m} \tag{40}$$

Putting (39) and (40) together,

$$\begin{aligned} \frac{\hbar^2 k_1^2}{2m} \gg \Delta E \gg \frac{\hbar^2}{8md^2} \\ \Rightarrow 2k_1 d \gg 1 \end{aligned} \tag{41}$$

This implies that,

$$\sinh(2k_1 d) \to \frac{e^{2k_1 d}}{2} \gg 1$$

From (38) we find tunneling probability in case of $V_0 > E$,

$$T = \frac{16k_1^2 k_2^2}{(k_1^2 + k_2^2)^2} e^{-4k_1 d} \tag{42}$$

Bardeen's Tunneling Theory: We are interested in applying Bardeen's Tunnelling Theory in this problem where we assume V_l potential due to left electrode is present at $t < 0$ and from $t \geq 0$, there is an appearance of second potential V_r by right electrode such that a perturbation Hamiltonian $H' = -\Theta(x - x_1)[V_0 - V_r(x)]$ is applied (x_1, an arbitrary point in the barrier). The potentials for the left and right electrodes are given by

$$V_l(x) = V_0\Theta(x + d) \quad \text{and} \quad V_r(x) = V_0\Theta(d - x) \tag{43}$$

In the region $x > -d$, the Schrodinger's equation for ψ_l can be written as

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}+V_0\right)\psi_l = E_l\psi_l \tag{44}$$

For $x \geq -d$, the solution is of the form

$$\psi_l(x) = Ae^{-k_1(x+d)} \tag{45}$$

where $k_1 = \sqrt{\frac{2m(V_0-E_l)}{\hbar^2}}$ and $V_0 - E_l > 0$.
Since $V_l = 0$ for $x < -d$, we have free-electron planar waves inside the left electrode of the form,

$$\psi_l(x) = e^{ik_2(x+d)} + re^{-ik_2(x+d)} \tag{46}$$

where $k_2 = \sqrt{\frac{2mE_l}{\hbar^2}}$

After this, we match $\psi_l(x)$ at the boundary ($x = -d$) in order to obtain A:

$$\psi_l(x=-d^-) = \psi_l(x=-d^+) \quad \text{and} \quad \left.\frac{d\psi_l}{dx}\right|_{x=-d^-} = \left.\frac{d\psi l}{dx}\right|_{x=-d^+} \tag{47}$$

Therefore, we have

$$\psi_l(x) = \frac{2k_2}{k_2+ik_1}e^{-k_1(x+d)} \quad \text{for } x \geq -d \tag{48}$$

In a similar fashion, for $x \leq d$, we can write the Schrodinger's equation

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}+V_0\right)\psi_r = E_l\psi_r \tag{49}$$

By Fermi's golden rule, we know, at $t \to \infty$, transition mainly happens between the initial state(ψ_l and final continuum states (ψ_r) where only $E_l \approx E_r$. So, we can conclude $k_{l(1)} \approx k_{r(1)} = k_1$ and $k_{l(2)} \approx k_{r(2)} = k_2$. We follow the same approach as the one we followed to determine $\psi_l(x)$ and obtain

$$\psi_r(x) = \frac{2k_1}{k_2+ik_1}e^{k_1(x-d)} \quad \text{for } x \leq d \tag{50}$$

Plugging in (48) and (50) into (34), we get

$$J_{lr}(x_1) = \frac{4\hbar^2k_1k_2^2}{m(k_1^2+k_2^2)}e^{-2k_1d} \tag{51}$$

where $-d < x_1 < d$ is an arbitrary point within the barrier.Now, assuming box normalization here for the plane waves where the box is extended from $-L$ to L, we can write the normalization coefficient for ψ_r,

$$N = \int_{-\infty}^{\infty}|\psi_r|^2dx \approx 2L \text{ length of the box} \tag{52}$$

Considering the plane waves $\psi_r = \frac{1}{2L}e^{-ik_2x}$ with energy $E = \frac{\hbar^2k_2^2}{2m}$, and hence to meet the boundary conditions, the 1D wave vector must have the form $k_2 = \frac{2\pi n}{2L}$. We can arrive at the density of states.

$$g_r(E) = \frac{dn_2}{dE} = \frac{2L}{2\pi}\frac{dk_2}{dE} = \frac{L}{\pi}\frac{m}{\hbar^2 k_2} \tag{53}$$

Using Fermi's golden rule and $H'_{lr} = J_{lr}(x_1)$, for this problem, we get transition rate,

$$\begin{aligned} w_{lr} &= \frac{2\pi}{\hbar}|J_{lr}(x_1)|^2\frac{g_r(E)}{N^2} \\ &= \frac{\hbar k_2}{2Lm}\frac{16k_1^2k_2^2}{(k_1^2+k_2^2)^2}e^{-4k_1d} = \frac{j}{N}T \end{aligned} \tag{54}$$

where $j = \frac{\hbar k_2}{m}$ = incident plane wave current density and T is the tunnelling probability (We are dividing by N because our incoming wavefunction is normalized, so $\psi \to \frac{1}{\sqrt{N}}\psi$ and $j \to \frac{1}{N}j$).

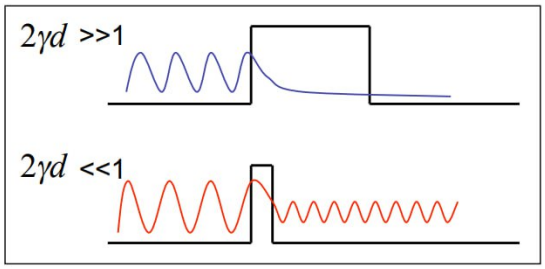

Figure 4: **Approximations used in tunneling approach. Figure from: [1]. Here, Bardeen's tunneling theory is valid for $2\gamma d \ll 1$ ($2k_1 d \ll 1$ in our case) (top figure).**

The expression for $T = \frac{16k_1^2k_2^2}{(k_1^2+k_2^2)^2}e^{-4k_1d}$ received from Bardeen's tunnelling theory is the same if the problem is solved in a time-independent way. **Therefore, we note that for a very low rate of tunneling across the barrier, Bardeen's theory and the time-independent method yield the same results.**

In α decay, $V_0 = 16$ MeV and energy of emitted α is $E = 6$ MeV and dimension of nucleus is $2d \sim 10^{-14}$ m. This suggests that with a value of $2k_1d \approx 10^{28}$, it follows that the probability of tunneling T is roughly around 10^{-24}, meaning that out of every 10^{24} collisions with the barrier, only one particle will tunnel through!

Conclusion

In this article, we have successfully derived Fermi's Golden Rule for a constant (time-dependent) perturbation, and have used it along with Bardeen's tunneling theory approach to obtain the tunneling probability of electrons across a potential barrier. We have solved the problem for a rectangular barrier using the time-dependent approach (Bardeen's tunneling theory), and using a time-independent approach as well. Both results agree in the case of weak tunneling rate of electrons.

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