Unit 2E: Matrix Operations

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Summary

1. Matrix operations
2. Two operations similar to vector operations
3. Two more operations on matrices
4. Transposition

Matrix operations

Definition

An $m \times n$ matrix is a rectangular array of entries, m high and n wide, i.e., with m horizontal rows and n vertical columns.

Of greatest interest is when the elements of a matrix are real numbers, but they could be other things, e.g., *Boolean values*, *integers*, *complex numbers*, *polynomials*, *other matrices*, etc.

Note that, from this perspective, vectors and numbers are simple matrices.

- A vector $x \in \mathbb{R}^m$, viewed as a column vector, is an $m \times 1$ matrix.
- Alternatively, a vector $x \in \mathbb{R}^n$, viewed as a row vector, is a $1 \times n$ matrix.
- A number $x \in \mathbb{R}$ is a 1×1 matrix.

Two operations similar to vector operations

Addition of two matrices: Let A and B be two matrices of the same size, say, two $m \times n$ matrices. The sum matrix $C = A + B$ is a matrix of the same size that has entries

$$C_{ij} = A_{ij} + B_{ij}$$

for $i \in \{1, 2, ..., m\}$ and $j \in \{1, 2, ..., n\}$.

Multiplication of a matrix by a scalar: Let A be an $m \times n$ matrix and $\alpha \in \mathbb{R}$. The product matrix $B = \alpha A$ is a matrix of the same size that has entries

$$B_{ij} = \alpha A_{ij}$$

for $i \in \{1, 2, ..., m\}$ and $j \in \{1, 2, ..., n\}$.

With these two operations on matrices, the set of all $m \times n$ matrices,

$$V = \{A : \; A \text{ is an } m \times n \text{ matrix}\},$$

is a **vector space**.

Two more operations on matrices

Multiplication Here we present two ways one can define the product of two matrices.

i. Hadamard product of matrices. This is a matrix product that is defined for any two matrices of the same size. Let A and B be two $m \times n$ matrices. In this case, the Hadamard matrix product $C = A \circ B$ is a matrix of the same size as matrices A and B that has entries

$$C_{ij} = B_{ij} A_{ij}$$

for $i \in \{1, 2, ..., m\}$ and $j \in \{1, 2, ..., n\}$.

Example.

If $A = \begin{pmatrix} 1 & 2 & 3 \\ -2 & 4 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 2 & 1 \\ 4 & 1 & 7 \end{pmatrix}$, then

$$A \circ B = \begin{pmatrix} 1 & 2 & 3 \\ -2 & 4 & 0 \end{pmatrix} \circ \begin{pmatrix} 3 & 2 & 1 \\ 4 & 1 & 7 \end{pmatrix}$$

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$$\begin{aligned} A \circ B &= \begin{pmatrix} 1 & 2 & 3 \\ -2 & 4 & 0 \end{pmatrix} \circ \begin{pmatrix} 3 & 2 & 1 \\ 4 & 1 & 7 \end{pmatrix} \\ &= \begin{pmatrix} 1 \times 3 & 2 \times 2 & 3 \times 1 \\ -2 \times 4 & 4 \times 1 & 0 \times 7 \end{pmatrix} \end{aligned}$$

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$$\begin{aligned} A \circ B &= \begin{pmatrix} 1 & 2 & 3 \\ -2 & 4 & 0 \end{pmatrix} \circ \begin{pmatrix} 3 & 2 & 1 \\ 4 & 1 & 7 \end{pmatrix} \\ &= \begin{pmatrix} 1 \times 3 & 2 \times 2 & 3 \times 1 \\ -2 \times 4 & 4 \times 1 & 0 \times 7 \end{pmatrix} \\ &= \begin{pmatrix} 3 & 4 & 3 \\ -8 & 4 & 0 \end{pmatrix}. \end{aligned}$$

ii. Matrix multiplication. This is a much more useful notion of the product of two matrices. Let A be an $m \times n$ matrix and B an $n \times p$ matrix. Then the product $C = AB$ is an $m \times p$ matrix with elements

$$C_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj}$$

for $i \in \{1, 2, ..., m\}$ and $j \in \{1, 2, ..., n\}$.

Row and column names. Let us adopt the following notations for row and column names.

$$\begin{array}{cc} & \begin{array}{cccc} A_{:1} & A_{:2} & A_{:3} & A_{:4} \end{array} \\ \begin{array}{c} A_{1:} \\ A_{2:} \\ A_{3:} \end{array} & \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix} \end{array}$$

Remark:

Requirement

To exists the product matrix AB, the number of columns in the first matrix A must be the same as the number of rows in the second matrix B.

Each element C_{ij} of the product matrix AB is determined as the dot product of $A_{i:}$ and $B_{:j}$:

$$C_{ij} = A_{i:} \cdot B_{:j} = \sum_{k=1}^{n} A_{ik} B_{kj}$$

for $i \in \{1, 2, ..., m\}$ and $j \in \{1, 2, ..., n\}$.

Transposition

Transposition: Given an $m \times n$ matrix A, the transpose A^T of A is the matrix of the size $n \times m$ obtained by interchanging the rows and columns, i.e.,

$$A = (A_{ij}) \Rightarrow A^T = (A_{ji}).$$

Here are two things that are good to know about transposes.

1. $(A^T)^T = A$.
2. $(AB)^T = B^T A^T$.

Examples of matrix multiplication

- If $A = \begin{pmatrix} 2 & -1 \\ 3 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 4 & -2 \\ 3 & 0 & 2 \end{pmatrix}$, then find AB.
  In this case, BA is not defined. In fact, the number of columns in B is 3. So, it is not equal to the number of rows in the second matrix A which is equals to 2.

- If $A = \begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix}, B = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, C = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix}$ and
  $D = \begin{pmatrix} 1 & 0 \\ 2 & 2 \\ 1 & 1 \end{pmatrix}$, then find

  $$AB,\ BA,\ AC,\ CD,\ DC.$$

  CA is not defined. Why ?

It is worthwhile that this example illustrates several things about the product of two matrices:

1. $AB \neq BA$. Hence matrix multiplication is not **commutative** in general.

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It is worthwhile that this example illustrates several things about the product of two matrices:

1. $AB \neq BA$. Hence matrix multiplication is not **commutative** in general.
2. AC is defined, but CA is not defined, so both need not be defined, and in particular both are not defined unless $m = p$ in the definition of matrix multiplication.
3. if both are defined, then their dimensions need not be the same and are not unless $m = n = p$.

Theorem.

Let A be an $m \times n$ matrix, B an $n \times p$ matrix, and C a $p \times q$ matrix, so that $(AB)C$ and $A(BC)$ are defined. Then $(AB)C = A(BC)$.

Proof.

Recall that $(AB)C$ is a matrix with $m \times q$ elements, indexed as $\alpha \in \{1, 2, ..., m\}$ and $\beta \in \{1, 2, ..., q\}$.

Proof ...

Let's consider the $(\alpha\beta)$ element:

$$((AB)C)_{\alpha\beta} = \sum_{l=1}^{p} (AB)_{\alpha l} C_{l\beta}$$

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$$\begin{aligned} ((AB)C)_{\alpha\beta} &= \sum_{l=1}^{p} (AB)_{\alpha l} C_{l\beta} \\ &= \sum_{l=1}^{p} \left(\sum_{k=1}^{n} A_{\alpha k} B_{kl} \right) C_{l\beta} \end{aligned}$$

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$$
\begin{aligned}
((AB)C)_{\alpha\beta} &= \sum_{l=1}^{p} (AB)_{\alpha l} C_{l\beta} \\
&= \sum_{l=1}^{p} \left(\sum_{k=1}^{n} A_{\alpha k} B_{kl} \right) C_{l\beta} \\
&= \sum_{l=1}^{p} \sum_{k=1}^{n} A_{\alpha k} B_{kl} C_{l\beta}.
\end{aligned}
$$

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Thus,

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\end{aligned}
$$

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Thus,

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\begin{aligned}
((AB)C)_{\alpha\beta} &= \sum_{k=1}^{n}\sum_{l=1}^{p} A_{\alpha k}B_{kl}C_{l\beta} \\
&= \sum_{k=1}^{n} A_{\alpha k}\left(\sum_{l=1}^{p} B_{kl}C_{l\beta}\right) \\
&= \sum_{l=1}^{p} A_{\alpha k}(BC)_{k\beta}
\end{aligned}
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Thus,

$$\begin{aligned}((AB)C)_{\alpha\beta} &= \sum_{k=1}^{n}\sum_{l=1}^{p} A_{\alpha k}B_{kl}C_{l\beta} \\ &= \sum_{k=1}^{n} A_{\alpha k}\left(\sum_{l=1}^{p} B_{kl}C_{l\beta}\right) \\ &= \sum_{l=1}^{p} A_{\alpha k}(BC)_{k\beta} \\ &= A(BC).\end{aligned}$$

$\square$