Unit 2I: Bases

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Summary

1. Spans
2. Linear independence
    - Testing for linear dependence and independence
3. Bases

Spans

We can't visualize the data in $\mathbb{R}^n$.

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However, we would like to be able to say that data points are roughly the same from the perspective of linear algebra in the sense that one could generate each data point from the others by performing vector addition as well as scalar multiplication.

A linear algebraic idea that permit one to do that is the idea of *linear dependence/independence*.

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To develop this, let's start with the following notion of linear combination.

Linear combination

If $a_1, \ldots, a_k \in \mathbb{R}$, then a linear combination of the vectors $v_1, \ldots, v_k \in \mathbb{R}^n$ is a vector $w \in \mathbb{R}^n$ such that

$$w = \sum_{i=1}^{k} a_i v_i.$$

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$$w = \sum_{i=1}^{k} a_i v_i.$$

This definition expresses the idea that one vector can be expressed in terms of others.

Example.

Let $\begin{pmatrix} 0 \\ 3 \\ 4 \end{pmatrix} \in \mathbb{R}^3$.

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$$v = 3 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + 4 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 3e_2 + 4e_3.$$

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Let $\begin{pmatrix} 0 \\ 3 \\ 4 \end{pmatrix} \in \mathbb{R}^3$. Then

$$v = 3\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + 4\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 3e_2 + 4e_3.$$

Thus v is a linear combination of e_2 and e_3.

Example

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 1 \cdot 5 + 2 \cdot 6 \\ 3 \cdot 5 + 4 \cdot 6 \end{pmatrix}$$

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In general,

Matrix-Vector product as a linear combination

Let A be an $m \times n$ matrix, and let x be an n-dimensional column vector. Then Ax is a linear combination of the columns of A.

Example

$$\begin{pmatrix} 7 & 8 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

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&= 7 \begin{pmatrix} 1 & 2 \end{pmatrix} + 8 \begin{pmatrix} 3 & 4 \end{pmatrix}.
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In general,

Vector-Matrix product as a linear combination

Let A be an $m \times n$ matrix, and let x be an m-dimensional column vector. Then $x^T A$ is a linear combination of the rows of A.

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Question

If we have a set of vectors, then which vectors can be computed from them with the operations of scalar multiplications and vector additions?

This gets us to the notion of *span*.

Span of vectors

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$$\text{span}\{v_1, v_2, ..., v_k\} = \left\{ \sum_1^k a_i v_i : \text{ each } a_i \in \mathbb{R} \right\}.$$

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If V is a vector space and $\text{span}\{v_1, v_2, ..., v_k\} = V$, we say that the vectors $v_1, v_2, ..., v_k$ **span** the vector space V.

Geometric Description of span$\{v\}$

Let v be a nonzero vector in $\mathbb{R}^3$. Then

$$\text{span}\{v\} = \{w : \; w = av, a \in \mathbb{R}\},$$

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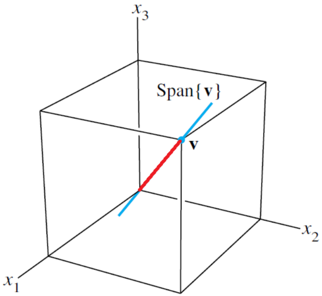
a line through the origin parallel to v.



Geometric Description of span$\{u, v\}$

If u and v are nonzero vectors in $\mathbb{R}^3$, with v not a multiple of u, then

$$\text{span}\{v\} = \{w : \ w = au + bv, \ a, b \in \mathbb{R}\},$$

a plane in $\mathbb{R}^3$ that contains u, v, and 0.

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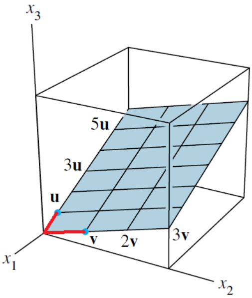

Example. Describe

1. The span of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ in $\mathbb{R}^2$,
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These sets are all lines through the origin on $\mathbb{R}^2$, and thus subspaces of $\mathbb{R}^2$.

Problem

Let e_1 and e_2 be the coordinate vectors for $\mathbb{R}^2$, i.e.,$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Prove that

$$\text{span}\{e_1, e_2\} = \mathbb{R}^2.$$

Similarly, prove that $\text{span}\{e_1, e_2, e_3\} = \mathbb{R}^3$..

Problem

Let
$$v_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}, \quad v_2 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}.$$
Prove that $\operatorname{span}\{v_1, v_2\} = \mathbb{R}^2$.

Proof.

We have

$$v_1 + v_2 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix} = \sqrt{2}\begin{pmatrix}1\\0\end{pmatrix}$$

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and

$$v_1 - v_2 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} - \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix} = \sqrt{2}\begin{pmatrix}0\\1\end{pmatrix}.$$

Proof...

If $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$, then

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

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If $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$, then

$$\begin{aligned} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= x_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= \frac{x_1}{\sqrt{2}}(v_1 + v_2) + \frac{x_2}{\sqrt{2}}(v_1 - v_2) \end{aligned}$$

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Therefore, $\text{span}\{v_1, v_2\} = \mathbb{R}^2$. $\square$

Example:

If $e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, then

$$\mathrm{span}\{e_1, e_2\} \neq \mathbb{R}^3.$$

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$$\operatorname{span}\{e_1, e_2\} \neq \mathbb{R}^3.$$

The reason is that the vector $e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ cannot be expressed as a linear combination of e_1 and e_2.

Example:

If $x_1 = \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix}$ and $x_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$, then

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The reason is that the vector $x = \begin{pmatrix} 0 \\ a \\ 0 \end{pmatrix}$ with $a \neq 0$ cannot be expressed as a linear combination of x_1 and x_2.

Example:

Let A be an $m \times n$ matrix, and let x vary over all possible n-dimensional column vectors. Then, the span of the columns of A is given by

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In particular, if $A_{:j}$ denotes the jth column of A, then this set is all vectors of the form

$$\sum_{i=1}^{n} x_j A_{:j},$$

as x is varied over all of $\mathbb{R}^n$.

Example:

Let A be an $m \times n$ matrix, and let y vary over all possible m-dimensional column vectors. Then, the span of the rows of A is given by

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It is clear that $\text{span}\{v_1, v_2, ..., v_k\} \subseteq V$.

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Proof.

It is clear that $\text{span}\{v_1, v_2, ..., v_k\} \subseteq V$. Let us demonstrate that it is closed under addition and scalar multiplication.

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$$x = \sum_1^k a_i v_i, \quad y = \sum_1^k b_i v_i,$$

where $a_i, b_i \in \mathbb{R}$. Thus,

$$x + y = \sum_1^k (a_i + b_i) v_i \in \text{span}\{v_1, v_2, ..., v_k\}.$$

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Scalar multiplication: Let $a \in \mathbb{R}$. Then

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$$\begin{aligned} ax &= a\sum_1^k a_i v_i \\ &= \sum_1^k (aa_i)v_i \in \operatorname{span}\{v_1, v_2, ..., v_k\}. \end{aligned}$$

We observe that $aa_i \in \mathbb{R}$ for all i.

Therefore, $\operatorname{span}\{v_1, v_2, ..., v_k\}$ is a subspace of V. $\square$

Problem. Consider the vectors $u = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $v = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

(a) Write the vector $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ in terms of the vectors u and v.

(b) Show that the vectors u and v span $\mathbb{R}^2$.

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Remark. It is a fact that $V = \text{span}\{v_1, ..., v_k\}$ is the smallest subspace of $\mathbb{R}^n$ that contains $v_1, ..., v_k$. We will not prove it.

Problem

Determine whether the vector $(3, -1, 11)$ lies in the subspace $\text{span}\{(-1, 5, 3), (2, -3, 4)\}$ of $\mathbb{R}^3$.

Problem

Let $v_1 = (2, -1, 0), v_2 = (1, 3, -2), v_3 = (1, 1, 4)$. Show that $v = (-4, 4, -6)$ lies in the $\text{span}\{v_1, v_2, v_3\}$.

Problems:

1. Determine whether the first vector is a linear combination of the other vectors.
   (a) (-1, 7); (1, -1), (2, 4)
   (b) (-3, 3, 7); (1, -1, 2), (2, 1, 0), (-1, 2, 1)
2. Determine whether the first vector is in the subspace of $\mathbb{R}^3$ generated by the other vectors.
   (a) (0, 10, 8); (-1, 2, 3), (1, 3, 1), (1, 8, 5)
   (b) (1, 4, -3); (1, 0, 1), (1, 1, 0), (3, 1, 2)

Orthogonal complement of a set

Let $S \subseteq \mathbb{R}^n$. Put
$$S^{\perp} = \{u \in \mathbb{R}^n : \ \forall\, v \in S\ u \cdot v = 0\}.$$

This set is called the **orthogonal complement** of S.

Problem. Prove that if $v_1, ..., v_k \in \mathbb{R}^n$ and $V = \text{span}\{v_1, ..., v_k\}$, then $V^{\perp}$ is a subspace of $\mathbb{R}^n$.

Linear independence

We observe that

$$\operatorname{span}\{e_1, e_2, e_3\} = \mathbb{R}^3$$

$$\operatorname{span}\{e_1, e_2, e_3, \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix}\} = \mathbb{R}^3$$

$$\operatorname{span}\{e_1, e_2, \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix}\} \neq \mathbb{R}^3.$$

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So, we face with the following question:
Which set of vectors span a vector space ?
To answer it, we need the following notion.

linearly independent

The vectors $v_1, ..., v_k$ are **linearly independent** if

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If a set of vectors is not linearly independent, then they are **linearly dependent**.

We know that

Standard basis vectors of $\mathbb{R}^2$

If e_1 and e_2 are standard basis vectors of $\mathbb{R}^2$, then

$$x_1e_1 + x_2e_2 = 0 \Leftrightarrow x_1 = x_2 = 0.$$

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In general,

Standard basis vectors of $\mathbb{R}^n$

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Problems:

1. Determine whether the set (1, 2, 0), (0, 1, -1), (1, 1, 2) is linearly independent in $\mathbb{R}^3$.
2. Let the set $\{v_1, v_2\}$ be linearly independent.Prove that $\{v_1 + v_2, v_1 - v_2\}$ is also linearly independent.
3. Determine whether the following sets of vectors are linearly dependent or independent.
   (a) {(-1,2),(2,-4)}
   (b) {(-1, 3), (2, 5)}
   (c) {(1, -2, 3), (-2, 4, 1), (-4, 8, 9)}

Nature of linearly dependent/independent vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$

Let u, v be two nonzero vectors in $\mathbb{R}^n$. Then they are linearly dependent if one is the scalar multiple of the other.

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Let u, v be two nonzero vectors in $\mathbb{R}^n$. Then they are linearly dependent if one is the scalar multiple of the other. For example, the vectors (3,1) and (6, 2) are linearly dependent, $(6, 2) = 2(3, 1)$. See the figure given below.

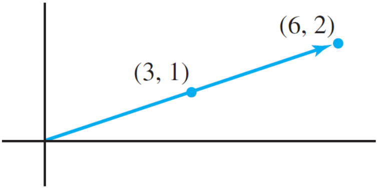

Figure: Linearly dependent

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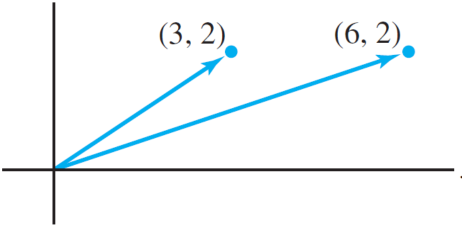

Figure: Linearly independent

Let u, v, w be three nonzero vectors in $\mathbb{R}^3$. Then they are linearly dependent if all three vectors lie on the same plane. See the figure given below.

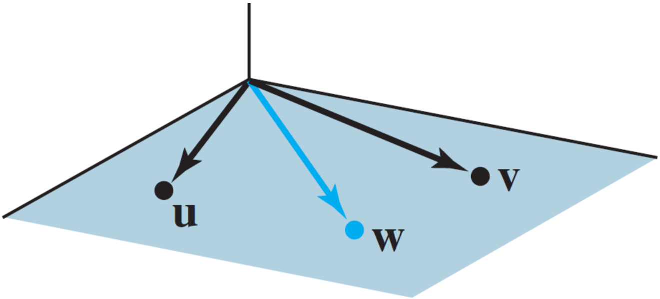

Figure: Linearly dependent $w \in \text{span}\{u, v\}$

Let u, v, w be three nonzero vectors in $\mathbb{R}^3$. Then they are linearly independent if one of the vectors does not lie on the plane containing the other two. See the figure given below.

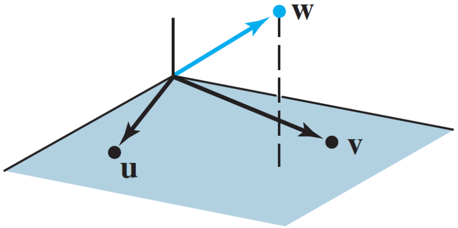

Figure: Linearly independent $w \notin \text{span}\{u, v\}$

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Let's see an important question:

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To get an idea,recall that

$$\begin{aligned} \operatorname{span}\{e_1\} &= \mathbb{R}, \\ \operatorname{span}\{e_1, e_2\} &= \mathbb{R}^2, \\ \operatorname{span}\{e_1, e_2, e_3\} &= \mathbb{R}^3. \end{aligned}$$

Let's see an important question:

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To get an idea,recall that

$$\operatorname{span}\{e_1\} = \mathbb{R},$$
$$\operatorname{span}\{e_1, e_2\} = \mathbb{R}^2,$$
$$\operatorname{span}\{e_1, e_2, e_3\} = \mathbb{R}^3.$$

Here is the theorem that generalizes that.

Theorem.

In $\mathbb{R}^n$,

(a) Any set of $n + 1$ vectors are never linearly independent.

(b) Any set of $n - 1$ vectors never span all of $\mathbb{R}^n$.

Example. Vectors $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix}$ don't span

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Example. Vectors $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix}$ don't span $\mathbb{R}^3$. Similarly, Vectors $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ don't span $\mathbb{R}^3$.

Example. Vectors $\begin{pmatrix}1\\0\\0\end{pmatrix}$ and $\begin{pmatrix}3\\4\\0\end{pmatrix}$ don't span $\mathbb{R}^3$. Similarly, Vectors $\begin{pmatrix}1\\1\\0\end{pmatrix}$ and $\begin{pmatrix}1\\-1\\0\end{pmatrix}$ don't span $\mathbb{R}^3$. In both cases, the span of these two linearly independent vectors is a two-dimensional plane corresponding to $x_3 = 0$, and so the span of these two vectors is a two-dimensional subspace of $\mathbb{R}^3$.

Vectors $\begin{pmatrix}1\\1\\0\end{pmatrix}$, $\begin{pmatrix}1\\-1\\0\end{pmatrix}$ $\begin{pmatrix}0\\0\\1\end{pmatrix}$ and $\begin{pmatrix}17\\12\\-2\end{pmatrix}$ are not linearly independent, but their span is all of $\mathbb{R}^3$.

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Testing for linear dependence and independence

1. A set containing only one vector, say, v, is linearly independent if and only if v is not the zero vector.

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   The zero vector is linearly dependent because any nonzero value of x_1 satisfies the equation $x_1 0 = 0$.

2. Two vectors v_1, v_2 are linearly dependent if one of the vectors is a multiple of the other. The set is linearly independent if and only if neither of the vectors is a multiple of the other.

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3. The vectors $v_1, ..., v_k \in \mathbb{R}^n$ are linearly dependent if one of the vectors is zero.

4. The vectors $v_1, ..., v_k \in \mathbb{R}^n$ are linearly dependent if $k \geq n$. For example, if $p > n$, the columns a $n \times p$ matrix are linearly dependent.

$$\begin{array}{cc} & p \\ n & \begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{bmatrix} \end{array}$$

Figure: The columns are Linearly independent.

Recall that the span of a set of elements in a vector space is the set of all their linear combinations.

Recall that the span of a set of elements in a vector space is the set of all their linear combinations.
The following proposition says, roughly, that linearly dependent vectors do not contribute to the span.

Theorem

Let $v_1, \ldots, v_k$ be vectors in a vector space V and v_{k+1} is a linear combination of the vectors $v_1, \ldots, v_k$. Then

$$\mathrm{span}(v_1, \ldots, v_k) = \mathrm{span}(v_1, \ldots, v_k, v_{k+1}).$$

Proof.

Let

$$A = \operatorname{span}(v_1, ..., v_k)$$
$$B = \operatorname{span}(v_1, ..., v_k, v_{k+1}).$$

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If $x \in A$, then there are scalars $c_1, c_2, ..., c_k$ such that

$$x = c_1 v_1 + ... + c_k v_k$$

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$$\begin{aligned} A &= \operatorname{span}(v_1, ..., v_k) \\ B &= \operatorname{span}(v_1, ..., v_k, v_{k+1}). \end{aligned}$$

If $x \in A$, then there are scalars $c_1, c_2, ..., c_k$ such that

$$\begin{aligned} x &= c_1 v_1 + ... + c_k v_k \\ &= c_1 v_1 + ... + c_k v_k + 0 v_{k+1} \in B. \end{aligned}$$

Proof.

Let

$$A = \text{span}(v_1, ..., v_k)$$
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If $x \in A$, then there are scalars $c_1, c_2, ..., c_k$ such that

$$x = c_1v_1 + ... + c_kv_k$$
$$= c_1v_1 + ... + c_kv_k + 0v_{k+1} \in B.$$

This shows that $A \subseteq B$.

Proof ...

On the other hand, by assumption, $v_{k+1} \in A$. So, then there are scalars $c_1, c_2, ..., c_k$ such that

$$v_{k+1} = c_1 v_1 + ... + c_k v_k.$$

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Now, if $y \in B$, then there are scalars $b_1, b_2, ..., b_{k+1}$ such that

$$y = b_1 v_1 + ... + b_k v_k + b_{k+1} v_{k+1}$$

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$$\begin{aligned} y &= b_1 v_1 + ... + b_k v_k + b_{k+1} v_{k+1} \\ &= b_1 v_1 + ... + b_k v_k + b_{k+1}(c_1 v_1 + ... + c_k v_k) \end{aligned}$$

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This shows that $B \subseteq A$. Therefore, $A = B$. $\square$

Bases

Basis

Let V be a subspace of $\mathbb{R}^n$. A set of vectors $v_1, ..., v_k \in V$ is called a **basis** for V if the following conditions are satisfied

(a) Vectors $v_1, ..., v_k$ in V are linearly independent.

(b) $\text{span}\{v_1, ..., v_k\} = V$.

The number of vectors in any basis for V is called the **dimension** of V, and is written as $\dim V$.

Examples.

- The set $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ is a basis for $\mathbb{R}^2$.
  Hence the plane $\mathbb{R}^2$ has dimension 2.

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- The set $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ is a basis for $\mathbb{R}^2$. Hence the plane $\mathbb{R}^2$ has dimension 2.
- Vectors $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ span $\mathbb{R}^2$ and provide a basis for $\mathbb{R}^2$.

Examples.

- The set $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ is a basis for $\mathbb{R}^2$. Hence the plane $\mathbb{R}^2$ has dimension 2.
- Vectors $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ span $\mathbb{R}^2$ and provide a basis for $\mathbb{R}^2$. Moreover, each pair of vectors that are not scalar multiples of each other provides a basis for $\mathbb{R}^2$.

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$$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \quad \ldots \quad, \quad e_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}.$$

- The columns of an invertible $n \times n$ matrix form a basis for all of $\mathbb{R}^n$ because they are linearly independent and span $\mathbb{R}^n$. One such matrix is the $n \times n$ identity matrix. Its columns are denoted by $e_1, \ldots, e_n$:

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The set $\{e_1, \ldots, e_n\}$ is the standard basis for $\mathbb{R}^n$.

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$$x_1 + x_2 + x_3 = 0, \text{ i.e. } x_3 = -x_1 - x_2.$$

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Put

$$V = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} : \; x_1 + x_2 + x_3 = 0 \right\}.$$

We claim that

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2. Each of the following sets:

$$\left\{\begin{pmatrix}1\\-1\\0\end{pmatrix},\begin{pmatrix}1\\0\\-1\end{pmatrix}\right\},\left\{\begin{pmatrix}-1\\0\\1\end{pmatrix},\begin{pmatrix}0\\1\\-1\end{pmatrix}\right\},$$

$$\left\{\begin{pmatrix}-1/2\\-1/2\\1\end{pmatrix},\begin{pmatrix}1\\-1\\0\end{pmatrix}\right\}$$

is a basis for V.

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   to a basis for V, then we get a basis for $\mathbb{R}^3$.
3. A basis for V is also a basis for the subspace orthogonal to any of the vector in (2).

Problem. Let

$$V = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 : \ x_1 + x_2 + x_3 = 0 \right\},$$

$$B = \left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}$$

Show that V is a subspace of $\mathbb{R}^3$, and B is a basis for V.

Solution.

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$$\begin{aligned} 1 - 1 + 0 &= 0 \\ 1 + 0 - 1 &= 0. \end{aligned}$$

   (b) $\operatorname{span}(B) = V$. In fact, let $\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \in V$. Then

$$\begin{aligned} & v_1 + v_2 + v_3 = 0 \\ \Rightarrow \quad & v_1 = -v_2 - v_3, \end{aligned}$$

and so

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} -v_2 - v_3 \\ v_2 \\ v_3 \end{pmatrix}$$

and so

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} -v_2 - v_3 \\ v_2 \\ v_3 \end{pmatrix}$$

$$= \begin{pmatrix} -v_2 \\ v_2 \\ 0 \end{pmatrix} + \begin{pmatrix} -v_3 \\ 0 \\ v_3 \end{pmatrix}$$

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$$= \begin{pmatrix} -v_2 \\ v_2 \\ 0 \end{pmatrix} + \begin{pmatrix} -v_3 \\ 0 \\ v_3 \end{pmatrix}$$

$$= -v_2 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} - v_3 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

Proof ...

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$$\Rightarrow \quad a_1 = a_2 = 0.$$

$\square$

Problem. Let

$$B' = \left\{ \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\}$$

Show that B' is a basis for V.

Problem. Let

$$B" = \left\{ \begin{pmatrix} -1/2 \\ -1/2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right\}$$

Show that $B"$ is a basis for V.

The definition of a basis for a vector space is designed so that

- the basis contains enough vectors
- the basis does not contain too many vectors.

In other words, a basis contains neither less nor more than the necessary number of vectors.

If a basis B for a vector space V contains less than the necessary number of vectors, then B will not span the vector space V and If a basis B for a vector space V contains too many vectors, then the vectors in B will not be linearly independent.

Thus, a set $B = \{v_1, ..., v_k\}$ is a basis for V, iff

1. the set B is a maximally linearly independent set, i.e., it is linearly independent, and if we add one more vector from V to it, then it will not be linearly independent.

Thus, a set $B = \{v_1, ..., v_k\}$ is a basis for V, iff

1. the set B is a maximally linearly independent set, i.e., it is linearly independent, and if we add one more vector from V to it, then it will not be linearly independent.
2. or the set B is a minimal spanning set, i.e.,B spans V and if we remove one vector from it, then it will no longer span V .

Theorem...

Let $B = \{v_1, ..., v_n\}$ be a basis for $\mathbb{R}^n$. Then every vector $w \in \mathbb{R}^n$ can be written uniquely as a linear combination of vectors in the basis B:

$$w = a_1 v_1 + ... + a_n v_n,$$

where $a_1, ..., a_n$ are real numbers.

Discussion:

- What does this theorem mean?
  Ans. There is only one way of expressing any vector as a linear combination of the basis vectors.
- What do we need to prove?

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By the definition of a basis,

$$\operatorname{span}(B) = \mathbb{R}^n.$$

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$$w = b_1 v_1 + ... + b_n v_n,$$

where $b_1, ..., b_n$ are real numbers.

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Then

$$(a_1 - b_1)v_1 + ... + (a_n - b_n)v_n = 0.$$

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$$\begin{aligned} & (a_1 - b_1) = ... = (a_n - b_n) = 0 \\ \Rightarrow \quad & a_1 = b_1, ..., a_n = b_n, \end{aligned}$$

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$$\begin{aligned} & (a_1 - b_1) = ... = (a_n - b_n) = 0 \\ \Rightarrow \quad & a_1 = b_1, ..., a_n = b_n, \end{aligned}$$

This shows that there is one and only one way to write w as a linear combination of the basis vectors $v_1, ..., v_n$. $\square$

Theorem.

Let V be an n-dimensional vector space. Then

1. Any set of n elements of V that spans V is linearly independent and thus is a basis for V.
2. Any linearly independent set of n elements of V spans V and thus is a basis for V.

Proof.

1. Suppose that $B = \{v_1, ..., v_n\}$ spans V.

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Proof.

1. Suppose that $B = \{v_1, ..., v_n\}$ spans V. If vectors $v_1, ..., v_n$ linearly dependent, then we know that some proper subset of B still spans V, making the dimension less than n, which is impossible. This proves Statement (1).

Proof...

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Proof...

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Proof...

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□

Problem

Any n non-zero orthogonal vectors in $\mathbb{R}^n$ form a basis for $\mathbb{R}^n$.

Usefulness of basis in data science.

(a) Data points on a two dimensional plane, scattered in a round manner.

Usefulness of basis in data science.

(a) Data points on a two dimensional plane, scattered in a round manner.

(b) Data points on a two dimensional plane, scattered in an elongated manner.

In Figure (a), different data points have different values for the two variables, x_1 and x_2, but both variables seem important to capture properties of the data.

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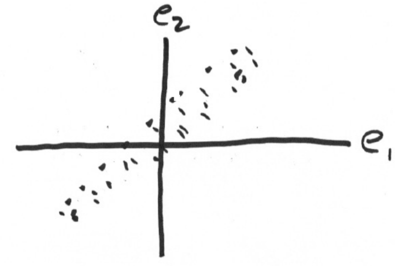
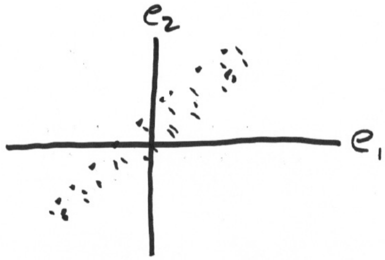
In Figure (b), the elongation is along the x-axis. It could be due to one feature being more "important" in some sense. That means, most of the information of interest is captured by x_1, while x_2 might be less important or simply random noise. In this case, we might hope or expect get very similar results by considering only (x_1), rather than (x_1, x_2), for each data point.



(c) Data points on a two dimensional plane, scattered in a different elongated manner.



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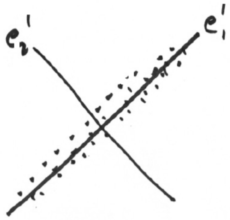

(d) Same data points on a two dimensional plane, scattered in an elongated manner, but with rotated axes.

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By changing the axes, the data set plotted in Figure (c) can be visualized as in Figure(d).

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By changing the axes, the data set plotted in Figure (c) can be visualized as in Figure(d).Now, the information from the data set can be obtained as in Figure (b) with respect to new axes e_1' and e_2'.

Not-standard basis vectors.

If we rotate e_1 by the angle θ, then

$$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \to \begin{pmatrix} 1 \\ \tan\theta \end{pmatrix} = \begin{pmatrix} 1 \\ a \end{pmatrix} = e_1'.$$

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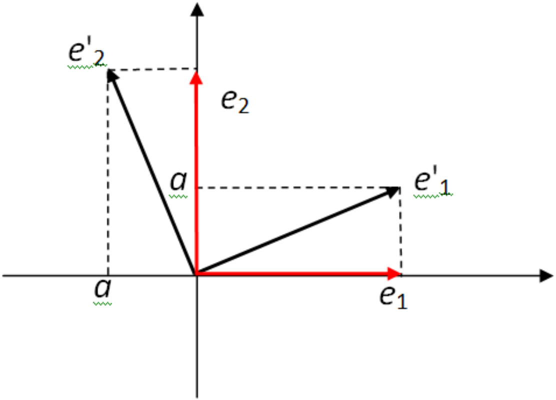
Note that just as $e_1 \cdot e_2 = 0$, since we have rotated both vectors by the same angle, so too $e_1' \cdot e_2' = 0$, i.e., e_1' and e_2' are also perpendicular.



e'2
e2
a
e'1
a
e1

With respect to these new basis vectors: any point on the line can be described by one number (the magnitude along e_1', and in the same way we might be able to consider only one coordinate axis, here we might be able to consider only one of the new coordinate axes e_1' and ignore the other e_2' and still be able to do something useful with the data.

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Summarizing this discussion, there are several points to note.

- The vector $\begin{pmatrix} 1 \\ a \end{pmatrix}$ seems more natural to describe the data in Figure (c) given above.
- Using this more natural description it takes 1 number rather than 2 numbers.
- The line through the origin defined by $\begin{pmatrix} 1 \\ a \end{pmatrix}$ – as well as the line through the origin defined by the $\begin{pmatrix} -a \\ 1 \end{pmatrix}$ perpendicular to it – is a one-dimensional subspace of $\mathbb{R}^2$.

In the same way that any point on the plane can be expressed in terms of the standard basis vectors, so too any point on the plane can be expressed in terms of the two vectors $\begin{pmatrix} 1 \\ a \end{pmatrix}$ and $\begin{pmatrix} -a \\ 1 \end{pmatrix}$. We will study this later in detail.

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