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Course Name : Mathematics & Statistics-III (BSC-M301)

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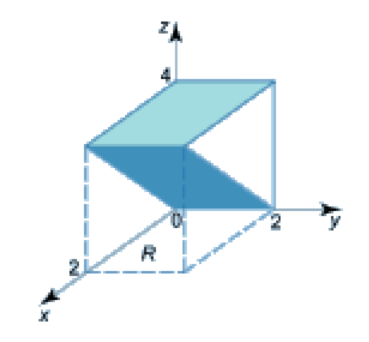
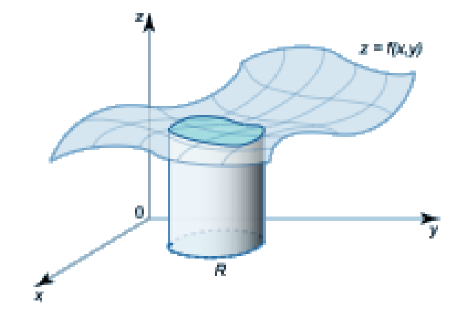
Multivariate Calculus (Integration)





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Double Integrals : Change of Variables

The formula of change of variables from (x, y) to (u, v) co-ordinate system is given by

$$\iint_R f(x,y)\,dx dy = \iint_S f[x(u,v), y(u,v)]\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du dv,$$

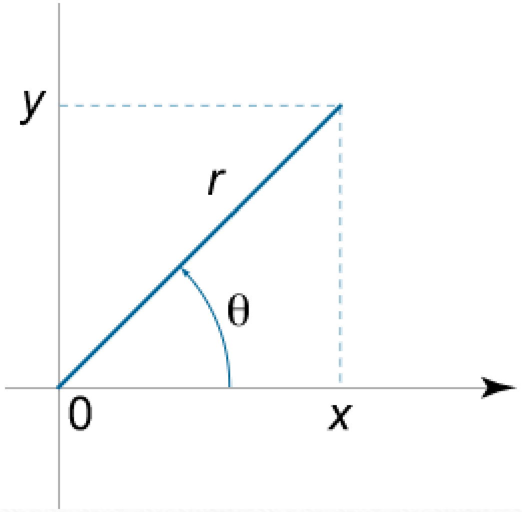
where $\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$ is called the ***Jacobian*** of the transformation.

Double Integrals in Polar Coordinates

One of the particular cases of change of variables is the transformation from **Cartesian** to **polar coordinate system**

$$x = r\cos\theta, \quad y = r\sin\theta.$$

The Jacobian determinant for this transformation is



$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \frac{\partial(r\cos\theta)}{\partial r} & \frac{\partial(r\cos\theta)}{\partial \theta} \\ \frac{\partial(r\sin\theta)}{\partial r} & \frac{\partial(r\sin\theta)}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = \cos\theta \cdot r\cos\theta$$

$$- (-r\sin\theta)\cdot\sin\theta = r\cos^2\theta + r\sin^2\theta = r\left(\cos^2\theta + \sin^2\theta\right) = r.$$

Double Integrals in Polar Coordinates

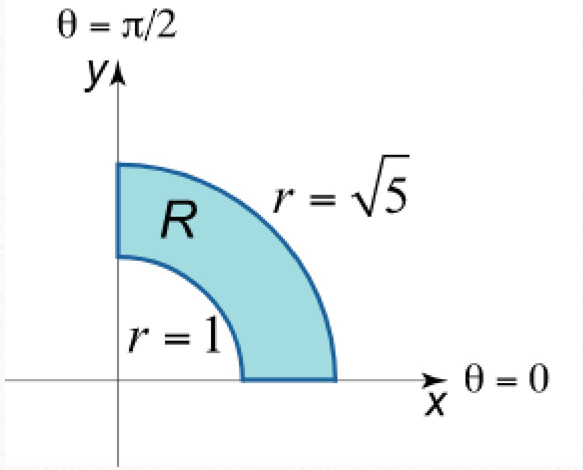
Evaluate the integral $\iint\limits_R xy\,dy\,dx$, where the region of integration R lies in the sector $0 \le \theta \le \frac{\pi}{2}$ between the curves $x^2+y^2=1$ and $x^2+y^2=5$.

Solution.

In polar coordinates, the region of integration

$$S = \left\{(r,\theta) \mid 1 \le r \le \sqrt{5},\ 0 \le \theta \le \frac{\pi}{2}\right\}.$$

Then, the given integral

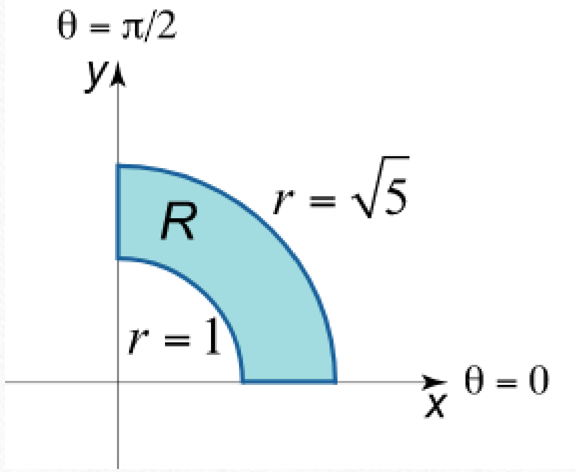


Double Integrals in Polar Coordinates

$$\iint\limits_R xy\,dy\,dx = \int\limits_0^{\frac{\pi}{2}} \int\limits_1^{\sqrt{5}} r\cos\theta r\sin\theta r\,dr\,d\theta = \int\limits_0^{\frac{\pi}{2}} \sin\theta\cos\theta\,d\theta \int\limits_1^{\sqrt{5}} r^3\,dr$$

$$= \frac{1}{2}\int\limits_0^{\frac{\pi}{2}} \sin 2\theta\,d\theta \int\limits_1^{\sqrt{5}} r^3\,dr = \frac{1}{2}\left(-\frac{\cos 2\theta}{2}\right)\Big|_0^{\frac{\pi}{2}} \cdot \left(\frac{r^4}{4}\right)\Big|_1^{\sqrt{5}}$$

$$= \frac{1}{4}(-\cos\pi + \cos 0)\cdot\frac{1}{4}(25-1) = \frac{1}{4}(1+1)\cdot 6 = 3.$$



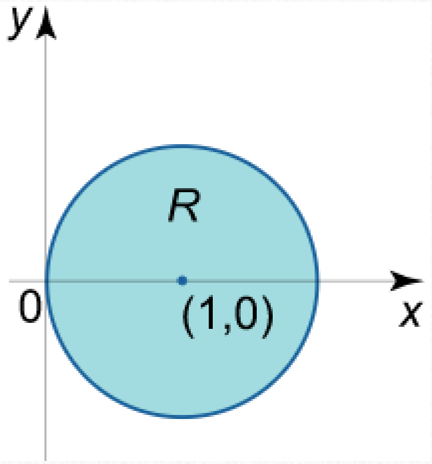
Double Integrals in Polar Coordinates

Calculate the double integral $\iint\limits_R \left(x^2+y^2\right) dxdy$ in the circle $x^2+y^2=2x$.

Solution.

The region of integration R is

$$x^2+y^2=2x, \quad \Rightarrow x^2-2x+1+y^2=1, \quad \Rightarrow (x-1)^2+y^2=1.$$



Substituting the expressions $x=r\cos\theta$, $y=r\sin\theta$, we obtain the equation of the circle in polar coordinates.

$$x^2+y^2=2x, \quad \Rightarrow r^2\cos^2\theta+r^2\sin^2\theta=2r\cos\theta, \quad \Rightarrow r^2\left(\cos^2\theta+\sin^2\theta\right)=2r\cos\theta,$$
$$\Rightarrow r=2\cos\theta.$$

Double Integrals in Polar Coordinates

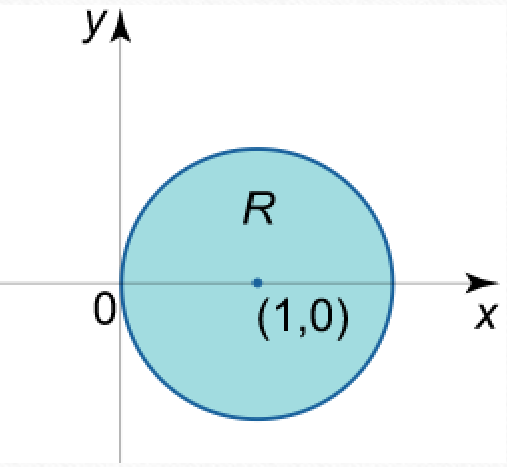
In polar coordinates, the region of integration

$$S = \left\{(r,\theta): 0 \le r \le 2cos\theta, -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}\right\}$$

After transition to polar coordinates we can calculate the double integral:

$$\iint\limits_R \left(x^2+y^2\right) dxdy = \iint\limits_S \left(r^2\cos^2\theta + r^2\sin^2\theta\right) rdrd\theta = \iint\limits_S r^3 drd\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\int_0^{2\cos\theta} r^3 dr\right] d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\left(\frac{r^4}{4}\right)\Big|_0^{2\cos\theta}\right] d\theta = 4\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^4\theta d\theta$$



Double Integrals in Polar Coordinates

$$= 4\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{1+\cos 2\theta}{2}\right)^2 d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(1 + 2\cos 2\theta + \cos^2 2\theta\right) d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{3}{2} + 2\cos 2\theta + \frac{1}{2}\cos 4\theta\right) d\theta = \left(\frac{3}{2}\theta + \sin 2\theta + \frac{1}{8}\sin 4\theta\right)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= \left(\frac{3}{2}\cdot\frac{\pi}{2} + \sin\pi + \frac{1}{8}\sin 2\pi\right) - \left(-\frac{3}{2}\cdot\frac{\pi}{2} - \sin\pi - \frac{1}{8}\sin 2\pi\right) = \frac{3\pi}{2}.$$

Thank You