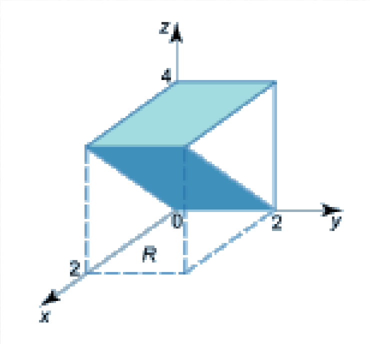
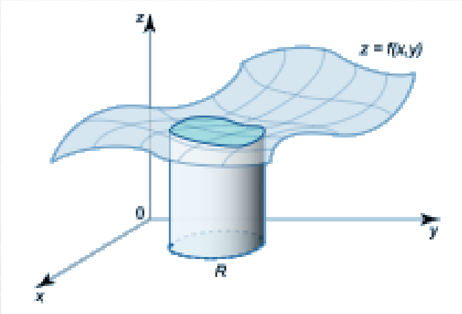
INSTITUTE OF ENGINEERING & MANAGEMENT, KOLKATA

Course Name : Mathematics-II (BSC-203)

INSTITUTE OF ENGINEERING & MANAGEMENT, KOLKATA

Multivariate Calculus (Integration)





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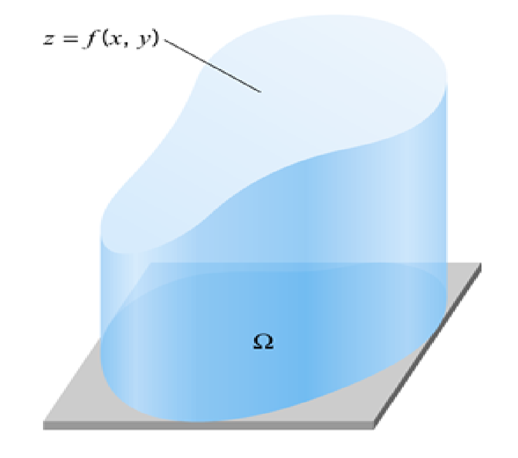
Double Integrals

The ***Double Integral*** of the function $\boldsymbol{f(x, y)}$ over a region $\boldsymbol{\Omega}$ is denoted as $\iint_{\Omega} f(x,y)dxdy$ and is given by the limit of the sum as $\lim_{N\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f_{ij} \Delta x_i \Delta y_j$.

Remember: **(i)** $\iint_{\Omega} f(x,y)dxdy$ gives the **volume of the solid** bounded below by $\boldsymbol{\Omega}$ and bounded above by the surface $z = f(x, y)$.



(ii) $\iint_{\Omega} dxdy$ represents the **area of the region $\boldsymbol{\Omega}$.**

Understanding Double Integrals

- Existence Condition for Double Integrals
- Properties of Double Integrals
- Evaluation of Double Integrals for a Rectangular Region

Condition for Existence of Double Integrals

Theorem : If $f(x, y)$ is **continuous** or **sectionally continuous** on the region $\boldsymbol{\Omega}$, then it is **integrable** on $\boldsymbol{\Omega}$.

Note:

(i) This is a **sufficient condition** for the existence of double integral.

(ii) The double integral exists if finite number of discontinuities are there in $\boldsymbol{\Omega}$, but the function should be bounded.

Properties of the Double Integral

Let f and g be assumed to be continuous functions on **Ω.**

I. ***Linearity*** **:** The double integral of a linear combination is the linear combination of the double integrals:

$$\iint_{\Omega}\left[\alpha f(x,y)+\beta g(x,y)\right]dx\,dy=\alpha\iint_{\Omega} f(x,y)\,dx\,dy+\beta\iint_{\Omega} g(x,y)\,dx\,dy$$

where α and β are constants.

II. ***Order*** **:** The double integral preserves order:

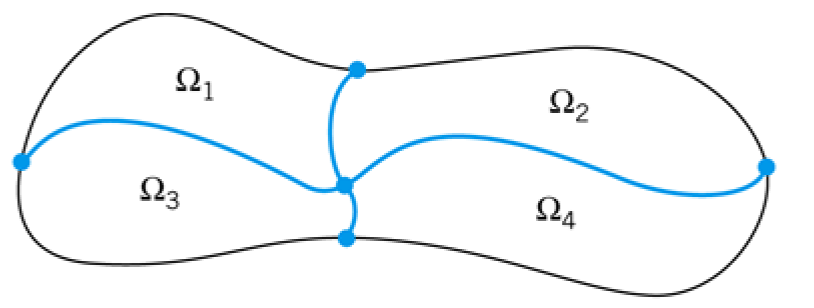
if $f \geq 0$ on Ω, then $\iint_{\Omega} f(x,y)\,dx\,dy \geq 0$

if $f \leq g$ on Ω, then $\iint_{\Omega} f(x,y)\,dx\,dy \leq \iint_{\Omega} g(x,y)\,dx\,dy$

Properties of the Double Integral

III. ***Additivity :*** If Ω is broken up into a finite number of non-overlapping regions $\Omega_1, \ldots, \Omega_n$, then

$$\boxed{\iint\limits_{\Omega} f(x,y)\,dx\,dy = \iint\limits_{\Omega_1} f(x,y)\,dx\,dy + \cdots + \iint\limits_{\Omega_n} f(x,y)\,dx\,dy}$$



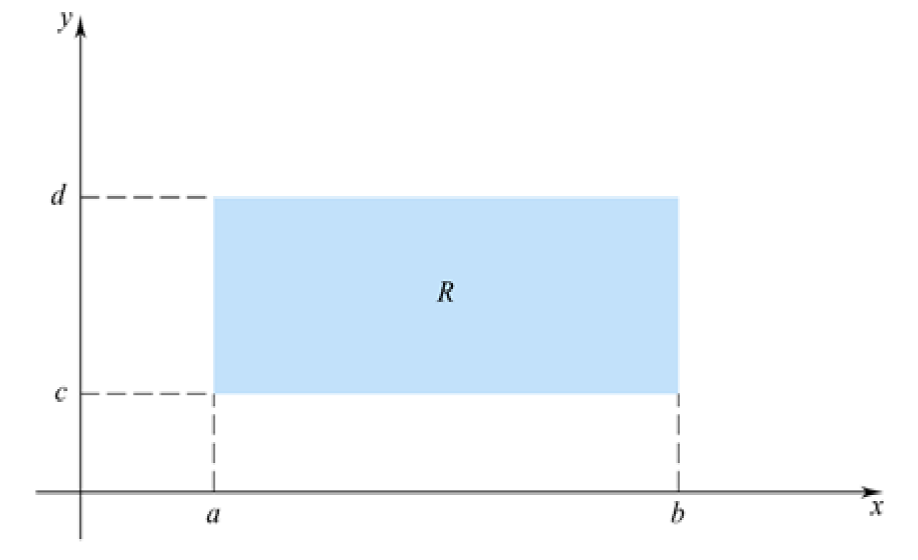
Evaluation of Double Integrals by Repeated Single Integrals

Rectangular Region : $R = \{(x, y): a \leq x \leq b, c \leq y \leq d\}$

Fubini's Theorem

$$\iint_R f(x,y)dxdy = \int_c^d \left\{ \int_a^b f(x,y)dx \right\} dy$$

$$\text{or } = \int_a^b \left\{ \int_c^d f(x,y)dy \right\} dx$$



Evaluation of Double Integrals by Repeated Single Integrals

Calculate the double integral $\iint\limits_R xy^2 dxdy$ over the region $R = \{(x, y) \mid 1 \le x \le 5,$
$0 \le y \le 2\}$.

Solution.

$$\iint\limits_R xy^2 dxdy = \int\limits_1^5 xdx \cdot \int\limits_0^2 y^2 dy = \left(\frac{x^2}{2}\right)\Big|_1^5 \cdot \left(\frac{y^3}{3}\right)\Big|_0^2 = \left(\frac{25}{2} - \frac{1}{2}\right)\left(\frac{8}{3} - 0\right) = 64.$$

Evaluation of Double Integrals by Repeated Single Integrals

Evaluate the integral $\iint\limits_R \cos(x+y)\,dx\,dy$ over the region $R = \left\{(x,y) \mid 0 \le x \le \frac{\pi}{4},\ 0 \le y \le \frac{\pi}{4}\right\}$.

Solution.

$$\iint\limits_R \cos(x+y)\,dx\,dy = \int_0^{\frac{\pi}{4}}\int_0^{\frac{\pi}{4}} \cos(x+y)\,dx\,dy = \int_0^{\frac{\pi}{4}}\left[\int_0^{\frac{\pi}{4}} \cos(x+y)\,dx\right]dy$$

$$= \int_0^{\frac{\pi}{4}}\left[\sin(x+y)\big|_{x=0}^{\frac{\pi}{4}}\right]dy = \int_0^{\frac{\pi}{4}}\left[\sin\left(\frac{\pi}{4}+y\right) - \sin y\right]dy$$

$$= \left[-\cos\left(\frac{\pi}{4}+y\right) + \cos y\right]\Big|_0^{\frac{\pi}{4}} = \left[-\cos\left(\frac{\pi}{4}+\frac{\pi}{4}\right) + \cos\frac{\pi}{4}\right]$$

$$- \left[-\cos\left(\frac{\pi}{4}+0\right) + \cos 0\right] = -\cos\frac{\pi}{2} + \cos\frac{\pi}{4} + \cos\frac{\pi}{4} - \cos 0$$

$$= 0 + \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} - 1 = \sqrt{2} - 1.$$

Thank You