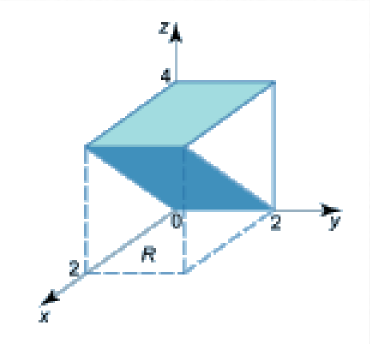
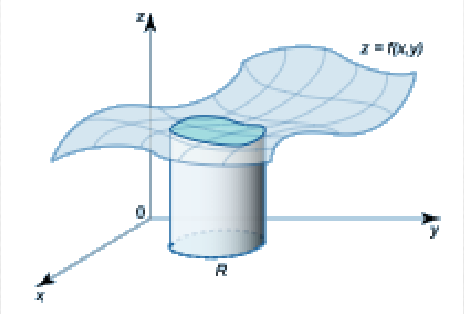
INSTITUTE OF ENGINEERING & MANAGEMENT, KOLKATA

Course Name : Mathematics & Statistics-III (BSC-M301)

INSTITUTE OF ENGINEERING & MANAGEMENT, KOLKATA

Multivariate Calculus (Integration)





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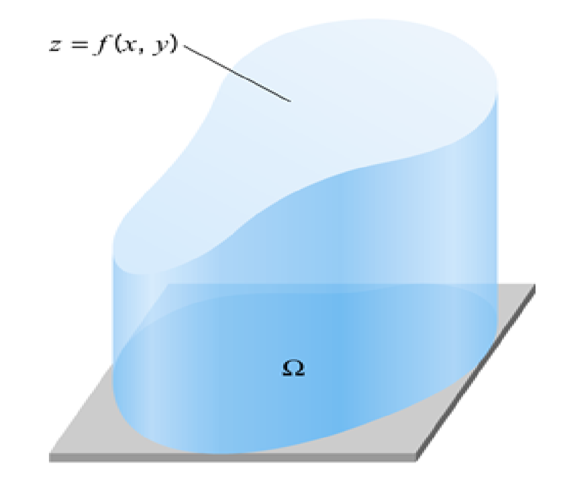
Application of Double Integrals : Volume

Try to remember:

The double integral

$$\boxed{\iint_{\Omega} f(x,y)\,dxdy,}$$

gives the **volume of the solid** that is bounded below by Ω and bounded above by the surface $z = f(x, y)$.



Application of Double Integrals : Volume

Find the volume of the tetrahedron bounded by the planes
$x = 0, z = 0, x = 2y, x + 2y + z = 2$.

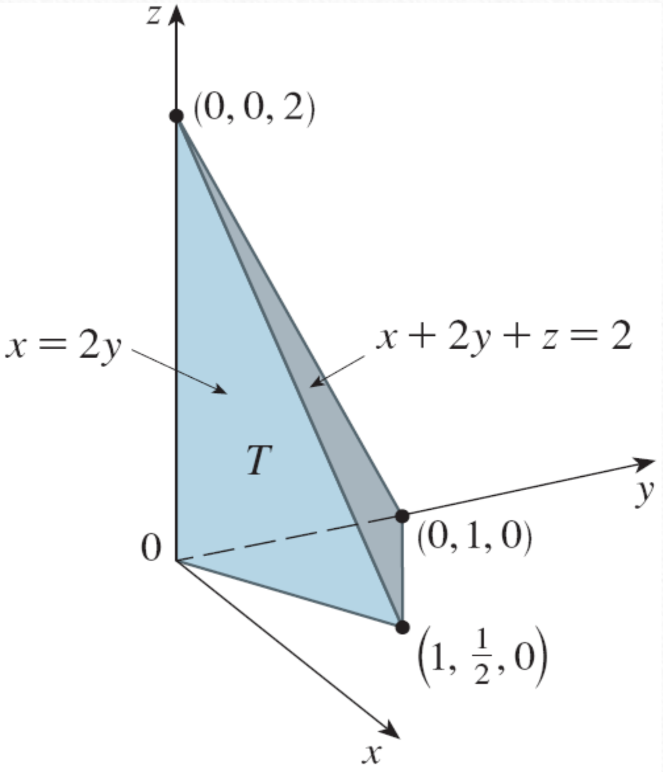
Solution:

The required volume is given by the double integral

$$\boxed{\iint_{\Omega} f(x, y)\,dxdy,}$$

where the base region Ω is a triangle bounded by the sides

$x = 0, x = 2y$ and $x + 2y = 2$ (as $z = 0$).



Application of Double Integrals : Volume

Now, the plane $x + 2y + z = 2$ can be written as

$$z = 2 - x - 2y.$$

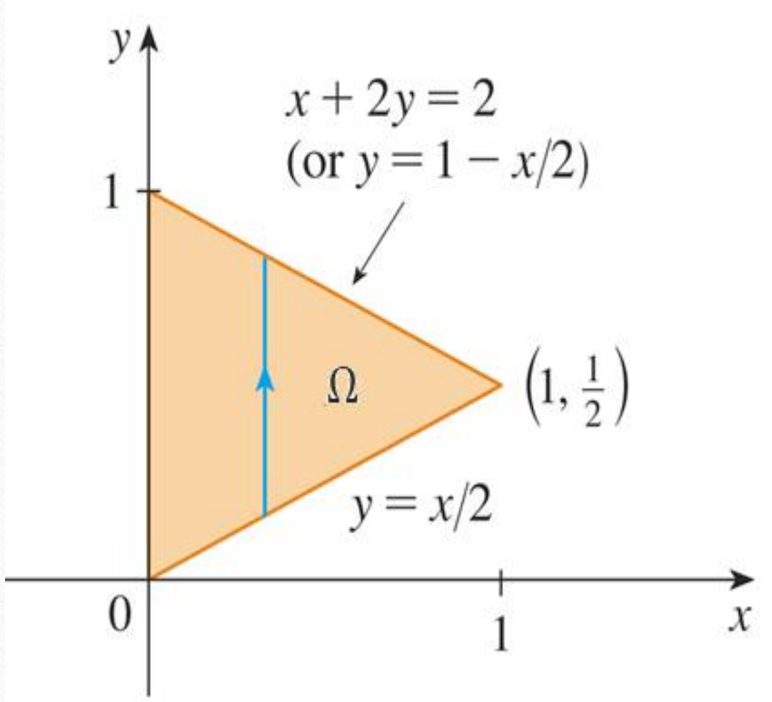
So, the required volume lies under the graph of the function $z = 2 - x - 2y$ and above the region

$$\Omega = \left\{(x, y): 0 \le x \le 1, \frac{x}{2} \le y \le 1 - \frac{x}{2}\right\}.$$

Then, $\boxed{\iint\limits_{\Omega} f(x,y)\,dx\,dy}$

$$= \int_0^1 \int_{x/2}^{1-x/2} (2 - x - 2y)\,dy\,dx$$

$$= \int_0^1 \left[2y - xy - y^2\right]_{y=x/2}^{y=1-x/2} dx$$



Application of Double Integrals : Volume

$$= \int_0^1 \left[2 - x - x\left(1 - \frac{x}{2}\right) - \left(1 - \frac{x}{2}\right)^2 - x + \frac{x^2}{2} + \frac{x^2}{4} \right] dx$$

$$= \int_0^1 \left(x^2 - 2x + 1\right) dx$$

$$= \left. \frac{x^3}{3} - x^2 + x \right]_0^1$$

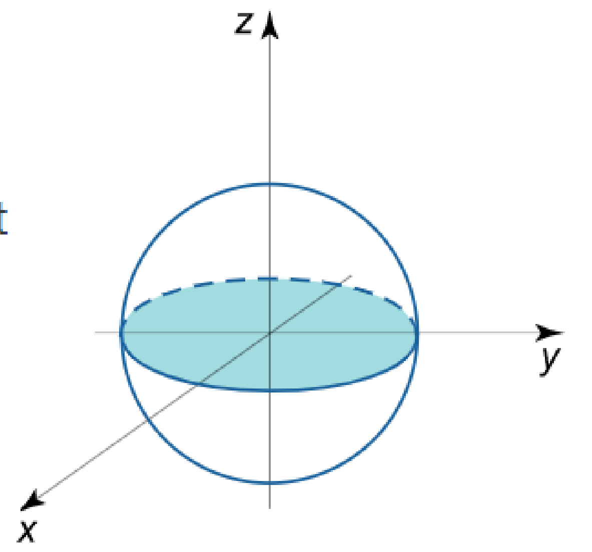
$$= \frac{1}{3}$$

Application of Double Integrals : Volume

Find the volume of the unit sphere.

Solution.

The equation of the sphere with radius 1 is $x^2 + y^2 + z^2 = 1$



Because of symmetry we find the volume of the upper hemisphere and then multiply the result by 2. The equation of the upper hemisphere is

$$z = \sqrt{1 - (x^2 + y^2)}.$$

Application of Double Integrals : Volume

Transforming to polar coordinates, we have

$$z(r,\theta) = \sqrt{1-r^2}.$$

In polar coordinates, the region of integration R is given by the set $R = \{(r,\theta) \mid 0 \le r \le 1,$ $0 \le \theta \le 2\pi\}$. Hence, the volume of the upper hemisphere is

$$V_{\frac{1}{2}} = \iint\limits_R \sqrt{1-r^2}\,r\,dr\,d\theta = \int\limits_0^{2\pi} d\theta \int\limits_0^1 \sqrt{1-r^2}\,r\,dr = 2\pi \int\limits_0^1 \sqrt{1-r^2}\,r\,dr.$$

Let $1 - r^2 = t$. Then, $-2rdr = dt$. Also note that, $t = 1$ when $r = 0$ and $t = 0$ when $r = 1$.

Application of Double Integrals : Volume

Thus,

$$V_{\frac{1}{2}} = 2\pi \int_0^1 \sqrt{1-r^2}\, r\,dr = 2\pi \int_1^0 \sqrt{t}\left(-\frac{dt}{2}\right) = -\pi \int_1^0 \sqrt{t}\,dt = \pi \int_0^1 t^{\frac{1}{2}}\,dt = \pi \left(\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right)\Bigg|_0^1$$

$$= \frac{2\pi}{3}.$$

Then, the volume of the unit sphere is

$$V = 2V_{\frac{1}{2}} = \frac{4\pi}{3}.$$

Thank You