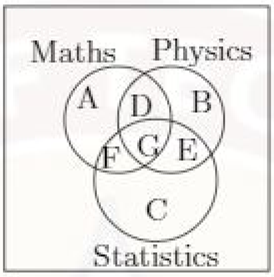
Week - 1

Solutions for Practice Assignment

Mathematics for Data Science - 1

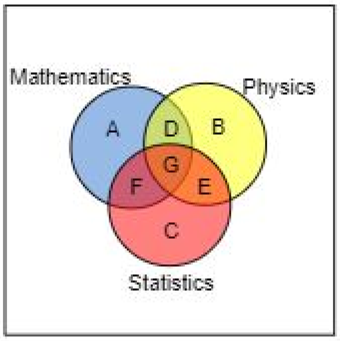
1. Given below is a Venn diagram for sets of students who take *Maths*, *Physics*, and *Statistics*. Which of the option(s) is(are) correct? [Notation: For sets P and Q, $P \setminus Q$ denotes the set of elements in P which are not in Q.]



- ○ D is the set of students who take both *Maths* and *Statistics*.
- ○ $D \cup E \cup F \cup G$ is the set of all students who take at least two subjects.
- ○ E is a subset of the set of the students who have not taken *Maths*.
- ○ $Maths \setminus D$ is the set of all students who have taken only *Maths*.
- ○ $Physics \setminus (D \cup G \cup E)$ is the set of all students who have taken only *Physics*.

Solution: According to Figure 1, D is the set of students who take both *Maths* and *Physics*. Hence the first statement is not valid.

The second option - $D \cup E \cup F \cup G$ is the set of all students who take at least two subjects - is correct. This is because D is the set of students who take both *Maths* and *Physics*, E is the set of students who take both *Physics* and *Statistics*, F is the set of students who take both *Maths* and *Statistics* and G is the set of students who take all three subjects.



PS-1.1: Figure for Question 1

Third option - E is a subset of the set of the students who have not taken $Maths$ - is also correct. E is the set of students who take both $Physics$ and $Statistics$ and G is the set of students who take $Maths$ in addition to $Physics$ and $Statistics$. $(B \cup E \cup C)$ is the set of students who have not taken $Maths$. Clearly, E is a subset of this set. As E and G are two different sets, this option is correct.

Fourth option - $Maths \setminus D$ is the set of all students who have taken only $Maths$ - is not correct. $Maths \setminus D$ represents the students of $Maths$ who have not taken $Physics$ and may or may not have taken $Statistics$. This implies that students who take only $Maths$ (set A), or the students who take both $Maths$ and $Statistics$ (set F) or the students who take all three subjects (set G) are also included in $Maths \setminus D$ set. Hence this option is not correct.

Fifth option - $Physics \setminus (D \cup G \cup E)$ is the set of all students who have taken only $Physics$ - is correct. $(D \cup G \cup E)$ represents the students who take only $Maths$ and $Physics$ or all three subjects or $Physics$ and $Statistics$. $Physics \setminus (D \cup G \cup E)$ represents B, which is the set of students who only take $Physics$. Hence this option is correct.

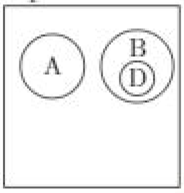
2. Let A be the set of natural numbers less than 6 and whose greatest common divisor with 6 is 1. Let B be the set of divisors of 6. What are the cardinalities of $A, B, A \cup B,$ and $A \cap B$?

- ◯ (1,5,6,0)
- ◯ (1,4,5,0)
- ◯ (2,4,5,1)
- ◯ (2,4,6,1)

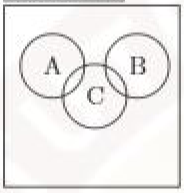
Solution: We have set A=$\{1, 5\}$, B=$\{1, 2, 3, 6\}$, $A \cup B$ =$\{1, 2, 3, 5, 6\}$ and $A \cap B$=$\{1\}$. It follows that the cardinalities (i.e. number of elements) of A, B, $A \cup B$ and $A \cap B$ are respectively 2, 4, 5 and 1. Hence, the third option - $\{2, 4, 5, 1\}$ - is correct.

3. Let A be the set of all even natural numbers (including zero), B be the set of all odd natural numbers, C be the set of all natural numbers which divide 100, and D be the set of all prime numbers less than 100. Which of the following is(are) correct representation of these sets? [Note: A region represented in a Venn diagram could be empty. Take the set of real numbers to be the universal set.]

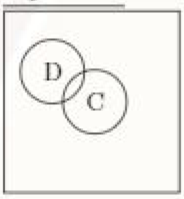
- ○ Option 1



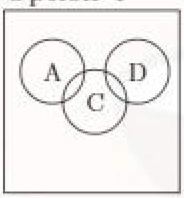
- ○ Option 2



- ○ Option 3



- ○ Option 4



Solution: By definition, $A = \{0, 2, 4, 6, 8,\ldots\}$, $B = \{1, 3, 5, 7,\ldots\}$, $C = \{1, 2, 4, 5, 10,\ldots, 100\}$ and $D = \{2, 3, 5, 7, 11,.., 97\}$.
Option 1 shows D as a subset of all odd natural numbers. But D contains element 2, whereas B does not. Hence, this option is wrong.
Option 2 has overlap between A and C and overlap between B and C, but no overlap between A and B. A and B are sets of even and odd natural numbers which have no overlap. C is the set of natural numbers which divide 100. $A \cap C = \{2, 4, 10, 20, 50, 100\}$ and $B \cap C = \{1, 5, 25\}$. Hence, this option is correct.
Option 3 represents C and D sets with an overlap between them. The overlapping area includes the set of all prime numbers which can divide 100. This is the set $\{2, 5\}$. Hence, option 3 is also correct.
$A \cap D = \{2\}$, but there is no overlap between A and D in Option 4. Hence, this option is wrong.

4. Let A be the set of natural numbers which are multiples of 5 strictly less than 100, and B be the set of natural numbers which divide 100. What are the cardinalities of the following sets?

$B \setminus A$ (the set of elements in B but not in A), $A \cap B$, and B

- ◯ (2, 5, 7)
- ◯ (4, 5, 9)
- ◯ (3, 4 , 7)
- ◯ (3, 5, 8)

Solution: By definition, $A = \{5, 10, 15, 20, 25, 30, 35, 40, 45, 50, 55, 60, 65, 70, 75, 80, 85, 90, 95\}$, $B = \{1, 2, 4, 5, 10, 20, 25, 50, 100\}$, $B \setminus A = \{1, 2, 4, 100\}$ and $A \cap B = \{5, 10, 20, 25, 50\}$. It follows that the cardinalities of sets $B \setminus A$, $A \cap B$ and B are, respectively, 4, 5 and 9. Hence, option 2 is correct.

5. Suppose the cardinality of set A is 2 and the cardinality of set B is 3, what are the cardinalities of the cartesian product $A \times B$ and the power set of $A \times B$?

- ◯ 6 and 36
- ◯ 5 and 32
- ◯ 6 and 64
- ◯ 5 and 25

Solution: Let the cardinality of set A be $\mathrm{n}(A)$ and the cardinality of set B be $\mathrm{n}(B)$. Then, the cardinality of the cartesian product $(A \times B)$, $\mathrm{n}(A \times B) = \mathrm{n}(A) \times \mathrm{n}(B) = 3 \times 2 = 6$. If a set A has cardinality n, then the cardinality of power set of A is 2^n. It follows that the cardinality of the power set of $(A \times B)$ is $2^6 = 64$. Hence, the third option is correct.

6. In a survey, it is found that in a particular locality 64 houses buy English newspapers, 94 houses buy Tamil newspapers, and 26 houses buy both English and Tamil newspapers. How many houses buy newspapers of only one language?
   Answer: 106
   Solution: Number of houses which buy only English newspapers is $(64 - 26) = 38$.
   Number of houses which buy only Tamil newspapers is $(94 - 26) = 68$.
   Therefore, number of houses which buy either English or Tamil newspaper is
   $(68 + 38) = 106$.

7. Which of the following numbers is(are) irrational?

- ○ $\underline{\sqrt{2+\sqrt{3}}}$
- ○ $(2+\sqrt{3})(2-\sqrt{3})$
- ○ $(2+\sqrt{3})+(2-\sqrt{3})$
- ○ $\underline{2\sqrt{3}+3\sqrt{2}}$

Solution: Since $\sqrt{3}$ is an irrational number, it follows that $(2+\sqrt{3})$ and hence $\sqrt{(2+\sqrt{3})}$ are also irrational.
In the second option, $(2+\sqrt{3})(2-\sqrt{3})$ = 4 - 3 = 1, which is a rational number.
In the third option, $(2+\sqrt{3})+(2-\sqrt{3})$ = 4, which is also a rational number.
Since both $\sqrt{3}$ and $\sqrt{2}$ are irrational numbers, we have $(2\sqrt{3}+3\sqrt{2})$ is an irrational number.

8. Which of the following is(are) true for the relation R given below?
$R = \{(a, b)|$ both a and b are even non-zero integers and $\frac{a}{b}$ is an integer $\}$

- ◯ R is a reflexive relation.
- ◯ R is a symmetric relation.
- ◯ R is a transitive relation.
- ◯ R is an equivalence relation.

Solution: A relation R on a set A is said to be reflexive if $(a, a) \in R$ for all $a \in A$. R is called symmetric if $(a, b) \in R$ implies $(b, a) \in R$, and R is called transitive if (a, b) and (b, c) is in R implies $(a, c) \in R$. If a relation R is reflexive, symmetric and transitive, then it is called equivalence relation.
For any non-zero even integer a, $\frac{a}{a} = 1$ is an integer. Hence, $(a, a) \in R$, which implies that R is reflexive.
Now, let a = 4, and b = 2. Then, $\frac{a}{b} = \frac{4}{2} = 2$ is an integer. Hence, $(a, b) \in R$. But $\frac{b}{a} = \frac{2}{4} = \frac{1}{2}$ is not an integer. Therefore, $(b, a) \notin R$. It follows that R is not symmetric.
Let $(a, b) \in R$ and $(b, c) \in R$. That is, both $\frac{a}{b}$ and $\frac{b}{c}$ are integers. Hence, their product $\frac{a}{b}.\frac{b}{c} = \frac{a}{c}$ is also an integer. It follows that $(a, c) \in R$. Therefore, R is transitive.
Although R is reflexive and transitive but not symmetric, it is not an equivalence relation.

9. Find the domain and range of the following real valued function.
$f(x) = \sqrt{3-x}$ (Note: $\sqrt{}$ denotes the positive square root)

- ◯ domain=$\{x \in \mathbb{R} \mid x \neq 3\}$
  range=$\{x \in \mathbb{R} | x \geq 3\}$
- ◯ domain=$\{x \in \mathbb{R} \mid x \geq 3\}$
  range=$\{x \in \mathbb{R} \mid x \geq 0\}$
- ◯ domain=$\{x \in \mathbb{R} \mid x \leq 3\}$
  range=$\{x \in \mathbb{R} \mid x \geq 0\}$
- ◯ domain=$\{x \in \mathbb{R} \mid x \leq 3\}$
  range=$\{x \in \mathbb{R} \mid x \leq 0\}$

Solution: The set of real numbers $\mathbb{R}$ includes all rational and irrational numbers.
$\sqrt{a}$ is real valued if $a \geq 0$. If f has to be real valued, then

$$3 - x \geq 0$$

$$\Rightarrow 3 \geq x$$

Hence, domain of the function f is $\{x \in \mathbb{R} \mid x \leq 3\}$.
Since $\sqrt{}$ denotes the positive square root (as given in the question statement), the range of function f is nothing but all the positive real numbers, i.e. $\{x \in \mathbb{R} \mid x \geq 0\}$.

10. Which of the following is(are) true for the given function?
$f : \mathbb{R} \to \mathbb{R}$
$f(x) = x^2 + 2$

- ○ f is not injective.
- ○ f is surjective.
- ○ f is not surjective.
- ○ f is bijective.

Solution: A function f is injective if $f(x_1) = f(x_2)$ implies $x_1 = x_2$, i.e. no two elements in the domain will have the same image. f is called surjective if for any element in the co-domain there is a pre-image in the domain, i.e. for any y in the co-domain, there exists an x in the domain such that $f(x) = y$. A function f is said to be bijective if it is both injective and surjective.
Since $f(x) = x^2 + 2$, we have $f(-1) = 3 = f(1)$. Hence, f is not injective. Now, the co-domain of the function is given as $\mathbb{R}$.
Now if f is surjective then codomain and the range should be same, that means every element in the codomain should have a preimage. Now let us try to find a preimage for 1 (observe that $1 \in \mathbb{R}$, as codomain of the function is given as $\mathbb{R}$). To find the preimage of 1, we have to find an element a from the domain for which $f(a) = 1$, i.e. $a^2 + 2 = 1$, i.e. $a^2 = -1$. Now we know that the square of any real number cannot be negative. Hence there cannot exist any real number a (in the domain) for which $f(a) = 1$. Hence 1 has no preimage. So codomain and range is not same. Hence f is not surjective. Also, $1 \in \mathbb{R}$. Let x be such that $x \in \mathbb{R}$, and $f(x) = 1$.
As the function is neither injective, nor surjective, therefore it is not bijective.

11. Find the domain of the following real valued function.
$f(x) = \frac{\sqrt{x+2}}{x^2-9}$
   - ◯ $\{x \in \mathbb{R} \mid x \geq -2, x \neq 3\}$
   - ◯ $\{x \in \mathbb{R} \mid x \leq -2, x \geq 3\}$
   - ◯ $\{x \in \mathbb{R} \mid x \neq -2, x \leq 3\}$
   - ◯ $\{x \in \mathbb{R} \mid x \neq -2, x \neq 3\}$

Solution: $f(x) = \frac{\sqrt{x+2}}{x^2-9}$. For f to be a well-defined function, the denominator must be non-zero. That is,

$$x^2 - 9 \neq 0$$
$$\Rightarrow x \neq \pm 3$$

Further, if f has to be real valued, then $\sqrt{x+2}$ has to be real valued. Hence $x+2$ must be non-negative. That is,

$$x + 2 \geq 0$$
$$\Rightarrow x \geq -2$$

It follows that the domain of the function $f(x)$ is $\{x \in \mathbb{R} \mid x \geq -2, x \neq 3\}$.

12. Let S be the set {January, February, March, April, May, June, July, August, September, October, November, December} of months in a year. Define the following three relations:

- $R_1 := \{(a, b) \mid a, b \in S,\ a \text{ and } b \text{ have same last four letters. }\}$
- $R_2 := \{(a, b) \mid a, b \in S,\ a \text{ and } b \text{ have same number of days. }\}$
- $R_3 := \{(a, c) \mid a, c \in S,\ \text{for some } b \in S, (a, b) \in R_1, (b, c) \in R_2\}$

For example,(December, June)$\in R_3$ since (December, September)$\in R_1$, (September, June)$\in R_2$.

(a) Choose the correct option(s).

○ R_3 is symmetric.
○ R_3 is reflexive.
○ R_3 is transitive.
○ None of the above.

(b) What is the cardinality of R_3?
Answer: 85

Solution: For definitions of types of relations, please refer to solution of Question 8.
Every month has the same last four letters as itself (except *May* which has only three letters). In Table 1, the months whose name has been shown in red color have the same last four letters as each other. Similarly, the months whose name has been shown in blue color also have the same last four letters as each other.

Name of the months (Elements of S)
January
February
March
April
May
June
July
August
September
October
November
December

Table 1: Question 12 : R_1 relation

Hence $R_1 = \{$(Jan, Jan), (Jan, Feb), (Feb, Jan), (Feb, Feb), (Mar, Mar), (April, April), (June, June), (July, July), (Aug, Aug), (Oct, Oct), (Sept, Sept), (Sept, Nov), (Sept, Dec), (Nov, Sept), (Nov, Nov), (Nov, Dec), (Dec, Sept), (Dec, Nov), (Dec, Dec)$\}$

The relation R_2 consists of the pairs of months with the same number of days. In Table 2, the months whose name has been shown in red color have 31 days each. The months whose name has been shown in black color have 30 days each.

Name of the months
January
February
March
April
May
June
July
August
September
October
November
December

Table 2: Question 12: R_2 relation

Observe that it is a equivalence relation. The partition formed by this equivalence relation is as follows:

Class 1: Jan, Mar, May, July, Aug, Oct, Dec [Months with 31 days each]

Class 2: April, June, Sept, Nov [Months with 30 days each]

Class 3: Feb [Month with 28 or 29 days]

Now, R_3 is defined as follows:

$$R_3 = \{(a, c) \mid a, c \in S, \text{ for some } b \in S, (a, b) \in R_1, (b, c) \in R_2\}$$

If $(a, c) \in R_3$, then there must exist some pair $(a, b) \in R_1$.
Let us list out the number of elements of R_3 by listing out pairs starting with as shown below :

January: (Jan, Jan) $\in R_1$, Now we assume three partitions in the set S, formed by the relation R_2. These partitions are class 1, class 2, class 3. Hence from these classes, 7 pairs will be there in R_3 starting with January. These are {(Jan, Jan), (Jan, Mar), (Jan, May), (Jan, July), (Jan, Aug), (Jan, Oct), (Jan, Dec)}. Moreover, (Jan, Feb) is in R_1, and Feb is in another partition in S due to R_2. So there are total 8 pairs (adding (Jan, Feb) with previous 7 elements) in R_3 starting with Jan.

February: Since (Feb, Jan) is in R_1, then due to class 1 there will be 7 pairs : {(Feb, Jan), (Feb, Mar), (Feb, May), (Feb, July), (Feb, Aug), (Feb, Oct), (Feb, Dec)}. The element (Feb, Feb) will be in R_3 due to class 3. Hence 8 pairs are there in R_3 starting with Feb.

March: Due to class 1, seven pairs {(Mar, Jan), (Mar, Mar), (Mar, May), (Mar, July), (Mar, Aug), (Mar, Oct), (Mar, Dec)}.

April: Due to class 2, four pairs {(April, April), (April, June), (April, Sept), (April, Nov)}.

May: No pair will start with May as there is no pair in R_1 starting with May.

June: Due to class 2, 4 pairs: {(June, April), (June, June), (June, Sept), (June, Nov)}

July: Due to class 1, 7 pairs. {(July, Jan), (July, March), (July, July), (July, Aug), (July, Oct), (July, Dec)}

August: Due to class 1, 7 pairs. {(Aug, Jan), (Aug, Mar), (Aug, May), (Aug, July), (Aug, Aug), (Aug, Oct), (Aug, Dec)}

September: As (Sept, Dec) is a pair in R_1, it will pair up will all months in class 1, and as (Sept, Sept) is in R_1, it will pair up with all months with class 2. Hence there are total 11 pairs in R_3 starting with Sept : {(Sept, Jan), (Sept, Mar), (Sept, May), (Sept, July), (Sept, Aug), (Sept, Oct), (Sept, Dec), (Sept, April), (Sept, June), (Sept, Sept), (Sept, Nov)}

October: Due to class 1, 7 pairs are there : {(Oct, Jan), (Oct, Mar), (Oct, May), (Oct, July), (Oct, Aug), (Oct, Oct), (Oct, Dec)}

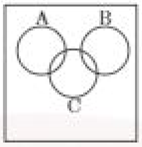
November: Due to both class 1 and class 2, 11 pairs : {(Nov, Jan), (Nov, Mar), (Nov, May), (Nov, July), (Nov, Aug), (Nov, Oct), (Nov, Dec), (Nov, April), (Nov, June), (Nov, Sept), (Nov, Nov)}

December: Due to both class 1 and class 2, 11 pairs: {(Dec, Jan), (Dec, Mar), (Dec, May), (Dec, July), (Dec, Aug), (Dec, Oct), (Dec, Dec), (Dec, April), (Dec, June), (Dec, Sept), (Dec, Nov)}

12. (b)
Hence cardinality of R_3 is $8 + 8 + 7 + 4 + 4 + 7 + 7 + 11 + 7 + 11 + 11 = 85$.
12. (a)

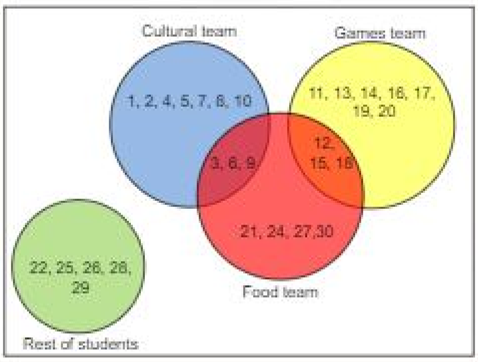
- (May, May) is not in R_3, hence R_3 is not reflexive.
- (Jan, May) is in R_3, but (May, Jan) is not in R_3, hence R_3 is not symmetric.
- (Mar, Dec) is in R_3, (Dec, Sept) is in R_3, but (Mar, Sept) is not in R_3. Hence R_3 is not transitive.

13. For a college event, thirty student volunteers were given id numbers from 1 to 30 such that each student gets a unique number. The students with id numbers from 1 to 10 are in Team 1 which coordinates the cultural program. The students with id numbers from 11 to 20 are in Team 2 which coordinates the games. The students whose roll numbers are multiples of 3 are in Team 3 which takes care of food. Now consider the following Venn diagram and choose the correct option(s).



- ◯ C, B , and A can represent Team 1, Team 2, and Team 3 respectively.
- ◯ A, B, and C can represent Team 1, Team 2, and Team 3 respectively.
- ◯ Roll number 15 has been assigned two jobs and is in both B and C.
- ◯ Roll number 25 is not in $A \cup B \cup C$.
- ◯ Roll number 10 is in both A and C.
- ◯ Cardinality of C is 20.

Solution:



PS-1.2: Venn diagram for Question 13

Figure PS-1.2 shows the Venn diagram corresponding to Question 13. Team 1, responsible for coordination of cultural programs, is represented by the blue circle. Team 2, responsible for game events, is represented by the yellow circle. Team 3, that takes care of food, is represented by the red circle. Rest of the students are represented using the green circle. Clearly, set A can correspond to the blue circle, B can denote the yellow circle and C can denote the red circle. That is, A, B, and C can represent Team 1, Team 2, and Team 3 respectively. Hence, option 2 is correct and option 1 is wrong. Roll number 15 is a common element between games team and food team, hence, option 3 is correct. Roll number 25 is located in the range of students with Roll number 21 to 30 but 25 is not divisible by 3. Hence, 25 does not belong to the set $A \cup B \cup C$ and so option 4 is correct. The number 10 is not divisible by 3, hence Roll number 10 is not in the set C. Therefore, option 5 is wrong. Further, since cardinality of C is 10, option 6 is also wrong.

Mathematics for Data Science - 1
Graded Assignment (week-1)
Sep 2022 term

1 Multiple Choice Questions (MSQ)

1. Which of the following are irrational numbers? (1 mark)

Set of correct options: **(Answer: (a),(b))(1 mark)**

- ○ $\mathbf{3^{1/3}}$
- ○ $(\sqrt{\mathbf{8}} + \sqrt{\mathbf{2}})(\sqrt{\mathbf{12}} - \sqrt{\mathbf{3}})$
- ○ $\dfrac{\sqrt{18}-3}{\sqrt{2}-1}$
- ○ $\dfrac{\sqrt{8}+\sqrt{2}}{\sqrt{8}-\sqrt{2}}$

Solution:

(a) $3^{1/3}$, this cannot be written in the form of p/q, where $p, q \in \mathbb{Z},\ q \neq 0$. So $3^{1/3}$ is an irrational number.

(b) $(\sqrt{8}+\sqrt{2})(\sqrt{12}-\sqrt{3}) = (2\sqrt{2}+\sqrt{2})(2\sqrt{3}-\sqrt{3}) = 3\sqrt{2}\sqrt{3} = 3\sqrt{6}$
We know that $\sqrt{6}$ is an irrational number. Hence, $(\sqrt{8}+\sqrt{2})(\sqrt{12}-\sqrt{3})$ is also an irrational number.

(c) $\dfrac{\sqrt{18}-3}{\sqrt{2}-1} = \dfrac{3\sqrt{2}-3}{\sqrt{2}-1} = \dfrac{3(\sqrt{2}-1)}{\sqrt{2}-1} = 3$
$\implies \dfrac{\sqrt{18}-3}{\sqrt{2}-1}$ is a rational number.

(d) $\dfrac{\sqrt{8}+\sqrt{2}}{\sqrt{8}-\sqrt{2}} = \dfrac{2\sqrt{2}+\sqrt{2}}{2\sqrt{2}-\sqrt{2}} = \dfrac{3\sqrt{2}}{\sqrt{2}} = 3$
$\implies \dfrac{\sqrt{8}+\sqrt{2}}{\sqrt{8}-\sqrt{2}}$ is a rational number.

Q2 :

Solution:

Material	Dielectric constant
Air	1
Vecuum	2
Paper	3
Glass	8
Nerve membrance	7
Silicon	13

Observe that elements in domain (Material) has unique output. And each element in codomain (Dieletric constant) has unique preimage. Hence this function is bijective.

Q3:

Solution: $A = \{x \in \mathbb{N} \mid x \bmod 2 = 0 \text{ and } 1 \le n \le 10\}$

$\Rightarrow A = \{2, 4, 6, 8, 10\}$

$B = \{x \in \mathbb{N} \mid x \bmod 5 = 0 \text{ and } 6 \le n \le 25\}$

$\Rightarrow B = \{10, 15, 20, 25\}$

$C = \{x \in \mathbb{N} \mid x \bmod 7 = 0 \text{ and } 7 \le n \le 29\}$

$\Rightarrow C = \{7, 14, 21, 28\}$

Now $A \setminus (B \cup C) = \{2, 4, 6, 8\}$

$B \setminus (A \cup C) = \{15, 20, 25\}$

$C \setminus (B \cup A) = \{7, 14, 21, 28\}$

$A \setminus (B \cup C) \cup B \setminus (A \cup C) \cup C \setminus (B \cup A)$

$= \{2, 4, 6, 8, 15, 20, 25, 7, 14, 21, 28\}$

so cardinality $= 11$.

Q 4

Soln:

Total number of people = 180

Number of people watched Dabang ($N(D)$) = 95

" " " " " Avatar ($N(A)$) = 100

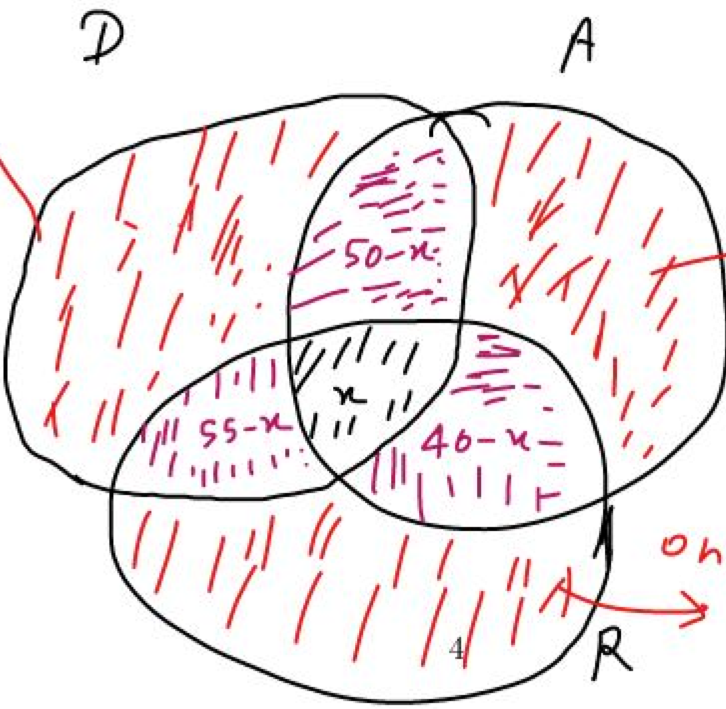
" " " " " RRR ($N(R)$) = 100

" " " " " Dabang and Avatar ($N(D \cup A)$) = 50

" " " " " Avatar and RRR ($N(A \cup R)$) = 40

" " " " " Dabang and RRR ($N(D \cup R)$) = 55

Let x number of people watched all 3 movies.



Only D

$95 - (x + 55 - x + 50 - x)$

$= x - 10$

Only A.

$100 - (50 - x + x + 40 - x)$

$= 10 + x$

Only R

$100 - (55 - x + x + 40 - x)$

$= 5 + x$

So $(x-10)+(10+x)+(5+x)+(55-x)+(50-x)+x$

$+(40-x) = 180$ (Total people)

$\Rightarrow \quad x + 150 = 180$

$\Rightarrow \quad x = 30$

So Number of people only watched RRR and Avatar $= 40 - x$

$= 40 - 30$

$= 10$

2 Numerical Answer Type (NAT)

5. Suppose $f : D \longrightarrow \mathbb{R}$ is a function defined by $f(x) = \dfrac{\sqrt{x^2-9}}{x+3}$, where $D \subset \mathbb{Z}$. Let A be the set of integers which are not in the domain of f, then find the cardinality of the set A. **(Answer:6)(2 marks)**

Solution:

The function $f(x) = \dfrac{\sqrt{x^2-9}}{x+3}$ is well defined if $x^2 - 9 \geq 0$ and $(x+3) \neq 0$.

Therefore, the domain of $f(x)$ is $\mathbb{Z} \setminus \{-3, -2, -1, 0, 1, 2\}$.

By definition, A is the set of integers which are not in the domain of f. Therefore $A = \{-3, -2, -1, 0, 1, 2\}$ and the cardinality of A is 6.

6. Consider a set $S = \{a \mid a \in \mathbb{N},\ a \leq 18\}$. Let R_1 and R_2 be relations from S to S defined as $R_1 = \{(x, y) \mid x, y \in S,\ y = 3x\}$ and $R_2 = \{(x, y) \mid x, y \in S,\ y = x^2\}$. Find the cardinality of the set $R_1 \setminus (R_1 \cap R_2)$. **(Answer: 5)(3 marks)**

Solution:

$S = \{0, 1, 2, 3, ..., 18\}$

$R_1 = \{(0, 0), (1, 3), (2, 6), (3, 9), (4, 12), (5, 15), (6, 18)\}$

$R_2 = \{(0, 0), (1, 1), (2, 4), (3, 9), (4, 16)\}$

$\therefore R_1 \cap R_2 = \{(0, 0), (3, 9)\}$.

Now, $R_1 \setminus (R_1 \cap R_2) = \{(1, 3), (2, 6), (4, 12), (5, 15), (6, 18)\}$.

Hence the cardinality of $R_1 \setminus (R_1 \cap R_2)$ is 5.

7. In a Zoo, there are 6 Bengal white tigers and 7 Bengal royal tigers. Out of these tigers, 5 are males and 10 are either Bengal royal tigers or males. Find the number of female Bengal white tigers in the Zoo. **(Answer: 3) (3 marks)**

Solution:

Let BW be the set of Bengal White tigers and BR be the set of Bengal Royal tigers, and M be the set of male tigers.

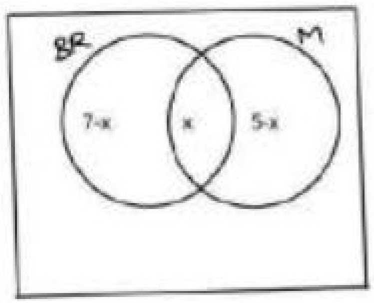


Figure Q7. Tigers in a Zoo

$$n(BR) = 7$$
$$n(M) = 5$$
$$n(BR \cup M) = 10$$

We know that,

$$n(BR \cup M) = n(BR) + n(M) - n(BR \cap M)$$

$$10 = 7 + 5 - n(BR \cap M)$$
$$\implies n(BR \cap M) = 2$$

$\implies$ The number of male Bengal Royal tigers is 2

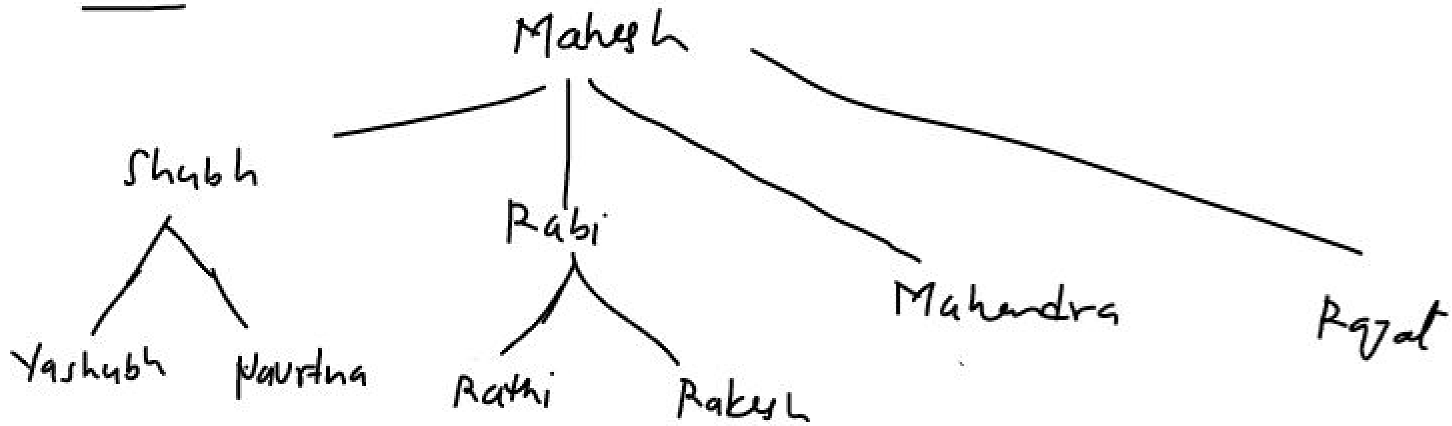
$\implies$ The number of male Bengal White tigers is $5 - 2 = 3$

$\therefore$ Out of 6 Bengal White tigers, 3 are male.

$\implies$ The number of female Bengal White tigers is $6 - 3 = 3$.

Question 8 (This question is wrong and will not be considered for marks)

Solu:



$R = \{(A,B) \mid A$ and B are cousins$\}$

so $R = \{$ (Yashubh, Rathi), (Yashubh, Rakesh), (Navrtha, Rathi), (Navrtha, Rakesh), (Rathi, Yashubh), (Rakesh, Yashubh), (Rathi, Navrtha), (Rakesh, Navrtha) $\}$

and $S = \{(A,B) \mid A$ is son of $B\}$

$S = \{$ (Shubh, Mahesh), (Rabi Mahesh), (Mahendra, Mahesh) (Rajat Mahesh), (Yashubh, Shubh), (Navrtha, Shubh), (Rathi, Rabi), (Rakesh, Rabi) $\}$

So from the elements of R and S, Every option can clearly decided that option is correct or not..

Q9: From solution of Q8, cardinality of $R(m)$ is 8 and of $S(n)$ is also

i.e $m = 8$, $n = 8$

so $m + n = 16$.

Q10: $f = \{(A, B) \mid A \text{ is son of } B\}$

$\subset P \times Q$ where $P, Q \subset M$.

Option 1: Observe, $f : P \to Q$, in set P, Mahesh $\in P$ does not have image as in Q. So f is not a function.

Option 2: Observe, $f : P \to Q$

$f = \{$(Yashubh, Shubh), (Navrtna, Shubh)

(Rathi, Rabi), (Rakesh, Rabi)} is a function but Yashubh and Navrtha has same image so f is not one-one.

Option-3: Observe $f: P \to Q$

$f = \{$ (Yashubh, shubh), (Navrtha, shubh) (Rathi, Rabi), (Rakesh, Rabi) $\}$

So f is a function and every element of codomain has preimage so f is onto.

Option 4: Observe $f: P \to Q$

$f = \{$ (Yashubh, shubh), (Rathi, Rabi) $\}$

So f is a function and for every element in domain has unique image and every element of codomain has preimage.

So f is bijective.

Practice Assignment Solutions

Mathematics for Data Science - 1

NOTE: There are some questions which have functions with discrete valued domains (such as month or year). For simplicity, we treat them as continuous functions.

1 Multiple Choice Questions (MCQ):

1. If R is the set of all points which are 5 units away from the origin and are on the axes then R is:

   - ◯ $R = \{(5,5), (-5,5), (-5,-5), (5,-5)\}$
   - ◯ $R = \{(5,0),\ (5,-5),\ (5,5),\ (-5,0)\}$
   - ◯ $R = \{(5,0),\ (0,5),\ (5,5),\ (0,-5)\}$
   - ◯ $\boldsymbol{R = \{(5,0),\ (0,5),\ (-5,0),\ (0,-5)\}}$
   - ◯ $R = \{(5,0),\ (0,5),\ (-5,0),\ (-5,5)\}$
   - ◯ There is no such set.

   Solution:

   The points on the $x-$axis are represented by $(\pm a,\ 0)$, and on the $y-$axis are represented by $(0,\ \pm b)$, where a and b are the distances of the points $(\pm a,\ 0)$ and $(0,\ \pm b)$, respectively, from the origin. Therefore, the points (5, 0), (0, 5), (-5, 0), (0, -5) lie on the axes and are 5 units away from the origin. See Figure PS-2.1 for reference.

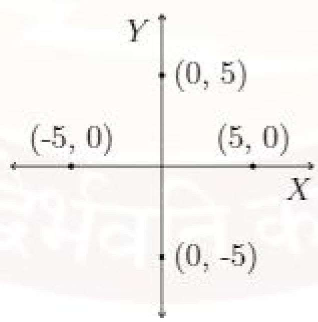


Figure PS-2.1

2. A point P divides the line segment MN such that $MP : PN = 2 : 1$. The coordinates of M and N are $(-2, 2)$ and $(1, -1)$ respectively. What will be the slope of the line passing through P and the point $(1, 1)$?

   ◯ $\frac{4}{3}$

   ◯ **1**

   ◯ Inadequate information.

   ◯ $-\frac{4}{3}$

   ◯ $\tan(\frac{4}{3})$

   ◯ None of the above.

Solution:

By the sectional formula, the coordinates of a point (x, y) that divides a line segment defined by two points $(x_1, y_1), (x_2, y_2)$ in the ratio $m : n$ is given by

$$x = \frac{m \times x_2 + n \times x_1}{m + n}$$

$$y = \frac{m \times y_2 + n \times y_1}{m + n}$$

Since point P divides the line segment formed by the points $M(-2, 2)$ and $N(1, -1)$ in the ratio 2:1, we obtain the coordinates of point P denoted by, say (x_p, y_p), using the sectional formula as follows.

$$x_p = \frac{2 \times 1 + 1 \times (-2)}{2 + 1} = 0$$

$$y_p = \frac{2 \times (-1) + 1 \times 2}{2 + 1} = 0$$

Hence point $P = (0, 0)$ denotes the origin as shown in Figure PS-2.2

Now, we compute the slope of the line passing through P and (1,1) as,

$$\text{Slope} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{1 - 0}{1 - 0} = 1$$

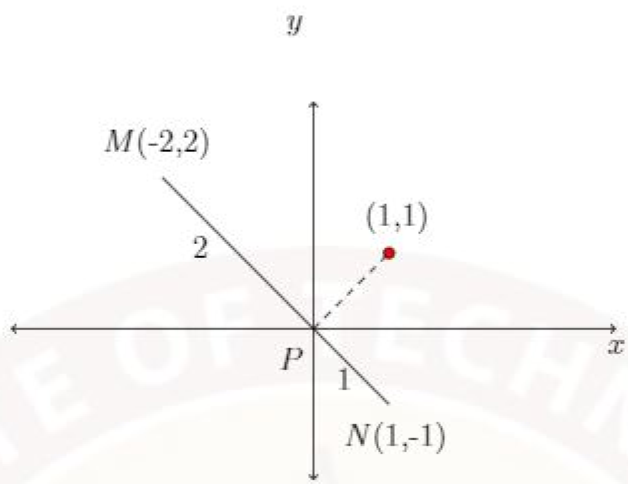


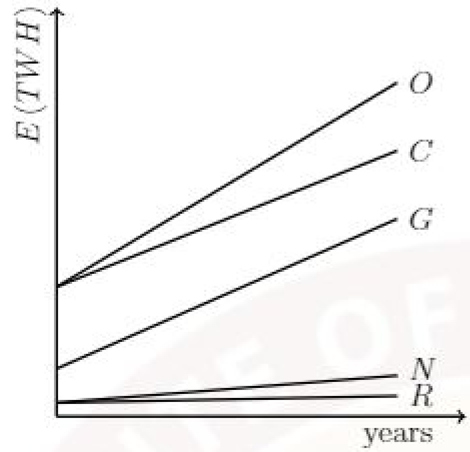
Figure PS-2.2

Use the following information to solve questions 3 and 4.
Table PS-2.1 shows the different types of energies consumed (approximate values) in years 1965 and 2015 across the world

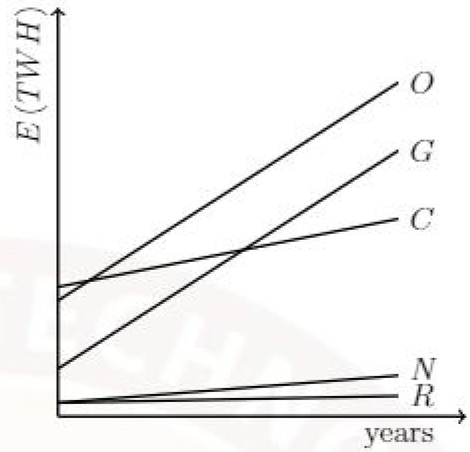
Energy type	Approximate energy used (TWH)	
	1965	2015
Oil (O)	19000	49000
Coal (C)	17000	38000
Gas (G)	7000	29000
Nuclear (N)	2000	6000
Renewable (R)	2000	3000

Table PS-2.1

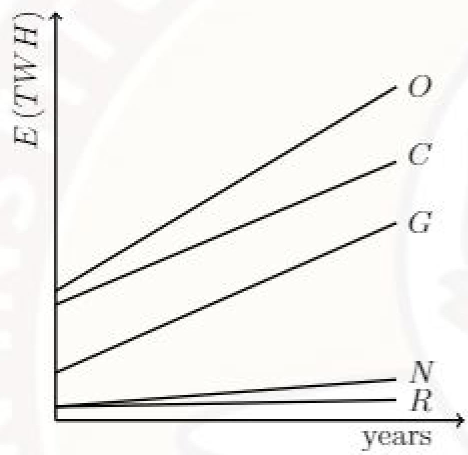
3. A student assumes a linear relationship between energy consumed (E) and the number of years after 1965. Choose the option which best represents the linear relationships assumed by the student (from 1965 to 2015). [Ans: Option C]



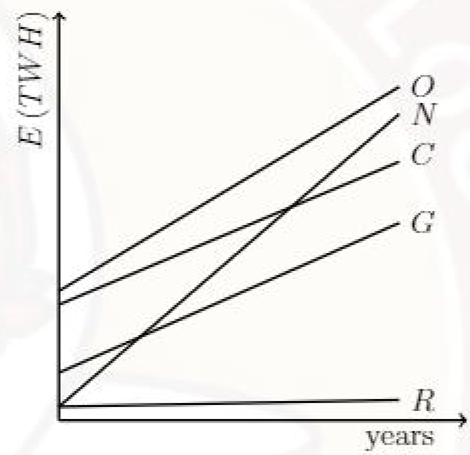
A.



B.



C.



D.

Solution:

Let x-axis and y-axis represent the years and the energy consumption respectively. The energy consumption in 2015 is in the order $O > C > G > N > R$, which is represented correctly in options (A) and (C). However, option (A) shows the energy consumption of O and C being same in the year 1965, which is not true. Hence, option (A) is not correct. Therefore, the correct answer is option (C).

4. The student estimated the energy consumption in 2025 and created Table PS-2.2. Choose the correct option.

Energy type	Approximate energy used (TWH)		
	1965	2015	2025
Oil (O)	19000	49000	o
Coal (C)	17000	38000	c
Gas (G)	7000	29000	g
Nuclear (N)	2000	6000	n
Renewable (R)	2000	3000	r

Table PS-2.2

- ◯ $o = 64000$
- ◯ $c = 48500$
- ◯ $g = 38500$
- ◯ $n = 8000$
- ◯ $r = 3500$
- ◯ **None of the above**.

Solution:

As earlier, let x-axis and y-axis represent the years and the energy consumption respectively. Using the data provided for two years, we can find the equation of the line in two-point form. Equation for the energy type *oil* (O) will be

$$(y - y_1) = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

$$\implies y - 19000 = \frac{49000 - 19000}{2015 - 1965}(x - 1965)$$

On solving the above equation with $x = 2025$,

$$\implies y - 19000 = \frac{49000 - 19000}{2015 - 1965}(2025 - 1965)$$

$$y = 55000$$

Equation for the energy type *coal* (C) will be

$$(y - y_1) = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

$$\implies y - 17000 = \frac{38000 - 17000}{2015 - 1965}(x - 1965)$$

On solving the above equation with $x = 2025$:

$$\implies y - 17000 = \frac{38000 - 17000}{2015 - 1965}(2025 - 1965)$$

$$y = 42200$$

Equation for the energy type *gas* (G) will be

$$(y - y_1) = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

$$\implies y - 7000 = \frac{29000 - 7000}{2015 - 1965}(x - 1965)$$

On solving the above equation with $x = 2025$,

$$\implies y - 7000 = \frac{29000 - 7000}{2015 - 1965}(2025 - 1965)$$

$$y = 33400$$

Equation for the energy type *nuclear* (N) will be

$$(y - y_1) = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

$$\implies y - 2000 = \frac{6000 - 2000}{2015 - 1965}(x - 1965)$$

On solving the above equation with $x = 2025$,

$$\implies y - 2000 = \frac{6000 - 2000}{2015 - 1965}(2025 - 1965)$$

$$y = 6800$$

Equation for the energy type *renewable* (R) will be:

$$(y - y_1) = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

$$\implies y - 2000 = \frac{3000 - 2000}{2015 - 1965}(x - 1965)$$

On solving the above equation with $x = 2025$:

$$\implies y - 2000 = \frac{3000 - 2000}{2015 - 1965}(2025 - 1965)$$

$$y = 3200$$

Thus, none of the options given is correct.

2 Multiple Select Questions (MSQ):

5. The elements of a relation R are shown as points in the graph in Figure P-2.3. Choose the set of correct options:

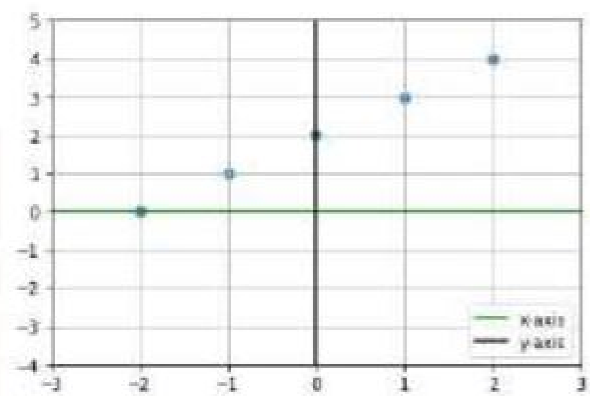


Figure PS-2.3

- ◯ **R can be represented as $R = \{(-2, 0), (-1, 1), (0, 2), (1, 3), (2, 4)\}$.**
- ◯ We can write R as $R = \{(a, b) | (a, b) \in X \times Y, b = a + 2\}$, where X is the set of all values on the $x - axis$, and Y is the set of all values on the $y - axis$.
- ◯ R cannot be a function because it is a finite set.
- ◯ R is a symmetric relation.
- ◯ **R is a function because it has only one output for one input.**
- ◯ If R is a function then it is a bijective function on $X \times Y$, where X is the set of all values on the $x - axis$, and Y is the set of all values on the $y - axis$.
- ◯ **We can write R as $R = \{(a, b) | (a, b) \in X \times Y, b = a + 2\}$, where $X = \{-2, -1, 0, 1, 2\}$ and $Y = \{0, 1, 2, 3, 4\}$.**

Solution:

- Option (a) is correct since the coordinates of the points in the Figure P-2.3 are as is defined by the function.
- Option (b) is incorrect. We can write R as $\{R = (a, b) \mid (a, b)\ in\ X \times Y,\ b = a + 2\}$, where $X = \{-2, -1, 0, 1, 2\}$ and $Y = \{0, 1, 2, 3, 4\}$. Here R is a finite set so we can not write for all values of x-axis or y-axis.
- Option (c) is incorrect since R can be a function of a finite set.
- Option (d) is incorrect since R is not a symmetric relation. For example, corresponding to the element (-2, 0), there is no element (0, -2) in R.

- Option (e) is correct since for every value of X there is single corresponding value in Y.
- Option (f) is incorrect since R as a function is not defined for all values on the x-axis, and Y is not the set of all values on the y-axis, whereas $X = \{-2, -1, 0, 1, 2\}$ and $Y = \{0, 1, 2, 3, 4\}$.
- Option (g) is correct, and explained in accordance with definition of function.

6. Find the values of a for which the triangle ΔABC is an isosceles triangle, where A, B, and C have the coordinates $(-1,1)$, $(1,3)$, and $(3,a)$ respectively.

- ◯ **If $\boldsymbol{AB = BC}$, then $\boldsymbol{a = 1}$.**
- ◯ If $AB = BC$, then $a = -1$ or -5.
- ◯ **If $\boldsymbol{BC = CA}$, then $\boldsymbol{a = -1}$.**
- ◯ If $BC = CA$, then $a = 1$.

Solution:

As we know, for an isosceles triangle two of its sides are equal. According to the question the vertices of C is $(3,a)$ therefore, depending on the value of a we can have length of $AB = BC$ or $BC = CA$

Since the vertices of triangle are given, we can find the length of each side using distance formula.

Value of a when length of $AB = BC$:

Length of any side of triangle is given by

$$\sqrt{(y_2 - y_1)^2 + (x_2 - x_1)^2}$$

$$\implies \sqrt{(3-1)^2 + (1-(-1))^2} = \sqrt{(a-3)^2 + (3-1)^2}$$

$$\implies \sqrt{8} = \sqrt{4 + (a-3)^2}$$

Squaring them on both sides, we have

$$\implies (a-3)^2 = 4 \implies a - 3 = \pm 2$$

Therefore,

$$a = 5,\ 1$$

But, if $a = 5$ then the three points will be co-linear therefore,

$$\boldsymbol{a = 1}$$

Value of a when length of $BC = CA$:

$$\sqrt{(a-3)^2 + (3-1)^2} = \sqrt{(a-1)^2 + (3-(-1))^2}$$

$$\implies \sqrt{4 + (a-3)^2} = \sqrt{16 + (a-1)^2}$$

Squaring on both sides of the equation, we get

$$\implies 4 + (a-3)^2 = 16 + (a-1)^2 \implies (a-3)^2 - (a-1)^2 = 12$$

$$\implies (2a-4)(-2) = \implies 12 \implies a = -1$$

Therefore,

$$\boldsymbol{a = -1}$$

7. A plane begins to land when it is at a height of 1500 metre above the ground. It follows a straight line path and lands at a point which is at a horizontal distance of 2700 metre away. There are two towers which are at horizontal distances of 900 metre and 1800 metre away in the same direction as the landing point. Choose the correct option(s) regarding the plane's trajectory and safe landing.

   - ◯ The trajectory of the path could be $\frac{y}{27} + \frac{x}{15} = 100$ if $x - axis$ and $y - axis$ are horizontal and vertical respectively.
   - ◯ The maximum safe height of the towers are 1000 metre and 1500 metre respectively.
   - ◯ **The trajectory of the path could be $\frac{y}{15} + \frac{x}{27} = 100$ if $x - axis$ and $y - axis$ are horizontal and vertical respectively.**
   - ◯ The maximum safe height of the towers are 1500 metre and 500 metre respectively.
   - ◯ **The maximum safe height of the towers are 1000 metre and 500 metre respectively.**
   - ◯ None of the above.

Solution:

Let us consider the height of plane from ground as $y-$axis and horizontal distance on ground as $x-$axis as shown in Figure PS-2.4
Then, the point P(0,1500) represents the position of the airplane when it began its descent and point Q(2700,0) represents the point where the plane landed.
The two towers which are 900m and 1800m away from the $y-$axis are represented by A and B respectively.
The equation of a straight line path traced by plane from $P(0, 1500)$ to $Q(2700, 0)$ can be obtained using the intercept-form.

$$\frac{x}{a} + \frac{y}{b} = 1$$

$$\implies \frac{x}{2700} + \frac{y}{1500} = 1$$

On rearranging:

$$\frac{y}{15} + \frac{x}{27} = 100$$

Now, to check the maximum safe height of towers:
For tower A at $X-$coordinate $= 900m$, the maximum safe height will be:

$$\frac{y}{15} + \frac{900}{27} = 100$$

$$\implies y = 1000m$$

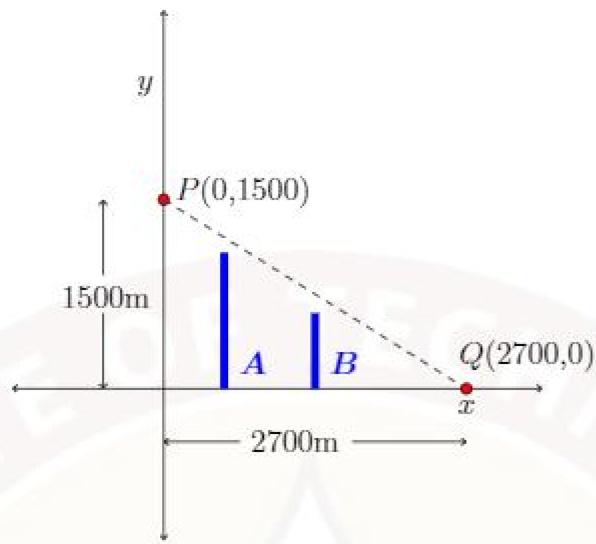


Figure PS-2.4

For tower B at $X-$coordinate $= 1800m$, the maximum safe height will be:

$$\frac{y}{15} + \frac{1800}{27} \;=\; 100$$
$$\implies y = 500m$$

3 Numerical Answer Type (NAT):

Use the following information to solve the question 1-2.
The coordinates of points A, B, C and E are shown in the figure PS-2.5 below. Points D and F are the midpoints of lines BC and AD respectively. Using the data given and Figure PS-2.5, answer the questions below.

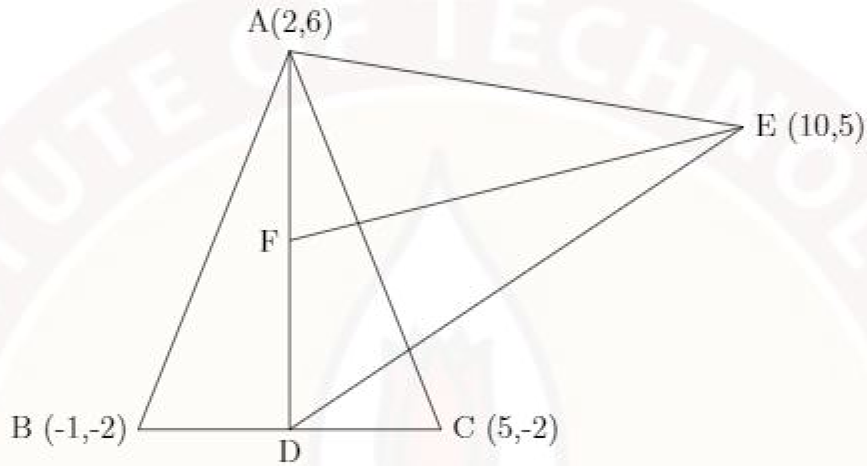


Figure PS-2.5

8. Find the area of triangle ADE. [Ans: 32]

Solution:

By the sectional formula, the coordinates of a point (x, y) that divides a line segment defined by two points $(x_1, y_1), (x_2, y_2)$ in the ratio $m : n$ is given by

$$x = \frac{m \times x_2 + n \times x_1}{m + n}$$

$$y = \frac{m \times y_2 + n \times y_1}{m + n}$$

Since point D is the midpoint of the line segment BC formed by the points $B(-1, -2)$ and $C(5, -2)$ so they are in the ratio 1:1. Thus, we can obtain the coordinates of the point D denoted by, say (x_d, y_d), using the sectional formula as follows.

$$x_d = \frac{1 \times 5 + 1 \times (-1)}{1 + 1} = 2$$

$$y_d = \frac{1 \times (-2) + 1 \times (-2)}{1 + 1} = -2$$

Therefore,

$$\implies D(2, -2)$$

Now, area of triangle ADE with vertices $A(2, 6)$, $D(2, -2)$ and $E(10, 5)$ can be obtained as:

$$= \frac{1}{2}|\ x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\ |$$

$$= \frac{1}{2}|\ 2(-2-5) + 2(5-6) + 10(6-(-2))\ |$$

$$= 32$$

9. Let the slope of a line F G be 2 and the coordinate of the point G be (a, 9). Then, what is the value of a? [Ans: 5.5]

Solution:

As seen earlier, the point F is the midpoint of the line segment AD formed by the points $A(2, 6)$ and $D(2, -2)$ so they are in the ratio 1:1. Thus we can obtain the coordinates of the point F denoted by, say (x_f, y_f), using the sectional formula as follows.

$$x_f = \frac{1 \times 2 + 1 \times 2}{1+1} = 2$$

$$y_f = \frac{1 \times (-2) + 1 \times 6}{1+1} = 2$$

Therefore,

$$\implies F(2, 2)$$

Now, the slope of FG will be $= \frac{9-2}{a-2} = 2$

On solving the above equation, we get $\boldsymbol{a} = \mathbf{5.5}$

10. Leo rents a motorcycle for 2 days. Hence, the rental company provides the motorcycle at Rs. 500 per day with 100 km free per day. The additional charges after 100 km are Rs. 2 per km Leo drives the motorcycle for a total of 500 km. How much (Rs.) will he have to pay to the rental company? [Ans: 1600]

Solution:

Leo has rented a motorcycle for 2 days, thus he has to pay Rs. 1,000 for free 200 km ride. Thereafter, he has to pay Rs. 2 per km. for rest of 300km, which accounts for Rs. 600. Thus, in total he has to pay Rs. 1,600.

Multiple Choice Questions (MCQ):

11. A vehicle is travelling on a straight line path and it passes through the points A(4, 2), $B(-1, 3)$, and $C(2, \mu)$. The value of μ is:

   - ○ 2
   - ○ 4
   - ○ -2
   - ○ 10

Solution:

Since the vehicle is travelling on a straight line path and passes through the points A, B, and C, it follows that A, B, and C are collinear. Hence the slope of the straight line path joining A and B will be equal to the slope of the straight line path joining B and C. Using the slope formula for two points, we have

$$\frac{3-2}{-1+4} = \frac{\mu - 3}{2+1}$$

$$\implies \boldsymbol{\mu = 4}.$$

Suppose two boats are starting their journey from the ferry ghat A (considered as the origin), one towards ferry ghat B along the straight line $y = -2x$ and the other towards the ferry ghat C along a straight line perpendicular to the path followed by B. The river is 1 km wide uniformly and parallel to the X-axis. Suppose Rahul wants to go to the exact opposite point of A along the river.

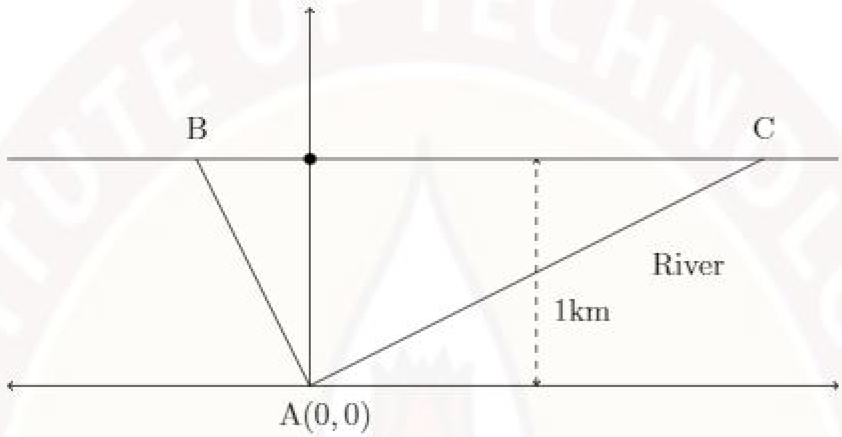


Figure PS-3.1

Then, answer the following questions.

12. How much total distance does Rahul have to travel to reach his destination if he takes the boat towards ferry ghat B?

- ○ $\sqrt{5}$
- ○ $\sqrt{5}+2$
- ○ $\frac{\sqrt{5}}{2}$
- ○ $\frac{\sqrt{5}+1}{2}$

Solution:

See the Figure PS-3.2 for reference:

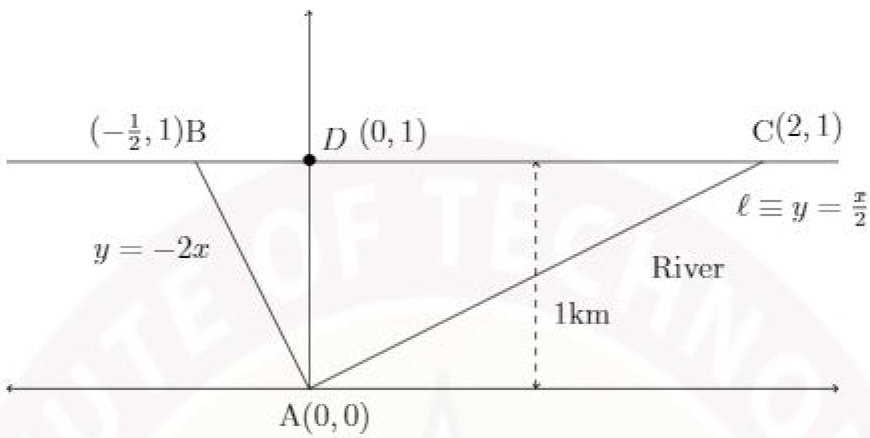


Figure PS-3.2

Since the point A is assumed to be the origin, the side of the river from which Rahul is starting his journey is considered to be the X-axis. The path towards Rahul's destination, which is perpendicular to the X-axis, is hence the Y-axis. Let D be Rahul's destination, which is 1 km away from the point A and is on the opposite side of the river. It follows that the point D is $(0,1)$.

Hence, the equation of the line representing the opposite side of the river is $y = 1$. Solution of the equations $y = 1$ and $y = -2x$ gives the location of ferry ghat B which is the point $(-\frac{1}{2}, 1)$.

Using the distance formula between two points, the distance between ferry ghat A and ferry ghat B is given by

$$\sqrt{(-\frac{1}{2} - 0)^2 + (1 - 0)^2} = \frac{\sqrt{5}}{2}\text{units}$$

Similarly, the distance between ferry ghat B and the point D is $\frac{1}{2}$ units.
Hence, the total distance that Rahul has to travel to reach his destination D if he takes the boat toward ferry ghat B is given by

$$\frac{\sqrt{5}}{2} + \frac{1}{2} = \frac{\sqrt{5}+1}{2} \text{ units}$$

13. How much total distance does Rahul have to travel to reach his destination if he takes the boat towards ferry ghat C?

- ◯ $\sqrt{5}$
- ◯ $\boldsymbol{\sqrt{5}+2}$
- ◯ $\frac{\sqrt{5}}{2}$
- ◯ $\frac{\sqrt{5}+1}{2}$

Solution: Let ℓ denote the path towards ferry ghat C from A. The equation of path ℓ will be $y = mx$ since it passes through the origin. Since ℓ is perpendicular to the line $y = -2x$, which has a slope $m_1 = -2$, it follows that

$$m = -\frac{1}{m_1} = \frac{1}{2}$$

$\implies$ the equation of ℓ is $y = \frac{x}{2}$.

Solution of the equations $y = \frac{x}{2}$ and $y = 1$ gives the location of ferry ghat C which is (2,1).

Using the distance formula between two points, the distance between ferry ghat A and ferry ghat C is

$$\sqrt{(2-0)^2 + (1-0)^2} = \sqrt{5} \text{ units}$$

Similarly, the distance between ferry ghat C and the destination point D is 2 units.

Hence, the total distance that Rahul has to travel to reach his destination D if he takes the boat towards ferry ghat C is $\sqrt{5} + 2$ units.

14. Suppose a bird is flying along the straight line 4x $- 5y = 20$ on the plane formed by the path of the flying bird and the line of eye point view of a person who shoots an arrow which passes through the origin and the point $(10, 8)$. What is the point on the co-ordinate plane where the arrow hits the bird?

- ○ $(20, 12)$
- ○ $(25, 16)$
- ○ **The arrow will miss the bird.**
- ○ Inadequate information.

Solution:
Using the two point form of a line, the equation of the path of arrow passing through the origin and the point (10,8) is

$$(y-0) = \frac{8-0}{10-0}(x-0) \implies 8x - 10y = 0$$

The slope intercept form of the above line is given by

$$y = \frac{8}{10}x$$

From the above line, we obtain the slope as

$$m_1 = \frac{8}{10} = \frac{4}{5}$$

Similarly, for the path of the bird along the straight line $4x - 5y = 20$, we get the slope

$$m_2 = \frac{4}{5}$$

Here, $m_1 = m_2$,
That is, the lines $8x - 10y = 0$ and $4x - 5y = 20$ have the same slope. Therefore, the path of flying bird and the path of the arrow are parallel to each other. Hence, the arrow will miss the bird.

15. We plot the displacement (S) versus time (t) for different velocities as it follows the equation $S = vt$, where v is the velocity. Identify the best possible straight lines in the Figure P-3.2 for the given set of velocities.

Table PS-3.1

v_1	v_2	v_3	v_4
1	-2	0.5	-1

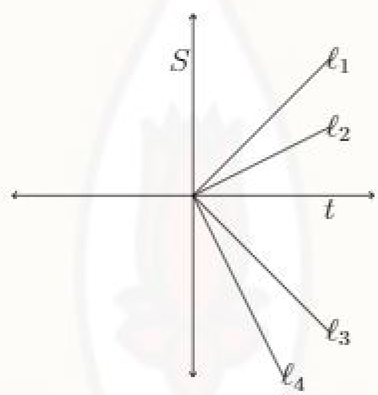


Figure PS-3.3

- ○ $v_1 \to \ell_1$, $v_2 \to \ell_2$, $v_3 \to \ell_3$, and $v_4 \to \ell_4$.
- ○ $v_1 \to \ell_1$, $v_2 \to \ell_4$, $v_3 \to \ell_3$, and $v_4 \to \ell_2$.
- ○ $\boldsymbol{v_1 \to \ell_1, v_2 \to \ell_4}$, $\boldsymbol{v_3 \to \ell_2}$, **and** $\boldsymbol{v_4 \to \ell_3}$.
- ○ $v_1 \to \ell_2$, $v_2 \to \ell_4$, $v_3 \to \ell_1$, and $v_4 \to \ell_3$.

Solution:

From Figure PS-3.3, ℓ_1 and ℓ_2 have positive slope and the slope of ℓ_1 is greater than the slope of ℓ_2. Similarly the slopes of ℓ_3 and ℓ_4 are negative and the slope of line ℓ_3 is greater than the slope of line ℓ_4.

Substituting the value of v in equation $s = vt$, we get the equations

$$s = t, s = -2t, s = 0.5t, s = -t$$

By comparing the above equations of lines and the lines in Figure PS-3.3, we conclude that v_1 corresponds to the line ℓ_1, v_2 corresponds to the line ℓ_4, v_3 corresponds to the line ℓ_2, and v_4 corresponds to the line ℓ_3.

2 Multiple Select Questions (MSQ):

16. A constructor is asked to construct a road which is at a distance of $\sqrt{2}$ km from the municipality office and perpendicular to a road which can be defined by the equation of the straight line $x-y=8$ (considering the municipality office to be the origin). Find out the possible equations of the straight lines to represent the new road to be constructed.

   ○ $x-y-2=0$
   ○ $\boldsymbol{x+y+2=0}$
   ○ $x-y+2=0$
   ○ $\boldsymbol{x+y-2=0}$

Solution:

See the Figure PS-3.4 for reference:

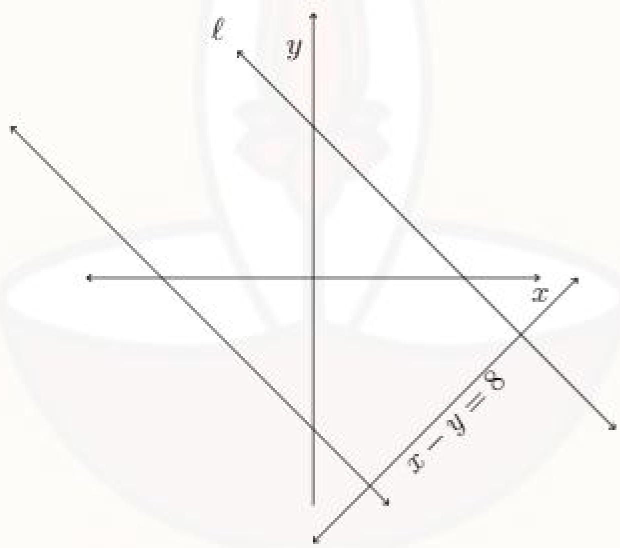


Figure PS-3.4

Let the new road constructed be denoted by ℓ. Given, ℓ is perpendicular to the straight line $x-y=8$. That is, ℓ is perpendicular to the line $y=x-8$ whose slope is $m_1=1$. Therefore, the slope of ℓ is

$$m_2=-\frac{1}{m_1}=-1$$

By the slope intercept form, the equation of ℓ is

$$y=m_2\,x\;+\;c$$

$$\implies y = -x + c \text{ , where } c \text{ is a constant}$$

That is, ℓ is the line given by

$$x + y - c = 0$$

It is given that the distance of ℓ from the municipality office is $\sqrt{2}$.
The distance formula of a point (x_1, y_1) from a line $(Ax + By + C = 0)$ is given by $\frac{|Ax_1+By_1+C|}{\sqrt{A^2+B^2}}$. Substituting $x_1 = 0, y_1 = 0, A = 1, B = 1, C = -c$ in formula, we will get the distance of the point(0,0) from the line ℓ,

$$\frac{|1 \times 0 + 1 \times 0 - c|}{\sqrt{1^2 + 1^2}}$$

which is equal to $\sqrt{2}$. So,

$$\frac{|0 + 0 - c|}{\sqrt{1+1}} = \sqrt{2}$$

$$\implies \frac{|c|}{\sqrt{2}} = \sqrt{2}$$

$$\implies |c| = 2$$

$\implies c = +2$ or $c = -2$.
Hence, the equation of the new road ℓ is

$$x + y + 2 = 0$$

or

$$x + y - 2 = 0.$$

17. Suppose there are two roads perpendicular to each other which are both at the same distance from Priya's house (considered as the origin). The meeting point of the two roads is on the x-axis and at a distance of 5 units from Priya's house.
Choose the correct possible equations representing the roads.

- ◯ Inadequate information.
- ◯ $y = \frac{1}{2}x + 5, y = -2x - 5$
- ◯ $\boldsymbol{y = -x - 5, y = x + 5}$
- ◯ $y = 2x - 10, y = -2x - 10$
- ◯ $y = 2x - 5, y = -\frac{1}{2}x - 5$
- ◯ $\boldsymbol{y = -x + 5, y = x - 5}$
- ◯ $x = 5, x = -5$

Solution:
Denote the two roads by ℓ_1 and ℓ_2. The meeting point of ℓ_1 and ℓ_2 are on the X-axis and at a distance of 5 units from Priya's house (origin) i.e x-intercepts of the roads are 5 or -5 and passing through the points (5,0) or (-5,0) respectively.

Case 1: when x-intercept is 5 and passes through (5,0)
Using intercept form of a line on the axes, the equation of line ℓ_1 is

$$\frac{x}{5} + \frac{y}{b} = 1$$

where b is a constant.

That is, ℓ_1 is

$$bx + 5y - 5b = 0 \tag{1}$$

See Figure PS-3.5 for reference.

The slope of the road ℓ_1 is $m_1 = -\frac{b}{5}$.
Since the road ℓ_2 is perpendicular to ℓ_1, the slope of road ℓ_2 is

$$m_2 = -\frac{1}{m_1} = \frac{5}{b}$$

Using the slope intercept form, the equation of the road ℓ_2 is

$$y = \frac{5}{b}x + c \implies by - 5x - bc = 0 \text{ where } b \text{ and } c \text{ are constant}$$

The roads ℓ_1 and ℓ_2 are at the same distance from Priya's house (origin).
Using distance formula of a line from a point, we get

$$\frac{|-5b|}{\sqrt{b^2+25}} = \frac{|-bc|}{\sqrt{b^2+25}} \implies |c| = |5| \implies c = 5 \quad or - 5$$

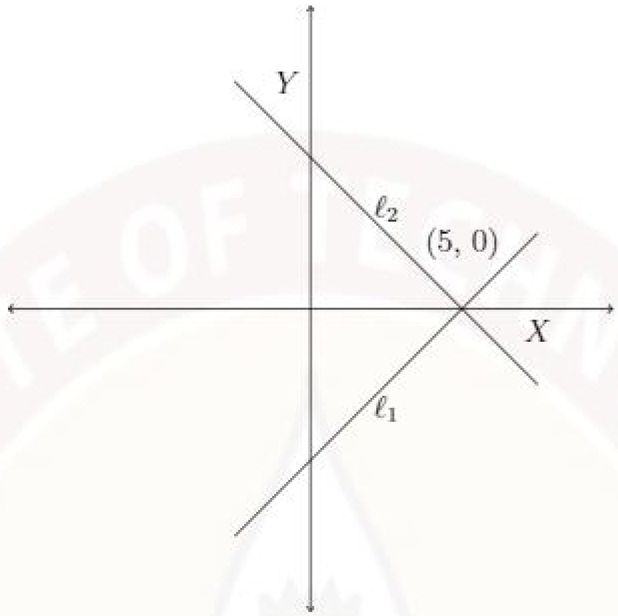


Figure PS-3.5

When $c = 5$, the equation of road ℓ_2 becomes $by - 5x - 5b = 0$. Since ℓ_2 passes through $(5, 0)$, we get $b = -5$.
Therefore, the equation of the road ℓ_2 is $y = -x + 5$.
Substituting $b = -5$ in Equation (1), we will get the equation of the road ℓ_1 as $y = x - 5$.
When $c = -5$, we will get the same equation alternatively.

Case 2: when x-intercept is -5 and passing through (-5,0)
We follow the same process as in Case 1 and we get the equation of the road ℓ_2 as $y = x + 5$ and the equation of the road ℓ_1 as $y = -x - 5$.
See Figure PS-3.6 for reference.

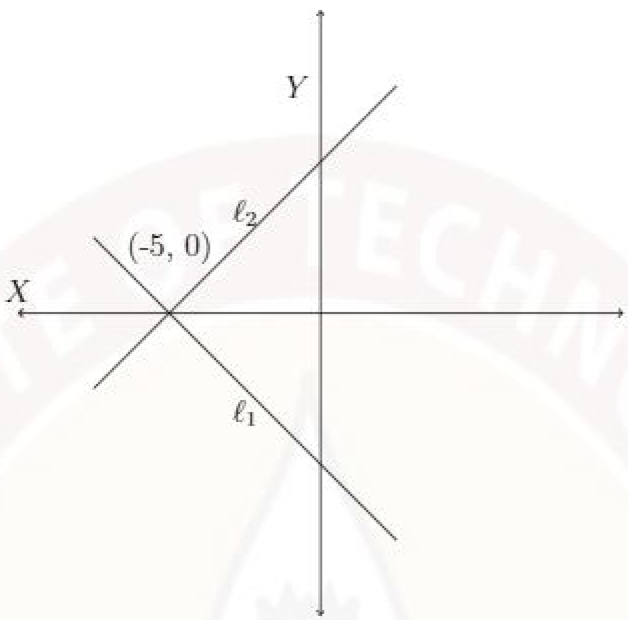


Figure PS-3.6

18. Consider the following two diagrams.

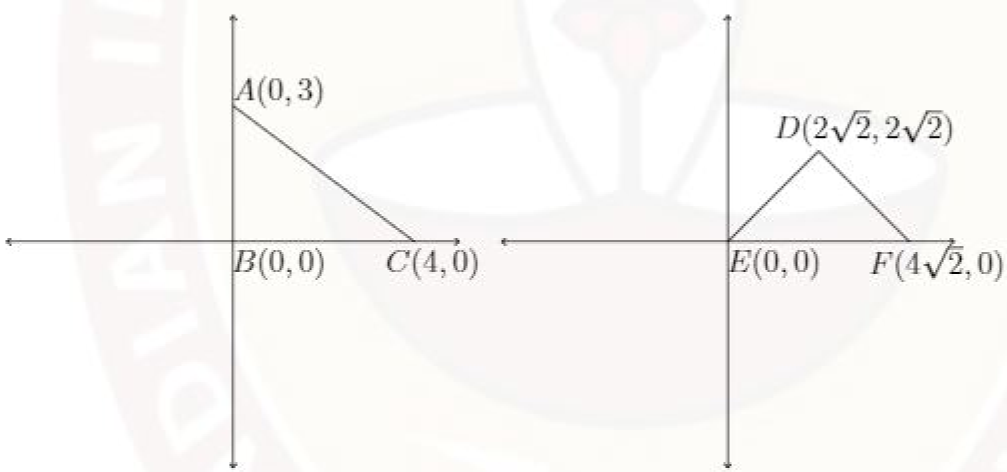


Figure PS-3.7

Which of the following option(s) is(are) true about the triangles ΔABC and ΔDEF given in Figure PS-3.7?

- ◯ Only ΔABC is a right angled triangle while ΔDEF is not.
- ◯ **Both ΔABC and ΔDEF are right angled triangles.**
- ◯ The area of ΔABC is greater than the area of ΔDEF.
- ◯ Both the triangles have the same area.

○ **The area of ΔDEF is 8 sq.unit.**

Solution:

In Figure PS-3.7, vertices A and C are on $Y-$axis and $X-$ axis respectively and the vertex B is at the origin itself.
Therefore, $\triangle ABC$ is a right angle triangle.
The distance formula between two points $(x_1, y_1), (x_2, y_2)$ is given by

$$\sqrt{(y_2 - y_1)^2 + (x_2 - x_1)^2}$$

Using the above formula, in $\triangle DEF$, the length of side DE is

$$\sqrt{(2\sqrt{2} - 0)^2 + (2\sqrt{2} - 0)^2} = \sqrt{(2\sqrt{2})^2 + (2\sqrt{2})^2} = 4$$

Similarly, the length of side DF is

$$\sqrt{(4\sqrt{2} - 2\sqrt{2})^2 + (0 - 2\sqrt{2})^2} = 4$$

The length of side EF is $4\sqrt{2}$. We have

$$DE^2 + DF^2 = 16 + 16 = 32 = (4\sqrt{2})^2 = EF^2$$

Hence, by the Pythagoras theorem, $\triangle DEF$ is also a right angled triangle.
Area of the right angled $\triangle ABC = \frac{1}{2} \times 4 \times 3 = 6$ sq. unit.
Area of the right angled $\triangle DEF = \frac{1}{2} \times 4 \times 4 = 8$ sq. unit.

19. Let the diagonals of a quadrilateral with one vertex at $(0,0)$ bisect each other perpendicularly at the point $(1,2)$. Further, let one of the diagonals be on the straight line $y=2x$. Then, which of the following is (are) correct statements?

- ◯ **The diagonally opposite vertex of $(0,0)$ is $(2,4)$.**
- ◯ The other diagonal is on the straight line $y=-\frac{1}{2}x$.
- ◯ **The other diagonal is on the straight line $y=-\frac{1}{2}x+\frac{5}{2}$.**
- ◯ The diagonally opposite vertex of $(0,0)$ is $(\frac{3}{2},3)$.

Solution:

Figure PS-3.8 shows a sketch of the quadrilateral.

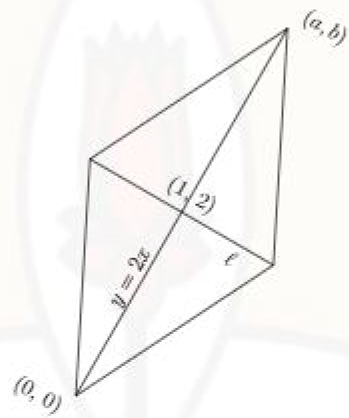


Figure PS-3.8

The diagonal $y=2x$ has slope $m_1=2$.
Let the other diagonal, perpendicular to the line $y=2x$, be on the line ℓ.
So, the slope of the line ℓ is

$$m_2=-\frac{1}{m_1}=-\frac{1}{2}$$

From the slope intercept form, the equation of the line ℓ is $y=-\frac{x}{2}+c$, where c is a constant.
Since both the diagonals intersect at the point (1,2) and one diagonal is on line ℓ, the point (1,2) belongs to ℓ and hence $c=\frac{5}{2}$.
Hence, the equation of the line ℓ is $\boldsymbol{y=-\frac{1}{2}x+\frac{5}{2}}$.

Let the opposite vertex of $(0,0)$ be (a,b).
Since the point (1, 2) is the bisection point of the both diagonals, it follows that the point $(1,2)$ is mid-point of the line segment joining the points $(0,0)$ and (a,b).
Using the section formula of a line segment,

$$\frac{a}{2} = 1 \implies a = 2$$

$$\frac{b}{2} = 2 \implies b = 4$$

Hence the diagonally opposite vertex of $(0,0)$ is $(2,4)$.

A woman is reported missing in a locality. The police department finds a human femur bone during their investigation. They estimate the height H of a female adult (in cm) using the relationship $H = 1.8f + 70$, where f is the length (in cm) of the femur bone. The length of the femur found is 35 cm, and the missing woman is known to be 130 cm tall. In the particular locality, maximum height of a female is 195 cm and the minimum length of a female femur bone is 15 cm. Based on the given data answer the following questions.

20. Choose the set of correct options.

- ○ If an error of 1 cm is allowed, bone could belong to missing female.
- ○ **If an error of 3 cm is allowed, bone could belong to missing female.**
- ○ If the height as a function of femur length is known to be accurate, the range of the function is [70, 195].
- ○ **If the height as a function of femur length is known to be accurate, the range of the function is [97, 195].**
- ○ **If the height as a function of femur length is known to be accurate, the domain of the function is $[15, \frac{625}{9}]$.**

Solution:

The relationship between height of a woman H and the length of her femur bone f is given by

$$H = 1.8f + 70. \tag{2}$$

Since the length of the femur bone found during the investigation is 35 cm, we have

$$H = 1.8 \times 35 + 70 = 133 \text{ cm}$$

The height of missing woman is known to be 130cm. Since $133 - 130 = 3 \leq 3$ and by our assumption, an error of 3 cm is allowed, it is possible that the femur bone found during the investigation belongs to the missing woman.

Given that the maximum height of a female in that location is 195 cm.
Substituting $H = 195$ in Equation (2) , we get the maximum length of female femur bone in that location i.e maximum $f = \frac{625}{9}$ cm.
Since the minimum length of of femur bone known in that location is 15 cm and if height as a function of femur length is known to be accurate then the domain of the function is $[15, \frac{625}{9}]$.

Given that the minimum length of the female femur bone in that location is 15 cm.
The minimum height of a female in that location is $H = 1.8 \times 15 + 70 = 97$ cm.
Since the maximum height of a female in that location is 195 cm, the range of the height function is $[97, 195]$

21. A new detective agency came up with a relationship H = mf + 70, where H is the height of a male adult (in cm) and f is the length (in cm) of the femur bone. They have used the following sample set given below in the Table P-3.2 , such that the sum squared error is minimum.

height(H) (in cm)	150	160	170	180
length of femur bone(f) (in cm)	40	42	48	56

Table PS-3.2

Choose the correct option (only one option is correct).

- ◯ $m = 1$
- ◯ $m = 1.5$
- ◯ $\boldsymbol{m = 2}$
- ◯ $m = 2.5$

Solution:

From Table PS-3.3, we can see that the minimum SSE is for $m = 2$.

H (in cm)	f (in cm)	$(H - mf - 70)^2$			
		$m = 1$	$m = 2$	$m = 1.5$	$m = 2.5$
150	40	1600	0	400	400
160	42	2304	36	729	225
170	48	2704	16	784	400
180	56	2916	4	676	900
SSE		$\sum = 9524$	$\sum = 56$	$\sum = 2589$	$\sum = 1925$

Table PS-3.3

3 Numerical Answer Type (NAT):

22. What will be the slopes of the straight lines perpendicular to the following lines?

a) $2x + 5y - 9 = 0$

Answer:2.5

Solution:

Using the slope intercept form, the slope of the line $2x + 5y - 9 = 0$ is $m_1 = -\frac{2}{5}$.
Let the slope of the perpendicular line be m_2. Then

$$m_1.m_2 = -1$$
$$\implies m_2 = \frac{5}{2} = 2.5$$

23. $-5x + 25y + 28 = 0$

Answer:-5

Solution:

Using the slope intercept form, the slope of the line $-5x + 25y + 28 = 0$ is $m_1 = \frac{1}{5}$.
Let the slope of the perpendicular line be m_2, then

$$m_1.m_2 = -1$$
$$\implies m_2 = -5.$$

Week 2

Graded Assignment Solution.

Mathematics for Data Science - 1

1. A ray of light passing through the point $A(1, 2)$ is reflected at a point B on X-axis and then passes through the point $(5, 3)$. Then the equation of straight line AB is (1 marks)

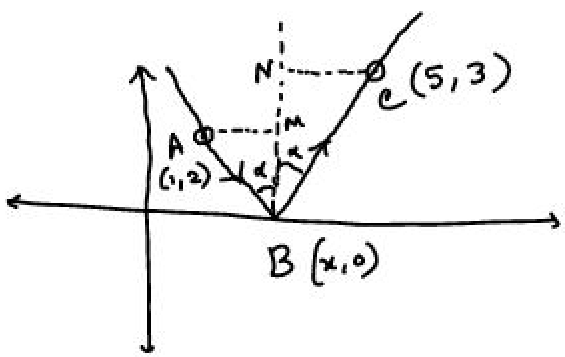
   ◯ **Option 1:** $5x + 4y = 13$

   ◯ Option 2: $5x - 4y = -3$

   ◯ Option 3: $4x + 5y = 14$

   ◯ Option 4: $4x - 5y = -6$

Soln.



$$\tan\alpha = \frac{AM}{BM} = \frac{CN}{BN}$$

$$\Rightarrow \frac{x-1}{2} = \frac{5-x}{3}$$

$$\Rightarrow 3x - 3 = 10 - 2x$$

$$\Rightarrow 5x = 13$$

$$\Rightarrow x = \frac{13}{5}$$

coordinate of B is $\left(\frac{13}{5}, 0\right)$

Hence equation of AB is,

$$\frac{y-0}{2-0} = \frac{x - \frac{13}{5}}{1 - \frac{13}{5}}$$

$$\Rightarrow \frac{y}{2} = \frac{5x - 13}{-8} \Rightarrow y = \frac{5x-13}{-4}$$

$$\Rightarrow -4y = 5x - 13$$

$$\Rightarrow \underline{5x + 4y = 13}$$

2. If p is the length of perpendicular from the origin to the line whose intercept of the axes are a and b, then which of the following options always holds? (1 marks)

- ◯ Option 1: $\frac{1}{p^2} = \frac{1}{a^2} - \frac{1}{b^2}$
- ◯ Option 2: $\frac{1}{p^2} = \frac{1}{(a+b)^2} + \frac{1}{(a-b)^2}$
- ◯ **Option 3:** $\frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2}$
- ◯ Option 4: $\frac{1}{p^2} = (\frac{1}{a} + \frac{1}{b})^2$

Soln. Eqn of the line whose intercept of the axes are a and b is,

$$\frac{x}{a} + \frac{y}{b} = 1$$

$$\Rightarrow \frac{x}{a} + \frac{y}{b} - 1 = 0$$

Length of perp. from (0,0) to the line $\frac{x}{a} + \frac{y}{b} - 1 = 0$ is,

$$\frac{\left|\frac{0}{a} + \frac{0}{b} - 1\right|}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2}}}$$

given that

$$p = \frac{|-1|}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2}}}$$

$$\Rightarrow \sqrt{\frac{1}{a^2} + \frac{1}{b^2}} = \frac{|-1|}{p}$$

$$\Rightarrow \frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{p^2}.$$

3. If p and q are the lengths of the perpendiculars from the origin to the lines $x\ cos\theta - y\ sin\theta = k\ cos2\theta$ and $x\ sec\theta + y\ cosec\theta = k,\ k \neq 0$, repectively. Which of the following options always holds? [Hint: You may have to use the following formulas: $sin2\theta = 2sin\theta\ cos\theta$ and $sin^2\theta + cos^2\theta = 1$] (1 marks)

- ◯ Option 1: $p^2 + q^2 = k^2$
- ◯ **Option 2:** $p^2 + 4q^2 = k^2$
- ◯ Option 3: $p^2 - q^2 = k^2$
- ◯ Option 4: $4p^2 + q^2 = k^2$

Soln.:

$$p = \frac{|0 \cdot \cos\theta - 0 \cdot \sin\theta - k\cos 2\theta|}{\sqrt{\cos^2\theta + \sin^2\theta}}.$$

$$\Rightarrow p^2 = k^2 \cos^2 2\theta.$$

$$q = \frac{|0 \cdot \sec\theta + 0 \cdot \mathrm{cosec}\,\theta - k|}{\sqrt{\sec^2\theta + \mathrm{cosec}^2\theta}}$$

$$\Rightarrow q^2 = \frac{k^2}{\sec^2\theta + \mathrm{cosec}^2\theta}$$

$$\Rightarrow q^2 = \frac{k^2 \cos^2\theta \sin^2\theta}{1}$$

$$\Rightarrow 4q^2 = k^2 \cdot (2\sin\theta\cos\theta)^2 = k^2 \sin^2 2\theta.$$

Hence, $p^2 + 4q^2 = k^2$.

4. A line l is such that its segment between the lines $x - y + 2 = 0$ and $x + y - 1 = 0$ is internally bisected at the point $(1, 1.5)$. What is the equation of the line l? (1 marks)

- ◯ Option 1: $x + 2y = 1$
- ◯ Option 2: $x - 2y = 3$
- ◯ Option 3: $y = 3x$
- ◯ **Option 4:** $x = 1$

Soln. Let l intersects $x - y + 2 = 0$ at $A(x_1, y_1)$ and $x + y - 1 = 0$ at $B(x_2, y_2)$

Hence, $\frac{x_1 + x_2}{2} = 1$ and $\frac{y_1 + y_2}{2} = 1.5$

$\Rightarrow x_1 + x_2 = 2$ and $y_1 + y_2 = 3$

Moreover, $x_1 - y_1 + 2 = 0$
and $x_2 + y_2 - 1 = 0$

$\Rightarrow (x_1 + x_2) - (y_1 - y_2) + 1 = 0$

$\Rightarrow 2 - (y_1 - y_2) + 1 = 0$

$\Rightarrow y_1 - y_2 = 3$

Hence, from $y_1 + y_2 = 3$ and $y_1 - y_2 = 3$, we get $y_1 = 3$ and $y_2 = 0$.

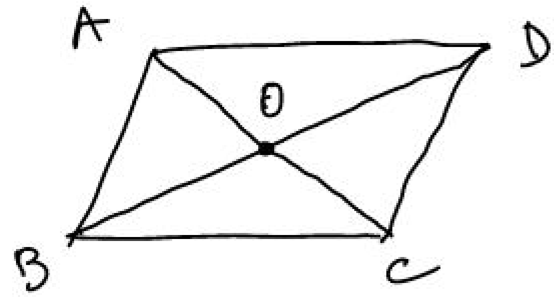
Hence, $x_1 = 1$ and $x_2 = 1$.

Hence eqn. of l is $\underline{x = 1}$

5. Let $ABCD$ be a parallelogram with vertices $A(x_1, y_1)$, $B(x_2, y_2)$, and $C(x_3, y_3)$. Which of the following always denotes the coordinate of the fourth vertex D? (1 marks)

- ◯ Option 1: $(x_1 + x_2 + x_3, y_1 + y_2 + y_3)$
- ◯ **Option 2:** $(x_1 - x_2 + x_3, y_1 - y_2 + y_3)$
- ◯ Option 3: $(x_1 + x_2 - x_3, y_1 + y_2 - y_3)$
- ◯ Option 4: $(x_1 - x_2 - x_3, y_1 - y_2 - y_3)$

Soln:



For a parallelogram ABCD, the diagonals AC and BD intersect each other at the midpoint.

Hence the coordinate of O is,

$$\left(\frac{x_1 + x_3}{2}, \frac{y_1 + y_3}{2}\right)$$

Let the coordinate of D is, (h, k)

then,

$$\frac{x_2 + h}{2} = \frac{x_1 + x_3}{2}$$

and $$\frac{y_2 + k}{2} = \frac{y_1 + y_3}{2}$$

Hence, $h = x_1 - x_2 + x_3$ and $k = y_1 - y_2 + y_3$.

6. A bird is flying along the straight line $2y - 6x = 20$. In the same plane, an aeroplane starts to fly in a straight line and passes through the point $(4, 12)$. Consider the point where aeroplane starts to fly as origin. If the bird and plane collides then enter the answer as 1 and if not then 0.
Note: Bird and aeroplane can be considered to be of negligible size.
(Answer: 0) [Marks: 1]

Soln. The eqn. of st. line along which the aeroplane flies is,

$$\frac{y-0}{12-0} = \frac{x-0}{4-0}$$

$$\Rightarrow \frac{y}{12} = \frac{x}{4} \Rightarrow y = 3x$$

The bird in flying along the st. line

$$2y - 6x = 20$$

$$\Rightarrow y = 3x + 10$$

Both the st. lines have the same slopes.
Hence, they are parallel. So bird and plane donot collide.

7. To determine the gas constant R, two students A and B perform an experiment based on the ideal gas equation given as $Pv = RT$. Both use the same gaseous sample having $v = 16.6\ m^3/mol$ and reported the approximate value of R as $8.3\ J/(Kmol)$ using the minimisation of sum squared error. The data collected by both the students are reported below. Choose the correct options: (1 marks)

$T(K)$	274	276	278	282	290
$P(Pa)$	137	139	142	141	142

Data collected by student A.

$T(K)$	276	280	284	288	290
$P(Pa)$	137	141	142	148	145

Data collected by student B.

- ◯ Option 1: A has better fit than B. ✗
- ◯ **Option 2:** B has better fit than A. ✓
- ◯ Option 3: A and B both have same fit. ✗
- ◯ **Option 4:** SSE calculated by B is 18. ✗
- ◯ Option 5: SSE calculated by A is 14. ✗
- ◯ Option 6: SSE calculated by both A and B is 18. ✗

Soln: $Pv = RT$

$\Rightarrow P(16.6) = (8.3)T \Rightarrow 2P = T$

For A

P	T collected	T original	Error	$(\text{Error})^2$
137	274	274	0	0
139	276	278	2	4
142	278	284	6	36
141	282	282	0	0
142	290	284	−6	36

Sum squared error $= 76$.

For B

P	T (collected)	T original	Error	$(Error)^2$
137	276	274	-2	4
141	280	282	2	4
142	284	284	0	0
148	288	296	8	64
145	290	290	0	0

Sum squared error

= 72

8. A carpenter has a call out fee (basic charges) of ₹100 and also charges ₹90 per hour. Which of the following are true? (1 marks)

 - ◯ Option 1: Following the same notations of y, x, equation of the total cost is represented by $y = 100x + 90$. ✗
 - ◯ **Option 2:** If y is the total cost in (₹) and x is the total number of working hours, then the equation of the total cost is represented by $y = 90x + 100$.
 - ◯ **Option 3:** The total charges, if the carpenter has worked for 4 hours, would be ₹420. ✗
 - ◯ Option 4: If the carpenter charged ₹350 for fixing a L-stand and changing door locks, then the number of working hours would be approximately one hour and 53 minutes. ✗

Soln:

Total cost (y)

$y = 90x + 100$.

9. A line perpendicular to the line segment joining the points $A\,(1,0)$ and $B\,(2,3)$, divides it at C in the ratio of $1:3$. Then the equation of the line is (1 marks)

- ◯ $2x + 6y - 9 = 0$
- ◯ $\mathbf{2x + 6y - 7 = 0}$
- ◯ $2x - 6y - 9 = 0$
- ◯ $2x - 6y + 7 = 0$

Soln:. Slope of the line segment AB is $= \frac{3-0}{2-1}$

$= 3$

Hence Slope of the line perp. to it is $= -\frac{1}{3}$

Let the eqn. of the line be

$$y = -\frac{1}{3}x + c.$$

The coordinate of C is $\left(\frac{1\times 2 + 3\times 1}{1+3}, \frac{1\times 3 + 3\times 0}{1+3}\right)$

$$= \left(\frac{5}{4}, \frac{3}{4}\right)$$

$$y = -\frac{1}{3}x + c$$

$$\Rightarrow \frac{3}{4} = -\frac{1}{3}\left(\frac{5}{4}\right) + c$$

$$\Rightarrow c = \frac{3}{4} + \frac{5}{12} = \frac{9+5}{12} = \frac{14}{12} = \frac{7}{6}$$

$$y = -\frac{1}{3}x + \frac{7}{6} \Rightarrow 6y = -2x + 7$$

$$\Rightarrow 2x + 6y - 7 = 0.$$

10. A rock is thrown in a pond, and creates circular ripples whose radius increases at a rate of 0.2 meter per second. What will be the value of $\frac{A}{\pi}$, where A is the area (in square meter) of the circle after 5 seconds?

<u>Hint:</u> The area of a circle $= \pi r^2$, where r is the radius of the circle. (1 marks)

Soln:

$r = 0.2\,t$ After 5 seconds,

$r = 0.2 \times 5$

Hence after P_1 seconds,

$A = \pi r^2 = \pi(0.04 \times 5^2)$

$\Rightarrow \frac{A}{\pi} = 1$

Mathematics for Data Science - 1

Practice Assignment Solutions

Week-3

1. Multiple Choice Questions (MCQ):

1. What will be the equation of the tangent to the curve $f(x) = 2x^2 + 9x + 20$ at point (-3,11)?

   - ◯ $y = 3x$
   - ◯ $\boldsymbol{y = -3x + 2}$
   - ◯ $y = -3x + 20$
   - ◯ $y = -\frac{x}{3} + 2$
   - ◯ $y = \frac{x}{3} + 20$
   - ◯ $y = -\frac{x}{3}$

   Solution:

   A rough diagram is given in the Figure PS-4.1 .

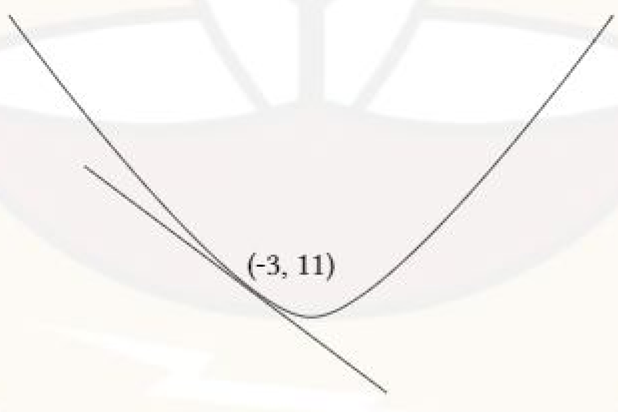


Figure PS-4.1

Let the equation of the tangent be $y = mx + c$, where m is the slope of the tangent line. Note that m is also the slope of f at (-3, 11).
The slope of any quadratic function $g(x) = ax^2 + bx + c$, where $a \neq 0$ at x will be $2ax + b$.

Therefore, at $x = -3$,

$$m = 2ax + b \implies m = 2 \times 2 \times (-3) + 9 \implies m = -3$$

Since the tangent passes through the point (-3, 11), it should satisfy the equation of the tangent.

$$y = mx + c \implies 11 = -3 \times (-3) + c \implies c = 2.$$

So, the equation of the tangent will be $\boldsymbol{y = -3x + 2}$.

2. Find the length of the line segment on the straight line $y = 2$ bounded by the curve $y = 4x^2$.

   - $\frac{1}{\sqrt{2}}$
   - $\boldsymbol{\sqrt{2}}$
   - $1 + \sqrt{2}$
   - $1 + \frac{1}{\sqrt{2}}$

Solution:

Given $y = 4x^2$. Observe that, on comparing the above with the the general form of a quadratic function $f(x) = ax^2 + bx + c$, we have $b = 0$ which means Y-axis is the axis of symmetry. Also $c = 0 \implies$ the curve represented by this function will pass through the origin.
$-b/2a = 0$ and at $x = 0 \implies y = 0$ which means the vertex is at the origin and $a > 0 \implies$ the parabola is upward opened.
$y = 2$ is a constant function and it will pass through the point (0, 2). A rough diagram is given in the Figure PS-4.2

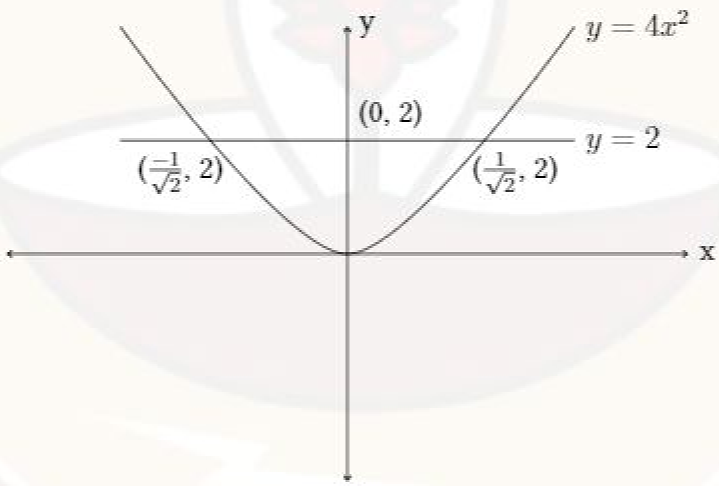
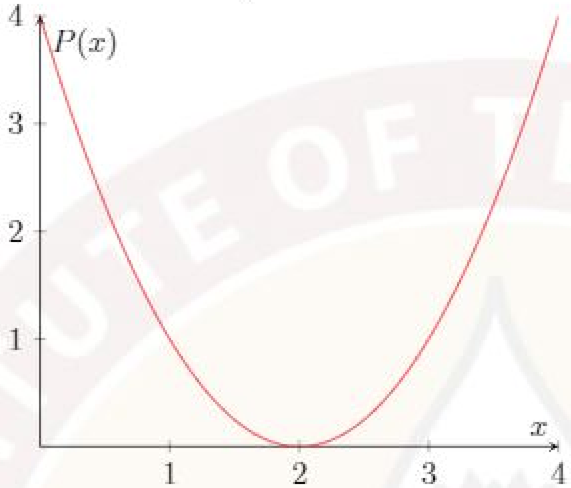
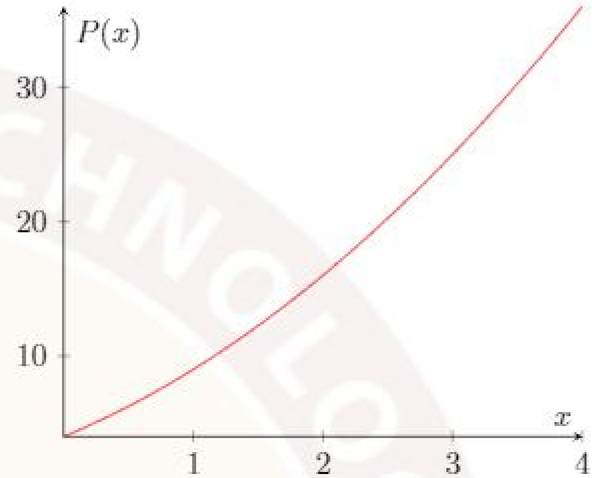


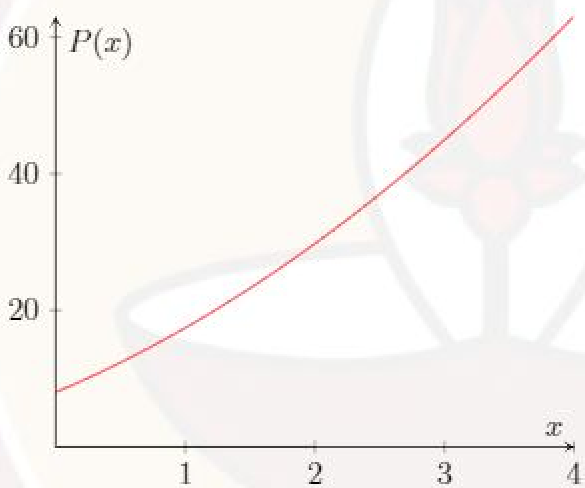
Figure PS-4.2

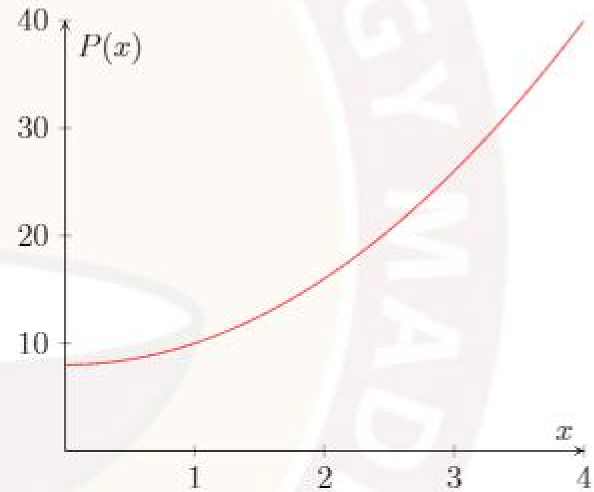
At the intersection points, $4x^2 = 2 \implies x = \pm\frac{1}{\sqrt{2}}$ which means the intersection points will be $(\frac{-1}{\sqrt{2}}, 2)$ and $(\frac{1}{\sqrt{2}}, 2)$.
Observe that these intersecting points will be the end points of the required line segment on the straight line $y = 2$.
Therefore, the length of the line segment on the straight line $y = 2$ bounded by the curve $y = 4x^2$ will be $\sqrt{(2-2)^2 + (\frac{1}{\sqrt{2}} - (\frac{-1}{\sqrt{2}})^2} = \sqrt{0 + (\frac{2}{\sqrt{2}})^2} = \sqrt{0 + (\sqrt{2})^2} = \boldsymbol{\sqrt{2}}$.

3. Mr. Mehta has two sons. Both sons send money to their father each month separately as $M_1(x) = (x-2)^2$ and $M_2(x) = (x+2)^2$ respectively. If x denotes the month, then choose the curve which best represents the total amount (P(x)) received by Mr. Mehta every month.

○ 

○ 

○ 

○ 

Solution:

Given,

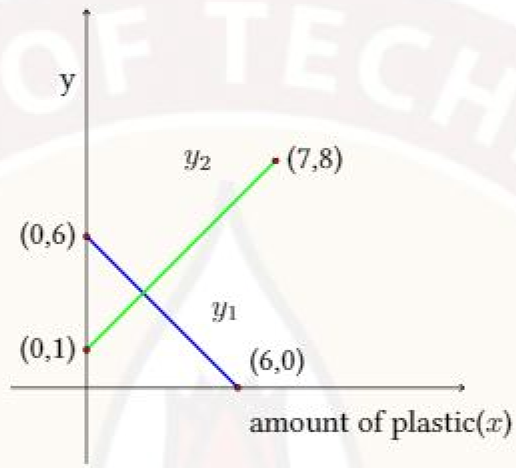
$$M_1(x) = (x-2)^2$$
$$M_2(x) = (x+2)^2.$$

So, the total amount received by Mr. Mehta is:

$$P(x) = M_1(x) + M_2(x) = (x-2)^2 + (x+2)^2 = x^2 - 4x + 4 + x^2 + 4x + 4$$
$$\Rightarrow P(x) = 2x^2 + 8.$$

In $P(x),\ b = 0$ which means Y-axis will be the axis of symmetry of the curve $p(x)$.
Now, the curve shown in the first option is not symmetric about the line $x = 0$. So, option 1 is incorrect.
The curve in the second option, passes through the origin but that is not the case for $P(x)$ as $x = 0 \implies P(x) = 8$. So, option 2 is incorrect.
The curve in the third option, does not pass through (4, 40). So, option 3 is also incorrect.
Now, the curve in the last option will pass through the points (0, 8), (1, 10), and (4,40). So, the curve in the fourth option will be the best curve that represents the total amount received by Mr.Mehta every month.

4. A civil engineer found that the durability d of the road she is laying depends on two functions y_1 and y_2 as follows: $d = ay_1y_2$ where $a > 0$. Functions y_1 and y_2 depend on the amount of plastic (x) mixed in bitumen, and their variations are shown in the graph given below. Find the values of functions y_1 and y_2 such that the durability of the road is maximum.



Solution:

Given, the durability of the road $d = ay_1y_2$.
From the given graph, the equations of the lines:

$$
\begin{aligned}
y_1 &= 6 - x \\
y_2 &= x + 1 \\
\implies d = ay_1y_2 &= a(6-x)(x+1) = -ax^2 + 5ax + 6a
\end{aligned}
$$

Here $a > 0 \implies -a < 0$ which means the curve represented by d is open downward and the durability d of the road is the maximum at $x = \frac{-b}{2a} = \frac{-5a}{2(-a)} = \frac{5}{2}$.
Therefore, the value of $y_1 = 6 - x = 6 - \frac{5}{2} = \frac{7}{2}$ and the value of $y_2 = x + 1 = \frac{5}{2} + 1 = \frac{7}{2}$.

5. Let A be the set of all points on the curve defined by the function $f_1(x) = x^2 - x - 42$ and let B be the set of all points on the curve f_2 defined by the reflection of the curve f_1 with respect to X axis. If C is the set of all points on the axes then choose the correct option regarding the cardinality of set D where $D = (A \cap B) \cup (A \cap C) \cup (B \cap C)$.

   - ○ infinite.
   - ○ 8
   - ○ 4
   - ○ 6
   - ○ 2
   - ○ zero.

Solution:

For the function $f_1(x) = x^2 - x - 42$, $a > 0 \implies$ opening upward, $-\frac{b}{2a} = \frac{1}{2} \implies x = \frac{1}{2}$ is the axis of symmetry.
$x = 0 \implies f_1(0) = -42$ so, it will pass through the point (0, -42).
The reflection of $f_1(x)$ with respect to $X-$ axis i.e. $f_2(x)$ will pass through the point (0, 42).
For intersection points of both curves:
Both the curves will be intersecting on same place on $X-$ axis as they are mirror image of each other around $X-$ axis. A rough diagram is given in the Figure PS-4.3

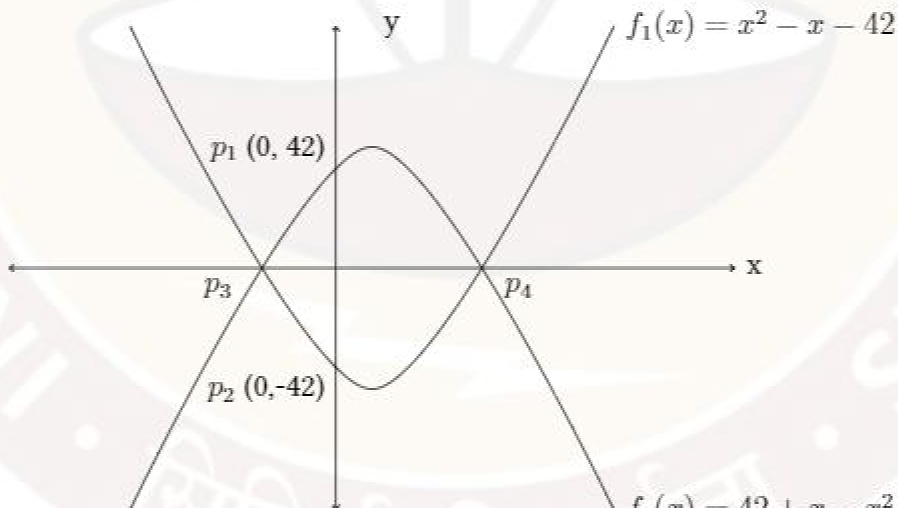


Figure PS-4.3

Since A is the set of all points on the curve f_1, B will be the set of all points on the curve f_2 and C will be the set of all points on the X-axis or Y-axis.

From Figure PS-4.3,
$A \cap B$ is the set of all points which are on f_1 and f_2. Therefore, $A \cap B = \{p_3, p_4\}$.
$A \cap C$ is the set of all points which are on the curve f_1 and on the X-axis or Y-axis. Therefore, $A \cap C = \{p_3, p_4, p_2\}$.
$B \cap C$ is the set of all points which are on the curve f_2 and on the $X-$axis or $Y-$axis. Therefore, $B \cap C = \{p_3, p_1, p_4\}$.
Now, $D = (A \cap B) \cup (A \cap C) \cup (B \cap C) = \{p_1, p_1, p_1, p_4\}$ and therefore, the cardinality of D is 4.

6. Let $f_1(x) = x^2 - 25$. Let A be the set of all points inside the region by the curves representing $f_1(x)$ and its reflection $f_2(x)$ with respect to $X-$ axis (excluding the points on curve). Choose the correct option.

   - ○ The cardinality of A is 2.
   - ○ The cardinality of A is 4.
   - ○ **$Y-$ coordinates of the points in set A belong to the interval $(-25, 25)$.**
   - ○ $Y-$ coordinates of the points in set A belong to the interval $[-25, 25]$.
   - ○ $X-$ coordinates of the points in set A belong to the interval $[-5, 5]$.
   - ○ $X-$ coordinates of the points in set A will be all real numbers because f_1 is a quadratic function.

Solution:

A rough diagram is shown in the Figure PS-4.4

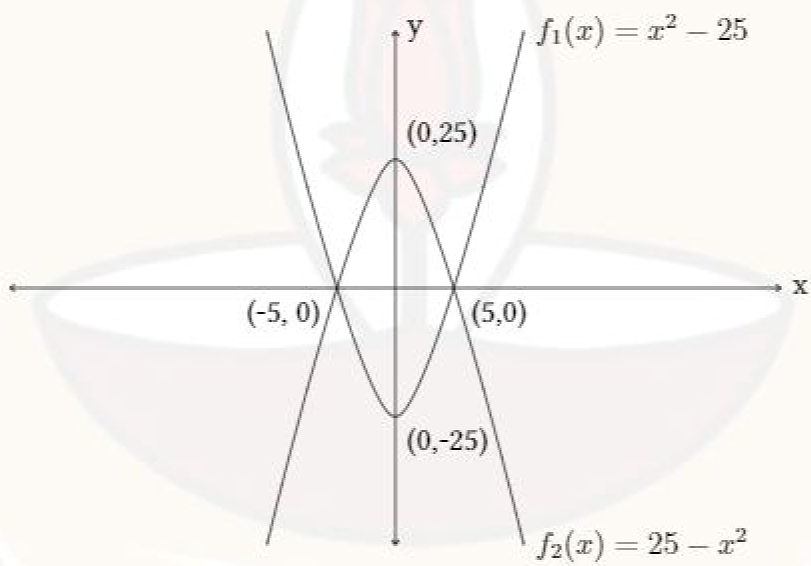


Figure PS-4.4

From the Figure PS-4.4, observe that the set A is infinite, because the region between the two curves f_1 and f_2 has infinitely many points. Therefore, the cardinality of A is not finite. So, options 1 and 2 are wrong.
Also, the region is in between the lines y = +25 and y = -25. Therefore, Y-coordinates of all the points in set A lie between -25 and +25 (-25 and +25 are excluded because they are points on the curves). So, option 3 is correct and option 4 is incorrect because -25 and +25 are included.

Also, the points in A are in between the lines x = -5 and x = +5 (-5 and +5 are excluded because they are points on the curves). Therefore, the X-coordinates of the points in set A belong to the interval (-5, 5). So, options 5 and 6 are incorrect.

2. Multiple Select Questions (MSQ):

7) **Choose the correct set of options regarding the function $\mathbf{f(x) = x^2 + 6x + 8}$**

- ◯ $y = -3$ is the axis of symmetry.
- ◯ **-2 and -4 are the zeroes of the above function**.
- ◯ The maximum value of the above function is -1.
- ◯ **Slope of the function at $(-3, -1)$ is zero**.
- ◯ **$2x + 6$ is the slope of this curve at any given x**.
- ◯ The function is symmetric around $x = 3$.

Solution:

Given, $f(x) = x^2 + 6x + 8$.
The axis of symmetry of $f(x)$ is $x = \frac{-b}{2a} = \frac{-6}{2} = -3$.
Therefore, $x = -3$ is the axis of symmetry of curve $f(x)$. So, options 1 and 6 are incorrect.

For zeros:

$$f(-2) = (-2)^2 + 6(-2) + 8 = 4 - 12 + 8 = 0$$
$$f(-4) = (-4)^2 + 6(-4) + 8 = 16 - 24 + 8 = 0$$

Hence, -2 and -4 are the zeros of the given function. So, option 2 is correct.

As $f(x)$ is an upward parabola, the maximum value of the function is + ∞ at $x = +\infty$. So, option 3 is incorrect.

Now, at $x = -3$, $f(x) = f(-3) = (-3)^2 + 6(-3) + 8 = 9 - 18 + 8 = -1$.
Therefore, the point (-3, -1) is the vertex of the given function. Also, the slope of the function at vertex is always 0. So, option 4 is correct.

We know that the slope of any given quadratic function $g(x) = ax^2 + bx + c; a, b, c \in \mathbb{R}$ at point $(x, g(x))$ is $2ax + b$. Here, $a = 1, b = 6$ and $c = 8$
Therefore, the slope of $f(x)$ is $2x + 6$ at any given x. So, option 5 is correct.

8) . A quadratic function f is such that its value decreases over the interval $(-\infty, -2)$ and increases over the interval $(-2, \infty)$, and $f(0) = f(-4) = 23$. Then, f can be

- ○ $-3x^2 - 12x + 23$
- ○ $\mathbf{3x^2 + 12x + 23}$
- ○ $5(x-2)^2 + 3$
- ○ $\mathbf{5(x+2)^2 + 3}$
- ○ $\boldsymbol{ax^2 + 4ax + 23,\ a > 0}$
- ○ $ax^2 + 4ax + 23,\ a < 0$

Solution:

Given, the values of f decreases over $(-\infty, -2)$ and increases over interval $(-2, \infty)$. Also, $f(0) = f(-4) = 23$.
The curve f is roughly shown in the Figure PS-4.5.

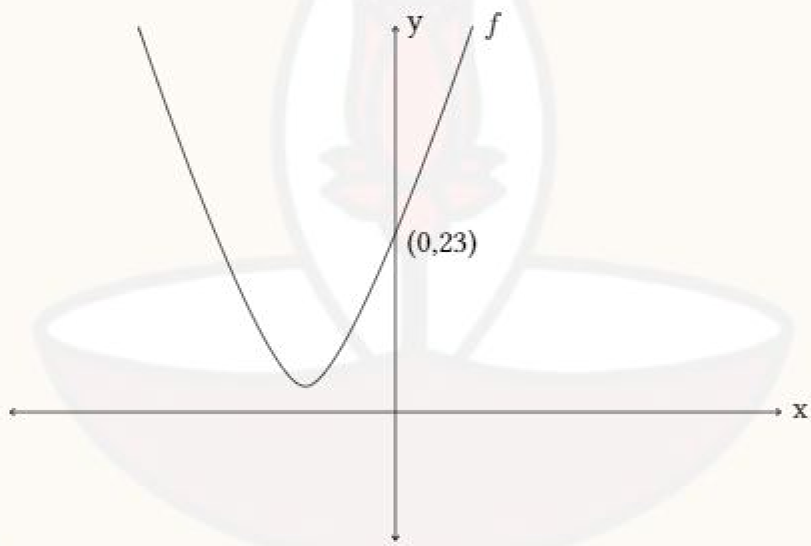


Figure PS-4.5

Suppose $f(x) = ax^2 + bx + c$, for any $a, b, c \in \mathbb{R}$.
We have $f(0) = 23 = a(0)^2 + b(0) + c = c \Rightarrow c = 23$.
Now, $f(-4) = 23 = a(-4)^2 + b(-4) + 23 \Rightarrow 16a - 4b = 0 \Rightarrow b = 4a$.
As the curve f which is shown in the Figure PS-4.5 is an upward parabola, the value of a should be positive.

Therefore, the quadratic function that satisfies the given conditions will be of the form $f(x) = ax^2 + 4ax + 23$, for all $a > 0$. So, option 5 is correct and option 6 is incorrect.

If $a = 3$, then f can be $3x^2 + 12x + 23$. So, option 2 is correct.

If $a = 5$, then f can be $5x^2 + 20x + 23 = 5(x + 2)^2 + 3$. So, option 4 is correct.

In option 1, the leading coefficient of the given function is $-3 = a < 0$. So, it is incorrect.

9). Suppose one root of a quadratic equation of the form $ax^2 + bx + c = 0$, with $a, b, c \in \mathbb{R}$, is $2 + \sqrt{3}$. Then choose the correct set of options.

- ◯ **There can be infinitely many such quadratic equations.**
- ◯ There is no such quadratic equation.
- ◯ There is a unique quadratic equation satisfying the properties.
- ◯ $\boldsymbol{x^2 - 4x + 1 = 0}$ **is one such quadratic equation**.
- ◯ $x^2 - 2x - 3 = 0$ is one such quadratic equation.

Solution:

Given, $2 + \sqrt{3}$ is a root of $ax^2 + bx + c = 0$. One root of the quadratic equation is known. The other root can be any real number k.

For each value of k we will have a different quadratic equation. Therefore, there can be infinitely many quadratic equations that have $2 + \sqrt{3}$ as a root. So, option 1 is correct and options 2,3 are incorrect.

Now, option 4 is correct because the function value (at $x = 2 + \sqrt{3}$) is

$$(2 + \sqrt{3})^2 - 4(2 + \sqrt{3}) + 1 = 4 + 4\sqrt{3} + 3 - 8 - 4\sqrt{3} + 1 = 0$$
$$\Rightarrow 2 + \sqrt{3} \text{ is a root of } x^2 - 4x + 1 = 0.$$

Option 5 is incorrect because the function value (at $x = 2 + \sqrt{3}$) is

$$(2 + \sqrt{3})^2 - 2(2 + \sqrt{3}) - 3 = 4 + 4\sqrt{3} + 3 - 4 - 2\sqrt{3} - 3 = 2\sqrt{3} \neq 0$$
$$\Rightarrow 2 + \sqrt{3} \text{ is not a root of } x^2 - 2x - 3 = 0.$$

10). A company's profits are known to be dependent on the months of a year. The profit pattern (in lakhs of Rupees) from January to December is $P(x) = -2x^2 + 25x$. Here, x represents the month number, starting from 1 (for January) and ending at 12 (for December). On this basis, choose the correct option.

- ◯ **The maximum profit in a month is Rs.78 lakhs.**
- ◯ The maximum profit in a month is Rs.78.125 lakhs.
- ◯ The maximum profit in a month is Rs.77 lakhs.
- ◯ **The maximum profit is recorded in June.**
- ◯ The profit in December is 144 lakhs.
- ◯ None of the above.

Solution:

The profit of the company is given as $P(x) = -2x^2 + 25x$. Observe $P(x)$ is downward open. So, the maximum profit will be recorded at vertex.
The X-Coordinate of the vertex is $x = \frac{-b}{2a} = \frac{-25}{2(-2)} = 6.25$
So, the vertex lies between the lines $x = 6$ and $x = 7$
Therefore, the maximum profit will be recorded in the month of June($x = 6$) or July($x = 7$).
The profit(in lakhs of Rupees) in June is

$$P(6) = -2(6)^2 + 25(6) = -72 + 150 = 78$$

and profit(in lakhs of Rupees) in July is

$$P(7) = -2(7)^2 + 25(7) = -98 + 175 = 77$$

Therefore, the maximum profit of Rs.78 lakhs is recorded in the month of June. So, options 1 and 4 are correct.

The profit (in lakhs of Rupees) in December is

$$P(12) = -2(12)^2 + 25(12) = -288 + 300 = 12$$

So, option 5 is incorrect.

11). Raghav sells 2000 packets of bread for Rs. 20000 each day, and makes a profit of Rs. 4,000 per day. He finds that if the cost price increases by Rs. x per packet, he can increase the selling price by Rs. $2x$ per packet. However, when this price increase happens, he loses $200x$ of his customers. Choose the correct options.

- ◯ **For the maximum profit per day, cost price is Rs. 12 per packet**.
- ◯ For the maximum profit per day, cost price is Rs. 4 per packet.
- ◯ For the maximum profit per day, the sale price increases by Rs. 4 per packet.
- ◯ For the maximum profit per day, Raghav will lose 400 customers.
- ◯ **The maximum difference in profit per day could be Rs. 3200**.
- ◯ The maximum difference in profit per day could be Rs. 7200.

Solution:

The selling price of bread $\frac{20000}{2000}$ = 10 Rupees per packet.
We know that, selling price - cost price = profit ⇒ 20000 - cost price = 4000 ⇒ cost price per day= 16000.
Therefore, the cost price is = $\frac{16000}{2000}$ = 8 Rupees per packet.

Now, if the cost price of each packet increases to $8 + x$ and the selling price of each packet is increased to $10 + 2x$, then the customers left will be $2000 - 200x$.

So, the total profit (say P) in terms of x:

profit = (selling price of each packet- cost price of each packet) x (number of customers)

$$\implies P(x) = \{(10 + 2x) - (8 + x)\} \times (2000 - 200x)$$
$$\implies P(x) = (2 + x)(2000 - 200x)$$
$$\implies P(x) = 4000 + 1600x - 200x^2.$$

The maximum profit occurs at $x = -\frac{b}{2a} = -\frac{1600}{2(-200)} = 4$.

Hence, for the maximum profit per day:

cost price per packet = $8 + x$ = 8 + 4 = **12**.
sale price per packet = $10 + 2x$ = 10 + 8 = **18**
The customers he loses = $200x$ = 200(4) = **800**.
Maximum profit = $4000 + 1600x - 200x^2$ = $4000 + 1600(4) - 200(4)^2$ = **7200**

Therefore, maximum difference in profit = 7200 - 4000 = **3200 Rupees**.
So, the options 1 and 5 are correct.

3. Numerical answer type(NAT):

12) A farmer has a wire of length 576 metres. He uses it to fence his rectangular field to protect it from animals. If he fences his field with four rounds of wire, and the field has the maximum area possible to accommodate such a fencing, what is the area (in square metres) of the field?

Solution:

Suppose, the length of the rectangular field is 'l' metres and breadth of the rectangular field is 'm' metres. So, the perimeter of the rectangular field will be $2(l+m)$.
Now, as he fences his field with four rounds of wire, we have four times the perimeter of the field which, in turn, is equal to the length of the wire. i.e,

$$4(2(l+m)) = 576$$
$$\Rightarrow l + m = \frac{576}{8} = 72$$
$$\Rightarrow m = 72 - l$$

Area of field (A) = lm

$$\Rightarrow A = l(72 - l) = 72l - l^2$$

The maximum area of the field (A_{max})= $-\frac{b^2}{4a} + c$

$$A_{max} = -\frac{72^2}{4 \times (-1)} + 0$$

$$\Rightarrow \boldsymbol{A_{max} = 1296 \textbf{ square metres}}$$

Consider the quadratic function $f(x) = x^2 - 2x - 8$. Two points P and Q are chosen on this curve such that they are 2 units away from the axis of symmetry. R is the point of intersection of axis of symmetry and the $X - axis$. And S is the vertex of the curve. Based on this information, answer the following:

13) What is the height of $\triangle PQR$ taking PQ as the base?

14) What is the height of $\triangle PQS$ taking PQ as the base?

Solution:

The axis of symmetry of $f(x)$ is $x = \frac{-b}{2a} = \frac{-(-2)}{2(1)} = 1$ and two units away points will be $x = 1 + 2 = 3$ and $x = 1 - 2 = -1$.
At $x = 3 \Rightarrow f(x) = -5$ and at $x = -1 \Rightarrow f(x) = -5$. Also, the vertex of the curve is (1, -9).
A rough diagram can be drawn with this information as shown in Figure PS-4.6.

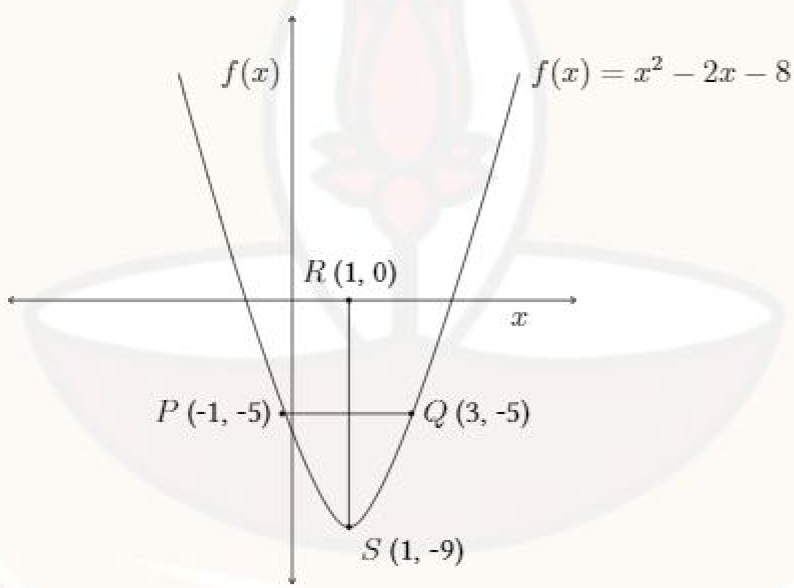


Figure PS-4.6

13) From the above figure, the height of $\triangle PQR$ taking PQ as the base will be the distance between lines $y = 0$ and $y = -5$ and that is equal to 0 - (-5) = 5 units.

14) From the above figure, the height of $\triangle PQS$ taking PQ as the base will be the distance between lines $y = -5$ and $y = -9$ and that is equal to (-5) - (-9) = 4 units.

NOTE:

- There are some questions which have functions with discrete-valued domains (such as month or year). For simplicity, we treat them as continuous functions.
- For a given quadratic equation $ax^2 + bx + c = 0, \ a \neq 0$:
  - ○ Sum of roots $= -\frac{b}{a}$.
  - ○ Product of roots $= \frac{c}{a}$.

1 Multiple Choice Questions (MCQ):

15) What will be the value of parameter k, if the discriminant of equation $4x^2 + 9x + 10k = 0$ is 1?

- ○ $\frac{82}{80}$
- ○ $\frac{41}{80}$
- ○ $\frac{1}{2}$
- ○ $\frac{41}{160}$
- ○ 1
- ○ None of the above.

Solutions:
Comparing the given equation $4x^2 + 9x + 10k = 0$ with the standard quadratic equation $ax^2 + bx + c = 0$:

$$a = 4, \ b = 9, \text{ and } c = 10k$$
$$\text{Discriminant } (d) = b^2 - 4ac$$
$$d = 9^2 - 4 \times 4 \times 10k$$
$$1 = 81 - 160k$$
$$\boldsymbol{k = \frac{1}{2}}$$

16) . A boat has a speed of 30 km/hr in still water. In flowing water, it covers a distance of 50 km in the direction of flow and comes back in the opposite direction. If it covers this total of 100 km in 10 hours, then what is the speed of flow of the water (in km/hr)?

- ◯ $5 - 5\sqrt{37}$
- ◯ $-10\sqrt{6}$
- ◯ $\mathbf{10\sqrt{6}}$
- ◯ $20\sqrt{3}$
- ◯ $-20\sqrt{3}$
- ◯ 2

Solutions:

Total time taken by the boat = time taken by the boat in the direction of flow + time taken by the boat in the opposite direction of flow.

We know that:

$$time(t) = \frac{distance}{net\ speed}$$

Considering the direction of flow of water to be positive:

The net speed in the direction of flow (v_f) = speed of the boat in still water + speed of flow.

The net speed in the opposite direction of flow (v_b) = speed of the boat in still water - speed of flow.

Let the speed of flow be x then,

$$10 = \frac{50}{v_f} + \frac{50}{v_b}$$

$$10 = \frac{50}{30+x} + \frac{50}{30-x}$$

$$1 = \frac{5}{30+x} + \frac{5}{30-x}$$

$$1 = \frac{5(30-x+30+x)}{(30+x)(30-x)}$$

$$(30+x)(30-x) = 300$$

$$30^2 - x^2 = 300$$

$$x^2 = 600x = \pm 10\sqrt{6}$$

Speed of flow can not be negative therefore, the correct answer is $10\sqrt{6}$.

17) A stunt man performs a bike stunt between two houses of the same height as shown in Figure 1. His bike (lowest part of the bike) makes an angle of θ at house A with the horizontal at the beginning of the stunt, follows a parabolic path and lands at house B with an angle of $(180 - \theta)$ with the horizontal.

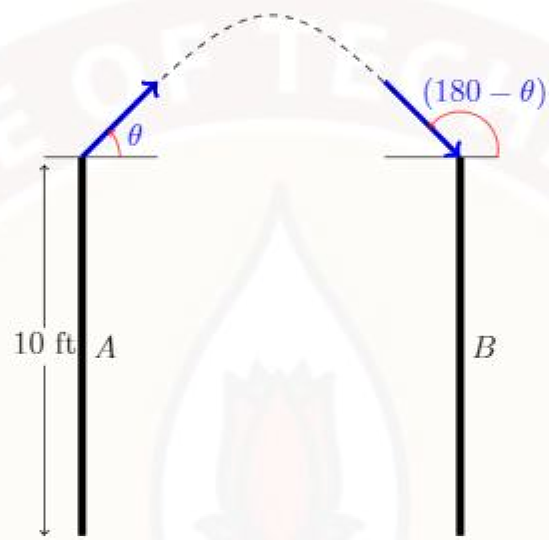


Figure PA-5.1

If the maximum height achieved by the bike is 12.5 ft from the ground and $tan\ \theta = 1$, then find the distance between the two houses.

- ○ 1 ft
- ○ 2.5 ft
- ○ 5 ft
- ○ **10 ft**
- ○ 15 ft
- ○ 20 ft

Solution:
Assuming the top of the house A to be origin, the horizontal direction as $X-$ axis, and the vertical direction as $Y-$ axis.

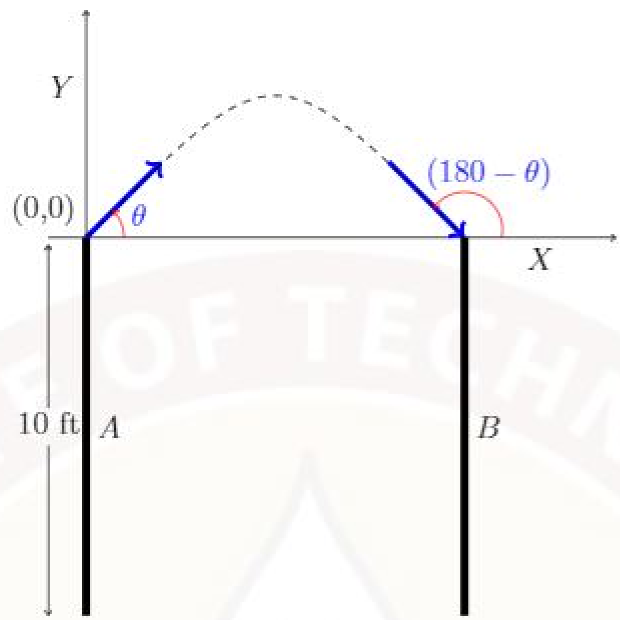


Figure M1W5PAS-3.1

Let the quadratic function representing the above curve be $f(x) = ax^2 + bx + c$.
Since the curve passes through the origin, we have $c = 0$.
The curve is making an angle θ with respect to positive $X-$ axis which means the slope of the tangent at the curve is $\tan\theta$.
We also know that the slope of the curve represented by quadratic function at $x = x$ is $2ax + b$. Therefore,

$$\begin{aligned} 2ax + b &= \tan\theta \\ 2a \times 0 + b &= 1 \\ b &= 1 \end{aligned}$$

The maximum height achieved by the bike is 12.5 ft which means the $y-$ coordinate of the vertex is $12.5 - 10 = 2.5$.
The $x-$ coordinate of the vertex for a curve represented by function $ax^2 + bx + c$ is

$$-\frac{b}{2a} = -\frac{1}{2a}$$

Therefore,

$$\begin{aligned}
f(x) &= ax^2 + bx + c \\
f(-\frac{1}{2a}) &= 2.5 \\
a \times (-\frac{1}{2a})^2 + 1 \times (-\frac{1}{2a}) + 0 &= 2.5 \\
\frac{1}{4a} - \frac{1}{2a} &= 2.5 \\
-\frac{1}{4a} &= 2.5 \\
a &= -\frac{1}{10}
\end{aligned}$$

Axis of symmetry,

$$x = -\frac{b}{2a} = -\frac{1}{2 \times (-1/10)} = 5$$

Because of symmetricity, the coordinate of landing point will be (10, 0).
Therefore two houses A and B are 10 ft apart.

2 Multiple Select Question (MSQ):

18) Given that $f_1(x) = -x^2 - 6x$ and $f_2(x) = x^2 + 6x + 10$. Let $f(x)$ be a function such that the domain of $f(x)$ is $[\alpha,\ \beta]$, where $f_1(\alpha) = f_2(\alpha)$ and $f_1(\beta) = f_2(\beta)$, then choose the set of correct options.

- ◯ Range of $f(x)$ is $[-1, 3]$.
- ◯ Range of $f(x)$ is $[0, 5]$.
- ◯ Domain of $f(x)$ is $[-5, 5]$.
- ◯ **Domain of $f(x)$ is $[-5, -1]$.**
- ◯ **Inadequate information provided for finding the range of $f(x)$.**
- ◯ Inadequate information provided for finding the domain of $f(x)$.

Solution:
Since $f_1(\alpha) = f_2(\alpha)$ and $f_1(\beta) = f_2(\beta)$, we have α and β are the abscissa of intersection points of both the curves.
To find the intersection points of the curves represented by $f_1(x)$ and $f_2(x)$:

$$\begin{gathered} f_1(x) = f_2(x) \\ -x^2 - 6x = x^2 + 6x + 10 \\ 2x^2 + 12x + 10 = 0 \end{gathered}$$

Here,

$$a = 2, b = 12, \text{ and } c = 10$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-12 \pm \sqrt{12^2 - 4 \times 2 \times 10}}{2 \times 2}$$

$$x = \frac{-12 \pm 8}{4} = -1, -5$$

Therefore,

$$\alpha = -5 \text{ and } \beta = -1.$$

Since the Domain of $f(x)$ is $[\alpha, \beta]$ domain of $f(x)$ is $[-5, -1]$.

The figure below gives a rough pictorial representation of $f_1(x)$ and $f_2(x)$ (drawn with smooth lines).
$f(x)$ can have any shape. An example is shown in the figure (drawn with dashed lines) for $f(x)$.

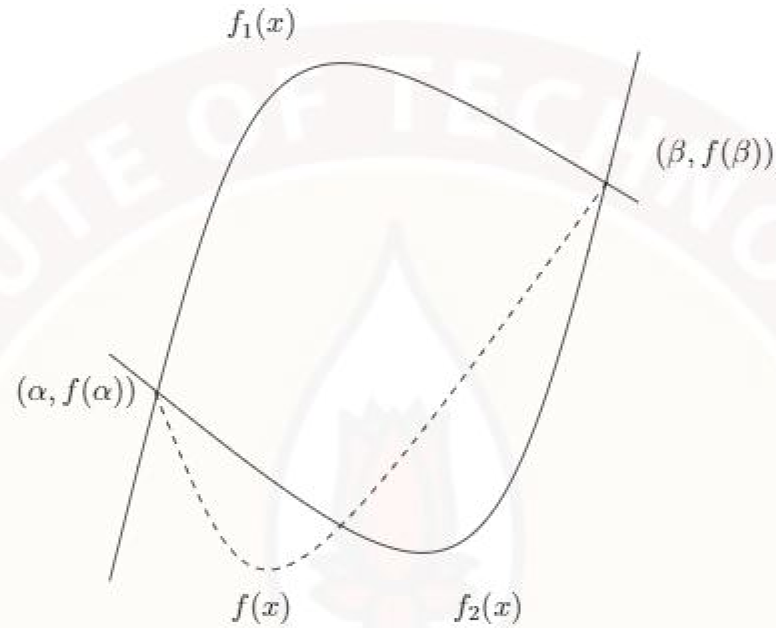


Figure M1W5PAS-4.1

As it is clear from figure that we do not know the minimum and maximum value of $f(x)$, we do not have proper data to comment on the range.

19) If $f(x) = 2x^2 + (5+k)x + 7$, $g(x) = 5x^2 + (3+k)x + 1$, $h_1(x) = f(x) - g(x)$, and $h_2(x) = g(x) - f(x)$, then choose the set of correct options.

- ◯ **Roots for $h_1(x) = 0$ and roots for $h_2(x) = 0$ are real, distinct, and the roots are the same for $h_1(x) = 0$ and $h_2(x) = 0$.**
- ◯ Roots for $h_1(x) = 0$ and roots for $h_2(x) = 0$ are real and distinct but the roots are not the same for $h_1(x) = 0$ and $h_2(x) = 0$.
- ◯ **Sum of roots of quadratic equation $h_1(x) = 0$ will be $\frac{2}{3}$.**
- ◯ **Product of roots of quadratic equation $h_2(x) = 0$ will be -2.**
- ◯ **Axis of symmetry for both the functions $h_1(x)$ and $h_2(x)$ will be the same.**
- ◯ Vertex for both the functions $h_1(x)$ and $h_2(x)$ will be the same.

Solution:
Given that,

$$h_1(x) = f(x) - g(x)$$
$$h_1(x) = -(g(x) - f(x))$$
$$h_1(x) = -h_2(x)$$

Negative sign before any function does not make any changes on zeros of the function. Therefore, roots of $h_1(x) = 0$ and roots of $h_2(x) = 0$ will be same.

Now, for the properties of $h_1(x)$:

$$h_1(x) = f(x) - g(x) = 2x^2 + (5+k)x + 7 - (5x^2 + (3+k)x + 1)$$
$$h_1(x) = -3x^2 + 2x + 6$$
$$d = 2^2 - 4(-3) \times 6 > 0$$

It means the roots of $h_1(x)$ are real and distinct.
The roots of $h_1(x) = 0$ has the same as the roots of $h_2(x) = 0$, which means the roots for $h_2(x) = 0$ will also be real and distinct.

Sum of the roots of $h_1(x) = -3x^2 + 2x + 6$ will be $-\frac{b}{a} = -\frac{2}{(-3)} = \frac{2}{3}$.

Product of the roots of $h_1(x) = -3x^2 + 2x + 6$ will be $\frac{c}{a} = \frac{6}{(-3)} = -2$.

Multiplying a quadratic function by the minus sign does not make any changes in the

axis of symmetry.
The answer to all the above questions can be seen in the given figure.

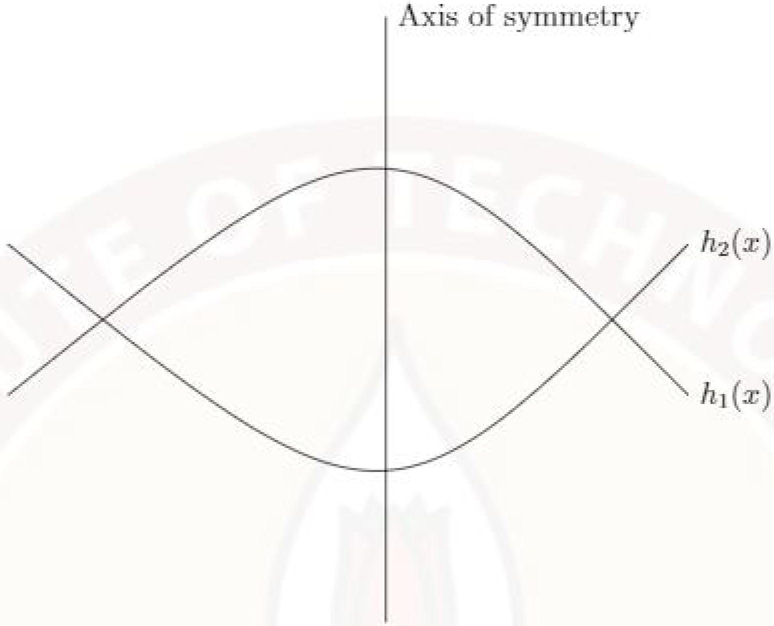


Figure M1W5PAS-5.1

Use following information for questions 6-8.

Vaishali wants to set up a small plate making machine in her village. Table P-5.1 shows the different costs involved in making the plates. Figure 5 shows her survey regarding the demand (number of packets of the plate) versus selling price of plate per packet (in ₹) per day.

Cost type	Cost
Electricity	₹1.5 per packet
Miscellaneous	₹6.5 per packet
Raw material	₹10 per packet

Table P-5.1

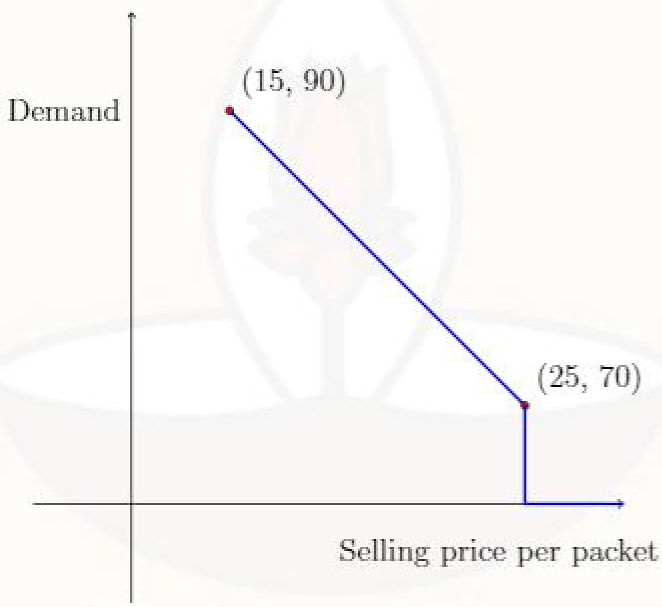


Figure PA-5.2

10) Choose the correct option which shows the profit obtained by Vaishali per day. Here, x is the selling price per packet.

- ◯ $2(60-x)$
- ◯ $x(x-18)$
- ◯ **$2(x-18)(60-x)$**
- ◯ $2(x+18)(60-x)$
- ◯ Inadequate information.

Solution:
From the figure, it is clear that the demand is dependent on the selling price of plates. Let y be the demand of the numbers of packets, then from two-points form of a line,

$$y - 90 = \frac{70 - 90}{25 - 15}(x - 15)$$

$$y - 90 = -2(x - 15)$$

$$y = -2x + 120$$

From the table, total cost per packet (in ₹)$= 1.5 + 6.5 + 10 = 18$

Per day profit = Demand per day×(Selling price per packet - Cost per packet)

$$Profit = y(x - 18)$$
$$Profit = (-2x + 120)(x - 18)$$
$$Profit = 2(x - 18)(60 - x)$$

21.) Choose the set of correct options.

- ◯ **Vaishali should sell a packet with a minimum price of ₹18 so as not to incur any loss.**
- ◯ Vaishali should sell a packet with a minimum price of ₹12 so as not to incur any loss.
- ◯ To make maximum profit per day, the selling price per packet should be ₹39.
- ◯ **To make maximum profit per day, the selling price per packet should be ₹25.**
- ◯ Vaishali should sell a packet with maximum price of ₹60 so as not to incur any loss.
- ◯ **Vaishali should sell a packet with a maximum price of ₹25 so as not to incur any loss.**

Solution:
From question 6,

$$Profit = 2(x-18)(60-x)$$

$$Profit = -2x^2 + 156x - 2160 \tag{1}$$

To get minimum selling price with no loss, profit should be zero. Therefore,

$$\begin{gathered} 2(x-18)(60-x) = 0 \\ x = 18 \text{ or } 60 \end{gathered}$$

From the graph given in question, it is clear that we can not sell a packet at ₹60, because the demand will be zero.
Therefore, the minimum selling price will be ₹18 per packet.

Since the profit is a quadratic function of the selling price (x) in equation (1) with negative coefficient of x^2.
Therefore, the maximum profit will occur at

$$x = -\frac{b}{2a} = -\frac{156}{2\times(-2)} = 39$$

A rough pictorial representation is shown in Figure below,

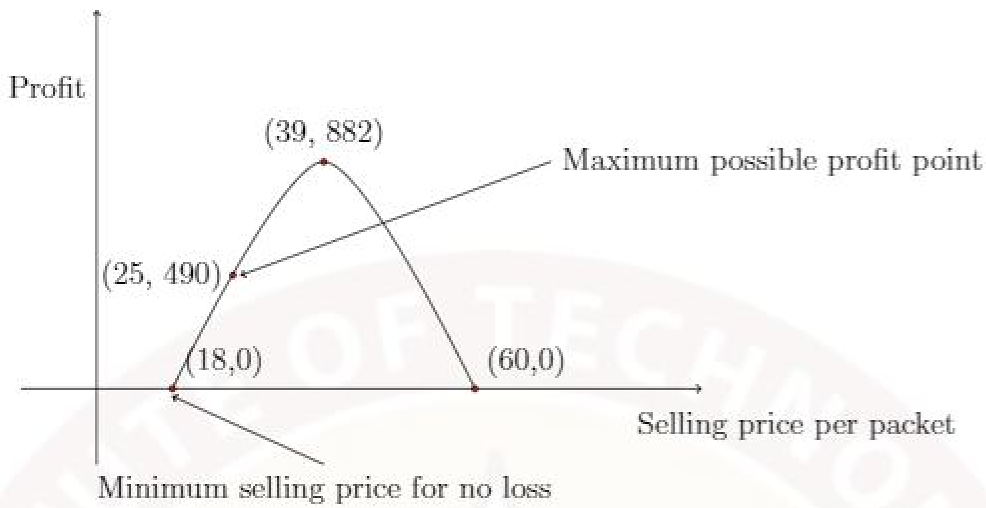


Figure M1W5PAS-7.1

The increase in selling price will result in profit increment till 39. But the maximum acceptable selling price is ₹25, therefore the maximum profit will occur at a selling price of ₹25.

So, from the figure it is clear that the maximum selling price for no loss is ₹60 but we can not increase the price beyond ₹25. Therefore, the maximum profit to incur any loss will be ₹25.

3 Numerical Answer type (NAT):

12) What should be the price of plate per packet (₹) to make a profit of ₹490 per day?
[Hint: $(x-53)$ a factor of $2(-x^2+78x-1325)$.] [Ans: 25]

Solution:
From equation (1) $Profit = -2x^2+156x-2160$

$$-2x^2+156x-2160 = 490$$
$$-2x^2+156x-2650 = 0$$
$$2(-x^2+78x-1325) = 0$$

It is given that $(x-53)$ a factor of $2(-x^2+78x-1325)$. So dividing $2(-x^2+78x-1325)$ by $(x-53)$ we will get $-2x+50$.
Therefore,

$$2(-x^2+78x-1325) = 0$$

$$(x-53)(2x-50) = 0$$

If

$$x-53 = 0$$

$$x = 53$$

But selling price can not go beyond 25.
Now if,

$$2x-50 = 0$$
$$x = \frac{50}{2}$$
$$x = 25$$

Therefore, the selling price of plate should be ₹25 .

23) What will be the value of $m + n$ if the sum of the roots and the product of the roots of equation $(5m + 5)x^2 - (4n + 3)x + 10 = 0$ are 3 and 2 respectively?

Solution:

We know that the sum of the roots of an equation $ax^2 + bx + c = 0$ is $\frac{-b}{a}$ and the product of its roots is $\frac{c}{a}$.

Here, $a = 5m + 5, b = -(4n + 3), c = 10$. Substituting these values we get,
The product of the roots of the given equation

$$\frac{c}{a} = \frac{10}{5m + 5} = 2$$

$$5m + 5 = 5$$

$$m + 1 = 1$$

$$\boldsymbol{m = 0}$$

The sum of the roots as

$$\frac{-b}{a} = \frac{-(-(4n + 3))}{5m + 5} = 3$$

$$4n + 3 = 3(5m + 5)$$

For $m = 0$

$$4n + 3 = 3 \times 5$$

$$4n = 12$$

$$\boldsymbol{n = 3}$$

Therefore,

$$m + n = 0 + 3 = 3.$$

24). What will the sum of two positive integers be if the sum of their squares is 369 and the difference between them is 3?. **Solution:**

Let a and b be the two positive integers. Given that

$$a^2 + b^2 = 369 \tag{2}$$
$$a - b = 3 \tag{3}$$

Squaring equation (3) on both sides, we get

$$\begin{aligned}(a-b)^2 &= 3^2\\ a^2 - 2ab + b^2 &= 9\\ 369 - 2ab &= 9\\ 2ab &= 369 - 9\end{aligned}$$

$$2ab = 360$$

Now, to find the sum of the integers

$$\begin{aligned}(a+b)^2 &= a^2 + 2ab + b^2\\ (a+b)^2 &= 369 + 360\\ (a+b)^2 &= 729\\ a+b &= \pm\sqrt{729} = \pm 27\end{aligned}$$

As a and b are positive integers, their sum should also be a positive integer.
Therefore, $\boldsymbol{a+b=27}$.

Mathematics for Data Science - 1
Week 3 GA Solution

1 Instructions:

1. Find out the points where the curve $y = 4x^2 + x$ and the straight line $y = 2x - 3$ intersect with each other.

   - ◯ $(\frac{3}{2}, 0)$ and $(\frac{3}{2}, \frac{21}{2})$.
   - ◯ Only at the origin.
   - ◯ The curve and the straight line do not intersect.
   - ◯ $(1, -1)$ and $(1, 5)$.

Solution: Suppose $y = 4x^2 + x$ & $y = 2x - 3$ are intersecting at the point (a, b). So the point (a, b) should satisfy both the equations.

$$b = 4a^2 + a \quad \& \quad b = 4a - 3.$$

$$\Rightarrow 4a^2 + a = 4a - 3$$

$$\Rightarrow 4a^2 - a + 3 = 0.$$

The discriminant of the above quadratic equation is $-47 < 0$. Therefore it has no real root & both the curves can not meet in the real plane.

2. Let a and b two consecutive positive odd natural numbers such that $a^2 + b^2 = 394$. Then find the value of $a + b$.

Solution: Let "x" and "$x+2$" be the two Consecutive Positive odd natural number.

Given, $$x^2 + (x+2)^2 = 394$$

$$\Rightarrow x^2 + x^2 + 4 + 4x = 394$$

$$\Rightarrow 2x^2 + 4x - 390 = 0$$

$$\Rightarrow x^2 + 2x - 195 = 0 \quad \text{(dividing by 2)}$$

$$\Rightarrow (x+15)(x-13) = 0$$

$$\Rightarrow x = 13 \text{ or } x = -15$$ (not possible because x is positive)

$$\Rightarrow x = 13 = a \ \& \ b = x + 2 = 15.$$

Sum: $a + b = 28$.

3. If the slope of parabola $y = ax^2 + bx + c$, where $a, b, c \in \mathbb{R} \setminus \{0\}$ at points (3,2) and (2,3) are 32 and 17 respectively, then find the value of a.

Solution: Slope of Parabola at "x" is $2ax+b$.

By using the given information, we get

$6a+b=32$ at Point $(3,2)$ — ①

$4a+b=17$ at Point $(2,3)$. — ②

eq① − eq② : $2a=15 \Rightarrow \boxed{a=7.5}$.

4. A class of 352 students are arranged in rows such that the number of students in a row is one less than thrice the number of rows. Find the number of students in each row.

Solution: Let x be the total number of rows. Then the number of students in each row is

$$3x-1.$$

Therefore total no. of student is

$$x(3x-1)=352.$$

$$\Rightarrow 3x^2-x-352=0$$

$$\Rightarrow (x-11)\left(x+\frac{32}{3}\right)=0$$

$$\Rightarrow x=11 \text{ or } x=-32/3 \text{ (not possible)}$$

$$\Rightarrow x=11 \text{ (no. of rows)}.$$

Number of students in each row is $3x-1=32$.

In order to cover a fixed distance of 48 km, two vehicles start from the same place. The faster one takes 2 hrs less and has a speed 4 km/hr more than the slower one. Using the given information, answer the following sub-questions (5 and 6).

5. What is the speed (in km/hr) of the slower vehicle?

Solution: Let the speed of the slower vehicle be x km/hr. The time taken by the slower one to cover 48km is $48/x$ hr.

The speed of the faster one is $x+4$ km/hr. So the time taken by the faster one to cover 48 km is $48/x+4$.

It is given that the faster one takes 2hrs less than the slower one to cover the distance. Therefore we have,

$$\frac{48}{x} - \frac{48}{x+4} = 2.$$

$$\Rightarrow \frac{x+4-x}{x(x+4)} = \frac{2}{48}$$

$$\Rightarrow x^2+4x-96=0$$

$$\Rightarrow (x+12)(x-8)=0 \quad \Rightarrow x=8.$$

Hence the speed of the slower vehicle is 8 km/hr.

6. What is the time (in hrs) taken by the faster one?

The speed of the faster one is $x+4 = 12 km/hr$.

So the time taken by the faster vehicle is $48/12 = 4$ hrs.

7. The maximum value of a quadratic function f is -3, its axis of symmetry is $x = 2$ and the value of the quadratic function at $x = 0$ is -9. What will be the coefficient of x^2 in the expression of f?

- ○ -1
- ○ 1
- ○ -1.5
- ○ -0.5

<u>Solution:</u> Let $f(x) = ax^2+bx+c$ be the quadratic equation. We know that f attains its maximum value at $x = -b/2a$ & the line $x = -b/2a$ is the axis of symmetric.

- Given that $x=2$ is the axis of sy. $\Rightarrow -b/2a = 2 \Rightarrow b = -4a$.
- value of f at $x=0$ is -9 $\Rightarrow f(0) = c = -9$.

The function f attains its maximum value at $x=2$ and the max. value is "-3". Therefore $(2,-3)$ should satisfy the quadratic eq. & we have

$4a+2b-9=-3$ [by putting $(2,-3)$ & $c=-9$ in the eq. of $f(x)$]

$\Rightarrow 4a+2b=6$

$\Rightarrow 4a-8a=6$ [using the relation $b=-4a$]

$$\Rightarrow -4a = 6$$

$$\Rightarrow a = -3/2 = -1.5$$

So the coefficient of x^2 is -1.5.

8. A ball is thrown from 3 m off the ground and reaches a maximum height of 5 m. Assume that the ball was released from the point $(0,3)$ in the xy-plane as shown in the Figure M1W3GA-3. The ball returns to a height of 3 m after 2 seconds. Let $h(t) = at^2 + bt + c$ be the quadratic function which represents the height of the ball after t seconds. What is the value of a ?

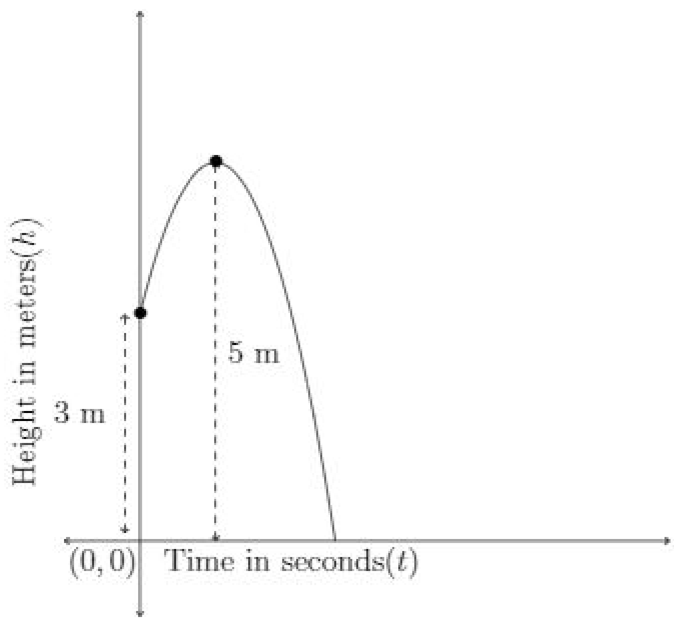
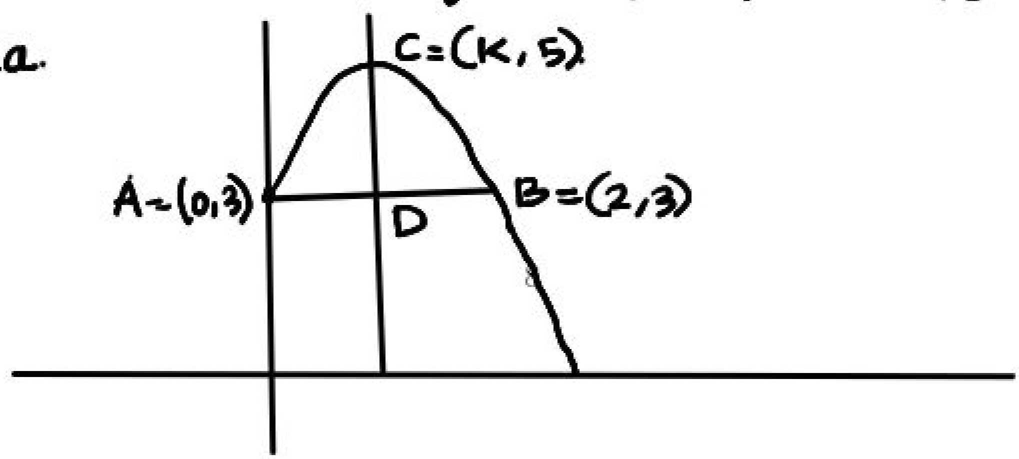


Figure M1W3GA-3

Solution: The ball was released from the point $(0,3)$ and returns to height of 3-met after 2-seconds. So $(0,3)$ & $(3,2)$ are on the Parabola.

Let the ball reaches a maximum height of 5cm after K seconds. Therefore $(K,5)$ is also on the Parabola.



$x=k$ is the axis of Symmetry and "D" is the middle Point of AB. Therefore $k=1$.

Finally, we have three points $(0,3)$, $(1,5)$ & $(2,3)$ on the parabola $h(t)=at^2+bt+c$.

$$h(0)=3 \Rightarrow c=3$$

$$h(1)=5 \Rightarrow a+b+3=5. \quad \text{——} ①$$

$$h(2)=3 \Rightarrow 4a+2b+3=3. \quad \text{——} ②$$

From eq ① and ②, we get $\boxed{a=-2.}$.

9. The product of two consecutive odd natural numbers is 255. Find the largest number among them.

Solution: Let x and $x+2$ be two consecutive odd natural numbers.

$$x(x+2) = 255$$

$$\Rightarrow x^2+2x-255=0$$

$$\Rightarrow (x-15)(x+17)=0$$

$$\Rightarrow x=15 \quad \text{or} \quad x=-17 \text{ (not possible).}$$

$$x=15 \text{ \& } x+2=17.$$

Therefore the largest number is 17.

10. The slope of a parabola $y = 3x^2 - 11x + 10$ at a point P is 1. Find the y-coordinate of the point P.

<u>Solution</u>: The slope of parabola $y = ax^2 + bx + c$ at 'x' is given by $2ax + b$.

Here, $y = 3x^2 - 11x + 10$ so the slope is $6x - 11$.

Let $P = (x, y)$, then slope of the parabola at P is

$$6x - 11 = 1$$

$$\Rightarrow x = 2.$$

The "y"-coordinate of P is

$$y = 3 \times 2^2 - (11 \times 2) + 10$$

$$= 22 - 22 = 0.$$

Week - 4
Practice Assignment Solution
Algebra of polynomials
Mathematics for Data Science - 1

1 Multiple Choice Questions (MCQ):

(Use the following data for the Question 1 and Question 2 only).

Let x be the number of years since the year 2000 (i.e., $x = 0$ denotes the year 2000). The total amount of profit (in ₹) on books in a shop is given by the function $T(x) = 5x^3 + 3x + 1$. The shop sells books of four languages English, Bengali, Hindi, and Tamil. The profits from selling English and Bengali books are given by $E(x) = 3x^3 - 5x^2 + x$ and $B(x) = x^2 + 4x + 5$ respectively. The profit from selling Hindi and Tamil books are found to be the same.

1. Which of the following polynomial functions represents the profit from selling Tamil books?

   ○ $2x^3 + 4x^2 - 2x - 4$
   ○ $x^3 - 2x^2 - x + 2$
   ○ $\boldsymbol{x^3 + 2x^2 - x - 2}$
   ○ $2x^3 - 4x^2 - 2x + 4$

2. In which year was the profit from Hindi books zero?

   ○ **2001**
   ○ 2002
   ○ 2004
   ○ 2010

Solution:

(a) The total profit from selling English and Bengali books is= $E(x) + B(x) = (3x^3 - 5x^2 + x) + (x^2 + 4x + 5) = 3x^3 - 4x^2 + 5x + 5$. Hence the total profit from selling Hindi and Tamil books is= $T(x) - (3x^3 - 4x^2 + 5x + 5) = 5x^3 + 3x + 1 - 3x^3 + 4x^2 - 5x - 5 = 2x^3 + 4x^2 - 2x - 4$.
As the profit from selling Hindi and Tamil books are found to be the same, the profit from selling Tamil books is= $\frac{1}{2}(2x^3 + 4x^2 - 2x - 4) = x^3 + 2x^2 - x - 2$

(b) Profit from selling Hindi books (which is same as the profit from selling Tamil books) is $x^3 + 2x^2 - x - 2$.

$$x^3 + 2x^2 - x - 2 = x^2(x + 2) - 1(x + 2) = (x + 2)(x^2 - 1) = (x + 2)(x + 1)(x - 1)$$

So the profit will be zero if $(x+2)(x+1)(x-1) = 0$, i.e., at $x = -2, -1, 1$ the profit can be 0. But in this context, x cannot be negative. So $x = 1$ is the only possibility. Hence in the year 2001 the profit from Hindi books was zero.

3. Find the quadratic polynomial which when divided by x, $x-1$, and $x+1$ gives the remainders 7, 14, and 8 respectively.

   - ◯ $4x^2 - 3x + 7$
   - ◯ $x^2 + 7x + 7$
   - ◯ $7x^2 + x + 7$
   - ◯ $\mathbf{4x^2 + 3x + 7}$

   Solution: Let the quadratic polynomial which is satisfying the given condition be $p(x) = ax^2 + bx + c$.
   When it is divided by x the remainder is 7. It implies that if we substitute $x = 0$ in $p(x)$ we will get 7, i.e., $p(0) = 7$. Similarly we have $p(1) = 14$ and $p(-1) = 8$.
   Hence we have the following equations:

$$\begin{aligned}
p(0) &= a(0)^2 + b(0) + c \\
&= c \\
&= 7 \\
p(1) &= a.(1)^2 + b.1 + c \\
&= a + b + c \\
&= 14 \\
p(-1) &= a(-1)^2 + b(-1) + c \\
&= a - b + c \\
&= 8
\end{aligned}$$

   So, we have $c = 7$, and substituting c in the second and third equation we get, $a + b = 7$, and $a - b = 1$. By solving these two equations we get $a = 4$ and $b = 3$.
   Hence the quadratic polynomial is $4x^2 + 3x + 7$.

4. Box A has length x unit, breadth $(x+1)$ unit, and height $(x+2)$ unit. Box B has length $(x+1)$ unit, breadth $(x+1)$ unit, and height $(x+2)$ unit. There are two more boxes C and D of cubic shape with side x unit. The total volume of A and B is y cubic unit more than the total volume of C and D. Find y in terms of x.

   - ○ $x^3 + 7x^2 + 7x + 2$
   - ○ $\mathbf{7x^2 + 7x + 2}$
   - ○ $7x^2 - 7x - 2$
   - ○ $x^3 + 7x^2 - 7x - 2$

   Solution: The volume of box A is $x(x+1)(x+2) = x^3 + 3x^2 + 2x$ cubic unit.
   The volume of box B is $(x+1)(x+1)(x+2) = (x^2+2x+1)(x+2) = x^3 + 4x^2 + 5x + 2$ cubic unit.
   The volume of box C and D is x^3 cubic unit each. So the total volume of A and B is $2x^3 + 7x^2 + 7x + 2$ and the total volume of C and D is $2x^3$.
   Hence $y = (2x^3 + 7x^2 + 7x + 2) - 2x^3 = 7x^2 + 7x + 2$.

5. The population of a bacteria culture in laboratory conditions is known to be a function of time of the form $p(t) = at^5 + bt^2 + c$, where p represents the population (in lakhs) and t represents the time (in minutes). Suppose a student conducts an experiment to determine the coefficients a, b, and c in the formula and obtains the following data:

   - $p(0) = 3$
   - $p(1) = 5$
   - $p(2) = 39$

   Which of the following options is correct?

   ◯ $p(t) = 3t^5 - t^2 + 3$

   ◯ $p(t) = 4t^5 - 2t^2 + 3$

   ◯ $\boldsymbol{p(t) = t^5 + t^2 + 3}$

   ◯ $p(t) = 39t^5 + 5t^2 + 3$

   Solution: Given that, $p(t) = at^5 + bt^2 + c$.
   $p(0) = c = 3$
   $p(1) = a + b + c = 5$, putting $c = 3$, we get $a + b = 2$.
   $p(2) = a(2)^5 + b(2)^2 + c = 32a + 4b + c = 39$, substituting $c = 3$, we get $32a + 4b = 36$, implies, $8a + b = 9$ (cancelling 4 from both sides)
   By solving these two equations we get $a = 1$, and $b = 1$.
   Hence, $p(t) = t^5 + t^2 + 3$.

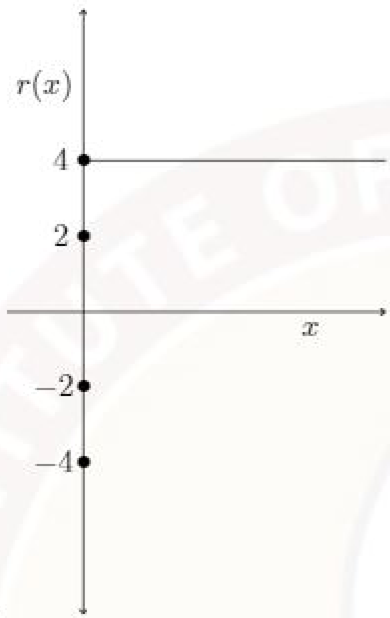
6. If the polynomials $x^3 + ax^2 + 5x + 7$ and $x^3 + 2x^2 + 3x + 2a$ leave the same remainder when divided by $(x - 2)$, then the value of a is:

   - ○ $\frac{3}{2}$
   - ○ $-\frac{3}{2}$
   - ○ $\frac{5}{2}$
   - ○ $-\frac{5}{2}$

   Solution: Given that both the polynomials leave same remainder when divided by $(x - 2)$. By substituting $x = 2$ both the polynomial should have same value.
   By substituting $x = 2$ in $x^3 + ax^2 + 5x + 7$, we get $8 + 4a + 10 + 7 = 4a + 25$.
   By substituting $x = 2$ in $x^3 + 2x^2 + 3x + 2a$, we get $8 + 8 + 6 + 2a = 2a + 22$.
   So we have,

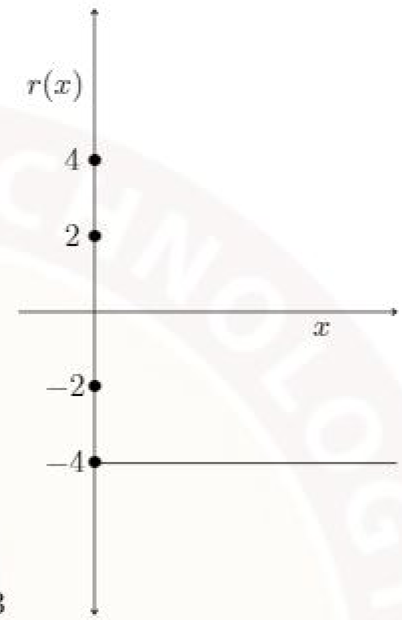
$$\begin{aligned} 4a + 25 &= 2a + 22 \\ 2a &= -3 \\ a &= -\frac{3}{2} \end{aligned}$$

7. Let $r(x)$ be a polynomial function which is obtained as the remainder after dividing the polynomial $2x^3 + x^2 - 5$ by the polynomial $2x - 3$. Choose the correct option which represents the polynomial $r(x)$ most appropriately.



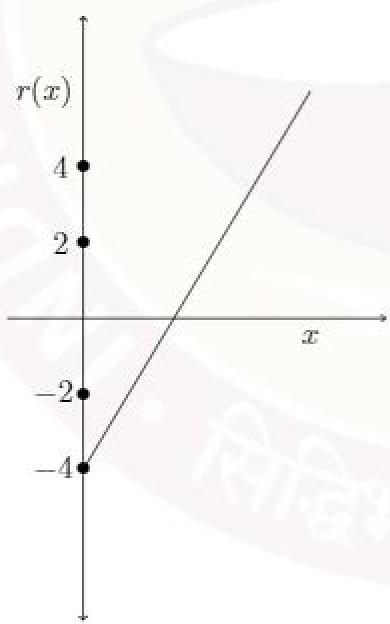
◯ **Option A**

Fig P-6.2



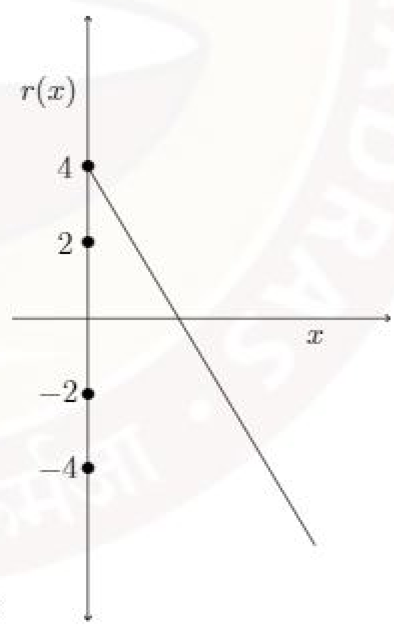
◯ Option B

Fig P-6.3



◯ Option C

Fig P-6.4



◯ Option D

Fig P-6.5

Solution We get 4 as the remainder if $2x^3 + x^2 - 5$ is divided by the polynomial $2x - 3$.

$$2x^3 + x^2 - 5 = (2x - 3)(x^2 + 2x + 3) + 4$$

Hence $r(x) = 4$, which is a constant polynomial. Hence, the first option is the correct.

2 Multiple Select Questions (MSQ):

(Use the following data for the Question 8 and Question 9 only).
By dividing a polynomial $p(x)$ with another polynomial $q(x)$ we get $h(x)$ as the quotient and $r(x)$ as the remainder.

8. The maximum degree of $r(x)$ can be,

   ○ $deg\ p(x)$

   ○ $deg\ (p(x)) - 1$

   ○ $deg\ q(x)$

   ○ $\boldsymbol{deg\ (q(x)) - 1}$

9. If $deg\ p(x) < deg\ q(x)$, then choose the set of correct answers:

   ○ $\boldsymbol{h(x) = 0}$

   ○ $deg\ h(x) = deg\ q(x)$

   ○ $deg\ r(x) = deg\ q(x)$

   ○ $\boldsymbol{deg\ r(x) = deg\ p(x)}$

Solution:

(a) The degree of the remainder $r(x)$ should be strictly less than the degree of the polynomial $q(x)$. So the maximum degree of $r(x)$ is $deg\ (q(x)) - 1$.

(b) If $deg\ p(x) < deg\ q(x)$, then quotient will be zero polynomial, hence $deg\ h(x) = 0$. The remainder will be $p(x)$ itself. So $deg\ r(x) = deg\ p(x)$.

3 Numerical Answer Type (NAT):

(Use the following data for the Question 10 and Question 11 only).
An open box can be made from a piece of cardboard of length $7x$ unit and breadth $5x$ unit, by cutting squares of side x unit out of the corners of the rectangular cardboard, then folding up the sides as shown in the Figure P-6.1.

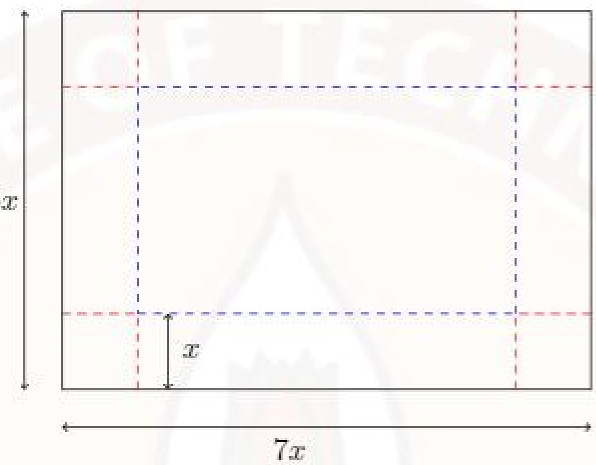


Figure P-6.1

10. What will be the coefficient of x^3 in the polynomial representing the volume of the box? [Answer:15]

11. What will be the coefficient of x^2 in the polynomial representing the volume of the box? [Answer:0]

    Solution: As the sides of the piece of the cardboard has been cut out, the length of the box made will be $7x - (x + x) = 5x$ unit and the breadth of the box made will be $5x - (x + x) = 3x$ unit, and the height will be x unit.
    Hence the volume of the box will be $5x \times 3x \times x = 15x^3$ cubic unit.

    (a) The coefficient of x^3 in the polynomial representing the volume of the box is 15.

    (b) The coefficient of x^2 in the polynomial representing the volume of the box is 0.

12. What should be subtracted from the polynomial $P(x) = 6x^4 + 5x^3 + 4x - 4$ to make it divisible by $2x^2 + x - 1$?

    ○ $4x$
    ○ $\mathbf{4x - 3}$
    ○ $6x - 3$
    ○ $2x - 3$

Solution:

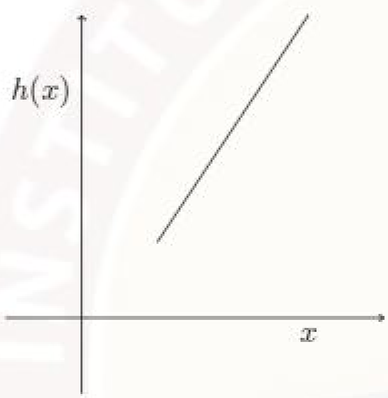
Using 4 step division algorithm, we find the remainder when $P(x)$ is divided by $2x^2+x-1$. If we subtract the obtained remainder from $P(x)$ then the resultant polynomial will be divisible by $2x^2 + x - 1$.
Now,

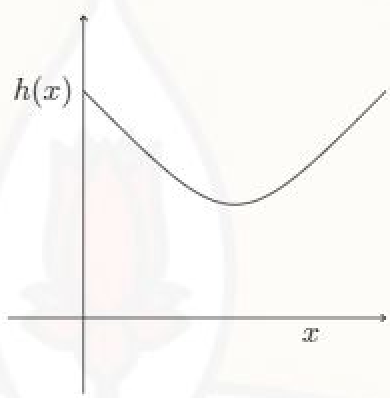
$6\text{x}^4 + 5x^3 + 0x^2 + 4x - 42x^2 + x - 1$ Therefore, when $P(x)$ is divided by $2x^2 + x - 1$, we get $4x - 3$ as the remainder.
Hence, $\boldsymbol{4x - 3}$ should be subtracted from the polynomial $P(x) = 6x^4 + 5x^3 + 4x - 4$ to make it divisible by $2x^2 + x - 1$.

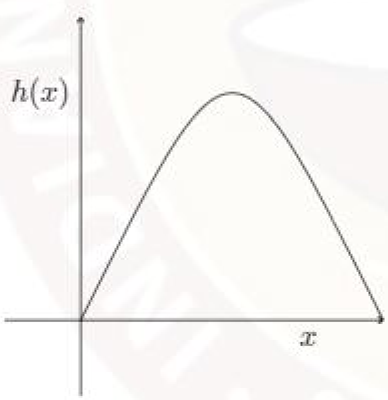
13. Table 1 provides the information regarding some polynomials. Which is the most suitable (not exact) representation of $h(x)$ where $h(x)$ is known to be a polynomial in x, and if $h(x) = \frac{P(x)Q(x)-R(x)S(x)+S(x)P(x)}{P(x)+P(x)Q^2(x)}$? [Ans: option D]

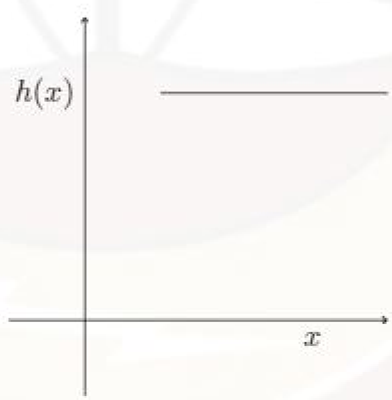
Polynomial	Degree	Condition
$P(x)$	m	$m > 0$
$Q(x)$	n	$m > 2n > 0$
$R(x)$	k	$k = m - n$
$S(x)$	t	$t = 2n$

Table 1

A. 

B. 

C. 

D. 

Solution:

Given, the degree of $P(x)$ is 'm' where $m > 0$, the degree of $Q(x)$ is 'n' where $m > 2n > 0$, the degree of $R(x)$ is 'k' where $k = m - n$, and the degree of $S(x)$ is 't' where $t = 2n$.

Also, $h(x) = \frac{P(x)Q(x)-R(x)S(x)+S(x)P(x)}{P(x)+P(x)Q^2(x)}$ and $h(x)$ is known to be a polynomial. The degree of $h(x)$ will be the difference between the degree of the numerator and the degree of the denominator. The degree of the numerator will be the degree of the term which has the highest degree in the numerator. Similarly, the degree of the denominator will be the degree of the term which has the highest degree in the denominator.

Now, the degree of the polynomial $P(x)Q(x)$ will be '$m+n$', the degree of $R(x)S(x)$ will be '$k+t = m-n+2n = m+n$', and the degree of $S(x)P(x)$ will be '$t+m = 2n+m$'.

Therefore, the degree of the numerator (polynomial $P(x)Q(x) - R(x)S(x) + S(x)P(x)$) will be '$m + 2n$'.
Similarly, the degree of the denominator (polynomial $P(x) + P(x)Q^2(x)$) will be $m+2n$.

As $h(x)$ is given to be a polynomial and also the degrees of the polynomials in the numerator and the denominator are same, we can conclude that the degree of $h(x)$ is zero i.e. $h(x)$ should be a constant.
So, option D is the most suitable representation of $h(x)$.

Use the following information to solve questions 14-16.

A manufacturing company produces three type of products A, B, and C from one raw material in a single continuous process. This process generates total solid wastes (W) (in kg) as $W(r) = -0.0001r^3 + 0.1r^2 + r$, where r is the amount of raw material used in kg. If instead, the company uses three different batch-processes (one batch process for one product) to produce the above products, then the amount of waste generated because of products A, B, and C are given as $W_A = -0.00001r^4 + 0.015r^3$, $W_B = -0.005r^3 + 0.05r^2$ and $W_C = 0.05r^2$ respectively.

(See the Figure A-5.1 for the reference.)

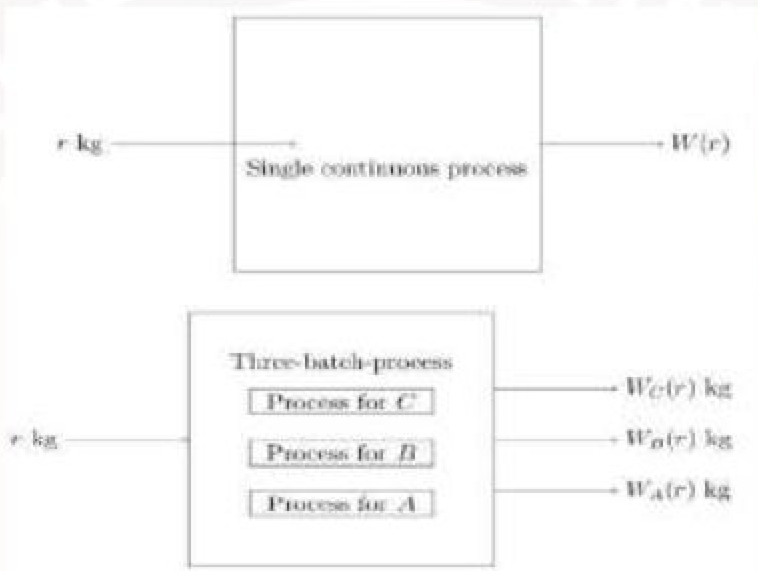


Figure A-5.1

14. What is the total amount of waste generated because of the three different batch-processes?

- ○ $-0.00001r^4 + 0.01r^3 + 1.5r^2$
- ○ $-0.00001r^4 + 0.015r^3 + 1.5r^2$
- ○ $\mathbf{-0.00001r^4 + 0.01r^3 + 0.1r^2}$
- ○ $-0.00001r^4 + 0.01r^3 + 0.5r^2$
- ○ $-0.00001r^4 + r^3 + 1.5r^2$
- ○ $0.0001r^4 + 0.01r^3 + 1.5r^2$

Solution:

The total amount of waste generated because of the three different batch-processes is

$$\begin{aligned} W_A + W_B + W_C &= -0.00001r^4 + 0.015r^3 - 0.005r^3 + 0.05r^2 + 0.05r^2 \\ &= \mathbf{-0.00001r^4 + 0.01r^3 + 0.1r^2} \end{aligned}$$

15. What is the ratio of the total waste generated by the three-batch-processes with respect to the single continuous process?

- ◯ $-0.001r^2$
- ◯ $-0.001r$
- ◯ $-0.01r$
- ◯ $-0.1r$
- ◯ $\mathbf{0.1r}$
- ◯ $0.01r$

Solution:

The total waste generated by the three-batch-processes is
$W_A + W_B + W_C = -0.00001r^4 + 0.01r^3 + 0.1r^2$.
The waste generated in the single continuous process is $W(r) = -0.0001r^3 + 0.1r^2 + r$.
The ratio of the total waste generated by the three-batch-processes with respect to the single continuous process is

$$\begin{aligned}\frac{W_A + W_B + W_C}{W(r)} &= \frac{-0.00001r^4 + 0.01r^3 + 0.1r^2}{-0.0001r^3 + 0.1r^2 + r} \\ &= \frac{(0.1r)(-0.0001r^3 + 0.1r^2 + r)}{-0.0001r^3 + 0.1r^2 + r} \\ &= \mathbf{0.1r}\end{aligned}$$

16. Let the company wastes Rs. 5,000 in waste treatment when it uses the single continuous process by consuming 100 kg of raw material. If instead of continuous process the company uses the three-batch-processes, then how much extra amount (in Rs.) will the company have to pay for waste treatment with respect to the continuous process?

    - ○ 50,000
    - ○ 500
    - ○ **45,000**
    - ○ 5,000
    - ○ 4,000

    Solution:

    As the ratio for waste generation (continuous to batch) is 10 we can calculate cost for waste management from batch process will be ten times of the continuous process.
    Therefore the cost for waste management from the batch process will be $5,000 \times 10 = 50,000$.
    So the the extra amount required is $50,000 - 5,000 = 45,000$.

4 Multiple Select Questions (MSQ):

17. Given $P(x)$ and $Q(x)$ be two non zero polynomials of degrees m and n respectively. If $f(x) = P(x) + Q(x)$, $g(x) = P(x)Q(x)$, and $h(x) = P(x)\{P(x)Q(x) + \frac{P(x)}{Q(x)}\}$, If $h(x)$ is known to be a polynomial in x,then choose the set of correct options.

    ○ The degree of $f(x)$ is $m + n$.

    ○ **The degree of $g(x)$ is $m + n$.**

    ○ **The degree of $f(x)$ is $max\{m, n\}$ if $m \neq n$, where $max\{m, n\}$ represents the maximum value from m and n.**

    ○ The degree of $h(x)$ is m^3.

    ○ The degree of $h(x)$ is n^3.

    ○ **The degree of $h(x)$ is $2m + n$.**

Solution:

Given, $P(x)$ and $Q(x)$ are two non zero polynomials of degree m and n respectively.
Also, $f(x) = P(x) + Q(x)$.
If $m > n$, then the degree of the polynomial $f(x)$ will be m, else if $m < n$, then the degree of the polynomial $f(x)$ will be n, else if $m = n$, then the degree of the polynomial will be less than or equal to m(or n).
Therefore, we can conclude that the degree of the polynomial $f(x)$ is $max\{m, n\}$ if $m \neq n$, where $max\{m, n\}$ represents the maximum value from m and n.
Hence, option 1 is incorrect, and option 3 is correct.

Now, $g(x) = P(x)Q(x)$, the degree of the polynomial $g(x)$ will be the sum of the degrees of the polynomials $P(x)$ and $Q(x)$.
Therefore, the degree of $g(x)$ is $m + n$. Hence, option 2 is correct.

Finally, $h(x) = P(x)\{P(x)Q(x) + \frac{P(x)}{Q(x)}\} = (P(x))^2Q(x) + \frac{(P(x))^2}{Q(x)}$.
The degree of the polynomial $(P(x))^2Q(x)$ will be $2m + n$ and as given that $h(x)$ is a polynomial implies $Q(x)$ divides $(P(x))^2$, so the degree of the polynomial $\frac{(P(x))^2}{Q(x)}$ will be $2m - n$.
Since $2m + n > 2m - n$, the degree of the polynomial $h(x)$ is $2m + n$. Hence, options 4 and 5 are incorrect, and option 6 is correct.

18. Given a polynomial $P(x) = (2x+5)(1-3x)(x^2+3x+1)$, then choose the set of correct options.

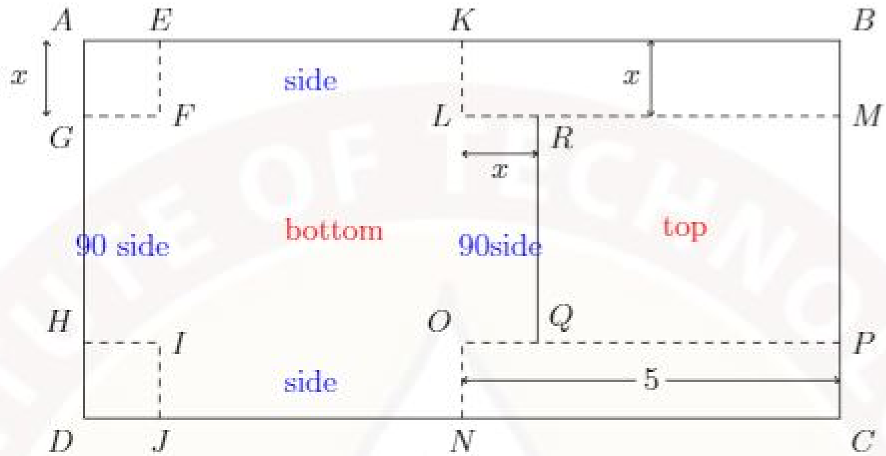
- ◯ **Coefficient of x^5 is 0.**
- ◯ Coefficient of x^3 is -18.
- ◯ **Degree of P is 4.**
- ◯ Coefficient of x^3 is -13.
- ◯ Degree of P is 7.
- ◯ All of the above.

Solution:

$$\begin{aligned}
\text{Given, } P(x) &= (2x+5)(1-3x)(x^2+3x+1) \\
&= (2x+5-6x^2-15x)(x^2+3x+1) \\
&= (5-6x^2-13x)(x^2+3x+1) \\
&= 5x^2-6x^4-13x^3+15x-18x^3-39x^2+5-6x^2-13x \\
&= -6x^4-31x^3-40x^2+2x+5
\end{aligned}$$

Option 1 is correct, because there is no x^5 term in the polynomial $P(x)$. So, the coefficient of x^5 is 0.
The degree of the polynomial $P(x)$ is 4. Hence, option 3 is correct and option 5 is incorrect.
The coefficient of x^3 is -31. Hence, options 2 and 4 are incorrect.

19. A sheet $ABCD$ of dimensions 10 ft x 3 ft is shown in Figure 7. A box is made by removing two squares of equal dimensions $AEFG$ and $DHIJ$ and two rectangles of equal dimensions $BKLM$ and $CNOP$ respectively.



- ◯ The volume of the box is $2x^2 - 23x + 30$.
- ◯ **The volume of the box is $\boldsymbol{2x^3 - 13x^2 + 15x}$.**
- ◯ If $x = 0.5$, then the volume of the box is 5.625 cubic ft.
- ◯ To create the box, value of x should always be greater than 0 but less than 1.5.

Solution:
From Figure A-6.1, the length of the box will be $EK = AB - KB - AE = 10 - 5 - x = 5 - x$, the breadth of the box will be $GH = AD - AG - HD = 3 - x - x = 3 - 2x$, and the height of the box will be $AE = x$.
Therefore, the volume of the box V given by length $\times$ breadth $\times$ height will be

$$V = (5 - x)(3 - 2x)(x)$$
$$V = (15 - 3x - 10x + 2x^2)(x)$$
$$V = 2x^3 - 13x^2 + 15x$$

Hence, options 1 is incorrect, and option 2 is correct.

If $x = 0.5$, then the volume of the box

$$V = 2x^3 - 13x^2 + 15x$$
$$V = 2(0.5)^3 - 13(0.5)^2 + 15(0.5)$$
$$V = 2(0.625) - 13(0.25) + 7.5$$
$$V = 1.25 - 3.25 + 7.5 = 5.5$$

Hence, option 3 is incorrect.

Now, given the volume of the cubical bar of soap is $(5 - x)$ cubic ft. and we know the volume of the box is $2x^3 - 13x^2 + 15x$.
So, the maximum bars of soap that can be packed in the box $= \frac{2x^3-13x^2+15x}{5-x}$

$2x^3 - 13x^2 + 15x5 - x$ Therefore, at most $-2x^2 + 3x = \boldsymbol{2x(1.5 - x)}$ bars of soap can be packed in the box. Hence, option 4 is correct.

5 Numerical Answer Type (NAT):

20. A curious student created a performance profile of his favourite cricketer as $R = -x^5 + 6x^4 - 30x^3 + 80x^2 + 70x + c$, where R is the total cumulative runs scored by the cricketer in x matches. He picked three starting values shown in Table 2 and tried to find the value of c. If he uses Sum Squared Error method, then what will be the value of c?[Ans: -2]

No. of matches	Total score
1	120
2	285
3	361

Table 2

Solution:

Let us calculate the predicted cumulative runs scored by the player in the first three matches.
Substituting $x = 1, 2, 3$ in the given function, we get

$$\begin{aligned}
R(1) &= -(1)^5 + 6(1)^4 - 30(1)^3 + 80(1)^2 + 70(1) + c \\
&= -1 + 6 - 30 + 80 + 70 + c \\
&= 125 + c \\
R(2) &= -(2)^5 + 6(2)^4 - 30(2)^3 + 80(2)^2 + 70(2) + c \\
&= -32 + 96 - 240 + 320 + 140 + c \\
&= 284 + c \\
R(3) &= -(3)^5 + 6(3)^4 - 30(3)^3 + 80(3)^2 + 70(3) + c \\
&= -243 + 486 - 810 + 720 + 210 + c \\
&= 363 + c
\end{aligned}$$

Now, let us find the sum squared error of cumulative score for these three matches.

$$\begin{aligned}
\text{SSE} &= \sum_{n=1}^{3}(R(n) - y_n)^2, \text{where } y_n \text{ is the total cumulative score in } n \text{ matches.} \\
&= (R(1) - y_1)^2 + (R(2) - y_2)^2 + (R(3) - y_3)^2 \\
&= (125 + c - 120)^2 + (284 + c - 285)^2 + (363 + c - 361)^2 \\
&= (5 + c)^2 + (c - 1)^2 + (2 + c)^2 \\
&= 25 + 10c + c^2 + c^2 - 2c + 1 + 4 + 4c + c^2 \\
&= 3c^2 + 12c + 30
\end{aligned}$$

We have to find the value of c such that SSE becomes minimum, this is equal to the minimum value of the quadratic equation $3c^2 + 12c + 30$.
We know that the minimum value of any quadratic function of form $f(x) = Ax^2+Bx+D$, occurs at $x = \frac{-B}{2A}$. Here, $A = 3, B = 12$
So, the minimum value of the quadratic equation $3c^2 + 12c + 30$, occurs at
$c = \frac{-B}{2A} = \frac{-12}{2(3)} = -2$
Therefore, the minimum SSE is obtained when the value of c is $\mathbf{-2}$.

21. What is the minimum value of $x-$coordinate for the points of intersection of functions $f(x) = -x^5 + 5x^4 - 7x - 2$ and $g(x) = -x^5 + 5x^4 - x^2 - 2$?

Solution:

At the points of intersection, observe that $f(x) = g(x)$.
Here, $f(x) = -x^5 + 5x^4 - 7x - 2$ and $g(x) = -x^5 + 5x^4 - x^2 - 2$.
Equating the functions we get,

$$\begin{aligned}
-x^5 + 5x^4 - 7x - 2 &= -x^5 + 5x^4 - x^2 - 2 \\
-7x &= -x^2 \\
x^2 - 7x &= 0 \\
x(x - 7) &= 0 \\
\implies x &= 0 \text{ (or) } x = 7
\end{aligned}$$

Therefore, the minimum value of x- coordinate for the points of intersection of functions $f(x)$ and $g(x)$ is $\mathbf{0}$.

Week 4 Graded assignment Solution
Mathematics for Data Science - 1

1 Multiple Choice Questions (MSQ):

1. Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be two functions, defined as $f(x) = x^3 - 8x^2 + 7$ and $g(x) = -2f(x)$ respectively. Choose the correct option(s) from the following

   - ○ f has two turning points and there are no turning points with negative $y-$coordinate.
   - ○ g has two turning points and $y-$coordinate of only one turning point is positive.
   - ○ g has two turning points and there are no turning points with negative $y-$coordinate.
   - ○ f is strictly increasing in $[10, \infty)$.

$f(x) = x^3 - 8x^2 + 7$

By hit and trial Method : $f(1) = 1 - 8 + 7 = 0$.

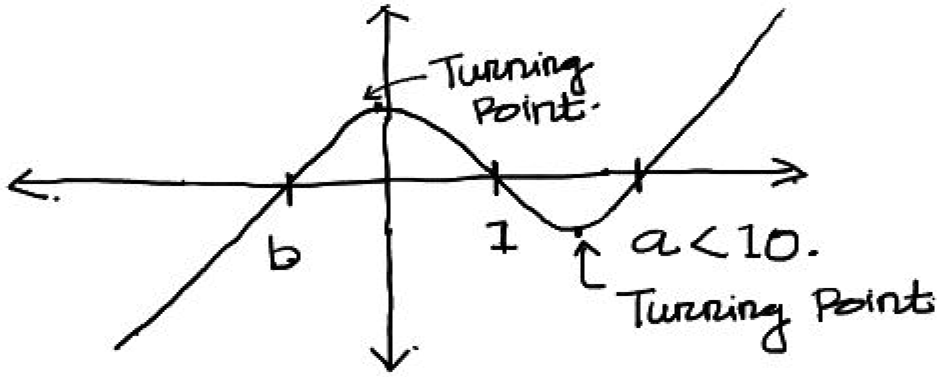
$$x^3 - 8x^2 + 7 = x^3 - x^2 - 7x^2 + 7$$
$$= x^2(x-1) - 7(x^2-1) = (x-1)(x^2 - 7(x+1))$$
$$= (x-1)(x^2 - 7x - 7).$$

$f(x)$ - has three real distinct roots: Say, 1, a, b.

Note: a, b are less than 10.

If $x \mapsto +\infty \Rightarrow x^3 \mapsto +\infty \Rightarrow f(x) \mapsto +\infty$.

Then the graph of $f(x)$ is Similar to;



(Graph of F(x)).

It is clear from the graph that "f" has two turning Point.

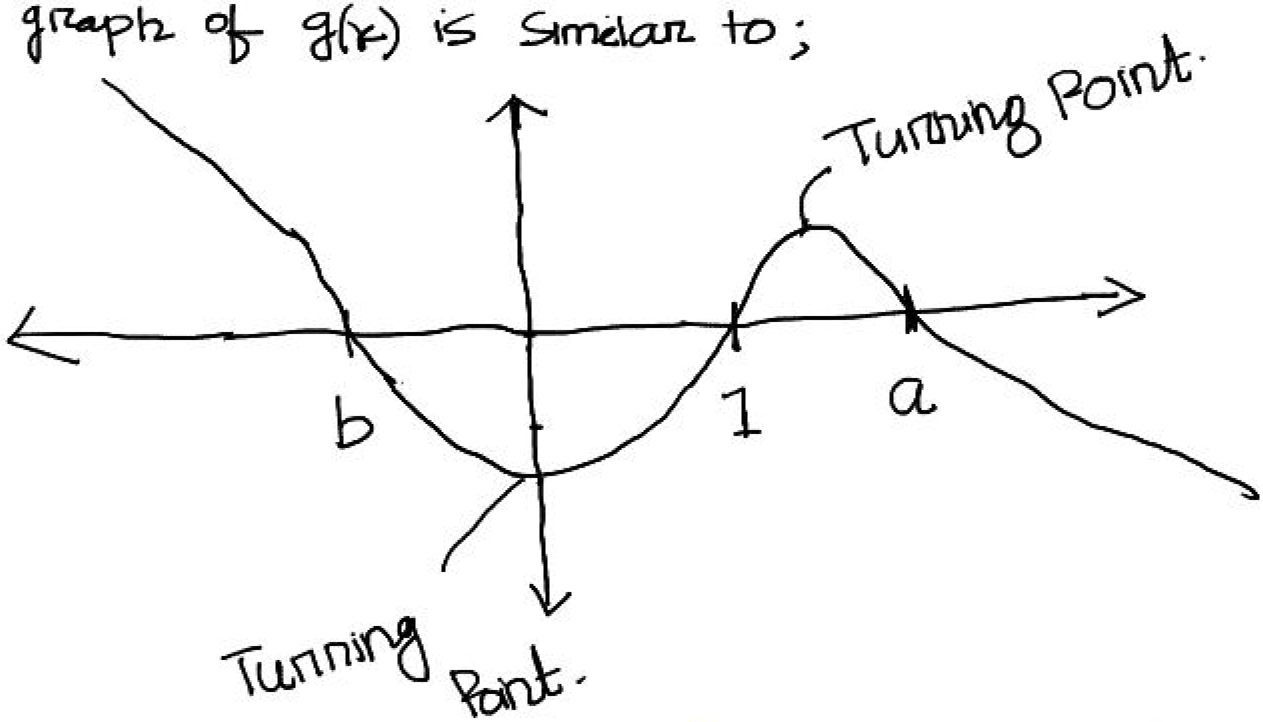
→ Y-coordinate of one of its turning Point is +ve.

→ Y-coordinate of one of its turning Point is -ve.

• Hence option-1 is incorrect and option-4 is correct.

$g(x) = -2f(x)$.

The graph of g(x) is Similar to;



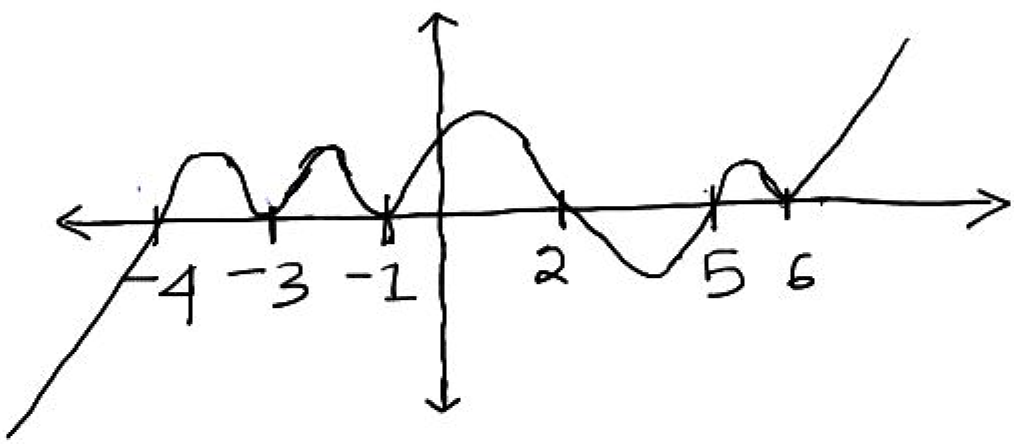
Hence option-2 is correct & option-3 is incorrect

2. Which among the following function first increases and then decreases in all the intervals $(-4,-3)$ and $(-1,2)$ and $(5,6)$.

- ○ $\frac{-1}{10000}(x+1)^2(x-2)(x+3)^2(x+4)(5-x)(x-6)^2$.
- ○ $\frac{1}{10000}(x+1)^2(x-2)(x+3)(x+4)(x-5)^2(x-6)^2(x+7)^2$.
- ○ $\frac{1}{10000}(x+1)^2(x-2)(x+3)^2(x+4)(x-5)^2(x-6)^2$
- ○ $\frac{-1}{10000}(x+1)^2(x-2)(x+3)^2(x+4)^2(5-x)^2(x-6)^2(3-x)$

We will solve this question using the graph of functions given in each options.

Option 1:



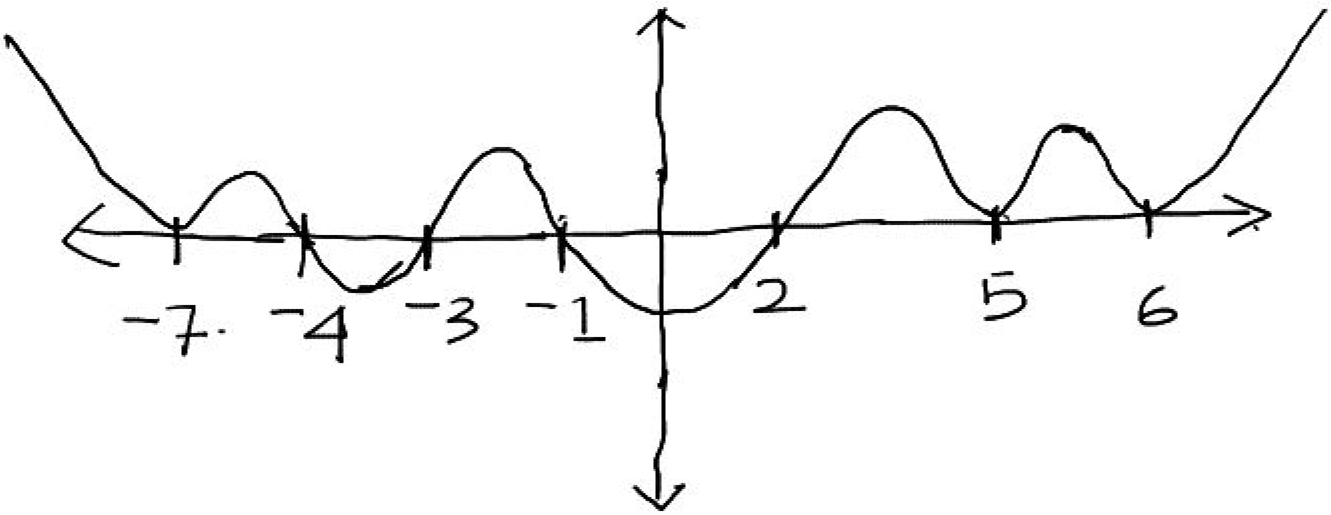
In this case $f(x)$ increases and then decreases in all the intervals

$$(-4,-3) \cup (-1,2) \cup (5,6).$$

Hence option-1 is correct.

Option 2:

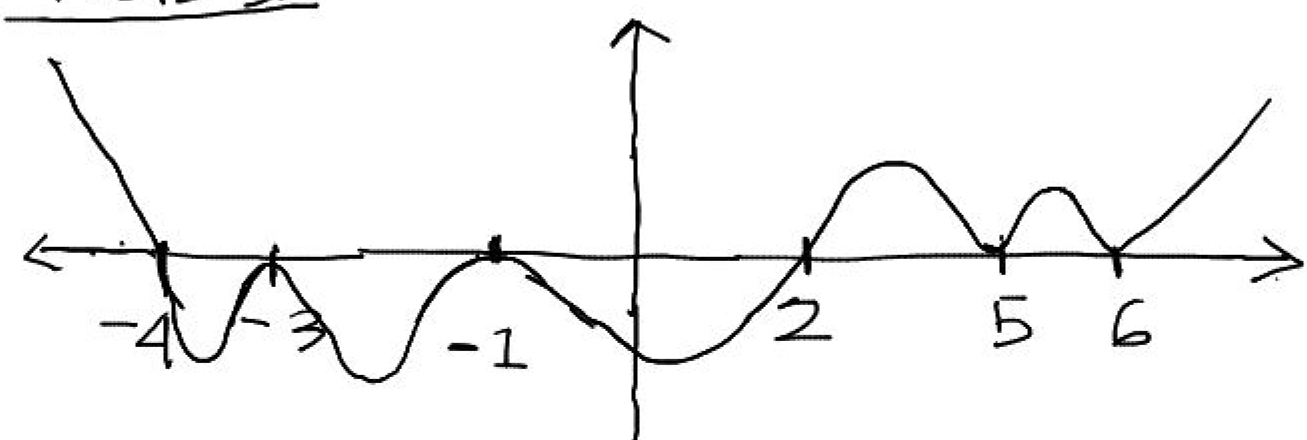
$-1, 2, -3, -4, 5, 6, -7.$



In $(-4,-3)$ & $(-1,2)$, the function first decreases and then increases.

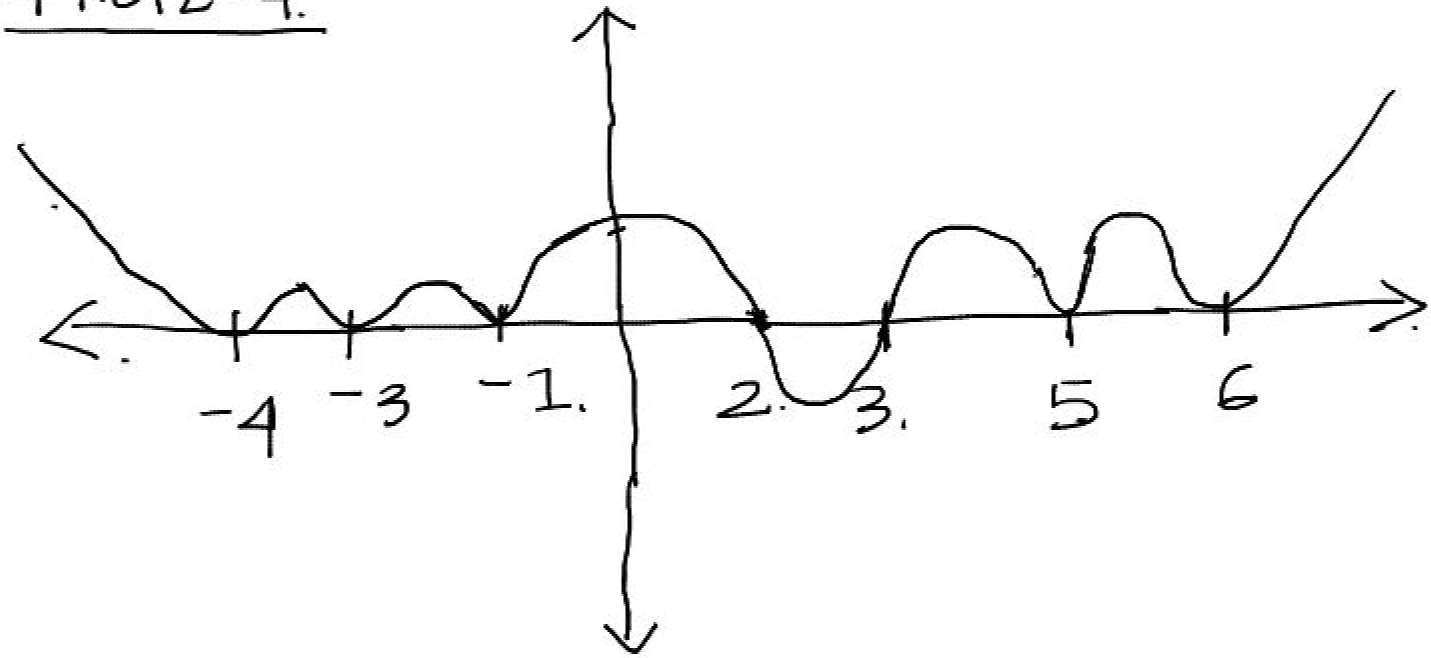
Hence Option-2 is incorrect.

Option-3



Option-3 is incorrect.

Option-4.



Hence option-4 is correct.

Question 3
5014

line $l(x) = x$

$f(x) = x^4 - 13x - 42$

$g(x) = x^2 - x - 6$

lets find $r(x)$ which is quotient polynomial when $f(x)$ is divided by $g(x)$

$$
\begin{array}{r|l}
 & x^2 + x + 8 \\
\hline
x^2 - x - 6 & \cancel{x^4} - 13x - 42 \\
 & \cancel{x^4} \underset{+}{-} x^3 \underset{+}{-} 6x^2 \\
 & - \\
\hline
 & \cancel{x^3} + 6x^2 - 13x - 42 \\
 & \cancel{x^3} \underset{+}{-} x^2 \underset{+}{-} 6x \\
 & - \\
\hline
 & \cancel{8x^2} - 7x - 42 \\
 & \cancel{8x^2} \underset{+}{-} 8x \underset{+}{-} 48 \\
 & - \\
\hline
 & x + 6
\end{array}
$$

So $r(x) = x^2 + x + 8$

Now two find the intersection point

$$l(x) = r(x)$$

$$\Rightarrow \quad x^2 + x + 8 = x$$

$$\Rightarrow \quad x^2 + 8 = 0$$

Observe that this equation has

no root.

So, there is no intersection point.

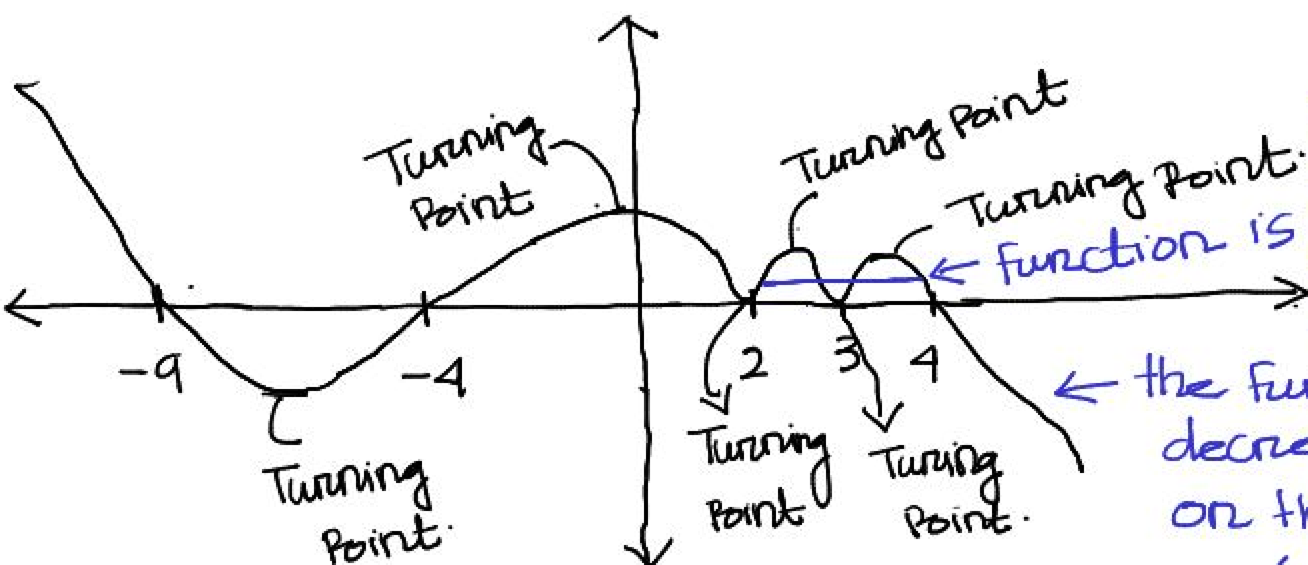
4. Consider a polynomial function $p(x) = -\left(x^2 - 16\right)(x-3)^2(2-x)^2(x+9)$. Choose the set of correct options.

- ○ There are exactly 7 points on $p(x)$ where the slope of the tangent is 0.
- ○ There are exactly 6 points on $p(x)$ where the slope of the tangent is 0.
- ○ $p(x)$ is strictly decreasing when $x \in (4, \infty)$.
- ○ $p(x)$ is one-one function when $x \in (2, 4)$.

- $P(x) = -(x^2 - 16)(x-3)^2(2-x)^2(x+9)$

  $= -(x+4)(x-4)(x-3)^2(x-2)^2(x+9)$

The graph of $P(x)$ is Similar to ; $\quad 4, -4, 3, 2, -9$



[Horizontal line test] ← function is not one-one on the interval (2, 4).

← the function decreases on the interval $(4, \infty)$.

- It is clear from the graph that the function has '6' turning points and there are exactly 6 points Where the Slope of the function is zero.

Hence option-2 is correct & option-1 is incorrect.

- Clear from the graph that the function is strictly decreasing on $(4, \infty)$.

Hence option-3 is correct.

- By Horizontal line test function is not one-one on $(2, 4)$.

option-4 is incorrect.

Question 5.

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$$

has following prop:-

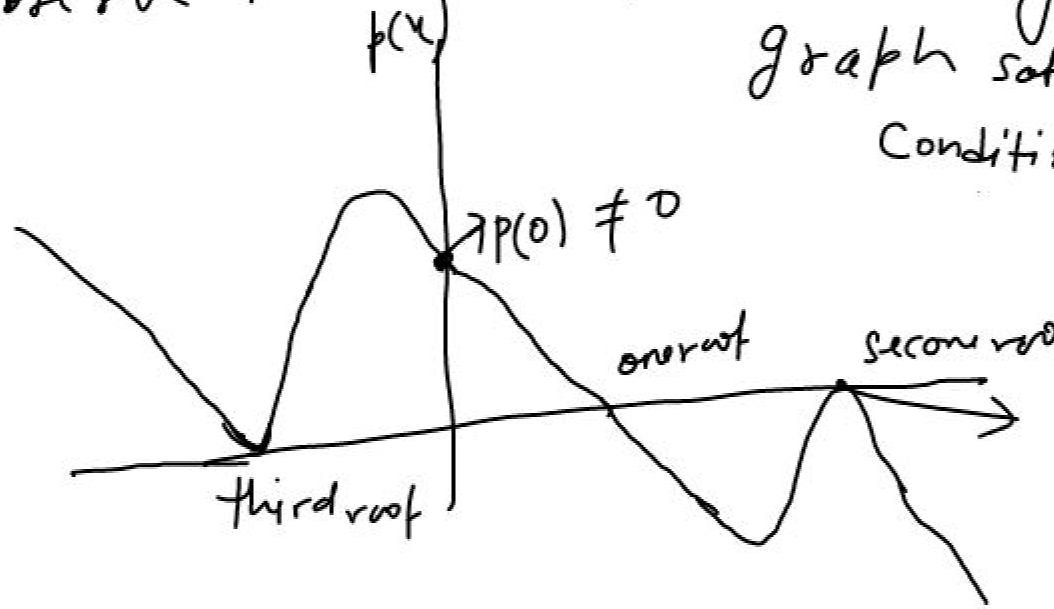
(I) $p(x)$ is an odd degree polynomial with at least three distinct roots.

(II) $p(x)$ has exactly two distinct positive real roots.

(III) $(x-5)^2$ is a factor of $p(x)$

(IV) $p(0) \neq 0$

Observe that $p(x)$ is only in this graph satisfies all conditions.

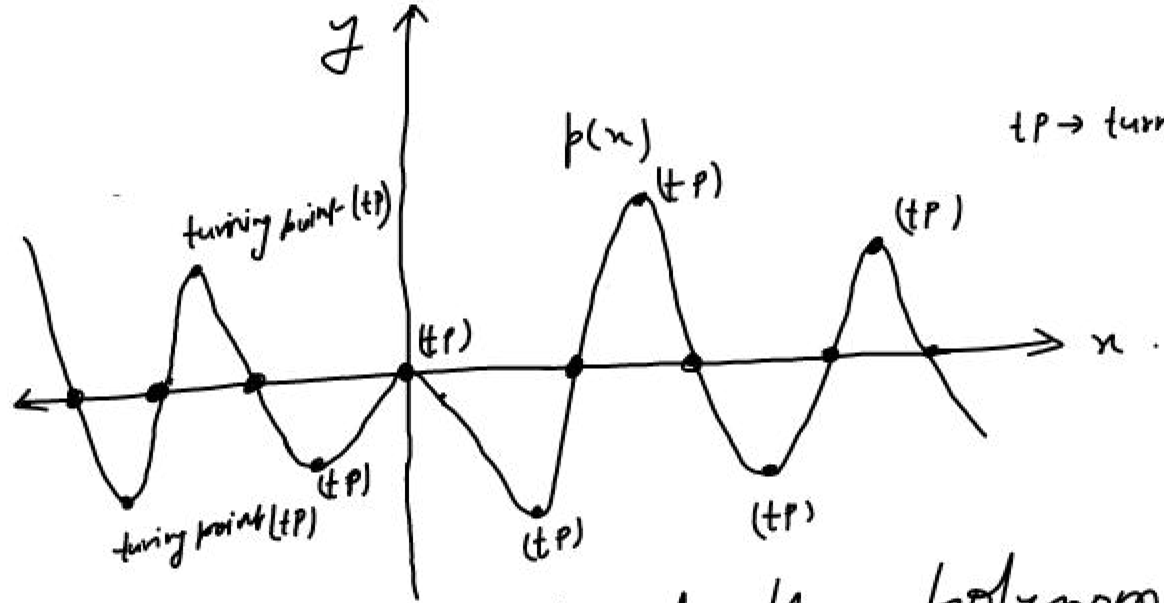


As 5 is a root with even multiplicity

Question 6

Sol^n

tP → turning point.



Option 1

Observe from the graph of the polynomial

Except at origin rest roots have only one multiplicity as graph of the polynomial is linear when it is intersecting the x-axis. And at the origin graph touches the and bounce back so it 0 is a root with even multiplicity (at least 2 multiplicity).

So p(x) has 8 roots and as 0 is a root with even multiplicity (at least 2 multiplicity) and so degree of p(x) is atleast 9. (Number of roots decides that, that number of factors are there in $p(x)$)).

Option2 As we calculated above degree of $p(x)$ is atleast 9. In this $p(x)$ is an odd degree polynomial.

Suppose $p(x)$ is an even degree polynomial, In this case surly any one of roots other than zero is also have even multiplicity But except zero root there is no root which have even multiplicity. Hence $p(x)$ is an odd degree polynomial.

Option3 Turning point are nothing but points where function value is changing from increasing to decreasing or from decreasing to increasing. And from graph it is clear that it have 8 turning points..

Option4: Observe that multiplicity of root zero is even (as we discussed in option1) and none of the root has this even multiplicity

7. An ant named B, wants to climb an uneven cliff and reach its anthill (i.e., home of ant). On its way home, B makes sure that it collects some food. A group of ants have reached the food locations which are at x−intercepts of the function $f(x) = (x^2 - 19)\left((x-9)^3 - 1\right)$. As ants secrete pheromones (a form of signals which other ants can detect and reach the food location), B gets to know the food location. Then the sum of the x-coordinates of all the food locations is

• x-coordinate of all the food locations are same as the roots of the polyno-mial

$$P(x) = (x^2 - 19)\left((x-9)^3 - 1\right).$$

$P(x) = 0 \Rightarrow x^2 - 19 = 0$ or $(x-9)^3 - 1 = 0$

$\Rightarrow x = \pm\sqrt{19}$

only real root of $(x-9)^3 = 1$ is 1.

$\Rightarrow (x-9) = 1.$

$\Rightarrow x = 9 + 1 = 10$

Sum of x-coordinates of the food locations

= sum of roots of the polynomial $P(x)$.

$= \sqrt{19} - \sqrt{19} + 10 = 10.$

8. The Ministry of Road Transport and Highways wants to connect three aspirational districts with two roads r_1 and r_2. Two roads are connected if they intersect. The shape of the two roads r_1 and r_2 follows polynomial curve $f(x) = (x-19)(x-17)^2$ and $g(x) = -(x-19)(x-17)$ respectively. What will be the x-coordinate of the third aspirational district, if the first two are at x-intercepts of $f(x)$ and $g(x)$.

- We need find the x-coordinate of the points where the two roads r_1 and r_2 intersect. This is same as the values of x for which

$$f(x) = g(x).$$

$$\Rightarrow (x-19)(x-17)^2 = -(x-19)(x-17).$$

$$\Rightarrow (x-19)(x-17)^2 + (x-19)(x-17) = 0$$

$$\Rightarrow (x-19)(x-17)(x-17+1) = 0$$

$$\Rightarrow (x-19)(x-17)(x-16) = 0$$

$$\Rightarrow x = 19, 17, 16.$$

x-coordinates of the x-intercepts of $f(x)$ & $g(x)$ are 19, 17.

Hence the x-coordinate of the third aspirational district is 16.

Question 9 $M(n) = -\left(\frac{n^2}{1000}\right)(n^3-15n^2+50n)+40, \; n \in \{1,2\cdots 12\}$

To get number of times Ritwik score 40

$M(n) = 40$

$\Rightarrow -\left(\frac{n^2}{1000}\right)(n^3-15n^2+50n)+40 = 40$

$\Rightarrow n^3-15n^2+50n = 0$

$\Rightarrow n(n^2-15n+50) = 0$

$\Rightarrow n(n-10)(n-5) = 0$

$\Rightarrow n=0$ or $n=10$ or $n=5$

but $n \neq 0$ as $n \in \{1,2,\cdots 12\}$

So two times Ritwik got excatly 40 marks.

Question 10 To get pass Ritwik marks should be greater than 40 i.e

$M(n) \geq 40$.

$\Rightarrow \left(-\frac{n^2}{1000}\right)(n^3-15n^2+50n)+40 \geq 40$

$\Rightarrow \left(\frac{-n^2}{1000}\right)(n^3-15n^2+50n) \geq 0$

$\Rightarrow n^3(n-10)(n-5) \leq 0$

From inequality, we can conclude that if

$n \in \{5,6,7,8,9,10\}$ then $n^3(n-10)(n-5) \leq 0$

So total in 6 mocks test Ritwik passed.

Weekly Mock Week 1-2 (Solution).

Time: 2 Hours: Full Marks: 25

Mathematics for Data Science - 1

1. If A and B are sets and $A \cup B = A \cap B$ then which of the following is(are) true? [Marks: 1]

   - ◯ Either one of A or B is empty set.
   - ◯ B is proper subset of A.
   - ◯ $\boldsymbol{A = B}$.
   - ◯ A is proper subset of B.

Soln :- Option 1 :- let $A = \phi$ (Empty set) and $B = \{1\}$ then $A \cup B = \{1\}$

but $A \cap B = \phi \cap \{1\} = \phi$. hence $A \cup B \neq A \cap B$.

option 2, 4: Let $A = B = \{1\}$. Then $A \cup B = A \cap B = \{1\}$

but B is not a proper set of A also A is not a proper subset of B.

option 3 :- Given $A \cap B = A \cup B$

$A \subseteq A \cup B \Rightarrow A \subseteq A \cap B \subseteq B \quad (\because A \cap B \subseteq B)$

$\Rightarrow A \subseteq B$.

again $B \subseteq A \cup B \Rightarrow B \subseteq A \cap B \subseteq A \quad (\because A \cap B \subseteq B)$

$\Rightarrow B \subseteq A$

Hence $A = B$

2. Points A $(4,4)$, B $(-3,-3)$, and C (m,n) are collinear. If points D $(-2,2)$, E $(-5,5)$, and C are also collinear, then what will $m-n$ be? [Answer: 0] [Marks: 1]

Solu: Given A, B & C are collinear.

So slope of the line joining A and B is $\frac{-3-4}{-3-4} = 1$

and slope of the line joining B and C is $\frac{n+3}{m+3}$

Since A, B and C are collinear so $\frac{n+3}{m+3} = 1 \Rightarrow n+3 = m+3$

$\Rightarrow n = m \Rightarrow n-m=0$

Now, slope of the line joining D and E is $\frac{5-2}{-5+2} = -1$

Slope of the line joining E and C is $\frac{n-5}{m+5}$

Since D, E & C are collinear so $\frac{n-5}{m+5} = -1$

$\Rightarrow n-5 = -m-5$

$\Rightarrow n+m = 0$

Hence $m = n = 0 \Rightarrow m-n = 0$

3. Which of the following pairs of straight lines are perpendicular to each other?[Marks: 2]

- ◯ $\boldsymbol{x + y = 1}$ **and** $\boldsymbol{x - y = -1}$
- ◯ $x = 0$ and $x = y$
- ◯ $\boldsymbol{2x + 3y = 9}$ **and** $\boldsymbol{\frac{y}{12} - \frac{x}{8} = 3}$
- ◯ $\frac{x}{8} + \frac{y}{9} = 1$ and $\frac{x}{8} - \frac{y}{9} = 1$

Soln! option 1!. Slope of the line $x+y=1$ is $m_1 = -1$

Similarly, slope of the line $x-y=-1$ is $m_2 = 1$

Now $m_1 \cdot m_2 = (-1)\cdot(1) = -1$

Hence lines $x+y=1$ & $x-y=-1$ are perpendicular to each other.

To get slope! change $x+y=1$ in intercept form $y=-x+1$ and compare it with $y=mx+c$, will get slope.

Similarly we can check for other options.

4. Let R be a relation on a collection of sets defined as follows,

$$R = \{(A, B) \mid A \subseteq B\}$$

Then pick out the correct statement(s). [Marks: 2]

- ◯ **R is reflexive and transitive.**
- ◯ R is symmetric.
- ◯ **R is antisymmetric.**
- ◯ R is reflexive but not transitive.

Soln:- Given $R = \{(A,B) \mid A \subseteq B\}$

Let $\mathcal{V}$ be ~~the~~ a collection of sets and $A \in \mathcal{V}$.

We know $A \subseteq A$ itself, so $(A, A) \in R, \forall A \in \mathcal{V}$

Hence R is a reflexive relation.

Let $(A,B) \in R \Rightarrow A \subseteq B$, where $A, B \in \mathcal{V}$ but $B \subseteq A$ is not true in general. So, (B,A) may be an element of R. Hence R is not a symmetric relation.

Let $(A,B) \in R$, $(B,C) \in R \Rightarrow A \subseteq B$ and $B \subseteq C$, where $A, B, C \in \mathcal{V}$

$$\Rightarrow A \subseteq C$$

$$\Rightarrow (A,C) \in R$$

Hence R is a transitive relation.

Let (A,B) and $(B,A) \in R \Rightarrow A \subseteq B$ and $B \subseteq A$, where $A, B \in \mathcal{V}$

$$\Rightarrow A = B$$

So R is an antisymmetric relation.

5. A study conducted at a factory manufacturing chemicals C_1 and C_2 yielded the following results. If a worker is exposed to either C_1 or C_2 (but not both), then the worker is at a low risk of developing a skin disorder. However, if the worker is exposed to both C_1 and C_2, then he is at a high risk of contracting the skin disorder. If out of 370 workers in the factory, 320 are exposed to C_1 and 90 are exposed to C_2, then how many workers are at high risk of developing the skin disorder? (Assume that every worker is exposed to at least one of the two chemicals C_1, C_2.) [Answer: 40] [Marks: 2]

Solu:- Total number of workers are $n(C_1 \cup C_2) = 370$

Number of workers exposed to C_1 is $n(C_1) = 320$.

Number of workers exposed to C_2 is $n(C_2) = 90$

Now, $n(C_1 \cup C_2) = n(C_1) + n(C_2) - n(C_1 \cap C_2)$

$\Rightarrow \quad 370 = 320 + 90 - n(C_1 \cap C_2)$

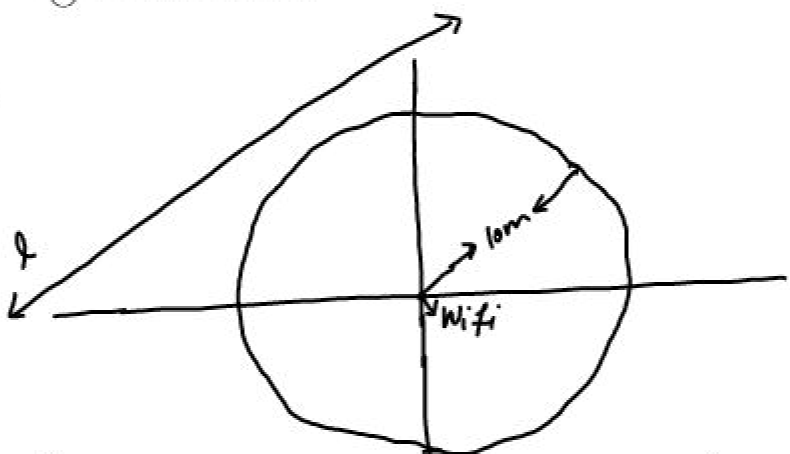
$\Rightarrow \quad n(C_1 \cap C_2) = 410 - 370 = 40$

Hence, number of workers exposed to high risk of developing the skin disorder is $n(C_1 \cap C_2) = 40$.

6. A Wi-Fi router is kept at position $(0, 0)$ and its maximum transmission range is 10m. If Rohan is walking along a straight line ℓ and his phone is in the transmission range of the Wi-Fi router for a brief duration, then ℓ could be (assume $x \in [-30, 30]$ for all the options and unit length is 1m): [Marks: 3]
(MSQ)

- ◯ $x - y - 5 = 0$
- ◯ $x - y - 10 = 0$
- ◯ $x - y - 15 = 0$
- ◯ $y = 9$
- ◯ All of the above.

Soln: -



Let l be the line and range of wifi router is shown in the figure above.

Rohan is walking along the line l and will get at least wifi transimission if distance of the line l from origin is less than 10m.

option1: Let Rohan be walking along the line $x-y-5=0$.
Then distance of line $x-y-5=0$ from origin

$$\text{is} \quad \frac{|-5|}{\sqrt{1^2+1^2}} = \frac{5}{\sqrt{2}} \leq 10$$

Hence, Rohan will get wifi transmission.

Similarly, we can check for others.

7. Let us define a function $f : \mathbb{Z} \to \mathbb{Z}$ as follows, [Marks: 2]

$$f(x) = \begin{cases} \frac{x}{2}, & \text{if } x \text{ is even} \\ 0, & \text{if } x \text{ is odd} \end{cases}$$

- ◯ **onto but not one to one.**
- ◯ one to one but not onto.
- ◯ one to one and onto.
- ◯ neither one to one nor onto.

Soln :- We have $3, 5 \in \mathbb{Z}$

Now $f(3) = 0$ and $f(5) = 0$ also,

so for different inputs 3, 5 from the domain, we get the same output. Hence f is not one-one.

Again let $a \in \mathbb{Z}$ (codomain), then we have $2a \in \mathbb{Z}$ (domain) such that $f(2a) = \frac{2a}{2} = a$

Hence f is onto.

8. Let us define a function $f : \mathbb{N} \to \mathbb{N}$ such that $f(0) = 0$, $f(1) = 1$, and $f(n) = f(n-1) + f(n-2)$ for all $n \geq 2$. Define the following sets

- $S_1 := \{n \mid f(n) \text{ is even}, n \leq 10\}$
- $S_2 := \{n \mid f(n) \text{ is multiple of } 3, n \leq 10\}$

Which of the following is (are) true? [Marks: 3]
(MSQ)

- ◯ Cardinality of $S_1 \setminus S_2$ is 1.
- ◯ **Cardinality of $S_1 \cap S_2$ is 1.**
- ◯ **Cardinality of $S_2 \setminus S_1$ is 2.**
- ◯ **Cardinality of $S_1 \cup S_2$ is 6.**

Solu: We have $f(0) = 0, \; f(1) = 1$

So $f(2) = f(2-1) + f(2-2) = f(1) + f(0) = 1 + 0 = 1$

$f(3) = f(2) + f(1) = 1 + 1 = 2$

$f(4) = f(3) + f(2) = 2 + 1 = 3$

$f(5) = f(4) + f(3) = 3 + 2 = 5$

$f(6) = f(5) + f(4) = 5 + 3 = 8$, $\quad f(8) = f(7) + f(6) = 13 + 8 = 21$

$f(7) = f(6) + f(5) = 8 + 5 = 13$ $\quad f(9) = f(8) + f(7) = 21 + 13 = 34$

$f(10) = f(9) + f(8) = 34 + 21 = 55$

So $S_1 = \{f(0), f(3), f(6), f(9)\} = \{0, 2, 8, 34\}$

$S_2 = \{f(0), f(4), f(8), f(10)\} = \{0, 3, 21\}$

So $S_1 \cap S_2 = \{0\} \Rightarrow |S_1 \cap S_2| = 1$

$S_2 \setminus S_1 = \{3, 21\} \Rightarrow |S_2 \setminus S_1| = 2$

$S_1 \cup S_2 = \{0, 2, 3, 8, 21, 34\} \Rightarrow |S_1 \cup S_2| = 6$

($|\cdot|$ denote the cardinality of the set).

9. Which of the following is (are) rational number(s)? [Marks: 2]

- ○ $(3+3\sqrt{5})(2-2\sqrt{5})$
- ○ $\frac{\sqrt{64}}{\sqrt{25}}$
- ○ $\frac{3+\sqrt{5}}{3-\sqrt{5}}$
- ○ $2^{\frac{1}{3}}$

Soln :- option 1: $(3+3\sqrt{5})(2-2\sqrt{5}) = 6 - \cancel{6\sqrt{5}} + \cancel{6\sqrt{5}} - 30 = -24$ a rational number.

option 2: $\frac{\sqrt{64}}{\sqrt{25}} = \frac{8}{5}$ a rational number

Option 3: $\frac{3+\sqrt{5}}{3-\sqrt{5}} = \frac{(3+\sqrt{5})(3+\sqrt{5})}{(3-\sqrt{5})(3+\sqrt{5})} = \frac{(3+\sqrt{5})^2}{3^2-(\sqrt{5})^2}$

$$= \frac{9+5+6\sqrt{5}}{9-5}$$

$$= \frac{14}{4} + \frac{6\sqrt{5}}{4}$$ is not a rational number as it contains $\sqrt{5}$.

option 3: Let $2^{1/3}$ be rational number.

i.e $2^{1/3} = p/q$ where $\gcd(p,q) = 1$

Taking cube both sides, we get

$$2 = p^3/q^3$$

$$\Rightarrow p^3 = 2q^3$$

i.e p is an even number.

Let $p = 2m$, $m \in \mathbb{Z}$ $\Rightarrow p^3 = 8m^3$

$\Rightarrow 2q^3 = 8m^3 \Rightarrow q^3 = 4m^3$ i.e q is an even number

so $\gcd(p,q) = 2$ which is a contradiction. Hence $2^{1/3}$ is not a rational number.

Use the following information for solving the questions 10 to 12.

Anthropological detectives measure the lengths of dried bones to estimate the living height (the height of the subject when he or she was alive) of their fossilised subject. Table 1 shows the formulas to calculate the height (H) (in inches) based on the lengths (in inches) of the bones: femur (f), humerus (h), radial bone (r), and tibia (t).

Male	Female
$H = 1.9f + 32$	$H = 1.9f + 28$
$H = 2.9h + 27$	$H = 2.7h + 28$
$H = 3.2r + 33.8$	$H = 3.3r + 32$
$H = 2.3t + 31$	$H = 2.3t + 29$

Table 1: Formulas to calculate the height

10. A 14-inch humerus bone from a male subject has been found. What will the error be if the bone belongs to a 66-inch tall male subject? [Answer: 1.6] [Marks: 2]

Soln: Formula for height when h is male humerus bone $H = 2.9h + 27$.

Given $h = 14 \Rightarrow H = 2.9 \times 14 + 27$

$= 67.6$

Since bone belongs to 66-inch tall male subject

So error $= 67.6 - 66 = 1.6$

11. A student calculates the living height of a person using the formula $H = 0.3H_f + 0.2H_h + 0.2H_r + 0.3H_t$, where H_f, H_h, H_r, and H_t are the heights calculated by f, h, r, and t respectively. Choose the correct set of options. [Marks: 3]
(MSQ)

- ◯ $H = 0.57f + 0.58h + 0.64r + 0.69t + 122.8$ for male.
- ◯ $H = 0.57f + 0.54h + 0.66r + 0.69t + 117$ for female.
- ◯ $\boldsymbol{H = 0.57f + 0.58h + 0.64r + 0.69t + 31.06}$ **for male.**
- ◯ $\boldsymbol{H = 0.57f + 0.54h + 0.66r + 0.69t + 29.1}$ **for female.**
- ◯ $H = 0.57f + 0.54h + 0.66r + 0.69t$ for male.
- ◯ $H = 0.57f + 0.58h + 0.64r + 0.69t + 122.8$ for female.

Soln:

$$H = 0.3H_f + 0.2H_h + 0.2H_r + 0.3H_t$$

So for male:

$$H = 0.3(1.9f + 32) + 0.2(2.9h + 27) + 0.2(3.2r + 33.8) + 0.3(2.3t + 31)$$

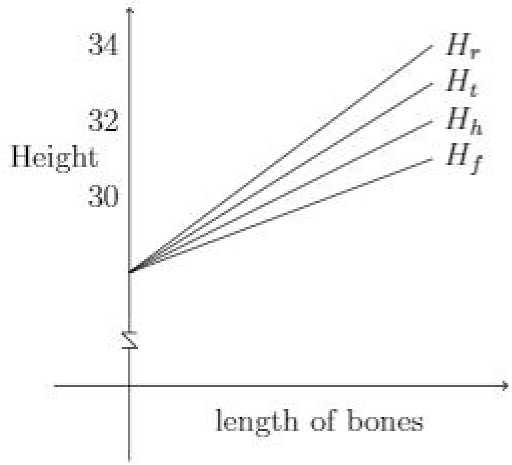
$$\Rightarrow H = 0.57f + 0.58h + 0.64r + 0.69t + 31.06$$

Similarly we can check for female.

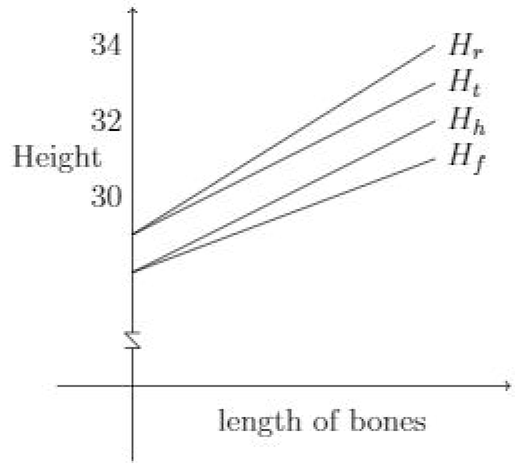
12. Choose the best representation of height (H) of a female person (calculated with respect to the length of the bones) versus the length of the bones (f, h, r, and t) for a female. (MCQ) [Marks: 2]

Answer: Option C

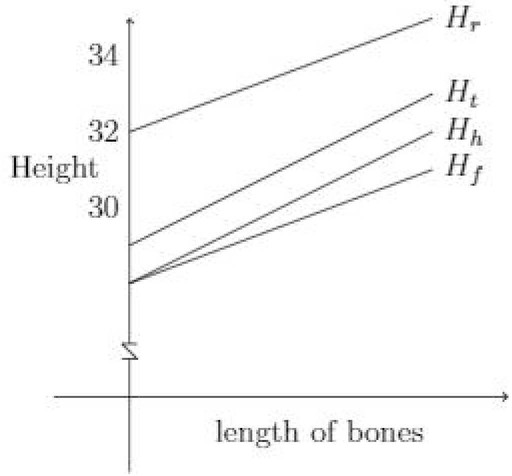
A



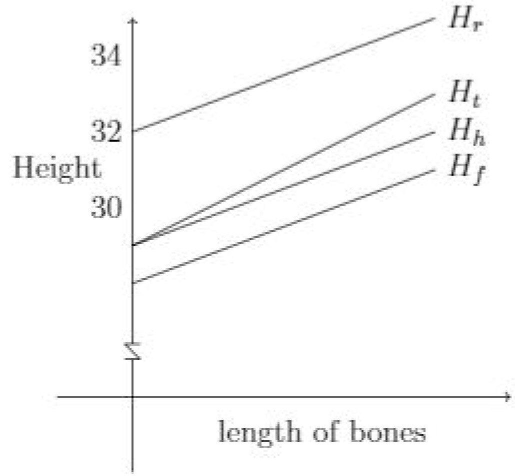
B



C



D



Solu: Observe that H_f, H_h, H_r, H_t are equations of lines. (From Table 1)

line $H_f = 1.9f + 28$ has Y-intercept is 28.

line $H_h = 2.7h + 28$ has Y-intercept is 28.

line $H_r = 3.3r + 32$ has Y intercept is 32

line $H_t = 2.3t + 29$ has Y-intercept 29

and $28 < 29 < 32$

So only figure in option C is matching with whatever we have concluded. So option C is true.

Mock 3 (Week 1 - 4) Solution

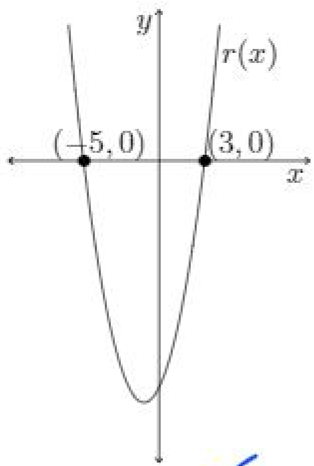
Mathematics for Data Science - 1

Max Marks: 50

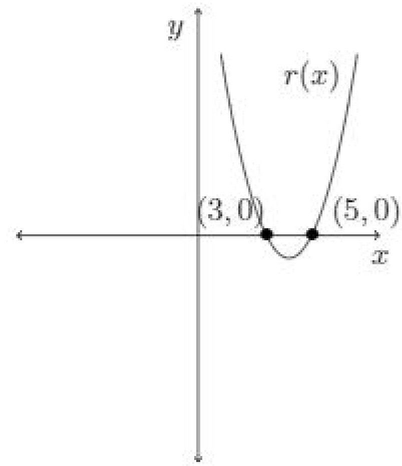
1 Instructions:

- There are some questions which have functions with discrete valued domains (such as day, month, year etc). For simplicity, we treat them as continuous functions.
- For NAT type question, enter only one right answer even if you get multiple answers for that particular question.
- **Notations:**
  - $\mathbb{R}$= Set of real numbers
  - $\mathbb{Q}$= Set of rational numbers
  - $\mathbb{Z}$= Set of integers
  - $\mathbb{N}$= Set of natural numbers
- The set of natural numbers includes 0.

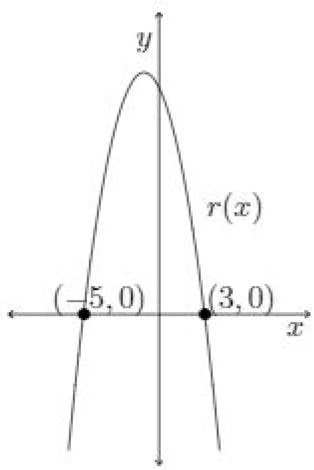
1. Let $r(x)$ be a polynomial function which is obtained as the quotient after dividing the polynomial $p(x) = -(x+5)(x-3)(x^2-16)$ by the polynomial $q(x) = -(x-4)(x+4)$. Choose the correct option(s) which represent(s) the polynomial $r(x)$ most appropriately. (MSQ)(Ans: Option 1) [Marks:3]



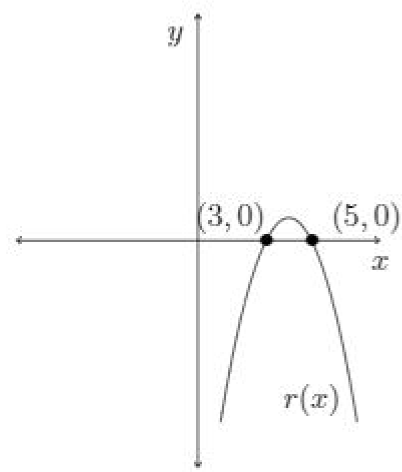
Option 1



Option 2



Option 3



Option 4

Solution

Given that $p(x) = -(x+5)(x-3)(x^2-16)$

$q(x) = -(x-4)(x+4)$

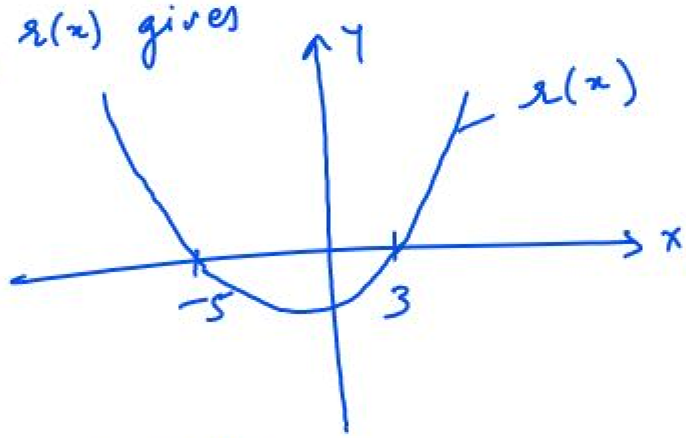
$r(x)$ is the quotient when $p(x)$ is divided by $q(x)$

$\Rightarrow \quad p(x) = -(x+5)(x-3)(x+4)(x-4)$ (On factorizing (x^2-16))

$$\frac{p(x)}{q(x)} = \frac{\cancel{-}(x+5)(x-3)\cancel{(x+4)}\cancel{(x-4)}}{\cancel{-}\cancel{(x-4)}\cancel{(x+4)}}$$

$\Rightarrow \quad r(x) = (x+5)(x-3)$

graphing of $r(x)$ gives



Thus option 1 is correct

2. Let $R = \{(a,c),(d,b),(b,d),(b,c),(c,a)\}$ be a relation on the set $A = \{a,b,c,d\}$. The relation R is
(MSQ)(Answer: Option(d)) [Marks: 3]

- ◯ a function
- ◯ reflexive
- ◯ transitive
- ◯ not symmetric

Solution:-

option 1:- R is not a function because there are two outputs for element 'b' i.e; (b,d) & (b,c)

option 2: R is not reflexive because it does not contains elements such as (a,a), (b,b), (c,c), (d,d)

option 3: R is not transitive relation as for the pairs such as (a,c), (c,a) there is no (a,a) ∈ R

✓ option 4: It is not symmetric relation because for the element (b,c) there is no (c,b)

3. For the purposes of a research, a survey of 1000 students is conducted in a IITM BSc Degree Program. The results show that 52% liked CT, 45% liked Statistics and 60% liked Mathematics. In addition, 25% liked both CT and Statistics, 28% liked both Mathematics and Statistics and 30% liked both CT and Mathematics. 6% liked none of these subjects. Based on this information answer the following questions

Solution:- Step 4 :-

(a) How many students like all the three subjects? $\rightarrow 20\%$ of $1000 = \frac{20}{100} \times 1000 = \underline{200}$
(NAT)(Answer: 200) [Marks: 3]

(b) Find the number of students who like only one of the three subjects.
(NAT)(Answer: 510) $\rightarrow (17\% + 12\% + 22\%)$ of 1000 [Marks: 3]
$= \frac{51}{100} \times 1000 = 510$

(c) Find the number of students who like at least two of the given subjects.
(NAT)(Answer: 430) $\rightarrow (10\% + 5\% + 8\% + 20\%)$ of $1000 = \underline{430}$ [Marks: 3]

Step 1:

C $n(CT) = \%$ of students who like CT $= 52\%$

F $n(S) =$ " " statistics $= 45\%$

R $n(M) =$ " " mathematics $= 60\%$

given that, $n(CT \cap S) = 25\%$

$n(S \cap M) = 28\%$

$n(CT \cap M) = 30\%$

Since 6% like non of the given subjects so, $n(CT \cup S \cup M) = 94\%$

Step 2:

Formula:

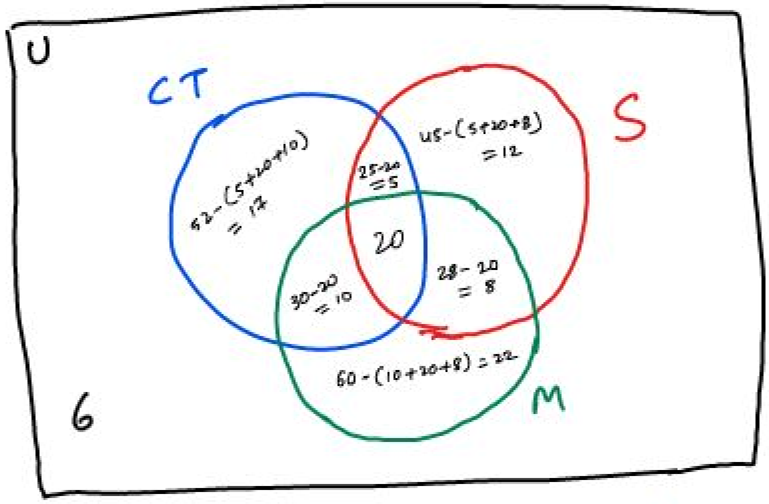
$$n(CT \cup S \cup M) = n(CT) + n(S) + n(M) - n(CT \cap S) - n(S \cap M) - n(CT \cap M) + n(CT \cap M \cap S)$$

$$96\% = 52\% + 45\% + 60\% - 25\% - 28\% - 30\% + n(CT \cap M \cap S)$$

$$\Rightarrow n(CT \cap M \cap S) = 20\%$$

Step 3:

Venn Diagram related to the above situation is shown below



Step 4:- (see above in subquestions)

4. Consider a triangle ABC (See Figure 1), the line segments AB, BC, and CA represent the sides of the triangle ABC. It is given that angle $\angle ABC = \angle BCA = \alpha$, where α is an acute angle.

Solution:-

Steps: Let l_1, l_2 and l_3 represent the line segment AB, BC and CA respectively.

⊙ Slope of l_1 is $\boxed{1/2}$

⊙ slope of l_2 is $\boxed{2}$

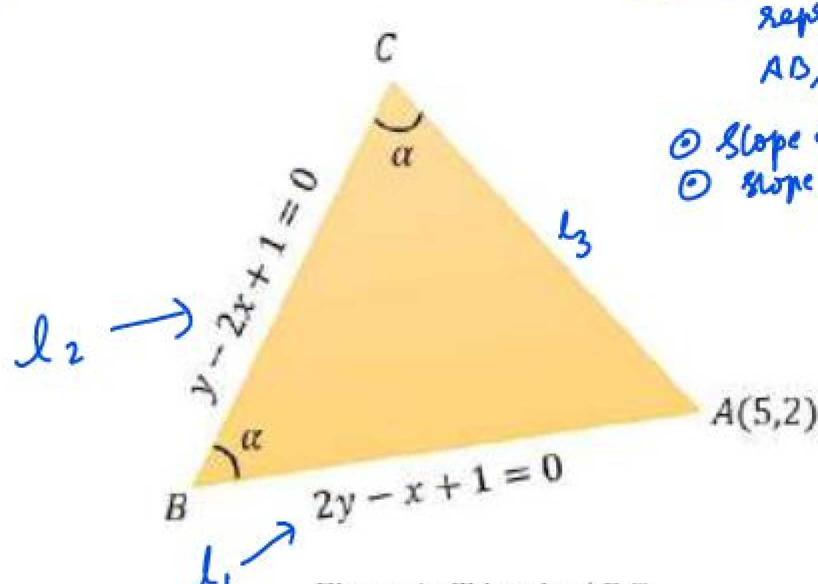


Figure 1: Triangle ABC

Based on this information answer the following questions

(a) Which of the following options are correct w.r.t triangle ABC
(MSQ)(Answer: Option(a)(d)) [Marks: 3]

- ○ Slope of the line segment AB is $\frac{1}{2}$
- ○ Slope of the line segment BC is $-\frac{1}{2}$
- ○ $\tan\alpha = -\frac{1}{2}$
- ○ $\tan\alpha = \frac{3}{4}$

(b) Which of the following options represents the equation of the line AC
(MSQ)(Answer: Option(a)(d)) [Marks: 3]

- ○ $2y + 11x - 59 = 0$
- ○ $6y = 2x + 2$
- ○ $3y - x - 1 = 0$
- ○ $y = -5.5x + 29.5$

Step 2: Angle b/w two lines l_1 and l_2 is α

So,
$$\tan\alpha = \frac{|m_1 - m_2|}{1 + m_1 m_2} = \frac{|1/2 - 2|}{1 + \frac{1}{2}\times 2} = \left|-\frac{3}{4}\right| = \boxed{\frac{3}{4}}$$

Again, angle b/w lines l_2 and l_3 is α

So, $$\tan\alpha = \frac{|m_2 - m_3|}{1 + m_2 m_3} = \frac{3}{4}$$

Note: m_1, m_2 and m_3 are the slopes of the line l_1, l_2 & l_3 resp.

$$\Rightarrow \quad \frac{3}{4} = \frac{|2 - m_3|}{1 + 2m_3}$$

$$\Rightarrow \frac{2 - m_3}{1 + 2m_3} = \frac{3}{4}$$

$$8 - 4m_3 = 3 + 6m_3$$

$$\Rightarrow \quad 10m_3 = 5$$

$$\boxed{m_3 = 1/2}$$

or

$$\frac{2 - m_3}{1 + 2m_3} = \frac{-3}{4}$$

$$8 - 4m_3 = -3 - 6m_3$$

$$2m_3 = -11$$

$$\boxed{m_3 = -11/2}$$

Step 3: As l_3 passes through $(5,2)$ and slope of l_3

Equation of the line l_3 is

$$y - 2 = \frac{1}{2}(x - 5)$$

$$\Rightarrow 2y - 4 = x - 5$$

$$\boxed{2y = x - 1}$$

Note this is line l_1

Equation of the line is

$$y - 2 = -\frac{11}{2}(x - 5)$$

$$2y - 4 = -11x + 55$$

$$\boxed{2y + 11x - 59 = 0}$$

or

$$\boxed{y = -5.5x + 29.5}$$

5. Rizwan wants to cross the river represented by $r(x) = 0.05(x-1)(x-5)(x-10) + k$, where k is an integer constant (Refer Figure 2). He chooses to cross the river $r(x)$ using the bridge. In order to cross the river he has to identify the coordinates of the point A. He realises that the coordinates of the point A can be identified in such a way that the curve $r(x)$ can be best fitted according to the data given in the table (refer Table 1). Consider the river and the bridge to be of negligible thickness. Based on these information answer the following questions

x	-10	-5	1	10	15
y	-150	-30	15	15	50

Table 1

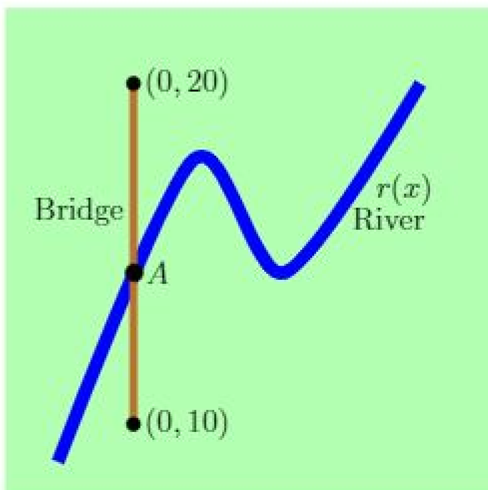


Figure 2: River and bridge

(a) Using the above information, find the value of k.
(NAT)(Answer: 15) [Marks: 4]

(b) Using the above information, what will be the $y-$coordinate of the point A?
(NAT)(Answer: 12.5) [Marks: 3]

Solution

It is given that $r(x)$ best fit the data given in the Table1.
Thus, SSE should be minimum.

x	y	$[y - f(x)]^2$
-10	-150	$\left[-150 - \{0.05(-10-1)(-10-5)(-10-10) + k\}\right]^2 = (15-k)^2$
-5	-30	$(15-k)^2$
1	15	$(15-k)^2$
10	15	$(15-k)^2$
15	50	$(15-k)^2$

$$SSE = 5(15-k)^2$$

This is quadratic equation in k, & it will be minimum at its vertex.

Thus $\boxed{k = 15}$

The equation of $r(x) = 0.05(x-1)(x-5)(x-10) + 15$

From the figure it is clear that x-coordinate of 'A' is 0

So, y-coordinate will be

$r(0) = 0.05(-1)(-5)(-10) + 15 = \underline{\underline{12.5}}$

6. Which of the following statements is (are) correct?
(MSQ)(Answer: Option(a)(b)(c)) [Marks: 4]

- The product of the minimum value of the function $f(x) = 5|x| + 10$ and the maximum value of the function $g(x) = 10 - |x + 12|$ is 100.
- There are infinitely many polynomial $p(x)$ of degree four such that $p(4) = 0$, $p(5) = 0$, $p(6) = 0$.
- $y - 4 = (x + 5)^2$ is an equation of a parabola whose vertex is at $(-5, 4)$.
- Elements in Cartesian product will only be pairs. → No, can be triplets etc. as well.

Solution:-

option 1: Minimum value of modules function is zero.
Thus, minimum value of $f(x)$ is 10
maximum value of $g(x)$ is 10

So, minimum value of $f(x) \cdot g(x)$ is 100

option 2: Given that factors of $p(x)$ are $(x-4)(x-5)(x-6)$ and it is four degree polynomial. so $P(x)$ can be any polynomial depending on the other factor & the stretch factor.

option 3: The equation of quadratic equation in vertex form is given as $y-K = a(x-h)^2$ where (h,K) are the vertex of the quadratic equation.

comparing, $y-4=(x+5)^2$ gives $h=-5$, $K=4$

Thus vertex is $(-5,4)$

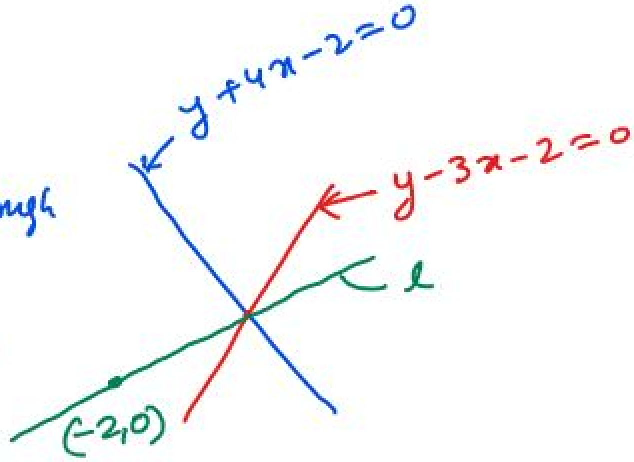
7. Choose the most appropriate option for the statement given below:

"The equation of the line joining the point $(-2, 0)$ to the point of intersection of the lines $y + 4x - 2 = 0$ and $y - 3x - 2 = 0$ is equidistant from the points $(0, 0)$ and $(2, 2)$."

(MSQ)(Answer: Option(a)) [Marks: 4]

- ◯ True
- ◯ False
- ◯ Data insufficient
- ◯ None of the above

Solution:-



consider l passes through $(-2,0)$ and point of intersection of the given two lines.

(see rough sketch)

Step1: Finding point of intersection of

$y+4x-2=0 \rightarrow$ Eq 1 & $y-3x-2=0 \Rightarrow y = 3x+2 \rightarrow$ Eq 2

substitute Eq 2 in Eq 1 we get

$3x+2+4x-2=0 \Rightarrow 7x=0 \Rightarrow \boxed{x=0}$

substitute $x=0$ in Eq 1 we get $\boxed{y=2}$

Thus point of intersection of two lines is $(0,2)$

Step2: Eqn of line 'l' will be $y-x-2=0$ (using two point form)

Step3: Distance of point from a line is $d = \dfrac{|Ax_1+By_1+c|}{\sqrt{A^2+B^2}}$

$d = \dfrac{|-2|}{\sqrt{2}} = \dfrac{2}{\sqrt{2}}$ (for point $(0,0)$) | $d = \dfrac{2}{\sqrt{2}}$ for $(2,2)$

Thus distance are same, so option (a) is correct

8. A total of ₹300 is raised by a group A, by collecting equal amounts from a certain number of people. Another group B, contains 5 more people and each person contributes ₹10 less to raise the same amount as that of group A. Based on this information answer the following questions

(a) How many people actually contributed in group A?
(NAT)(Answer: 10) [Marks: 3]

(b) What is the contribution (₹) made by each person in group B?
(NAT)(Answer: 20) [Marks: 1]

Solution:

Group A: Let the no. of people be 'x'
Contribution of each person will be $\frac{300}{x}$

Group B: contribution of each person is $\frac{300}{x} - 10$
No. of people is $x+5$

Group B too raises same amount i.e, 300.

so, $\left(\frac{300}{x} - 10\right)(x+5) = 300$

$\Rightarrow (300 - 10x)(x+5) = 300x$

$\Rightarrow \cancel{300x} - 10x^2 + 1500 - 50x = \cancel{300x}$

$\Rightarrow x^2 + 5x - 150 = 0$

$\Rightarrow x^2 + 15x - 10x - 150 = 0$

$\Rightarrow x(x+15) - 10(x+15) = 0 \Rightarrow (x-10)(x+15) = 0$

$x = 10$ or $x = -15$ (Not possible)

Thus, no. of people who actually contributed in group A is $\underline{10}$

contribution made by each person in group B is $\frac{300}{10} - 10 = \underline{\underline{20}}$

9. Suppose the function $f(x) = -x^2 + 4x - k$ and $g(x) = x^2 - kx + 4$ intersects at most at one point, where $k \in \mathbb{Z}$. Based on this information answer the following questions

(a) Which of the following could be the value of k.
(MSQ)(Answer: Option(a)) [Marks: 2]

- ✓ 4
- ○ −5
- ○ 10
- ○ −10

(b) Find the possible number of values of k? (Also given full marks for $|K| = 2$)
(NAT)(Answer: 9) [Marks: 2]

Solution: Given that $f(x)$ & $g(x)$ intersects thus

$$f(x) = g(x)$$

$$\Rightarrow -x^2 + 4x - K = x^2 - Kx + 4$$

$$\Rightarrow 2x^2 - (K+4)x + K + 4 = 0 \quad \text{— Eq 1}$$

Given that they intersects at most at one point, therefore, two possibilities.

(i) Both the curves do not intersect at all.

(ii) Both the curves intersect at exactly one point.

Thus, discriminant (D) of Eq 1 should be $D \leq 0$

$$(K+4)^2 - 4(2)(K+4) \leq 0$$

$$(K+4)(K+4-8) \leq 0$$

$$(K+4)(K-4) \leq 0 \quad \Rightarrow \text{rough graph}$$

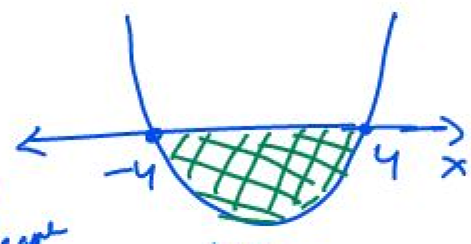


fig 1

Thus, $K \in [-4, 4]$ or

$$K = \{-4, -3, -2, -1, 0, 1, 2, 3, 4\} \quad (\because K \in \mathbb{Z})$$

$$|K| = 9$$

10. Consider two polynomial function $p(x) = 0.1\,(x-1)\,(x-5)\,(x-10)$ and $q(x) = 0.4\,(x-1)\,(x-10)$ defined in the interval $(1, 10)$. A line $l(x)$ passes through $x-$intercept of $p(x)$ and the intersection point of $p(x)$ and $q(x)$.

(a) Which of the following are correct.
(MSQ)(Answer: Option(a)(b)(c)(d)) [Marks: 4]

- The $x-$intercept of $p(x)$ in the given domain is 5. } see fig below
- In the given domain $p(x)$ has 2 turning points. }
- The $x-$coordinate of the vertex of $q(x)$ is 5.5. $\longrightarrow \frac{-b}{2a} = \frac{-(-9)}{2} = +5.5$
- The $x-$coordinate of the intersection point of $p(x)$ and $q(x)$ is 9.

(b) What is the slope of the line $l(x)$?
(NAT)(Answer: -0.8) [Marks: 2]

Solution: A rough graph of $p(x)$ & $q(x)$ is shown below

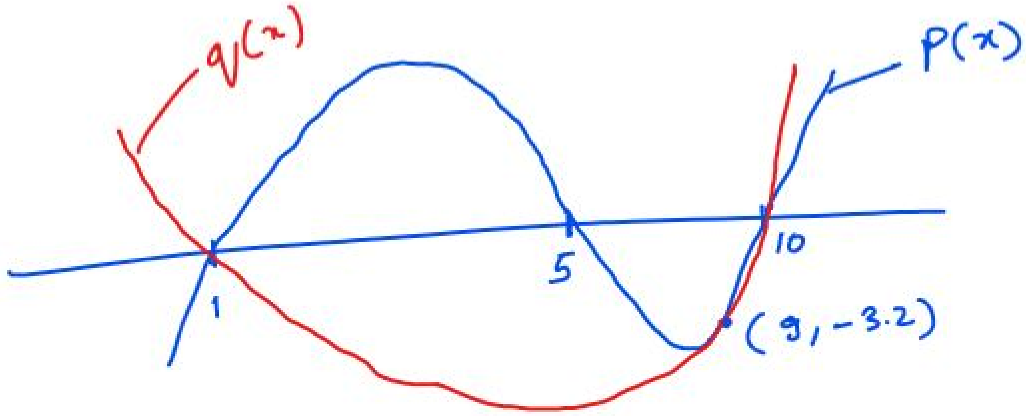


Fig 1.

Note the domain of both the f^n is $(1, 10)$

Intersection point of $p(x)$ & $q(x)$ can be found as follow

$$p(x) = q(x)$$
$$\Rightarrow 0.1(x-1)(x-5)(x-10) = 0.4(x-1)(x-10)$$
$$\Rightarrow (x-1)(x-5)(x-10) - 4(x-1)(x-10) = 0$$
$$(x-1)(x-10)\{(x-5)-4\} = 0$$
$$x = 1, 9, 10 \quad (1, 10 \text{ are not in domain})$$

Thus x-coordinate of point of intersection of $p(x)$ & $q(x)$ is 9.
y-coordinate of " " is -3.2

Slope of the line will be $\frac{-3.2-0}{9-5} = -0.8$

Week - 5

Practice assignment Solution

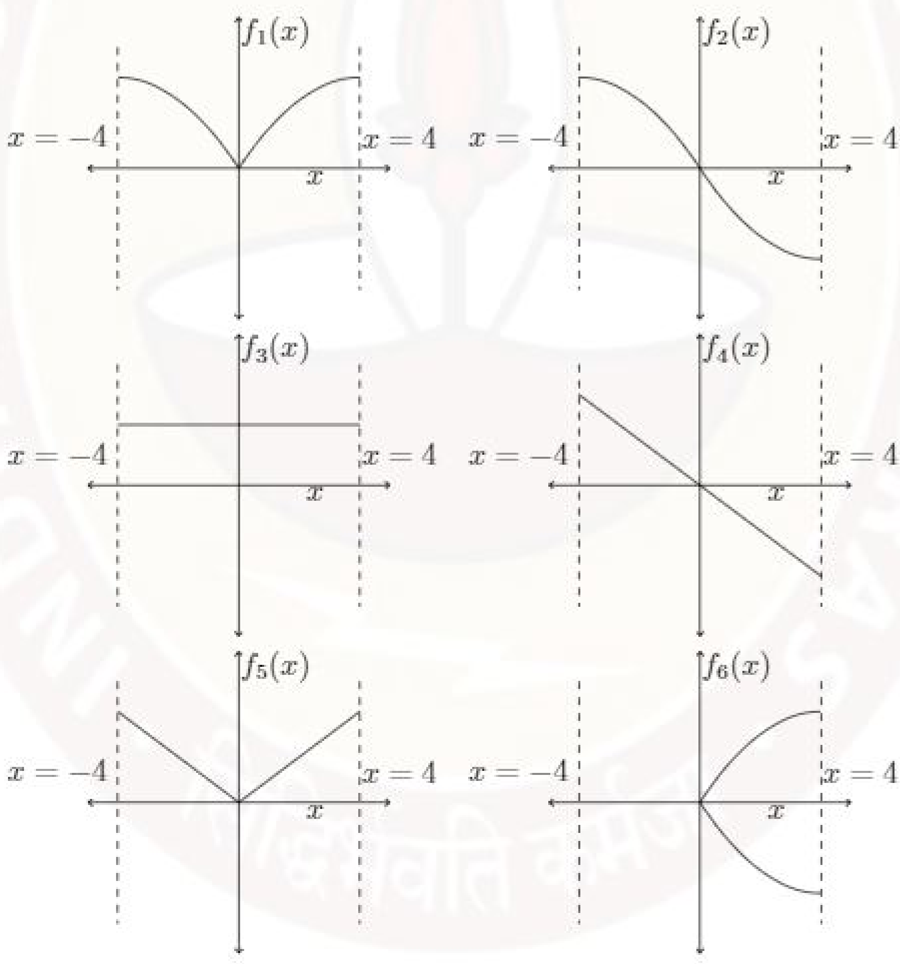
Exponential Functions

Mathematics for Data Science - 1

NOTE:
There are some questions which have functions with discrete valued domains (such as month or year). For simplicity, we treat them as continuous functions.

1 Multiple Choice Questions (MCQ):

Answer the questions 1, 2, and 3 based on the given graphs.



Domain for each one is $[-4, 4]$.

1. Choose the correct option.

   ◯ f_3 is not a function.

   ◯ **f_6 is not a function.**

   ◯ f_5 is not a function.

   ◯ All of the above are functions.

   Solution:

   Vertical line test fails only for f_6 and therefore $f_6(x)$ is not a function.

2. Choose the correct option.

   ◯ f_1 and f_3 are one-one functions in the given domain.

   ◯ **f_2 and f_4 are one-one functions in the given domain.**

   ◯ f_3 and f_5 are one-one functions in the given domain.

   ◯ f_5 is one-one function in the given domain.

   Solution:

   The function f_2 and f_4 are strictly decreasing function in the domain [-4, 4], therefore these are one to one functions.

   Or

   The functions f_2 and f_4 are the only functions which satisfy the conditions of horizontal and vertical line tests in the domain [-4, 4], therefore these are one to one functions.

3. Choose the correct option.

   ◯ f_1 and f_5 are strictly increasing functions in the given domain.

   ◯ **f_2 and f_4 are strictly decreasing functions in the given domain.**

   ◯ f_4 and f_5 are strictly decreasing functions in the given domain.

   ◯ f_5 is strictly increasing function in the given domain.

   Solution:

   A function $f(x)$ is said to be strictly decreasing on a given interval if $f(b) < f(a)$ for all $b > a$, where a, b belong to the domain. On the other hand, if $f(b) <= f(a)$ for all $b > a$, then the function is said to be simply decreasing function.
   Clearly from the given graph, f_2 and f_4 are strictly decreasing functions in the domain [-4, 4].

Use the following information for the questions 4 and 5.

Let N_0 be the number of atoms of a radioactive material at the initial stage i.e., at time $t = 0$, and $N(t)$ be the number of atoms of the same radioactive material at a given time t, which is given by the equation $N(t) = N_0 e^{-\lambda t}$, where λ is the decay constant.

4. If at time t_1, the number of atoms reduces to the half of N_0 and at the time t_2 the number of atoms reduces to the one fourth of N_0, then which one of the following equations is correct?

   - ◯ $e^{\frac{t_1}{t_2}} = 2$
   - ◯ $e^{\frac{t_2}{t_1}} = 2$
   - ◯ $\boldsymbol{e^{\lambda(t_2 - t_1)} = 2}$
   - ◯ $e^{\lambda(t_1 - t_2)} = 2$

   Solution:

   According to the question, at t_1,

   $$N(t) = \frac{N_0}{2}$$

   According to the equation,

   $$N(t) = N_0 \, e^{-\lambda t}$$

   Therefore for $t = t_1$,

   $$\frac{1}{2} \times N_0 = N_0 \, e^{-\lambda t_1}$$

   $$\frac{1}{2} = e^{-\lambda t_1} \tag{1}$$

   It is also given that at t_2, $N = \frac{N_0}{4}$

   $$\frac{1}{4} \times N_0 = N_0 \, e^{-\lambda t_2}$$

   $$\frac{1}{4} = e^{-\lambda t_2} \tag{2}$$

   On dividing (1) by (2) we get,

   $$e^{\lambda(t_2 - t_1)} = 2$$

5. If $N_{\frac{1}{\lambda}}$ is the number of atoms at time $t = \frac{1}{\lambda}$, then what is the ratio of N_0 to $N_{\frac{1}{\lambda}}$?

   - ◯ $1 : e$
   - ◯ $\boldsymbol{e : 1}$
   - ◯ $1 : e^{-\lambda}$

○ $1 : e^{\lambda}$

Solution:

It is given that at $t = \frac{1}{\lambda}$, $N = N'$

$$N' = N_0\ e^{-\frac{\lambda}{\lambda}}$$

$$N' = \frac{N_0}{e}$$

$$\frac{N_0}{N'} = \frac{e}{1}$$

Therefore,

$N_0 : N' = e : 1$

2 Multiple Select Questions (MSQ):

6. Selvi deposits ₹P in a bank A which provides an interest rate of 10% per year. After 10 years, she withdraws the whole amount from bank A and deposits it in another bank B for n years which provides an interest rate of 12.5% per year. $M_A(x)$ represents the amount in Selvi's account after x years of depositing in bank A. $M_B(y)$ represents the amount in Selvi's account after y years of depositing in bank B. If the interests are compounded yearly, then choose the set of correct options.

    - ◯ **$M_A(x)$ is an one-one function of x, for $x \in (0, 10)$.**
    - ◯ **$M_B(y)$ is an one-one function of y.**
    - ◯ $M_A(12) = P \times 1.1^{12}$
    - ◯ **$M_A(12) = 0$**
    - ◯ **$M_A(x)$ is a strictly increasing function of x, for $x \in (0, 10)$.**
    - ◯ $M_B(y)$ is a decreasing function of y.
    - ◯ **$M_B(n) = (P \times 1.1^{10}) \times (1.125)^n$**
    - ◯ $M_B(n) = (P \times 1.1^n) \times (1.125)^{10}$

    Solution:

    When the principal amount P is compounded annually, the amount M after q years is given by

    $$M = P \times \left(1 + \frac{\text{Interest rate}}{100}\right)^q$$

    Amount $M_A(x)$ after x years in bank A will be

    $$M_A(x) = P \times \left(1 + \frac{10}{100}\right)^x$$

    So after 10 years the amount $M_A(10)$ will be

    $$M_A(10) = P \times (1.1)^{10}$$

    As Selvi has withdrawn all the amounts from bank A after 10 years so amount left in bank A after 12 years will be $M_A(12) = 0$.

    After 10 years the new principal amount $P \times (1.1)^{10}$ is deposited in another bank B, so for any years y the amount will be $M_B(y)$ which is given by

    $$M_B(y) = P \times (1.1)^{10} \times \left(1 + \frac{12.5}{100}\right)^y$$

So for n years

$$M_B(n) = P \times (1.1)^{10} \times (1.125)^n$$

Clearly $M_A(x)$ and $M_B(y)$ are strictly increasing functions therefore both are one-to-one functions of x and y respectively.

Use the following information for questions 7 and 8.
There are two offers in a shop. In the first offer, the discount in total payable amount is $M(n)\%$ if the number of products bought at a time is n. The second offer involves a discount of ₹1000 on the total payable amount. If Geeta shops of ₹15,000, then answer the following questions.

7. If the total payable amounts after applying the first and second offers (one at a time) are represented by the functions $f(n)$ and $g(n)$ respectively and the total payable amount after applying both the offers together is represented by $T(n)$, then choose the set of correct options.

   - ○ $f(n) = (100 - M(n)) \times 15000$ and $g(n) = 14000$
   - ○ $f(n) = (100 - M(n)) \times 1500$ and $g(n) = (100 - M(n)) \times 15000 - 1000$
   - ○ $\boldsymbol{f(n) = (100 - M(n)) \times 150}$ **and** $\boldsymbol{g(n) = 14000}$
   - ○ $T(n) = (100 - M(n)) \times 15000$ is the total payable amount when the first offer is applied after the second.
   - ○ $\boldsymbol{T(n) = (100 - M(n)) \times 140}$ **is the total payable amount when the first offer is applied after the second.**
   - ○ $\boldsymbol{T(n) = (100 - M(n)) \times 150 - 1000}$ **is the total payable amount when the second offer is applied after the first.**

Solution:

It is given that total payable amount without any offer is ₹15,000. Then, total payable amount after first offer is

$$f(n) = \frac{(100 - M(n))}{100} \times 15,000 = (100 - M(n)) \times 150$$

And total payable amount if second offer is applied will be

$$g(n) = 15,000 - 1000 = ₹14,000.$$

Now, total payable amount when the first offer is applied after the second will be

$$T(n) = \frac{100 - M(n)}{100} \times g(n)$$

$$T(n) = \frac{(100 - M(n))}{100} \times 14000 = (100 - M(n)) \times 140$$

And total payable amount when the second offer is applied after the first will be

$$T(n) = f(n) - 1000$$

$$T(n) = \frac{(100 - M(n))}{100} \times 15000 - 1000 = (100 - M(n)) \times 150 - 1000$$

8. If Geeta is allowed to use the offer in any sequence and $M(n) = -n^2 + 18n - 72$, where $n \in \{6, 7, 8, 9\}$, then choose the set of correct options which minimizes the total payable amount.

   ◯ Total payable amount is same irrespective of the order in which the offers are applied.

   ◯ **She should choose offer one and then offer two i.e., $gof(M(n))$.**

   ◯ She should choose offer two and then offer one i.e. $gof(M(n))$.

   ◯ **If she chooses offer one and then offer two, the minimum payable amount will be ₹12650.**

Solution:

Total payable amount when she choose offer one and then offer two is

$$T_1(n) = (100 - M(n)) \times 150 - 1000$$

It is given that $M(n) = -n^2 + 18n - 72$, so

$$T_1(n) = (100 - (-n^2 + 18n - 72)) \times 150 - 1000$$

On solving we get,

$$T_1(n) = 150n^2 - 2700n + 24800$$

And total payable amount when she chooses offer two and then offer one is

$$T_2(n) = (100 - M(n)) \times 140$$

On substituting $M(n)$ and solving we get,

$$T_2(n) = 140n^2 - 2520n + 24080$$

Since the coefficient of n^2 is positive for both $T_1(n)$ and $T_2(n)$ therefore minimum value i.e., minimum payable amount of these function can be calculated as follows
For $T_1(n)$

$$\text{Vertex}(n) = \frac{-b}{2a} = \frac{-(-2700)}{2 \times 150} = 9$$

The minimum payable amount will be

$$T_1(9) = 150(9)^2 - 2700(9) + 24800 = ₹12{,}650$$

For $T_2(n)$

$$\text{Vertex}(n) = \frac{-b}{2a} = \frac{-(-2520)}{2 \times 140} = 9$$

The minimum payable amount will be

$$T_2(9) = 140(9)^2 - 2520(9) + 24080 = ₹12{,}740$$

Thus if she chooses offer one and then offer two, the minimum payable amount will be ₹12,650.

n	$T_1(n)$ ₹	$T_2(n)$ ₹
6	14000	14000
7	13250	13300
8	12800	12880
9	12650	12740

Table: M1W8PAS-1

From Table: M1W8PAS-1, it is clear that for all the values of n the total payable amount is lower for $T_1(n)$ as compared to $T_2(n)$ therefore she should choose offer one and then offer two.
Note: This can be also identified by plotting the graph for $T_1(n)$ and $T_2(n)$.

3 Numerical Answer Type (NAT):

Use the following information for questions 9-15.
Given two real valued functions $f(x) = \frac{5x+9}{2x}$, $g(y) = \sqrt{y^2 - 9}$. If $h(x) = f(g(x))$, then answer the following questions.

9. If domain of $f(x)$ and $g(x)$ are $(-\infty, m) \cup (m, \infty)$ and $\mathbb{R} \setminus (-n, n)$ respectively, then find the value of $m + n$. [Ans: 3]
Solution:

At $x = 0$ the function $f(x) \to \infty$ or the function is undefined at $x = 0$ thus the domain of $f(x)$ is $\mathbb{R} \setminus 0$.
We can also write the domain as $(-\infty, 0) \cup (0, \infty)$ therefore, $m = 0$.

It is given that $g(y) = \sqrt{y^2 - 9}$ on changing the variable in terms of x we get $g(x) = \sqrt{x^2 - 9}$.

$g(x)$ will be defined when $x^2 - 9 \geq 0$. On solving

$$x^2 \geq 9$$
$$x \geq 3$$

or

$$x \leq -3$$

Thus the domain will be $\mathbb{R} \setminus (-3, 3)$, hence $n = 3$. So, $m + n = 0 + 3 = 3$

10. If range of $f(x)$ and $g(x)$ are $(-\infty, m) \cup (m, \infty)$ and $[n, \infty)$ respectively, then find the value of $2(m + n)$. [Ans: 5]

Solution:
As $f(x)$ is defined everywhere except 0, therefore there will be an asymptote at $x = 0$.
If we draw a graph of $f(x)$:
End behaviour:
As $x \to \infty$, $f(x) \to \frac{5}{2}$.
As $x \to -\infty$, $f(x) \to \frac{5}{2}$.
The end behaviours show that the function has another asymptote at $f(x) = y = \frac{5}{2}$.

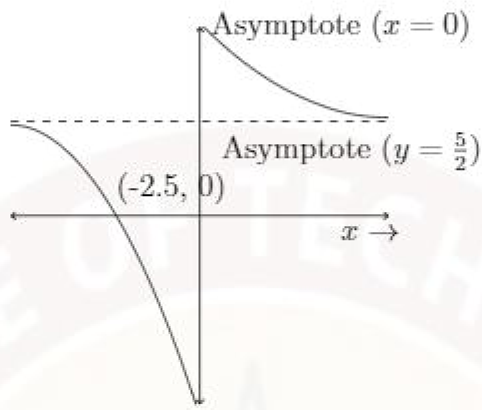
Intercept:

$$f(x) = 0 \implies \frac{5x + 9}{2x} = 0$$
$$x = -\frac{9}{5}$$

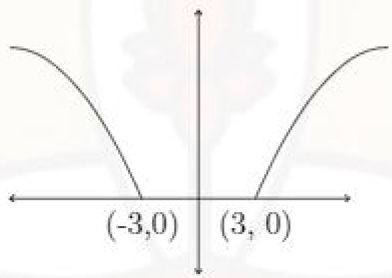
It means $f(x)$ might change the sign at $x = -\frac{9}{5}$.
For $-\infty < x < 0$, $f(x)$ will have value from $-\infty$ to $\frac{5}{2}$.

Similarly for $0 < x < \infty$, $f(x)$ will have value from $\frac{5}{2}$ to ∞.

Therefore the range of $f(x)$ is $(-\infty, \frac{5}{2}) \cup (\frac{5}{2}, \infty)$. A rough diagram of $f(x)$ is shown below.



As $g(x) = \sqrt{x^2 - 9}$ is a positive square root function so it will have only the positive values including zero at $x = 3$ and $x = -3$.
A rough diagram is created using the facts that the $g(x)$ is not defined from (-3, 3) and at $x = 3$ the function gives the value zero. At ∞ the function provides the value ∞. As the quadratic function involved and the $b = 0$ the function will be symmetric about $y-$axis.



Therefore the range will be $[0, \infty)$. Thus $m = 2.5$ and $n = 0$, so,

$$2(m + n) = 2(2.5 + 0) = 5$$

11. If domain of $h(x)$ is $(-\infty, -3) \cup (m, \infty)$, then find the value of m. [Ans: 3]

Solution:
Given,

$$h(x) = f(g(x))$$
$$h(x) = f(\sqrt{x^2 - 9})$$
$$= 2.5 + \frac{4.5}{\sqrt{x^2 - 9}}$$

There are two possibilities when the function is undefined. Firstly when the denominator is zero and secondly when the function in square root provides negative value. It means

$\sqrt{x^2-9} \neq 0$ and $x^2 - 9 \geq 0$.
Combining both the conditions we can say the function is defined only when

$$x^2 - 9 > 0$$

$$x^2 > 9 \implies -3 < x < 3$$

Thus the domain will be $(-\infty, -3) \cup (3, \infty)$, hence $m = 3$.

12. If domain of $f^{-1}(x)$ is $(-\infty, m) \cup (m, \infty)$, then find the value of $2m$. [Ans: 5]

Solution:
Given that $f(x) = \frac{5x+9}{2x}$ let us say $f(x) = y$ so $y = \frac{5x+9}{2x}$ on rearranging,

$$y = \frac{5}{2} + \frac{9}{2x}$$

$$\frac{2y-5}{2} = \frac{9}{2x}$$

$$x = \frac{9}{2y-5}$$

Therefore $f^{-1}(x) = \frac{9}{2x-5}$. This function will be defined when

$$2x - 5 \neq 0$$

$$x \neq \frac{5}{2}$$

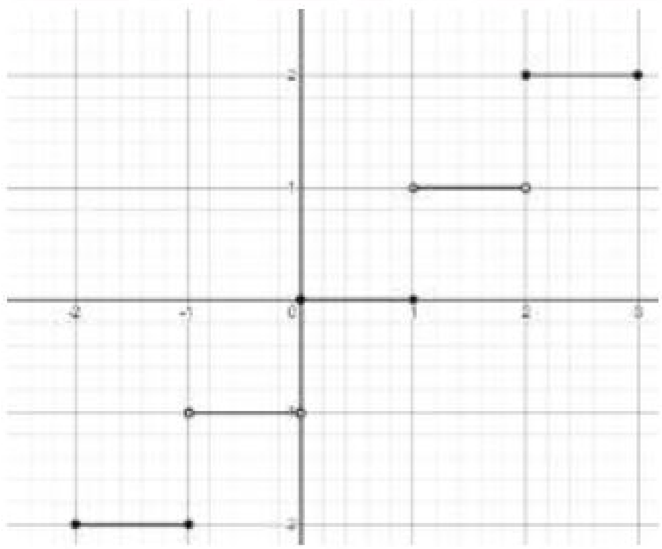
The domain of this function is $(-\infty, 2.5) \cup (2.5, \infty)$ thus $m = 2.5$ therefore $2m = 5$

13. If $f^{-1}(5) = 9/m$, then find the value of m. [Ans: 5]

Solution:
$f^{-1}(5) = \frac{9}{2\times5-5} = \frac{9}{5}$, thus $m = 5$.

Week 5 Graded Assignment Question
Mathematics for Data Science - 1

1. A graph is shown in Figure M1W8A-8.1, $\circ$ symbol signifies that the straight line does not touch the point and the $\bullet$ symbol signifies that the line touches the point.
Choose the correct option.



M1W8A-8.1

- ◯ The graph cannot be a function, because it fails the vertical line test.
- ◯ The graph cannot be a function, because it passes the horizontal line test but fails the vertical line test.
- ◯ **The graph can be a function, because it passes the vertical line test.**
- ◯ The graph cannot be a function, because it passes the vertical line test but fails the horizontal line test

Solution
First, to check the given graph represents a function, we have to use vertical line test. Now, $x = c$ where c is a constant, shown already(in grid vertical lines) in the figure M1W8A-8.1, crosses the graph once (including $\bullet$ and $\circ$ per definition). Therefore, the given graph represents a function.
Set of correct options:

- The graph can be of a function, because it passes the vertical line test.
- The graph fails the horizontal line test.

- The graph represents the graph of neither even function nor odd function.
- The given graph is not invertible in the given domain.

Set of incorrect options:

- The graph cannot be of a function, because it passes the vertical line test but fails the horizontal line test.
- The graph cannot be of a function, because it fails the vertical line test.
- The graph cannot be of a function, because it passes the horizontal line test but fails the vertical line test.
- The graph fails the horizontal line test thus it can be an injective function.
- The graph represents the graph of either even function or odd function.

2. For $y = x^n$,where n is a positive integer and $x \in \mathbb{R}$, which of the following statement is true?

   - ◯ For all values of n, y is not a one-to-one function.
   - ◯ For all values of n, y is an injective function.
   - ◯ y is not a function.
   - ◯ **If n is an even number, then y is not an injective function. If n is an odd number, then y is an injective function.**

Solution:
To check y is a function for all positive integer, let there exist two elements in the domain $x_1, x_2 \in \mathbb{R}$ such that

$$x_1 = x_2$$

taking the power n on both sides, we get,

$$\implies x_1^n = x_2^n$$

$$\implies y(x_1) = y(x_2).$$

Observe that for single input we get a unique output. So, $y = x^n$ is a function for all $n \in \mathbb{Z}^+$.(For vertical line test, see Figure M1W8AS-8.1 and M1W8AS-8.2.
If $n > 1$ is odd positive integer then graph of the function is similar to Figure M1W8AS-8.1, where vertical and horizontal lines are for vertical and horizontal line test respectively.

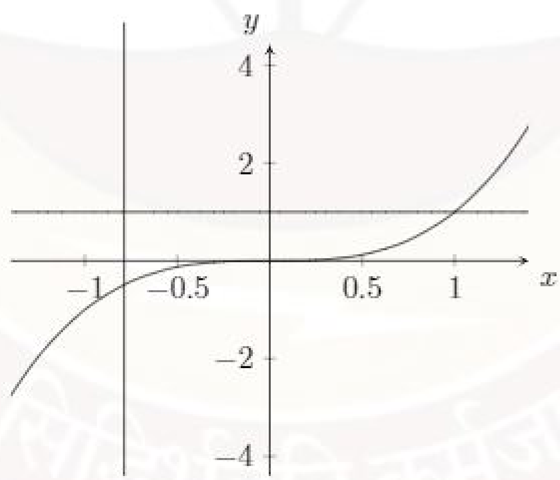


Figure M1W8AS-8.1

If n is even positive integer then graph of the function is similar to Figure M1W8AS-8.2, where vertical and horizontal lines are for vertical and horizontal line test respectively).

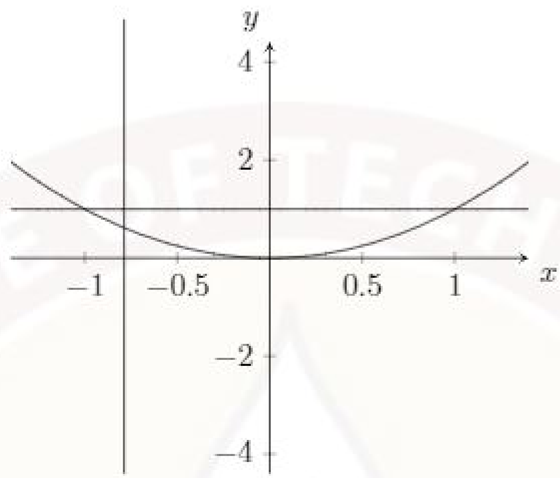


Figure M1W8AS-8.2

So, third option is not correct.
If $n = 1$ which is a positive integer then given function becomes $y = x$.
Using horizontal line test, this function is one-to-one.
So, first option is not correct.
If $n = 2$ which is an even positive integer then given function becomes $y = x^2$.
Using the horizontal line test, this function is not one-to-one(For horizontal line test, see the Figure M1W8AS-8.2).
So, second option is not true.
Now, to check for odd positive integer n, $y = x^n$ is injective. Let $n = 2m + 1$ be an odd integer and $m \in \mathbb{Z}$ (For horizontal line test, see the Figure M1W8AS-8.1). Let there exist two distinct elements in the domain $x_1, x_2 \in \mathbb{R}$ such that

$$f(x_1) = f(x_2)$$
$$\implies x_1^{2m+1} = x_2^{2m+1}$$

Taking the power $\frac{1}{2m+1}$ on both sides we get,

$$x_1 = x_2.$$

From above we see that as no two distinct elements in the domain give the same image. Hence $x_1, x_2 \in \mathbb{R}$ can't be distinct. So, this shows that this function is injective.
Now, to check for even positive integer $n, y = x^n$ is injective or not. Let $n = 2m$ be an even integer and $m \in \mathbb{Z}$, $y = x^{2m} = (x^2)^m$ (See Figure M1W8AS-8.2).
Now, for $x = a, -a$, we get same output $y = a^m$.
Therefore, for even positive integer $n, y = x^n$ not one to one function.
Hence, fourth option is correct.

3. If $4m - n = 0$, then the value of

$$\left(\frac{16^m}{2^n} + \frac{27^n}{9^{6m}}\right)$$

is

Answer: **2**

Solution:

Given $4m - n = 0$.

Now

$$\begin{aligned}
&\frac{16^m}{2^n} + \frac{27^n}{9^{6m}} \\
&= \frac{(2^4)^m}{2^n} + \frac{(3^3)^n}{(3^2)^{6m}} \\
&= \frac{2^{4m}}{2^n} + \frac{3^{3n}}{3^{2\times 6m}} \\
&= 2^{4m-n} + 3^{3n-12m} \\
&= 2^{4m-n} + 3^{-3(4m-n)} \\
&= 2^0 + 3^0 \\
&= 1 + 1 = 2
\end{aligned}$$

4. Half-life of an element is the time required for half of a given sample of radioactive element to change to another element.The rate of change of concentration is calculated by the formula $A(t) = A_o(\frac{1}{2})^{(\frac{t}{\gamma})}$ where γ is the half-life of the material, A_o is the initial concentration of the radioactive element in the given sample, $A(t)$ is the concentration of the radioactive element in the sample after time t.
If Radium has a half-life of 1600 years and the initial concentration of Radium in a sample was 100%, then calculate the percentage of Radium in that sample after 2000 years.

   ○ 35%
   ○ **42%**
   ○ 19%
   ○ 21%

Solution
Given $A(t) = A_o(\frac{1}{2})^{(\frac{t}{\gamma})}$ and half life of radium is 1600 i.e $\gamma = 1600$

$$\implies A(2000) = A_o(\frac{1}{2})^{(\frac{2000}{1600})}$$

$$= A_o(\frac{1}{2})^{(\frac{5}{4})}$$

At initial time($t = 0$) the concentration of Radium is 100%
$\implies$ At initial time the concentration of Radium is A_o
So, the percentage of Radium in that sample after 2000 years is

$$(\frac{A_o(\frac{1}{2})^{(\frac{5}{4})}}{A_0} \times 100)\% = ((\frac{1}{2})^{(\frac{5}{4})} \times 100)\% \approx 42\%$$

5. If $f(x) = (1-x)^{\frac{1}{2}}$ and $g(x) = (1-x^2)$, then find the domain of the composite function $g \circ f$.

- ◯ $\mathbb{R}$
- ◯ $\boldsymbol{\big((-\infty, 1] \cap [-2, \infty)\big) \cup (-\infty, -2)}$
- ◯ $[1, \infty)$
- ◯ $\boldsymbol{\mathbb{R} \setminus (1, \infty)}$

Answer: Option 2, Option 4

Solution:

Given $f(x) = (1-x)^{\frac{1}{2}}$.
To define $f(x)$, $1 - x \geq 0 \implies x \leq 1$.
So, the domain of $f(x)$ is $(-\infty, 1] = \mathbb{R} \setminus (1, \infty)$.
Now, the domain of $g(x) = (1-x^2)$ is $\mathbb{R}$ and the range of $f(x)$ is $[0, \infty)$.
Hence, when we use the two rules as in the video lecture to determine the domain of $g \circ f$.
Here, the domain of $g \circ f =$ The domain of $f = (-\infty, 1] = \mathbb{R} \setminus (1, \infty)$.
Now, $\big((-\infty, 1] \cap [-2, \infty)\big) \cup (-\infty, -2) = [2, 1] \cup (-\infty, -2) = (-\infty, 1]$.
Hence, second and fourth option is true.

6. Find the domain of the inverse function of $y = x^3 + 1$.

   - ○ $\mathbb{R}$
   - ○ $\mathbb{R} \setminus \{1\}$
   - ○ $[1, \infty)$
   - ○ $\mathbb{R} \setminus [1, \infty)$

   Answer: Option 1

   Solution:

   We know that the domain of the inverse of a given function is the range of the given function.

   Since the range of the function $y = x^3 + 1$ is $\mathbb{R}$, domain of the inverse function of $y = x^3 + 1$ is $\mathbb{R}$.

   Hence, first option is correct.

7. If $f(x) = x^3$, then which of the following options is the set of points where the graphs of the functions $f(x)$ and $f^{-1}(x)$ intersect each other?

   - ◯ {(-1,1),(0,0),(1,-1)}
   - ◯ {(-2,-8),(1,1),(2,8)}
   - ◯ **{(-1,-1),(0,0),(1,1)}**
   - ◯ {(-2,-8),(0,0),(2,8)}

   Solution:
   Let $g(x) = x^{\frac{1}{3}}$ be a function such that $f \circ g = (x^{\frac{1}{3}})^3 = x$ and $g \circ f = (x^3)^{\frac{1}{3}} = x$
   Hence, g is the inverse function of f.
   To get intersection point

$$
\begin{aligned}
& f = g \\
\implies & x^3 = x^{\frac{1}{3}} \\
\implies & x^9 = x \\
\implies & x^9 - x = 0 \\
\implies & x(x^8 - 1) = 0 \\
\implies & x((x^4)^2 - 1) = 0 \\
\implies & x(x^4 + 1)(x^4 - 1) = 0 \\
\implies & x(x^4 - 1)(x + 1)(x - 1)(x^2 + 1) = 0
\end{aligned}
$$

   Observe that for real value 0, -1, 1, $f = g$ and $g(0) = 0, g(-1) = -1, g(1) = 1$.
   It follows that the set of points where the graphs of the functions $f(x)$ and $f^{-1}(x)$ intersect each other is {(-1,-1),(0,0),(1,1)}

8. In a survey, the population growth in an area can be predicted according to the equation $\alpha(T) = \alpha_o(1 + \frac{d}{100})^T$ where d is the percentage growth rate of population per year and T is the time since the initial population count α_o was taken. If in 2015, the population of Adyar was 30,000 and the population growth rate is 4% per year, then what will be the approximate population of Adyar in 2020?($T = 0$ corresponds to the year 2015, $T = 1$ corresponds to the year 2016 and so on..)

   - ○ 60251
   - ○ 71255
   - ○ 91000
   - ○ **36500**

Solution:
Given $\alpha(T) = \alpha_o(1 + \frac{d}{100})^T$ and initial population count $\alpha_o = 30000$.
The population growth rate $d = 4\%$ per year and $T = 5$.
Hence, the approximate population of Adyar in 2020 is $\alpha(5) = 30000 \times (1 + \frac{4}{100})^5 \approx 36500$

9. An ant moves along the curve whose equation is $f(x) = x^2 + 1$ in the restricted domain $[0, \infty)$. Let a mirror be placed along the line $y = x$. If the reflection of the ant with respect to the mirror moves along the curve $g(x)$, then which of the following options is(are) correct?

   ○ $\boldsymbol{g(x) = f^{-1}(x)}$
   ○ $g(x) = f(x)$
   ○ $\boldsymbol{g(x) = \sqrt[2]{(x-1)}}$
   ○ $g(x) = \sqrt[2]{(x+1)}$

   Solution:
   Given $f(x) = x^2 + 1$ in the restricted domain $[0, \infty)$.
   Using horizontal test, given function f is one to one function(See Figure M1W8AS-8.3).

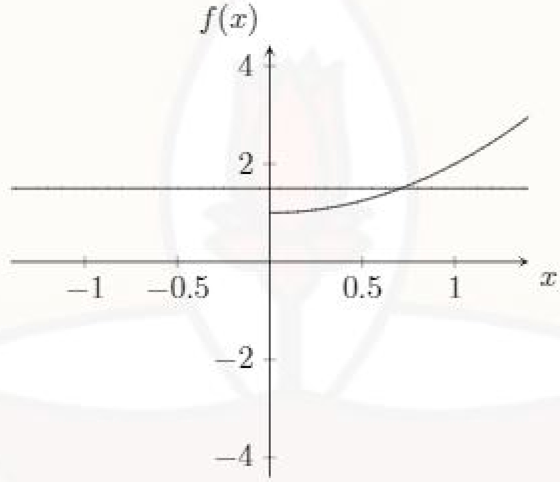


Figure M1W8AS-8.3

   Hence, f is reversible function.
   We know that the graph of the inverse of a function is symmetric along the line $y = x$ and the mirror placed along the the line $y = x$.
   It follows that $g(x)$ is inverse of the function f i.e $g(x) = f^{-1}(x)$.
   Now, consider a function $f^{-1}(x) = \sqrt{x-1}$.
   Since, $f \circ f^{-1} = (\sqrt{x-1})^2 + 1 = x = I$, similarly, $f^{-1} \circ f = x = I$, where I is the identity function.
   Therefore, inverse function of $f(x) = x^2 + 1$ is $f^{-1}(x) = \sqrt{x-1}$,
   where f^{-1} is just a notation of inverse function.
   Hence, first and third options are correct.

10. Suppose a textile shop provides two different types of offers during a festival season. The first offer(D_1) is "shop for more than ₹14,999 and pay only ₹9,999". The second offer(D_2) is "avail 30% discount on the total payable amount". If Shalini wants to buy two dresses each of which costs more than ₹8,000 and she is given the choice to avail both offers simultaneously, then which of the following options is(are) correct?

- ◯ The minimum amount she should pay after applying two offers cannot be determined because the exact values of the dresses she wanted to buy are unknown.
- ◯ **The minimum amount she should pay after applying the two offers simultaneously is approximately ₹6,999.**
- ◯ The amount she is supposed to pay after applying D_2 only is approximately ₹11,199.
- ◯ **The amount she is supposed to pay after applying D_1 only is approximately ₹9,999.**
- ◯ **Suppose the total payable amount is ₹17,999 for the two dresses. In order to pay minimum amount Shalini should avail offer D_1 first and offer D_2 next.**
- ◯ **Suppose the total payable amount is ₹17,999 for the two dresses. If Shalini avails offer D_2 first, then she cannot avail offer D_1.**
- ◯ Suppose the total payable amount is ₹17,999 for the two dresses. In order to pay minimum amount Shalini should avail offer D_2 first and offer D_1 next.

Solution:
Given each dress cost more than ₹8000.
If Shalini wants to buy two dresses then total payable amount is greater than ₹16000 which is greater than ₹14999.
So she can avail offer D_1.
If she avails offer D_1 first and offer D_2 next, then the total payable amount = ₹9999$(1 - \frac{30}{100}) \approx$ ₹6999
If she avails offer D_2 first, then the total payable amount can be less than ₹14999 or greater than or equal to ₹14999. After that she may or not avail the offer D_1. If she avails offer D_1 after D_2, then also the total payable amount can not be less than ₹6999.
In any case the minimum amount she should pay after applying the two offers(without any order) simultaneously is at least ₹6,999.
Hence, first option is not correct and second option is correct.
Since the total payable amount is unknown, therefore we can not say how much she needs to pay after applying the offer D_2.
Hence, third option is not correct.
As we see above if Shalini avails offer D_1 only, then payable amount = ₹9999.
Hence, fourth option is correct.
Suppose the total payable amount is ₹17,999 for the two dresses (which is greater than

₹14999).
If Shalini avails offer D_2 first then payable amount $= ₹17999(1 - \frac{30}{100}) \approx$ ₹12599 which is less than ₹14999.
So, she can not avail offer D_1 next and she has to pay $\approx$ ₹12599.
Hence, sixth option is correct and seventh option is not correct.
And if Shalini avails offer D_1 first then payable amount $=$ ₹9999
and then offer D_2 then total payable amount $= ₹9999(1 - \frac{30}{100}) \approx$ ₹6999
Hence, from above in order to pay minimum amount Shalini should avail offer D_1 first and offer D_2 next.
So, fifth option is correct.

11. If $f(x) = x^2$ and $h(x) = x - 1$, then which of the following options is(are) incorrect?

- ◯ $f \circ h$ is not an injective function.
- ◯ **$\boldsymbol{f \circ h}$ is an injective function**
- ◯ $\boldsymbol{f(f(h(x))) \times h(x) = (x-1)^4}$
- ◯ $f(f(h(x)) \times h(x) = (x-1)^5$

Solution:
Given $f(x) = x^2$ and $h(x) = x - 1$.
Using horizontal line test, $f \circ h = (x-1)^2$ is not a injective function.
Again, $f(f(h(x))) \times h(x) = ((x-1)^2)^2 \times (x-1) = (x-1)^{2\times2+1} = (x-1)^5$.
Hence, second and third options are incorrect.

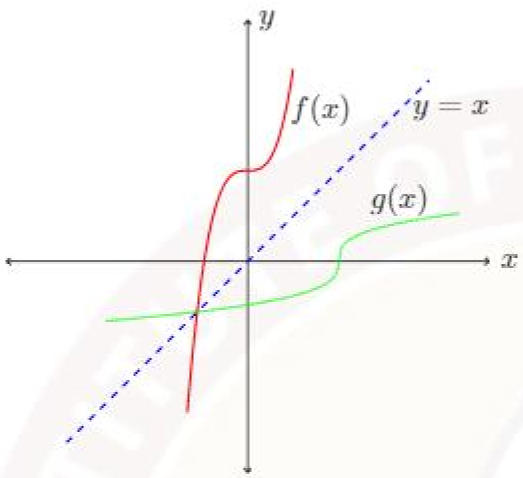
Likewise, the set of other options given below can be solved accordingly:
Set of correct options:

- $f \circ h$ is not an injective function.
- $f(f(h(x)) \times h(x) = (x-1)^5$
- $h \circ f$ is not an injective function.
- There are two distinct solution for $h(h(f(x)) = 0$.

Set of incorrect options:

- $f \circ h$ is an injective function.
- $f(f(h(x))) \times h(x) = (x-1)^4$.
- $h \circ f$ is an injective function.
- There is only one solution for $h(h(f(x)) = 0$.
- Let $q(x) = f(x) - h(x)$, then $q(x)$ is bijective function if $q(x) : \mathbb{R} \to [-1.25, \infty)$
- Let $q(x) = f(x) + h(x)$, then $q(x)$ is bijective function if $q(x) : \mathbb{R} \to [0.75, \infty)$
- Let $q(x) = f(x) + h(x)$, then $q(x)$ is bijective function if $q(x) : \mathbb{R} \to [-1.25, \infty)$
- Let $q(x) = f(x) - h(x)$, then $q(x)$ is bijective function if $q(x) : \mathbb{R} \to [0.75, \infty)$

12. Let $f(x)$, $g(x)$, $p(x)$ and $q(x)$ be the functions defined on $\mathbb{R}$. Refer Figure 3 (A and B) and choose the correct option(s) from the following.
(MSQ), (Answer: Option (a)(b)(c)(d))



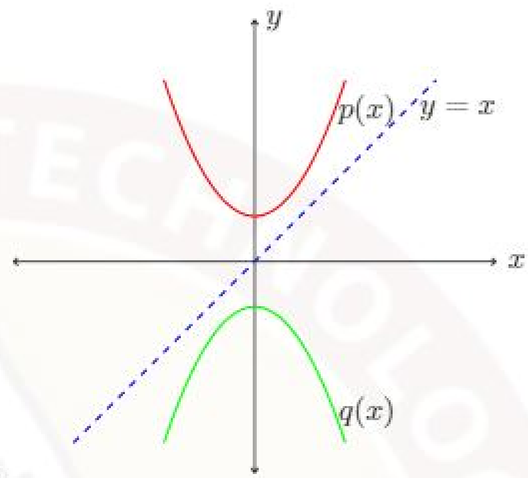


Figure 3

- ◯ $g(x)$ may be the inverse of $f(x)$.
- ◯ $p(x)$ and $q(x)$ are even functions but $f(x)$ and $g(x)$ are neither even functions nor odd functions.
- ◯ $q(x)$ could not be the inverse function of $p(x)$.
- ◯ $p(x)$, $q(x)$ can be an even degree polynomial functions and $f(x)$ can be an odd degree polynomial functions.

Solution: {See Figure 3 for reference}

• $f(x)$ and $g(x)$ are approximately symmetric across $y = x$. Therefore, $g(x)$ may be inverse of $f(x)$.

• $p(x)$ and $q(x)$ are approximately symmetric across $y-$axis. Therefore, $p(x)$ and $q(x)$ are even functions.

• $p(x)$ and $q(x)$ are not symmetric across $y = x$. Therefore, $q(x)$ could not be the inverse of $p(x)$.

• From the end behaviors of the graphs, we can claim that $p(x)$ and $q(x)$ are even-degree polynomials and $f(x)$ can be an odd-degree polynomial.

Week 6
Practice Assignment
Mathematics for Data Science - 1

1. If $b > 0$ and $4\log_x b + 9\log_{b^5x} b = 1$, then the possible value(s) of x is (are)

(Ans: (a), (c))

- ○ b^{10} ✓
- ○ b^9
- ○ b^{-2} ✓
- ○ b^5
- ○ b^4

Solution:
Given that: $4\log_x b + 9\log_{b^5x} b = 1$ and $b > 0$

$$\begin{aligned} 4\log_x b + 9\log_{b^5x} b = 1 &\implies \frac{4}{\log_b x} + \frac{9}{\log_b(b^5x)} = 1 \\ &\implies \frac{4}{\log_b x} + \frac{9}{\log_b b^5 + \log_b x} = 1 \\ &\implies \frac{4}{\log_b x} + \frac{9}{5\log_b b + \log_b x} = 1 \end{aligned}$$

Let $p = \log_b x$

$$\begin{aligned} \text{Now, } 4\log_x b + 9\log_{b^5x} b = 1 &\implies \frac{4}{p} + \frac{9}{5+p} = 1 \\ &\implies \frac{4(5+p)+9p}{p(5+p)} = 1 \\ &\implies 20 + 4p + 9p = 5p + p^2 \\ &\implies p^2 - 8p - 20 = 0 \\ &\implies p^2 - 10p + 2p - 20 = 0 \\ &\implies p(p-10) + 2(p-10) = 0 \\ &\implies (p+2)(p-10) = 0 \\ &\implies p = -2, 10 \end{aligned}$$

We know that $p = \log_b x$
Case 1: $p = -2 \implies -2 = \log_b x \implies x = b^{-2}$

Case 2: $p = 10 \implies 10 = \log_b x \implies x = b^{10}$

Note: $\log_{a^b} c = \dfrac{1}{b} \log_a c$
Proof:

$$\begin{aligned} LHS &= \log_{a^b} c \\ &= \frac{1}{\log_c a^b} \\ &= \frac{1}{b \log_c a} \\ &= \frac{1}{b} \times \frac{1}{\log_c a} \\ &= \frac{1}{b} \log_a c \\ &= RHS \end{aligned}$$

2. George deposits ₹ $5L$ in a bank that compounded quarterly at the rate of 20% per year. How long will it take to increase his money to 16 times the principal amount (in a year)?

(Ans: (b), (c), (d))

- ○ $\frac{\ln 16}{4}$
- ○ $\frac{\ln 16}{4 \ln \frac{21}{20}}$ ✓
- ○ $\frac{\ln 2}{\ln \frac{21}{20}}$ ✓
- ○ $\log_{\frac{21}{20}} 2$ ✓
- ○ $\frac{\ln 2^{\frac{1}{4}}}{\ln \frac{21}{20}}$

Solution:

The formula for Compound Interest is given by:

$$A = P\left(1 + \frac{R}{100}\right)^{nt}; \text{ where}$$

$t =$ time period (years)
$R =$ Interest year per year
$P =$ Principal on initial deposit
$A =$ Amount after t years
$n =$ No. of times it compounded in a year

$$\text{Now, } A = P\left(1 + \frac{R}{n \times 100}\right)^{nt}$$

At $A = 16P$, $n = 4$ and $R = 20$,

$$\begin{aligned}
A = 16P &= P\left(1 + \frac{20}{400}\right)^{4t} \\
\implies 16P &= P\left(1 + \frac{1}{20}\right)^{4t} \\
\implies 16 &= \left(\frac{21}{20}\right)^{4t} \\
\implies \ln 16 &= 4t \ln\left(\frac{21}{20}\right) \\
\implies t &= \frac{1}{4}\left(\frac{\ln 16}{\ln \frac{21}{20}}\right) \text{ (using Formula: } \ln a^b = b \ln a \text{)} \\
\implies t &= \frac{\ln(16)^{\frac{1}{4}}}{\ln \big) \frac{21}{20} \big(} \\
\implies t &= \frac{\ln(2^4)^{\frac{1}{4}}}{\ln \big) \frac{21}{20} \big(} \\
\implies t &= \frac{\ln 2}{\ln \big) \frac{21}{20} \big(} \\
\implies t &= \log_{\frac{21}{20}} 2 \text{ (change of base formula)}
\end{aligned}$$

3. Choose the set of correct options.

- ○ $\log_5 2$ is a rational number.
- ○ If $0 < b < 1$ and $0 < x < 1$ then $\log_b x < 0$
- ○ If $\log_3(\log_5 x) = 1$ then $x = 125$ ✓
- ○ If $0 < b < 1$, $0 < x < 1$ and $x > b$ then $\log_b x > 1$

◯ If $0 < b < 1$ and $0 < x < y$ then $\log_b x > \log_b y$ ✓

(Ans: (c), (e)) Solution:

- Suppose $\log_5 2$ is a rational number.
Then, it can be represented in the $\frac{p}{q}$ form, where p and q are integers.
Now,
$$\log_5 2 = \frac{p}{q} \implies 2 = 5^p q \implies 2^q = 5^p$$
But 5 cannot divide 2 which is a contradiction to our assumption.
Therefore, our assumption is wrong. $\log_5 2$ is not a rational number.
Therefore, option (a) is not true.

- Given that: $\log_3(\log_5 x) = 1$
$\log_3(\log_5 x) = 1 \implies \log_3(\log_5 x) = \log_3 3 \implies \log_5 x = 3$
$\implies \log_5 125 = \log_5 5^3 = 3$
Thus, $x = 125$. Therefore, option (c) is true.

- Let $x > 0$. We know that: $\log_a x$ is increasing when $a > 1$ and $\log_a x$ is decreasing when $0 < a < 1$.
Therefore, for $0 < b < 1$, $\log_b x$ is a decreasing function.
For $0 < x < 1, \log_b x > \log_b 1 \implies \log_b x > 0$.
Similarly, for $0 < x < 1$ and $x > b, \log_b x < \log_b b \implies \log_b x < 1$
Similarly, for $0 < x < y, \log_b x > \log_b y$
Therefore, option (e) is true and options (b) and (d) are not true.

4. Suppose that two types of insects are found in a pond. Their growth in number after t seconds is given by the equations $f(t) = 5^{3t-2}$ and $h(t) = 3^{2t-1} (t \neq 0)$. For what value of t will both insects be of same number in the pond?

(Ans: (a), (b), (c))

◯ $\frac{\ln 3 + 2\ln 5}{3\ln 5 - 2\ln 3}$ ✓

◯ $\frac{\ln 75}{\ln \frac{125}{9}}$ ✓

◯ $\log_{\frac{125}{9}} 75$ ✓

◯ $\frac{\ln 5 + 2\ln 3}{3\ln 3 - 2\ln 5}$

Solution:

Insects number will be same when:

$$f(t) = h(t) \implies 5^{3t-2} = 3^{2t-1}$$

$$\implies (3t-2)\ln 5 = (2t+1)\ln 3$$

$$\implies (3\ln 5)t - 2\ln 5 = (2\ln 3)t + \ln 3$$

$$\implies (3\ln 5)t - (2\ln 3)t = \ln 3 + 2\ln 5$$

$$\implies t(3\ln 5 - 2\ln 3) = \ln 3 + 2\ln 5$$

$$\implies t = \frac{\ln 3 + 2\ln 5}{3\ln 5 - 2\ln 3}$$

$$\implies t = \frac{\ln 3 + \ln 5^2}{\ln 5^3 - \ln 3^2}$$

$$\implies t = \frac{\ln 3 \times 25}{\ln\left(\frac{125}{9}\right)}$$

$$\implies t = \frac{\ln 75}{\ln\left(\frac{125}{9}\right)}$$

$$\implies t = \log_{\frac{125}{9}} 75$$

By Using Formulas:
(1) $\ln a^b = b\ln a$
(2) $\ln(ab) = \ln a + \ln b$
(3) $\ln \frac{a}{b} = \ln a - \ln b$
(4) $\log_b a = \frac{\log a}{\log b}$

5. Which of the following is/are true?
(MSQ) **(Ans: (a), (c))**

○ If m and n are positive real numbers, then $m^{\log(n)} = n^{\log(m)}$. ✓

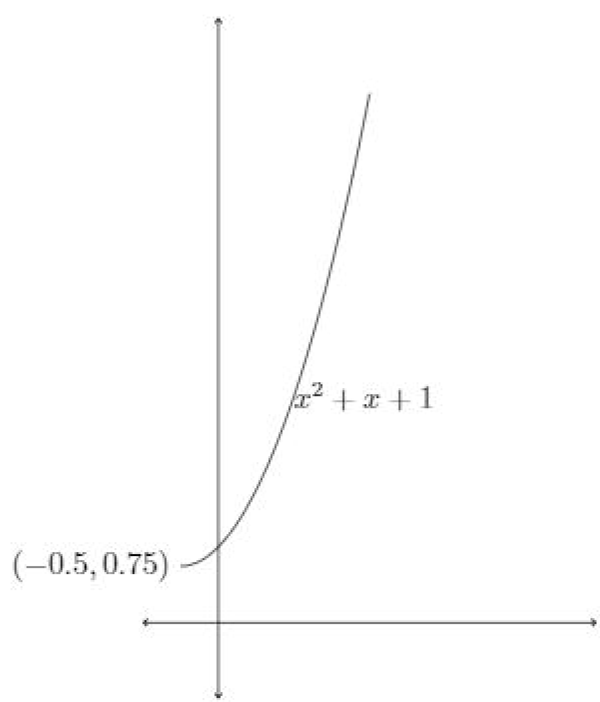
○ $\log_5 123456789999999999999$ is a rational number.

○ The function $f(x) = \log_{10}(x^2 + x + 1)$ is one-one on the interval (-0.5, ∞). ✓

○ None of the above.

Solution:

- Given that m and n are positive real numbers. Let $a = m^{\log(n)}$ and $b = n^{\log(m)}$.
  Now, $\log a = \log(n) \times \log(m)$ and $\log b = \log(m) \times \log(n)$
  But $\log(n) \times \log(m) = \log(n) \times \log(m) \implies \log a = \log b$
  Since, log is a one-one function, $a = b$.
  Therefore, option (a) is true.
- Consider $a = 123456789999999999999$. Suppose $\log_5 a$ is a rational number.
  Then, it can be represented in the $\frac{p}{q}$ form, where p and q are integers.
  Now,
  $$\log_5 a = \frac{p}{q} \implies a = 5^p q \implies a^q = 5^p$$
  But 5 cannot divide a which is a contradiction to our assumption.
  Therefore, our assumption is wrong. $\log_5 a$ is not a rational number.
  Therefore, option (b) is not true.
- For $x^2 + x + 1$,



As shown in above figure, $x^2 + x + 1$ is one-one (strictly increasing) in the interval $(-0.5, \infty)$. Also, $log_1 0x$ is a one-one (strictly increasing) function on its entire domain.
Therefore, $f(x) = \log_{10}(x^2 + x + 1)$ is one-one on the interval (-0.5, ∞).
Therefore, option (c) is true.

6. Which of the following is/are true?
(MSQ) **(Ans: (a), (b), (c), (d))**
   - ◯ Suppose D is an arbitrary subset of $\mathbb{R}$ and f is one-one function on D. $\log f(x)$ whenever defined is also an one-one function on D. ✓
   - ◯ $(14!)^{\frac{1}{14}} < (15!)^{\frac{1}{15}}$, where ! denotes the factorial function, and for a non-negative integer n, the value of $n!$ is $n \times (n-1) \times ... \times 2 \times 1$.✓
   - ◯ The function $f(x) = 2^x + 3^x + ... + 100^x$ is one-one function on $\mathbb{R}$.✓
   - ◯ There exists a function $f(x)$ on $\mathbb{R}$, such that $\log(f(x)) \geq 100$ for all $x \in \mathbb{R}$. ✓

Solution:

(a) Given: f is one-one function on D.
Also, $\log f(x)$ is defined on D.
We, know that the log function is a one-one function and therefore *log* of a one-one function (strictly increasing or decreasing) will give a one-one function.
Therefore, option (c) is true.

(b) To verify: $(14!)^{\frac{1}{14}} < (15!)^{\frac{1}{15}}$
Suppose that the above statement is true.
Taking $\log_{14!}$ on both sides, we have:

$$
\begin{aligned}
(14!)^{\frac{1}{14}} < (15!)^{\frac{1}{15}} &\iff \frac{1}{14}\log_{14!} 14! < \frac{1}{15}\log_{14!} 15! \\
&\iff \frac{1}{14} < \frac{1}{15}\log_{14!}(15 \times 14!) \\
&\iff \frac{1}{14} < \frac{1}{15}\left[\log_{14!} 15 + \log_{14!} 14!\right] \\
&\iff \frac{15}{14} < \log_{14!} 15 + 1 \\
&\iff \frac{15}{14} - 1 < \log_{14!} 15 \\
&\iff \frac{1}{14} < \log_{14!} 15 \\
&\iff 1 < 14\log_{14!} 15 \\
&\iff 1 < \log_{14!}(15)^{14} \\
&\iff 14! < 15^{14}, \text{which is true}
\end{aligned}
$$

Therefore, option (b) is true.

(c) We know that, the exponential function of a natural number (except 0 and 1) is strictly increasing function and thus one-one function. Suppose $x \neq y; x, y \in \mathbb{R}$ such that $0 < x < y$.

$$
\begin{aligned}
x \neq y \text{ and } 0 < x < y &\implies 2^x < 2^y, 3^x < 3^y, ..., 100^x < 100^y \\
&\implies 2^x + 3^x + ... + 100^x < 2^y + 3^y + ... + 100^y \\
&\implies f(x) < f(y) \\
&\implies f(x) \neq f(y)
\end{aligned}
$$

Thus, $f(x)$ is a one-one function.
Therefore, option (c) is true.

(d) Let $c \in \mathbb{R}$ such that $\log c > 100$. There exists such a c because the range of log is $(-\infty, \infty)$.
Define $f : \mathbb{R} \to \mathbb{R}, f(x) = c$, for all $x \in \mathbb{R}$. Thus, $\log(f(x)) \geq 100$ for all $x \in \mathbb{R}$.
Therefore, option (d) is true.

7. If $\log_{\sqrt{2}}(x+4) - \log_2 \left(\frac{1}{2}x + 2\right) = 1$ then x is

(Ans: (a))

- ○ -3 ✓
- ○ 1
- ○ -4

◯ 5

Solution:

Given that $\log_{\sqrt{2}}(x+4) - \log_2 \big)\frac{1}{2}x + 2 \big(= 1$

$$\implies \frac{1}{2}\log_2(x+4) - \log_2\left(\frac{x}{2}+2\right) = 1$$

$$\implies \log_2(x+4)^{\frac{1}{2}} - \log_2\left(\frac{x+4}{2}\right) = 1$$

$$\implies \log_2\left(\frac{(x+4)^{\frac{1}{2}}}{\big)\frac{x+4}{2}\big(}\right) = 1$$

$$\implies \frac{(x+4)^{\frac{1}{2}}}{\big)\frac{x+4}{2}\big(} = 2$$

$$\implies (x+4)^{\frac{1}{2}} = 2\left(\frac{x+4}{2}\right)$$

Squaring on both sides,

$$(x+4) = (x+4)^2$$

$$(x+4)^2 - (x+4) = 0$$

$$(x+4)((x+4)-1) = 0$$

We get:

$x+4=0$ or $x+4-1=0$
$x=-4$ or $x=-3$

From Question we have:

$$\log_{\sqrt{2}}(x+4) - \log_2\left(\frac{x}{2}+2\right) = 1$$

When $x=-4$,

$$\log_{\sqrt{2}}(-4+4) - \log_2\left(\frac{-4}{2}+2\right) = 1$$

Notice: $-4+4=0$
Thus -4 is out of domain of log function because $\log 0$ is undefined.
Now, when $x=-3$,

$$\begin{aligned}
\log_{\sqrt{2}}(-3+4) - \log_2\left(\frac{-3}{2}+2\right) &= \log_{\sqrt{2}}(1) - \log_2\left(\frac{1}{2}\right) \\
&= 0 - \log_2\left(\frac{1}{2}\right) \\
&= 0 - [\log_2 1 - \log_2 2] \\
&= -[0-1] = 1
\end{aligned}$$

Thus $x=-3$ is the right option.

8. Seismologists use the Richter scale to measure and report the magnitude of an earthquake as given by the equation $R = \ln I - \ln I_0$, where I is the intensity of an earthquake with respect to a minimal or reference intensity I_0 (i.e. $I = cI_0$, where c is a constant). The reference intensity is the smallest earth movement that can be recorded on a seismograph. If an earthquake in a city A recorded of magnitude 8.0 in the Richter scale and the intensity of the earthquake in the city B is the reference intensity, then what is the ratio of the intensity of the earthquake in city A with respect to city B?

(Ans: (c))

- ○ $e^0 : 1$
- ○ $e^1 : 2$
- ○ $e^8 : 1$ ✓
- ○ $e^5 : 1$
- ○ $e^8 : 2$

Solution:

Given: R = 8.0, intensity of the earthquake in city $B = I_0$

Using the given equation, $R = \ln I - \ln I_0$

$$\implies R = \ln \frac{I}{I_0} = 8$$

$$\implies e^8 = \frac{I}{I_0}$$

$$\implies \frac{I}{I_0} = \frac{e^8}{1}$$

$$\implies e^8 : 1$$

Therefore the ratio of the intensity of earthquake in city A w.r.t. city B is $e^8 : 1$.

9. Suppose that the number of bacteria present in a loaf of rotten bread after t minutes is given by the equation $G(t) = G_0 3^{kt}$, where G_0 represents the number of bacteria at $t = 0$, k is a constant (Given $\ln 730 = 6.59$ and $\ln 3 = 1.09$). If the initial number of bacteria is 1000 and it takes 1 min to increase to 9000 then how long (in minutes) would it take for the bacteria count to grow to 730000 (integer value of t)?

(Ans: (c))

- ◯ 2
- ◯ 1
- ◯ 3 ✓
- ◯ 6

Solution:

Given that: $G_0 = 1000$. At $t = 1$ min, $G(t) = 9000$
To find: At what time (t), $G(t) = 7,30,000$

$$G(t) = G_0 3^{kt} \implies \frac{G(t)}{G_0} = 3^{kt}$$

At $t = 1$ min,

$$\begin{aligned}\frac{G(t)}{G_0} = 3^{kt} &\implies \frac{9000}{1000} = 9 = 3^{kt}\\ &\implies \log_3 3^2 = kt \log_3 3\\ &\implies 2 \log_3 3 = kt \log_3 3\\ &\implies k = 2\end{aligned}$$

On substituting the values of k and G_0, equation 1 becomes:

$$G(t) = 1000 \times 3^{2t}$$

When $G(t) = 730000$,

$$\begin{aligned} G(t) = 1000 \times 3^{2t} &\implies \frac{730000}{1000} = 3^{2t} \\ &\implies 730 = 3^{2t} \\ &\implies \ln 730 = 2t \ln 3 \\ &\implies t = \frac{\ln 730}{2 \ln 3} \\ &\implies t = \frac{6.59}{2 \times 1.09} = 3 \text{ min} \end{aligned}$$

Thus at $t = 3$ min (integer value) bacteria count will be 7,30,000.

Read the following paragraph and answer questions 8 and 9

Let c_A and c_B be the luminosity (measure of brightness) (luminous efficacy) of the bulbs A and B respectively. The bulb A is $f(x)$ times brighter than the B, if $f(x) = 3^{x^2+1}$ (i.e. $c_A = f(x) \times c_B$), where x is the difference of the magnitude of supply voltage between the bulb A and the bulb B. Answer questions 8 and 9 based on the above information.

10. If the bulb A is 10 times brighter than the bulb B, then the difference of the magnitude of supply voltage between the two bulbs is

(Ans: (d))

- ○ $\sqrt{\log_3 5 - 1}$
- ○ $\sqrt{\log_3 10}$
- ○ $\sqrt{\frac{\ln 10}{\ln 3}}$
- ○ $\sqrt{\log_3 \frac{10}{3}}$ ✓

Solution:

Given that: $c_A = 10\, c_B; c_A = c_B \times f(x); c_A = c_B \times 3^{x^2+1}$

$$\begin{aligned} 10\, c_B = c_B \times 3^{x^2+1} &\implies \log_3 10 = (x^2+1)\log_3 3 \\ &\implies \log_3 10 = x^2 + 1 \\ &\implies x^2 = \log_3 10 - 1 \\ &\implies x = \sqrt{\log_3 10 - 1} \\ &\implies x = \sqrt{\log_3 10 - \log_3 3} \\ &\implies x = \sqrt{\log_3 \frac{10}{3}} \end{aligned}$$

The difference between the magnitude of 2 bulbs is $\sqrt{\log_3 \frac{10}{3}}$

11. If 4 voltage and 3 voltage are the supply voltages for the bulbs A and B respectively then how many times the bulb A is brighter than the bulb B?

(Ans: 9)

Solution:
Since x is the difference between the supply voltage of A and B, thus $x = 4 - 3 = 1$
We know that $c_A = c_B \times f(x)$
We have to find $f(x)$.

$$f(x) = 3^{x^2+1} \implies f(x) = 3^{1+1} = 3^2 = 9$$

$$\text{Now, } c_A = 9\, c_B$$

Therefore, bulb A is 9 times brighter than bulb B.

12. Find the number of values of x satisfying the equation $(5x)^{\log_{(5x)^{\frac{1}{5}}}(6x^3-36x^2+66x-35)^{\frac{1}{5}}} = 1.$

(Ans: 3)

Solution:
Given: $(5x)^{\log_{(5x)^{\frac{1}{5}}}(6x^3-36x^2+66x-35)^{\frac{1}{5}}} = 1$

$$
\begin{aligned}
&(5x)^{\log_{(5x)^{\frac{1}{5}}}(6x^3-36x^2+66x-35)^{\frac{1}{5}}} = 1\\
&\implies \log_{(5x)^{\frac{1}{5}}}(6x^3-36x^2+66x-35)^{\frac{1}{5}} = 0\\
&\implies 6x^3-36x^2+66x-35 = 1\\
&\implies 6x^3-36x^2+66x-36 = 0\\
&\implies 6(x^3-6x^2+11x-6) = 0\\
&\implies (x-1)(x^2-5x+6) = 0\\
&\implies (x-1)(x-3)(x-2) = 0\\
&\implies x=1, x=3, x=2
\end{aligned}
$$

Thus, there are three values of x satisfying the given equation.

Mathematics for Data Science - 1
Exponential and Logarithm
Assignment

1 Multiple Choice Questions(MCQ)

1. If $18^x - 12^x - (2 \times 8^x) = 0$, then the value of x is.

   1. $\frac{\ln 2}{\ln 3 - \ln 2}$
   2. $\frac{\ln 18}{\ln 12 - \ln 8}$
   3. $\ln 2$
   4. $\ln 18$

   Answer: Option 1

Solution: $18^x - 12^x - (2 \times 8^x) = 0$

Domain $= \mathbb{R}$ as all the terms are exponential functions.

we can write: $2^x . 9^x - 2^x . 6^x - 2^x(2 \times 4^x) = 0$

2^x is a positive number. Dividing by $2^x \Rightarrow$

$9^x - 6^x - (2 \times 4^x) = 0$ —— ①

$\boxed{\text{Let } a = 3^x \text{ and } b = 2^x}$

then $9^x = (3^2)^x = 3^{2x} = (3^x)^2 = a^2$

$6^x = 2^x 3^x = ba$

$4^x = (2^x)^2 = a^2$

Therefore equation ① would be

$a^2 - ab - 2b^2 = 0$

$\Rightarrow a^2 - 2ab + ab - 2b^2 = 0$

$a(a-2b) + b(a-2b) = 0$

$(a-2b)(a+b) = 0$

If $a - 2b = 0 \Rightarrow a = 2b \Rightarrow 3^x = 2 \times 2^x$

taking log $\Rightarrow$ $x \log 3 = \log 2 + x \log 2$

$\boxed{x = \frac{\log 2}{\log 3 - \log 2}}$ — Answer.

If $a + b = 0$

$3^x + 2^x = 0 \Rightarrow$ Not possible

2. Suppose three distinct persons A, B and C are standing on the X- axis of the XY- plane (as shown in the figure M1W9G-1) and the distance between B and A is same as the distance between C and B. The coordinates of A, B and C are $(\log_5 3, 0)$, $(\log_5(3^x - \frac{9}{2}), 0)$ and $(\log_5(3^x - \frac{9}{4}), 0)$ respectively. What is the distance between C and B?
(MCQ), (Ans:(a))

1. $\log_5(2)$ units.
2. $\log_5(\frac{5}{4})$ units.
3. $\log_5(\frac{3}{2})$ units
4. $\log_5(\frac{7}{3})$ units.

$AB = \sqrt{\left\{\log_5(3^x - 9/2) - \log_5 3\right\}^2 + 0} = \log_5 \frac{(3^x - 9/2)}{3} \Rightarrow 3^x > 9/2$

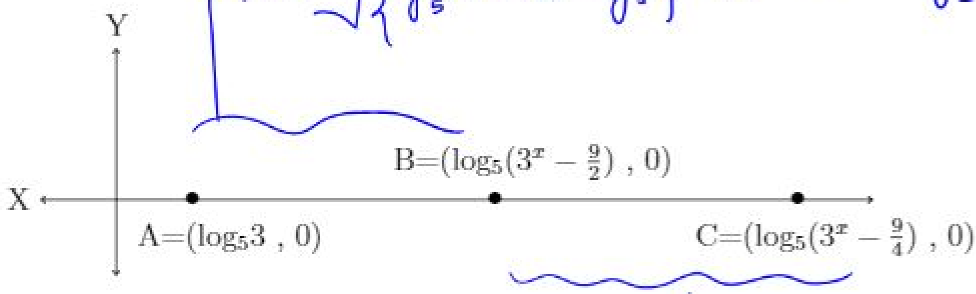


Figure M1W9G-1

$BC = \sqrt{\left\{\log_5(3^x - 9/4) - \log_5(3^x - 9/2)\right\}^2 + 0}$

Answer: Option 3

$= \log_5 \left\{\frac{3^x - 9/4}{3^x - 9/2}\right\} \Rightarrow 3^x - 9/4 > 0$

Given: $AB = BC$

$\log_5 \left\{\frac{3^x - 9/2}{3}\right\} = \log_5 \left\{\frac{3^x - 9/4}{3^x - 9/2}\right\}$

$\frac{3^x - 9/2}{3} = \frac{3^x - 9/4}{3^x - 9/2} \Rightarrow (3^x - 9/2)^2 = 3(3^x - 9/4)$

$\Rightarrow (3^x)^2 + (9/2)^2 - 2(3^x)(9/2) = 3(3^x) - 3(9/4)$

take $3^x = a \Rightarrow a^2 + (\frac{9}{2})^2 - 9a = 3a - 3(9/4) \Rightarrow a^2 - 12a + \frac{81}{4} + 27/4 = 0$

$\Rightarrow a^2 - 12a + \frac{108}{4} = 0 \Rightarrow a^2 - 12a + 27 = 0 \Rightarrow (a-3)(a-9) = 0$

If $a = 3 \Rightarrow 3^x = 3 \Rightarrow x = 1$ but $3^1 \not> 9/2$. Therefore, $\boxed{x = 2}$

If $a = 9 \Rightarrow 3^x = 9 \Rightarrow x = 2$ Now $3^2 = 9 > 9/2$

$$BC = \log_5 \left\{ \frac{3^x - 9/4}{3^x - 9/2} \right\} = \log_5 \left\{ \frac{3^2 - 9/4}{3^2 - 9/2} \right\} = \log_5 \left\{ \frac{3/4}{1/2} \right\}$$

$$\boxed{BC = \log_5 \left\{ 3/2 \right\}} \Longrightarrow \underline{\text{Answer.}}$$

3. In a city, a rumour is spreading about the safety of corona vaccination. Suppose N number of people live in the city and $f(t)$ is the number of people who **have not** yet heard about the rumour after t days. Suppose $f(t)$ is given by $f(t) = Ne^{-kt}$, where k is a constant. If the population of the city is 1000, and suppose 40 have heard the rumor after the first day. After how many days (approximately) half of the population would have heard the rumor?

   1. 20
   2. 17
   3. 13
   4. 12

   Answer: Option 2

After first day $\Rightarrow$ $t=1$

40 have heard therefore, $1000 - 40 = 960$ people

have not heard i.e.,

$$960 = Ne^{-kt}$$

$$t=1 \Rightarrow \quad 960 = 1000e^{-k} \Rightarrow \boxed{e^{-k} = \frac{960}{1000}}$$

Half of population will heard then $f(t) = \frac{1000}{2} = 500$

therefore,

$$500 = 1000e^{-kt} \Rightarrow \frac{500}{1000} = \left(e^{-k}\right)^t$$

$$\frac{500}{1000} = \left(\frac{960}{1000}\right)^t$$

taking log:

$$\ln\left\{\frac{500}{1000}\right\} = t \ln\left\{\frac{960}{1000}\right\}$$

$$\ln(1/2) = t \ln(.96) \Rightarrow \quad t = \frac{\ln(0.5)}{\ln(.96)}$$

$$t \simeq 16.97$$

$\boxed{t \text{ can be condisered as } 17 \text{ days.}}$

4. Consider the function $f(x) = \log_2(12+4x-x^2)$. Choose the correct set of option(s)

× a. f(x) is strictly increasing when x is in [2, infinity).
× b. The range of 'f' is (0,log 12].
× c. The minimum value of f(x) is 4.
× d. The range of 'f' is (-infinity, infinity).
✓ e. f(x) is one-one function when x is in (-infinity,2].
✓ f. The range of 'f' is (-infinity,4].

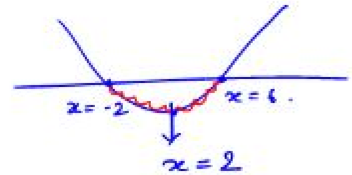
Solution:- Finding domain

$12+4x-x^2 > 0$

$x^2-4x-12 < 0$

$(x-6)(x+2) < 0$

$\boxed{x \in (-2,6)}$



at $x=2$

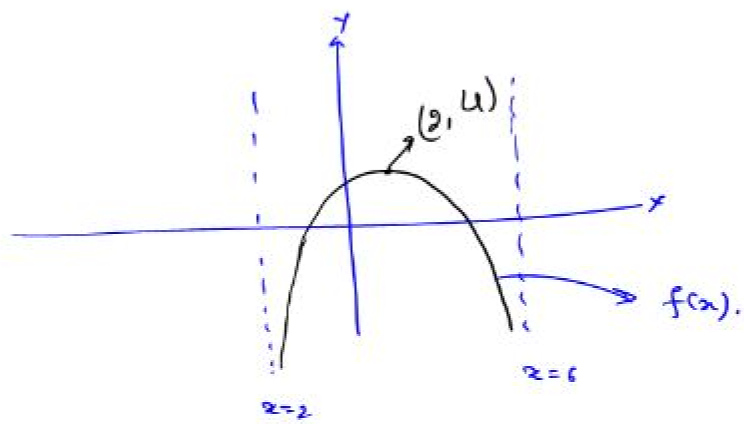
$12+4x-x^2 = 12+8-4 = 16$

$f(x) = \log_2(16) = 4$

→ $x=-2$ and $x=6$ will work as assymptotes.

→ As at $x=0$, $\log x$ tends towards $-\infty$, similarly at $x=-2$ and $x=6$, $f(x)$ will tends towards $-\infty$.

→ $f(x)$ will be symmetric Arround $x=2$.



graph 1.

$\boxed{\text{Range: } (-\infty, 4]}$ — option f is correct & options b & d are incorrect.

× option a: From graph it is clear that $f(x)$ is strictly decreasing when $x \in [2,\infty)$

× option c: The maximum value of $f(x)$ is 4.

✓ option e: $f(x)$ passes the horizontal line test when $x \in (-\infty, 2]$

Use the following information for the questions 5 and 6.

Consider the function $f(x) = \frac{2e^x}{3e^x+1}$ from $\mathbb{R}$ to $\mathbb{R}$.

5. Which of the following is true about f?

    1. f is not a one to one function.
    2. f is a one to one function.
    3. Range of f is $\mathbb{R}$.
    4. f is a bijective function.

    Answer: Option 2

6. The inverse of f would be

    1. $\ln(\frac{2x}{2-3x})$
    2. $\ln(\frac{2x}{2x-x})$
    3. $\ln(\frac{x}{2-3x})$
    4. $\ln(\frac{x}{2x-x})$

    Answer: Option 3

$$f(x) = \frac{2e^x}{3e^x+1}$$

To find one to one nature:

take $x_1 > x_2$.

$$f(x_1) = \frac{2e^{x_1}}{3e^{x_1}+1}$$

$$f(x_2) = \frac{2e^{x_2}}{3e^{x_2}+1}$$

let $f(x_1) > f(x_2)$

then
$$\frac{\cancel{2}e^{x_1}}{3e^{x_1}+1} > \frac{\cancel{2}e^{x_2}}{3e^{x_2}+1}$$

$$\cancel{3e^{(x_1+x_2)}} + e^{x_1} > \cancel{3e^{(x_1+x_2)}} + e^{x_2}$$

$$e^{x_1} > e^{x_2}$$

we know that: e^x is an exponetial and increasing function.

therefore if $e^{x_1} > e^{x_2} \Rightarrow x_1 > x_2$

which is true with our assumption.

$\boxed{\text{Therefore, } f(x) \text{ is an increasing function and thats why one to one function.}}$

Now for Range: $f(x) = \frac{2e^x}{3e^x+1}$ — $2e^x$ always positive, $3e^x+1$ always positive $\rightarrow$ $f(x)$ is always positive

which means, codain $\neq$ Range.

$\boxed{f(x) \text{ is not Onto function.}}$

As $f(x)$ is one to one function, inverse of $f(x)$ is possible:

$$f(x) = \frac{2e^x}{3e^x+1}$$

Replace x by $f^{-1}(x)$ and $f(x)$ by x:

$$x = \frac{2e^{f^{-1}(x)}}{3e^{f^{-1}(x)}+1}$$

$$3x\,e^{f^{-1}(x)} + x = 2e^{f^{-1}(x)}$$

$$3x\,e^{f^{-1}(x)} - 2e^{f^{-1}(x)} = -x$$

$$e^{f^{-1}(x)}\left\{3x-2\right\} = -x$$

$$e^{f^{-1}(x)} = \frac{x}{2-3x}$$

$$\boxed{f^{-1}(x) = \ln\left\{\frac{x}{2-3x}\right\}}$$ — Answer.

2 Multiple Select Questions (MSQ)

Use the following information for the questions 7 and 8.

The amount of gold (in kilograms) sold by a jeweler on the mth day of 2019 is given by the function $f(m) = \log_{10}(m+1) - \frac{1}{2}\log_{m+1}(0.01)$ (where $m = 1$ corresponds to the 1st January, 2019, and $m = 365$ corresponds to the 31st December, 2019). Find the correct set of options.

7. If $m > n > 9$, then choose the correct option(s).

    1. $f(m) > f(n)$
    2. $f(m) < f(n)$
    3. $f(m) = f(n)$
    4. $f(m) \leq f(n)$

    Answer: Option 1

8. Choose the correct option(s).

    1. The jeweler sold at least 540 kg gold in 2019.
    2. The jeweler sold at least 730 kg gold in 2019.
    3. The jeweler sold at least 2 kg gold daily throughout the year 2019.
    4. The jeweler sold at least 10 kg gold daily throughout the year 2019.

    Answer: Options 2 and 3

Solution Given $f(m) = \log_{10}(m+1) - \frac{1}{2}\log_{m+1}(0.01)$

$$= \log_{10}(m+1) - \frac{1}{2}\log_{m+1}10^{-2}$$

$$= \log_{10}(m+1) - (-2)\times\frac{1}{2}\log_{m+1}10$$

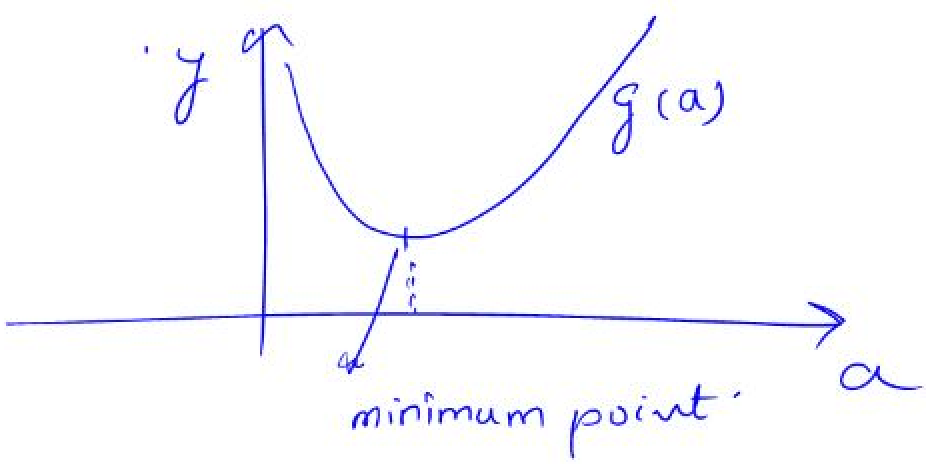
$$= \log_{10}(m+1) + \frac{1}{\log_{10}(m+1)}$$

Let $\log_{10}(m+1) = a$ then $f(m) = a + \frac{1}{a} = g(a)$ (let)

If $a \to \infty$ the $g(a) \to \infty$

if $a \to 0$ then $g(a) \to \infty$ as $\frac{1}{a} \to \infty$.

therefore the curve will look like

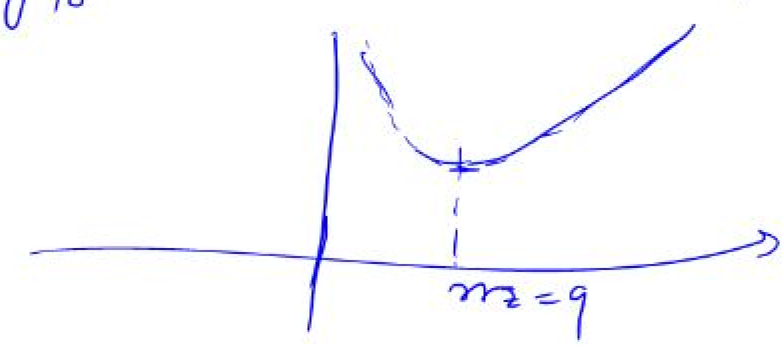


The minimum point will occure at $a = 1$.

We can use Desmos to see the behaviour.

$a = 1 \Rightarrow \log_{10} m+1 = 1 \Rightarrow m = 9$

Therefore:



Therefore After $m=9$ $f(m)$ is an increasing function that's why $f(m) > f(n)$ for $m > n > 9$. ⟶ Answer Question 7.

The minimum value of $f(m)$ would be at $m=9 \Rightarrow$ $$f(m)\Big|_{min} = \log_{10}(9+1) + \frac{1}{\log_{10}(9+1)}$$

$$= 2$$

$\Rightarrow$ Therefore jeweller sells atleast 2 kg gold per day. ⟶ Answer.

$\Rightarrow$ And a year contains 365 days, therefore atleast $365 \times 2 = 730$ kg gold will be sold in a year. ⟶ Answer.

9. The stock market chart of a tourism company (A) is shown roughly in the Figure M1W9G-2. This company was listed in February ($x = 2$) and experiences a logarithmic fall after the COVID−19 outbreak which is given by $y = -a\ log(x-h)+a$. x represents the number of months since the beginning of the year and y represents the stock price in ₹(1000). During the 10^{th} month the pharmacy company announced that the vaccine is made for the COVID−19. Thereafter, the stock price of the company (A) is raised exponentially $y = 10^{\frac{x}{5}} - b$. Choose the correct set of options. (Note: a is any positive real number, b is a positive integer and h is a constant.)

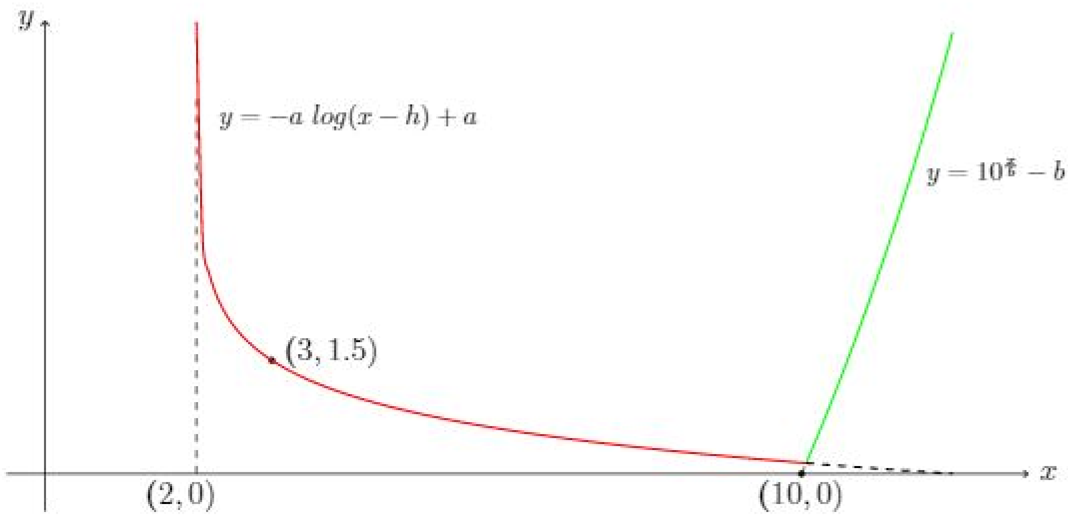


Figure M1W9G-2

1. For logarithmic fall the value of $a = 1.5$ and $h = 2$.
2. For exponential rise passing through $(10, 0)$ the value of $b = 10$.
3. The stock price in 12^{th} month is ₹4000.
4. If the vaccine was not made and the stock price just followed the same logarithmic function through out, then the investor would have lost his/her entire investment on the 12^{th} month.

Answer: Options 1, 2, and 4

The assymptote will occure when $x - h = 0$

$\Rightarrow x = h = 2.$

At $x = 3 \Rightarrow y = -a \log(3-2) + a = 1.5$

$= -a \log 1 + a = 1.5$

$= 0 + a = 1.5 \Rightarrow \boxed{a = 1.5}$

Given $y = 10^{x/b} - b = 0$ at $x = 10$

$\Rightarrow 10^{10/b} - b = 0 \Rightarrow 10^{10/b} = b$

Taking log at base 10 $\Rightarrow$ $\frac{10}{b} \log_{10} 10 = \log_{10} b$

$\Rightarrow \frac{10}{b} = \log_{10} b \Rightarrow 10 = b \log_{10} b$

Taking Antilog:

$\Rightarrow 10^{10} = b^b \Rightarrow \boxed{b = 10}$

for 12^{th} month:

$y = -a \log_{10}(12-2) + a = -1.5 \log_{10} 10 + 1.5$

$y = 0$

option (4) is correct.

10. If $m^{\log_3 2} + 2^{\log_3 m} = 16$. Then, what is the value of m ?
(NAT)
Answer: 27

$$m^{\log_3 2} + 2^{\log_3 m} = 16$$

$$2^{\log_3 m} + 2^{\log_3 m} = 16$$

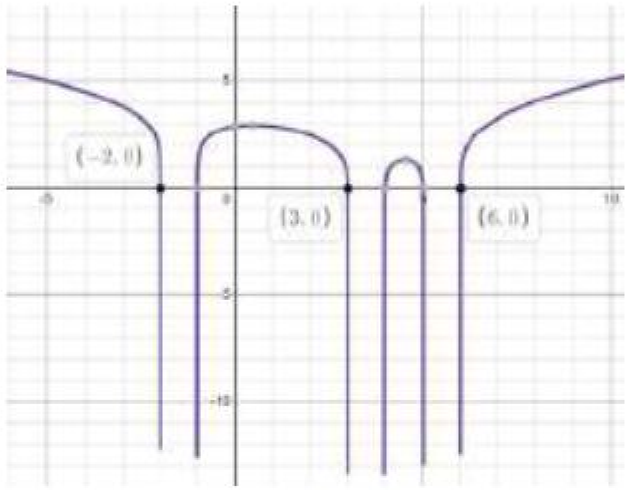
$$2\left\{2^{\log_3 m}\right\} = 16$$

$$2^{\log_3 m} = 8 = 2^3$$

$$\log_3 m = 3$$

$$m = 3^3 = 27 \text{ Ans.}$$

11. Choose the correct options with respect to the graph of a function $f(x)$ shown below. (MSQ) (Ans. Option (a), (d), (e))



- The given function is not defined in the restricted domain $(-2,-1) \cup (3,4) \cup (5,6)$. → It is clear from the graph
- ~~The given function is invertible in the restricted domain $(-\infty,-2) \cup (1,3) \cup (4,5) \cup (6,\infty]$~~ → It fails the horizontal line test, therefore in this domain the f^n is not injective. Thus, it is not invertible
- $f(x)$ is discontinuous graph ← ~~The given graph is a graph of a polynomial.~~
- The range of the given function could be $(-\infty,\infty)$. → It may be because there are multiple vertical asymptotes in which $y \to -\infty$. Also note that $x \to \infty$ it appears $f(x) \to \infty$
- The graph of $f(x)$ could be a graph of $\log_{10}(1+(x+2)(x+1)(x-3)(x-4)(x-5)(x-6))$
- ~~The function is invertible in restricted domain $[5,\infty)$~~

$f(x)$ is not defined when $x \in (5,6)$.

→ Analyse by substituting the values.

MATHEMATICS FOR DATA SCIENCE - 1 - WEEK-7 - PRACTICE ASSIGNMENT.

Note Title 09-09-2021

(1) From the graph, clearly the values of the function tends to 0 as x tends to 0. (option 2)

The graph is smooth (no sharp edges or sharp turns) and so the graph has a unique tangent at all points (at $x = \pi$ and $x = -\pi$ too). (option 5)

The function values are decreasing (slope of the curve is negative) in $[-0.5\pi, 0]$ (option 7).

(2) Statement 1 : If $\{a_n\}$ converges to a and $\{b_n\}$ converges to b, then $\{a_n b_n\}$ converges to ab. If $a, b \neq 0$, then so is ab.

Statement 2 : Let $a_n = (-1)^n$ and $b_n = (-2)^n$. Then $\{a_n\}$ and $\{b_n\}$ do not converge. But $\{a_n b_n\} = \{2\}$. This sequence converges to 2.

Statement 3 : $\lim_{n \to \infty} a_n = \lim_{n \to \infty} c = c$.

All 3 statements are true.

(3) Note that $\left\{ \frac{2n+1}{((2n+1)!)^{\frac{1}{2n+1}}} \right\}$ and $\left\{ \frac{3n}{((3n)!)^{\frac{1}{3n}}} \right\}$ are subsequences of the sequence $\left\{ \frac{n}{(n!)^{1/n}} \right\}$ and hence have the same limit as $\{a_n\}$.

$$\lim_{n\to\infty} \ln\left(\frac{2n+1}{((2n+1)!)^{\frac{1}{2n+1}}}\right) = \ln\left(\lim_{n\to\infty}\left(\frac{2n+1}{((2n+1)!)^{\frac{1}{2n+1}}}\right)\right) = \ln(e) = 1$$

$|||^{ly}$ $\lim_{n\to\infty} \ln\left(\frac{3n}{((3n)!)^{\frac{1}{3n}}}\right) = 1.$

$$\therefore \lim_{n\to\infty} \ln\left(\frac{2n+1}{((2n+1)!)^{\frac{1}{2n+1}}}\right) + \lim_{n\to\infty} \ln\left(\frac{3n}{((3n)!)^{\frac{1}{3n}}}\right) = 1+1 = 2 \;//$$

(4) $a_n = \frac{1}{n^2}$ is a subsequence of $\{\frac{1}{n}\}$.

$0 \notin \{\frac{1}{n}\}$. So $b_n = 0$ cannot be a subsequence of $\{\frac{1}{n}\}$.

$c_1 = 0 \notin \{\frac{1}{n}\}$. So $\{c_n\}$ cannot be a subsequence of $\{\frac{1}{n}\}$.

Only one sequence is a subsequence of $\{\frac{1}{n}\}$.

(5) $a_n = n^{1/n}$ converges to 1.

$b_n = n^{\frac{2}{n}} + 5n^{\frac{1}{n}} - 1.$

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty}\left(n^{\frac{2}{n}} + 5n^{\frac{1}{n}} - 1\right) = \lim_{n\to\infty} n^{\frac{2}{n}} + \lim_{n\to\infty} 5n^{\frac{1}{n}} + \lim_{n\to\infty}(-1)$$

$$= \left(\lim_{n\to\infty} n^{1/n}\right)^2 + 5\lim_{n\to\infty} n^{1/n} - 1$$

$$= (1)^2 + 5(1) - 1 = 5$$

(6) $a_n = \frac{2n+1}{3n+2}$. $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n(2+1/n)}{n(3+2/n)} = \frac{2}{3}$.

$b_n = 27a_n^2 + 9a_n + \frac{2n^2}{n^3+1}$.

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty}\left(27a_n^2 + 9a_n + \frac{2n^2}{n^3+1}\right) = \lim_{n\to\infty}(27a_n^2) + \lim_{n\to\infty}(9a_n) + \lim_{n\to\infty}\left(\frac{2n^2}{n^3+1}\right)$$

$$= 27\left(\lim_{n\to\infty} a_n\right)^2 + 9\lim_{n\to\infty} a_n + \lim_{n\to\infty} \frac{2n^2}{n^2(n+1)}$$

$$= \left(27 \times \frac{4}{9}\right) + \left(9 \times \frac{2}{3}\right) + \lim_{n\to\infty}\left(\frac{2}{n+1}\right)$$

$$= 12 + 6 + 0 = \underline{\underline{18}}.$$

(7) If there is no sharp turn at a, then the graph has a unique tgt at a. Curves 1, 3 and 4 have sharp turns at 0 (Curve 4 has a jump at 0). Hence these three curves do not have a unique tgt at 0.

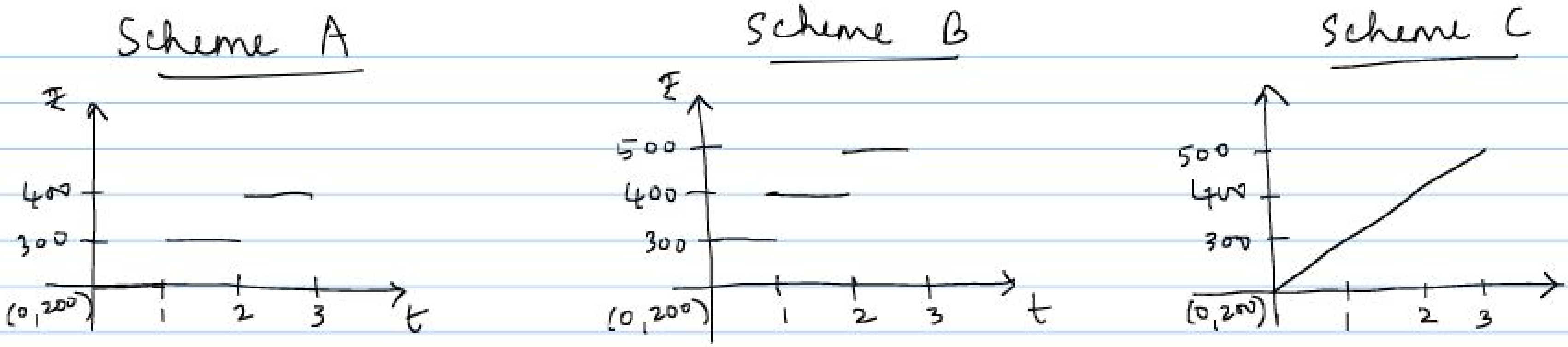
(8) $p_1(2.6) = (100 \times 2) + 200 = 400$

$p_2(2.6) = (100 \times 3) + 200 = 500$

$p_3(2.6) = (100 \times 2.6) + 200 = 460$.

The lowest cost is Rs 400 by availing scheme A.

(9)



Clearly tangents of $p_1(t)$ and $p_2(t)$ do not exist at $n = 1, 2, 3, \ldots$ (jumps)

Tangent exists for all values of t for $p_3(t)$.

Options 1, 2, 4 are right.

(10) Figure 2 (see (9)). $p_2(0) = 200$, $p_2(0.5) = 300$, ...

Week 7 - graded assignment Solution

(The numbers in your assignment may differ)

1) Let $\{a_n\}$ and $\{b_n\}$ be two sequences of real numbers.

Let $a_n = 1$ for all n and $b_n = -1$ for all n.

$\{a_n\}$ converges to 1, $\{b_n\}$ converges to -1

$a_n + b_n = 0$ for all n.

$\{a_n + b_n\}$ converges to 0.

Hence option 1 is not correct.

Let $\{a_n\}$ be an increasing sequence.

$a_n = n$ for all n.

$(-1)^n a_n = (-1)^n n$.

Hence $\{(-1)^n a_n\}$ is not a decreasing sequence in this case.

Hence option 2 is not correct.

If $\{a_n\} \to a$, $\{b_n\} \to b$

then, $\{a_n b_n\} \to ab$.

If both a and b are non-zero, then ab must be non-zero.

Hence option 3 is correct.

If $\{a_n\} \to a$ and $\{b_n\} \to a$

then $\{a_n - b_n\} \longrightarrow a - a = 0$.

Hence option 4 is correct.

Let $a_n = \begin{cases} n & \text{when } n = \text{odd.} \\ 1 & \text{When } n = \text{even.} \end{cases}$

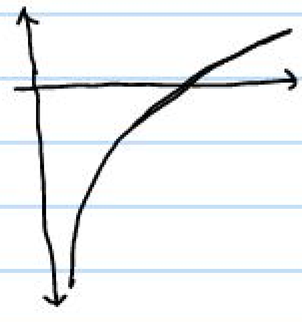
Hence, a_n is a divergent sequence.

But $\{a_{2n}\}$ is a constant sequence $\{1\}$,

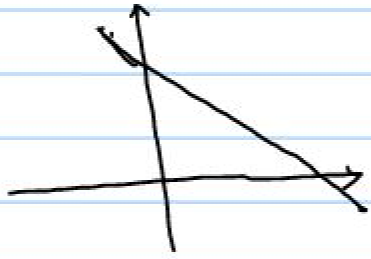
which is obviously a convergent subsequence of $\{a_n\}$.

Hence option 5 is not correct.

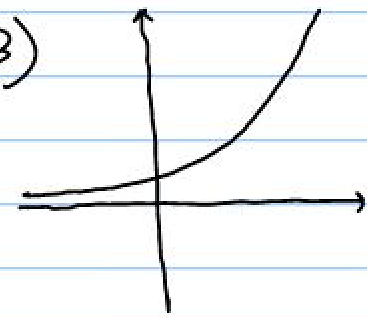
2) i) $f(x) = 3\ln x - 2$ d) logarithmic function 3)



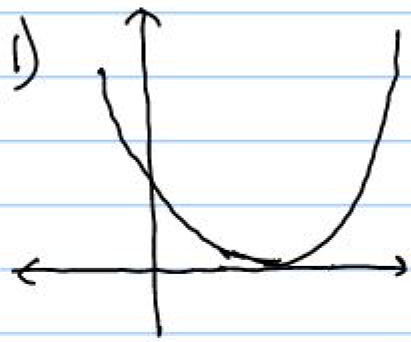
ii) $f(x) = 10 - 4x$ c) linear function 4)



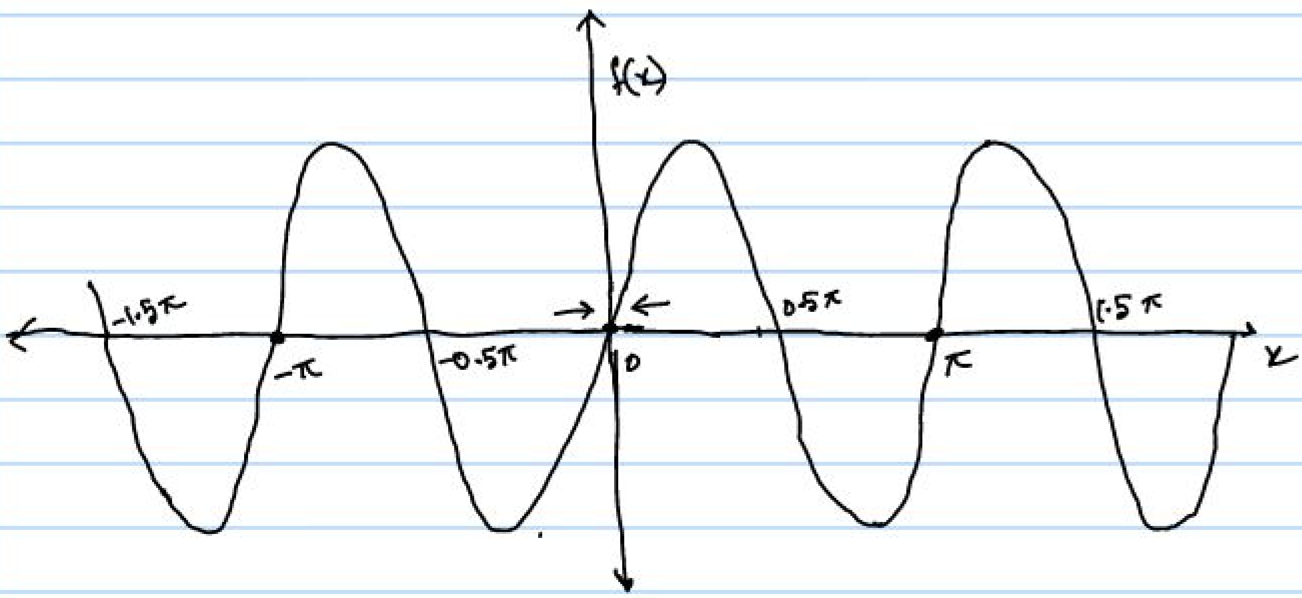
iii) $f(x) = 2^x + 7$ a) exponential function 3)



iv) $f(x) = x^2 - 4x + 4$ b) quadratic function 1)



3)



$\lim_{x \to 0^+} f(x) = 0 = \lim_{x \to 0^-} f(x)$. Hence option 2 is correct.

At $x = \pi$ and $x = -\pi$ there are no sharp corner at the given curve. So, option 5 is correct.

In the interval $[-0.5\pi, 0.5\pi]$ the function is oscillatory (neither monotonically increasing nor monotonically decreasing).

4) $$\lim_{x\to 0} \frac{\log(1+x)}{\sin x} = \lim_{x\to 0} \frac{\frac{\log(1+x)}{x}}{\frac{\sin x}{x}} = \frac{\lim_{x\to 0} \frac{\log(1+x)}{x}}{\lim_{x\to 0} \frac{\sin x}{x}}$$

$$= \frac{1}{1} = 1.$$

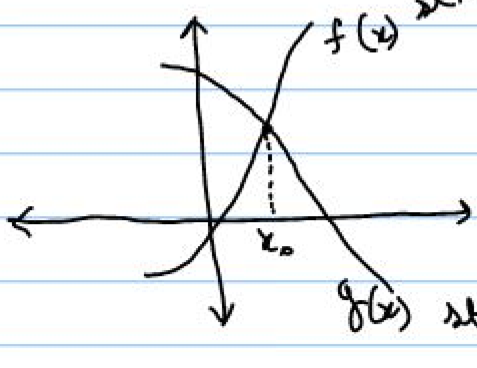
$$\lim_{x\to 0} \frac{\sin 5x}{x} = \lim_{x\to 0} \frac{\sin 5x}{5x} \times 5 = 5 \lim_{x\to 0} \frac{\sin 5x}{5x}$$

$$= 5 \times 1 = 5$$

$$\lim_{x\to 0} \frac{e^{x/2} - 1}{\sin 2x} = \lim_{x\to 0} \frac{\frac{e^{x/2}-1}{x/2} \times \frac{x}{2}}{\frac{\sin 2x}{2x} \times 2x}$$

$$= \lim_{x\to 0} \frac{\frac{e^{x/2}-1}{x/2}}{\frac{\sin 2x}{2x}} \times \frac{1}{4}$$

$$= 1 \times 1/4 = 1/4$$

5) 

$f(x) \geq g(x)$ for all $x \geq x_0$.

$f(x) \leq g(x)$ for $x \leq x_0$.

We have, $f(x_0) = g(x_0)$. But for any $x > x_0$, $f(x)$ and $g(x)$ will never intersect. So, Option 2 is incorrect.

6) $$a_n = \frac{12n^2}{3n+5} - \frac{4n^2+7}{n+3}$$

$$= \frac{12n^2(n+3) - (4n^2+7)(3n+5)}{(3n+5)(n+3)}$$

$$= \frac{\cancel{12n^3} + 36n^2 - \cancel{12n^3} - 21n - 20n^2 - 35}{3n^2+5n+9n+15}$$

$$= \frac{16n^2 - 21n - 35}{3n^2+14n+15}$$

$$= \frac{16 - \frac{21}{n} - \frac{35}{n^2}}{3 + \frac{14}{n} + \frac{15}{n^2}}$$

as, $n \to \infty$, $a_n \to \frac{16}{3}$.

7) $$R(\omega) = \frac{50e^{\omega}}{10+e^{\omega}} = \frac{50}{\frac{10}{e^{\omega}}+1}$$

If $\omega_1 \geqslant \omega_2$ then, $e^{\omega_1} \geqslant e^{\omega_2}$

$$\Rightarrow \frac{10}{e^{\omega_1}} \leq \frac{10}{e^{\omega_2}}$$

$$\Rightarrow \frac{10}{e^{\omega_1}}+1 \leq \frac{10}{e^{\omega_2}}+1$$

$$\Rightarrow \frac{50}{\frac{10}{e^{\omega_1}}+1} \geqslant \frac{50}{\frac{10}{e^{\omega_2}}+1}$$

$$\Rightarrow R(\omega_1) \geqslant R(\omega_2)$$

$R(\omega)$ is increasing function.

$$\lim_{\omega\to\infty} R(\omega) = 50$$

Hence, the minimum possible value of n such that $R(\omega) < n$, for all ω, where $n \in \mathbb{Z}$, is 50.

8) $$\lim_{n\to\infty} e\,\sqrt[n]{n!}\left[\log(1+6/n) - \frac{e^{1/n}-1}{\left(\sqrt{2\pi n}\right)^{1/n}}\right]$$

$$= \lim_{n\to\infty} e\,\sqrt[n]{n!}\left[\frac{\log(1+6/n)}{6/n}\times\frac{6}{n} - \frac{\frac{e^{1/n}-1}{1/n}\times 1/n}{\frac{n\left(\sqrt{2\pi n}\right)^{1/n}}{(n!)^{1/n}}\cdot\frac{(n!)^{1/n}}{n}}\right]$$

$$= \lim_{n\to\infty} e\left[\frac{\log(1+6/n)}{6/n}\times\frac{6}{\frac{n}{\sqrt[n]{n!}}} - \frac{\frac{e^{1/n}-1}{1/n}}{n\left(\frac{\sqrt{2\pi n}}{n!}\right)^{1/n}}\times\frac{1/n\cdot\sqrt[n]{n!}}{\frac{\sqrt[n]{n!}}{n}}\right]$$

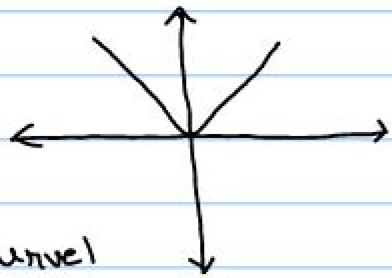
let $1/n = x$. As $n\to\infty$, $x\to 0$

$$= e\left[\lim_{x\to 0}\frac{\log(1+6x)}{6x}\times\frac{6}{\lim_{n\to\infty}\frac{n}{\sqrt[n]{n!}}}\right.$$

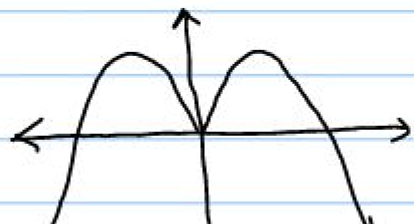
$$\left. - \frac{\lim_{x\to 0}\frac{e^x-1}{x}}{\lim_{n\to\infty} n\left(\frac{\sqrt{2\pi n}}{n!}\right)^{1/n}}\times 1\right]$$

$$= e\left[1\times\frac{6}{e} - \frac{1}{e}\times 1\right] = 6-1 = 5$$

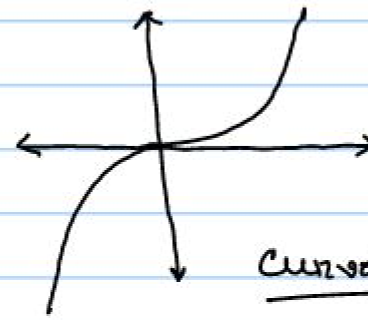
9) Both curve 1 and curve 2 have sharp corners at the origin (0,0). Hence, at the origin these two curves do not have tangents at the origin.



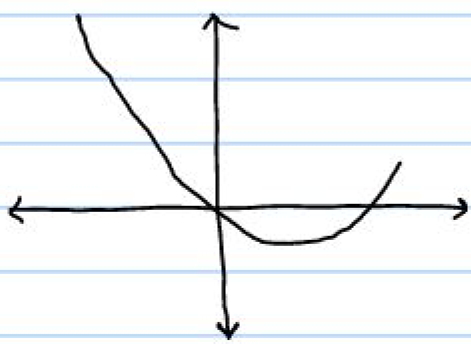
curve1



curve 2



curve3



curve 4

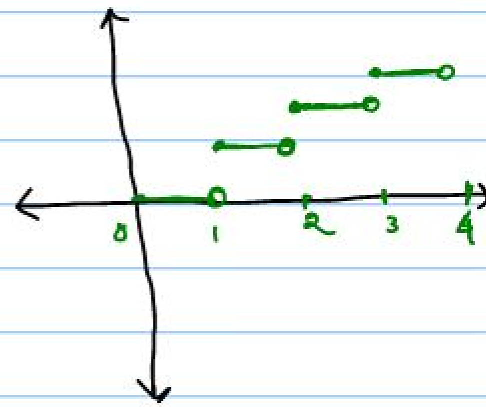
10) $$a_n = \frac{9+15+21+\cdots+3(2n-1)}{n^2}$$

$$= \frac{3\left(3+5+7+\cdots+(2n-1)\right)}{n^2}$$

$$= \frac{3\,(n^2-1)}{n^2} = 3\left(1-\frac{1}{n^2}\right)$$

$$\lim a_n = \lim_{n\to\infty} 3\left(1-\frac{1}{n^2}\right) = 3$$

11) $5 \lim_{x \to 3^+} \lfloor x \rfloor - 3 \lim_{x \to 1^-} \lfloor x \rfloor$



$= 5 \times 3 - 3 \times 0 = 15$

Comprehension Type Question:

12) Error in estimation by Algorithm 1:

$$a_n = \frac{n^2+5n}{6n^2+1}$$

$$\lim a_n = \lim_{n \to \infty} \frac{n^2+5n}{6n^2+1}$$

$$= \lim_{n \to \infty} \frac{1 + 5/n}{6 + 1/n} = \frac{1}{6} \approx 0.166$$

Error in estimation by Algorithm 2:

$$b_n = \frac{1}{8} + (-1)^n \frac{1}{n}$$

$$\lim b_n = \frac{1}{8} + \lim_{n \to \infty} (-1)^n \cdot \frac{1}{n}$$

$$= \frac{1}{8} + 0 = \frac{1}{8} = 0.125 \left(\text{As, } -\frac{1}{n} \le \frac{(-1)^n}{n} \le \frac{1}{n} \right)$$

Error in estimation by Algorithm 3:

$$c_n = \frac{e^n + 4}{7e^n}$$

$$\lim c_n = \lim_{n\to\infty} \frac{e^n+4}{7e^n} = \lim_{n\to\infty} \frac{1+\frac{4}{e^n}}{7}$$

$$= \frac{1}{7} \approx 0.143$$

Maximum error in estimation will be given by Algorithm 1.

Minimum error in estimation will be given by Algorithm 2.

13) Error in estimation by the new algorithm:

$$\lim (a_n - b_n) = \lim a_n - \lim b_n$$

$$= \frac{1}{6} - \frac{1}{8}$$

$$= \frac{4-3}{24} = \frac{1}{24}$$

The error in estimation using the new algorithm is less than the error in estimation using any of the Algorithm 1, Algorithm 2 and Algorithm 3.

14) $$c_n' = ne^{1/8n} - n$$

$$= \frac{e^{1/8n} - 1}{1/8n} \times \frac{1}{8}$$

$$\lim c_n' = \lim_{n \to \infty} \frac{e^{1/8n} - 1}{1/8n} \times \frac{1}{8} = 1/8$$

$$= 0.125$$

Week-8
Mathematics for Data Science - 1
Limits, Continuity, Differentiability, and the derivative
Practice Assignment Solutions

1 Multiple Select Questions (MSQ)

1. Table M2W2P1 gives functions in Column A with the equation of their tangents at the origin $(0, 0)$ in column B and the plotted graphs and tangents in Column C.

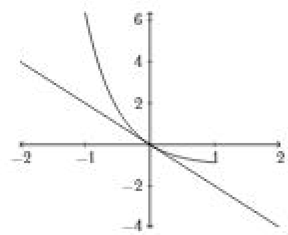
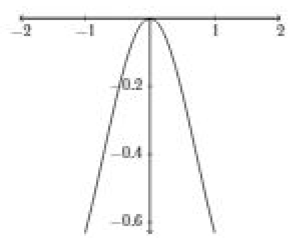
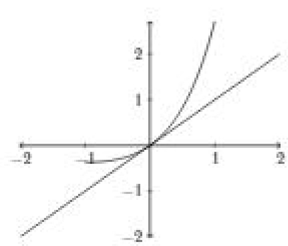
	Function (Column A)		It's tangent at (0,0) (Column B)		Graph (Column C)
i)	$f(x) = xe^x$	a)	$y = -2x$	1)	
ii)	$f(x) = e^{-2x} - 1$	b)	$y = x$	2)	
iii)	$f(x) = e^{-x^2} - 1$	c)	$y = 0$	3)	

Table: M2W2P1

Based on the given Table M2W2P1, Choose the options which represent the correct matching of a given function with its tangent at (0,0) and its graph.

- ○ **Option 1:** i) → b) → 3)
- ○ Option 2: ii) → c) → 2)
- ○ **Option 3:** iii) → c) → 2)
- ○ Option 4: iii) → b) → 3)
- ○ **Option 5:** ii) → a) → 1)
- ○ Option 6: i) → a) → 1)

Solution:

1. Given

$$f(x) = xe^x$$
$$f(1) = 1e^1 = e > 0$$

Only figure 3 has this property. Now differentiating the function,

$$f'(x) = 1e^x + xe^x$$
$$f'(0) = 1 + 0 = 1$$

Let the equation of tangent is $y = mx + c$. As the tangent passes through (0,0) therefore, c=0. And the slope of tangent is $m = f'(0) = 1$, then the equation of tangent

$$y = x$$

Which is b) in column B. Therefore, $i) \to b) \to 3$.

2. Given

$$f(x) = e^{-2x} - 1$$
$$f(-1) = e^2 - 1 > 0$$

Only figure 1 has this property. Now differentiating the function,

$$f'(x) = e^{-2x}(-2) = -2e^{-2x}$$
$$f'(0) = -2$$

Let the equation of tangent is $y = mx + c$. As the tangent passes through (0,0) therefore, c=0. And the slope of tangent is $m = f'(0) = -2$, then the equation of tangent

$$y = -2x$$

Which is a) in column B. Therefore, $ii) \to a) \to 1$.

3. Given

$$f(x) = e^{-x^2} - 1$$

$$f(-x) = e^{-x^2} - 1 = f(x)$$

The function is even and only figure 2 has this property. Now differentiating the function,

$$f'(x) = e^{-x^2}(-2x) = -2xe^{-x^2}$$

$$f'(0) = 0$$

Let the equation of tangent is $y = mx + c$. As the tangent passes through (0,0) therefore, c=0. And the slope of tangent is $m = f'(0) = 0$, then the equation of tangent

$$y = 0$$

Which is c) in column B. Therefore, $iii) \to c) \to 2$.

2. Consider the graphs given below:

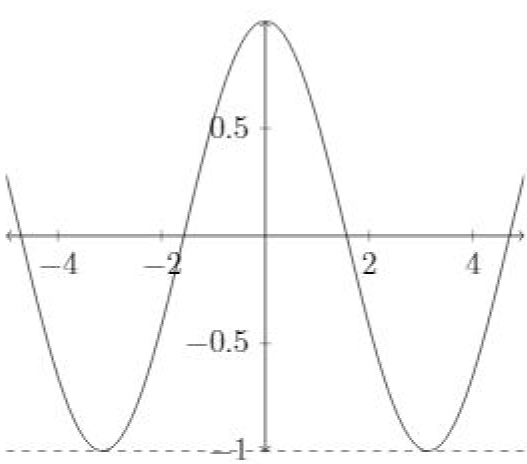


Figure: Curve 1

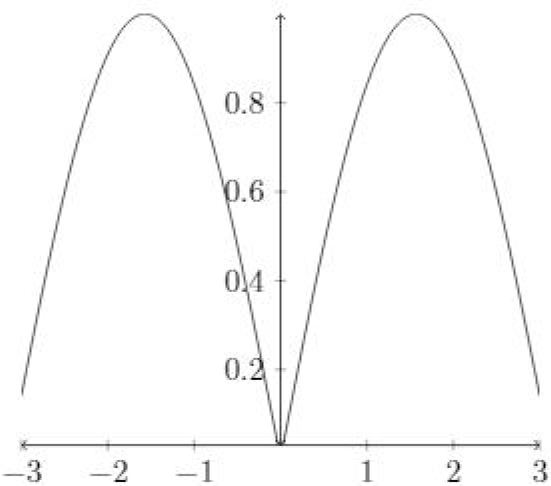


Figure: Curve 2

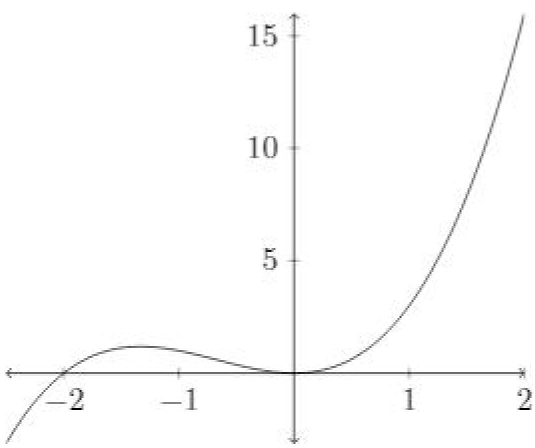


Figure: Curve 3

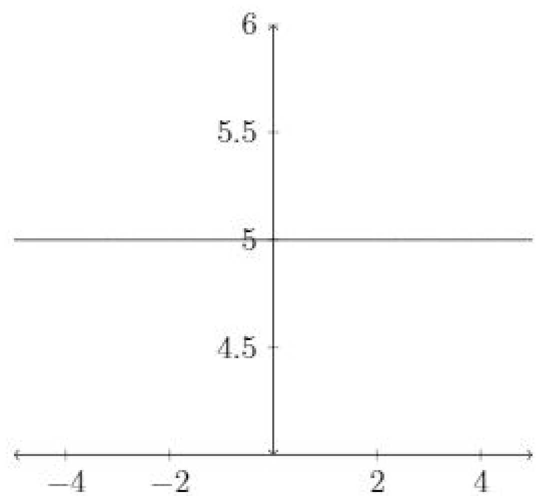


Figure: Curve 4

Choose the set of correct options:

- ◯ **Option 1:** There are at least two points on Curve 1, where the derivatives of the function corresponding to Curve 1, are equal.
- ◯ **Option 2:** At the origin the derivative of the function corresponding to Curve 2 does not exist.
- ◯ Option 3: The derivative of the function corresponding to Curve 3, at the origin and at the point $(-2, 0)$ are the same.

◯ Option 4: The derivative of the function corresponding to Curve 4 does not exist at any point.

Solution:
Option 1: There are at least two points on Curve 1, where the derivatives of the function corresponding to Curve 1, are equal.

As it is shown in the figure, the straight line $y = -1$ is tangent at two point of the curve. So at those two points on Curve 1, the derivatives of the function corresponding to Curve 1, as slope of the tangents at those two points are the same.

Option 2: At the origin the derivative of the function corresponding to Curve 2 does not exist.

Curve 2 has a sharp corner at $x = 0$, which shows the derivative of the function corresponding to Curve 2 does not exist.That's why option 2 is correct.

Option 3: The derivative of the function corresponding to Curve 3, at the origin and at the point $(-2, 0)$ are the same.

At origin the derivative of the function corresponding to Curve 3 is zero as the X-axis is the tangent of the curve at the origin. But at $x = -2$ the tangent is not parallel to the X-axis, hence the slope of the tangent at $x = -2$ must be different from 0. So the derivative of the function corresponding to Curve 3, at the origin and at the point $(-2, 0)$ are not the same.

Option 4: The derivative of the function corresponding to Curve 4 does not exist at any point.

The function corresponding to Curve 4 is a constant function, therefore, the derivative of the function corresponding to Curve 4 always exists and is 0.

3. Let f be a function and the Figure M2W2P1 represent the graph of function f. The solid points denote the value of the function at the points, and the values denoted by the hollow points are not taken by the functions.

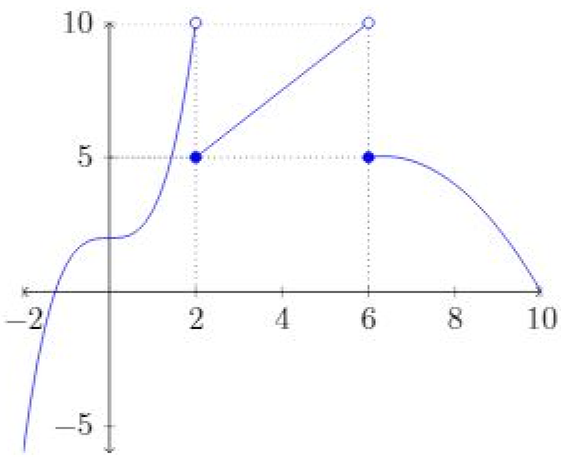


Figure: M2W2P1

Choose the set of correct options.

- ◯ **Option 1:** $\lim_{t\to 2-} f(t) = 10$
- ◯ **Option 2:** $\lim_{t\to 2+} f(t) = 5$
- ◯ **Option 3:** $\lim_{t\to 6-} f(t) = 10$
- ◯ Option 4: $\lim_{t\to 6+} f(t) = 10$
- ◯ Option 5: f is continuous at $x = 2$.
- ◯ **Option 6:** f is continuous at $x = 4$

Solution:

We can see that the curve is discontinuous at $t = 2$ and $t = 6$ only.

As t is approaching to 2 from left side, f is approaching to the value 10, which means the LHL (left hand limit) i.e., $t \to 2^-$ is 10.
As t is approaching to 2 from right side, f is approaching to the value 5, which means the RHL (right hand limit) i.e., $t \to 2^+$ is 5.
Hence limit at the function f does not exist at $t = 2$. So f is discontinuous at $t = 2$. Similar explanation can be given for $t = 6$.

4. Define a function f as follows:

$$f(x) = \begin{cases} x^3 & \text{if } x > 1, \\ x^2 & \text{if } 0 < x \leq 1 \\ x & \text{if } x < 0, \\ 0 & \text{if } x = 0 \end{cases}$$

Choose the set of correct options.

- ◯ **Option 1:** f is continuous, but not differentiable at $x = 1$.
- ◯ Option 2: f is both continuous and differentiable at $x = 1$.
- ◯ **Option 3:** f is continuous, but not differentiable at $x = 0$.
- ◯ Option 4: f is both continuous and differentiable at $x = 0$.
- ◯ Option 5: f is not continuous at $x = 0$.
- ◯ Option 6: f is not continuous at $x = 1$.

Solution:

For $x = 1$:

Left Hand Limit:

$$\lim_{x \to 1-} f(x) = \lim_{x \to 1-} x^2 = 1$$

Right Hand Limit:

$$\lim_{x \to 1+} f(x) = \lim_{x \to 1+} x^3 = 1$$

Moreover $f(1) = 1$. Hence

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = f(1) = 1$$

Therefore, the function is continuous at $x = 1$.

For differentiability at $x = 1$:

$$\lim_{h \to 0^+} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0^+} \frac{(1+h)^3 - 1}{h} = \lim_{h \to 0^+} \frac{(1 + h^3 + 3h^2 + 3h) - 1}{h}$$

$$\lim_{h \to 0^+} \frac{h^3 + 3h^2 + 3h}{h} = \lim_{h \to 0^+} (h^2 + 3h + 3) = 3$$

Similarly,

$$\lim_{h \to 0^-} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0^+} \frac{(1+h)^2 - 1}{h} = \lim_{h \to 0^+} \frac{(1 + h^2 + 2h) - 1}{h}$$

$$\lim_{h\to 0^+} \frac{h^2 + 2h}{h} = \lim_{h\to 0^+} (h+2) = 2$$

Hence,

$$\lim_{h\to 0^+} \frac{f(1+h) - f(1)}{h} \neq \lim_{h\to 0^-} \frac{f(1+h) - f(1)}{h}$$

So $\lim\limits_{h\to 0} \frac{f(1+h)-f(1)}{h}$ does not exist. Therefore the function f is not differentiable at $x = 1$. Similar argument can be given for $x = 0$.

5. Let f and g be two real valued functions defined as:

$$f : \mathbb{R} \to \mathbb{R}$$

$$f(x) = e^x - 1$$

$$g : \mathbb{R} \to \mathbb{R}$$

$$g(x) = x$$

Choose the set of correct options.

- ◯ **Option 1:** The linear function $ex - 1$ is the best linear approximation of the function $f(x)$ at the point $x = 1$.
- ◯ Option 2: In this case, $\lim_{x\to 0} \frac{f(x)}{g(x)} = \frac{\lim_{x\to 0} f(x)}{\lim_{x\to 0} g(x)}$.
- ◯ **Option 3:** In this case, $(f+g)$ (where, $(f+g)(x)$ is defined by $f(x)+g(x)$) is continuous at $x = 0$.
- ◯ **Option 4:** $\lim_{x\to 0} f(x)g(x) = 0$.

Solution:

Given,

$$f(x) = e^x - 1$$

The linear approximation for this function at $x = 1$ would be

$$y = f'(1)(x-1) + f(1) = e^1(x-1) + e^1 - 1 = ex - 1$$

In option 2 given,

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \frac{\lim_{x\to 0} f(x)}{\lim_{x\to 0} g(x)}$$

LHS:

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{e^x - 1}{x}$$

This is 0 divided by 0 case, therefore, we can use L'Hôpital's rule,

$$LHS = \lim_{x\to 0} \frac{f'(x)}{g'(x)} = \lim_{x\to 0} \frac{e^x}{1} = 1$$

Now

$$RHS = \frac{\lim_{x\to 0} f(x)}{\lim_{x\to 0} g(x)}$$

$RHS = \frac{\lim_{x\to 0} e^x - 1}{\lim_{x\to 0} x}$ is in indeterminate form, as both the numerator and denominator are 0.
Therefore,

$$LHS \neq RHS$$

In option 3,

$$(f+g)(x) = e^x - 1 + x$$

$$LHL = \lim_{x\to 0^-}(f+g)(x) = \lim_{x\to 0^-}(e^x - 1 + x) = 0$$

$$RHL = \lim_{x\to 0^+}(f+g)(x) = \lim_{x\to 0^+}(e^x - 1 + x) = 0$$

$$(f+g)(0) = 0$$

Therefore, $(f+g)(x)$ is continuous at $x = 0$.

In option 4,

$$\lim_{x\to 0} f(x)g(x) = \lim_{x\to 0}(e^x - 1)(x) = 0$$

2 Numerical Answer Type (NAT)

6. Let f be a differentiable function at $x = 1$. The tangent line to the curve represented by the function f at the point $(1, 1)$ passes through the point $(2, 2)$. What will be the value of $f'(1)$?
Solution:
The slope of the tangent line (as the line passes through $(1, 1)$ and $(2, 2)$) is

$$f'(1) = \frac{2-1}{2-1} = 1$$

7. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$ and $g : \mathbb{R} \to \mathbb{R}$ be defined by $g(x) = x - 5$. Find the value of $(fg)'(1) - (f \circ g)'(1)$, where $f \circ g(x) = f(g(x))$ and $fg(x) = f(x)g(x)$.
Solution:

Take,

$$\begin{aligned}
(fg)(x) &= f(x)g(x) = x^2(x-5) = x^3 - 5x^2 \\
(fg)'(x) &= 3x^2 - 10x \\
(fg)'(1) &= 3(1)^2 - 10(1) = -7
\end{aligned}$$

Now,

$$\begin{aligned}
(f \circ g)(x) &= (x-5)^2 \\
(f \circ g)'(x) &= 2(x-5)(1) \\
(f \circ g)'(1) &= 2(1-5) = -8
\end{aligned}$$

Therefore,

$$(fg)'(1) - (f \circ g)'(1) = -7 - (-8) = 1$$

3 Comprehension Type Question:

The profit of Company A with respect to time (in months) is given by the function $f(t)$ (in lakhs) as follows:

$$f(t) = \begin{cases} \frac{(t-2)^n-1}{t-3} & \text{if } 0 \le t < 3, \\ \lfloor t \rfloor & \text{if } t \ge 3 \end{cases}$$

for some integer n.

The profit of Company B with respect to time (in months) is given by the function $g(t)$ (in lakhs) as follows:

$$g(t) = \begin{cases} \frac{t^3-3^3}{t-3} & \text{if } 0 \le t < 3, \\ 3t^m & \text{if } t \ge 3 \end{cases}$$

for some integer m.

Use the information given above answer Questions 8,9 and 10.

8. If the functions $f(t)$ and $g(t)$ denoting the profits of Company A and Company B, respectively, are known to be continuous at $t = 3$, then which of the following options is(are) true? (MSQ)

   ◯ **Option 1:** $n = 3$

   ◯ Option 2: $n = 2$

   ◯ **Option 3:** $m = 2$

   ◯ Option 4: $m = 3$

   Solution:

   Given,

$$f(t) = \begin{cases} \frac{(t-2)^n-1}{t-3} & \text{if } 0 \le t < 3, \\ \lfloor t \rfloor & \text{if } t \ge 3 \end{cases}$$

   For $f(t)$ to be continuous at $t = 3$,

$$\lim_{t \to 3^-} f(t) = \lim_{t \to 3^+} f(t) = f(3)$$

$$\lim_{t \to 3^-} f(t) = \lim_{t \to 3^-} \frac{(t-2)^n - 1}{t-3}$$

   Using L'Hospital rule we get,

$$\lim_{t \to 3^-} \frac{n(t-2)^{(n-1)}}{1} = n$$

   Further we have,

$$\lim_{t \to 3^+} f(t) = \lim_{t \to 3^+} \lfloor t \rfloor = 3$$

Hence we have $n = 3 = f(3)$.

Now,

$$g(t) = \begin{cases} \frac{t^3-3^3}{t-3} & \text{if } 0 \leq t < 3, \\ 3t^m & \text{if } t \geq 3 \end{cases}$$

For $g(t)$ to be continuous at $t = 3$,

$$\lim_{t \to 3^-} g(t) = \lim_{t \to 3^+} g(t) = g(3)$$

$$\lim_{t \to 3^-} g(t) = \lim_{t \to 3^-} \frac{t^3 - 3^3}{t - 3}$$

Using L'Hospital rule we get,

$$\lim_{t \to 3^-} \frac{3t^2}{1} = 27$$

Further we have,

$$\lim_{t \to 3^+} g(t) = \lim_{t \to 3^+} 3t^m = 3^{(m+1)}$$

Again we have $f(3) = 3^{(m+1)}$.
Therefore, $3^{(m+1)} = 27 \implies m + 1 = 3 \implies m = 2$.
So, options (1) and (3) are correct.

9. Assuming g to be continuous at $t = 3$, choose the set of correct options. (MSQ)

- ◯ Option 1: $\lim\limits_{t\to 3-} \frac{g(t)-g(3)}{t-3} = 18$
- ◯ **Option 2:** $\lim\limits_{t\to 3+} \frac{g(t)-g(3)}{t-3} = 18$
- ◯ Option 3: $\lim\limits_{t\to 3+} \frac{g(t)-g(3)}{t-3} = 9$
- ◯ **Option 4:** $\lim\limits_{t\to 3-} \frac{g(t)-g(3)}{t-3} = 9$
- ◯ **Option 5:** g is not differentiable at $t = 3$.
- ◯ Option 6: g is differentiable at $t = 3$.

Solution:

As g is continuous at $t = 3$, we have

$$g(t) = \begin{cases} \frac{t^3-3^3}{t-3} & \text{if } 0 \le t < 3, \\ 3t^2 & \text{if } t \ge 3 \end{cases}$$

$$\lim_{t\to 3^-} \frac{g(t)-g(3)}{t-3} = \lim_{t\to 3^-} \frac{\frac{t^3-3^3}{t-3} - (3\times 3^2)}{t-3}$$
$$\lim_{t\to 3^-} \frac{g(t)-g(3)}{t-3} = \lim_{t\to 3^-} \frac{t^3-27-27t+81}{(t-3)^2}$$

Using L'Hospital rule two times consecutively,

$$\lim_{t\to 3^-} \frac{g(t)-g(3)}{t-3} = \lim_{t\to 3^-} \frac{3t^2-27}{2(t-3)} = \lim_{t\to 3^-} \frac{6t}{2} = 9$$

So, option 4 is correct.
Similarly we can calculate,

$$\lim_{t\to 3^+} \frac{g(t)-g(3)}{t-3} = \lim_{t\to 3^+} \frac{3t^2-27}{t-3} = \lim_{t\to 3^+} \frac{6t}{1} = 18$$

So, option 2 is correct.
As,

$$\lim_{t\to 3^-} \frac{g(t)-g(3)}{t-3} \neq \lim_{t\to 3^+} \frac{g(t)-g(3)}{t-3},$$

g is not differentiable at $t = 3$.
So, option 5 is correct.

10. Let $L_f(t)$ be the best linear approximation of the function $f(t)$ at the point $t = 1$, assuming f to be continuous at $t = 3$, then find the value of $L_f(1)$(or $L_f(2)$). (NAT)

Solution:

Continuous at $t = 3$ means $n = 3$, then

$$f(t) = \frac{(t-2)^3 - 1}{t-3}$$

$$L_f(t) = f'(1)(t-1) + f(1)$$

$$f'(t) = \frac{3(t-2)^2(t-3) - ((t-2)^3 - 1)}{(t-3)^2}$$

$$f'(1) = \frac{-6 - (-2)}{4} = -1$$

And

$$f(1) = 1$$

Therefore,

$$L_f(t) = -1(t-1) + f(1)$$

$$L_f(t) = 2 - t$$

Hence,

$$L_f(1) = 2 - 1 = 1$$

$$L_f(2) = 2 - 2 = 0$$

Week- 8

Mathematics for Data Science - 1

Limits, Continuity, Differentiability, and the derivative

Graded Assignment

Note: Numbers may differ for some questions, but solution pattern will be the same.

1 Multiple Select Questions (MSQ)

1. Match the given functions in Column A with the equations of their tangents at the origin $(0, 0)$ in column B and the plotted graphs and the tangents in Column C, given in Table M2W2G1.

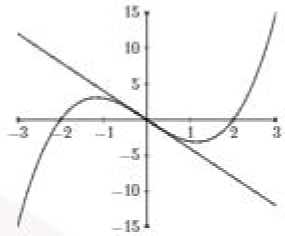
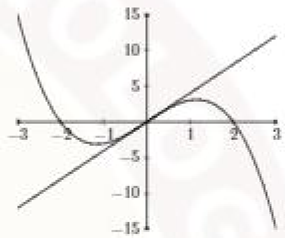
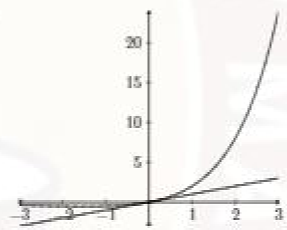
	Function (Column A)		It's tangent at (0,0) (Column B)		Graph (Column C)
i)	$f(x) = x2^x$	a)	$y = -4x$	1)	
ii)	$f(x) = x(x-2)(x+2)$	b)	$y = x$	2)	
iii)	$f(x) = -x(x-2)(x+2)$	c)	$y = 4x$	3)	

Table: M2W2G1

◯ **Option 1:** ii) $\rightarrow$ a) $\rightarrow$ 1.

◯ **Option 2:** i) $\rightarrow$ b) $\rightarrow$ 3.

◯ Option 3: iii) $\rightarrow$ b) $\rightarrow$ 1.

◯ **Option 4:** iii) $\rightarrow$ c) $\rightarrow$ 2.

◯ Option 5: i) $\rightarrow$ a) $\rightarrow$ 1.

Solution:
i) Given $f(x) = x2^x \implies f'(x) = 2^x + x2^x \ln 2$. So, $f(0) = 0$ and $f'(0) = 1$.
Hence the equation of the tangent at the origin is

$$y - 0 = 1.(x - 0) \implies y = x.$$

In Column C, figure 3 has the line $y = x$ and exponential graph.
Hence i) $\to$ b) $\to$3).
ii) Given $f(x) = x(x-2)(x+2) = x^3 - 4x \implies f'(x) = 3x^2 - 4$.
So, $f(0) = 0$ and $f'(0) = -4$
Hence the equation of the tangent at the origin is

$$y - 0 = -4(x - 0) \implies y = -4x.$$

In Column C, figure 1 has the line $y = -4x$.
Hence ii) $\to$ a) $\to$1).
iii) Given $f(x) = -x(x-2)(x+2) = -x^3 + 4x \implies f'(x) = -3x^2 + 4$.
So, $f(0) = 0$ and $f'(0) = 4$
Hence the equation of the tangent at the origin is

$$y - 0 = 4(x - 0) \implies y = 4x$$

In Column C, figure 2 has the line $y = 4x$.
Hence iii) $\to$ c) $\to$2).

2. Consider the following two functions $f(x)$ and $g(x)$.

$$f(x) = \begin{cases} \frac{x^3-9x}{x(x-3)} & \text{if } x \neq 0, 3 \\ 3 & \text{if } x = 0 \\ 0 & \text{if } x = 3 \end{cases}$$

$$g(x) = \begin{cases} |x| & \text{if } x \leq 2 \\ \lfloor x \rfloor & \text{if } x > 2 \end{cases}$$

Choose the set of correct options.

- ○ Option 1: $f(x)$ is discontinuous at both $x = 0$ and $x = 3$.
- ○ Option 2: $f(x)$ is discontinuous only at $x = 0$.
- ○ **Option 3:** $f(x)$ is discontinuous only at $x = 3$.
- ○ Option 4: $g(x)$ is discontinuous at $x = 2$.
- ○ **Option 5:** $g(x)$ is discontinuous at $x = 3$.

Solution:
(Options 1,2,3)
Given

$$f(x) = \begin{cases} \frac{x^3-9x}{x(x-3)} & \text{if } x \neq 0, 3 \\ 3 & \text{if } x = 0 \\ 0 & \text{if } x = 3 \end{cases}$$

Now, $\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0} \frac{x^3-9x}{x(x-3)} = \lim\limits_{x\to 0} \frac{x(x-3)(x+3)}{x(x-3)} = \lim\limits_{x\to 0} x + 3 = 3 = f(0)$.
So $f(x)$ is continuous at $x = 0$.
Similarly, $\lim\limits_{x\to 3} f(x) = \lim\limits_{x\to 3} \frac{x^3-9x}{x(x-3)} = \lim\limits_{x\to 3} \frac{x(x-3)(x+3)}{x(x-3)} = \lim\limits_{x\to 3} x + 3 = 6 \neq f(3)$.
So $f(x)$ is not continuous at $x = 3$.
Also observe that $f(x) = \frac{x^3-9x}{x(x-3)}$ if $x \neq 0, 3$, is continuous at all points except at $x = 3$.
Hence $f(x)$ is discontinuous only at $x = 3$.
(Option 5)

Given

$$g(x) = \begin{cases} |x| & \text{if } x \leq 2 \\ \lfloor x \rfloor & \text{if } x > 2 \end{cases}$$

Observe that, as $x > 2$, $g(x) = \lfloor x \rfloor$. And $\lim\limits_{x\to 3^+} g(x) = 3 \neq 2 = \lim\limits_{x\to 3^-} g(x)$, i.e, $\lim\limits_{x\to 3} g(x)$ does not exist.
Hence $g(x)$ is discontinuous at $x = 3$.
(Option 4)
Observe that $\lim\limits_{x\to 2^+} g(x) = \lim\limits_{x\to 2^+} \lfloor x \rfloor = 2$
and $\lim\limits_{x\to 2^-} g(x) = \lim\limits_{x\to 2^-} |x| = 2$.

Hence, $\lim\limits_{x\to 2^+} g(x) = 2 = \lim\limits_{x\to 2^-} g(x)$
i.e., $\lim\limits_{x\to 2} g(x) = 2 = g(2)$.
So $g(x)$ is continuous at $x = 2$.

3. Consider the graphs given below:

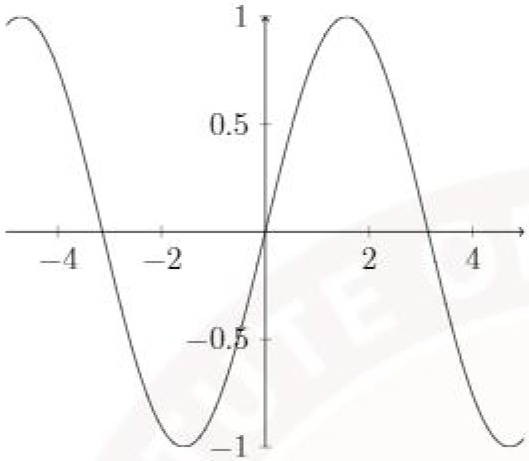


Figure: Curve 1

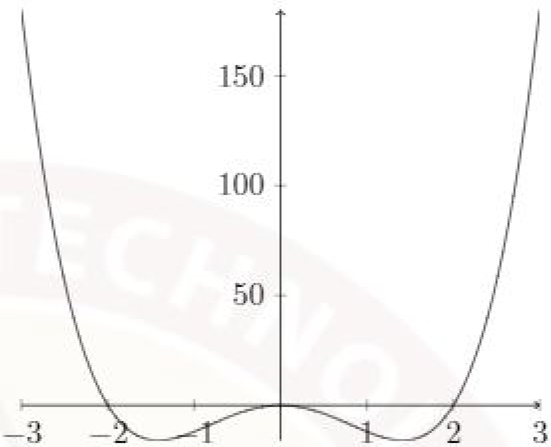


Figure: Curve 2

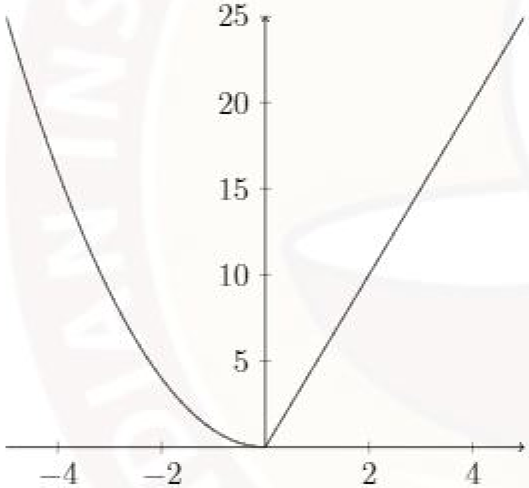


Figure: Curve 3

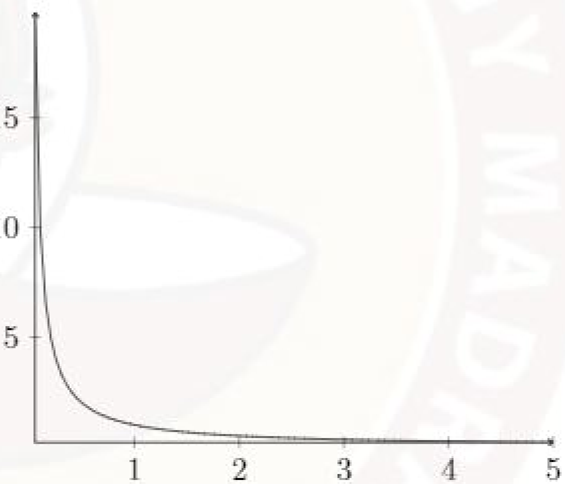


Figure: Curve 4

Choose the set of correct options.

- ◯ **Option 1:** Curve 1 is both continuous and differentiable at the origin.
- ◯ Option 2: Curve 2 is continuous but not differentiable at the origin.
- ◯ **Option 3:** Curve 2 has derivative 0 at $x = 0$.
- ◯ **Option 4:** Curve 3 is continuous but not differentiable at the origin.
- ◯ Option 5: Curve 4 is not differentiable anywhere.
- ◯ Option 6: Curve 4 has derivative 0 at $x = 0$.

Solution:
Option 1: Observe that if x approaches 0 from the left or from the right the value of the function represented by Curve 1 approaches 0. So, the limit of the function exists at $x = 0$ which is 0. Since $f(0) = 0$, the function represented by Curve 1 is continuous at $x = 0$.
We can draw a unique tangent to Curve 1 at the origin as shown in Figure M2W2GS (also observe that at $x = 0$, the graph has no sharp corner).
Hence function is differentiable at $x = 0$.

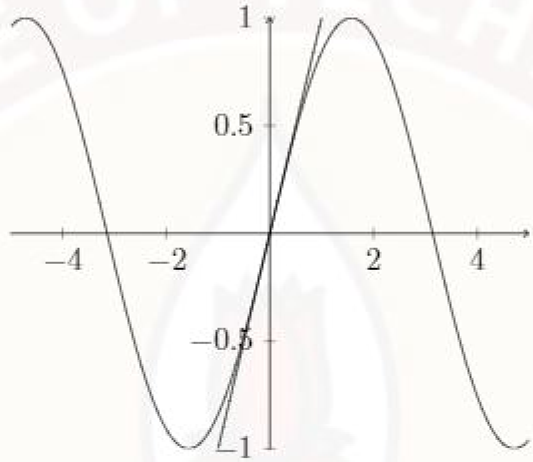


Figure M2W2GS

Options 2, 3: Observe that there is a unique tangent to the curve at the origin which is the $X-$axis itself and we know that slope of the $X-$axis is zero. Hence the function represented by Curve 2 is differentiable at $x = 0$ with derivative 0.
And we know that a differentiable function is continuous.
Hence function represented by Curve 2 is continuous at the origin.

Option 4: Observe that there is a sharp corner on Curve 3 at the origin. So function represented by Curve 3 is not differentiable at the origin.
But if x approaches 0 from the left or from the right the value of the function represented by Curve 3 approaches 0. So, the limit of the function exists at $x = 0$ which is 0. Since the value of the function $f(x)$ is 0 at $x = 0$, the function represented by Curve 3 is continuous at $x = 0$.
Option 6: If the derivative of the function represented by Curve 4 is 0 at the origin then at the origin the slope of of the tangent must be 0 i.e., the tangent must be parallel to the $X-$axis. For Curve 4, the tangent (if exists) at the origin can never be parallel to the $X-$axis. Hence this statement is not true.

Option 5: Observe that at $x = 1$, there does not exist any sharp corner and at that

point, there exists a unique tangent (which is not vertical).
Hence the function represented by Curve 4 is differentiable at $x = 1$.
Hence option 5 is not true.

4. Choose the set of correct options considering the function given below:

$$f(x) = \begin{cases} \frac{sin\ x}{x} & \text{if } x \neq 0, \\ 1 & \text{if } x = 0 \end{cases}$$

- ◯ Option 1: $f(x)$ is not continuous at $x = 0$.
- ◯ **Option 2:** $f(x)$ is continuous at $x = 0$.
- ◯ Option 3: $f(x)$ is not differentiable at $x = 0$.
- ◯ **Option 4:** $f(x)$ is differentiable at $x = 0$.
- ◯ **Option 5:** The derivative of $f(x)$ at $x = 0$ (if exists) is 0.
- ◯ Option 6: The derivative of $f(x)$ at $x = 0$ (if exists) is 1.

Solution:
We know that $\lim\limits_{x\to 0} \frac{\sin x}{x} = 1 = f(0)$. So $f(x)$ is continuous at $x = 0$.
Hence option 2 is true.
Now, $\lim\limits_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim\limits_{h\to 0} \frac{f(h)-1}{h} = \lim\limits_{h\to 0} \frac{\frac{\sin h}{h}-1}{h} = \lim\limits_{h\to 0} \frac{\sin h - h}{h^2} = \lim\limits_{h\to 0} \frac{\cos h - 1}{2h} = \lim\limits_{h\to 0} \frac{-\sin h}{2} = 0$
(using L'Hopital's rule twice).
Hence the derivative of $f(x)$ at $x = 0$ is 0.
So options 4 and 5 are true.

5. Let f be a polynomial of degree 5, which is given by

$$f(x) = a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0.$$

Let $f'(b)$ denote the derivative of f at $x = b$. Choose the set of correct options.

- ◯ **Option 1:** $a_1 = f'(0)$
- ◯ **Option 2:** $5a_5 + 3a_3 = \frac{1}{2}(f'(1) + f'(-1) - 2f'(0))$
- ◯ **Option 3:** $4a_4 + 2a_2 = \frac{1}{2}(f'(1) - f'(-1))$
- ◯ Option 4: None of the above.

Solution:
Given $f(x) = a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 \implies f'(x) = 5a_5x^4 + 4a_4x^3 + 3a_3x^2 + 2a_2x + a_1$
So $f'(0) = a_1$, $f'(1) = 5a_5 + 4a_4 + 3a_3 + 2a_2 + a_1$, and $f'(-1) = 5a_5 - 4a_4 + 3a_3 - 2a_2 + a_1$
Hence $5a_5 + 3a_3 = \frac{1}{2}(f'(1) + f'(-1) - 2f'(0))$ and $4a_4 + 2a_2 = \frac{1}{2}(f'(1) - f'(-1))$

2 Numerical Answer Type (NAT)

6. Let f be a differentiable function at $x = 3$. The tangent line to the graph of the function f at the point $(3, 0)$, passes through the point $(5, 4)$. What will be the value of $f'(3)$?
[Answer: 2]
Solution: Slope of the line passing through two points (x_1, y_1) and (x_2, y_2) is $\frac{y_2 - y_1}{x_2 - x_1}$.
So slope of the tangent at $x = 3$ is $\frac{4-0}{5-3} = 2$.
Since derivative of a function at a point equals the slope of the tangent at that point.
Hence $f'(3) = 2$

7. Let f and g be two functions which are differentiable at each $x \in \mathbb{R}$. Suppose that, $f(x) = g(x^2 + 5x)$, and $f'(0) = 10$. Find the value of $g'(0)$. [Answer: 2]
Solution:
Given $f(x) = g(x^2 + 5x) \implies f'(x) = g'(x^2 + 5x)(2x + 5)$
So $f'(0) = 5g'(0) \implies g'(0) = \frac{10}{5} = 2$

3 Comprehension Type Questions:

The population of a bacteria culture of type A in laboratory conditions is known to be a function of time of the form

$$p : \mathbb{R} \to \mathbb{R}$$

$$p(t) = \begin{cases} \frac{t^3-27}{t-3} & \text{if } 0 \le t < 3, \\ 27 & t = 3 \\ \frac{1}{e^{81}(t-3)}(e^{27t} - e^{81}) & \text{if } t > 3 \end{cases}$$

where $p(t)$ represents the population (in lakhs) and t represents the time (in minutes).
The population of a bacteria culture of type B in laboratory conditions is known to be a function of time of the form

$$q : \mathbb{R} \to \mathbb{R}$$

$$q(t) = \begin{cases} (5t-9)^{\frac{1}{t-2}} & \text{if } 0 \le t < 2, \\ e^4 & t = 2 \\ \frac{e^{t+2}-e^4}{t-2} & \text{if } t > 2 \end{cases}$$

where $q(t)$ represents the population (in lakhs) and t represents the time (in minutes).
Using the above information, answer the following questions .

8. Consider the following statements (a function is said to be continuous if it is continuous at all the points in the domain of the function). (MCQ)

   - **Statement P:** Both the functions $p(t)$ and $q(t)$ are continuous.
   - **Statement Q:** $p(t)$ is continuous, but $q(t)$ is not.
   - **Statement R:** $q(t)$ is continuous, but $p(t)$ is not.
   - **Statement S:** Neither $p(t)$ nor $q(t)$ is continuous.

   Find the number of the correct statements. [Ans: 1]
   Solution:
   Given

$$p(t) = \begin{cases} \frac{t^3-27}{t-3} & \text{if } 0 \le t < 3, \\ 27 & t = 3 \\ \frac{1}{e^{81}(t-3)}(e^{27t} - e^{81}) & \text{if } t > 3 \end{cases}$$

   and

$$q(t) = \begin{cases} (5t-9)^{\frac{1}{t-2}} & \text{if } 0 \le t < 2, \\ e^4 & t = 2 \\ \frac{e^{t+2}-e^4}{t-2} & \text{if } t > 2 \end{cases}$$

   It is enough to check the continuty of $p(t)$ at $t = 3$ and of $q(t)$ at $t = 2$.

So right limit, $\lim\limits_{t\to 3^+} p(t) = \lim\limits_{t\to 3^+} \frac{1}{e^{81}(t-3)}(e^{27t} - e^{81}) = \lim\limits_{t\to 3^+} \frac{27e^{27t}}{e^{81}} = 27$ (Using L'Hopital's rule).
Left limit, $\lim\limits_{t\to 3^-} p(t) = \lim\limits_{t\to 3^-} \frac{t^3-27}{t-3} = \lim\limits_{t\to 3^-} 3t^2 = 27$
Hence, $\lim\limits_{t\to 3^-} p(t) = \lim\limits_{t\to 3^+} p(t) = 27 = p(3)$.
So $p(t)$ is continuous at $x = 3$.

Now right limit, $\lim\limits_{t\to 2^+} q(t) = \lim\limits_{t\to 2^+} \frac{e^{t+2}-e^4}{t-2} = \lim\limits_{t\to 2^+} e^{t+2} = e^4$ (using L'Hopital's rule).
Left limit, $\lim\limits_{t\to 2^-} q(t) = \lim\limits_{t\to 2^-} (5t-9)^{\frac{1}{t-2}}$, to get the left limit,
let $y = (5t-9)^{\frac{1}{t-2}}$.
Taking log with base e on both sides and $t > \frac{9}{5}$,
we get, $\ln y = \frac{\ln(5t-9)}{t-2} \implies \lim\limits_{t\to 2^-} \ln y = \lim\limits_{t\to 2^-} \frac{\ln(5t-9)}{t-2} = \lim\limits_{t\to 2^-} \frac{5}{5t-9} = 5$ (using L'Hopital's rule)
Hence, $\lim\limits_{t\to 2^-} \ln y = 5 \implies \lim\limits_{t\to 2^-} y = e^5$.
So $\lim\limits_{t\to 2^-} (5t-9)^{\frac{1}{t-2}} = e^5$.
Since $\lim\limits_{t\to 2^+} q(t) \neq \lim\limits_{t\to 2^-} q(t)$ i.e., $\lim\limits_{t\to 2} q(t)$ does not exists, $q(t)$ is not continuous at $t = 2$.

9. If $L_p(t) = At + B$ denotes the best linear approximation of the function $p(t)$ at the point $t = 1$, then find the value of $2A + B$. [Ans: 18]

Solution:

$p(t) = \frac{t^3-27}{t-3}$ if $0 \leq t < 3 \implies p(1) = 13$

$p'(t) = \frac{(t-3)(3t^2)-(t^3-27)}{(t-3)^2} \implies p'(1) = 5.$

Therefore the best linear approximation $L_p(t)$ of the function $p(t)$ at the point $t = 1$ is $L_p(t) = p(1) + p'(1)(t-1) = 13 + 5(t-1) = 5t + 8.$

Observe that $A = 5, B = 8,$

So $2A + B = 18.$

10. If $L_p(t) = e^4(At + B) + Ce^5$ denotes the best linear approximation of the function $q(t)$ at the point $t = 3$, then find the value of $A + B + C$. [Ans: -2]

Solution:

$q(t) = \frac{e^{t+2} - e^4}{t-2}$ if $t > 2 \implies q(3) = e^5 - e^4$

$q'(t) = \frac{(t-2)e^{t+2} - (e^{t+2} - e^4)}{(t-2)^2} \implies q(3) = e^4$

Therefore the best linear approximation $L_q(t)$ of the function $q(t)$ at the point $t = 3$ is $L_q(t) = q(3) + q'(3)(t-3) = e^5 - e^4 + e^4(t-3) = e^4 t + e^5 - 4e^4 = e^4(t-4) + e^5$.

Observe that $A = 1, B = -4, C = 1$,

So $A + B + C = -2$.

11. Consider a function $f : \mathbb{R} \to \mathbb{R}$ defined as

$$f(x) = \begin{cases} \frac{\sin 14x + A\sin x}{19x^3} & \text{if } x \neq 0, \\ B & \text{if } x = 0. \end{cases}$$

If $f(x)$ is continuous at $x = 0$, then find the value of $114B - A$. [Ans: -2716]

Solution:

Given that the function is continuous that at $x = 0 \implies \lim_{x\to 0} f(x) = f(0) = B$.

$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{\sin 14x + A\sin x}{19x^3} = \lim_{x\to 0} \frac{14\cos 14x + A\cos x}{57x^2}$ (using L'Hopital's rule)

Observe that $\lim_{x\to 0} \frac{14\cos 14x + A\cos x}{57x^2}$ exist, if $(14\cos 14x + A\cos x) \to 0$ and $(57x^2) \to 0$ as $x \to 0$

Now, $14\cos 14x + A\cos x \to 0$ as $x \to 0 \implies 14 + A = 0 \implies A = -14$

So $\lim_{x\to 0} \frac{14\cos 14x + A\cos x}{57x^2} = \lim_{x\to 0} \frac{14\cos 14x - 14\cos x}{57x^2} = \lim_{x\to 0} \frac{-196\sin 14x + 14\sin x}{114x}$ (using L'Hopital's rule)

Now, $\lim_{x\to 0} \frac{-196\sin 14x + 14\sin x}{114x} = \lim_{x\to 0} \frac{-2744\cos 14x + 14\cos x}{114} = \frac{-2744+14}{114} = \frac{-2730}{114}$ (using L'Hopital's rule)

So $B = \frac{-2730}{114}$

Hence $114B - A = -2716$.

12. The distance (in meters) traveled by a car after t minutes is given by the function $d(t) = g(4t^3 + 2t^2 + 5t + 2)$, where g is a differentiable function with domain $\mathbb{R}$. Find the instantaneous speed of the car after 5 min, where $g'(577) = 2$. [Ans: 650]

Solution:

The instantaneous speed of the car after t min $= d'(t) = g'(4t^3+2t^2+5t+2)(12t^2+4t+5)$.
(use derivative property of composition of two functions)

So the instantaneous speed of the car after 5 min $= g'(577) \times 325 = 2 \times 325 = 650$

13. Consider the following two functions

$$p : \mathbb{R} \to \mathbb{R}$$

$$p(t) = \begin{cases} \frac{2e^{(t-2)}-2}{t-2} & \text{if } 0 \leq t < 2, \\ 2 & t = 2 \\ 2(t^2-4)^{\frac{1}{\ln(t-2)}} & \text{if } t > 2 \end{cases}$$

and

$$q : \mathbb{R} \to \mathbb{R}$$

$$q(t) = |t(t-7)(t-8)|$$

and the following statements (a function is said to be continuous (respectively differentiable) if it is continuous (respectively differentiable) at all the points in the domain of the function).

- **Statement P:** Both the functions $p(t)$ and $q(t)$ are continuous.
- **Statement Q:** Both the functions $p(t)$ and $q(t)$ are not differentiable.
- **Statement R:** $p(t)$ is continuous, $q(t)$ is differentiable.
- **Statement S:** $q(t)$ is continuous, $p(t)$ is not differentiable.
- **Statement T:** Neither $p(t)$ nor $q(t)$ is continuous.

Find the number of correct statements. [Ans : 2]

Solution:

Right limit of $p(t)$ at 2, $\lim\limits_{t \to 2^+} p(t) = \lim\limits_{t \to 2^+} 2(t^2-4)^{\frac{1}{\ln(t-2)}} = 2 \lim\limits_{t \to 2^+} (t^2-4)^{\frac{1}{\ln(t-2)}}$

Let $y = (t^2-4)^{\frac{1}{\ln(t-2)}}$

taking ln both sides,

$\ln y = \ln(t^2-4)\frac{1}{\ln(t-2)}$

Now, $\lim\limits_{t \to 2^+} \ln y = \lim\limits_{t \to 2^+} \frac{\ln(t^2-4)}{\ln(t-2)} = \lim\limits_{t \to 2^+} \frac{2t(t-2)}{(t+2)(t-2)} = 1$ (using L'Hopital's rule)

So as $t \to 2^+$, $y \to e^1$

hence $\lim\limits_{t \to 2^+} p(t) = \lim\limits_{t \to 2^+} 2(t^2-4)^{\frac{1}{\ln(t-2)}} = 2e^1 = 2e \neq 2 = p(2)$

So function $p(t)$ is not continuous and so $p(t)$ is not differentiable.

Now, consider the function $q(t)$,

$$q(t) = |t(t-7)(t-8)| = \begin{cases} -t(t-7)(t-8) & \text{if } t < 0, \\ t(t-7)(t-8) & \text{if } 0 \leq t < 7, \\ -t(t-7)(t-8) & \text{if } 7 \leq t < 8, \\ t(t-7)(t-8) & \text{if } t \geq 8, \end{cases}$$

So discontinuity can be possible at $x = 0, 7, 8$ but observe that $\lim_{t \to 0^-} q(t) = \lim_{t \to 0^+} q(t) = q(0)$,
$\lim_{t \to 7^-} q(t) = \lim_{t \to 7^+} q(t) = q(7)$
and $\lim_{t \to 8^-} q(t) = \lim_{t \to 8^+} q(t) = q(8)$.
Hence $q(t)$ is continuous.

For differentiabilty of $q(t)$,
observe that left derivative,
$\lim_{h \to 0^-} \frac{q(0+h)-q(0)}{h} = \lim_{h \to 0^+} \frac{q(-h)-0}{-h} = \lim_{h \to 0^+} \frac{-(-h)(-h-7)(-h-8)-0}{-h} = -56$
and right derivative
$\lim_{h \to 0^+} \frac{q(0+h)-q(0)}{h} = \lim_{h \to 0^+} \frac{q(h)-0}{h} = \lim_{h \to 0^+} \frac{h(h-7)(h-8)-0}{h} = 56$.
So, Left derivative $\neq$ Right derivative.
Hence $q(t)$ is not differentiable.

14. Consider the following function

$$p : \mathbb{R} \to \mathbb{R}$$

$$p(t) = \begin{cases} \frac{2e^{(t-2)}-2}{t-2} & \text{if } 0 \le t < 2, \\ 2 & t = 2 \\ 2(t^2-4)^{\frac{1}{\ln(t-2)}} & \text{if } t > 2 \end{cases}$$

If linear function $L_p(t) = At + B$ denotes the best linear approximation of the function $p(t)$ at the point $t = 1$, find the value of $\frac{-2}{e^{-1}-1}(A + B)$. [Ans: 4]

Solution:

Observe that $p(t) = \frac{2e^{(t-2)}-2}{t-2}$ if $0 \le t < 2$.

Linear approximation of the $p(t)$ at $t = 1$ is $L_p(t) = p'(1)(t-1)+p(1) = p'(1)t-p'(1)+p(1)$

So here $A = p'(1), B = -p'(1) + p(1)$.

Therefore $A + B = p(1)$

Hence $\frac{-2}{e^{-1}-1}(A + B) = \frac{-2}{e^{-1}-1}p(1) = 4$

15. Consider the following function

$$q : \mathbb{R} \to \mathbb{R}$$

$$q(t) = |t(t-7)(t-8)|.$$

If m is slope of the tangent of the function $q(t)$ at point $t = \frac{3}{2}$, find the value $m - \frac{27}{4}$.
[Ans: 11]

Solution:

From question 13, observe that $q(t) = t(t-7)(t-8) = t^3 - 15t^2 + 56t$ if $0 \leq t < 7$.
So $q'(t) = 3t^2 - 30t + 56 \implies q'(\frac{3}{2}) = \frac{27}{4} - 45 + 56 = \frac{27}{4} - 11$.
Now, slope of the tangent of the function $q(t)$ at point $t = \frac{3}{2}$ is $q'(\frac{3}{2})$.
Hence $m = q'(\frac{3}{2})$.
So $m - \frac{27}{4} = 11$

Mock 4 Solutions (Week 5-6) Sept 2022
Mathematics for Data Science - 1
Max Marks: 25

1 Instructions:

- There are some questions which have functions with discrete valued domains (such as day, month, year etc). For simplicity, we treat them as continuous functions.
- For NAT type question, enter only one right answer even if you get multiple answers for that particular question.
- **Notations:**
  - $\mathbb{R}$= Set of real numbers
  - $\mathbb{Q}$= Set of rational numbers
  - $\mathbb{Z}$= Set of integers
  - $\mathbb{N}$= Set of natural numbers

1. On an average, a video lecture in our online degree course has 200 views on the same day that it is posted. It is verified that the total number of views increases exponentially according to the function $y = 200 \times 5^{0.1t}$, where t represents the number of days since the video was posted ($t = 0$ on the day of posting the video). How many days does it take for 1000 people to view the video?
(NAT)(Answer: 10) [Marks: 3]

Solution:

Sol:-

$$25000 = 200 \times 5^{0.15 \times t}$$

$$\Rightarrow \overset{125}{\cancel{250}} = \cancel{2} \times 5^{15t/100}.$$

$$\Rightarrow 5^{\frac{15t}{100}} = 125$$

$$\Rightarrow \frac{15t}{100} = \log_5 125$$

$$\Rightarrow t = \frac{100}{15} \times \log_5 5^3$$

$$= \frac{\cancel{100}^{20}}{\cancel{15}_{\cancel{3}}} \times \cancel{3}. = 20.$$

$$\boxed{t = 20.}$$

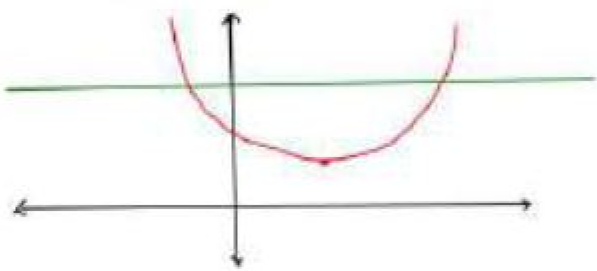
2. Let $f(x) = e^{a\left(x^2-7x+6\right)}$, $a \in \mathbb{R}$ then choose the set of correct options.
(MSQ) (Ans: (a),(b)) [Marks: 3]

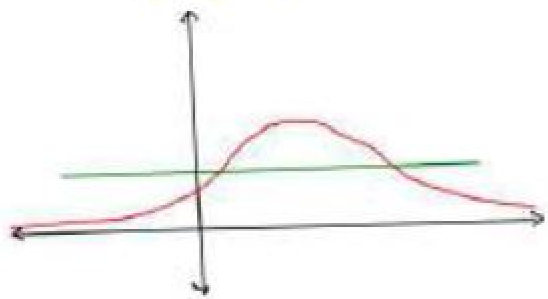
- ◯ $f(x)$ will be positive for $x \in (-20, 10)$ if $a > 0$.
- ◯ $f(x)$ will be positive for $x \in (10, 20)$ if $a < 0$.
- ◯ $f(x)$ is a one-to-one (injective) function.
- ◯ If $a = 1$, then $f(x)$ will have two X-intercepts.

Solution:

Soln:-

If $a>0$: If $a<0$:





NOTE:- "$e^{g(x)}$" is always positive, for any function $g(x)$.

$\therefore$ $f(x) > 0$ for all $x \in \mathbb{R}$ and independent of sign of a.

$\Rightarrow$ $f(x) > 0$ for $x \in (-10, 5)$ and $a>0$

and $f(x) > 0$ for $x \in (5, 10)$ and $a<0$.

* Horizontal line test is failed $\Rightarrow$ Not one-to-one.

* If $a=1$, then $f(x) = e^{x^2-7x+6}$.

  * for X-intercepts, $f(x)=0$.

  $\Rightarrow$ $0 = e^{x^2-7x+6}$.

  but for no "x" this above equation is satisfied.

  $\Rightarrow$ No X-intercepts.

3. If a is the number of solutions of the given equation $x^{\left(\frac{3}{4}(\log_2 x)^2+\log_2 x-\frac{5}{4}\right)} = \sqrt{2}$, then the value a is

(NAT)(Answer: 3) [Marks: 3]

Solution:

Soln:- $x^{\left(\frac{3}{4}(\log_2 x)^2+\log_2 x-\frac{5}{4}\right)} = \sqrt{2}$

Taking "$\log_2$" on both sides, we get

$$\left(\frac{3}{4}(\log_2 x)^2+\log_2 x-\frac{5}{4}\right)(\log_2 x) = \log_2 \sqrt{2}$$

Put $\boxed{p = \log_2 x}$,

$$\left(\frac{3}{4}p^2+p-\frac{5}{4}\right)(p) = \frac{1}{2}$$

$$(3p^2+4p-5)(p) = 2$$

$$\Rightarrow 3p^3+4p^2-5p-2 = 0$$

$$\Rightarrow (p-1)\underbrace{(3p^2+7p+2)}_{} = 0$$

$$\begin{array}{r} 3p^2+7p+2 \\ p-1\overline{)\,3p^3+4p^2-5p-2} \\ 3p^3-3p^2 \\ \hline 7p^2-5p \\ 7p^2-7p \\ \hline 2p-2 \\ 2p-2 \\ \hline 0 \end{array}$$

Discriminant $= 49-4\times3\times2 = 25 > 0$

$\therefore$ Total 3 roots $\Rightarrow \boxed{a=3}$

4. Simplify the expression $(\frac{a^x}{a^y})^{(x+y-z)}.(\frac{a^y}{a^z})^{(y+z-x)}.(\frac{a^z}{a^x})^{(z+x-y)}$ (Answer (c))(2 marks)

- a^{x+y+z}
- $a^{x^2+y^2+z^2-xy-yz-zx}$
- 1
- a

Solution:

Solution :-

$$= a^{(x-y)(x+y-z)} . a^{(y-z)(y+z-x)} . a^{(z-x)(z+x-y)}$$

$$= a^{x^2-xz-y^2+yz} . a^{y^2-xy-z^2+xz} . a^{z^2-yz-x^2+xy}$$

$$= a^{x^2-xz-y^2+yz+y^2-xy-z^2+xz+z^2-yz-x^2+xy}$$

$$= a^0 = 1$$

So, correct option is (c).

5. Which of the following statements are correct? (Answer: (b),(c))(3 marks)

- ◯ The functions $f(x) = -\sqrt{ln(x)}$ and $g(x) = e^{x^2}$ are inverses to each other.
- ◯ The domain of the real-valued function $f(x) = \sqrt{e^{x^2-8x} - 1}$ is $(-\infty, 0] \cup [8, \infty)$.
- ◯ The line $x = 3$ is a vertical asymptote of the function $f(x) = ln(x^2 + 5x - 24)$.
- ◯ None of the above

Solution:

Solution :-

option ⓐ :- To check this,

$$(f \circ g)(x) = x \ \& \ (g \circ f)(x) = x$$

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= f(e^{x^2}) \\ &= -\sqrt{\ln(e^{x^2})} \\ &= -\sqrt{x^2} \\ &= -|x|\end{aligned}$$

$$\begin{aligned}(g \circ f)(x) &= g(f(x)) = g(-\sqrt{\ln(x)}) \\ &= e^{(-\sqrt{\ln x})^2} \\ &= e^{\ln x} = x\end{aligned}$$

So we can see that $(f \circ g)(x) \neq x$

$\therefore$ $f(x)$ & $g(x)$ are not inverse to each other.

option ⓑ :-

$$f(x) = \sqrt{e^{x^2-8x} - 1}$$

for all real value,

$$e^{x^2-8x} - 1 \geq 0$$

$$e^{x^2-8x} \geq 1$$

$$x^2 - 8x \geq 0$$

$$x(x-8) \geq 0$$

$$x = 0, 8$$

The domain will be $(-\infty, 0] \cup [8, \infty)$

Hence, the given statement is correct.

option ⓒ :-

$$f(x) = \ln(x^2 + 5x - 24)$$

for asymptote,

$$x^2 + 5x - 24 = 0$$

$$(x+8)(x-3) = 0$$

$$x = -8, 3$$

$\therefore x = 3$ is a vertical asymptote.

Hence, the given statement is correct.

6. Suppose $f(x) = \frac{x+5}{x-3}$ and $g(x) = \sqrt{x^2-1}$ are functions on their respective domains. Which of the following statements are correct? (Answer: (a),(c),(d))(3 marks)

   - ◯ The domain of the composite function $(f \circ g)(x)$ is $(-\infty, -\sqrt{10}) \cup (-\sqrt{10}, -1] \cup [1, \sqrt{10}) \cup (\sqrt{10}, \infty)$.
   - ◯ The domain of the composite function $(f \circ g)(x)$ is $\mathbb{R} \setminus \{-\sqrt{10}, \sqrt{10}\}$.
   - ◯ $(f \circ g)(x) = \frac{\sqrt{x^2-1}+5}{\sqrt{x^2-1}-3}$.
   - ◯ $(g \circ f)(x) = \frac{4\sqrt{x+1}}{|x-3|}$.

Solution:

Solution :-

$$f(x) = \frac{x+5}{x-3}$$

$$g(x) = \sqrt{x^2-1}$$

$$(f \circ g)(x) = f(g(x))$$

$$= \frac{g(x)+5}{g(x)-3}$$

$$(f \circ g)(x) = \frac{\sqrt{x^2-1}+5}{\sqrt{x^2-1}-3}$$

domain of $(f \circ g)(x) = ?$

$\sqrt{x^2-1} - 3 \neq 0$ and $x^2 - 1 \geq 0$

$x^2 - 1 \neq 9$ | $(x-1)(x+1) \geq 0$

$x^2 \neq 10$ | +ve $\;-1\;$ −ve $\;1\;$ +ve

$x \neq \pm\sqrt{10}$ | $x \leq -1$ and $x \geq 1$

3

On combining the both cases,

$(-\infty, -\sqrt{10}) \cup (-\sqrt{10}, -1] \cup [1, \sqrt{10}) \cup (\sqrt{10}, \infty)$

Hence, option (a) is correct.

option (c) is correct

$$(g \circ f)(x) = g(f(x))$$

$$= g\left(\frac{x+5}{x-3}\right)$$

$$= \sqrt{\left(\frac{x+5}{x-3}\right)^2 - 1}$$

$$= \frac{\sqrt{(x+5)^2 - (x-3)^2}}{\sqrt{(x-3)^2}}$$

$$= \frac{\sqrt{\cancel{x^2} + 10x + 25 - \cancel{x^2} + 6x - 9}}{|x-3|}$$

$$= \frac{\sqrt{16x + 16}}{|x-3|}$$

$$\therefore (g \circ f)(x) = \frac{4\sqrt{x+1}}{|x-3|}$$

Hence, option (d) is also correct.

7. Stock price (y) (in ₹) for a motor cycle company (A) is predicted by the equation

$$y = -7log_2(x + a) + 35,$$

where x represents the number of months since January of the year 2022 (note: for January, consider $x = 0$) and $a \in \mathbb{N}$. If the stock price of the company goes to zero in November of the year 2022, following the same trend, then find the value of a. (Answer: 22)(4 marks)

Solution:

Solution:- $y = -7\log_2(x+a) + 35$

$x = 10$ (for November)

$0 = -7\log_2(10+a) + 35$

$\Rightarrow -7\log_2(10+a) = -35$

$\Rightarrow \log_2(10+a) = 5$

$\Rightarrow 10 + a = 2^5$

$\therefore a = 32 - 10$

$\boxed{a = 22}$

Consider the following information and answer the questions 8 - 10

A group of Biotechnology students were creating a Genetically Modified Plant (GMP). They found that the expression $f(x) = \frac{a}{1+e^{-0.5x}}$ gives the increase in the number of leaves on the plant as a function of days. On 0^{th} day there were 10 leaves. By the end of 36^{th} day, the number of leaves started decreasing as function of $g(x) = -10 \times 2^{\frac{x}{b}} + 100$ and eventually there were no leaf on that plant after some days (Refer Figure 2). Consider $f(x)$ and $g(x)$ represents the number of leaves on that plant by the end of x^{th} day.

Note:

(1) Take 19.9… as 20.

(2) For simplicity consider a leaf is fully grown when $f(x)$ is an integer value.

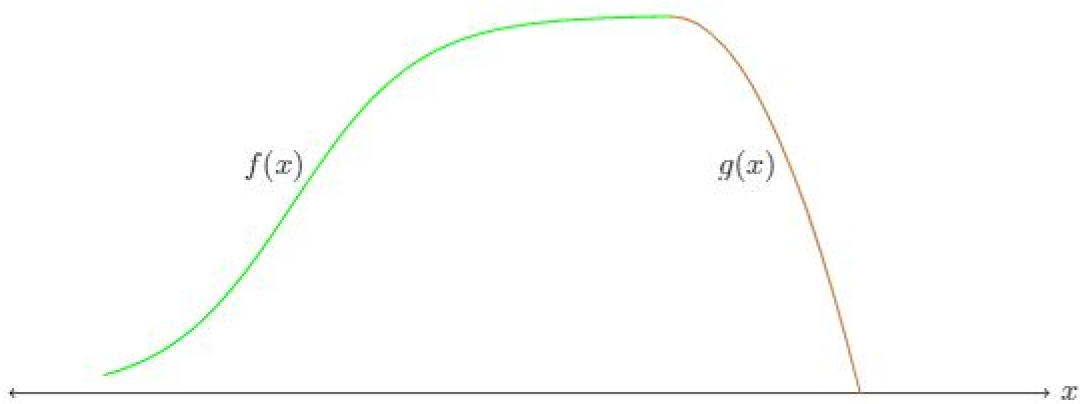


Figure 2

8. What will be the value of a.
   (NAT) (Answer: 20), [1 marks]

9. Which of the following statements is (are) correct?
   (MSQ) (Answer: Option (a), (c)), [2 marks]

   ◯ b can be found using $f(36) = g(36)$

   ◯ b cannot be determined.

   ◯ By the end of 36^{th} day, there are roughly 20 leaves.

   ◯ By the end of 36^{th} day, there are roughly 30 leaves.

10. What will be the value of b.
    (NAT) (Answer: 12), [1 marks]

Solution:

Given that $f(x) = \frac{a}{1+e^{-0.5x}}$, at $x = 0, f(0) = 10$

a) $\therefore$ at $x=0$, $\frac{a}{1+e^0} = 10 \Rightarrow \frac{a}{1+1} = 10$ or, $a = 20$ [Ans]

b) At end of 36 th day, we have,

$\frac{20}{1+e^{-x/2}} = -10 \times 2^{x/b} + 100$

For $x = 36$, we have,

$\frac{20}{1+e^{-18}} = -10 \times 2^{36/b} + 100$. Now, $e^{-18} \approx 0$

$\therefore 20 = -10 \times 2^{36/b} + 100$ or, $10 \times 2^{36/b} = 80$

or, $2^{36/b} = 8 = 2^3$.

$\therefore 36/b = 3$ or, $b = 36/3 = 12$.

Therefore b can be found from $f(36) = g(36)$

At the end of 36th day, $f(36) \approx 20$. So option (c) is correct. Option (a) is also correct.

c) $b = 12$ from previous calculation.

M1T5_MOCK 2_Solution

Mathematics for Data Science - 1

Set-1 solutions

Max. Marks: 50

1 Multiple Select Questions (MSQ)

1. Simplify the expression $(\frac{a^x}{a^y})^{(x+y-z)}.(\frac{a^y}{a^z})^{(y+z-x)}.(\frac{a^z}{a^x})^{(z+x-y)}$ (Answer (c))(3 marks)

◯ a^{x+y+z}

◯ $a^{x^2+y^2+z^2-xy-yz-zx}$

◯ 1

◯ a

Solution :-

$$= a^{(x-y)(x+y-z)} . a^{(y-z)(y+z-x)} . a^{(z-x)(z+x-y)}$$

$$= a^{x^2-xz-y^2+yz} . a^{y^2-xy-z^2+xz} . a^{z^2-yz-x^2+xy}$$

$$= a^{x^2-xz-y^2+yz+y^2-xy-z^2+xz+z^2-yz-x^2+xy}$$

$$= a^0 = 1$$

So, correct option is (c).

2. Which of the following statements are correct? (Answer: (b),(c),(d))(5 marks)

- ○ The functions $f(x) = -\sqrt{ln(x)}$ and $g(x) = e^{x^2}$ are inverses to each other.
- ○ The domain of the real-valued function $f(x) = \sqrt{e^{x^2-8x} - 1}$ is $(-\in, 0] \cup [8, \in)$.
- ○ The line $x = 3$ is a vertical asymptote of the function $f(x) = ln(x^2 + 5x - 24)$.
- ○ f may be continuous at the point $x = a$ even if f is not differentiable at a point $x = a$.

Solution :-

option (a) :- To check this,

$$(f \circ g)(x) = x \ \& \ (g \circ f)(x) = x$$

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= f(e^{x^2}) \\ &= -\sqrt{\ln(e^{x^2})} \\ &= -\sqrt{x^2} \\ &= -|x|\end{aligned}$$

$$\begin{aligned}(g \circ f)(x) &= g(f(x)) = g(-\sqrt{\ln(x)}) \\ &= e^{(-\sqrt{\ln x})^2} \\ &= e^{\ln x} = x\end{aligned}$$

So we can see that $(f \circ g)(x) \neq x$

$\therefore$ $f(x)$ & $g(x)$ are not inverse to each other.

option (b) :-

$$f(x) = \sqrt{e^{x^2-8x} - 1}$$

for all real value,

$$e^{x^2-8x} - 1 \geq 0$$

$$e^{x^2-8x} \geq 1$$

$$x^2 - 8x \geq 0$$

$$x(x-8) \geq 0$$

$$x = 0, 8$$

The domain will be $(-\infty, 0] \cup [8, \infty)$

Hence, the given statement is correct.

option (c) :-

$$f(x) = \ln(x^2 + 5x - 24)$$

for asymptote,

$$x^2 + 5x - 24 = 0$$

$$(x+8)(x-3) = 0$$

$$x = -8, 3$$

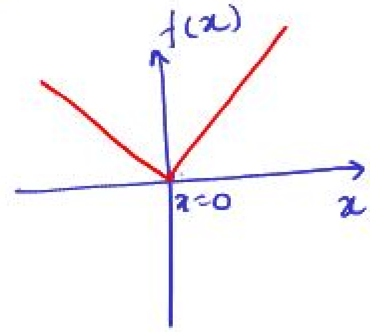
$\therefore x = 3$ is a vertical asymptote.

Hence, the given statement is correct.

option (d) :-

A function f can be continuous at $x=a$ even if f is not differentiable at $x=a$

for example: $f(x) = |x|$



f(x) is contenuous at $x=0$
but not differenteable at $x=0$.

Hence, options (b), (c) and (d) are correct.

3. Suppose $f(x) = \frac{x+5}{x-3}$ and $g(x) = \sqrt{x^2-1}$ are functions on their respective domains. Which of the following statements are correct? (Answer: (a),(c),(d))(5 marks)

- ◯ The domain of the composite function $(f \circ g)(x)$ is $(-\in, -\sqrt{10}) \cup (-\sqrt{10}, -1] \cup [1, \sqrt{10}) \cup (\sqrt{10}, \in)$.
- ◯ The domain of the composite function $(f \circ g)(x)$ is $\mathbb{R} \setminus \{-\sqrt{10}, \sqrt{10}\}$.
- ◯ $(f \circ g)(x) = \frac{\sqrt{x^2-1}+5}{\sqrt{x^2-1}-3}$.
- ◯ $(g \circ f)(x) = \frac{4\sqrt{x+1}}{|x-3|}$.

<u>Solution</u> :-

$$f(x) = \frac{x+5}{x-3}$$

$$g(x) = \sqrt{x^2-1}$$

$$(f \circ g)(x) = f(g(x))$$

$$= \frac{g(x)+5}{g(x)-3}$$

$$(f \circ g)(x) = \frac{\sqrt{x^2-1}+5}{\sqrt{x^2-1}-3}$$

domain of $(f \circ g)(x) = ?$

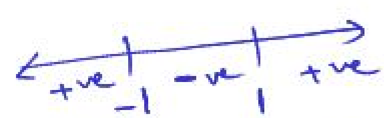
$\sqrt{x^2-1} - 3 \neq 0$ and $x^2-1 \geq 0$

$x^2-1 \neq 9$

$x^2 \neq 10$

$x \neq \pm\sqrt{10}$

$(x-1)(x+1) \geq 0$



$x \leq -1$ and $x \geq 1$

On combining the both cases,

$(-\infty, -\sqrt{10}) \cup (-\sqrt{10}, -1] \cup [1, \sqrt{10}) \cup (\sqrt{10}, \infty)$

Hence, option ⓐ is correct.

option ⓒ is correct

$$(g \circ f)(x) = g(f(x))$$

$$= g\left(\frac{x+5}{x-3}\right)$$

$$= \sqrt{\left(\frac{x+5}{x-3}\right)^2 - 1}$$

$$= \frac{\sqrt{(x+5)^2 - (x-3)^2}}{\sqrt{(x-3)^2}}$$

$$= \frac{\sqrt{\cancel{x^2} + 10x + 25 - \cancel{x^2} + 6x - 9}}{|x-3|}$$

$$= \frac{\sqrt{16x+16}}{|x-3|}$$

$$\therefore (g \circ f)(x) = \frac{4\sqrt{x+1}}{|x-3|}$$

Hence, option ⓓ is also correct.

4. Consider a sequence $\{a_n\}$ defined as $a_n = \frac{a_{n-1}+a_{n-2}}{2}$ for all $n \geq 3$ and $a_1 = 0, a_2 = 1$. Which of the following statements are correct? (Answer: (b),(d))(5 marks)

- ○ The sequence $\{a_n\}$ is not convergent.
- ○ $\lim_{n\to\infty} a_n = \frac{2}{3}$.
- ○ $\sum_{i=3}^{n} a_i = \frac{a_2 + a_{n-1}}{2} + \sum_{i=3}^{n-2} a_i$.
- ○ $\sum_{i=3}^{n} a_i = \frac{a_{n-1}}{2} + \sum_{i=2}^{n-2} a_i$.

Solution:-

$$a_n = \frac{a_{n-1} + a_{n-2}}{2}, \quad n \geq 3$$

given, $a_1 = 0$, $a_2 = 1$

$$a_3 = \frac{a_2 + a_1}{2}$$

$$a_4 = \frac{a_3 + a_2}{2}$$

$$\cdots\cdots$$

$$a_n = \frac{a_{n-1} + a_{n-2}}{2}$$

On addition,

$$a_3 + a_4 + \cdots + a_n = \frac{a_2 + a_1}{2} + \frac{a_3 + a_2}{2} + \cdots + \frac{a_{n-1} + a_{n-2}}{2}$$

$$\sum_{i=3}^{n} a_i = \frac{a_2 + a_1}{2} + \frac{a_3 + a_2}{2} + \cdots + \frac{a_{n-1} + a_{n-2}}{2}$$

$$\sum_{i=3}^{n} a_i = \frac{a_1}{2} + a_2 + a_3 + \cdots + a_{n-2} + \frac{a_{n-1}}{2}$$

$$= 0 + \sum_{i=2}^{n-2} a_i + \frac{a_{n-1}}{2}$$

$$\therefore \sum_{i=3}^{n} a_i = \frac{a_{n-1}}{2} + \sum_{i=2}^{n-2} a_i$$

So, option 4 is correct.

Now,

$$\sum_{i=3}^{n} a_i = \frac{a_{n-1}}{2} + \sum_{i=2}^{n-2} a_i$$

$$\Rightarrow \cancel{a_3} + \cancel{a_4} + \cdots + \cancel{a_{n-2}} + a_{n-1} + a_n = \frac{a_{n-1}}{2} + a_2 + \cancel{a_3} + \cdots + \cancel{a_{n-2}}$$

$$\Rightarrow a_{n-1} + a_n = \frac{a_{n-1}}{2} + 1 \qquad (\because a_2 = 1)$$

$$a_n = 1 - \frac{a_{n-1}}{2}$$

observe that, as $n \to \infty$, $a_{n-1} \to a_n$

$$\boxed{\lim_{n\to\infty} a_{n-1} = \lim_{n\to\infty} a_n}$$

Now, $a_n = 1 - \frac{a_{n-1}}{2}$

$$\Rightarrow \lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(1 - \frac{a_{n-1}}{2}\right)$$

$$\Rightarrow \lim_{n\to\infty} a_n + \lim_{n\to\infty}\frac{a_{n-1}}{2} = 1$$

$$\Rightarrow \lim_{n\to\infty} a_n + \frac{1}{2}\cdot\lim_{n\to\infty} a_{n-1} = 1$$

$$\Rightarrow \frac{3}{2}\cdot\lim_{n\to\infty} a_n = 1$$

$$\boxed{\therefore \lim_{n\to\infty} a_n = \frac{2}{3}}$$

So option (2) is correct.

2 Numerical Answer Type(NAT)

5. Stock price (y) (in ₹) for a motor cycle company (A) is predicted by the equation

$$y = -7log_2(x + a) + 35,$$

where x represents the number of months since January of the year 2022 (note: for January, consider $x=0$) and $a \in \mathbb{N}$. If the stock price of the company goes to zero in November of the year 2022, following the same trend, then find the value of a. (Answer: 22)(4 marks)

Solution :-

$$y = -7\log_2(x+a) + 35$$

$x = 10$ (for November)

$$0 = -7\log_2(10+a) + 35$$

$$\Rightarrow -7\log_2(10+a) = -35$$

$$\Rightarrow \log_2(10+a) = 5$$

$$\Rightarrow 10 + a = 2^5$$

$$\therefore a = 32 - 10$$

$$\boxed{a = 22}$$

6. Ravi borrowed ₹3,000 and ₹12,000 from his friends Vinay and Bhumi respectively. Vinay lent the money at 7 percent simple interest per annum for 4 years and Bhumi lent the money at 10 percent compound interest per annum for x years. The compound interest which Bhumi received after x years is thrice the value of the simple interest which Vinay received after 4 years. What is the value of x?

[Note: Simple interest $= \frac{PTR}{100}$ and Compound Interest $= P(1+\frac{R}{100})^T - P$, where P is the principle amount, T is time (in years) and R is the interest rate per annum, i.e., if $x\%$ is the interest rate per annum then $R = x$] (Answer: 2), (5 marks)

Solution :-

Simple interest recieved by Vinay $= \frac{3000 \times 7 \times 4}{100}$

Compound interest recieved by Bhumi

$$= 12000\left(1 + \frac{10}{100}\right)^x - 12000$$

According to question,

$$3 \times \left(\frac{\cancel{3000} \times 7 \times \cancel{4}}{100}\right) = \cancel{12000}\left[(1+0.1)^x - 1\right]$$

$$\Rightarrow \frac{21}{100} = (1.1)^x - 1$$

$$\Rightarrow (1.1)^x = 1.21$$

$$\therefore x = \frac{\ln(1.21)}{\ln(1.1)} \approx 2$$

$$\boxed{\therefore x = 2 \text{ years}}$$

Use the following information for questions 7-9:

Consider the function defined as follows with $p, q, r \in \mathbb{R}$:

$$f(x) = \begin{cases} pe^x - 4x + 3 & \text{if } x < 0 \\ q - 5 & \text{if } x = 0 \\ rsin(x) + 9cos(x) & \text{if } x > 0 \end{cases}$$

7. If the limit exists at $x = 0$ for the given function $f(x)$, then what will be the value of p? (Answer: 6)(2 marks)
8. If f is continuous at $x = 0$, then find the value of $\frac{q}{2}$. (Answer: 7)(3 marks)
9. If f is differentiable everywhere, then find the value of r. (Answer: 2)(4 marks)

Solution of ⑦ :-

If the limit exists at $x=0$ for function $f(x)$ then

left hand limit = Right hand limit

i.e. $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x)$

$\Rightarrow \lim_{x \to 0} (pe^x - 4x + 3) = \lim_{x \to 0} r\sin(x) + 9\cos(x)$

$\Rightarrow p + 3 = 9$

$\boxed{\therefore p = 6}$

Solution of (8):

If fn $f(x)$ is said to contenuous at $x=0$

then,
$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) = f(0)$$

$$\therefore \lim_{x\to 0^-} f(x) = f(0)$$

$$\Rightarrow \lim_{x\to 0^-} pe^{x} - 4x + 3 = q-5$$

$$\Rightarrow p+3 = q-5$$

$$\Rightarrow 6+3 = q-5$$

$$\therefore q = 14$$

The value of $\frac{q}{2} = 7$.

Solution of (9):-

$$f(x) = r\sin(x) + q\cos x \quad \text{(for left hand side)}$$

$$f'(x) = r\cos x - q\sin x$$

Put $x=0$

$$f'(0) = r - 0$$

$\therefore \quad \gamma = f'(0) \quad \text{—} (i)$

$f(x) = pe^{x} - 4x + 3$ (for right hand side)

$f'(x) = pe^{x} - 4$

$f'(0) = p - 4 = 6 - 4 = 2 \quad \text{—} (ii)$

from (i) & (ii)

$\boxed{\gamma = 2}$

Use the following information for questions 10-11:

Suppose f is a real valued function defined on domain D. let $f(x+y) = f(x)f(y)$ for all $x, y \in D$ and $f(1) = 5$, $f'(0) = 3$.

10. What is the value of $f(0)$? (Answer: 1)(2 marks)

11. What is the value of $f'(1)$? (Answer: 15)(4 marks)

Solution of 10 :-

given, $f(x+y) = f(x) \cdot f(y)$

Put $x = 0$

$$f(0+y) = f(0) \cdot f(y)$$

$$f(y) = f(0) \cdot f(y)$$

$$\boxed{\therefore \; f(0) = 1}$$

Solution of 11 :-

$$f'(1) = ?$$

As we know that,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Let $x = 1$

$$f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h \to 0} \frac{f(1+h) - f(1+0)}{h}$$

$$= \lim_{h \to 0} \frac{f(1) \cdot f(h) - f(1) \cdot f(0)}{h}$$

$$= f(1) \lim_{h \to 0} \frac{f(h) - f(0)}{h}$$

$$= f(1) \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

$$= f(1) \cdot f'(0)$$

$$f'(1) = 5 \times 3$$

$$\boxed{\therefore f'(1) = 15}$$

Use the following information for questions 12-14:

Consider a sequence $\{a_n\}$ defined as

$$a_n = \begin{cases} \frac{3n - \lfloor \frac{n}{2} \rfloor}{7+n} & \text{when } n \text{ is odd} \\ \frac{5n^2-4n+1}{6n+2n^2} & \text{when } n \text{ is even} \end{cases}$$

where $\lfloor x \rfloor$ is the greatest integer that is less than or equal to a real number x.

12. Which of the following statements are correct? (Answer: (a),(d))(2 marks)
    - ◯ If n is odd, then $\lfloor \frac{n}{2} \rfloor = \frac{n-1}{2}$.
    - ◯ If n is even, then $\lfloor \frac{n}{2} \rfloor = \frac{n}{2} + 1$.
    - ◯ If n is odd, then $\lfloor \frac{n}{2} \rfloor = \frac{n+1}{2}$.
    - ◯ If n is even, then $\lfloor \frac{n}{2} \rfloor = \frac{n}{2}$.

13. Find the limit of the sequence $\{4a_n\}$. (Answer: 10)(3 marks)

14. Find the limit of the sequence $\{b_n\}$ defined as $b_n = 4a_n^2 - 10a_n$. (Ans: 0)(3 marks)

Solution of 12 :-

option (a) :- $\lfloor \frac{n}{2} \rfloor = \frac{n-1}{2}$, n is odd

Let $n = 1$

$\therefore$ L.H.S $= \lfloor \frac{1}{2} \rfloor = \lfloor 0.5 \rfloor = 0$

RHS $= \frac{n-1}{2} = \frac{1-1}{2} = 0$

$\therefore$ LHS = RHS

Therefore, option (a) is correct.

option (b) :- $\lfloor \frac{n}{2} \rfloor = \frac{n}{2} + 1$, n is even

Let $n = 2$

$$LHS = \left\lceil \frac{2}{2} \right\rceil = \lceil 1 \rceil = 1$$

$$RHS = \frac{n}{2} + 1 = \frac{2}{2} + 1 = 2$$

$$LHS \neq RHS$$

Therefore, option is incorrect.

option ⓒ :- It is incorrect. since we checked in option ⓐ

option ⓓ :-

$$\left\lceil \frac{n}{2} \right\rceil = \frac{n}{2}$$

Let $n = 2$

$$LHS = \left\lceil \frac{n}{2} \right\rceil = \left\lceil \frac{2}{2} \right\rceil = \lceil 1 \rceil = 1$$

$$RHS = \frac{n}{2} = \frac{2}{2} = 1$$

$$\therefore LHS = RHS$$

Therefore, option ⓓ is correct.

Solution of ⑬ :-

$$\lim_{n \to \infty} 4a_n = ?$$

for that we have to find a_n

$$a_n = \begin{cases} \dfrac{3n - \frac{n-1}{2}}{7+n}, & \text{when } n \text{ is odd} \\ \dfrac{5n^2 - 4n + 1}{6n + 2n^2}, & \text{when } n \text{ is even} \end{cases}$$

<u>when n is odd</u>

$$\lim_{n\to\infty} 4\left(\frac{3n - \frac{n-1}{2}}{7+n}\right)$$

$$= \lim_{n\to\infty} 4\left(\frac{5n+1}{14+2n}\right)$$

$$= \lim_{n\to\infty} 4\left(\frac{5+\frac{1}{n}}{\frac{14}{n}+2}\right)$$

$$= 4\times\left(\frac{5+0}{0+2}\right) = \frac{5}{2}\times 4 = 10$$

<u>when n is even</u>

$$\lim_{n\to\infty} 4\left(\frac{5n^2 - 4n + 1}{6n + 2n^2}\right)$$

$$= \lim_{n\to\infty} 4\left(\frac{5 - \frac{4}{n} + \frac{1}{n^2}}{\frac{6}{n} + 2}\right)$$

$$= 4\left(\frac{5 - 0 + 0}{0 + 2}\right) = 10$$

$$\therefore \lim_{n\to\infty} \{4a_n\} = 10$$

Solution of 14:

$$\lim_{n\to\infty}\{b_n\} = \lim_{n\to\infty}(4a_n^2 - 10a_n)$$

$$\lim_{n\to\infty}\{b_n\} = \lim_{n\to\infty} 4a_n^2 - \lim_{n\to\infty} 10a_n$$

$\therefore$ when n is odd $\rightarrow$

$$\lim_{n\to\infty}\{b_n\} = \lim_{n\to\infty} 4\cdot\left(\frac{5n+1}{14+2n}\right)^2 - \lim_{n\to\infty} 10\cdot\left(\frac{5n+1}{14+2n}\right)$$

$$= \lim_{n\to\infty} 4\cdot\left(\frac{5+\frac{1}{n}}{\frac{14}{n}+2}\right)^2 - \lim_{n\to\infty} 10\left(\frac{5+\frac{1}{n}}{\frac{14}{n}+2}\right)$$

$$= 4\cdot\left(\frac{5+0}{0+2}\right)^2 - 10\cdot\left(\frac{5+0}{0+2}\right)$$

$$= 4\times\frac{25}{4} - 10\times\frac{5}{2}$$

$$= 25 - 25$$

$$\boxed{\therefore \lim_{n\to\infty}\{b_n\} = 0}$$

When n is even :-

$$\lim_{n\to\infty}\{b_n\} = \lim_{n\to\infty} 4\cdot\left(\frac{5n^2-4n+1}{6n+2n^2}\right)^2 - \lim_{n\to\infty} 10\left(\frac{5n^2-4n+1}{6n+2n^2}\right)$$

$$= 4\cdot\lim_{n\to\infty}\left(\frac{5-\frac{4}{n}+\frac{1}{n^2}}{\frac{6}{n}+2}\right)^2 - 10\cdot\lim_{n\to\infty}\frac{5-\frac{4}{n}+\frac{1}{n^2}}{\frac{6}{n}+2}$$

$$= 4\cdot\left(\frac{5-0+0}{0+2}\right)^2 - 10\left(\frac{5-0+0}{0+2}\right)$$

$$= 4\times\frac{25}{4} - 10\times\frac{5}{2}$$

$$\boxed{\therefore \lim_{n\to\infty}\{b_n\} = 0}$$

Week-9

Mathematics for Data Science - 1

Practice Assignment Solution

Note: The numbers given in your assignment may differ in **question-7**. Use the same idea and do it accordingly.

1 Multiple Select Questions (MSQ)

1. Match the given functions in Column A with the (signed) area between its graph and the interval $[-1, 1]$ on the X-axis in column B and the pictures of their graphs and the highlighted region corresponding to the area computation in Column C, given in Table M2W3P1.

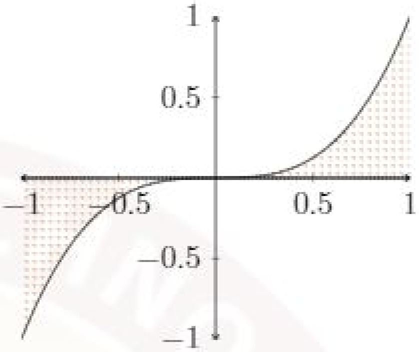
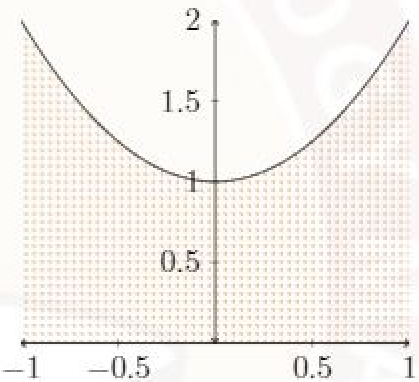
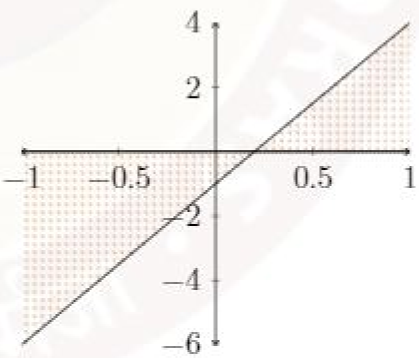
	Functions (Column A)		Area under the curve (Column B)		Graphs (Column C)
i)	$f(x) = 5x - 1$	a)	$\dfrac{\pi}{2}$	1)	
ii)	$f(x) = x^3$	b)	0	2)	
iii)	$f(x) = \dfrac{1}{x^2+1}$	c)	-2	3)	

Table: M2W3P1

◯ **Option 1:** i) $\rightarrow$ c) $\rightarrow$ 3).

◯ Option 2: ii) → a) → 1).

◯ **Option 3:** ii) → b) → 1).

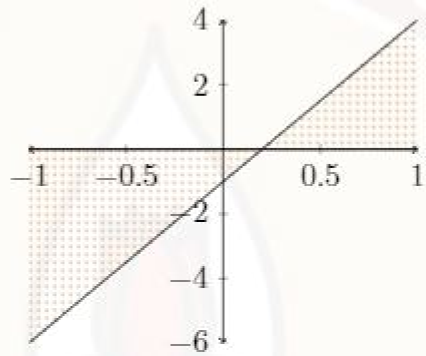
◯ **Option 4:** iii) → a) → 2).

◯ Option 5: iii) → c) → 2)

◯ Option 6: i) → b) → 1)

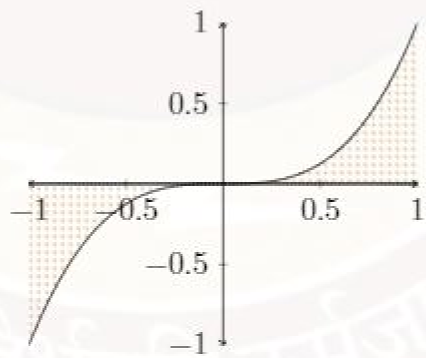
Solution:

- $f(x) = 5x - 1$



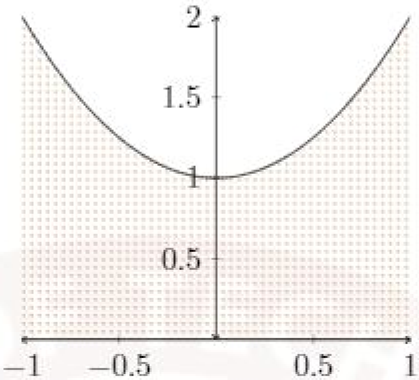
$$\int_{-1}^{1} f(x)\ dx = \int_{-1}^{1} (5x-1)\ dx = \int_{-1}^{1} 5x\ dx - \int_{-1}^{1} 1\ dx = (5 \times \frac{x^2}{2})\Big|_{-1}^{1} - x\Big|_{-1}^{1} = -2.$$

- $f(x) = x^3$



$$\int_{-1}^{1} f(x)\ dx = \int_{-1}^{1} x^3\ dx = \frac{x^4}{4}\Big|_{-1}^{1} = 0.$$

- $f(x) = \dfrac{1}{x^2+1}$



$$\int_{-1}^{1} f(x)\, dx = \int_{-1}^{1} \frac{1}{x^2+1}\, dx = \tan^{-1}(x)\Big|_{-1}^{1} = \tfrac{\pi}{2}.$$

2. Suppose $\int x\ln(1+x)dx = f(x)\ln(x+1) - \frac{x^2}{4} + Ax + B$, where B is the constant of integration. Which of the following are correct?

- ○ **Option 1:** $f(x) = \frac{x^2-1}{2}$
- ○ Option 2: $f(x) = \frac{x^2-1}{4}$
- ○ Option 3: $A = \frac{1}{4}$
- ○ **Option 4:** $A = \frac{1}{2}$

Solution: By using integration by parts:

$$\begin{aligned}
\int x\ln(1+x)\ dx &= \ln(1+x)\int x\ dx - \int\left\{\frac{d(\ln(1+x))}{dx}\int x\ dx\right\} dx \\
&= \frac{x^2\ln(1+x)}{2} - \int \frac{x^2}{2(1+x)}\ dx \\
&= \frac{x^2\ln(1+x)}{2} - \int \frac{x^2+x-x}{2(1+x)}\ dx \\
&= \frac{x^2\ln(1+x)}{2} - \int \frac{x^2+x}{2(1+x)}\ dx + \int \frac{x}{2(1+x)}\ dx \\
&= \frac{x^2\ln(1+x)}{2} - \frac{x^2}{4} + \int \frac{x+1-1}{2(1+x)}\ dx \\
&= \frac{x^2\ln(1+x)}{2} - \frac{x^2}{4} + \int \frac{x+1}{2(1+x)}\ dx - \int \frac{1}{2(1+x)}\ dx \\
&= \frac{x^2\ln(1+x)}{2} - \frac{x^2}{4} + \frac{x}{2} - \frac{1}{2}\ln(1+x) + B \\
&= \frac{x^2-1}{2}\ln(1+x) - \frac{x^2}{4} + \frac{x}{2} + B.
\end{aligned}$$

If we equate the above expression with $f(x)\ln(x+1) - \frac{x^2}{4} + Ax + B$, then we have $f(x) = \frac{x^2-1}{4}$ and $A = \frac{1}{2}$.

3. Consider the function $f(x) = x^3 - 6x$. Which of the following options are correct?

   - ◯ Option 1: f has neither local maxima nor local minima.
   - ◯ **Option 2**: $\sqrt{2}$ is a local minimum.
   - ◯ Option 3: $\sqrt{2}$ is a local maximum.
   - ◯ **Option 4:** $-\sqrt{2}$ is a local maximum.
   - ◯ Option 5: $-\sqrt{2}$ is a local minimum.
   - ◯ **Option 6:** f has two critical points.

Solution: The function f is differentiable on $\mathbb{R}$. Number of critical points will be same as the number of solutions of the following equation:

$$f'(x) = 3x^2 - 6 = 0 \implies x^2 - 2 = 0 \implies (x - \sqrt{2})(x + \sqrt{2}) = 0.$$

Hence, the number of critical points is 2.

<u>when x = $\sqrt{2}$:</u> $f''(\sqrt{2}) > 0$. Therefore, $\sqrt{2}$ is a local minimum.

<u>when x = $-\sqrt{2}$:</u> $f''(-\sqrt{2}) < 0$. Therefore, $-\sqrt{2}$ is a local maximum.

4. Choose the set of correct options.

- ○ **Option 1:** The left Riemann sum of the function $f(x) = x + 5$ on the interval $[1, 10]$ divided into three sub-intervals of equal length is 81.
- ○ **Option 2:** The middle Riemann sum of the function $f(x) = x^2$ on the interval $[0, 8]$ divided into four sub-intervals of equal length is 168.
- ○ **Option 3:** The left Riemann sum of the function $f(x) = x + 5$ on the interval $[3, 6]$ divided into n sub-intervals of equal length is $\frac{57}{2}$, as n tends to ∞.
- ○ Option 4: The right Riemann sum of the function $f(x) = \frac{1}{x}$ on the interval $[1, 9]$ divided into four sub-intervals of equal length is $\frac{16}{15}$.

Solution:

Option 1: If we divide $[1, 10]$ in three sub-intervals of equal length, we get the partition: $\{1, 4, 7, 10\}$. The left Riemann sum of the function $f(x) = x + 5$ is:

$$(4-1)f(1)+(7-4)f(4)+(10-7)f(7) = 3\times(f(1))+f(4)+f(7)) = 3\times(6+9+12) = 81$$

Option 2: If we divide $[0, 8]$ in four sub-intervals of equal length, we get the partition: $\{0, 2, 4, 6, 8\}$. The middle Riemann sum of the function $f(x) = x^2$ is:

$$(2-0)f(1)+(4-2)f(3)+(6-4)f(5)+(8-6)f(7) = 2\times(f(1))+f(3)+f(5)+f(7)) = 2 \times (1 + 9 + 25 + 49) = 168$$

Option 3: If we divide $[3, 6]$ in n sub-intervals of equal length, we get the partition: $\{3, 3 + \frac{3}{n}, 3 + \frac{6}{n} \ldots 6 - \frac{3}{n}, 6\}$. The left Riemann sum of the function $f(x) = x + 5$ is:

$$\frac{3}{n}f(3)+\frac{3}{n}f\Big(3+\frac{3}{n}\Big)+\frac{3}{n}f\Big(3+\frac{6}{n}\Big)+\cdots+\frac{3}{n}f\Big(6-\frac{3}{n}\Big)$$

$$=\frac{3}{n}\Bigg[(3+5)+(3+\frac{3}{n}+5)+(3+\frac{6}{n}+5)+\cdots+(6-\frac{3}{n}+5)\Bigg]$$

$$=\Big(\frac{3}{n}\times 5n\Big)+\frac{3}{n}\Bigg[3+(3+\frac{3}{n})+(3+\frac{6}{n})+\cdots+(6-\frac{3}{n})\Bigg]$$

$$=15+\frac{3}{n}\Bigg[3+(3+\frac{3}{n})+(3+\frac{6}{n})+\cdots+\Big(3+(3-\frac{3}{n})\Big)\Bigg]$$

$$=15+\Big(\frac{3}{n}\times 3n\Big)+\frac{3}{n}\Bigg[\frac{3}{n}+\frac{6}{n}+\cdots+\frac{3n-3}{n}\Bigg]$$

$$=15+9+\frac{9}{n^2}\Bigg[1+2+\cdots+(n-1)\Bigg]$$

$$=24+\frac{9}{n^2}\frac{(n-1)n}{2}$$

As n tends to ∞, the above sum converges to $24+\frac{9}{2}=\frac{57}{2}$.

Option 4: If we divide $[1,9]$ in four sub-intervals of equal length, we get the partition: $\{1,3,5,7,9\}$. The right Riemann sum of the function $f(x)=\frac{1}{x}$ is:

$(3-1)f(3)+(5-3)f(5)+(7-5)f(7)+(9-7)f(9)=2\times(f(3))+f(5)+f(7)+f(9))=$

$2\times(\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{9})=\frac{496}{315}$

2 Numerical Answer Type (NAT)

5. The value of $\displaystyle\int_0^{\frac{\pi^2}{4}} \sin\sqrt{x}\; dx$ is [Answer: 2]

Solution:

We make the substitution $t^2 = x \implies \dfrac{dx}{dt} = 2t \implies dx = 2tdt$, and the limits change to $t = \sqrt{0} = 0$ and $t = \sqrt{\frac{\pi^2}{4}} = \frac{\pi}{2}$. The integral becomes

$$2\int_0^{\frac{\pi}{2}} t\sin t\; dt.$$

Now, $2\displaystyle\int_0^{\frac{\pi}{2}} t\sin t\; dt = \Big[-tcos(t) + sin(t)\Big]_0^{\frac{\pi}{2}} = 2$ [By using integration by parts].

6. Suppose $x + y = 16$. What is the value of xy when $x^3 + y^3$ is minimum? [Answer: 64]

Solution: It is given that $x + y = 16 \implies y = 16 - x$. so, $x^3 + y^3 = x^3 + (16 - x)^3$. Let $f(x) = x^3 + (16 - x)^3 = x^3 + 16^3 - x^3 - 768x + 48x^2 = 16^3 - 768x + 48x^2$. To get a minima we can equate $\frac{df}{dx}$ to 0.

$$\frac{df}{dx} = -768 + 96x = 0 \implies x = 8.$$

The second derivative is $\frac{d^2f}{dx^2} = 96 > 0$. Therefore $x^3 + y^3$ is minimum when $x = 8$ and $y = 16 - x = 8$ and the value of xy is 64.

7. Suppose a wire of length 31 is cut into two pieces. One part is bent into a circle and other into a square. Let x be the minimum value of the combined area of the circle and the square. Then the value of $4x(\pi + 4)$ is

Solution: Let the piece that is bent into a circle have length y and the remaining piece of wire that bent into a square have length $31-y$. Then the radius of the circle is $r = \dfrac{y}{2\pi}$ and side of the square is $a = \dfrac{31-y}{4}$.

$$\begin{aligned} \text{A} = \text{Total area} &= \text{Area of the circle} + \text{Area of the square} \\ &= \pi r^2 + a^2 \\ &= \pi \times \left(\frac{y}{2\pi}\right)^2 + \left(\frac{31-y}{4}\right)^2 \\ &= \frac{y^2}{4\pi} + \frac{(31-y)^2}{16} \end{aligned}$$

To get the minima we can equate $\dfrac{dA}{dy}$ to 0.

$$\frac{dA}{dy} = \frac{y}{2\pi} - \frac{(31-y)}{8} = 0 \implies \frac{y}{2\pi} = \frac{(31-y)}{8} \implies y = \frac{31\pi}{4+\pi}$$

The second derivative $\dfrac{d^2A}{dy^2} = \dfrac{1}{2\pi} + \dfrac{1}{8}$ is always greater than 0. Hence, $y = \dfrac{31\pi}{4+\pi}$ is a point of minimum, that is, the combined area is minimum when $y = \dfrac{31\pi}{4+\pi}$.

Then the minimum value of the combined area is

$$\begin{aligned}
x &= \pi r^2 + a^2 \\
&= \pi\Big(\frac{y}{2\pi}\Big)^2 + \Big(\frac{31-y}{4}\Big)^2 \\
&= \frac{(31\pi)^2}{4\pi(4+\pi)^2} + \frac{(4\times 31 + 31\pi - 31\pi)^2}{16(4+\pi)^2} \\
&= \frac{(31\pi)^2}{4\pi(4+\pi)^2} + \frac{(4\times 31)^2}{16(4+\pi)^2} \\
&= \frac{(31\pi)^2}{4\pi(4+\pi)^2} + \frac{31^2}{(4+\pi)^2} \\
&= \frac{31^2\pi}{4(4+\pi)^2} + \frac{31^2}{(4+\pi)^2} \\
&= \frac{31^2\pi + 4\times 31^2}{4(4+\pi)^2} \\
&= \frac{31^2(\pi+4)}{4(4+\pi)^2} \\
&= \frac{31^2}{4(4+\pi)}
\end{aligned}$$

The value of $4x(\pi+4)$ is $31^2 = 961$.

3 Comprehension Type Question:

A car manufacturer determines that in order to sell x number of cars, the price per car(in lakh) must be $f(x) = 1000 - x$, if $x \leq 800$, and the manufacturer also determines that the total cost(in lakh) of producing x number of cars is

$$g(x) = \begin{cases} 30000 + 300x & \text{if } x \leq 400, \\ 100x + 110000 & \text{if } 400 < x \leq 800 \end{cases}$$

Although in the above context, x can take only integer values, assume that x is a continuous variable in the interval $[0, 800]$ and that the functions $f(x)$ and $g(x)$ are defined as above on this entire interval.
Answer Questions 8,9, and 10 using the data given above.

8. Suppose the company can produce a maximum of 400 cars due to a production issue. The number of cars the company should produce and sell in order to maximize profit is

Solution: If the company sells x number of cars then the total income is $I(x) = x(1000 - x)$. Total profit of the company is:

$$\text{Profit} = \text{Total income - Total cost}$$
$$P(x) = I(x) - g(x)$$

$$P(x) = \begin{cases} x(1000 - x) - (30000 + 300x) & \text{if } x \leq 400, \\ x(1000 - x) - (100x + 110000) & \text{if } 400 < x \leq 800, \end{cases}$$

which is same as:

$$P(x) = \begin{cases} -x^2 + 700x - 30000 & \text{if } x \leq 400, \\ -x^2 + 900x - 110000 & \text{if } 400 < x \leq 800 \end{cases}$$

It is given that the company can produce a maximum of 400 cars due to a production issue, i.e, $x \leq 400$. The total profit is: $P(x) = -x^2 + 700x - 30000$.

To get a maxima we can equate $\frac{dP}{dx}$ to 0.

$$\frac{dP}{dx} = -2x + 700 = 0 \implies x = 350.$$

The second derivative is $\frac{d^2P}{dx^2} = -2 < 0$. Hence, $x = 350$ is a point of maximum. Therefore, company should produce and sell 350 number of cars in order to maximize its profit.

9. Suppose the company can produce a minimum of 401 cars and a maximum of 800 cars due to a production issue. The number of cars the company should produce and sell in order to maximize profit is

   Solution: It is given that the company can produce a minimum of 401 cars and a maximum of 800 cars due to a production issue, i.e, $401 \leq x \leq 800$. The total profit is: $P(x) = -x^2 + 900x - 110000$.

   To get a maxima we can equate $\dfrac{dP}{dx}$ to 0.

   $$\frac{dP}{dx} = -2x + 900 = 0 \implies x = 450.$$

   The second derivative is $\dfrac{d^2P}{dx^2} = -2 < 0$. Hence, $x = 450$ is a point of maximum. Therefore, company should produce and sell 450 numbers of cars in order to maximize its profit.

10. Let $P(x)$ denotes the function representing the profit of the company. Choose the set of correct statements.

   ◯ **Option 1:** $P(x)$ is continuous in the interval $[0, 800]$

   ◯ **Option 2:** The function $P(x)$ has two local maxima in the interval $[0, 800]$.

   ◯ Option 3: All the global maxima of $P(x)$ lie in the interval $[0, 400]$.

   ◯ **Option 4:** All the global maxima of $P(x)$ lie in the interval $[300, 500]$.

Solution:

$$P(x) = \begin{cases} -x^2 + 700x - 30000 & \text{if } x \leq 400, \\ -x^2 + 900x - 110000 & \text{if } 400 < x \leq 800 \end{cases}$$

Domain of P is $[0, 800]$. Clearly, P is continuous $[0, 400) \cup (400, 800]$. So, we need to check the continuity of the function only at $x = 400$.

LHL of $\mathbf{P(x)}$ at $\mathbf{x = 400}$:

$$\lim_{x \to 400^-} P(x) = \lim_{x \to 400^-} -x^2 + 700x - 30000 = -(400)^2 + 280000 - 30000 = 90000$$

RHL of $\mathbf{P(x)}$ at $\mathbf{x = 400}$:

$$\lim_{x \to 400^+} P(x) = \lim_{x \to 400^+} -x^2 + 900x - 110000 = -(400)^2 + 360000 - 110000 = 90000$$

Hence, $P(x)$ is continuous in the interval $[0, 800]$. From the solutions of Q8 and Q9, it is clear that the function $P(x)$ has two local maxima in the interval $[0, 800]$. Now,

$P(0) = -30000$, $P(350) = 92500$, $P(400) = 90000$, $P(450) = 92500$, and $P(800) = -30000$.

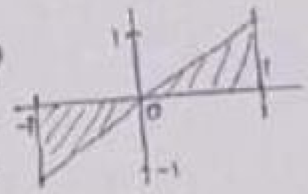
So, both $x = 350$ and $x = 450$ are global maxima of $P(x)$.

MATHEMATICS FOR DATA SCIENCE - 1

WEEK 9 GRADED ASSIGNMENT

(1) Match functions with graphs and area under the curve.

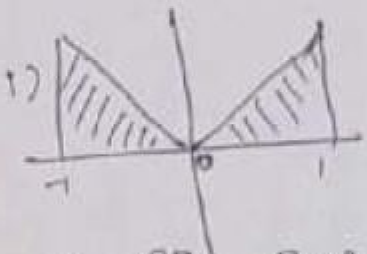
(i) $f(x) = x$

3) 

$$\int_{-1}^{1} f(x)\,dx = \int_{-1}^{1} x\,dx = \left[\frac{x^2}{2}\right]_{-1}^{1} = 0.$$

(i) $\longrightarrow$ (b) $\longrightarrow$ (3)

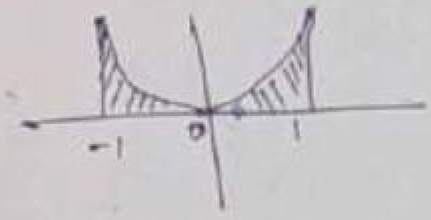
(ii) $f(x) = |x|$

1) 

$$\int_{-1}^{1} f(x)\,dx = \int_{-1}^{1} |x|\,dx$$

$$= \int_{-1}^{0} (-x)\,dx + \int_{0}^{1} x\,dx = \left[-\frac{x^2}{2}\right]_{-1}^{0} + \left[\frac{x^2}{2}\right]_{0}^{1} = 1$$

(ii) $\longrightarrow$ (c) $\longrightarrow$ (1)



(iii) $f(x) = x^2$

$$\int_{-1}^{1} x^2\,dx = \left[\frac{x^3}{3}\right]_{-1}^{1} = \frac{2}{3}$$

(iii) $\longrightarrow$ (a) $\longrightarrow$ (2)

(2) Which of the following curves enclose a negative area on the x-axis in the interval [0, 1]?

Area enclosed above x-axis (+ve direction of y-axis) is positive and area enclosed below x-axis (-ve direction of y-axis) is negative. So if the area enclosed below the x-axis is more than the area enclosed above, then the area enclosed by the curve is negative.

Curve 2 and Curve 4 enclosed negative area.

(3) cylinder of radius x and height $2h$ inscribed in a circle of radius R.

Soln From the right angled Δ^{le} OAB, we have $h^2+x^2=R^2$

For the volume, $V = 2\pi x^2 h = 2\pi(R^2-h^2)h = 2\pi R^2 h - 2\pi h^3$

$$\frac{dV}{dh} = 2\pi R^2 - 6\pi h^2$$

$$\frac{dV}{dh} = 0 \Rightarrow 2\pi R^2 = 6\pi h^2 \Rightarrow h^2 = \frac{R^2}{3} \Rightarrow h = \pm\frac{R}{\sqrt{3}}$$ are critical points

But since h is height of the cylinder, $h = \frac{R}{\sqrt{3}}$

$$\frac{d^2V}{dh^2} = -12\pi h. \quad \frac{d^2V}{dh^2}\left(\frac{R}{\sqrt{3}}\right) = -12\pi \frac{R}{\sqrt{3}} < 0.$$

$\therefore$ Max volume is attained when $\underline{\underline{h = \frac{R}{\sqrt{3}}}}$.

III^{ly} for the surface area, $S = 4\pi x h = 4\pi h\sqrt{R^2-h^2}$

$$\frac{dS}{dh} = 4\pi\left[\sqrt{R^2-h^2} + h\cdot\frac{1}{2\sqrt{R^2-h^2}}(-2h)\right]$$

$$= 4\pi\left[\sqrt{R^2-h^2} \cancel{+} - \frac{h^2}{\sqrt{R^2-h^2}}\right]$$

$$\frac{dS}{dh} = 0 \Rightarrow R^2-h^2 = h^2 \Rightarrow h^2 = \frac{R^2}{2} \Rightarrow h = \pm\frac{R}{\sqrt{2}}$$ are the critical pts

But, again since h is the height, $h = \frac{R}{\sqrt{2}}$

$$\frac{d^2S}{dh^2} = 4\pi\left[\frac{-2h}{\sqrt{R^2-h^2}} + \frac{h^2(2h)}{2(R^2-h^2)^{3/2}} - \frac{2h}{\sqrt{R^2-h^2}}\right] = 4\pi\left[\frac{-4h}{\sqrt{R^2-h^2}} - \frac{h^3}{(R^2-h^2)^{3/2}}\right]$$

$$\frac{d^2S}{dh^2}\left(\frac{R}{\sqrt{2}}\right) = 4\pi\left[\left(-4\times\frac{R}{\sqrt{2}}\times\frac{\sqrt{2}}{R}\right) - \left(\frac{R^3}{2\sqrt{2}}\times\frac{2\sqrt{2}}{R^3}\right)\right] = -20\pi < 0.$$

$\therefore$ Max surface area is attained when $\underline{\underline{h = \frac{R}{\sqrt{2}}}}$

$f_1(x) = x^3$, $f_2(x) = x$; $g_1(x) = \sqrt{x}$, $g_2(x) = e^x$

(4) Note that $f_2(x)$ and $g_2(x)$ are increasing functions. Thus the minimum is attained at 0 (in the interval [0,1])

$f_2(0) = 0$ and $g_2(0) = 1$

$\therefore$ The difference is 1

(5) Error in prediction for company A will be the difference in areas enclosed by curves f_1 and g_1

$$= \left| \int_0^1 (f_1 - g_1)(x) \right| = \left| \int_0^1 f_1(x)\,dx - \int_0^1 g_1(x)\,dx \right|$$

$$= \left| \int_0^1 x^3\,dx - \int_0^1 \sqrt{x}\,dx \right| = \left| \left[\frac{x^4}{4}\right]_0^1 - \left[\frac{x^{3/2}}{3/2}\right]_0^1 \right|$$

$$= \left| \frac{1}{4} - \frac{2}{3} \right| = \frac{5}{12} \,//$$

Error in prediction for company B will be the difference in areas enclosed by f_2 and g_2

$$= \left| \int_0^1 (f_2 - g_2)(x)\,dx \right| = \left| \int_0^1 f_2(x)\,dx - \int_0^1 g_2(x)\,dx \right|$$

$$= \left| \int_0^1 x\,dx - \int_0^1 e^x\,dx \right| = \left| \left[\frac{x^2}{2}\right]_0^1 - \left[e^x\right]_0^1 \right| = e - \frac{3}{2} \,//$$

Clearly $e - \frac{3}{2} > \frac{5}{12}$. So error in prediction for company B is greater than the error in prediction for company A.

(6) $f(x) = x^3 - 3x + 1 \Rightarrow f'(x) = 3x^2 - 3 \rightarrow f''(x) = 6x$

$f'(x) = 0 \Rightarrow x = \pm 1$ (critical points)

$f''(1) = 6 > 0$ – local minimum $\quad f(1) = 1 - 3 + 1 = -1 \,//$

$f''(-1) = -6 < 0$ – local maximum

(7) $f(x) = 2x^2 + \frac{5}{6}$, $0 \le x \le 6$

Dividing $[0,6]$ into 3 sub intervals of equal lengths

$[0,2]$, $[2,4]$, $[4,6]$

Riemann sum $= \sum_{i=1}^{3} f(x_i^*) \Delta x_i$, x_i^* - left end point of the interval

$$= 2f(0) + 2f(2) + 2f(4)$$

$$= 2\left[\frac{5}{6} + \left(8 + \frac{5}{6}\right) + \left(32 + \frac{5}{6}\right)\right] = 2\left(40 + \frac{5}{2}\right) = 85 //$$

(8) $f(x) = \begin{cases} -2x+3 & 0 \le x \le 10 \\ x^2 & 10 < x \le 20 \end{cases}$

$f'(x) = \begin{cases} -2 & 0 < x < 10 \\ 2x & 10 < x < 20 \end{cases}$

$f'(x) \neq 0$ for $x \in (0,10)$. similarly, $f'(x) \neq 0$ for $x \in (10,20)$.

$f(x)$ is not cont at $x = 10$ (hence not differentiable)

So $x = 10$ is a critical point.

$f(0) = 3$; $f(10) = -17$; $f(20) = 400$ (0 & 20 end pts, 10-critical p

Global min attained at $x = 10$. Min. value $= \underline{-17}$.

(9) $x - y = 5 \Rightarrow y = x - 5$

$f(x) = 2xy = 2x(x-5) = 2x^2 - 10x$

$f'(x) = 4x - 10$. $f'(x) = 0 \Rightarrow x = \frac{5}{2}$.

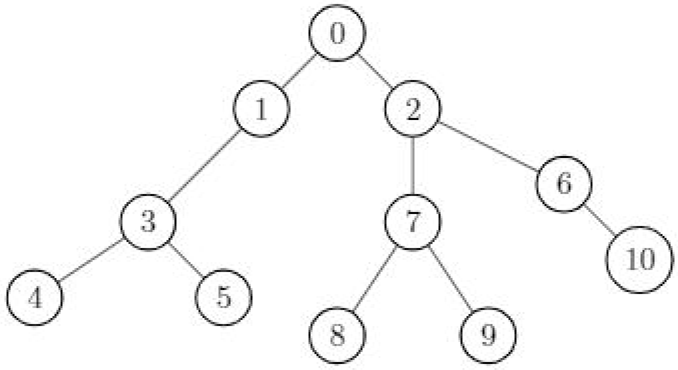
$f''\left(\frac{5}{2}\right) = 4 > 0 \Rightarrow x = \frac{5}{2}$ is a local minimum

$f\left(\frac{5}{2}\right) = 2 \times \frac{25}{4} - 10 \times \frac{5}{2} = \frac{25}{2} - 25 = -\frac{25}{2} //$.

Mathematics for Data Science - 1
Practice Assignment
Week -10

1 MULTIPLE CHOICE QUESTIONS:

1. Suppose we obtain the following DFS tree rooted at node 0 for an undirected graph with vertices $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.



Which of the following cannot be an edge in the original graph? [Ans: c]

(a) $(1, 4)$
(b) $(0, 4)$
(c) $(7, 10)$
(d) $(2, 9)$

Soln :- DFS tree for an undirected graph can have edges only in the same branch because if there is an edge between two vertices in G which are in different branches of DFS tree, then the neighbours of vertex 'u' must be visited in DFS in order to remove it from the stack.

Clearly from Figure 1, we have five branches (b1, b2, b3, b4, b5)

✓ Option (a) :- Vertices (1, 4) $\in$ branch(b1)

✓ Option (b) :- Vertices (0, 4) $\in$ branch(b1)

✗ Option (c) :- Vertex (7) $\in$ branch(b3) & vertex (10) $\in$ branch(b5) which cannot form a possible edge in original unrooted DFS tree.

✓ Option (d) :- Vertex (2, 9) $\in$ branch(b4)

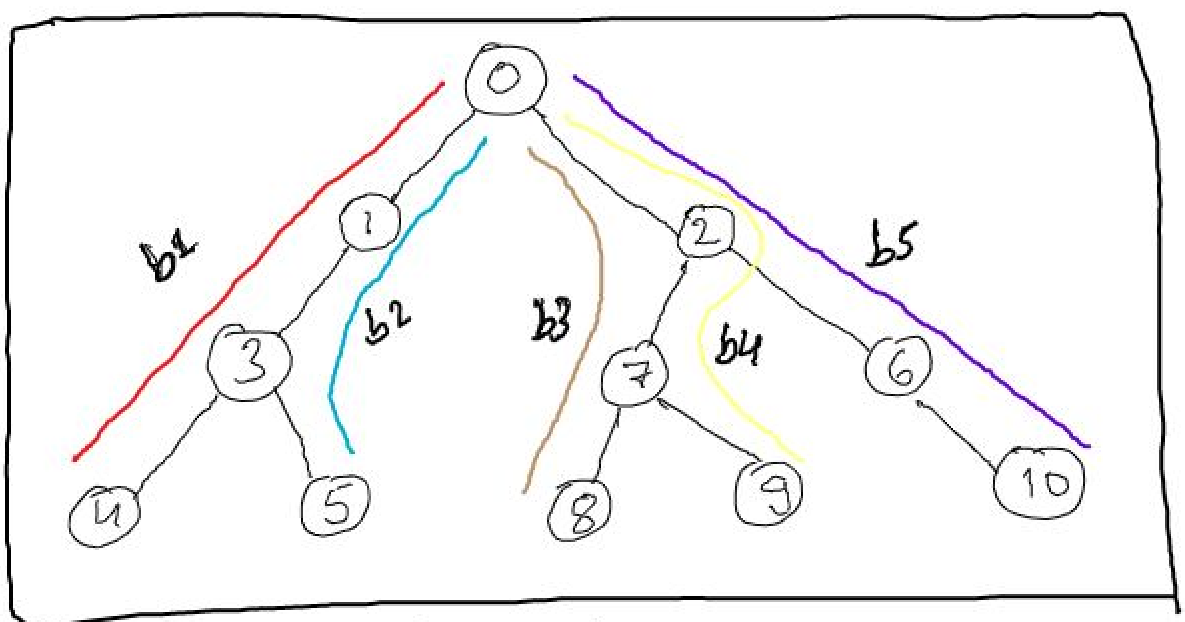
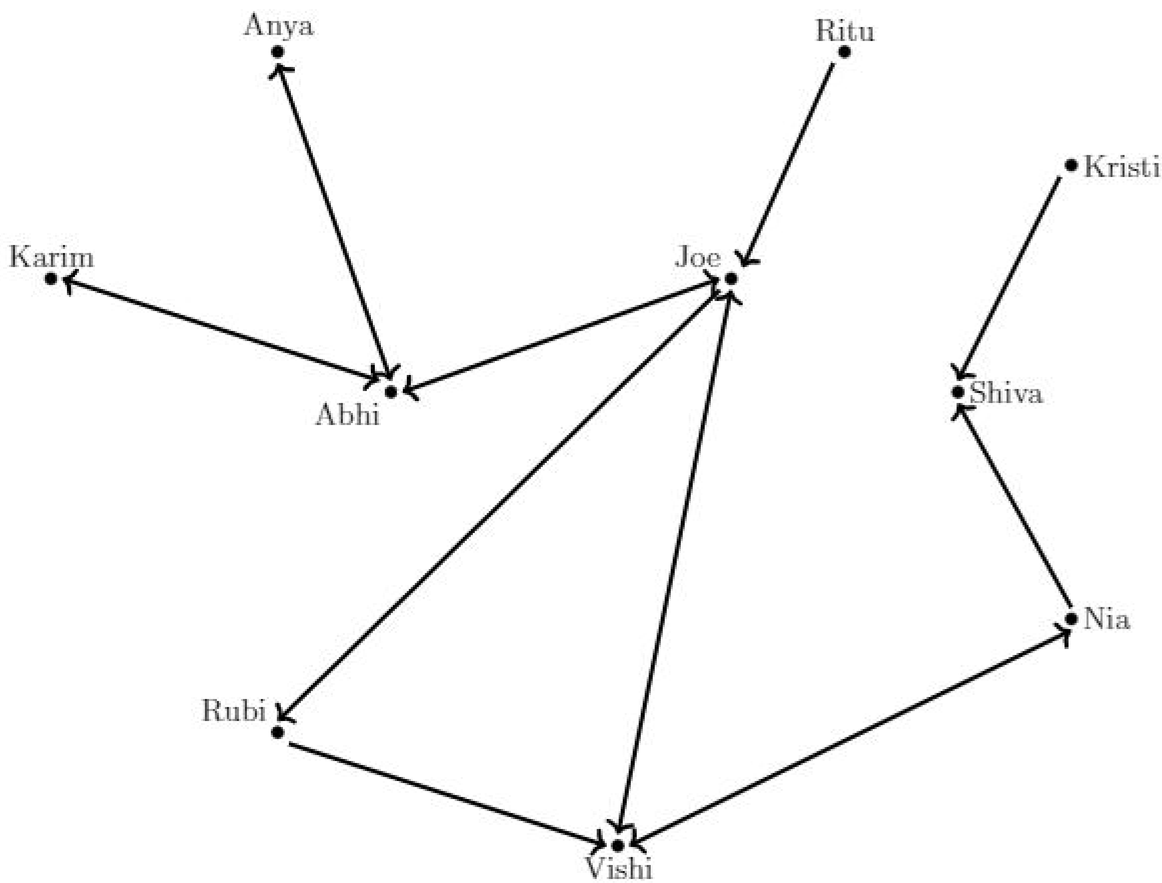


Figure 1

Use the following information for the questions 2-4:
Ten friends in a college decided to have a night party at the home of one of them. Unfortunately at D-day the government closes many of the routs of the city. The below graph shows the location of their homes and the open routs as the graph $G = (V, E)$, where V is the set of nodes and E is the edges representing the open routs.



2. The possible place for the party is. [Ans: a]

    (a) Shiva's house.
    (b) Abhi's house.
    (c) Joe's house.
    (d) Vishi's house.

Soln:-

Given that $G = (V, E)$ where V is set of nodes representing a person's house & E is the edge representing the open routes. Let us consider the rechability of each node

Note: Vertex (v) is rechable from vertex (u) if there is a path from u to v, where $u \& v \in V$

(i) Kristi's & Ritu's house are not recheable by anyone

(ii) Anya's, Karim's, Abhi's, Joe's, Rubi's, Vishi's and Nia's house are not recheable by Kristi and Shiva

(iii) Shiva's house is recheable by everyone so the best possible place for the party is Shiva's house as this node is recheable from every other nodes.

3. Let $V_1 = \{\text{Kristi, Shiva, Nia}\}$ and $E_1 = E \cap (V_1 \times V_1)$, that is, E_1 is the subset of edges of G with both end points in V_1. Choose the correct option. [Ans: d]

   (a) $G_1 = (V_1, E_1)$ is an undirected graph.

   (b) $G_1 = (V_1, E_1)$ is a cyclic graph.

   (c) $G_1 = (V_1, E_1)$ will not be a graph.

   (d) $G_1 = (V_1, E_1)$ is a directed graph.

Soln

Given that: $V_1 = \{\text{Kristi, Shiva, Nia}\}$

$V_1 \times V_1 = \{(\text{Kristi, Shiva}), (\text{Shiva, Kristi}), (\text{Kristi, Nia}), (\text{Nia, Kristi}), (\text{Shiva, Nia}), (\text{Nia, Kristi})\}$

Note: $V_1 \times V_1$ represents the interconnected path in V_1

$E = \{(\text{Kristi, shiva}), (\text{Nia, shiva}), \ldots\}$

$E_1 = E \cap (V_1 \times V_1) = \{(\text{Kristi, shiva}), (\text{Nia, shiva})\}$

So, clearly $G_1 = (V_1, E_1)$ is a directed graph as E_1 is directed. Figure 1.1 shows the subgraph G_1.

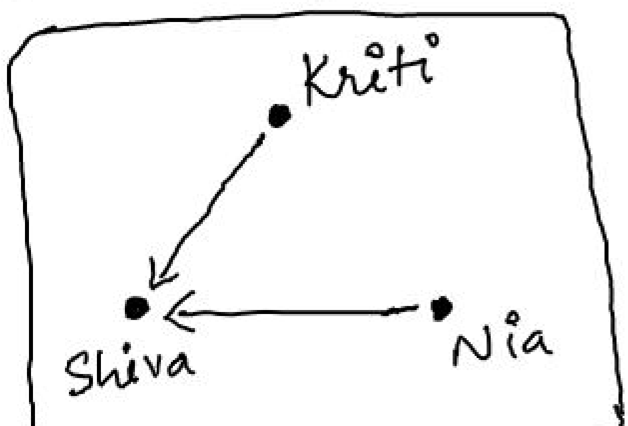


Figure 1-1: $G_1 = (V_1, E_1)$

4. If Joe wants to have the party on his home, then at most how many members can join the party. [Ans: d]

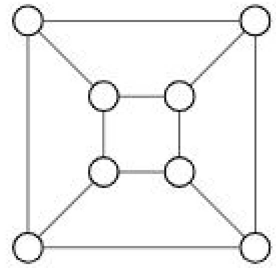
   (a) 5.
   (b) 6.
   (c) 7.
   (d) 8.

Soln

As seen earlier Joe's home was not reachable by kristi and Shiva & there are 10 members altogether. Thus 8 members in total will join the party.

2 MUTIPLE SELECT QUESTIONS:

5. Suppose G be a graph (shown in the below figure). Let V be the set of vertices of G, V_i be the maximum independent set and V_c be the minimum vertex cover. Which of the followings is(are) true? [Ans: a,d]



(a) Cardinality of V_i is 4.

(b) Cardinality of V_c is 3.

(c) Cardinality of V_i is 5.

(d) Cardinality of V_c is 4.

Soln Vertex cover:

In a graph G, vertex cover is the set of vertices that includes at least one end point of every edge of the graph.

So, minimum vertex cover (V_c) is a vertex cover having smallest possible number of vertices.

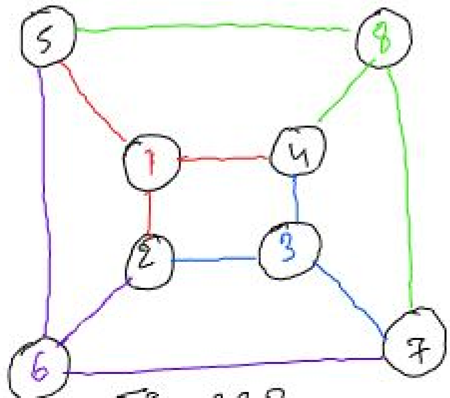


Figure 2

From Figure 2, one of the possible minimum vertex cover $(V_c) = \{1, 3, 6, 8\}$

Thus, cardinality of $(V_c) = 4$

Independent set:

Given a graph $G = (V, E)$, where V is vertex & E is edges, $F \subseteq V$ is an independent set if there are no edges between vertices in F.

Maximum independent set (V_i) is said to be maximal if no vertex of 'G' can be added to F

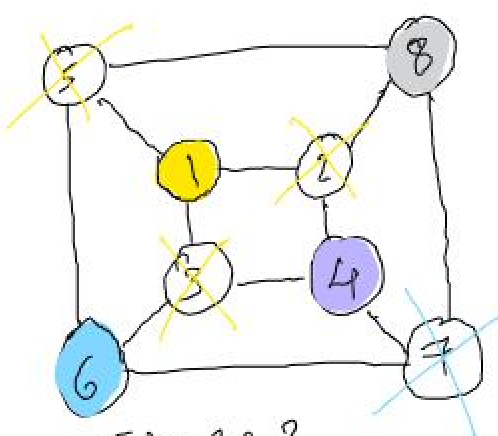
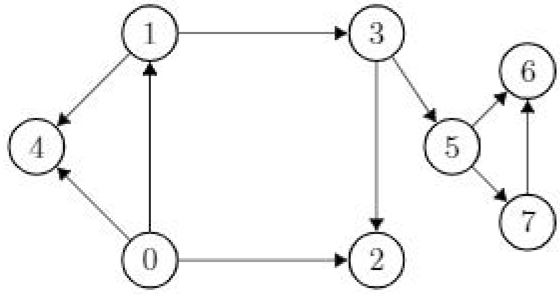


Figure 3

From Figure 3, one of the possible values of $V_i = \{1, 4, 6, 8\}$

Thus, Cardinality of $V_i = 4$

6. Consider the graph given below.



Suppose we perform BFS/DFS so that when we visit a vertex, we explore its unvisited neighbours in a random order. Which of the following options are correct? [a,c,d]

(a) If we perform Breadth First Search at node 0, then one of the possible order in which the nodes will be visited is 01423567.

(b) If we perform Depth First Search at node 0, then one of the possible order in which the nodes will be visited is 04123576

(c) If we perform Breadth First Search at node 0, then one of the possible order in which the nodes will be visited is 01423576.

(d) If we perform Depth First Search at node 0, then one of the possible order in which the nodes will be visited is 04132567.

Soln

BFS tree from node 0, for the given graph could be as follows

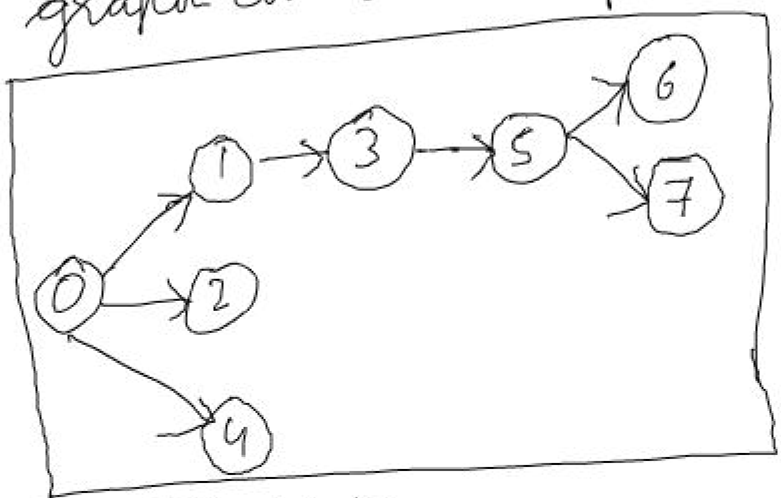


Figure 4

Explore 0, visit 1, 4, 2
Explore 1, visit 3
Explore 4
Explore 2
Explore 3, visit 5
Explore 5, visit 6, 7
Explore 6
Explore 7

$\boxed{0\ 1\ 4\ 2\ 3\ 5\ 6\ 7}$

Explore 0, visit 1, 4, 2
Explore 1, visit 3
Explore 4,
Explore 2
Explore 3, visit 5
Explore 5, visit 7, 6
Explore 7
Explore 6

$\boxed{0\ 4\ 1\ 2\ 3\ 5\ 7\ 6}$

These are the two possible orders however more orders are possible too.

DFS tree from the node 0, for the given graph could be as follows.

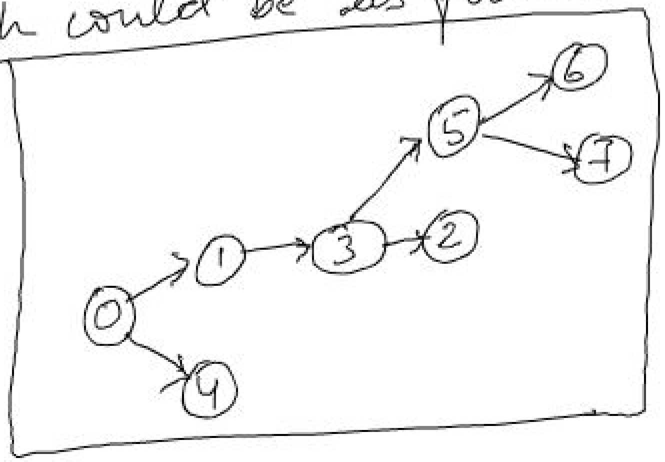


Figure 5

One of the possible order is 0 4 1 3 2 5 6 7.

Thus option (a), (c), (d) are right options

7. Which of the followings options are correct? [Ans: a,b,c,d]

(a) In Depth First search of a directed graph only back edges generate cycles.

(b) If the maximum independent set of a graph G contains only 1 element, then the graph must be connected.

(c) If we add an edge to a tree T, then the resulting graph becomes cyclic.

(d) In a connected graph G having n vertices, at least two vertices have same degree.

Soln

(a) In a DFS, the vertices $v_0, v_1, v_2, \ldots v_{n-1}$ are connected by outward edge.
Suppose there is a backward edge (v_i, v_j) where $j < i$ for some $i, j \in \{0, 1, \ldots n-1\}$ then $v_j \to v_{j+1} \to v_{j+2} \to \cdots v_{i-1} \to v_i$ is a cycle.

(b) Consider a graph G, which is disconnected, then atleast the graph has 2 components of connected graph. Then, the maximum idependent set of a graph G contains more than 2 elements in independent set.
Thus, for a graph with 1 element in maximum independent set must be a connected graph.

(c) Suppose in a tree there exists a path from vertex i to vertex j, for all $i, j \in V$

Now if we add an edge 'x' connecting vertex i to vertex j then there exists a path from vertex i to vertex j which forms cycle as shown below in Figure 6

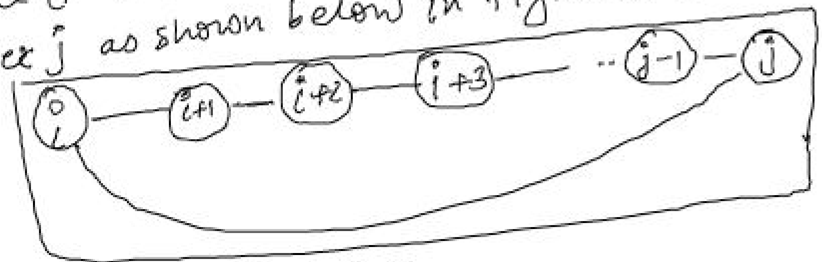
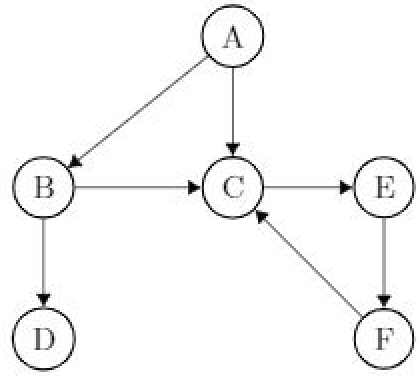


Figure 6

Thus if we add an edge to a tree T, then the resulting graph becomes cyclic.

(d) Let $V_1, V_2, V_3 \ldots V_n$ be the vertices and let V_1 has degree 1, V_2 has degree 2, V_3 has degree 3, … V_{n-1} has degree $n-1$. Now the degree of vertex V_n should choose from $\{1, 2, 3, \ldots n-1\}$ which is one of the degree of the above vertices i.e., if the degree of the vertex V_n is 'k', then V_k and V_n has the same degree 'k'.

8. Consider the following directed graph.



Suppose DFS of this graph is performed from node A, such that when we visit a vertex, we explore its unvisited neighbours in alphabetical order.

Which of the following options are correct? [Ans: a,d]

(a) AC is a forward edge.

(b) CE is a backward edge.

(c) BD is a forward edge.

(d) FC is a backward edge.

Soln DFS tree of the given graph when performed from node A is shown in Figure 7

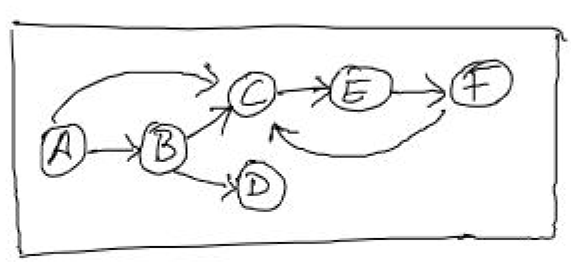


Figure 7

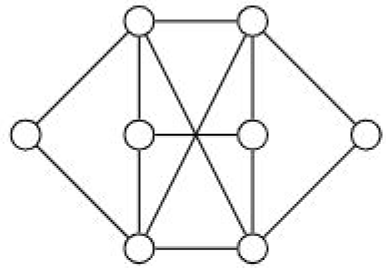
The only forward edge is A → C } Refer Figure
and only backward edge is F → C }

Rest all edges are the normal edge of DFS tree.

Therefore, AC forms forward edge & FC forms backward edge respectively.

3 NUMERICAL ANSWER TYPE:

9. The cardinality of the maximum independent set of the graph given below is [ans: 4]



Soln One of the possible way to find the maximum independent set is shown in Figure 8.

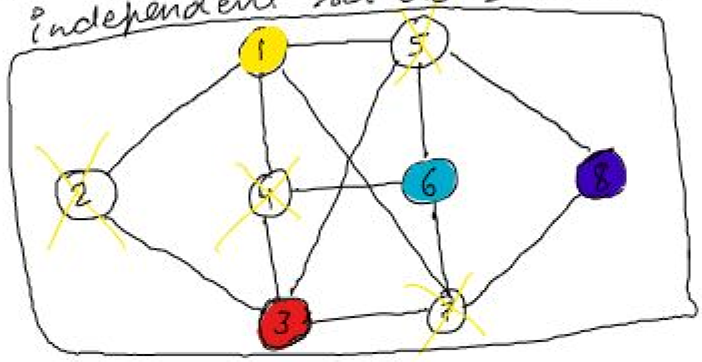
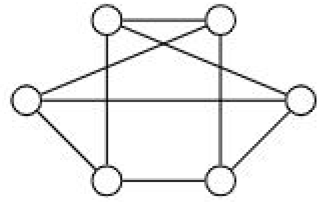


Figure 8

$V_i = \{1, 3, 6, 8\}$

Cardinality of $V_i = 4$

10. The minimum colouring of the below graph is [Answer: 2]



Soln We know that, graph $G = (V, E)$, set of colors C coloring is a function $c : V \to C$ such that $(u, v) \in E \Rightarrow c(u) \neq c(v)$

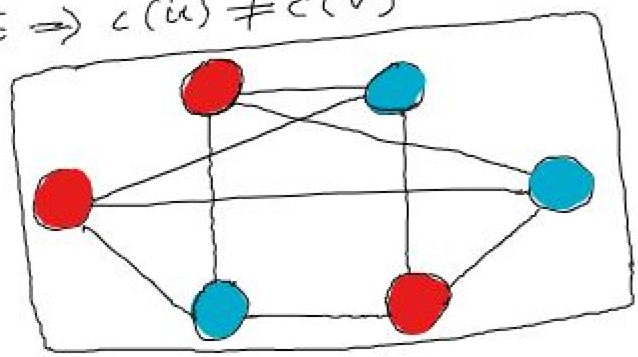


Figure 9

In simplest form, it is a way of coloring the vertices of a graph such that no two adjacent vertices are of the same color.

From Figure 9 the minimum coloring required is 2.

Mathematics for Data Science - 1
Graded Assignment
Week 10

1 MULTIPLE CHOICE QUESTIONS:

1. The maximum number of non-zero entries in an adjacency matrix of a simple graph having n vertices can be [option: d]

   (a) n^2

   (b) $\frac{n(n-1)}{2}$

   (c) $\frac{n(n-1)}{4}$

   (d) $n(n-1)$

   Solution:
   Number of non zero entry means number of ones in the adjacency matrix which is equal to the sum of the degrees of all vertices. In a graph a vertex can have maximum $n-1$ degree. In at most case if each vertex has the degree $n-1$, then the sum of degrees of all vertex will be $(n-1)+(n-1)+\ldots..n\ times$ which means $n \times (n-1)$.

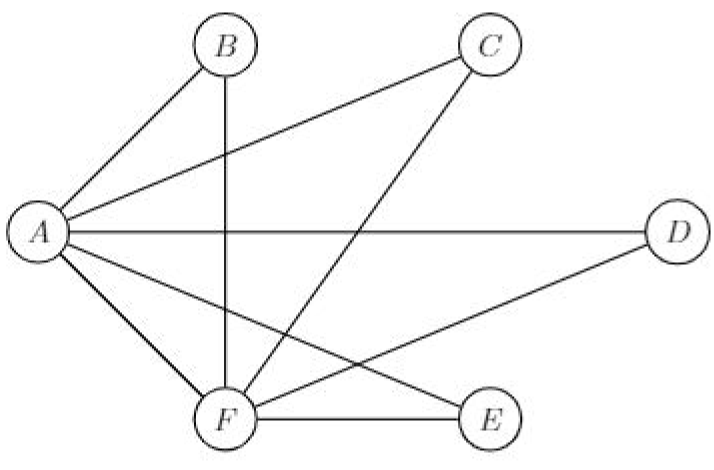
2. We have a graph G with 6 vertices. We write down the degrees of all vertices in G in descending order. Which of the following is a possible listing of the degrees? [option: c]

   (a) 6,5,4,3,2,1
   (b) 5,5,2,2,1,1
   (c) 5,3,3,2,2,1
   (d) 2,1,1,1,1,1

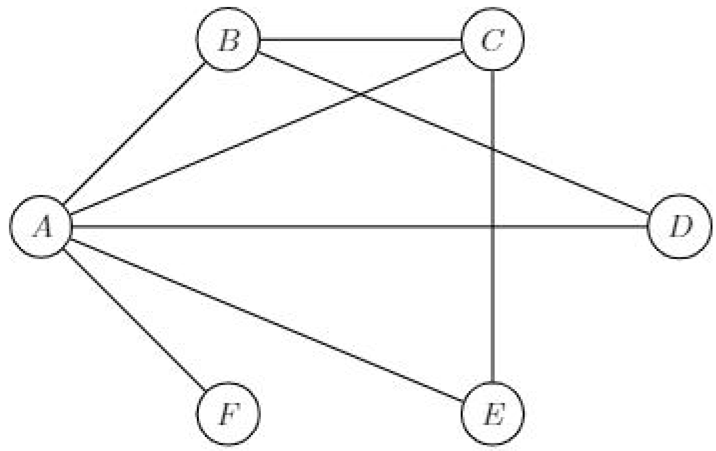
   Solution:
   Step 1:
   Any vertex in a graph can have maximum degree (n-1).
   Here n-1 = 6-1 = 5, therefore we do not need to check option 1.

   Step 2:
   Sum of degree of all vertices always be even therefore option 4 can not be correct option.

   Step 3:
   If two vertices in a graph have (n-1) degree it means there will be no vertex with degree 1 (shown in figure). Vertices A and F have degree 5.



   Step 4:
   Option 3 satisfies all the possible conditions therefore the correct option is option 3.



B
C
A
D
F
E

3. We are trying to find the correct path in a maze. We start at the entrance. At some points, we have to choose a direction to explore. If we reach a dead end, we come back to the most recent intersection where we still have an unexplored direction to investigate. What is a good data structure to keep track of the intersections we have visited? [option: b]

   (a) List
   (b) Stack
   (c) Queue
   (d) Array

   Solution:
   This is a recursive exploration of the maze, so intermediate stages should be stored on a stack.

4. Below table shows the adjacency list w.r.t outgoing edges of a directed graph G.

1	{2,4}
2	{3,5,6}
3	{7}
4	{3,5,6}
5	{6,7}
6	{1}
7	{1,2,6}

Table 1: adjacency list w.r.t forward edges

Which of the following tables shows the adjacency list w.r.t incoming edges of the graph G? [option: c]

1	{6,7}
2	{1,6}
3	{2,4}
4	{1}
5	{2,7}
6	{2,4,5,7}
7	{3,5}

(a)

1	{6,7}
2	{1,7}
3	{2,4}
4	{1,5}
5	{2,4}
6	{2,4,7}
7	{3,5}

(b)

1	{6,7}
2	{1,7}
3	{2,4}
4	{1}
5	{2,4}
6	{2,4,5,7}
7	{3,5}

(c)

1	{6,7}
2	{1,4}
3	{2,7}
4	{1,5}
5	{2,4}
6	{2,4,7}
7	{3,5}

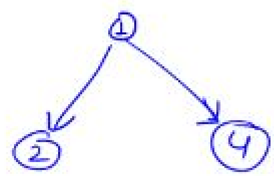
(d)

Solution

If outgoing edges are represented as

1	{2,4}

$\Rightarrow$



There is one edge coming from 1 to 2 and one from 1 to 4. And if we create an adjacency list for incoming edges we will find as:

2	{1}
4	{1}

If we apply the same approch we will get the list as shown in option c.

'1' will have incoming edge from 6 and 7.

'2' will have " " " 1 and 7

'3' " " " " " 2 and 4

'4' " " " " " 1.

'5' " " " " " 2 and 4.

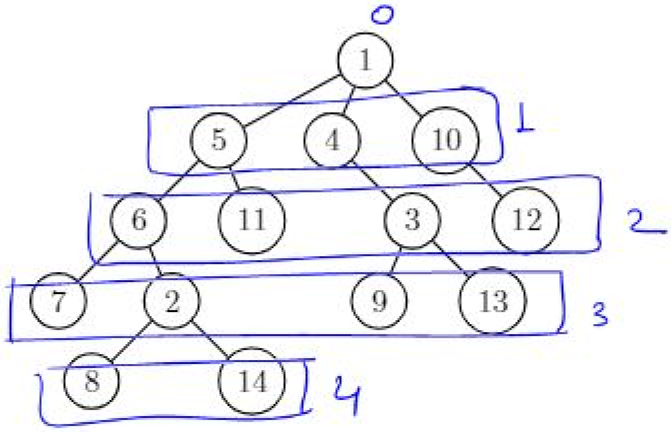
'6' " " " " " 2, 4, 5, and 7.

'7' " " " " " 3 and 5

Therefore:

1	{6, 7}
2	{1, 7}
3	{2, 4}
4	{1}
5	{2, 4}
6	{2, 4, 5, 7}
7	{3, 5}

5. Suppose we obtain the following BFS tree rooted at node 1 for an undirected graph with vertices (1,2,3,4,5,.....14).



Which of the following cannot be an edge in the original graph? [option: A]

(a) (8,11)
(b) (3,10)
(c) (4,5)
(d) (6,9)

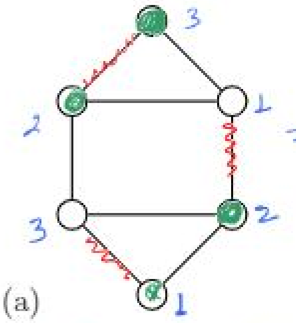
option ⓐ (8, 11) ⇒ level 2 and 4 ie level i and $i+2$. There fore it can not be an edge in original graph.

option ⓑ (3, 10) ⇒ level 2, level 1. ⇒ i and $i+1$ So could be connected

option ⓒ (4, 5) ⇒ level 1, level 1 ⇒ i and i So could be connected.

2 MUTIPLE SELECT QUESTIONS:

6. Which of the following graphs satisfies the below properties:
   1. $|VC(G)| = 3$, where $VC(G)$ is the minimum vertex cover of a graph G.
   2. $|PM(G)| = 3$, where $PM(G)$ is the perfect matching of a graph G.
   3. The graph is a 3-colouring. [option: c,d]

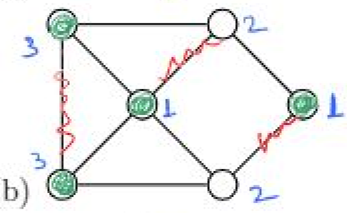


(a)

⇒ 3 colouring

⇒ 3 perfect matching

⇒ minimum vertex cover = 4

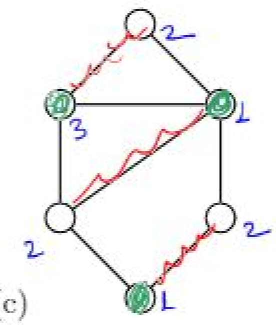


(b)

⇒ 3 colouring

⇒ 3 perfect matching

⇒ minimum vertex cover = 4

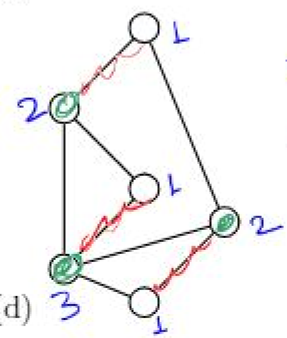


(c)

⇒ 3 colouring

⇒ 3 perfect matching

⇒ minimum vertex = 3



(d)

⇒ 3 colouring

⇒ 3 perfect matching

⇒ minimum vertex cover = 3

7. Which of the following statements is(are) true? [option: a,b]

(a) BFS can be used to identify the vertex which is at the farthest distance from v in any graph, in terms of number of edges, where vertex v is the starting vertex.

(b) BFS and DFS identifies all the vertices reachable from the starting vertex v.

(c) BFS cannot be used to check for cycles in the graph while DFS can be used to check for cycles in the graph.

(d) DFS can be used to identify the shortest distance from starting vertex v to every other vertex in the graph, in terms of number of edges.

Solution:

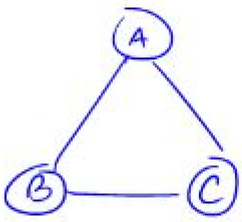
(a) BFS provides a tree level wise which means it shows how many minimum edges will be required to reach from source to a vertex. The levels can be used to identify the vertex which is at the farthest distance from v in any graph, in terms of number of edges, where vertex v is the starting vertex.

(b) BFS and DFS both show the connectivity of vertices from a source vertex which means we can use this to find the reach-ability too. Therefore, option b is correct.

(c) BFS and DFS removes the cycle from original graph so it can not be used to find the cycles.

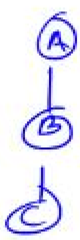
(d) DFS is based on the vertex where to start the search from. Therefore it can not be used to find the shortest distance.

for Example:



The shortest distance or number of edges required to reach from A to C is one.

But if we use DFS then we can find two ways:

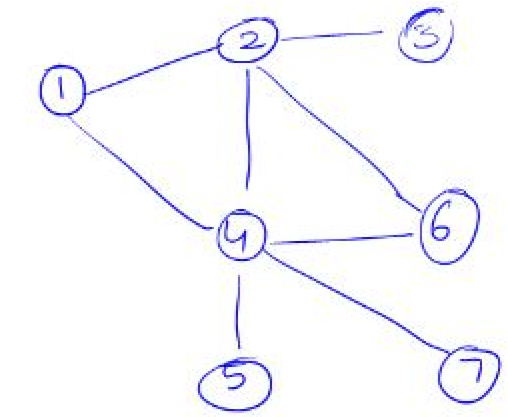


2 edges.

1 edge.

3 NUMERICAL ANSWER TYPE:

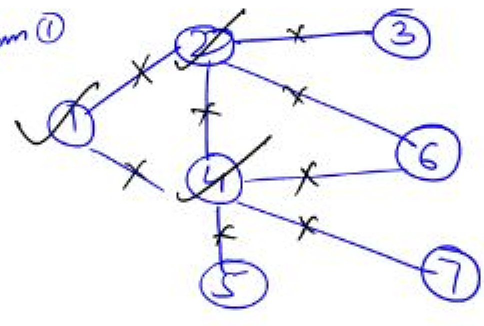
8. If $A = \begin{array}{cc} & \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} & \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$ represents adjacency matrix of a graph G, then the cardinality of the maximum independent set of the graph G is [Answer: 5]



To find the cardinality of maximum independent set we will first find the minimum vertex cover.

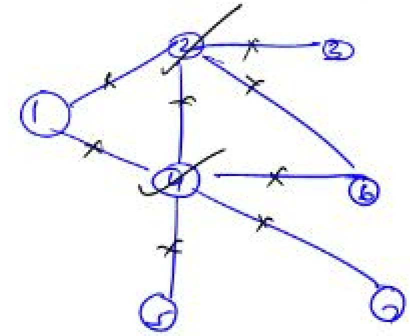
~~ap~~approach 1.

Start from 1



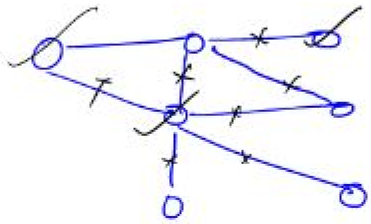
minimal vertex cover = 3.

approach 2 - Starting from 2



minimal vertex cover = 2

Approach 3: starting from (4)



minimal vertex cover = 3

There fore,

we can not get vertex cover less than 2.

Therefore the minimum vertex cover is 2 using the approach (2)

Cardinality of maximum independent set = 7 − 2

minimum vertex cover

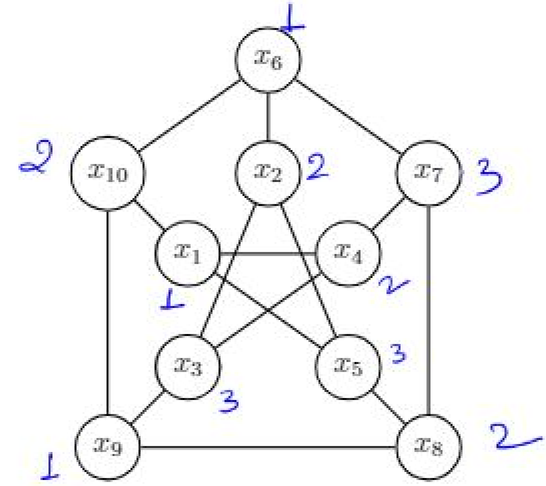
Number of total vertices

= 5

Independent (maximum) set = { 1, 3, 5, 6, 7 }

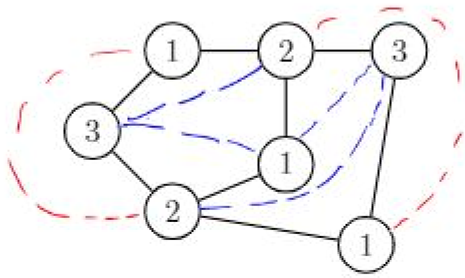
Ans.

9. A company manufactures 10 chemicals $x_1, x_2, x_3, ...x_{10}$. Certain pairs of these chemicals are incompatible and would cause explosions if brought into contact with each other. Below graph shows the incompatibility of the chemicals, each vertex represents the chemical and each edge between a pair of chemicals represents that those two chemicals are incompatible. As a precautionary measure the company wishes to partition its warehouse into compartments, and store incompatible chemicals in different compartments. What is the least number of compartments into which the warehouse should be partitioned?
[Answer: 3]



Minimum number of campartment can be found using the colouring of Graph i.e. 3 Ans.

10. An incomplete undirected graph is given below and the numbering on each vertex denotes the colouring of the graph('1' denotes color 1, '2' denotes color 2, and '3' denotes color 3). Find the number of maximum edges that can be added to the given graph such that the colouring is retained and the graph is planar.
NOTE: Planar graph is a graph that can be drawn on the plane in such a way that its edges intersect only at their endpoints. [Answer: 6]



Total 6 edges.

Mathematics for Data Science - 1
Practice Assignment Solution
Week 11

1 MULTIPLE CHOICE QUESTIONS

1. Consider the following weighted graph in Figure PA-12.1.

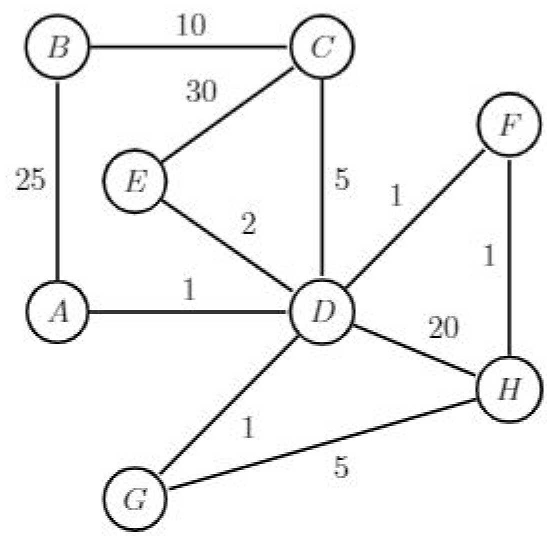


Figure PA-12.1

The shortest weighted path from H to B is **(Ans: d)**

(a) $H \to G \to D \to A \to B$
(b) $H \to D \to C \to B$
(c) $H \to F \to D \to E \to C \to B$
(d) $H \to F \to D \to C \to B$

Solution
As the edge weights are positive we can use Dijkstra's Algorithm.

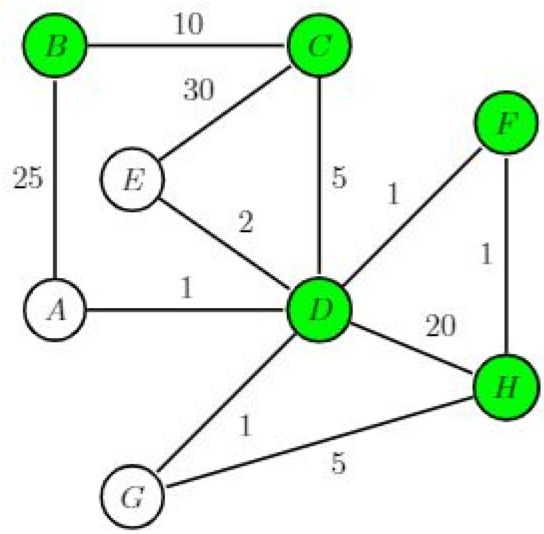


Figure 1

Clearly from Figure 1, the shortest path to reach vertex B from vertex H is $H \to F \to D \to C \to B$ as in this path the total weight is minimum which is 17.

2. Suppose Dijkstra's algorithm is run on the graph below (Figure PA-12.2), starting at node A. In what order do the shortest distances to the other vertices get finalized? **(Ans: a)**

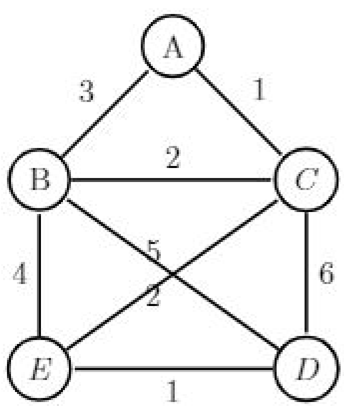


Figure PA-12.2

(a) A, C, B, D, E
(b) A, C, B, E, D
(c) A, C, D, B, E
(d) A, C, D, E, B

Solution

Using Dijkstra's Algorithm the shortest distances to reach other vertices are shown in Figure 2.

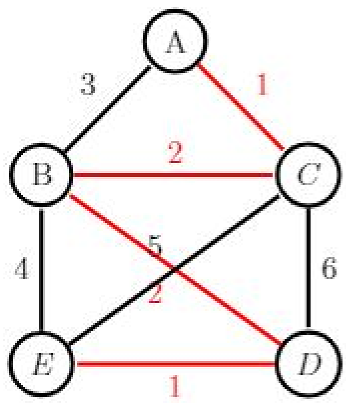


Figure 2

Notice that from source vertex A to vertex C the distance is shortest as compared with the distance from vertex A to vertex B. The same logic is applied for every other vertices and we get the shortest distance from vertex A to other vertices as follows:
$A \to C \to B \to D \to E$.

3. If we perform Floyd-Warshall algorithm for the graph shown below, then which of the following matrices represents SP^4? **(Ans: c)**

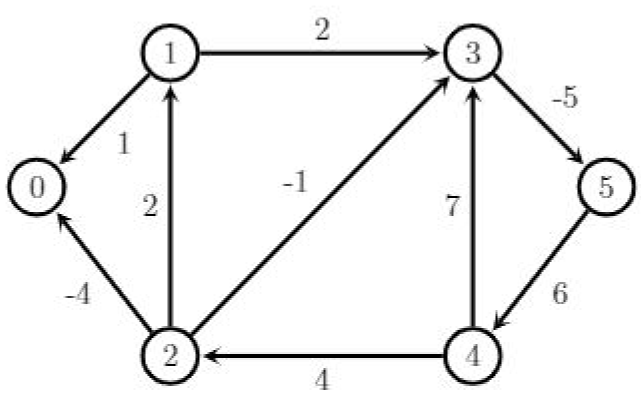


Figure PA-12.3

(a)

SP^4	0	1	2	3	4	5
0	∞	∞	∞	∞	∞	∞
1	1	∞	∞	2	∞	∞
2	-4	2	∞	-1	∞	∞
3	∞	∞	$-\infty$	∞	∞	-5
4	∞	∞	4	7	∞	∞
5	∞	∞	∞	∞	6	∞

(b)

SP^4	0	1	2	3	4	5
0	∞	∞	∞	∞	∞	∞
1	1	∞	∞	2	∞	∞
2	-4	2	∞	-1	∞	∞
3	∞	∞	∞	∞	∞	-5
4	0	6	4	3	∞	∞
5	∞	∞	∞	∞	6	∞

(c)

SP^4	0	1	2	3	4	5
0	∞	∞	∞	∞	∞	∞
1	1	∞	∞	2	∞	-3
2	-4	2	∞	-1	∞	-6
3	∞	∞	∞	∞	∞	-5
4	0	6	4	3	∞	-2
5	∞	∞	∞	∞	6	∞

(d)

SP^4	0	1	2	3	4	5
0	∞	∞	∞	∞	∞	∞
1	1	∞	∞	2	∞	-3
2	∞	2	∞	-1	∞	-6
3	∞	∞	∞	∞	∞	-5
4	0	6	4	3	∞	∞
5	∞	∞	∞	∞	6	∞

Solution

- **To find:** SP^4 adjacency matrix.
- **Approach:** Find SP^0, SP^1, SP^2, SP^3 and then find SP^4.

- **Floyd-Warshall Algorithm**
  Let $SP^K[i,j]$ be the length of the shortest path from i to j via vertices $\{0, 1, ..., k-1\}$

- Note: $SP^0[i,j] = W[i,j]$, where $W[i,j]$ is weight of an edge from i to j.

  For SP^1 find shortest path via vertex {0}.
  For SP^2 find shortest path via vertices {0,1}.
  For SP^3 find shortest path via vertices {0,1,2}.

SP^0	0	1	2	3	4	5
0	∞	∞	∞	∞	∞	∞
1	1	∞	∞	2	∞	∞
2	-4	2	∞	-1	∞	∞
3	∞	∞	∞	∞	∞	-5
4	∞	∞	4	7	∞	∞
5	∞	∞	∞	∞	6	∞

SP^0

SP^1	0	1	2	3	4	5
0	∞	∞	∞	∞	∞	∞
1	1	∞	∞	2	∞	∞
2	-4	2	∞	-1	∞	∞
3	∞	∞	∞	∞	∞	-5
4	∞	∞	4	7	∞	∞
5	∞	∞	∞	∞	6	∞

SP^1

SP^2	0	1	2	3	4	5
0	∞	∞	∞	∞	∞	∞
1	1	∞	∞	2	∞	∞
2	-4	2	∞	-1	∞	∞
3	∞	∞	∞	∞	∞	-5
4	∞	∞	4	7	∞	∞
5	∞	∞	∞	∞	6	∞

SP^2

SP^3	0	1	2	3	4	5
0	∞	∞	∞	∞	∞	∞
1	1	∞	∞	2	∞	∞
2	-4	2	∞	-1	∞	∞
3	∞	∞	∞	∞	∞	-5
4	0	6	4	3	∞	∞
5	∞	∞	∞	∞	6	∞

SP^3

SP^4	0	1	2	3	4	5
0	∞	∞	∞	∞	∞	∞
1	1	∞	∞	2	∞	-3
2	-4	2	∞	-1	∞	-6
3	∞	∞	∞	∞	∞	-5
4	0	6	4	3	∞	-2
5	∞	∞	∞	∞	6	∞

SP^4

2 MULTIPLE SELECT QUESTIONS

Using the graph below answer the following questions [Question 4 and 5]

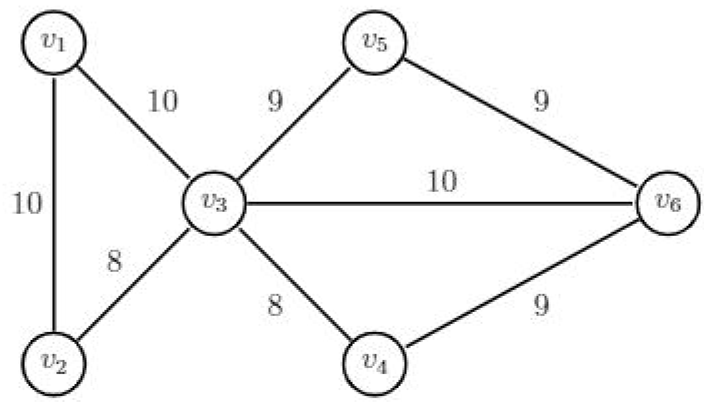
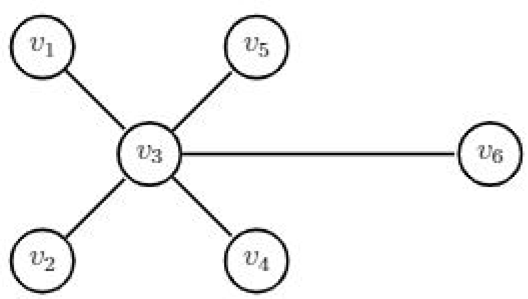


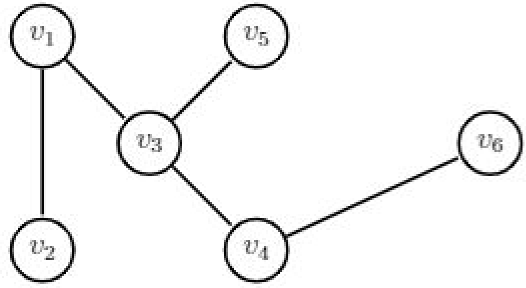
Figure PA-12.4

4. Which of the following could be the minimum cost spanning tree computed by running Prim's algorithm on the graph in Figure PA-12.4? **(Ans: c,d)**

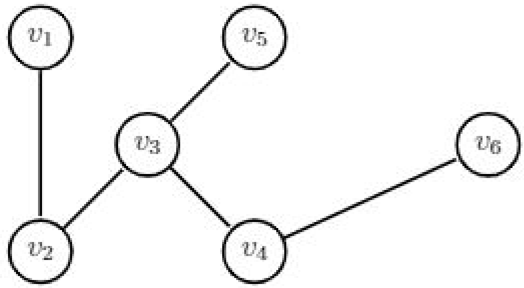
(a)



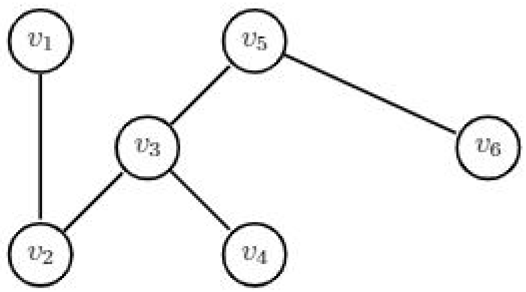
(b)



(c)

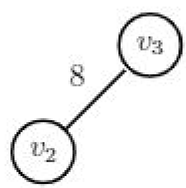


(d)

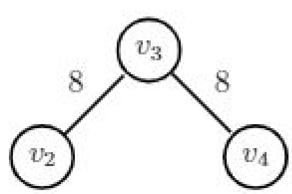


Solution

Prim's Algorithm
Step 1: Select the edge with minimum cost (edge weight). For example

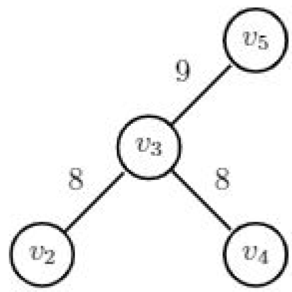


Step 2: Now check for all the edges adjacent to (v_1, v_2). Select the one which has the lowest weight and include it in the tree. For example

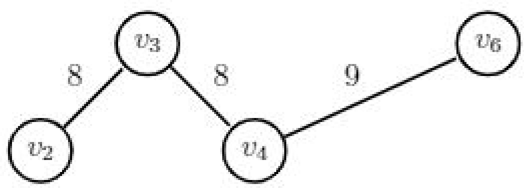


Note: Make sure no cycles are formed.

Step 3: Repeat Step 2 by adding one more edge [adjacent to (v_2, v_3) or (v_3, v_4)] which has the lowest weight edge. For example



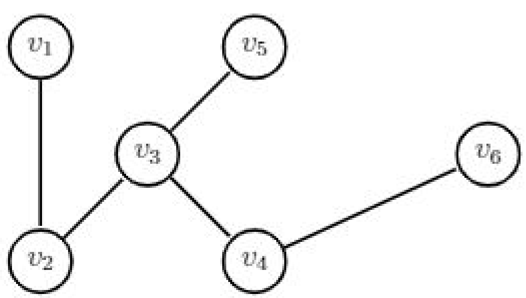
OR



Step 4: Repeat the above step to cover all the edges to obtain the Minimum Cost Spanning Tree.

Note: In the given graph (Figure PA-12.4) there are many edges which have same weights and thus we have multiple Minimum Cost Spanning Tree. Some of them are shown in Figure 3.

(Option c)



(Option d)

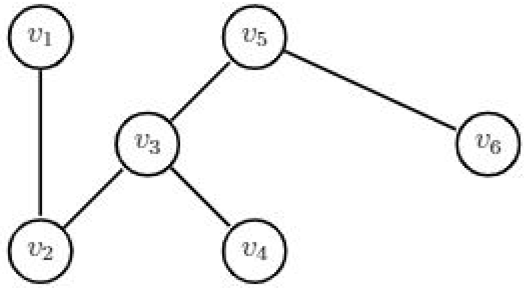
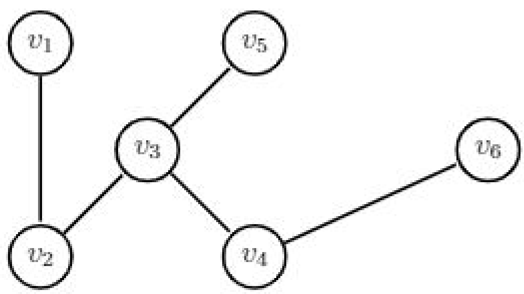


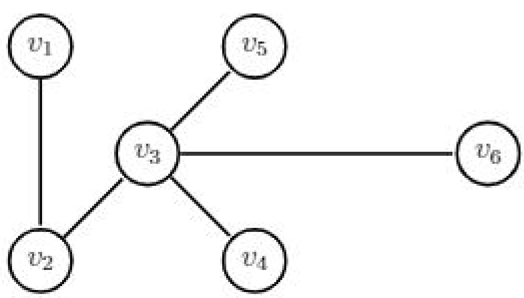
Figure 3 Minimum Cost Spanning Tree using Prim's Algorithm.

5. Which of the following could be the minimum cost spanning tree computed by running Kruskal's algorithm on the graph in Figure PA-12.4? **(Ans: a,d)**

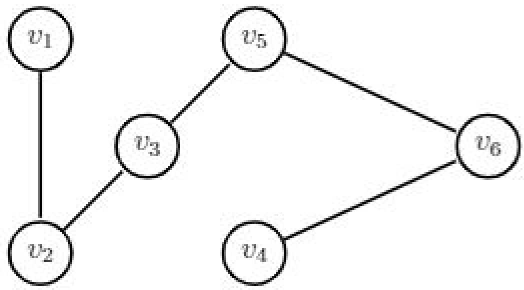
(a)



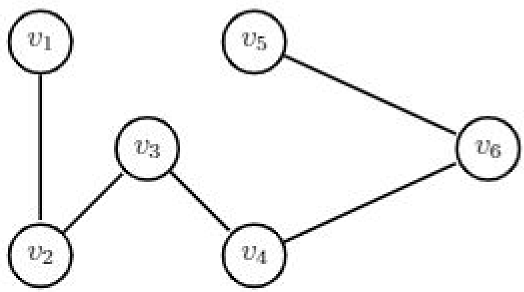
(b)



(c)



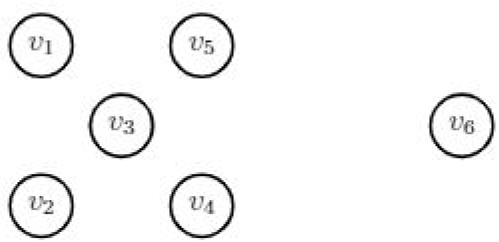
(d)



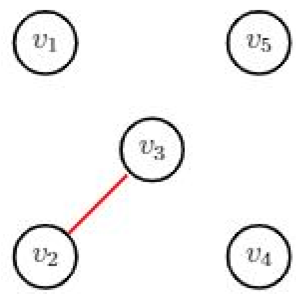
Solution

Note: Make sure no cycles are formed.

Step 1: For the graph given in (Figure PA-12.4) break them into n components. Here $n = 6$. For example



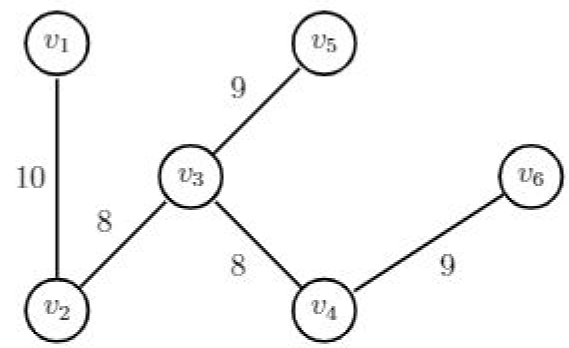
Step 2: Connect two components in ascending order of cost. For example



Note: We can connect either v_2 and v_3 or v_3 and v_4 as they have same edge weights. In above example we are connecting vertex v_2 and vertex v_3.

Step 3: Repeat Step 2 until all the edges are connected. See Figure 4.

(Option a)



(Option d)

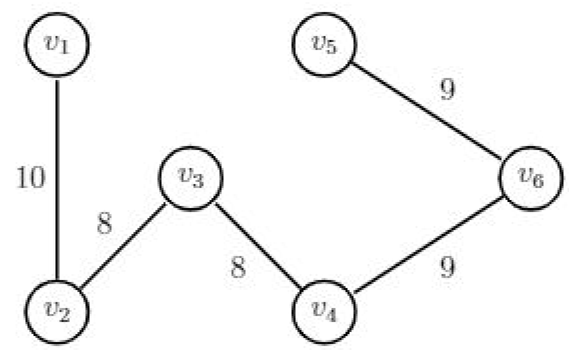


Figure 4 Minimum Cost Spanning Tree using Kruskal's Algorithm.

Note: In the given graph (Figure PA-12.4) there are many edges which have same weights and thus we have multiple Minimum Cost Spanning Tree. Some of them are shown in Figure 4.

6. While using Bellman-Ford Algorithm for the graph shown below (Figure PA-12.5), let $D(v)$ be the shortest distance of vertex v from the source vertex 4 after 7 iterations.

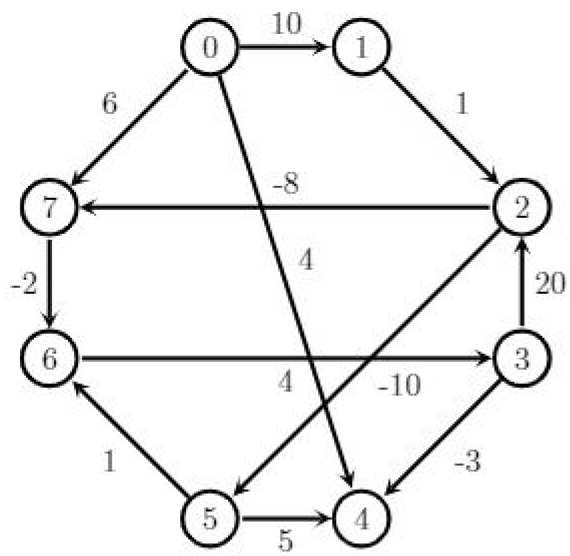


Figure PA-12.5

If the direction of edges in the graph are reversed, then which of the following is (are) CORRECT? **(Ans: a,d)**

(a) $D(0) = 2$
(b) $D(2) = 9$
(c) Bellman-Ford is not applicable for the new graph.
(d) $D(v)$ is negative for some vertex v.

Solution

Consider the new graph shown in Figure 5 after changing the directions of the edges of the graph given in Figure PA-12.5.

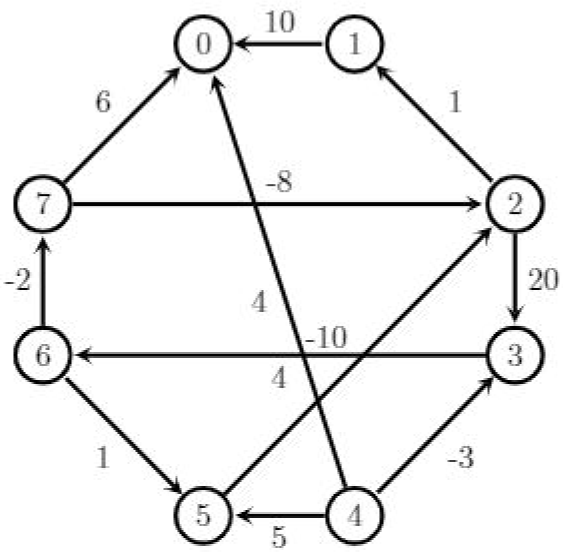


Figure 5

Bellman-Ford Algorithm is applicable for this new graph because there is no negative weight cycle.

Now, using Bellman-Ford Algorithm we get adjacency matrix as shown in the Table 1.

v	$D(v)$							
0	∞	4	4	4	4	4	2	2
1	∞	∞	∞	-4	-4	-8	-8	-8
2	∞	∞	-5	-5	-9	-9	-9	-9
3	∞	-3	-3	-3	-3	-3	-3	-3
4	0	0	0	0	0	0	0	0
5	∞	5	5	2	2	2	2	2
6	∞	∞	1	1	1	1	1	1
7	∞	∞	∞	-1	-1	-1	-1	-1

Table 1

Note after 7 iterations we get the values of $D(v)$ as shown in last column. Clearly $D(0) = 2$; $D(2) = -9$; $D(v)$ is negative for some vertex v.

7. Which of the following options are correct? **(Ans: c,d)**

(a) Let G be a weighted graph and in which the weights of all the edges are different. If we run a shortest path algorithm on G, then we will get a unique shortest path from the starting vertex to every other vertex.

(b) Suppose $G = (V, E)$ is a weighted graph, where $V = \{v_1, v_2, ..., v_n\}$. Let P be a shortest path from v_i to v_j $(i \neq j)$. If we increase the weight of each edge in the graph by one, then P will still be the shortest path from v_i to v_j.

(c) A graph G can have more than one spanning tree.

(d) Suppose $G = (V, E)$ is a weighted graph and the weights of all the edges are positive. Let P be a shortest path from $a \in V$ to $B \in V$. If we double the weight of every edge in the graph G, then the shortest path remains same but the total weight of path changes.

Solution

Option (a)
Let G be weighted graph as shown in Figure 6.

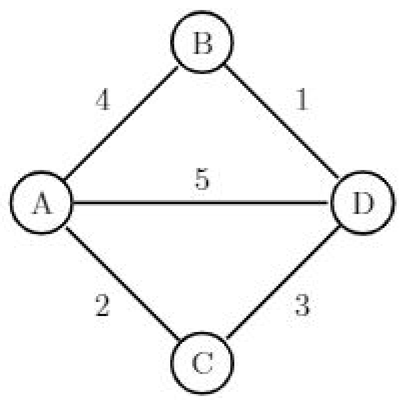


Figure 6

Note: All the edge weights are different.

There are more than one shortest path to reach the vertex D from the starting vertex A. Thus, the statement as in Option (a) is wrong.

Option (b)
Consider the graph shown in Figure 7.

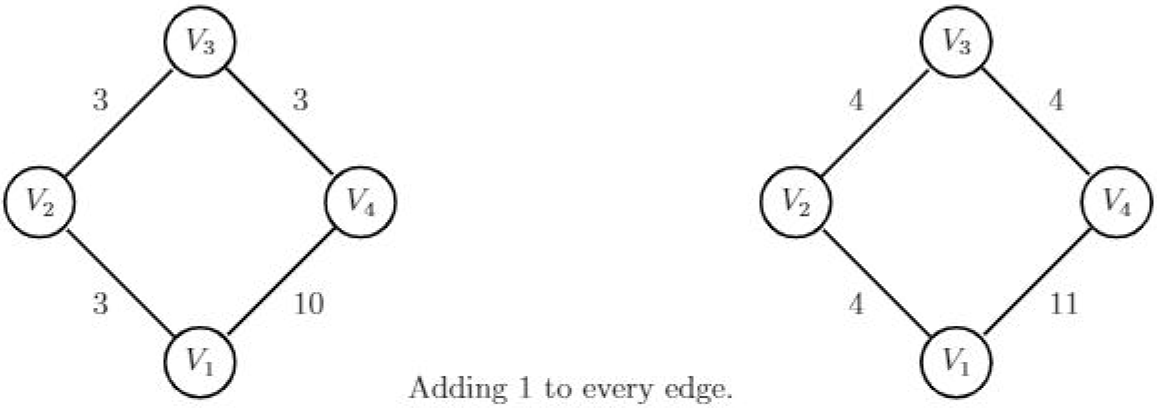


Figure 7

Initially, the shortest path, to reach vertex V_4 from vertex V_1 was $V_1 \to V_2 \to V_3 \to V_4$ (total cost is 9). If we increase the weights of each edge by one then the shortest path from vertex V_1 to vertex V_4 becomes $V_1 \to V_4$. So we are getting the new shortest path after the increment. So Option (b) is wrong.

Option (c)
A simplest example could be a graph G, when all the edges have same weights.

Option (d)
If we multiply all edge weights by 2, then the shortest path doesn't change. Because weights of all paths from a to b gets multiplied by 2. Here the number of edges in a path doesn't matter.

8. Which of the following options are correct? **(Ans: a,d)**

(a) Dijkstra's algorithm works for graphs having no negative weight edge.
(b) Floyd–Warshall algorithm works for graphs with negative weight cycles.
(c) Dijkstra's algorithm works on any graph without negative weight cycles.
(d) Shortest path problem is not applicable for a graph with a negative weight cycle.

Solution

(a) Dijkstra's Algorithm work for graph with non-negative edges, is the basic condition for this algorithm. So Option (a) is correct.

(b) Floyd–Warshall Algorithm doesn't works for graphs with negative weight cycles. So Option (b) is wrong.

(c) This option is incorrect. For Dijkstra's Algorithm to work, the edge weights must be non-negative. A graph can have negative edges even though there are no negative weight cycles. For example Figure 8.

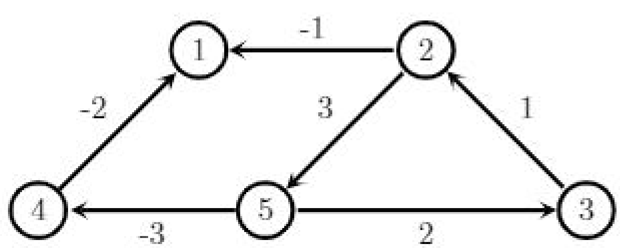


Figure 8

Vertices 2,3 and 5 forms a non-negative cycle but there are edges between vertices 2 and 1, 4 and 1, and 5 and 4 with negative weight edges.

(d) Shortest path problem is applicable for non-negative weight cycle. So Option (d) is correct.

3 NUMERICAL ANSWER TYPE

9. What is the weight of a minimum cost spanning tree of the graph given below (Figure PA-12.6)? **(Ans: 38)**

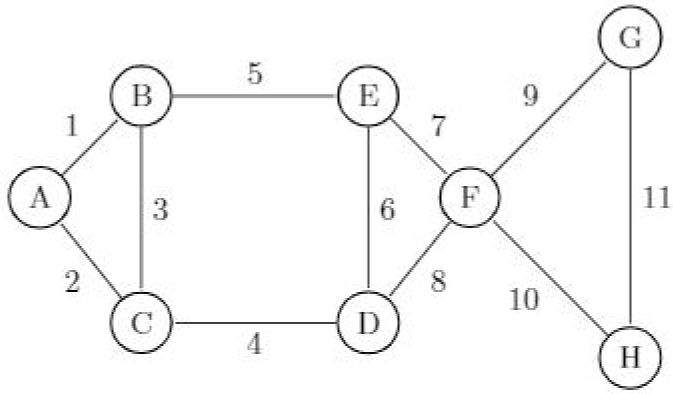


Figure PA-12.6

Solution

Using Prim's Algorithm, Minimum Cost Spanning Tree of graph shown in Figure PA-12.6 is (see Figure 9)

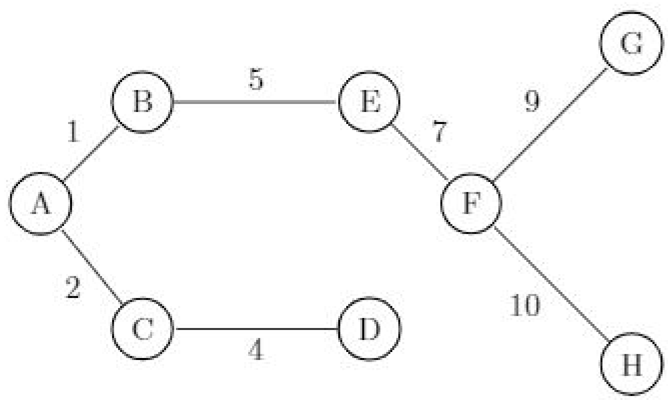


Figure 9 Minimum Cost Spanning Tree using Prim's Algorithm

Weight of Minimum Cost Spanning Tree will be sum of weights of all the edges in Minimum Cost Spanning Tree which is $1 + 2 + 4 + 5 + 7 + 9 + 10 = 38$.

Mathematics for Data Science - 1
Graded Assignment
Week 11

1 MULTIPLE CHOICE QUESTIONS:

1. An undirected graph G has 22 vertices and the degree of each vertex is at least 2. What is the minimum number of edges that graph G can have?

Sol:- we know that,
Sum of degree of all the vertices is twice the number of edges in the graph.

* Consider the minimum possible case: Suppose every vertex in the graph G has degree 2.

Now,
Sum of degree of all vertices = $22 \times 2 = 44$

$\Rightarrow$ Minimum number of edges that the graph G can have is $\frac{44}{2} = 22$

2. If G is a connected undirected graph such that every vertex has degree at most k, and the shortest path between any two vertices has length at most 2, then the number of vertices in G can be at most

(a) $k^2 - 1$

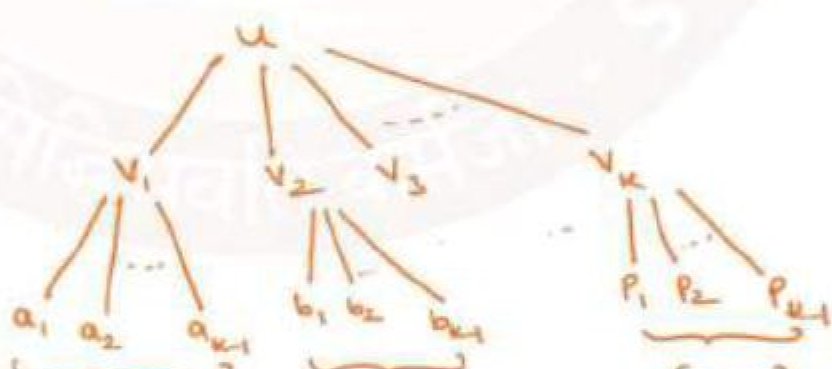
(b) $k^2 + 1$

(c) k^2

(d) $k^2 - k$

Sol:- Let 'u' be a Vertex in the graph G.
As the degree of every vertex is at most 'k', the number of vertices that are adjacent to 'u' can be at most 'k'.
Also, if we draw a BFS tree starting with vertex 'u', then the depth of the tree cannot be more than 2 because the length of the shortest path between any two vertices is at most 2.

Now each of v_i can have at most $k-1$ adjacent vertices because 'u' is already adjacent to each of v_i.
$\therefore$ BFS tree with at most vertices can be



$(k-1) + (k-1) + \cdots + (k-1)$ (k times)

$\therefore$ level 2 of BFS tree can have atmost $k(k-1)$ vertices.

$\Rightarrow$ the graph G can have atmost $\underset{(\text{level } 0)}{1} + \underset{(\text{level } 1)}{k} + \underset{(\text{level } 2)}{k(k-1)}$

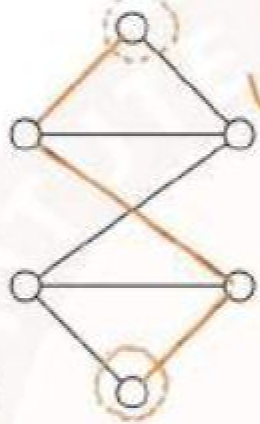
$$= 1 + \cancel{k} + k^2 - \cancel{k}$$
$$= k^2 + 1$$

Hence, the graph can have at most k^2+1 vertices.

option (b) is correct.

3. Suppose A is the adjacency matrix of a connected undirected graph G.

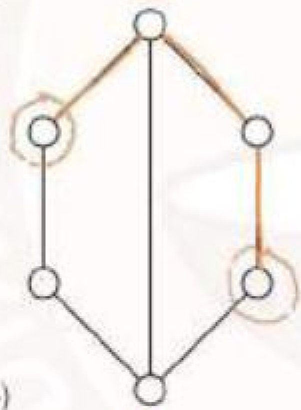
If $A^2 = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}$ and the shortest path between any two vertices has length at most 2, then which of the following may represent the graph G?



(a)

length of shortest path between the highlighted vertices is more than 2.

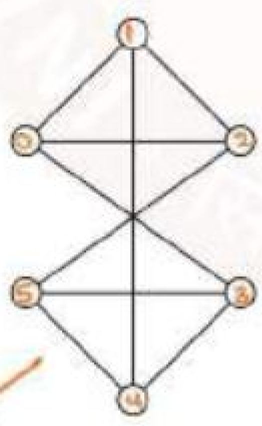
So, option (a) is incorrect.



(b)

length of shortest path between the highlighted vertices is more than 2.

So, option (b) is incorrect.



✓ (c)

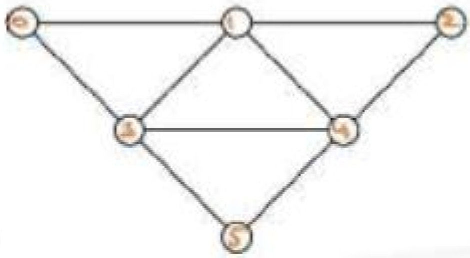
* shortest path between any two vertices is at most 2.

* $A = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$

$A^2 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \end{bmatrix} \end{array}$

$\therefore$ option (c) is correct.

(d)



$$A = \begin{array}{c|cccccc} & 0 & 1 & 2 & 3 & 4 & 5 \\ \hline 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 & 0 \\ 2 & 0 & 1 & 0 & 0 & 1 & 0 \\ 3 & 1 & 1 & 0 & 0 & 1 & 1 \\ 4 & 0 & 1 & 1 & 1 & 0 & 1 \\ 5 & 0 & 0 & 0 & 1 & 1 & 0 \end{array}$$

$$A^2 = \begin{array}{c|cccccc} & 0 & 1 & 2 & 3 & 4 & 5 \\ \hline 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 & 1 & 1 & 1 \\ 3 & 1 & 1 & 1 & 1 & 1 & 1 \\ 4 & 1 & 1 & 1 & 1 & 1 & 1 \\ 5 & 1 & 1 & 1 & 1 & 1 & 1 \end{array}$$

A^2 that we have obtained is different from the given information

so, option (d) is incorrect.

4. Suppose G is a graph with 6 vertices $0,1,2,3,4,5$ and the adjacency matrix of the graph G is $A = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. Which of the following statements is True?

(a) The graph G is a directed acyclic graph.

(b) From vertex 4, every other vertex is reachable.

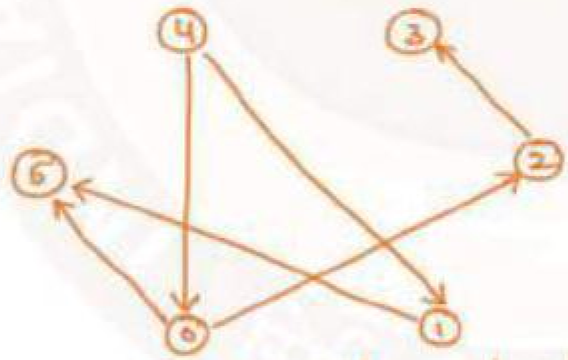
(c) The longest path in the graph G has length 4, in terms of number of edges.

(d) The longest path in the graph is $4 \longrightarrow 0 \longrightarrow 2 \longrightarrow 3$

Sol :- Given,

$$A = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

we first draw the graph G.

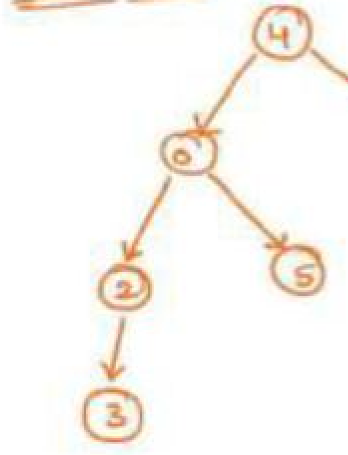


* As the obtained graph does not have any cycles and also it is a directed graph $\Rightarrow$ G is a directed acyclic graph.

So, option (a) is Correct.

* If we draw a BFS tree starting with Vertex '4', then we can get the Vertices that are reachable from Vertex '4'.

BFS tree :-



Every Vertex is present in the BFS tree

$\Rightarrow$ Every Vertex is reachable from Vertex '4'.

so, option (b) is Correct.

* we have

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

from this we find A^2, A^3, A^4.

so,

$$A^2 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad A^3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ From A^4, we can conclude that there is no path in the graph G that has length 4. So, option (c) is incorrect.

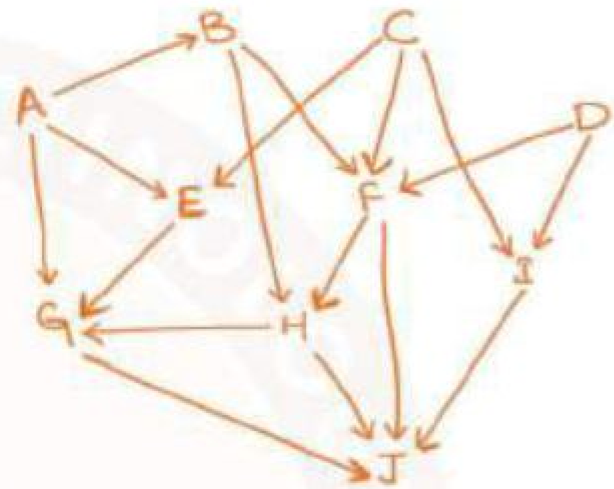
* From A^3, $4 \longrightarrow 3$ is the longest path in the graph G and the path is $4 \longrightarrow 0 \longrightarrow 2 \longrightarrow 3$.

so, option (d) is correct.

5-6

USE THE FOLLOWING INFORMATION FOR QUESTIONS [[redacted]]:
Shreya needs to perform 10 tasks namely $\{A, B, C, D, ...J\}$. Some tasks needs to be performed after performing a particular task. In the below table, column 1 shows the tasks and column 2 shows the sets of tasks that can be performed only after performing the particular task.

A	$\{B, E, G\}$
B	$\{F, H\}$
C	$\{E, F, I\}$
D	$\{F, I\}$
E	$\{G\}$
F	$\{H, J\}$
G	$\{J\}$
H	$\{G, J\}$
I	$\{J\}$
J	$\{\}$



5. Which of the following sequences may represent the possible order in which Shreya can perform the tasks?

(a) $A, C, B, D, E, I, F, H, G, J$
(b) $A, D, C, B, E, I, F, H, G, J$
(c) $C, A, D, E, B, I, F, G, H, J$
(d) $D, C, A, B, E, I, F, H, G, J$

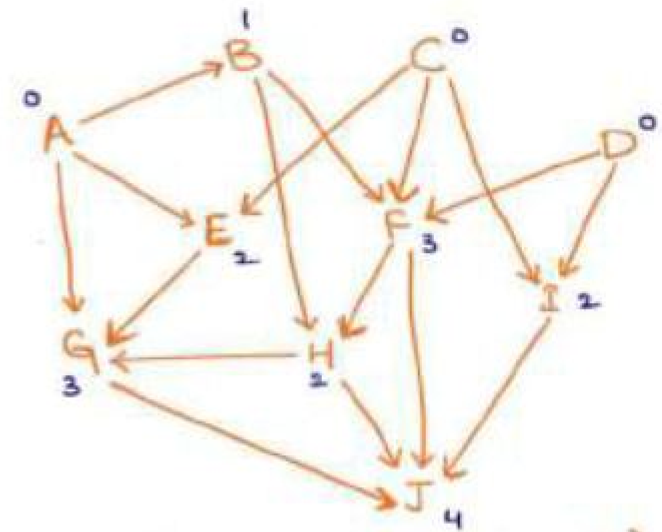
Sol:- Draw a directed graph 'K' that represents the given data such that each vertex represents a task and a directed edge from vertex 'i' to vertex 'j' if task 'j' can be performed only after task 'i'.

* observe that the obtained graph is a DAG (Directed acyclic graph).

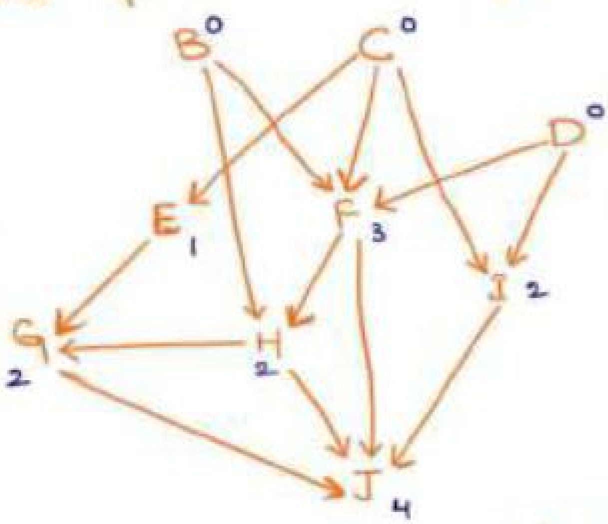
Now, if we find topological sequence of this DAG, then in the same order she can perform the tasks.

∴ our aim is to find topological sequence of the obtained DAG.

* Find indegree of each Vertex in the graph.



Now, we have three vertices (A, C, D) that has indegree 0.
∴ we can choose any one of them i.e, she can perform any of the three tasks
Suppose we choose Vertex 'A', remove it from the DAG and update the indegree of each of the remaining vertex.



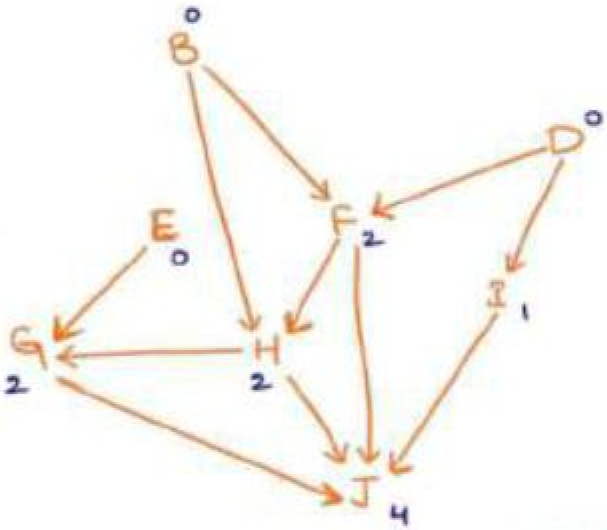
* indegree of vertex B has become 0 and indegree(E) = 1, indegree (G) = 2.

* Topological Sequence :

A,

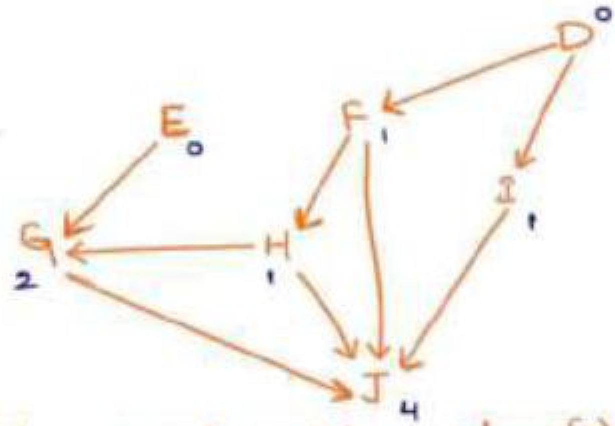
* Repeat the process till the last Vertex is removed and added to the topological sequence.

Now, we have again three Vertices (B, C, D) having indegree 0.
So we can choose any one of them. Suppose we choose vertex 'C'.
∴ Remove Vertex 'C' from the graph and update the indegree of each of the vertex, also add 'C' to the Topological Sequence.



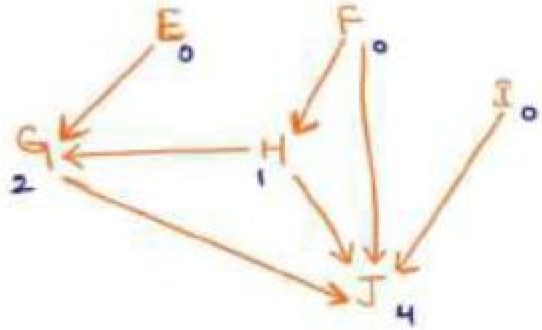
* indegree (E) = 0, indegree (F) = 2, indegree (I) = 1
* Topological Sequence :-

A, C,



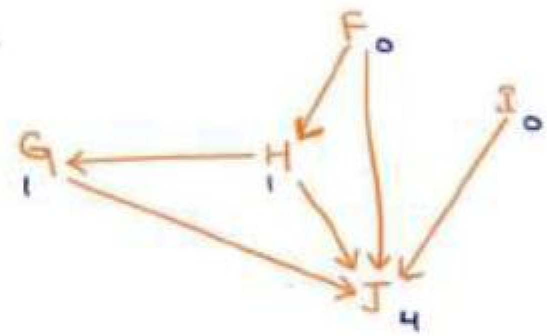
* indegree (F) = 1, indegree (H) = 1
* Topological Sequence :-

A, C, B,



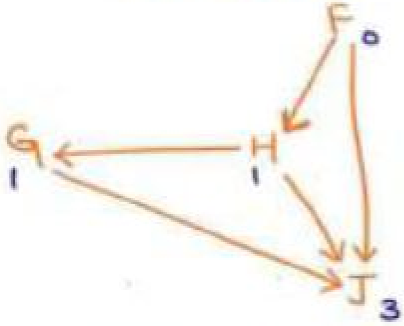
* indegree (F) = 0, indegree (I) = 0
* Topological Sequence :-

A, C, B, D,



* indegree (G) = 1
* Topological Sequence :-

A, C, B, D, E,



* indegree (J) = 3
* Topological Sequence :-

A, C, B, D, E, I,

* indegree (H) = 0, indegree of (J) = 2
* Topological Sequence :-

A, C, B, D, E, I, F

* indegree (G) = 0 , indegree (J) = 1
* Topological Sequence :-

A, C, B, D, E, I, F, H,

* indegree (J) = 0
* Topological Sequence :-

A, C, B, D, E, I, F, H, G,

* Finally we remove 'J' and add it to the topological Sequence.

∴ Topological Sequence :- A, C, B, D, E, I, F, H, G, J.

NOTE :- the obtained topological Sequence is not unique because instead of starting with A, we can also start with Vertex 'C' or Vertex 'D'.

∴ Shreya Can perform tasks in the order A, C, B, D, E, I, F, H, G, J
Hence, option (a) is Correct.

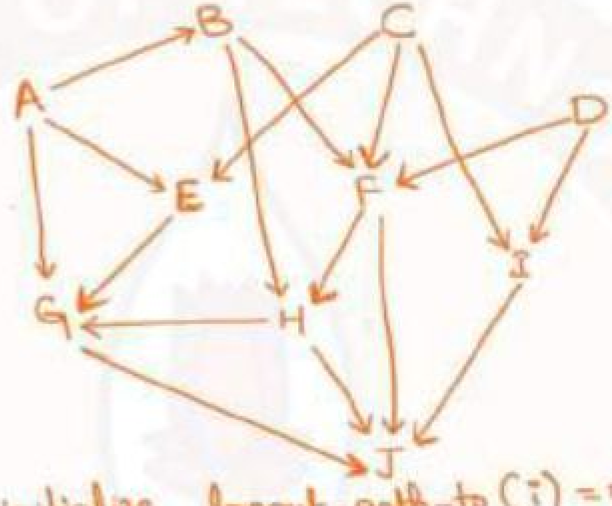
* Similarly, A, D, C, B, E, I, F, H, G, J and D, C, A, B, E, I, F, H, G, J are also a possible topological Sequence which means shreya can perform tasks in that order too.
Hence, options (b), (d) are Correct.

* option (c) is incorrect because task 'G' Cannot be performed before task 'H'. ∴ that is not a possible order.

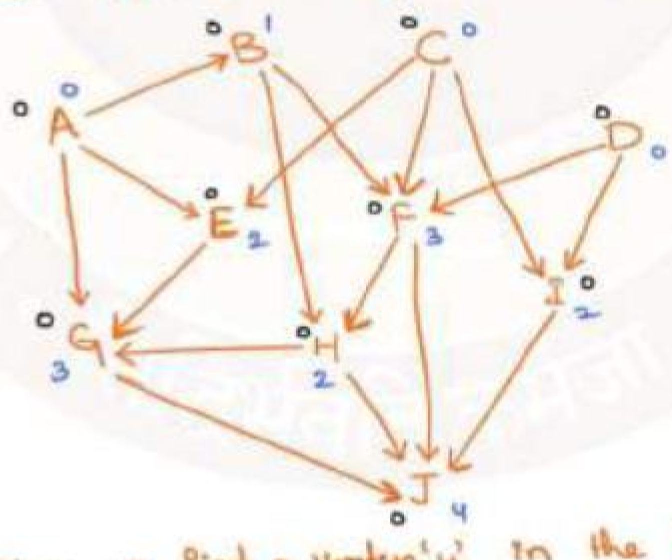
3 NUMERICAL ANSWER TYPE:

6. If each task takes 5 minutes to complete and she performs all the independent tasks simultaneously, then the time(in minutes) taken by Shreya to complete all the tasks is

Sol:- we Compute the longest path to each vertex in the DAG (that we got in the previous problem)



* we first initialize longest-path-to (i) = 0 for each vertex 'i' in the DAG.
Also find the indegree (i) for each vertex 'i' in the DAG.



longest-path-to (i).
indegree (i)

Now, we find a vertex 'u' in the DAG which has indegree 0.
* we remove vertex 'u' from the graph and update indegree(i) and longest-path-to(i) for every vertex 'i' that is adjacent to vertex 'u'.

* we update longest-path-to(i) as

$$\text{longest-path-to}(i) = \text{Max}\left\{\text{longest-path-to}(i),\ 1 + \text{longest-path-to}(u)\right\}$$

* Repeat the process by finding a new vertex 'v' that has indegree 0, removing it from the graph and update indegree(i) and longest-path-to(i) till all the vertices are removed from the graph.

Finally, after removing all the vertices and updating longest-path-to(i) for all vertices, we get

longest-path-to(A) = longest-path-to(C) = longest-path-to(D) = 0.

longest-path-to(B) = longest-path-to(E) = longest-path-to(I) = 1

longest-path-to(F) = 2

longest-path-to(H) = 3

longest-path-to(G) = 4

longest-path-to(J) = 5

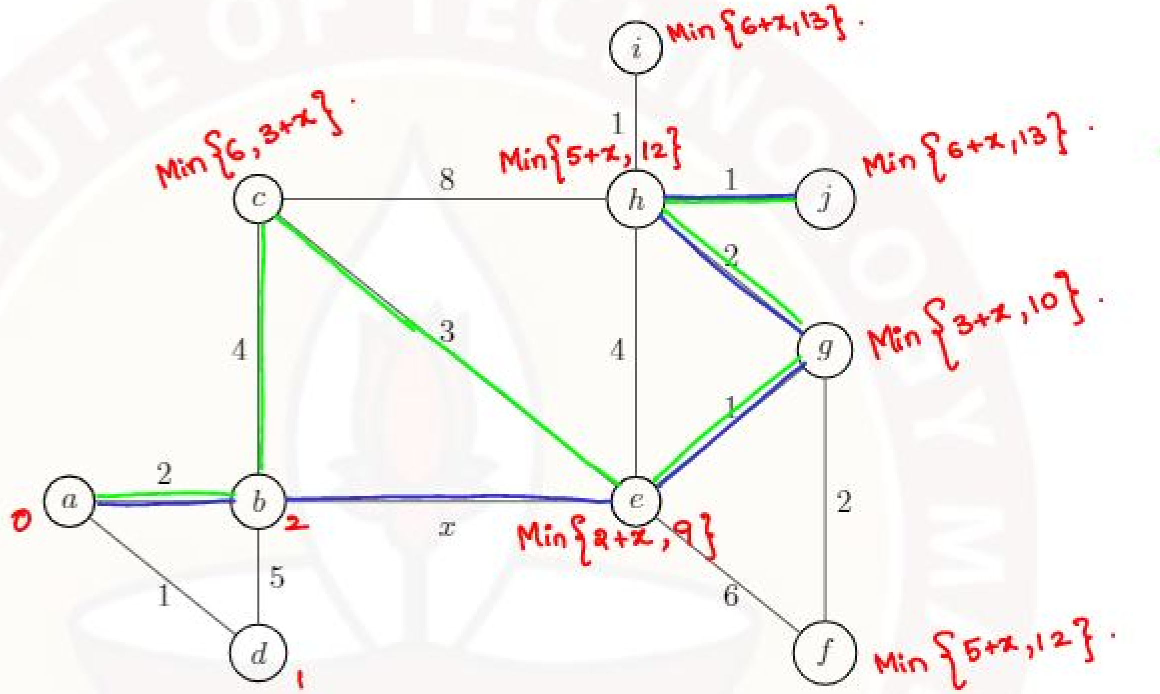
∴ She can first perform tasks 'A','C','D' at same time and next tasks 'B','E','I' at same time followed by task 'F' then by task 'H' then by 'G' and finally task 'J'

⇒ She takes 5 minutes (for tasks A, C, D) + 5 minutes (for tasks B, E, I) + 5 minutes (task F) + 5 minutes (task H) + 5 minutes (task G) + 5 minutes (task J)

$= 5 + 5 + 5 + 5 + 5 + 5 = 30$ minutes.

Answer :- 30

7 An undirected weighted graph G is shown below. Find the set of all positive integer values of x such that if we use Dijkstra's algorithm, there will be a unique shortest path from vertex a to vertex j that contains the edge (b, e).



(a) $\{x | x \in \mathbb{Z}, 0 < x < 8\}$

(b) $\{x | x \in \mathbb{Z}, 0 < x < 7\}$

(c) $\{x | x \in \mathbb{Z}, 0 < x < 6\}$

(d) $\{x | x \in \mathbb{Z}, 0 < x < 9\}$

Sol:- If (b,e) edge is not considered, then the green path shown in the figure will be the shortest path from vertex 'a' to vertex 'j'.

The length of the green path is $2+4+3+1+2+1 = 13$. —— ①

Now, it is given that the shortest path from vertex 'a' to vertex 'j' contains edge (b,e)

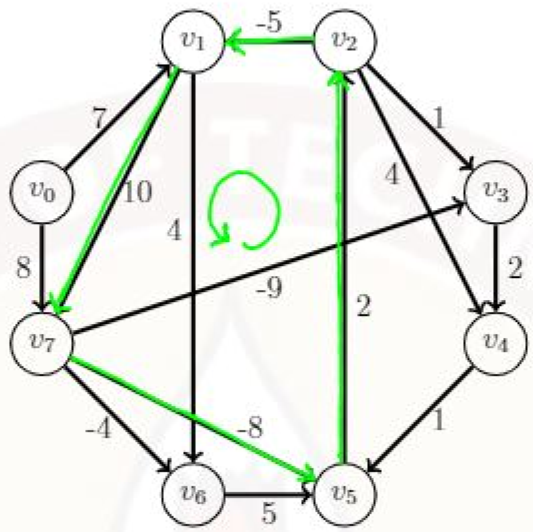
So from ①, the length of the shortest path should be less than 13.

$\Rightarrow$ The length of blue path $2+x+1+2+1 = x+6 < 13$

$\Rightarrow \boxed{x < 7}$.

Hence, the set of all positive integers values of x is $\{x \mid x \in \mathbb{Z}, 0<x<7\}$

8. An undirected graph G is shown below. Suppose we are trying to perform an algorithm to find the shortest path from vertex v_0 to v_4. Which of the following statements is (are) correct?



(a) Dijkstra's algorithm can be used to find the shortest path from v_0 to v_4.

(b) Bellman-Ford algorithm can be used to find the shortest path from v_0 to v_4 because there are negative weighted edges.

(c) The weight of the shortest path from v_0 to v_4 is 1.

(d) Bellman-Ford algorithm cannot be used to find the shortest path from v_0 to v_4 because there is a negative cycle in the given graph.

Sol:- observe that $V_1 \xrightarrow{10} V_7 \xrightarrow{-8} V_5 \xrightarrow{2} V_2 \xrightarrow{-5} V_1$ is a negative cycle

therefore, Bellman-Ford algorithm and Dijkstra's algorithm cannot be used.

9. Which of the following statements is (are) INCORRECT?

(a) In an undirected graph G, if all edges have different positive weights, then the minimum cost spanning tree of graph G is unique.

(b) If there is a cycle of weight 0 in a directed graph G, then we cannot find the shortest path using Bellman-Ford algorithm.

(c) Suppose an acyclic undirected graph G with n vertices has $n-1$ edges. Then the graph is connected.

(d) In a graph G, every edge with the minimum weight will be in the minimum cost spanning tree.

(a) let $G = (V, E)$ be the original graph.

* Suppose there are two different MCST's T_1 and T_2.
* As both the MCST's are different, the edges in the T_1 and edges in the T_2 are not same. This means that there is at least one edge that belongs to one MCST but not the other.
* Out of all the edges that are present in only one of the MCST, choose the minimum weight edge. let 'e_1' be the minimum weight edge and also assume that it is in T_1 (we can also assume it in T_2).
* Now, As T_2 is an MCST, adding this edge e_1 to T_2 creates a cycle 'C'
* T_1 is an MCST so it does not contain a cycle, therefore cycle 'C' must have an edge 'e_2' that is not in T_1.
* observe that weight of edge 'e_1' is less than weight of edge 'e_2'.
* Replacing 'e_2' with 'e_1' in T_2 yields a spanning tree with a smaller weight, which contradicts that T_2 is an MCST.

so, option (A) is correct.

(b) Bellman-ford algorithm cannot be performed only if the graph has a negative cycle.

so, Bellman-ford algorithm can be performed on a graph G having a cycle of weight 0.

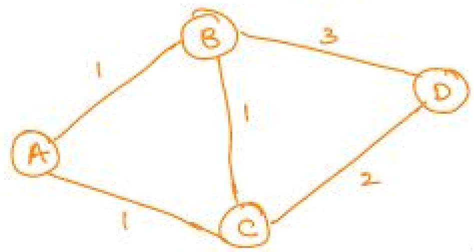
option (b) is incorrect.

(c) An acyclic undirected graph with n vertices and n-1 edges is a Tree and a Tree is a minimally connected graph

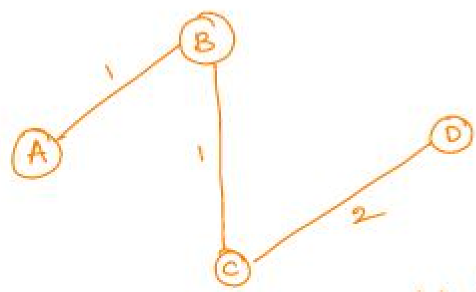
$\Rightarrow$ G is a connected graph.

so, option (C) is correct.

(d) Consider the below graph.



Here, minimum cost spanning tree is



Therefore, edge (A,C) has weight '1' which is the minimum weight in the given graph but it is not in the minimum cost spanning tree.

⇒ Every edge in the graph having the minimum weight may not be in the minimum cost spanning tree.

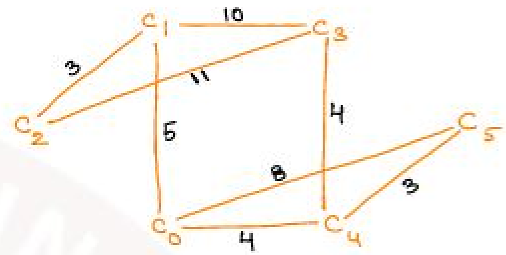
so, option (d) is incorrect.

10-11

Use the following information for questions [redacted]:

A company has branches in each of six cities $C_0, C_1, ..., C_5$. The fare (in thousands of rupees) for a direct flight from C_i to C_j is given by the (i,j)th entry in the following matrix ($\lfloor$ indicates that there is no direct flight):

$$\begin{array}{c} \\ C_0 \\ C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \end{array}\begin{array}{c} \begin{array}{cccccc} C_0 & C_1 & C_2 & C_3 & C_4 & C_5 \end{array} \\ \begin{bmatrix} 0 & 5 & \lfloor & \lfloor & 4 & 8 \\ 5 & 0 & 3 & 10 & \lfloor & \lfloor \\ \lfloor & 3 & 0 & 11 & \lfloor & \lfloor \\ \lfloor & 10 & 11 & 0 & 4 & \lfloor \\ 4 & \lfloor & \lfloor & 4 & 0 & 3 \\ 8 & \lfloor & \lfloor & \lfloor & 3 & 0 \end{bmatrix} \end{array}.$$

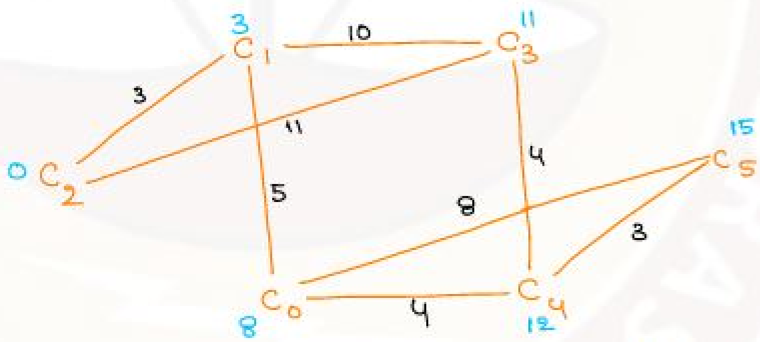


10. An employee of that company wanted to travel from the city C_2 to the city C_5. If he travelled by the cheapest route possible, then the total fare (in rupees) he paid for flight journey was

Sol:- We use Dijkstra's algorithm, to find the cheapest route possible from the city C_2 to the city C_5.

Start by assigning value 0 to C_2 which is the source vertex and compute values of each vertex using the edge weights.

After all the computations, we get the following values:



Finally we get, the value of C_5 is 15 that means he should pay ₹15,000 as fare if he has travelled by the cheapest route possible.

Cheapest route from C_2 to C_5 is

$$C_2 \longrightarrow C_1 \longrightarrow C_0 \longrightarrow C_4 \longrightarrow C_5.$$

Answer:- 15

11. If an inspection team member wanted to inspect all the branches of the company starting from C_2 and ending at C_5, visiting each brach exactly once, then which of the following routes should he choose in order to pay minimum fare for flight journey?

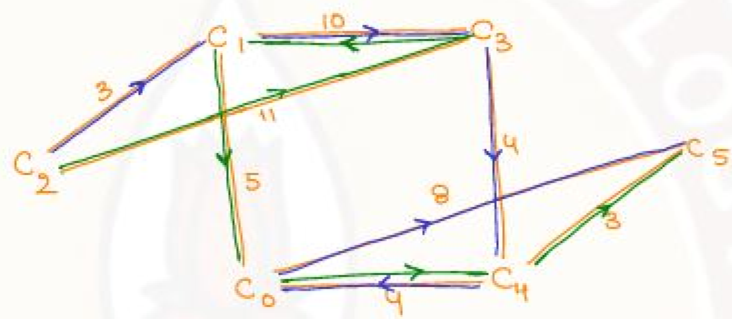
(a) C_2 –| C_3 –| C_1 –| C_0 –| C_4 –| C_5

(b) C_2 –| C_1 –| C_3 –| C_4 –| C_0 –| C_5

(c) C_2 –| C_3 –| C_1 –| C_4 –| C_0 –| C_5

(d) Such a route is not possible.

Sol :- the possible routes are shown in the following graph:



Green route : $C_2 \longrightarrow C_3 \longrightarrow C_1 \longrightarrow C_0 \longrightarrow C_4 \longrightarrow C_5$

Blue route : $C_2 \longrightarrow C_1 \longrightarrow C_3 \longrightarrow C_4 \longrightarrow C_0 \longrightarrow C_5$

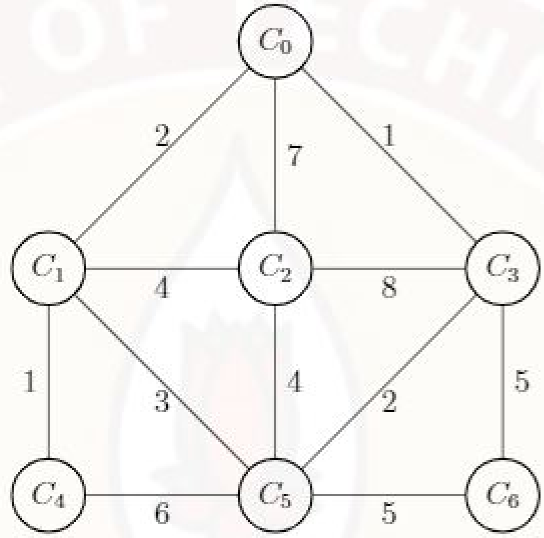
* the total fare he should pay if he choose green route is $11+10+5+4+3$ $= 33$.

* the total fare he should pay if he choose blue route is $3+10+4+4+8$. $= 29$.

Therefore, he has to choose blue route inorder to pay minimum fare.

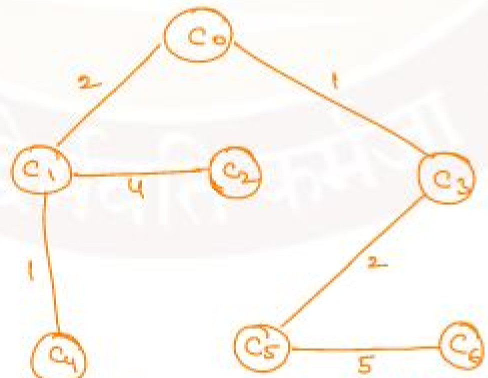
12-13

Use the following information for questions [redacted]:
Seven computers $\{C_0, C_2, ...C_6\}$ are linked by a network, and each link has a maintenance cost. The graph below shows how the computers are linked. Each node represents a computer, each edge represents a link between a pair of computers, and weights on the edges represent the maintenance cost (in hundreds of rupees). The goal is to pick a subset of links such that the total maintenance cost is minimum and the computers remain connected through the chosen links.



12. What is the total maintenance cost (in hundreds of rupees) of the optimum subset of links?

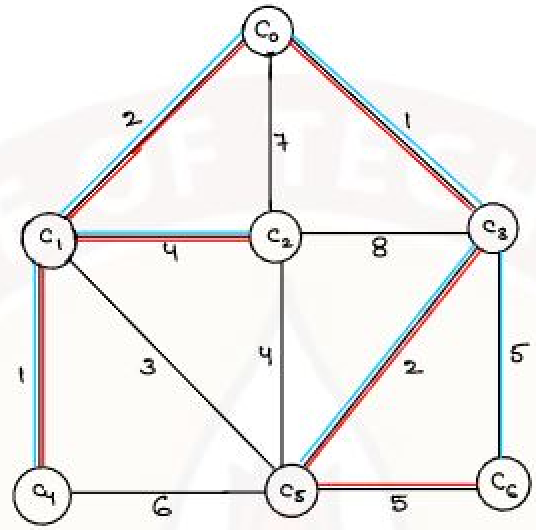
Sol:- We have to find the cost of minimum cost spanning tree (MCST) of this graph.
Now, perform prim's or Kruskals algorithm to find MCST.
* A possible MCST is.



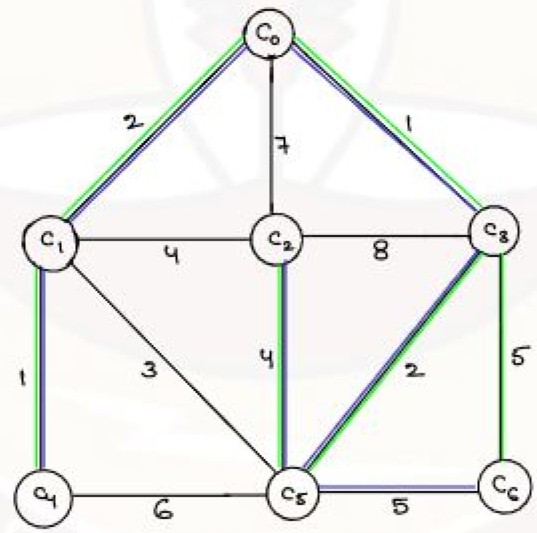
There fore the total maintenance cost (in hundreds of rupees) of the optimal subsets of links is $1+2+1+2+4+5 = 15$.

Answer:- 15 (NOT 1500)

13. Find the number of different ways of choosing an optimum subset of links for the given graph.



* Here the graph with red links and blue links shows two different possible MCST's.



* Here the green links and dark blue links shows two other different possible MCST's of the graph.

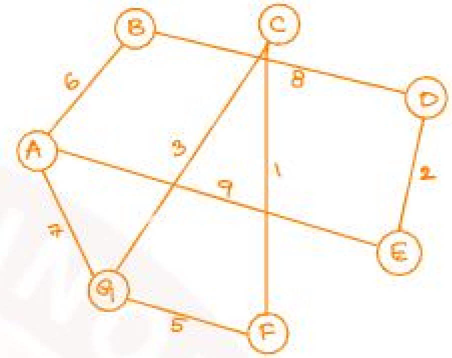
$\therefore$ The number of different ways of choosing the optimal subset of links for the given graph are $2+2=4$.

Answer :- 4.

14-15

Use the following information for questions [[redacted]]:
Consider a weighted graph G with 7 vertices $\{A, B, C, D, E, F, G\}$, which is represented by the following adjacency matrix.

$$
\begin{array}{c|c}
 & \begin{array}{ccccccc} A & B & C & D & E & F & G \end{array} \\
\begin{array}{c} A \\ B \\ C \\ D \\ E \\ F \\ G \end{array} &
\begin{bmatrix}
0 & 6 & 0 & 0 & 9 & 0 & 7 \\
6 & 0 & 0 & 8 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 3 \\
0 & 8 & 0 & 0 & 2 & 0 & 0 \\
9 & 0 & 0 & 2 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 5 \\
7 & 0 & 3 & 0 & 0 & 5 & 0
\end{bmatrix}.
\end{array}
$$



14[redacted]. Suppose we perform Prim's algorithm on the graph G starting from vertex A to find an MCST. Then the order in which the vertices are added is

(a) A, C, F, G, B, D, E

(b) A, B, D, E, G, C, F

(c) ✓ A, B, G, C, F, D, E

(d) A, C, F, G, E, D, B

Sol:- Suppose TV is the set of MCST Vertices and TE is the set of MCST edges.
Initialize TV = ϕ and TE = ϕ
Now, given that we start with Vertex 'A'. So, A is added into the set TV.
$\therefore$ TV = {A} and TE = ϕ.
choose an edge that is incident to vertex 'A' and has a minimum weight.
(A, B) edge has weight 6 which is the minimum among the edges (A, B), (A, E), and (A, G).
$\therefore$ edge (A, B) is added to TE and Vertex 'B' is added to TV.

* TV = {A, B}
TE = {(A, B)}.

Now, choose an edge that is incident to vertex 'A' & vertex 'B' and has a minimum weight.
(A, G) edge has weight 7 which is the minimum among edges (A, E), (A, G), and (B, D).
$\therefore$ edge (A, G) is added to TE and Vertex 'G' is added to TV.

* TV = {A, B, G}
TE = {(A, B), (A, G)}.

* Now we proceed further by choosing the edge which is incident to one of the Vertices A, B & G that has minimum weight.

$W((C,G)) = 3$. is the minimum weight edge.

$TV = \{A, B, G, C\}$.

$TE = \{(A,B), (A,G), (C,G)\}$.

* Like wise if we proceed further, finally we get

$TV = \{A, B, G, C, F, D, E\}$.

$TE = \{(A,B), (A,G), (C,G), (C,F), (B,D), (D,E)\}$

therefore the order in which the Vertices get added to the set TV is

A, B, G, C, F, D, E.

15 . Suppose we perform Kruskal's algorithm on the graph G starting from vertex C to find an MCST. Which of the following edges are not added to the minimum cost spanning tree?

(a) (A, E) ✓

(b) (B, D)

(c) (G, F) ✓

(d) (A, G)

Sol:- In Kruskal's algorithm, we arrange all the edges in ascending order with respect to weights.

so, (C,F), (D,E), (C,G), (G,F), (A,B), (A,G), (B,D), (A,E) will be the order.

Now, we start constructing the MCST by adding the edges in the ascending order w.r.t. weights.

NOTE: When we add edge make sure that a cycle is not formed.

∴ (C,F), (D,E), (C,G) are added first and (G,F) cannot be added because (C,F), (C,G) & (G,F) forms a cycle.

In the same way, (A,E) is not added because (A,B), (B,D), (D,E) & (A,E) forms a cycle.

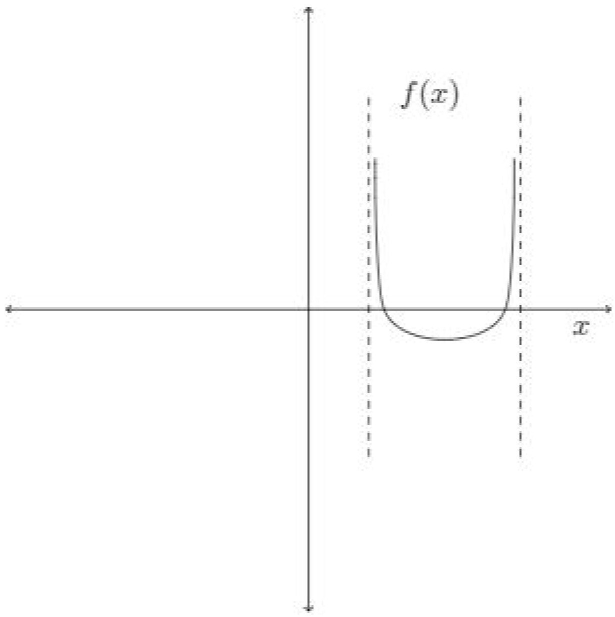
Therefore, edges (G,F) and (A,E) are not in the MCST.

End-term mock Solutions
Mathematics for Data Science - 1
Max Marks: 50

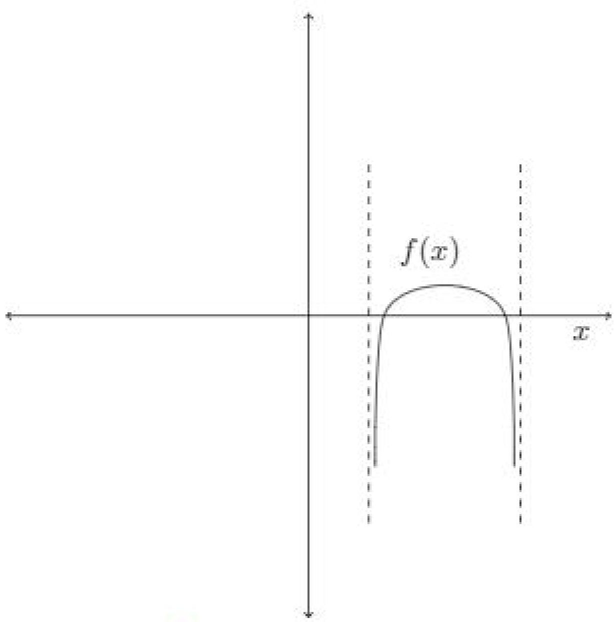
1 Instructions:

- There are some questions which have functions with discrete valued domains (such as day, month, year etc). For simplicity, we treat them as continuous functions.
- For NAT type question, enter only one right answer even if you get multiple answers for that particular question.
- **Notations:**
  - $\mathbb{R}$= Set of real numbers
  - $\mathbb{Q}$= Set of rational numbers
  - $\mathbb{Z}$= Set of integers
  - $\mathbb{N}$= Set of natural numbers
- The set of natural numbers includes 0.

1. Choose the most relevant representation of the following function from the given options $f(x) = \log(-x^2 + 14x - 45)$.
(MSQ)(Ans: (option 2)) [Marks:3]



1.



2.

Vertical asymptotes:-

$-x^2+14x-45 = 0.$

$x^2-14x+45 = 0.$

$x^2-9x-5x+45=0$

$x(x-9)-5(x-9)=0$

$x = 5, 9.$

Both asymptotes are +ve.

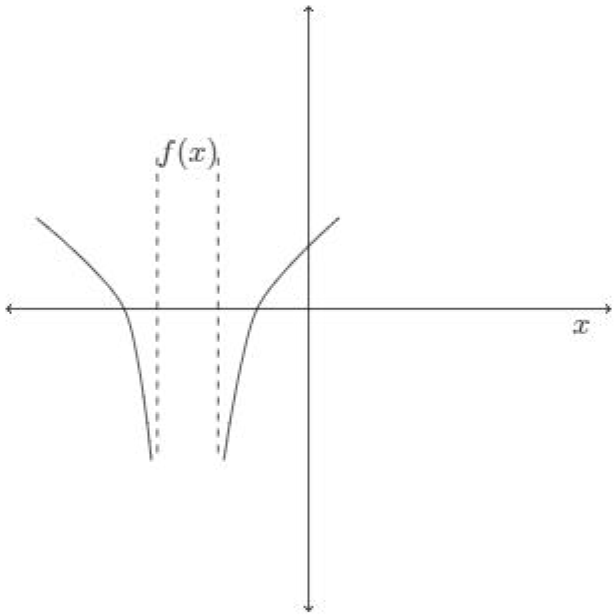
option (3) & (4) are incorrect.

As $x \to 5$ (or) $x \to 9$,

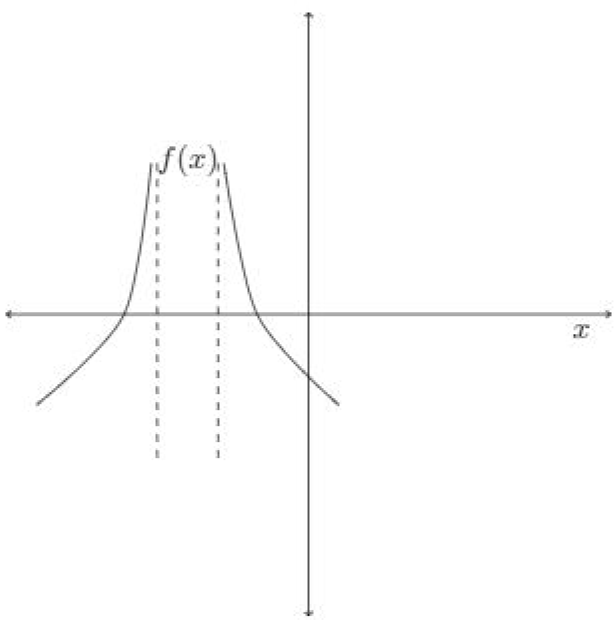
$-x^2+14x-45 \longrightarrow 0$

$\Rightarrow f(x) \longrightarrow -\infty.$

$\therefore$ option (2) is correct



3.



4.

2. Rahul wrote 12 assignments. His score in each assignment $R(n)$ is represented as $R(n) = -(\frac{n^2}{1000})\,(n^3 - 15n^2 + 50n) + 50$, where n represents the assignment number i.e., $n \in \{1, 2, ..., 11, 12\}$. He should score 50 or above in order to pass the assignment. Based on this information answer the following questions

(a) How many times did Rahul score exactly 50?
(NAT)(Answer: 2) [Marks: 2]

(b) In total, how many assignments did Rahul pass?
(NAT)(Answer: 6) [Marks: 2]

Soln:- (a) We have to find the number of assignments in which Rahul score Exactly 50 ie, all n's Such that $R(n) = 50$

$$R(n) = \left(\frac{-n^2}{1000}\right)(n^3 - 15n^2 + 50n) + 50.$$

$$\Rightarrow \cancel{50} = \left(\frac{-n^2}{1000}\right)(n^3 - 15n^2 + 50n) + \cancel{50}$$

$$\Rightarrow \frac{-n^2}{1000}(n^3 - 15n^2 + 50n) = 0.$$

$$\Rightarrow \frac{n^3}{1000}(n^2 - 15n + 50) = 0.$$

$$\Rightarrow n = \cancel{0}, 5, 10.$$

∴ In $\underline{\text{2 assignments}}$ he scored Exactly 50.

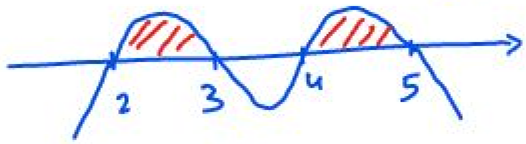
3. Which of the following statements are correct.
(MSQ)(Answer: Option 1, 2, 3, 4) [Marks: 3]

1. The domain of $f(x) = \sqrt{-(x-2)(x-3)(x-4)(x-5)}$ is $[2,3] \cup [4,5]$.
2. If $f \circ g(x) = g \circ f(x) = x$ then f and g are inverses of each other. $\rightarrow$ Its direct definition
3. $f(x) = x^3 + 5$ has a critical point which is not a point of local extremum.
4. If the polynomials $x^3 + ax^2 + 5x + 7$ and $x^3 + 2x^2 + 3x + 2a$ leave the same remainder when divided by $(x-2)$, then the value of a is $\frac{-3}{2}$.

Solution:-

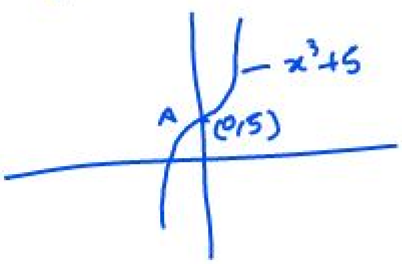
option 1: clearly in $f(x)$, $-(x-2)(x-3)(x-4)(x-5) \geq 0$

This will happen in shaded region



thus Domain of $f(x)$ is $[2,3] \cup [4,5]$

option 3:



$\Rightarrow$ Notice at 'A' the slope is zero & double derivative test is unconclusive. It is a critical point (saddle point). Also it is not local extremum

Option 4:

Given that both the polynomial leaves the same remainder when divided by $(x-2)$. Thus.

$$\cancel{(2)^3} + a(2)^2 + 5(2) + 7 = \cancel{(2)^3} + 2(2)^2 + 3(2) + 2a$$

$$\Rightarrow 4a + 17 = 2a + 14$$

$$\Rightarrow 2a = -3$$

$$\boxed{a = -3/2}$$

4. Two highways perpendicular to each other are considered as the coordinate axes. Arpita's house is at the point $(-3, a)$ and her friend Puja's house is at the point $(-3, 10)$. The distance between their houses is 6 km. Sofi's house is at $(b, 10)$. The distance between Puja's and Sofi's houses is 8 km. If a and b are real numbers, then what is the distance between Arpita's and Sofi's house? (Consider the unit length to be 1 km on both the axes)

(NAT)(Answer: 10) [Marks: 2]

Solution:-

Given : $A(-3,a)$, $P(-3,10)$, $S(b,10)$ are the coordinates of Arpita's, Puja's and Sofi's house respectively.

Dist. b/w Pooja & Arpita's house → $AP = 6\text{ Km} = \sqrt{(10-a)^2 + (-3\cancel{+3})^2}$ (0)

$6 = 10-a$

$\Rightarrow \boxed{a=4}$

Dist. b/w Pooja & Sofi's house (PS) is

$PS = 8\text{ Km} = \sqrt{(b+3)^2 + 0}$

$\Rightarrow 8 = (b+3)$

$\Rightarrow \boxed{b=5}$

Dist. b/w Arpita & Sofi's house (AS) is

$A(-3,4)$; $S(5,10)$

$AS = \sqrt{(10-4)^2 + (5+3)^2} = \sqrt{6^2+8^2}$

$\boxed{AS = 10\text{ Km}}$

5. A company wants to manufacture iron rods. Before they start production, they estimated the total cost(C) for producing x number of iron rods is $C(x) = 0.2x + 1000$ and the total revenue(R) for selling x number of iron rods is $R(x) = 2.5x$. They are also sure that every iron rod produced will be sold. Find the minimum number of iron rods the company should manufacture to get a profit.
(NAT) (Answer: 435) [Marks: 2]

Solution:- As per question

$$2.5x = 0.2x + 1000$$

$$2.3x = 1000$$

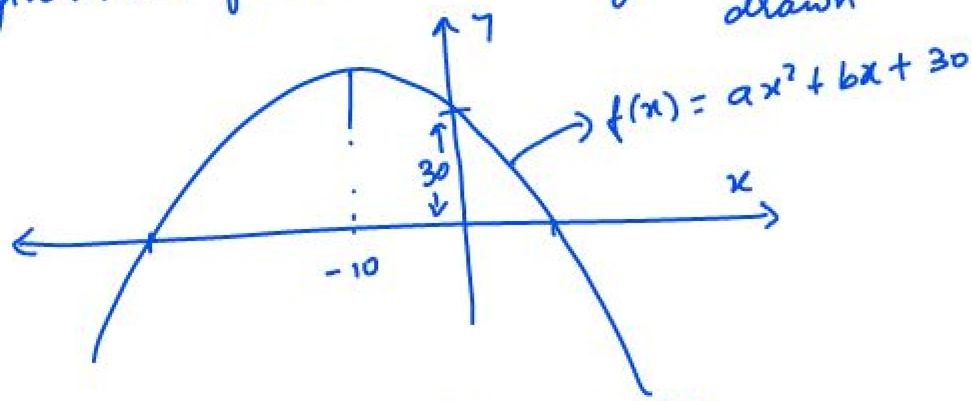
$$x = \frac{1000}{2.3} = 434.78 \simeq 435$$

6. The value of a quadratic function $f(x)$ increases over the interval $(-\infty, -10)$ and decreases over the interval $(-10, \infty)$. Also, $f(0) = f(-20) = 30$. Which of the following statements about $f(x)$ can be true?
(MSQ) (Answer: Option b, c) [Marks: 2]

- ○ $f(x)$ is an injective function.
- ✓ $D \geq 0$, where D is the discriminant of $f(x)$.
- ✓ $f(x)$ can be $-0.1\,(x+30)\,(x-10)$. → $f(-20) = 30$; Also if fulfills other condition
- ✗ $f(x)$ can be $-0.1\,(x-30)\,(x+10)$. → $f(-20) = -0.1(-50)(-10) = -50 \neq 30$

Solution:

Given that $f(0) = 30 \Rightarrow$ y-intercept is 30. And based on given condition figures can be drawn



Thus $D \geq 0$ as per given condition

7. Let A = maximum ($5\log_{10}(100|\sin 2x|)$), and B = minimum($\log_{10}(10x^2 - 40x + 50)$). If the functions are well defined, choose the set of correct options.
(MSQ)(Answer:(a), (b), (c)) [4 marks]

- ◯ $\log_{AB}(A-B) > 0$
- ◯ $\log_{A+10B}(A+B) > 0$
- ◯ $\log_{A+10B}(B-A)$ is not defined.
- ◯ $A + B = -11$.

Solution:-

$A = \max(5\log_{10}(100\times 1))$ (Max. value of sine f^n is 1)

$= \max(5\times 2) = 10$

$\boxed{A=10}$

$B = \min(\log_{10}(10\times 4 - 40\times 2 + 50))$ (Min value will be at $-\frac{b}{2a} = 2$)

$= \min(\log_{10}10) = 1$

$\boxed{B=1}$

✓ option 1: $\log_{10}(9) > 0$

✓ option 2: $\log_{20}(11) > 0$

✓ option 3: $\log_{20}(-9) \rightarrow$ Not defined

X option 4: $A+B = -11$

$11 \neq -11$

8. Define a function

$$f(x) = \begin{cases} \frac{|x^2-4x+3|}{x-3} & \text{if } x \neq 3 \\ 2 & \text{if } x = 3 \end{cases} = \begin{cases} -(x-1) & \text{if } x \in (1,3) \\ (x-1) & \text{if } x \in (-\infty,1) \cup (3,\infty) \\ 0 & \text{if } x = 1 \\ 2 & \text{if } x = 3 \end{cases}$$

(a) Choose the set of correct options.
(MSQ)(Answer: Option a, c) [Marks: 3]

- $\lim_{x\to 3^+} f(x) = f(3)$. $\Rightarrow \lim_{x\to 3^+} (x-1) = 2$; $f(3) = 2$
- $\lim_{x\to 3^-} f(x) = f(3)$. $\Leftarrow \lim_{x\to 3^-} -(x-1) = -2 \neq 2$
- $f(2) = -1$. $\Rightarrow f(2) = -(x-1) = -1$
- f is continuous at $x = 3$. $\Rightarrow LHL \neq RHL$

(b) Choose the set of correct options.
(MSQ)(Answer: Option a, b, d) [Marks: 3]

- $f'(5) = f'(-5) = 1$. $\Rightarrow$ At $x = \pm 5$, $f(x) = x-1 \Rightarrow f'(5) = 1$
- f is not differentiable at $x = 1$.
- f is differentiable at $x = 1$.
- $f'(2) = -1$. $\Rightarrow$ At $x = 2$; $f(x) = -(x-1) \Rightarrow f'(2) = -1$

$$\lim_{h\to 1^-} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 1^+} \frac{f(1+h) - f(1)}{h}$$

$$\Rightarrow \lim_{h\to 1^-} \frac{(1+h-1) - 0}{h} = \lim_{h\to 1^+} \frac{-(1+h-1) - 0}{h}$$

$$\Rightarrow 1 \neq -1$$

Thus not differentiable

9. Let $f(x) : [-11, 9]\backslash\{-10\} \longrightarrow \mathbb{R}$, given that there is only one saddle point. Answer the following questions based on the graph of $f(x)$ shown in Figure: 1 where the symbol $\circ$ signifies the point is excluded and the symbol $\bullet$ signifies the point is included.

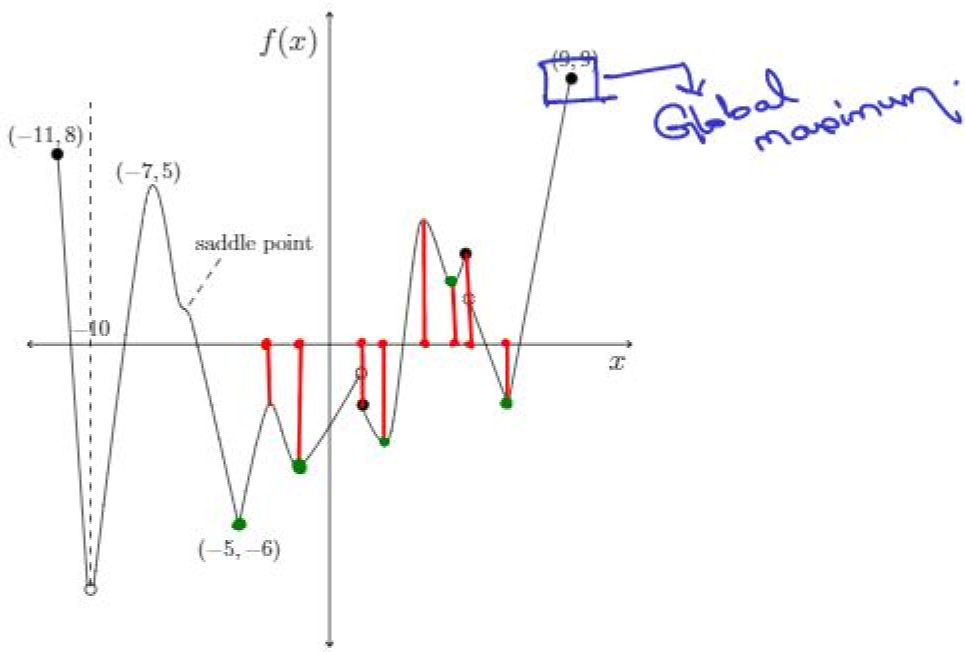


Figure: 1

(a) Choose the set of correct options.
(MSQ)(Answer: Option a, c) [Marks: 2]

- ◯ Global minimum value of the function doesn't exist.
- ◯ Global minimum value of the function is -7.
- ◯ Global maximum value of the function is 9.
- ◯ Global maximum value of the function doesn't exist.

(b) The number of critical points in restricted domain $(-5, 9)$ are
(NAT) (Answer: 8) [Marks: 2]

Soln :- there are 8 critical points.

(c) The number of points where $f(x)$ has local minimum in restricted domain $[-7, 9]$ are
(NAT)(Answer: 5) [Marks: 1]

Soln :- there are 5 local minimum points.

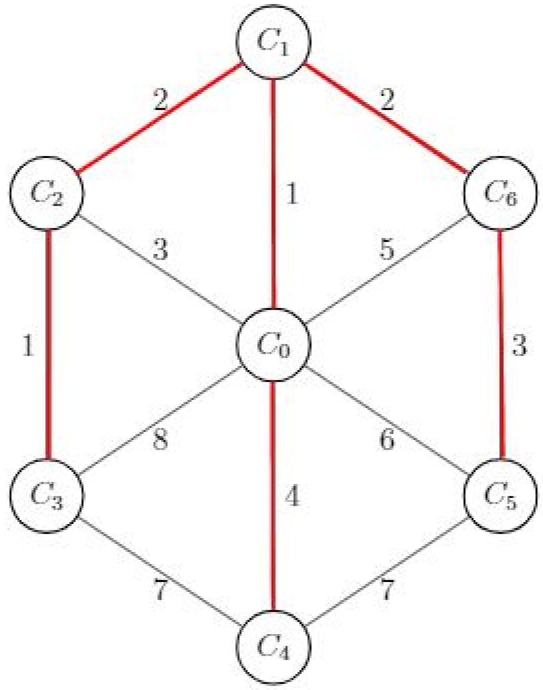
10. Let $f(x) = 4x^3 + 3x^2 + 2x$, then find the value of the integral $\int_0^2 f(x)\,dx$ using limit of Riemann sums as $n \to \infty$, for the given partition $P = \{0 = x_0, x_1 = \frac{2}{n}, \ldots, x_i = \frac{2\times i}{n}, \ldots, x_n = 2\}, i = 1, 2, \ldots, n$ and $x_i^* \in [x_{i-1}, x_i]$, where $x_i^* = \frac{2\times i}{n}$.
(NAT)(Answer: 28) [Marks: 2]

Solution:- Note $n \to \infty$; thus we can directly use

$$\int_0^2 4x^3 + 3x^2 + 2x$$

$$\Rightarrow \left[x^4 + x^3 + x^2\right]_0^2 = \underline{\underline{28}}$$

11. What is the weight of the minimum spanning tree of the graph given below?



(NAT)(Answer: 13) [Marks: 3]

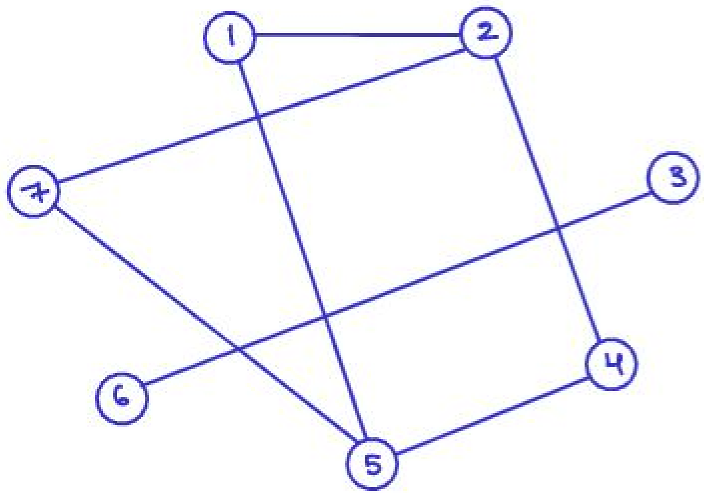
Soln:-

the weight of MCST is $1+1+2+2+3+4$

$= 13.$

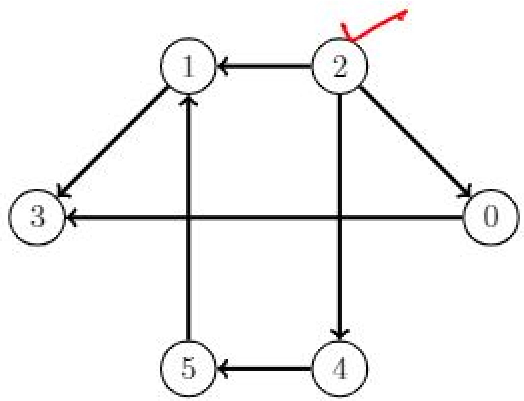
12. Let G be a simple graph with the vertex set $\{1, 2, 3, 4, 5, 6, 7\}$. Suppose two distinct vertices, say i and j, of G are adjacent if and only if $i + j$ is a multiple of 3. Find the number of edges in the graph G?
(NAT)(Answer: 7) [Marks: 3]

Soln:- Number of Vertices = 7
Edge set (E) = $\{(1,2), (2,4), (1,5), (2,7), (3,6), (4,5), (5,7)\}$.



$\therefore$ The number of edges in the graph is '7'.

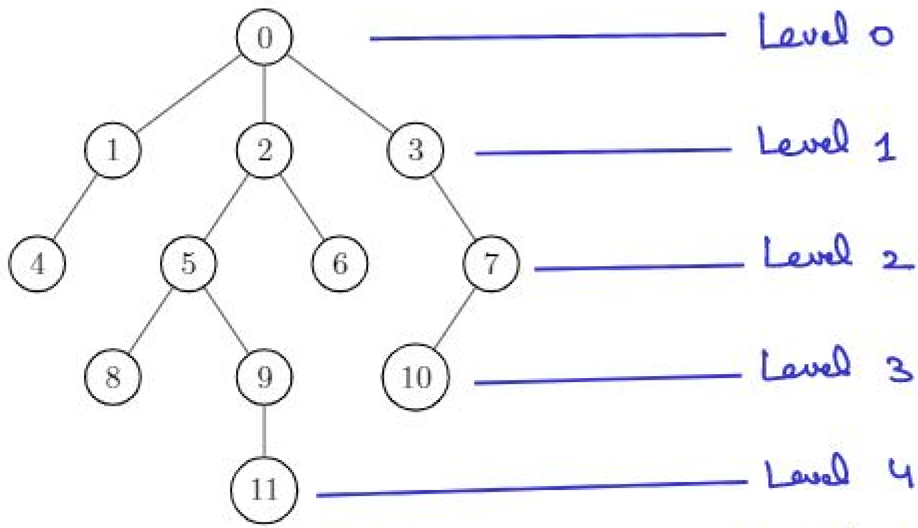
13. A vertex u is called a root vertex of a directed graph if there is a directed path from u to every other vertex in the graph. Below is a directed graph G with the vertex set $\{0, 1, 2, 3, 4, 5\}$. Find the root vertex of G.
(NAT)(Answer: 2) [Marks: 2]



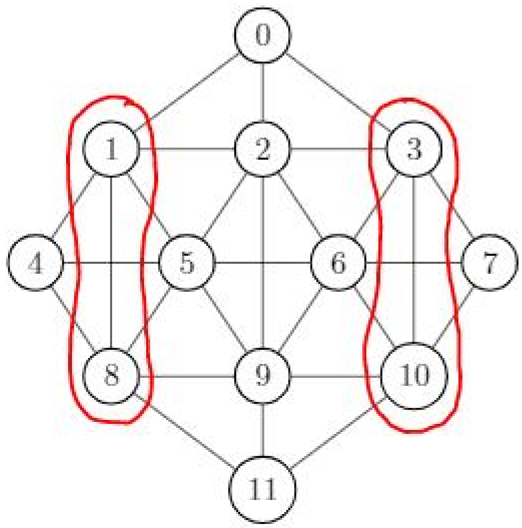
from Vertex 2, every other vertex is reachable.

$\therefore$ 2 is the root vertex.

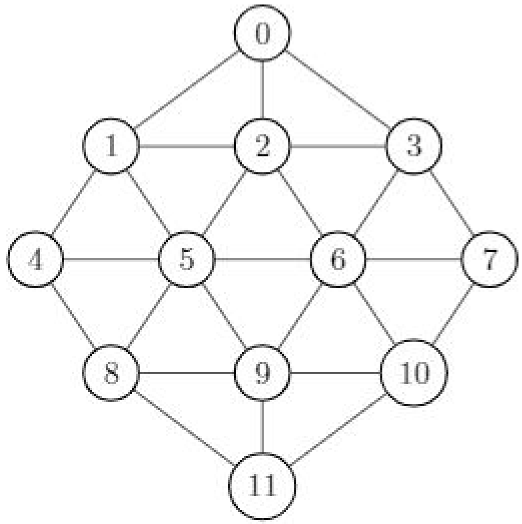
14. The BFS (Breadth First Search) tree of a graph G is shown below. Choose the option which may represent the original graph G.
(MCQ) [Marks: 3]



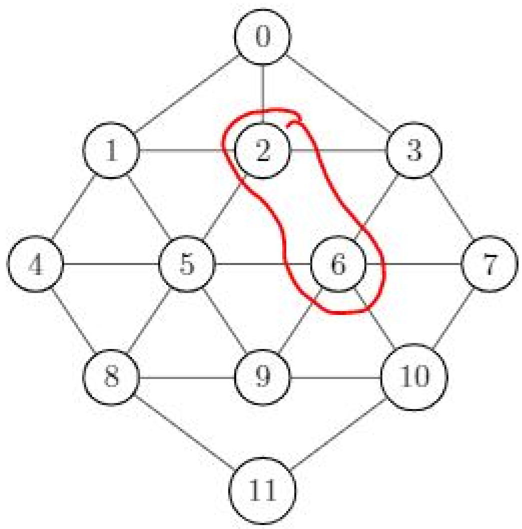
NOTE:- ~~Options:~~ Same level (or) one level above (or) one level below vertices can have an edge b/w them in the original graph.



1. Vertex 1 cannot have an edge with vertex 8 because they are two levels far.

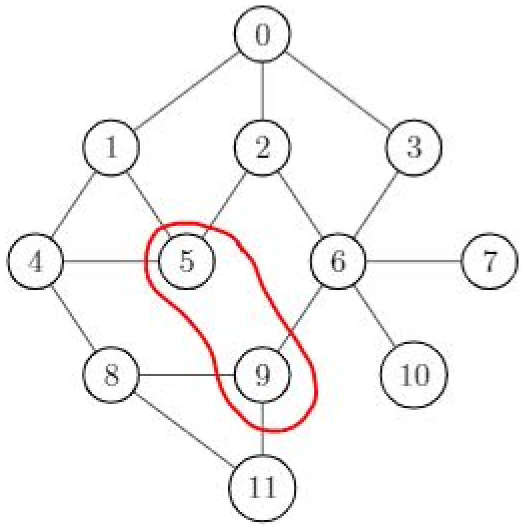


2.



(2,6) edge is missing.

3.

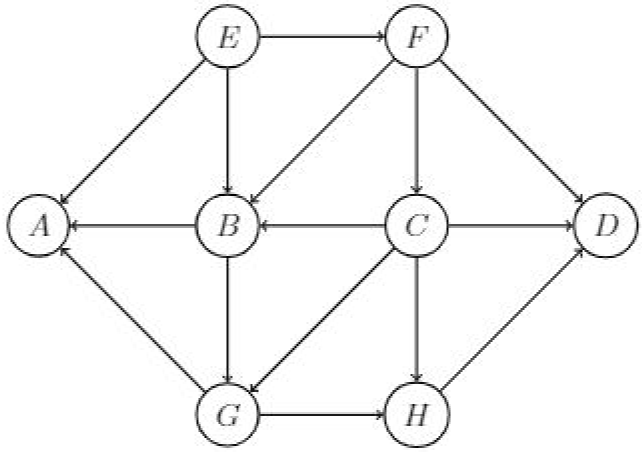


(5,9) edge is missing.

4.

Answer: Option 2.

15. Which of the following statements are correct w.r.t the below given DAG? (MSQ) [Marks: 3]

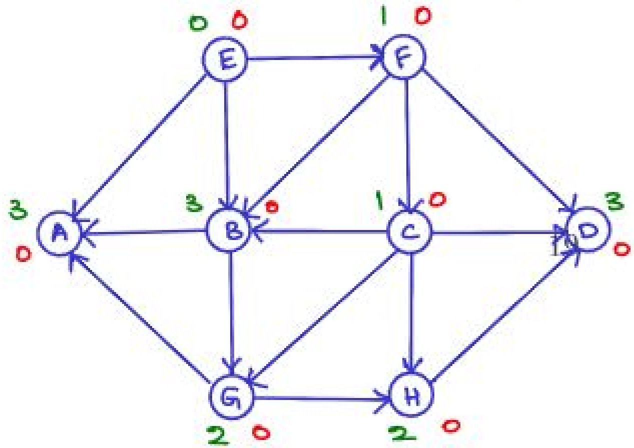


1. A possible topological sequence is $E,\ F,\ C,\ B,\ A,\ G,\ H,\ D$.
2. The longest path in the DAG has length 4, in terms of number of edges.
3. A possible topological sequence is $E,\ F,\ C,\ B,\ G,\ A,\ H,\ D$.
4. A possible topological sequence is $E,\ F,\ C,\ B,\ G,\ H,\ D,\ A$.

Answer: Option (3), Option (4)

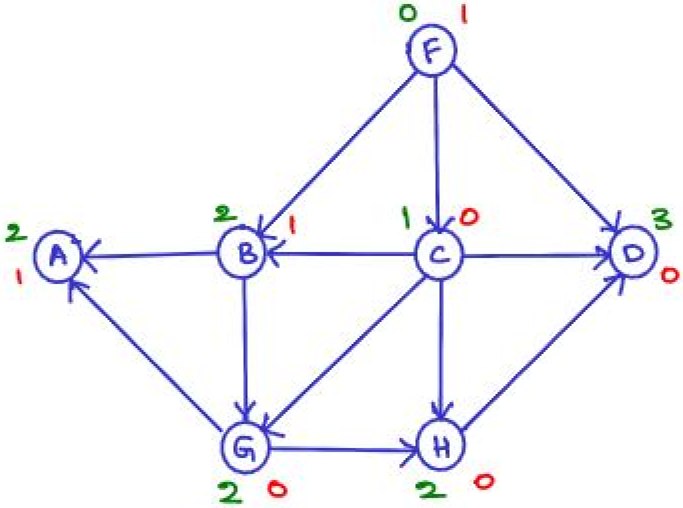
Soln :- Compute the indegrees of every vertex and initialize longest-path-to(i) = 0 for all vertices.

Next,
Remove vertex "E" and add it to the topological sequence and update the indegrees and
longest-path-to(i) = Max {longest-path-to(i), 1 + longest-path-to(E)}.
for all $i \in$ neighbours of vertex "E".

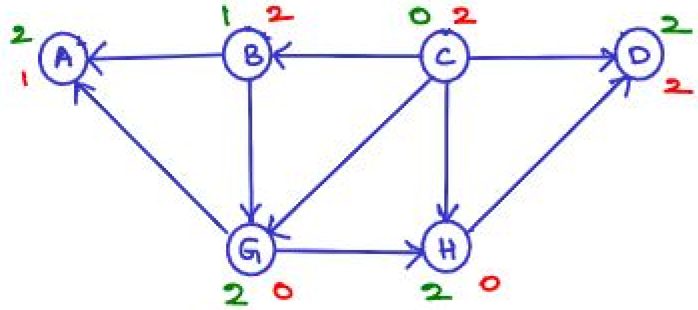


Topological sequence :- E,
longest-path-to(i) :- 0,

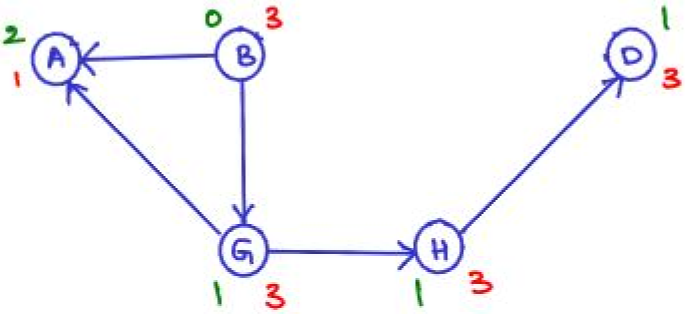
Remove 'E' and update.



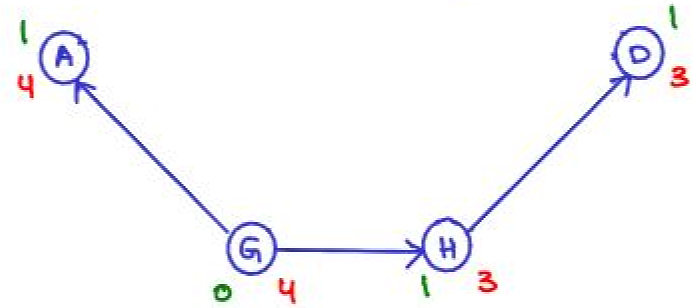
Topological sequence :- E, F
Longest-path-to(i) :- 0, 1,
Remove 'F' and update.



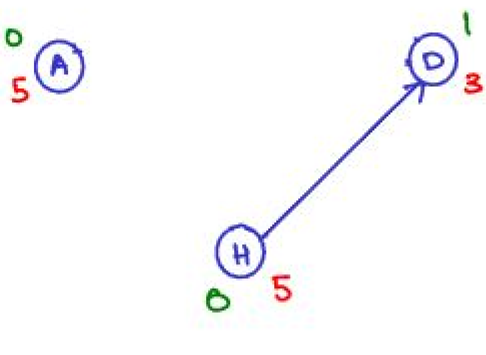
Topological sequence :- E, F, C,
Longest-path-to(i) :- 0, 1, 2,
Remove 'C' and update



Topological sequence :- E, F, C, B,
Longest-path-to(i) :- 0, 1, 2, 3,
Remove 'B' and update.



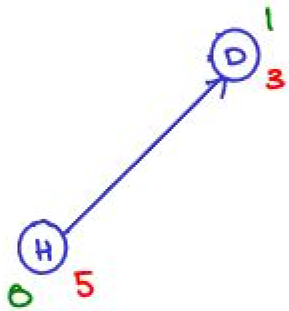
Topological sequence :- E, F, C, B, G,
Longest-path-to(i) :- 0, 1, 2, 3, 4,
Remove 'G' and update



Here, we can add 'A' or 'H' to the topological Sequence.

Case-I :- We add 'A'.

Topological Sequence : E, F, C, B, G, A
Longest path to(i) : 0, 1, 2, 3, 4, 5
Remove 'A' and update.



Topological sequence :- E, F, C, B, G, A, H
Longest-path-to(i) :- 0, 1, 2, 3, 4, 5, 5

Remove 'H' and update

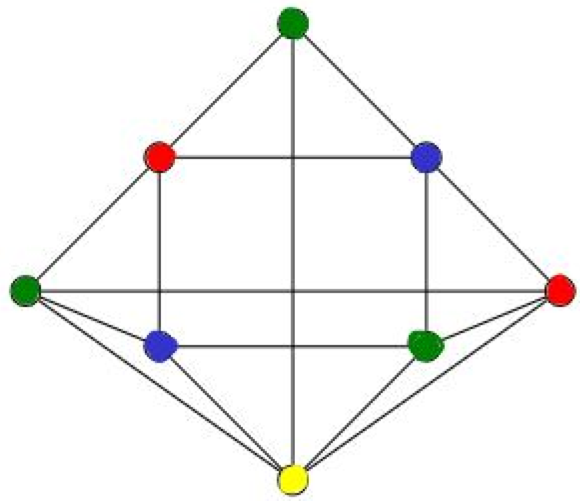
Topological sequence :- E, F, C, B, G, A, H, D
Longest-path-to(i) :- 0, 1, 2, 3, 4, 5, 5, 6

Case-II :- If we add 'H' before 'A' to the topological sequence, then we will get two possible topological sequences.

(1) Topological sequence :- E, F, C, B, G, H, A, D

(2) Topological sequence :- E, F, C, B, G, H, D, A

16. Find the vertex colouring of the below given graph?



(NAT)(Answer: 4) (3 marks)