1 课堂所留题目

求证：$\forall 0 \le \theta \le 1, H(\theta p_1 + (1-\theta)p_2) \ge \theta H(p_1) + (1-\theta)H(p_2)$

证明：

$$H(\theta p_1 + (1-\theta)p_2) = \theta p_1 log\frac{1}{\theta p_1} + (1-\theta)p_2 log\frac{1}{(1-\theta)p_2}$$

$$\theta H(p_1) = \theta p_1 log\frac{1}{p_1}$$

$$(1-\theta)H(p_2) = (1-\theta)p_2 log\frac{1}{p_2}$$

$$\because \theta \in [0,1]$$
$$\therefore \frac{1}{p_1} \le \frac{1}{\theta p_1}$$
$$\frac{1}{p_2} \le \frac{1}{(1-\theta)p_2}$$
$$\therefore \theta H(p_1) = \theta p_1 log\frac{1}{p_1} \le \theta p_1 log\frac{1}{\theta p_1}$$

$$(1-\theta)H(p_2) = (1-\theta)p_2 log\frac{1}{p_2} \le (1-\theta)p_2 log\frac{1}{(1-\theta)p_2}$$

$$\therefore \theta H(p_1) + (1-\theta)H(p_2) \le \theta p_1 log\frac{1}{\theta p_1} + (1-\theta)p_2 log\frac{1}{(1-\theta)p_2} = H(\theta p_1 + (1-\theta)p_2)$$

2 课后所留作业

2.1

解：天平每称一次共有三种结果，每种结果出现的概率均为$p = \frac{1}{3}$

则最小次数$k = \lceil log_3 32 \rceil = \lceil \frac{log_2 32}{log_2 3} \rceil = 4$

2.2

解：将$k = 2021$带入上题式子，得：

$2021 = \lceil log_3 n \rceil$

解得$n = 3^{2021}$

然而由于本题并不清楚假币是更轻或是更重，因此最终结果为$3^{2021}/2$

2.3

解：

(1) 由题意得：令$黑 = B$，$白 = W$

$$P = \begin{pmatrix} B & W \\ 0.3 & 0.7 \end{pmatrix}$$

$$
\begin{aligned}
\therefore H(X) &= \sum plog\frac{1}{p} \\
&= p(B)log\frac{1}{p(B)} + p(W)log\frac{1}{p(W)} \\
&= 0.3log\frac{1}{0.3} + 0.7log\frac{1}{0.7} \\
&= 0.881
\end{aligned}
$$

(2) 对于单次消息而言，其熵与前后条件概率无关。

$\therefore H_2(X) = H(X) = 0.881$

2.4

解：由题意得：

$p(X=i) = q^{i-1}p$

$$
\begin{aligned}
H(X) &= -\sum_{i-1}^{\infty} q^{i-1}plog(q^{i-1}p) \\
&= -p[\sum_{i=1}^{\infty} q^{i-1}logq^{i-1} + \sum_{i=1}^{\infty} q^{i-1}logp] \\
&= -p[logq\sum_{i=1}^{\infty}(i-1)q^{i-1} + logp\sum_{i=1}^{\infty}q^{i-1}]
\end{aligned}
$$

其中：

$$
\begin{aligned}
\sum_{i=1}^{\infty}(i-1)q^{i-1} &= \sum_{i=1}^{\infty} iq^{i-1} - \sum_{i=1}^{\infty} q^{i-1} \\
&= (\sum_{i=1}^{\infty} q^i)' - \lim_{n\to\infty}\frac{1-q^n}{1-q} \\
&= [\lim_{n\to\infty}\frac{q(1-q^n)}{1-q}]' - \frac{1}{1-q} \\
&= \frac{1}{(1-q)^2} - \frac{1}{1-q} \\
&= \frac{q}{(1-q)^2}
\end{aligned}
$$

$\sum_{i=1}^{\infty} q^{i-1} = \frac{1}{1-q}$

$$
\begin{aligned}
\therefore H(X) &= -p[logq\sum_{i=1}^{\infty}(i-1)q^{i-1} + logp\sum_{i=1}^{\infty}q^{i-1}] \\
&= -p[\frac{q}{(1-q)^2}logq + \frac{1}{1-q}logp]
\end{aligned}
$$

2.5

解：

(a) 当$f(X)$是X的一个单射的时候，$H(f(X)) = H(X)$

(b) 由于$Y = f(X) = g^X$是一个单射函数，所以$H(X) = H(Y)$

(c) 由于$Y = f(X) = 5cos(X)$不是一个单射函数，多个X值会对应一个Y值，映射后其不确定性会变小，因此$H(X) > H(Y)$

2.6

解:

(1)

$$
\begin{aligned}
H(Y|x_1) &= -\sum_i p(y_i|x_1)logp(y_i|x_1) \\
&= -(0.6*log0.6+0.4*log0.4) \\
&= 0.97
\end{aligned}
$$

(2)

$$
\begin{aligned}
H(Y|x_2) &= -\sum_i p(y_i|x_2)logp(y_i|x_2) \\
&= -(0.4*log0.4+0.6*log0.6) \\
&= 0.97
\end{aligned}
$$

(3)

$$p(x_1,y_1)=p(y_1|x_1)p(x_1)=0.6*0.75=0.45$$

$$p(x_1,y_2)=p(y_2|x_1)p(x_1)=0.4*0.75=0.3$$

$$p(x_2,y_1)=p(y_1|x_2)p(x_2)=0.4*0.25=0.1$$

$$p(x_2,y_2)=p(y_2|x_2)p(x_2)=0.6*0.25=0.15$$

$$
\begin{aligned}
\therefore H(Y|X) &= -\sum_i\sum_j p(x_i,y_j)logp(y_j|x_i) \\
&= -(0.45*log0.6+0.3*log0.4+0.1*log0.4+0.15*log0.6) \\
&= 0.97
\end{aligned}
$$

(4)

$$
\begin{aligned}
H(Y|X) &= -\sum_i\sum_j p(x_i,y_j)logp(x_i,y_j) \\
&= -(0.45*log0.45+0.3*log0.3+0.1*log0.1+0.15*log0.15) \\
&= 1.781
\end{aligned}
$$

(5)

由(3)得

$$p(y_1)=p(x_1,y_1)+p(x_2,y_1)=0.45+0.1=0.55$$

$$p(y_2)=p(x_1,y_2)+p(x_2,y_2)=0.3+0.15=0.45$$

$$p(x_1|y_1)=p(x_1,y_1)/p(y_1)=0.818$$

$$p(x_1|y_2)=p(x_1,y_2)/p(y_2)=0.666$$

$$p(x_2|y_1)=p(x_2,y_1)/p(y_1)=0.181$$

$$p(x_2|y_2)=p(x_2,y_2)/p(y_2)=0.333$$

$$
\begin{aligned}
\therefore H(X|Y) &= -\sum_i\sum_j p(x_i,y_j)logp(x_i|y_j) \\
&= -(0.45*log0.818+0.3*log0.666+0.1*log0.181+0.15*log0.333) \\
&= 0.788
\end{aligned}
$$