1 课堂所留题目

求证：
当输入分布$P(X)$固定时，$I(X;Y)$是条件概率分布$p(y|x)$的严格下凸函数。

证明：
令$P_1(X|Y)$和$P_2(X|Y)$是任意的两个条件概率分布，其平均互信息分别为$I_1(X;Y_1)$和$I_2(X;Y_2)$
令$P(X|Y)=\alpha P_1(X|Y)+\beta P_2(X|Y)$，其中$0<\alpha<1$，$\alpha+\beta=1$，该分布对应的平均互信息为$I(X;Y)$
要证严格下凸函数，即证$\alpha I_1(X;Y_1)+\beta I_2(X;Y_2)\geq I(X;Y)$

$$
\begin{aligned}
& I(X;Y)-\alpha I_1(X;Y_1)-\beta I_2(X;Y_2) \\
&=\sum_x\sum_y[\alpha p(x)p_1(y|x)+\beta p(x)p_2(y|x)]log\frac{p(x|y)}{p(x)}-\alpha\sum_x\sum_y p(x)p_1(y|x)log\frac{p_1(x|y)}{p(x)}-\beta\sum_x\sum_y p(x)p_2(y|x)log\frac{p_2(x|y)}{p(x)} \\
&=\alpha\sum_x\sum_y p(x)p_1(y|x)log\frac{p(x|y)}{p_1(x|y)}-\beta\sum_x\sum_y p(x)p_2(y|x)log\frac{p(x|y)}{p_2(x|y)} \\
&\leq\alpha log\sum_x\sum_y p(x)p_1(y|x)\frac{p(x|y)}{p_1(x|y)}-\beta log\sum_x\sum_y p(x)p_2(y|x)\frac{p(x|y)}{p_2(x|y)} \\
&=\alpha log\sum_x\sum_y p_1(xy)\frac{p(x|y)}{p_1(xy)/p_1(y)}-\beta log\sum_x\sum_y p_2(xy)\frac{p(x|y)}{p_2(xy)/p_2(y)} \\
&=\alpha log\sum_x\sum_y p(x|y)p_1(y)-\beta log\sum_x\sum_y p(x|y)p_2(y) \\
&=\alpha log\sum_y\frac{p_1(y)}{p(y)}\sum_x p(xy)-\beta log\sum_y\frac{p_2(y)}{p(y)}\sum_x p(xy) \\
&=\alpha log\sum_y\frac{p_1(y)}{p(y)}p(y)-\beta log\sum_y\frac{p_2(y)}{p(y)}p(y) \\
&=\alpha log\sum_y p_1(y)-\beta log\sum_y p_1(y) \\
&=\alpha log1-\beta log1 \\
&=0
\end{aligned}
$$

2 课后所留作业

2.1

解：

$$
\begin{aligned}
\because\ & p(x_1)=p(x_2)=1/2 \\
& p(y_1|x_1)=0.98 \quad p(y_2|x_1)=0.02 \\
& p(y_1|x_2)=0.2 \quad p(y_2|x_2)=0.8
\end{aligned}
$$

$$
\begin{aligned}
\therefore\ & p(x_1y_1)=p(x_1)p(y_1|x_1)=0.49 \\
& p(x_1y_2)=p(x_1)p(y_2|x_1)=0.01 \\
& p(x_2y_1)=p(x_2)p(y_1|x_2)=0.1 \\
& p(x_2y_2)=p(x_2)p(y_2|x_2)=0.4
\end{aligned}
$$

$$
\begin{aligned}
\therefore\ & p(y_1)=p(x_1y_1)+p(x_2y_1)=0.59 \\
& p(y_2)=p(x_1y_2)+p(x_2y_2)=0.41
\end{aligned}
$$

由$p(x|y)=p(xy)/p(y)$得:

$$
\begin{aligned}
& p(x_1|y_1)=0.831 \qquad p(x_2|y_1)=0.169 \\
& p(x_1|y_2)=0.024 \qquad p(x_2|y_2)=0.976
\end{aligned}
$$

$$
\begin{aligned}
& H(X)=-\sum_{i=1}^2 p(x_i)logp(x_i)=1(bits/symbol) \\
& H(Y)=-\sum_{i=1}^2 p(y_i)logp(y_i)=0.98(bits/symbol) \\
& H(XY)=-\sum_{i=1}^2\sum_{j=1}^2 p(x_iy_j)logp(x_iy_j)=1.43(bits/symbol)
\end{aligned}
$$

$$
\begin{aligned}
\therefore\ & I(X;Y)=H(X)+H(Y)-H(XY)=0.55(bits/symbol) \\
& H(X|Y)=H(X)-I(X;Y)=0.45(bits/symbol) \\
& H(Y|X)=H(Y)-I(X;Y)=0.43(bits/symbol)
\end{aligned}
$$

- 2.2

解：

由无记忆得，$p(x_ix_j)=p(x_i)p(x_j)$

$\therefore p(x_1x_1)=1/4 \quad p(x_1x_2)=1/8 \quad p(x_1x_3)=1/8$

$p(x_2x_1)=1/8 \quad p(x_2x_2)=1/16 \quad p(x_2x_3)=1/16$

$p(x_3x_1)=1/8 \quad p(x_3x_2)=1/16 \quad p(x_3x_3)=1/16$

二次扩展信源的概率空间：

$$\begin{pmatrix} X \\ p(X) \end{pmatrix} = \begin{pmatrix} & x_1 & x_2 & x_3 \\ x_1 & 1/4 & 1/8 & 1/8 \\ x_2 & 1/8 & 1/16 & 1/16 \\ x_3 & 1/8 & 1/16 & 1/16 \end{pmatrix}$$

信息熵：

$H(X,X)=2H(X)=-2\sum_{i=1}^{3}p(x_i)logp(x_i)=3(bits/symbol)$

- 2.3

解：

(1) 先计算M_1和输出第一个符号是0的互信息$I_1(y_1=0;M_1)$

$p(y_1=0|M_1)=p(y_1=0|x_1=0)=1-p$

$p(y_1=0)=1/2$

$\therefore I_1(y_1=0;M_1)=1+log(1-p)$

类推可得：

$I_2(y_1=0;M_2)=1+log(1-p)$

$I_3(y_1=0;M_3)=1+logp$

$I_4(y_1=0;M_4)=1+logp$

(2) 先计算M_1和输出第二个符号也是0的互信息$I_1(y_1=0,y_2=0;M_1)$

$p(y_1=0,y_2=0|M_1)=p(y_1=0|x_1=0)*p(y_2=0|x_2=0)=(1-p)^2$

$p(y_1=0,y_2=0)=p(y_1=0)*p(y_2=0)=1/4$

$\therefore I_1(y_1=0,y_2=0;M_1)=1+log(1-p)=2+2log(1-p)$

类推可得：

$I_2(y_1=0;M_2)=2+log(1-p)p$

$I_3(y_1=0;M_3)=2+log(1-p)p$

$I_4(y_1=0;M_4)=2+2logp$

- 2.4

解：

平均符号熵：

$$\begin{aligned} H_2(X) &= \frac{1}{2}H(X_1X_2) \\ &= -\frac{1}{2}\sum_i\sum_j p(x_ix_j)logp(x_ix_j) \\ &= -\frac{1}{2}[0.25log0.25+4*0.06log0.06+0.33log0.33+0.18log0.18] \\ &= 1.223(bits/symbol) \end{aligned}$$

极限熵：

由$p(x_j|x_i)=p(x_ix_j)/p(x_i)$得：

$p(x_0|x_0)=p(x_0x_0)/p(x_0)=0.25/0.31=0.806$

$p(x_1|x_0)=p(x_1x_0)/p(x_0)=0.06/0.31=0.193$

$p(x_2|x_0)=p(x_2x_0)/p(x_0)=0$

$p(x_0|x_1)=p(x_0x_1)/p(x_1)=0.06/0.45=0.133$

$p(x_1|x_1)=p(x_1x_1)/p(x_1)=0.33/0.45=0.733$

$$p(x_2|x_1) = p(x_2x_1)/p(x_1) = 0.06/0.45 = 0.133$$

$$p(x_0|x_2) = p(x_0x_2)/p(x_2) = 0$$

$$p(x_1|x_2) = p(x_1x_2)/p(x_2) = 0.06/0.24 = 0.25$$

$$p(x_2|x_2) = p(x_2x_2)/p(x_2) = 0.18/0.24 = 0.75$$

$$\begin{aligned}
\therefore H(X_2|X_1) &= -\sum_i \sum_j p(x_ix_j) logp(x_j|x_i) \\
&= -[0.25log0.806 + 0.06log0.193 + 0 \\
&\quad +0.06log0.133 + 0.33log0.733 + 0.06log0.133 \\
&\quad +0 + 0.06log0.25 + 0.18log0.75] \\
&= 0.909(bits/symbol)
\end{aligned}$$