Minimum Cost Flow Problem

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Abstract

This article describes the Min Cost Flow Problem in detail. Considering the scenario of the road map in a city and the traffic on each road as the cost, then using the Min Cost Flow problem we can find a route which has the least traffic. Given a flow network with cost of individual links and outgoing/ingoing flows from a particular node. It shows the method for formulating the LPP for the given network. Then the LPP can be solved using the Simplex Method. In this article, we have used 2-Phase method for solving the LPP. It also gives a brief idea to convert the Min-Cost Flow problem to Min-Cost Max Flow problem.

I. Problem Statement

For graph G, the number of edges is m and the number of vertices is n. Capacity and cost per unit flow of each edge are given. We have to transport the maximum amount of fluid/data from source S to sink T, such that the cost to achieve the maximum flow is minimum. In another variation of problem, supply and demand at each node could be given and we have to satisfy those constraints at minimum cost.

II. Introduction

The Minimum-cost flow problem is a combination of the shortest-path problem and the max-flow problem. The shortest path problem considers the cost in the path and takes which path cost is minimized or in other words selects the shortest path. The max-flow problem tries to maximize the flow in the network without considering the cost of the path. Our problem min-cost flow problem tries to find the path with minimum cost and also maintain the desired flow.

The Internet started in 1969 as ARPANET and has become the very basic need of today. We always need more bandwidth and performance. But the channel has limited capacity and bandwidth, and the router has limited buffer size. We can not transmit an infinite amount of packets to the channel. Queuing time, Propagation delay, Transmission delay can not be zero practically. And if we want the maximized use of the channel we must use the channel in a very efficient manner. Sometimes we also have constraints of money, like most of the people can't afford the internet which costs million or billion rupees. Cost can also be in the form of time like time taken by data packets to reach the destination. It can be in the form of practical constraints on Weather conditions like establishing a network in the mountain area or consider a case where weather conditions are too bad at the Bermuda Triangle. Cost can be in the form of political restrictions like the different country has different policies for the network service providers. Cost can also be in the form of availability of resources or it can be in the form of theoretical constraints such as maximum bandwidth rate defined by Shanon's Law. Therefore we must minimize the cost for getting the maximal output.

Consider a network consisting of several routers and several hosts. The exchange of data packets is ongoing between several hosts. To increase the speed of data transfer, data packets should be routed optimally. As each router to router link has the capacity and costs a certain amount of time per unit data as per channel medium. Thus we should figure out an optimal routing algorithm such that Host A can transfer B bytes of data to Host B in minimum time.

Consider a Water Transportation network. Where every pipeline has a limited capacity of water to flow and also cost associated with water per unit. We want to transport maximum water with respect to minimum cost.

Consider a goods transport network. Where we want to send goods to the destination as much as we can, there is the cost of transportation by the length of road and time taken.

In the transmission of electricity, as the distance between two power stations increases, power loss increases. Thus we should find a path such that the maximum amount of power is transmitted with minimum cost.

Below are the some problems that lead to minimum cost maximum flow problem, *A. Routing*

B. Water Transportation

C. Goods Transport Network

D. Energy Transmission

E. Logistic Supply chain

III. AIM OF THE MINIMUM COST FLOW

The minimum cost flow problem can be seen as a generalization of the shortest path and maximum flow problems. We can use this method to find the maximum flow which has the lowest cost among the maximum flow solutions. That is, by suitably choosing costs, capacities, and supplies we can solve the shortest path or maximum flow using any method which will solve the min-cost flow.

The aim of this min-cost flow problem is to reduce the cost or time for any problems that are required to minimize any variable of problems. Like we have seen above, this method is useful in real-life problems such as in routing, water or goods transportation, energy transmission, and also in other problems like logistics, passenger selection in ride sharing, instant water supply, etc. As a result, we can find the cheapest or optimal solution of the flow in the network and we can save resources or time.

IV. OBJECTIVE OF LOGISTIC CHAIN SUPPLY

The objective of problem supply chain logistics is, to optimize the cost of the logistic supply chain.

Respecting capacities, find link flows which balance supply and demand among sources and sinks, with minimum total cost. Each link (i, j) has a specified cost c_{ij} and we must determine its flow x_{ij} . Each node has a supply bi . If $b_i > 0$, the node is a source(or supply), if $b_i < 0$ it is a sink(or demand), and if $b_i = 0$, it is a transshipment node(or intermediate node).

We want to minimize the total transportation cost

$$T(x) = \sum_{(i,j)\in A} c_{ij}x_{ij}$$

.

For simplicity, assume that if there is a link (i, j), then there is no link in the reverse direction (j, i).

The objective is to minimize

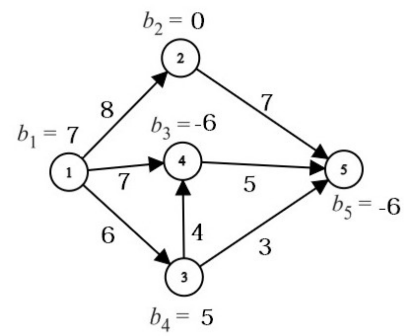

Fig. 1. Simple Graph for Problem

$$z(x) = \sum_{(i,j)\in A} x_{ij}$$

.

Such that,

$$\sum_{(i,j)\in A(i)} x_{ij} - \sum_{(h,i)\in B(i)} x_{hi} = b_i$$

There are three kinds of nodes:

- Manufacturing Plants
- Warehouses
- Retailers

Each manufacturing plant produces a certain quantity of product, which must be shipped first to a warehouse, and then from a warehouse to a retailer which will sell a pre-specified quantity of product.

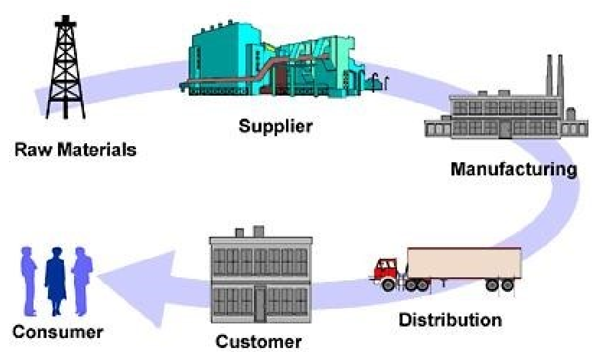

Fig. 2. Supply Chain Logistics Model

Our objective is to minimize transportation costs between each plant and warehouse, and between each warehouse and retailer.

These transportation links have a capacity; furthermore, each warehouse has a capacity.

Some assumptions for our objective are,

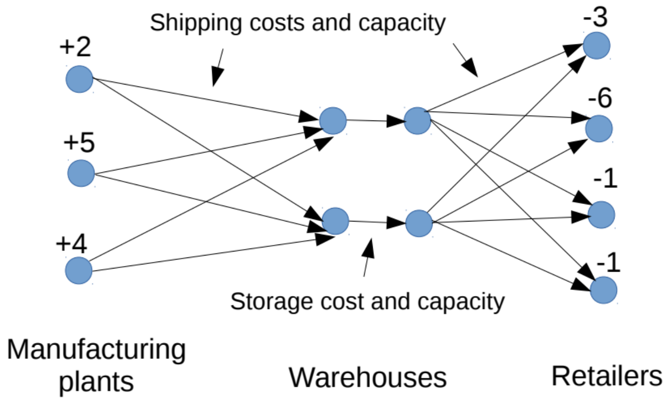

Fig. 3. Transportation Problem

- All data (c_{ij}, b_i) are integral.
- The network is directed.
- All costs associated with edges are nonnegative.
- The supply/demand at the vertexes satisfy the condition $\sum_{\forall i} b_i = 0$ and the minimum cost flow problem has a feasible solution.

V. LPP Formulation

We are given graph G(V, E) (figure 1), with demand and supply given on each node and cost per unit flow on each edge. Our goal is to minimize the total cost while satisfying the supply and demands of nodes.

In figure 1, positive value of b indicates that is a supply point and negative value indicates demand point. Problem should be balanced, in other words total supply = total demand.

C_{ij} is the cost per unit flow of edge and X_{ij} is the flow in edge between i and j.

$$x_{12} = x_1 = \text{edge from 1 to 2}$$
$$x_{13} = x_2 = \text{edge from 1 to 3}$$
$$x_{14} = x_3 = \text{edge from 1 to 4}$$

$$x_{25} = x_4 = \text{edge from 2 to 5}$$
$$x_{34} = x_5 = \text{edge from 3 to 4}$$
$$x_{35} = x_6 = \text{edge from 3 to 5}$$
$$x_{45} = x_7 = \text{edge from 4 to 5}$$

Objective function,

$$z = Minimize \sum_{i=1}^{i=n} \sum_{j=1}^{j=n} C_{ij} X_{ij}$$

$$Z = 8x_1 + 6x_2 + 7x_3 + 7x_4 + 4x_5 + 3x_6 + 5x_7$$

Constraints are as follows: (as per supply and demand conditions and flow of edges)

$$x_1 + x_2 + x_3 = 7$$
$$-x_1 + x_4 = 0$$
$$-x_2 + x_5 + x_9 = 5$$
$$-x_3 - x_5 + x_7 = -6$$
$$-x_4 - x_6 - x_7 = -6$$
$$x_i >= 0, \forall i$$

Thus, above is the LPP formulation which can be solved using simplex method.

VI. Implementation Using 2 - Phase Simplex Method

After formulating the LPP, the second step is solving it to obtain the optimized solution using any appropriate method. Here we have used a 2-phase method for calculating the solution of the Linear Programming problem.

First, we have introduced the artificial variables in all the constraints, so that we can easily get the initial basic feasible solution. In Phase 1, after getting the initial basic feasible solution we have run a simplex algorithm for cost function with 0 co-efficient for all variables of LPP and (-1) for all artificial variables. In Phase-2, we apply simplex with the original cost function to get the minimum cost.
Considering the below graph for the calculations:
For the above given graph, find the minimum value of z,

$$Z = 8x_1 + 6x_2 + 7x_3 + 7x_4 + 4x_5 + 3x_6 + 5x_7$$

Constraints of above graph are,

$$x_1 + x_2 + x_3 = 7$$
$$-x_1 + x_4 = 0$$
$$-x_2 + x_5 + x_9 = 5$$
$$-x_3 - x_5 + x_7 = -6$$
$$-x_4 - x_6 - x_7 = -6$$
$$\forall x_i >= 0$$

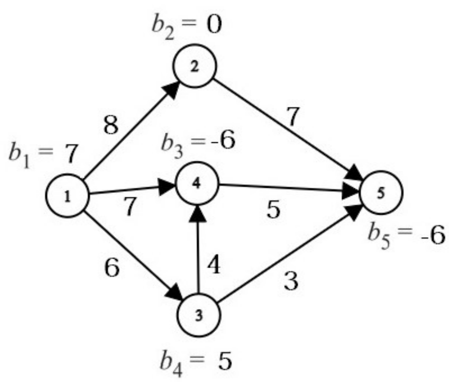

Fig. 4. Flow Network

The problem is converted to canonical form by adding slack, surplus and artificial variables as required.

Converting above equation into standard LPP by multiply 4^{th} and 5^{th} equation by -1,

$$x_1 + x_2 + x_3 = 7$$
$$-x_1 + x_4 = 0$$
$$-x_2 + x_5 + x_6 = 5$$
$$x_3 + x_5 - x_7 = 6$$
$$x_4 + x_6 + x_7 = 6$$
$$\forall x_i >= 0$$

Now, Introducing artificial variables,

- As the constraint-1 is of type $'='$ we should add artificial variable A_1
- As the constraint-2 is of type $'='$ we should add artificial variable A_2
- As the constraint-3 is of type $'='$ we should add artificial variable A_3
- As the constraint-4 is of type $'='$ we should add artificial variable A_4
- As the constraint-5 is of type $'='$ we should add artificial variable A_5

After adding artificial variables, equations are as follows:

$$x_1 + x_2 + x_3 + A_1 = 7$$
$$-x_1 + x_4 + A_2 = 0$$
$$-x_2 + x_5 + x_6 + A_3 = 5$$
$$x_3 + x_5 - x_7 + A_4 = 6$$
$$x_4 + x_6 + x_7 + A_5 = 6$$
$$x_i >= 0, \forall i$$
$$A_i > 0, \forall i$$

For 2-Phase method,

$$Z^* = -A_1 - A_2 - A_3 - A_4 - A_5$$

Phase-1

Initial Table:

B.V.	x_1	x_2	x_3	x_4	x_5	x_6	x_7	A_1	A_2	A_3	A_4	A_5	X_B
$Z_j - C_j$	0	0	2	2	2	2	0	0	0	0	0	0	Z=24
A_1	1	1	1	0	0	0	0	1	0	0	0	0	7
A_2	-1	0	0	1	0	0	0	0	1	0	0	0	0
A_3	0	-1	0	0	1	1	0	0	0	1	0	0	5
A_4	0	0	1	0	1	0	-1	0	0	0	1	0	6
A_5	0	0	0	1	0	1	1	0	0	0	0	1	6

Entering variable = x_3
Leaving variable = A_4
MinRatio = $min\{\frac{7}{1} = 7, \frac{6}{1} = 6\} = 6$
The pivot element is 1.

Row operations after this,
$R_2(new) = R_2(old) - R_5(new)$

B.V.	x_1	x_2	x_3	x_4	x_5	x_6	x_7	A_1	A_2	A_3	A_5	X_B
$Z_j - C_j$	0	0	0	2	0	2	2	0	0	0	0	Z=12
A_1	1	1	0	0	-1	0	1	1	0	0	0	1
A_2	-1	0	0	1	0	0	0	0	1	0	0	0
A_3	0	-1	0	0	1	1	0	0	0	1	0	5
x_3	0	0	1	0	1	0	-1	0	0	0	0	6
A_5	0	0	0	1	0	1	1	0	0	0	1	6

Entering variable = x_4
Leaving variable = A_2
MinRatio = $min\{\frac{0}{1} = 0, \frac{6}{1} = 6\} = 0$
The pivot element is 1.

Row operations after this,
$R_6(new) = R_6(old) - R_3(new)$

B.V.	x_1	x_2	x_3	x_4	x_5	x_6	x_7	A_1	A_3	A_5	X_B
$Z_j - C_j$	2	0	0	0	0	2	2	0	0	0	Z=12
A_1	1	1	0	0	-1	0	1	1	0	0	1
x_4	-1	0	0	1	0	0	0	0	0	0	0
A_3	0	-1	0	0	1	1	0	0	1	0	5
x_3	0	0	1	0	1	0	-1	0	0	0	6
A_5	2	0	0	0	0	2	2	0	0	0	0

Entering variable = x_1
Leaving variable = A_1
MinRatio = $min\{\frac{1}{1} = 1, \frac{6}{1} = 6\} = 1$
The pivot element is 1.

Row operations after this,
$R_3(new) = R_3(old) + R_2(new)$
$R_6(new) = R_6(old) - R_2(new)$

B.V.	x_1	x_2	x_3	x_4	x_5	x_6	x_7	A_3	A_5	X_B
$Z_j - C_j$	0	-2	0	0	2	2	0	0	0	Z=10
x_1	1	1	0	0	-1	0	1	0	0	1
x_4	0	1	0	1	-1	0	1	0	0	1
A_3	0	-1	0	0	1	1	0	1	0	5
x_3	0	0	1	0	1	0	-1	0	0	6
A_5	0	-1	0	0	1	1	0	0	1	5

Entering variable = x_5
Leaving variable = A_3
MinRatio = $min\{\frac{5}{1} = 5, \frac{6}{1} = 6, \frac{5}{1} = 5\} = 5$
The pivot element is 1.

Row operations after this,
$R_2(new) = R_2(old) + R_4(new)$
$R_3(new) = R_3(old) + R_4(new)$
$R_5(new) = R_4(old) - R_4(new)$
$R_6(new) = R_6(old) - R_4(new)$

B.V.	x_1	x_2	x_3	x_4	x_5	x_6	x_7	A_5	X_B
$Z_j - C_j$	0	0	0	0	0	0	0	0	Z=0
x_1	1	0	0	0	0	1	1	0	6
x_4	0	0	0	1	0	1	1	0	6
x_5	0	-1	0	0	1	1	0	0	5
x_3	0	1	1	0	0	-1	-1	0	1
A_5	0	0	0	0	0	0	0	1	0

Since all $Z_j - C - j \leq 0$.
Hence, optimal solution is,
$x_1 = 6, x_2 = 0, x_3 = 1, x_4 = 6,$
$x_5 = 5, x_6 = 0, x_7 = 0$
So,
$$Z = 0$$

Phase-2
Here,
$$Z = 8x_1 + 6x_2 + 7x_3 + 7x_4 + 4x_5 + 3x_6 + 5x_7$$
Initial table:

B.V.	x_1	x_2	x_3	x_4	x_5	x_6	x_7	X_B
$Z_j - C_j$	0	-3	0	0	0	9	3	Z=117
x_1	1	0	0	0	0	1	1	6
x_4	0	0	0	1	0	1	1	6
x_5	0	-1	0	0	1	1	0	5
x_3	0	1	1	0	0	-1	-1	1

Entering variable = x_6
Leaving variable = x_5
MinRatio = $min\{\frac{6}{1} = 6, \frac{6}{1} = 6, \frac{5}{1} = 5\} = 1$
The pivot element is 1.

Row operations after this,
$R_2(new) = R_2(old) - R_4(new)$
$R_3(new) = R_3(old) - R_4(new)$
$R_5(new) = R_5(old) + R_4(new)$

Entering variable = x_2
Leaving variable = x_1
MinRatio = $min\{\frac{1}{1} = 1, \frac{1}{1} = 1\} = 1$
The pivot element is 1.

B.V.	x_1	x_2	x_3	x_4	x_5	x_6	x_7	X_B
$Z_j - C_j$	0	6	0	0	-9	0	3	Z=72
x_1	1	1	0	0	-1	0	1	1
x_4	0	1	0	1	-1	0	1	1
x_5	0	-1	0	0	1	1	0	5
x_2	0	0	1	0	1	0	-1	6

Row operations after this,
$R_3(new) = R_3(old) - R_2(new)$
$R_4(new) = R_4(old) + R_2(new)$

B.V.	x_1	x_2	x_3	x_4	x_5	x_6	x_7	X_B
$Z_j - C_j$	-6	0	0	0	-3	0	-3	Z=66
x_2	1	1	0	0	-1	0	1	1
x_4	-1	0	0	1	0	0	0	0
x_6	1	0	0	0	0	1	1	6
x_3	0	0	1	0	1	0	-1	6

Since all $Z_j - C - j \leq 0$.
Hence, optimal solution is,
$x_1 = 0, x_2 = 1, x_3 = 6, x_4 = 0,$
$x_5 = 0, x_6 = 6, x_7 = 0$
So,
$$Z = 66$$

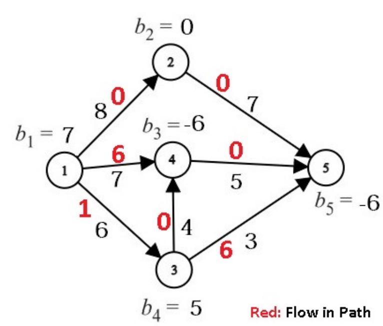

Fig. 5. Optimized Flow Network

In the above problem, demand and supply at each node was kept fixed. If we remove this constraint then it can be converted to Min Cost Max Flow problem. In Min Cost Max Flow problem given a source node and sink node, we have to find a path from all the paths having maximum flow from source to sink that gives minimum cost.

In order to obtain Min Cost Max flow problem, we calculate min cost for a particular value of flow. If for a particular flow, the solution does not exist then the algorithm terminates. If there exist a solution then we increase the value of flow as much as possible. If at some instant the flow reaches the desired value or the solution does not exist then we stop iterating further.

VII. APPLICATIONS

As we know, the min-cost flow problem is an optimization problem that helps find the cheapest way of acquiring the required flow. So, it is helpful in almost all areas of industries like agriculture, communications, defense, education, energy, health care, manufacturing, medicine, retailing, and transportation. So essential applications are as follows. Distribution and transportation problems take up a significant portion of the min cost-flow issues. They can be generalized as some production units, consumer units, and different paths that connect them. Here, we aim to satisfy our demand with the route that costs us the lowest.

The min-cost flow problem can also be helpful when encountering situations where we need to distribute the human resources while also ensuring the balanced ratio of gender and minimizing the distance they need to travel. Also, there are employment scheduling, Optimal capacity scheduling, equipment replacement, different linear programs, and many other essential problems that can be solved using the min cost flow problem. Here we can see that many real life problems can be solved using min-cost flow problem. We can convert the real life problem in mathematical model and then it can be solved using the best algorithm. Therefore solving ing min-cost flow problem is very useful for us.

VIII. CONCLUSION

In this paper, we considered the Min-Cost flow problem on a particular graph. For Solving the problem, we generated the LPP for the graph. Then we used the 2-Phase Simplex Method to solve the Linear Programming Problem. The advantage of this method is that it is easy to understand with the help of Simplex tables.

Our computations show that this problem can also be converted into a transportation and transshipment problem. However, we have not discussed those in detail.

There are many applications of Min-Cost Flow Problem. The most common is the Transportation Problem. Min-Cost Flow problem can be easily converted to it by adding a constraint of link capacity to the network/graph under consideration. Many modifications can be made to solve several problems in the real life.

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