50 years of spin glass theory

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In 1975, two papers were published that together sparked major new directions, conceptual, mathematical and practically applicable, in several previously disparate fields of science. In this short review, we expose key aspects of their thinking, implementations and implications, along with a selection of further crucial and consequential developments. These papers were "Theory of spin glasses" by S.F.Edwards and P.W.Anderson (EA)[1] and "Solvable Model of a Spin-Glass" by D.Sherrington and S.Kirkpatrick (SK)[2], both concerned with trying to understand recent experiments that suggested a new phase of matter.

The experiments date from around 1970. They were on substitutional alloys of magnetic transition and non-magnetic noble metals and indicated that at low but finite concentration of the magnetic ions and low temperature the magnetic moments (spins) are individually 'frozen' in their orientation over nuclear spin resonance timescales, but in a quasi-random, non-periodic, fashion. It was named *spin glass* by analogy with positional amorphousness in normal glasses. Initially, transition to this behaviour as temperature was reduced seemed gradual, but in 1972 Cannella and Mydosh [3] observed that it sharpened as the external field was reduced, suggesting a cooperative phase transition to a hitherto-unknown novel phase.

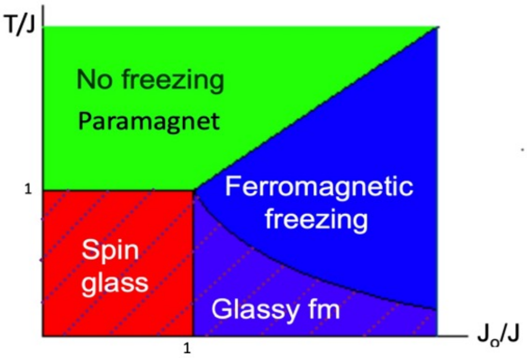
However, there was no potentially satisfying theoretical explanation until EA took an interest. They used a statistical physics model approach but with many inspirationally-new concepts and mathematical tools. Among them were a novel but irregular mathematical procedure for evaluating quenched averages of observables, involving a combination of replication and limit-taking, $\langle \ln Z \rangle = \lim_{n\to 0} n^{-1}\{Z^n - 1\}; Z^n = \Pi_{\alpha=1,..n} Z_\alpha$ where the α label replicas, a new type of order parameter, $q^{\alpha\beta} = N^{-1}\sum_i \langle S_i^\alpha S_i^\beta \rangle; \alpha \neq \beta$, as well as replacing (experimental) random site disorder by 'easier-to-deal-with' (theoretical) random bond disorder. With some further assumptions, approximations and ansätze they were able to demonstrate a phase transition to a new 'amorphous' cooperative spin ordering,

EA's paper immediately excited one of us (DS) by both its conceptually new ideas and its new methods of analysis, but he also found it challenging in evaluating the correctness of the assumptions and approximations. To test these further, he devised a related model that he expected should be solvable exactly, with an infinite-ranged and range-independent interaction distribution. Also, for calculational simplicity while maintaining conceptuality, he considered discrete Ising spins. Using the same two ansätze as EA, he derived Ising analogues of their equations and interested the other of us (SK) in further joint studies, both analytic and by computer simulation. These yielded a phase diagram in qualitative accord with experiment, but also led to discoveries that had major further consequence; (i) a prediction of a negative entropy at zero temperature, a result that is fundamentally forbidden for discrete variables and signalled a serious procedural error, (ii) evidence of a complex 'energy landscape' with many hills and valleys on many levels, hindering equilibration. More detailed computer simulations, reported in a subsequent longer paper (KS)[4], indicated a lower ground state energy than the calculation. There was a problem.

EA and SK had both made a 'natural' ansatz for the order parameter $q^{\alpha\beta}$ that it should have the same value for any non-identical pair of replicas $\alpha \neq \beta$, now known as 'replica symmetry' (RS). However, two years later, J.R.L. de Almeida and D.J Thouless [5] showed that this choice was unstable against fluctuations in replica space. Two further years later, Giorgio Parisi [6] devised a remarkable 'replica symmetry breaking' (RSB) ansatz that solved the negative entropy problem and also gave a lower ground state energy result close to the KS simulation. It involved radically new mathematical conceptualization and formulation, in which the order parameter is now a function $q(x); 0 \leq x \leq 1$. In 1983 he provided its physical explanation [7] and opened a new insightful physical conceptualization. Interest and activity exploded, leading to many further new discoveries [8], such as ultrametricity, non-selfaveraging, unusual dynamics, breakdown of the usual fluctuation-dissipation theorem, extensions to new theoretical models and further new concepts and perspectives, e.g. to simple glasses and multiple meta-equilibria, leading eventually to Parisi's Nobel Prize of 2021.

While EA chose to study the symmetric zero-mean random exchange distribution, SK included provision for a non-zero mean to emulate the experimental observations of transitions from ferromagnetism to spin glass with lowering of the concentration of magnetic constituent.

Later, in 1982, John Hopfield [10] proposed and demonstrated by computer simulation a model system for



SK-AT-Parisi-Toulouse[9]: $\overline{J_{ij}} = J_0/N, (\overline{J_{ij}^2 - \overline{J_{ij}}^2}) = {J_0}^2/N$

distributed attractor memory and associative retrieval, in which the stored memories can be interpreted as an extension of SK to include many gauge-transformed ferromagnets, with corresponding attractor basins. Through an extension of SK analysis, Amit, Gutfreund and Sompolinsky [11] demonstrated its resultant combination of 'basins of attraction' of stored memories, up to its critical retrieval capacity, their competitively constrained 'basin extents' and passage to 'spin glass complexity' beyond that capacity. Hopfield's model did not include explicitly the 'exchange-distribution' variance term of EA, J in SK. Rather, for each retrieved memory, the non-retrieved memory terms in the Hamiltonian act as potentially-disruptive quenched 'noise'.

Both biological brains and artificially intelligent neural networks require mechanisms for learning as well as recalling and generalizing. Within an extension of the SK conceptualisation this can be translated to gradually modifying the interactions/synapses in a learning stage, to be later recalled or generalized in faster neural dynamics, either in a recurrent version (as one might model the brain) or a directed layered one as in most AI/'deep learning' applications.

EA and SK are examples of optimization in the presence of stochasticity, temperature noise. At the time of our collaboration, Scott K, at IBM Research, was involved in understanding problems of designing computer systems and their components. He recognised that extensions of the computer-simulation methods developed for spin glasses would be a natural and powerful generalization of the highly specialised and more limited approaches then used in the placement of components and the routing of signals between them. The generalization to other optimization problems led to the concept and application of 'simulated annealing' [12].

EA's theoretical replication concept has also led to informative computer-simulational analogues, in which replica systems with identical quenched disorder and the same temperature level of noise are allowed to evolve independently and correlations between them studied to indicate spin glass-like transitions, critical properties and physical measures [13].

Another consequence within computer science has been the recognition of interest in 'typical' as well as 'worst-case' studies.

As well as both EA and SK stimulating much further research in theoretical physics [14], the SK model also provided new exciting challenges for mathematical physicists [15] [16] and probabilists [17] [18] and led to several new developments in both subjects. Its rigorous solution by Michel Talagrand played a significant role in his award of the 2024 Abel Prize.

So, the SK model is indeed 'solvable' but the solution has proven to be very subtle, as well as consequential.

Prior to SK, mean field/infinite-range theory was mostly considered to be fairly trivial . However, its study has demonstrated that it can be complex and highly non-trivial when it involves quenched disorder and frustration (competing interactions or instructions). It has spawned a plethora of conceptual extensions, some physics oriented but many also in other areas. Also, in issues concerning information communication, the internet has made distance essentially irrelevant.

The actual alloys that started the journey have not proven practically valuable, but one of us (DS) has argued that the pictures developed provide possible understanding of some other solid state alloys, experimentally discovered long ago and of practical value but with less theoretical investigation, such as relaxor ferroelectrics and martensitic shape-memory alloys. [19].

Finally, we may note that blue sky experimental studies of some alloys at low temperature have, by a combination of attempts at theoretical modelling, observations of puzzling anomalies, curiosity, tenacity and brilliance, led to a 'cornucopia'[20] of unanticipated major conceptual, theoretical, mathematical and practical-computational advances, with more expected.

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