Generalized Differential Geometry*

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Abstract

Generalized Functions play a central role in the understanding of differential equations containing singularities and nonlinearities. Introducing infinitesimals and infinities to deal with these obstructions leads to controversies concerning the existence, rigor and the amount of nonstandard analysis needed to understand these theories. Milieus constructed over the generalized reals sidestep them all. A Riemannian manifold M embeds discretely into a generalized manifold M^* on which singularities vanish and products of nonlinearities make sense. Linking this to an already existing global theory provides an algebra embedding $\kappa: \hat{\mathcal{G}}(M) \longrightarrow \mathcal{C}^{\infty}(M^*, \widetilde{\mathbb{R}}_f)$. Generalized Space-Time is constructed and its possible effects on Classical Space-Time are examined.

1 Introduction

One of the main problems of mathematics is to construct environments in which equations have solutions. They need to be constructed together with a set of mathematical tools that facilitates the constructions of solutions and guarantee a way of obtaining classical solutions if they do exists. For differential equations involving singularities and nonlinearities the construction of such milieus involves the introduction of infinitesimals and infinities and hence the endeavor is not only challenging but also leads to misconceptions about the amount of non-standard analysis one must master.

This does not need to be the case at all! For example, in [15, 16, 19] such an endeavor is undertaken. An environment, whose basic underlying structure is ${}^{\bullet}\mathbb{R}$, the Fermat reals, a totally ordered topological ring containing infinitesimals, is constructed in which nearly all classical features of Newtonian Calculus hold. Basically, ${}^{\bullet}\mathbb{R}$ is the union of the halos of elements ${}^{\circ}x \in \mathbb{R}$ with $halo({}^{\circ}x) = \{{}^{\circ}x + dt_a : a \in \mathbb{R}, a \geq 1\}$ and each dt_a is a nilpotent element of order $min\{p \in \mathbb{N} : \frac{a}{p} < 1\}$. In particular, $Inv({}^{\bullet}\mathbb{R})$, the group of invertible elements of ${}^{\bullet}\mathbb{R}$, consists of the non-infinitesimals and hence, $Inv({}^{\bullet}\mathbb{R})$ is a group which is open but not dense in ${}^{\bullet}\mathbb{R}$. Moreover, \mathbb{R} is discrete in ${}^{\bullet}\mathbb{R}$ and, because of its structure, ${}^{\bullet}\mathbb{R}$ contains no infinities. This should pose some limitation on the Calculus over ${}^{\bullet}\mathbb{R}$ when dealing with problems involving certain singularities and nonlinearities both being very common in Physics, Fluid Dynamics and other areas. In such areas, infinitesimals and infinities have to coexist. Hence any milieu aiming to deal with physical reality and in which infinitesimals and infinities do coexist, some of the latter and the former must be invertible elements. The following questions then arises: Can such an environment be constructed? And if so, what is

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the amount of non-standard analysis needed to master the basics of the construction? If it does exists, how does the Calculus constructed in this environment compare to the already familiar and so successful Newtonian Calculus and Schwartz's premier work on generalized functions? A reasonable way or tools to obtain classical solutions from the generalized solutions must also be given.

In this paper we present environments \mathbb{R}^n whose underlying structure is an ultrametric non-Archimedean partially ordered ring \mathbb{R} containing \mathbb{R} . Hence, our proposal is in the realm of non-Archimedean function theory born in 1961 in a Harvard seminar of J. Tate. It is a specialized and advanced field of mathematics, requiring a solid foundation in mathematical analysis and familiarity with nonstandard analysis. Tate's remarkable discoveries led to a theory which made possible analytic continuation over totally disconnected ground fields. The original context of the construction of \mathbb{R} can be found in [2, 11, 30], but these original sources do not contain any of the algebraic and topological ingredients which now do exists.

To understand the construction of the non-Archimedean milieus that we propose, only working knowledge in Algebra, Calculus and Distribution Theory is needed. In spite of the nonstandard flavor, absolutely no familiarity with nonstandard analysis is required. The ring \mathbb{R} , coined the generalized reals, has very nice properties some of which we enumerate: it is a partially ordered ultrametric ring, has no nilpotent elements and its group of invertible elements, $Inv(\mathbb{R})$, is open and dense. Given $x \in \mathbb{R} - Inv(\mathbb{R})$, there exist $f, e \in \mathcal{B}(\mathbb{R})$, the Boolean algebra of idempotents of $\widetilde{\mathbb{R}}$, such that $e \cdot x = 0$ and $f \cdot x \in Inv(f \cdot \widetilde{\mathbb{R}})$. Moreover, \mathbb{R} embeds into $\overline{B}_1(0) \subset \widetilde{\mathbb{R}}$ as a grid of equidistant points. The unit ball $B_1(0)$ consists of infinitesimals, as expected, and \mathbb{R} contains an isomorphic copy of the group $(\mathbb{R},+)$ which we denote by $\{\alpha_r : \alpha_r \cdot \alpha_s = \alpha_{r+s}, r \in \mathbb{R}\}$ all of which, except for $1 = \alpha_0$, are either infinities or infinitesimals. In a subset of \mathbb{R} , Dirac's infinity, the delta Dirac function, becomes a differentiable function such that $\delta(0) = \alpha_{-1}$, an infinity, and the zero distribution $f(x) = x\delta$ becomes a non-zero differentiable function with f(0) = 0. Also, differently than in [19], the existence of primitives follows easily. So the milieus we referred to do exist and the Calculus developed extends in a very natural way classical calculus and distribution theory. In this framework theories of generalized functions are brought back to Newton's most fundamental insight: the notion of variation; the single most important idea in Science and the essence of Nature. One of the mathematical tools to obtain a classical solution from a solution obtained in these generalized environments is the tool of support which can be defined in any such environment. It naturally extends the notion of association or shadow from distribution theory.

The layout of the paper is thought out to first construct these environments in a simple and clear way so that the reader can get acquainted with its construction. For this, we first construct what are known as the simple environments. After this, we switch gear and look at what are known as the full environments with underlying basic structure \mathbb{R}_f . The latter are needed if one looks for a canonical embedding of $\mathcal{D}'(\Omega)$. It is in them that we prove a fixed point theorem and construct a generalized differential manifold from a classical Riemannian manifold. In particular, we obtain Generalized Space-Time, a generalized version of Classical Space-Time in which infinities and infinitesimals naturally coexists and which contains Classical Space-Time as a discrete and bounded grid of equidistant points.

In the next section, we explain the construction of the generalized reals \mathbb{R} and give its main properties. In the third section, we introduce functions defined on subsets of \mathbb{R}^n and show how to embed classical functions in a natural way into these new algebras. In the fourth section, we define the notion of differentiability being a natural extension of Newton's Calculus. In the fifth section new ideas and machinery are proposed, constructed and used. As an application we prove a Fixed

Point Theorem and the *Down Sequencing Argument*.

Theorem 1.1 [Fixed Point Theorem]

Let $\Omega \subset \mathbb{R}^N$, $A = [(A_{\varphi})_{\varphi}] \subset B_{r_0}(0) \cap \mathcal{G}_f(\Omega)$, $r_0 < 2$, be an internal set, and $T : A \to A$ be given with representative $(T_{\varphi} : A_{\varphi} \to A_{\varphi})_{\varphi \in \mathcal{A}_0}$. If there exists $k = [(k_{\varphi})_{\varphi}] \in \widetilde{\mathbb{N}}$ such that $T^k = (T_{\varphi}^{k_{\varphi}})$ is a λ -contraction, then T is well defined, continuous and has a unique fixed point in A.

Theorem 1.2 [Down Sequencing Argument] Let $f \in \mathcal{G}_f(\Omega)$, $f \in W^0_{l,4^k r}[0]$ with r > 0. Then $f \in W^k_{l,r}[0]$, i.e., $W^0_{l,4^k r}[0] \subset W^k_{l,r}[0]$.

In section six, we continue in the full version of these milieus and show how to construct a generalized manifold M^* from a classical Riemannian manifold M and prove an embedding result extending results of [3] by linking them to results obtained in [20, 21]. The construction of M^* involves several developments in the field. Consequently, we extend some former results, obtained for open subsets of \mathbb{R}^n , to abstract manifolds.

Theorem 1.3 [Embedding Theorem] Let M be an n-dimensional orientable Riemannian manifold. There exists an n-dimensional \mathcal{G}_f -manifold M^* and an algebra monomorphism

$$\kappa: \widehat{\mathcal{G}}(M) \longrightarrow \mathcal{C}^{\infty}(M^*, \widetilde{\mathbb{R}}_f)$$

which commutes with derivation. Moreover, $ssupp(M^*) = \overline{M}$, the topological closure of M, and equations defined on M, whose data have singularities or nonlinearities, naturally extend to equations on M^* where these data become C^{∞} -functions.

Finally, we extend Generalized Differential Calculus to modules over the generalized reals and the generalized complex numbers in the full setting. The effects of Generalized Space-Time on Classical Space-Time is considered as is the effect of the existence of generalized solutions of differential equations on numerical solutions of these equations. Absolutely no claim is made that Generalized Space-Time corresponds to physical reality. References on which this introduction is based are [1, 2, 3, 5, 9, 10, 11, 23, 24, 25, 30, 32, 36]. Notation used is standard.

2 The underlying structure $\widetilde{\mathbb{R}}$

In this section we construct the ring \mathbb{R} which is the basic underlying structure of the environments to be created. Its construction is without the use of nonstandard methods and basically mimics the sequential construction of the reals starting from the rationals, the reason to coin it the *generalized reals*. It is constructed in such a way that whatever norm used in \mathbb{R}^n results in the same norm in \mathbb{R}^n .

Let us first explain the construction in terms of the sequential construction of \mathbb{R} starting from \mathbb{Q} , but enlarging a bit the initial ring and factoring out by an ideal so that infinities and infinitesimals are introduced in the resulting quotient ring. Our initial ring consists of rational sequences of polynomial growth. There is a reasonable explanation for the why of this model: this has to do with the notion of computability which in turn relates to the notion of time, thus making the model more in sync with physical reality since infinities whose growths are bigger then polynomial growth should not exist in physical reality, but, on the other hand, infinitesimals can be used to perform calculations of exponential growth. Let's look at the math. We consider rational sequences and

the partial order $(x_n) \leq (y_n)$ if there exists $n_0 \in \mathbb{N}$ such that $x_n \leq y_n, \ \forall \ n \geq n_0$. Consider the sequence $\tau = (\frac{1}{n}), \ \tau_k = \tau^k = (\frac{1}{n^k}), k \in \mathbb{Z}$ and the algebra $\mathcal{A} = \{(x_n)_{n \in \mathbb{N}} : x_n \in \mathbb{Q}, \exists \ N \in \mathbb{Z} \}$ \mathbb{N} , such that $(|x_n|) \leq \tau_{-N}$. Then $\mathcal{J} = \{(x_n) \in \mathcal{A} : (|x_n|) \leq \tau_N, \ \forall \ N \in \mathbb{N}\}$ is a radical ideal of \mathcal{A} . The image of the Cauchy sequences in the quotient algebra $\mathcal{A} = \mathcal{A}/\mathcal{J}$ is a copy of \mathbb{R} and the image of the bounded sequences (compactly supported sequences) is a subring containing the copy of \mathbb{R} . The nontrivial elements of the Boolean algebra of idempotents, $\mathcal{B}(\hat{\mathcal{A}})$, are characteristic functions of special subsets of $\mathbb N$ and the image of the powers of τ are infinities and infinitesimals. Infinitesimals have representatives which converge to zero. Given an element x with a bounded representative, there exist idempotent $e \in \mathcal{B}(\mathcal{A})$ and $x_e \in \mathbb{R}$ such that $ex - ex_e$ is an infinitesimal. The element exis nothing more than a subsequence of x, i.e., multiplying an element by an idempotent corresponds to taking a subsequence of that element and x_e is an accumulation point of x. For any element $x \in \mathcal{A}$, define $supp(x) = \{x_e \in \mathbb{R} : ex - ex_e \text{ is an infinitesimal}, e \in \mathcal{B}(\mathcal{A})\}$. If the support consists of only one element x_{e_0} we say that x_{e_0} is the shadow of x, i.e., the sequence converges. It is not hard to see that in this algebra elements are either units or zero divisors and that zero divisors have idempotents in their annihilators. The prime and maximal ideals are related to ultrafilters in $\mathcal{B}(\mathcal{A})$. So it is the parameter space N that gives rise to idempotents, filters, the zero set of elements and how elements are interlinked. Compactification is not needed. The reader should have this example in mind when reading the rest of this section. Let's now continue with the construction of \mathbb{R} which mimics what we just did.

Let I =]0, 1] and $\mathcal{E}(\mathbb{R}) = \mathcal{F}(I, \mathbb{R})$ whose elements are called nets and denoted either by (x_{ε}) or $(x_{\varepsilon})_{\varepsilon}$. The algebra $\mathcal{E}(\mathbb{R})$ has a natural partial order induced by the comparison of values of functions, more precisely: we say that $(x_{\varepsilon}) \leq (y_{\varepsilon})$ if there exists $\eta \in I$ such that $x_{\varepsilon} \leq y_{\varepsilon}, \forall \varepsilon \in I_{\eta} :=]0, \eta]$. Given a net $x = (x_{\varepsilon})$, we denote by |x| the net $(|x_{\varepsilon}|)$. We denote by $\alpha = (\varepsilon)_{\varepsilon}$, coining it the natural gauge or standard gauge, and by $\alpha_r := \alpha^r, r \in \mathbb{R}$. Since $\alpha_r \cdot \alpha_s = \alpha_{r+s}$, it follows that the map $(\mathbb{R}, +) \ni r \longrightarrow \alpha_r \in \mathcal{E}(\mathbb{R})$ is a group monomorphism. This fundamental gauge will be key in all the constructions. All growth will be measured by this gauge, resulting in the rejection of certain infinities and the avoidance of some infinitesimals. At the end of this section we shall see that it is related to the way we measure in Classical Analysis. At first, one could think of using other gauges, but, just like in the case of the Principle of Computational Equivalence, one should look for the gauge that permits interpretation of physical reality without overcomplicating.

An element $x \in \mathcal{E}(\mathbb{R})$ is moderate if $|x| < \alpha^r$, for some $r \in \mathbb{R}$. Denote the set of moderate nets by \mathcal{E}_M and by $\mathcal{I} = \{x : x \in \mathcal{E}_M, |x| < \alpha^n, \forall n \in \mathbb{N}\}$. For $x \in \mathcal{E}_M$, denote by $V(x) = Sup\{r \in \mathbb{R} : |x| < \alpha^r\}$ and set $||x|| = e^{-V(x)}$. It is easily seen that \mathcal{I} is a radical ideal of the ring \mathcal{E}_M and setting $\widetilde{\mathbb{R}} := \frac{\mathcal{E}_M}{\mathcal{I}}$, we have that $(\widetilde{\mathbb{R}}, || ||)$ is a partially ordered ultra-metric ring containing a copy of \mathbb{R} . It is common to denote elements of \mathcal{E}_M by \hat{x} and its class in $\widetilde{\mathbb{R}}$ by x and call \hat{x} a representative of x. We shall call $\widetilde{\mathbb{R}}$ the ring of generalized reals (or Colombeau reals) and, to understand it better, one should think in terms of germs at 0, although $0 \notin I$ (see [11, 30]). The set of invertible elements of $\widetilde{\mathbb{R}}$ is denoted by $Inv(\widetilde{\mathbb{R}})$ and its boolean algebra of idempotents is denoted by $\mathcal{B}(\widetilde{\mathbb{R}}) = \{e \in \widetilde{\mathbb{R}} : e^2 = e\}$. The following theorem summarizes all basic properties one needs to know about $\widetilde{\mathbb{R}}$. Just as one rarely uses the way \mathbb{R} is constructed from \mathbb{Q} , in general, one does not need to remember the construction of $\widetilde{\mathbb{R}}$. Most proofs can be done intrinsically without appealing to representatives.

Theorem 2.1 [The Extended Fundamental Theorem of $\widetilde{\mathbb{R}}$]

1. \mathbb{R} is a partially ordered non-Archimedean ultrametric algebra such that $||x \cdot y|| \leq ||x|| \cdot ||y||$.

In particular, $||x + y|| \le max\{||x||, ||y||\}$, ||rx|| = ||x||, $r \in \mathbb{R}^*$ and $\widetilde{\mathbb{R}}$ is a totally disconnected topological ring.

- 2. $Inv(\widetilde{\mathbb{R}})$ is open and dense in $\widetilde{\mathbb{R}}$. In particular, maximal ideals are closed and rare.
- 3. $x \in Inv(\widetilde{\mathbb{R}})$ if and only if $|x| \geq \alpha_r$, for some $r \in \mathbb{R}$.
- 4. $\mathcal{B}(\widetilde{\mathbb{R}}) = \{\chi_S : S \subset \mathcal{S}\}$, where χ_S is the characteristic function of S and $S \in \mathcal{S}$ if and only if 0 is in the topological closure, in \mathbb{R} , of both S and $S^c = I S$.
- 5. $x \in \widetilde{\mathbb{R}} Inv(\widetilde{\mathbb{R}})$ if and only if there exists an idempotent $e \in \mathcal{B}(\widetilde{\mathbb{R}}) \{0, 1\}$ such that $e \cdot x = 0$.
- The Jacobson Radical of R is {0}. In particular, R has no nontrivial nilpotent elements and is embeddable into a product of integral domains.
- 7. The unit ball $B_1(0) = \{x \in \mathbb{R} : ||x|| < 1\}$ consists of infinitesimals, i.e., if $x \in B_1(0)$ then $|x| < \frac{1}{n}, \ \forall \ n \in \mathbb{N}$. In particular, $\alpha_r \in B_1(0), \ \forall \ r > 0$.
- 8. For each r < 0, α_r is an infinity, i.e., $\alpha_r > n$, $\forall n \in \mathbb{N}$.
- 9. If $x \in \widetilde{\mathbb{R}} \{0\}$ then there exists $e \in \mathcal{B}(\widetilde{\mathbb{R}})$ such that $e \cdot x \in Inv(e \cdot \widetilde{\mathbb{R}})$.
- 10. The Biagioni-Oberguggenberger topology of $\widetilde{\mathbb{R}}$, the sharp topology, is generated by the balls with generalized radii $V_r(0) = \{x \in \widetilde{\mathbb{R}} : |x| < \alpha_r\}$.

The reals, \mathbb{R} , are embedded as a grid of equidistant points in the generalized reals $\widetilde{\mathbb{R}}$, the common distance being equal to one (one can adjust the ultra-metric so that this common distance is at the scale of uncertainty in physical reality). The ideals of $\widetilde{\mathbb{R}}$ are convex (an ideal J is convex if $x \in J$, $y \in \widetilde{\mathbb{R}}$ and $|y| \leq |x|$ implies that $y \in J$), its Krull dimension is infinite and it has a minimal prime which is also a maximal ideal. Ultrafilters of \mathcal{S} partially parametrize prime and maximal ideals of $\widetilde{\mathbb{R}}$. The halo of $x \in \widetilde{\mathbb{R}}$ is defined as $halo(x) = B_1(x) = x + B_1(0)$. We say that x and y are associated (denoted by $x \approx y$) if x - y is an infinitesimal or equivalently $\lim_{\varepsilon \to 0} (\hat{x} - \hat{y})(\varepsilon) = 0$. In particular, if $y \in halo(x)$ then $y \approx x$.

The environments we propose are $\widetilde{\mathbb{R}}^n$ with $n \in \mathbb{N}$. Note that, in some sense, points in these environments are not static since one must think of them in terms of germs when $\varepsilon \downarrow 0$. Since norms in \mathbb{R}^n are all equivalent, they all induce the same topology in \mathbb{R}^n . We say that a point $p = (p_1, \dots, p_n) \in \widetilde{\mathbb{R}}^n$ is compactly supported if there exists $L \in \mathbb{R}$ such that $||p||_1 \leq L$, in $\widetilde{\mathbb{R}}$. We also call them finite points. The set of finite points is denoted by $\widetilde{\mathbb{R}}^n_c$ and is an $\widetilde{\mathbb{R}}_c$ module. Thus, equipping $\widetilde{\mathbb{R}}^n$ with the product topology, it follows that $\widetilde{\mathbb{R}}^n_c \subset \overline{B}_1(0) = \{x \in \widetilde{\mathbb{R}}^n : ||x|| \leq 1\}$.

Given $\Omega \subset \mathbb{R}^n$, define $\widetilde{\Omega} := \{ p \in \widetilde{\mathbb{R}}^n : \exists \eta \in I, \hat{p} = (p_{\varepsilon}), p_{\varepsilon} \in \Omega, \text{ for } \varepsilon \in I_{\eta} \} \text{ and } \widetilde{\Omega}_c := \{ p \in \widetilde{\mathbb{R}}^n : \exists \eta \in I, \hat{p} = (p_{\varepsilon}), p_{\varepsilon} \in \Omega, \text{ for } \varepsilon \in I_{\eta} \} = \widetilde{\Omega} \cap \widetilde{\mathbb{R}}^n_c.$ Given a point $p \in \widetilde{\mathbb{R}}^n$ we define its support as $supp(p) = \{ q \in \mathbb{R}^n : \exists e \in \mathcal{B}(\widetilde{\mathbb{R}}), \text{ such that } e \cdot p \approx e \cdot q \}$. For a subset $X \subset \widetilde{\mathbb{R}}^n_c$ we define its support as $supp(X) = \bigcup_{p \in X} supp(p)$. If $\|p\| < 1$ then $supp(p) = \{\vec{0}\}, supp(\alpha_{-1}) = \emptyset$,

 $supp([sin(\alpha_{-1})]) = [-1,1]$ and if $p \in \mathbb{R}^n_c$, then $supp(p) \neq \emptyset$. It is clear that $supp(\Omega_c) = \overline{\Omega}$, the topological closure of Ω in \mathbb{R}^n . These notions are related to *interleaving*, membranes and internal sets for which we refer the reader to [23, 29]. The notion of support can be extended to operators acting on these environments and be used to obtain classical solutions, if they exist,

for classical operators in the support of an operator in these environments: If L is a generalized operator, $L_0 \in supp(L)$, F a generalized solution of the equation $L(\tau) = 0$, then $L_0(f_0) = 0$, for $f_0 \in supp(F)$. In fact, there exists an idempotent $e \in \mathcal{B}(\mathbb{R})$ such that $f_0 = e \cdot F$. So this serves as a mathematical tool to obtain classical solutions. This shows the advantage of $\mathcal{B}(\mathbb{R})$ not to be trivial, which is not the case when the underlying structure is a fields or the Fermat reals.

We finish this section comparing the classical topology in Analysis with the sharp topology. This is best done looking at the definition of continuity. Let $f: \mathbb{R} \longrightarrow \mathbb{R}$ be continuous at x_0 . Classically this means that for each ε there exists δ_{ε} such that whenever $|x - x_0| \leq \delta_{\varepsilon}$ we have that $|f(x) - f(x_0)| \leq \varepsilon$ ($|x - x_0| \leq \delta_{\varepsilon} \Longrightarrow |f(x) - f(x_0)| \leq \varepsilon$). Translated to the sharp topology this can be written as: f is continuous at x_0 if given α , there exists $\delta = [(\delta_{\varepsilon})]$ such that $x \in V_{\delta}(x_0) \Longrightarrow f(x) \in V_1(f(x_0))$, where $V_{\delta}(x_0) = \{x \in \mathbb{R} : |x - x_0| < \delta\}$. This proves that classically we stop measuring at scale α , this being the reason why we coined it our natural gauge. References on which this section is based are [2, 5, 11, 23, 25, 28, 31, 36].

3 Functions on $\widetilde{\Omega}_c$

We now define functions on subsets of \mathbb{R}^n_c in such a way that it naturally extends our classical definitions, but at the same time permits that we can see both classical functions and distributions as functions in these new environments, i.e., we are looking for a domain for these objects. More important, composition of functions, if the classical conditions are satisfied, must be defined. The way to achieve this is to use our natural gauge.

Let $\Omega \subset \mathbb{R}^n$ be an open subset and let (Ω_m) be an exhaustion of compact subsets of Ω . Then $(\widetilde{\Omega}_{mc})$, where $\widetilde{\Omega}_{mc} = (\widehat{\Omega}_m)_c$, is an exhaustion of $\widetilde{\Omega}_c$ by these subsets, called *principle membranes*, of $\widetilde{\Omega}_c$.

Given a net $\hat{f} = (\hat{f}_{\varepsilon}) \in \mathcal{C}^{\infty}(\Omega, \mathbb{R})$ and $\hat{p} = (p_{\varepsilon})$, consider the net $\hat{f}(\hat{p}) = (\hat{f}_{\varepsilon}(p_{\varepsilon})) \in \mathcal{E}(\mathbb{R})$. For $\hat{q} = (q_{\varepsilon})$, we have that $|\hat{f}(\hat{p}) - \hat{f}(\hat{q})| \leq ||\nabla \hat{f}(\hat{p}_1)||_2 \cdot ||\hat{p} - \hat{q}||_2$, which will make \hat{f} a well defined function on $\tilde{\Omega}_c$ if imposed a growth condition on both \hat{f} and $\nabla \hat{f}$.

We say that f is α -bounded (or of moderate growth) if for each $m \in \mathbb{N}$ there exists $r = r(m) \in \mathbb{R}$ such $\{\hat{f}(\hat{p}), \partial_{x_i} \hat{f}(\hat{p}) : i = 1, n, \ \hat{p} \in \Omega_m\} \subset V_{r(m)}(0)$. It follows that if \hat{f} is α -bounded then it defines a function $\hat{f}: \widetilde{\Omega}_c \longrightarrow \widetilde{\mathbb{R}}, \ \hat{f}(p) = [\hat{f}(\hat{p})]$ and $\hat{f}(\widetilde{\Omega}_{mc}) \subset V_{r(m)}(0)$. Let $\mathcal{E}(\Omega, \mathbb{R}) = \{\hat{f} \in \mathcal{C}^{\infty}(\Omega, \mathbb{R}) : \hat{f} \text{ is } \alpha$ -bounded}, i.e., we look at the set of all nets which define functions on $\widetilde{\Omega}_c$ taking values in $\widetilde{\mathbb{R}}$. Clearly, $J = \{\hat{f} \in \mathcal{E}(\Omega, \mathbb{R}) : \hat{f} = 0 \text{ on } \widetilde{\Omega}_c\}$ is an ideal of $\mathcal{E}(\Omega, \mathbb{R})$ and $\mathcal{E}(\Omega, \mathbb{R})/J$ embeds into $\mathcal{F}(\widetilde{\Omega}_c, \widetilde{\mathbb{R}})$. With the notation of the previous section, one can write $J = \{\hat{f} \in \mathcal{E}(\Omega, \mathbb{R}) : \hat{f}(\hat{p}) \in \mathcal{I}$, for all $\hat{p} \in \widetilde{\Omega}_c\}$. For example, take n = 1, let $\varphi \in \mathcal{D}(\mathbb{R})$ and consider the net $\hat{f} = (\varphi_{\varepsilon})$, where $\varphi_{\varepsilon}(x) = \frac{1}{\varepsilon}\varphi(\frac{x}{\varepsilon})$, a contraction of φ . Then it is easily seen that $\hat{f} \in \mathcal{E}(\Omega, \mathbb{R})$ thus proving that $\mathcal{F}(\widetilde{\Omega}_c, \widetilde{\mathbb{R}})$ is nonempty. In fact, r = -2.1 serves to prove that \hat{f} is α -bounded.

Define $\mathcal{E}_M(\Omega,\mathbb{R})=\{\hat{f}\in\mathcal{E}(\Omega,\mathbb{R}):\partial^{\beta}\hat{f}\in\mathcal{E}(\Omega,\mathbb{R}),\ \forall\ \beta\in\mathbb{N}^n\}$ and $\mathcal{N}(\Omega)=\{\hat{f}\in J:\partial^{\beta}\hat{f}\in J,\ \forall\ \beta\in\mathbb{N}^n\}$. Then the former is a subalgebra of $\mathcal{E}(\Omega,\mathbb{R})$ containing the latter as a radical ideal and we have an embedding of the quotient algebra $\kappa:\mathcal{G}(\Omega)=\frac{\mathcal{E}_M(\Omega,\mathbb{R})}{\mathcal{N}(\Omega)}\longrightarrow\mathcal{F}(\widetilde{\Omega}_c,\widetilde{\mathbb{R}})$. Let $\tau\in\mathcal{S}(\mathbb{R})$ be an element of the Schwartz Space which is a nonzero constant in an interval containing 0 and denote by ρ its inverse Fourier transform. Then (ρ_{ε}) defines an element $\kappa(\rho)\in\mathcal{G}(\mathbb{R})$ which can be identified with δ , Dirac's delta function. So here $\delta=\kappa(\rho)=\alpha_{-1}\cdot\rho\circ\alpha_{-1}$ becomes a function! and $\delta(0)=\kappa(\rho)(0)=[(\frac{1}{\varepsilon}\rho(0))]=\rho(0)\cdot\alpha_{-1}$, an infinity. We have that $T=x\delta=0$ in $\mathcal{D}'(\mathbb{R})$,

but $\kappa(x)\kappa(\delta)$ is a non-zero function. In fact, $\kappa(T)(x_0\alpha)=x_0\rho(x_0)$. The function ρ is called a mollifier because of the following two moments properties which follow directly from properties of the Fourier transform: $\int\limits_{\mathbb{R}} \rho dx=1$ and $\int\limits_{\mathbb{R}} x^k \rho dx=0$, for all $k\in\mathbb{N}, k\neq 0$. These properties are

crucial to embed $\mathcal{D}'(\Omega)$ into $\mathcal{G}(\Omega)$ and thus into $\mathcal{C}^{\infty}(\widetilde{\Omega}_c, \widetilde{\mathbb{R}})$, i.e., distributions become infinitely differentiable functions. It is now clear that to compose two such functions f and g, one must have that, just like in the classical sense, $Im(g) \subset Domain(f)$. In particular, there exists a real number L > 0 such that $||g(x)|| \leq L$ in $\widetilde{\mathbb{R}}$ for all $x \in Domain(g)$, i.e., $Domain(g) \subset \widetilde{\mathbb{R}}_c^n$. We shall now introduce our notion of derivation, thus going back to Newton's and Nature's most basic idea of viewing physical reality: the notion of variation. References on which this section is based are [2, 9, 11, 25, 28, 30].

4 Differentiability in $\widetilde{\mathbb{R}}^n$

In this section we shall define differentiability in the milieus constructed. Milieus which somehow are the union of each single computable path or event we can imagine. The challenge is how to capture variation in such environments whose underlying structure was turned into a totally disconnected topological algebra with zero divisors. It must be done intrinsically to stand on its own. At the same time, it must extends, in a natural way, the classical notion of variation. J. Tate must have had the same challenge when he founded non-Archimedean function theory on totally disconnected topological fields. How to achieve such a thing in a totally disconnected ultrametric ring where spheres are open sets and zero divisors do exists? The solution is to use the natural gauge, just as it was used in the previous sections to define the concepts of moderateness, domains and functions in these milieus. We basically start to measure where classical measurement ends: at scale α ! Without further ado, here is our definition.

Let $\Omega \subset \mathbb{R}^n$ be open. We say that $f \in \mathcal{F}(\widetilde{\Omega}_c, \widetilde{\mathbb{R}})$ is differentiable at a point $p \in \widetilde{\Omega}_c$ if there exists a vector $v \in \widetilde{\mathbb{R}}^n$ such that

$$\lim_{h \longrightarrow 0} \frac{f(p+h) - f(p) - \langle h \mid v \rangle}{\alpha_{-\ln(\parallel p - h \parallel)}} = 0$$

Note that

$$\left\| \frac{f(p+h) - f(p) - \langle h \mid v \rangle}{\alpha_{-\ln(\|p-h\|)}} \right\| = \frac{\|f(p+h) - f(p) - \langle h \mid v \rangle\|}{\|p-h\|}$$

showing that our definition is very close to the Newtonian notion of differentiability. The last equality holds because the α_r 's are multiplicative elements of $\widetilde{\mathbb{R}}$ in the sense of [10]. The vector v, if it exists, is unique and differentiability at a point implies also continuity at that point. The resemblance to Newton's derivation is even closer than meets the eye.

Example 4.1 Let
$$\hat{f}_{\varepsilon} = f_0 \in \mathcal{C}^{\infty}(\mathbb{R}), \hat{g}_{\varepsilon} = f'_0, \ \forall \varepsilon \in I \ and \ set \ f = \kappa(\hat{f}) \ and \ g = \kappa(\hat{g}).$$
 Then $f' = g$, i.e., $(\kappa(\hat{f}))' = \kappa(\hat{f}')$.

The example not only shows that our notion of Calculus extends naturally that of classical Calculus but also gives the existence of primitives. This will become more clearer from our Embedding Theorem. For the Calculus developed in the Ring of Fermat reals this is not as direct and obvious

as in our case (see [19]). Partial derivatives are defined in an obvious way as is integration on Ω . Integration on other sets, such as membranes and internal sets, is also defined in a natural and obvious way. The following result shows that our notion of variation and that of Newton are a perfect mirror match. Moreover, it shows the existence of an algebra of functions in which the linear space of Schwartz's distributions can be embedded as infinitely differentiable functions defined on a principal membrane, an example of an internal set, both of which are subsets of a Cartesian product of the generalized reals. In particular, the product of distributions does not only makes sense, but is well defined in a classical sense as the product of functions.

Theorem 4.2 [Embedding Theorem] There exists an algebra embedding

 $\kappa: \mathcal{G}(\Omega) \longrightarrow \mathcal{C}^{\infty}(\widetilde{\Omega}_c, \widetilde{\mathbb{R}})$ which commutes with derivation, i.e., $\kappa \circ \partial^{\beta} = \partial^{\beta} \circ \kappa$, for all $\beta \in \mathbb{N}^n$. When restricted to the image of $\mathcal{D}'(\Omega)$ in $\mathcal{G}(\Omega)$, κ is \mathbb{R} -linear and, when restricted to the image of $\mathcal{C}^{\infty}(\Omega)$, it is an algebra monomorphism. Moreover $\operatorname{supp}(\widetilde{\Omega}_c) = \overline{\Omega}$.

As a consequence of the theorem, we have that if H is the Heaviside function, then $\kappa(H)$ is a \mathcal{C}^{∞} -function, with $H' = \delta$ and $\kappa(H) \notin \mathcal{B}(\widetilde{\mathbb{R}})$, i.e., $\kappa(H)^2 \neq \kappa(H)$. For n = 1, the function $g(x) = \alpha_{\ln(\|x\|^{-2})}, g(0) = 0$, is differentiable, non-constant and g'(x) = 0 for all x. However, if $f \in \kappa(\mathcal{G}(\mathbb{R}))$ and $f' \equiv 0$, then f is constant. Hence this fundamental classical rule is not violated if we restrict to function coming from $\mathcal{G}(\mathbb{R})$. It is natural to call g a quanta, since $F(x) = x\alpha_{\ln(\|x\|^2)}, F(0) = \infty$ changes the sphere on which x was. It acts like the inversion in the unit sphere: $\|x\| \cdot \|F(x)\| = 1$. Other quanta are the multiplication by a fixed α_r .

If $T \in \mathcal{D}'(\Omega)$ has compact support, then $\kappa(T)$ is constructed by convolution with the mollifier ρ which immediately gives us the following remarkable result. In these milieus, the action of a distributions on $\mathcal{D}(\Omega)$ is represented by a globally defined function! Thus revealing its true linearity.

Lemma 4.3 Let $T \in \mathcal{D}'(\Omega)$ and $\varphi \in \mathcal{D}(\Omega)$. Then, in $\widetilde{\mathbb{R}}$, we have that

$$\langle T \mid \varphi \rangle = \int_{\Omega} \kappa(\varphi) \kappa(T) d\Omega$$

This lemma was proved for an open subset $\Omega \subset \mathbb{R}^n$ in [25] and used in [13] to embed $\mathcal{D}'(\Omega)$ into an Aragona algebra, a sub-algebra of $\mathcal{C}^{\infty}(\widehat{\Omega}, \mathbb{L})$, $\widehat{\Omega} \subset \mathbb{L}^n$, where \mathbb{L} is a totally ordered non-Archimedean ultra-metric field of characteristic zero. Differentiability readily extends to vector valued functions and thus we have this new Calculus available with the same classical features and which, although \mathbb{R}^n is discretely embedded into $\widetilde{\mathbb{R}}^n$, extends the latter in a very natural way. References on which this section is based are [3, 4, 7, 13, 25].

5 A Fixed Point Theorem

In this section we shall be working in what are known as the full environments which permit $\mathcal{D}'(\Omega)$ to be embedded canonically into $\mathcal{G}(\Omega)$. Those of the previous sections are known as the simple milieus and the embedding of $\mathcal{D}'(\Omega)$ into $\mathcal{G}(\Omega)$ depends on the mollifier ρ . This can be a feature since one may adapt ρ to the specific problem one is interested in. Although the construction in the full case is more elaborate, everything is very similar to the simplified construction and the results on the basic underlying, in this case denoted by $\widetilde{\mathbb{K}}_f$, are identical to what we already saw. We also do have an embedding theorem and the lemma of the previous section also holds. The

reader can find more in [23, 24] where results in this and the next sections are proved in \mathbb{R} . The best way to master the math in these milieus is to master their algebraic theory and understand well how the topology interlinks with the former. Nearly everything else works as in the classical theory. That is why we start by reviewing the *sharp topology* in these full milieus.

The sequence $(\mathcal{A}_q)_{q\in\mathbb{N}}$ stands for a decreasing chain of sets, with empty intersection, consisting of functions φ (recall the definition of its contraction φ_{ε}), of compact support, with the same property at the origin as the function that defined the mollifier ρ . The natural gauge in $\widetilde{\mathbb{K}}_f$ is denoted by α^{\bullet} . One of its representatives is $\hat{\alpha}^{\bullet}$: $\mathcal{A}_0 \longrightarrow \mathbb{K}$ is given by $\hat{\alpha}^{\bullet}(\varphi) = i(\varphi)$, the diameter of the support of φ . We thus have that $\hat{\alpha}^{\bullet}(\varphi_{\varepsilon}) = i(\varphi)\varepsilon$. A part from these technicalities, definitions and results mimic those from the previous sections. Details can be found in the references mentioned at the end of this section. Here $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$.

Given $x \in \widetilde{\mathbb{K}}_f$ set $A(x) := \{r \in \mathbb{R} : \alpha_r^{\bullet} x \approx 0\}$ and define the valuation of x as $V(x) = \sup(A(x))$. We have that $D : \widetilde{\mathbb{K}}_f \times \widetilde{\mathbb{K}}_f \to \mathbb{R}_+$ defined by $D(x,y) = e^{-V(x-y)}$ is a translation invariant ultrametric on $\widetilde{\mathbb{K}}_f$. It determines a uniform structure on $\widetilde{\mathbb{K}}_f$, the sharp uniform structure, and the topology resulting from D is the sharp topology. Setting ||x|| = D(x,0) $x \in \widetilde{\mathbb{K}}_f$, the distance between $x,y \in \widetilde{\mathbb{K}}_f$ is given by D(x,y) = ||x-y||. This distance extends to $\widetilde{\mathbb{K}}_f^n$ in the obvious way.

Let $\Omega \subset \mathbb{R}^n$ and let $(\Omega_l)_l$ be an exhaustion of Ω . For $f \in \mathcal{G}_f(\Omega)$, define the family of sets $A_{lp}(f) := \{r \in \mathbb{R} : \alpha_r^{\bullet} || f ||_{lp} \approx 0\}$ and define the family of pseudo-valuations $V_{lp}(f) := \sup(A_{lp}(f))$. It follows that $D_{lp} : \mathcal{G}_f(\Omega) \times \mathcal{G}_f(\Omega) \to \mathbb{R}_+$ defined by $D_{lp}(x,y) = e^{-V_{lp}(x-y)}$ is an family of pseudo-ultrametrics on $\mathcal{G}_f(\Omega)$, where $l, p \in \mathbb{N}$ denote the index of the exhaustion (Ω_l) and the order of the derivative of f, respectively. The uniform structure it determines on $\mathcal{G}_f(\Omega)$ is called the sharp uniform structure and the resulting topology is called the sharp topology. If $x = [(x_{\varphi})_{\varphi}] \in \widetilde{\mathbb{K}}_f$ we set $|x| = [(|x_{\varphi}|)_{\varphi}]$. If $x_0 \in \widetilde{\mathbb{K}}_f$ and $r \in \mathbb{R}$, let $V_r[x_0] := \{x \in \widetilde{\mathbb{K}}_f : |x - x_0| \le \alpha_r^{\bullet}\}$ and $\mathcal{B}_f := \{V_r[0] : r \in \mathbb{R}\}$. Similarly, for every $f \in \mathcal{G}_f(\Omega)$, and fixed $\beta \in \mathbb{N}_0^n$ and $l \in \mathbb{N}$, let $||f||_{\beta,l} := [(||\partial^{\beta} f_{\varphi}(\cdot)||_{l})_{\varphi \in \mathcal{A}_0}]$, where $||\cdot||_{l}$ denotes the supremum norm over Ω_l . Finally, for $f_0 \in \mathcal{G}_f(\Omega)$ and $r \in \mathbb{R}$, define $W_{l,r}^{\beta}[f_0] := \{f \in \mathcal{G}_f(\Omega) : ||f - f_0||_{\sigma,l} \le \alpha_r^{\bullet} \ \forall \ \sigma \le \beta\}$ and $\mathcal{B}_{f\Omega} := \{W_{l,r}^{\beta}[0] : \beta \in \mathbb{N}_0^n, \ l \in \mathbb{N} \ e \ r \in \mathbb{R}\}$.

Theorem 5.1

- 1. \mathcal{B}_f is a system of 0-neighborhoods of $\widetilde{\mathbb{K}}_f$ that induces a topology compatible with the ring structure of $\widetilde{\mathbb{K}}_f$ and coincides with the sharp topology.
- 2. $\mathcal{B}_{f\Omega}$ is a system of 0-neighborhoods of $\mathcal{G}_f(\Omega)$ that induces a topology compatible with the ring structure of $\mathcal{G}_f(\Omega)$ and coincides with the sharp topology.

Moreover $\widetilde{\mathbb{K}}_f$ is precisely the set of constant generalized functions on $\mathcal{G}_f(\Omega)$ and the topology of the former is induced by that of the latter.

Definition 5.2 [Hypernatural Numbers and Hypersequences]

- 1. The subset $\widetilde{\mathbb{N}} := \{n = [(n_{\varphi})_{\varphi \in A_0}] \in \widetilde{\mathbb{R}} : n_{\varphi} \in \mathbb{N} \ \forall \ \varphi \in A_0\}$ is called the set of hypernatural numbers. In short, hypernaturals are generalized numbers with representatives in $\mathcal{E}_{M_f}(\mathbb{N})$.
- 2. A hypersequence is a function $f: \widetilde{\mathbb{N}} \to \mathcal{G}_f$ and will be denoted by $(f_n)_{n \in \widetilde{\mathbb{N}}}$.
- 3. A hypersequence $(f_n)_{n\in\widetilde{\mathbb{N}}}$ converges to $f\in\mathcal{G}_f$ if for any neighborhood of zero $W_{l,r}^{\beta}[0]$ there exists $n_0\in\widetilde{\mathbb{N}}$ such that if $n\geq n_0$ then $(f_n-f)\in W_{l,r}^{\beta}[0]$.

If $f(\widetilde{\mathbb{N}}) \subset \widetilde{\mathbb{K}}_f$ then the equality $\widetilde{\mathbb{K}}_f \cap W_{l,r}^{\beta}[0] = V_r[0]$ implies that $f_n \to f$ if under the same conditions of the above definition we have $(f_n - f) \in V_r[0]$. A hypersequence $(f_n)_{n \in \widetilde{\mathbb{N}}}$ is Cauchy if for any neighborhood of zero $W_{l,r}^{\beta}[0]$ there exists $n_0 \in \widetilde{\mathbb{N}}$ such that if $n, m \geq n_0$ then $(f_n - f_m) \in W_{l,r}^{\beta}[0]$. Similarly, we can replace the condition $(f_n - f_m) \in W_{l,r}^{\beta}[0]$ by $(f_n - f_m) \in V_r[0]$ if $f(\widetilde{\mathbb{N}}) \subset \widetilde{\mathbb{K}}_f$. Since \mathcal{G}_f is complete, we have that $(f_n)_{n \in \widetilde{\mathbb{N}}}$ is convergent if, and only if, it is Cauchy. Note that $\widetilde{\mathbb{N}}$ is uncountable and not totally ordered. Hence converging via a hypersequence is a matter of choices but all leading to the same point. Recall that in our model \mathbb{K}^n is embedded into $\widetilde{\mathbb{K}}_f^n$ as a grid of equidistant points, this common distance being equal to 1. Hence the concept of hypersequences helps to understand how physical reality behaves when distances are smaller than a certain constant. Here this distance is 1, which is just a symbol and thus replaceable by any other constant, multiplying the ultrametric by this constant.

Example 5.3 The sequence $(\frac{1}{n})_{n\in\mathbb{N}}$ does not converge in $\widetilde{\mathbb{K}}_f$ but the hypersequence $(\frac{1}{n})_{n\in\widetilde{\mathbb{N}}}$ does converges to 0 in $\widetilde{\mathbb{K}}_f$. In fact, for an arbitrarily chosen r>0, it suffices to take $n_0=[(\lfloor i(\varphi)^{-r}+1\rfloor)_{\varphi}]$. Then, if $n=[(n_{\varphi})_{\varphi}]>n_0$, we have $\frac{1}{n_{\varphi}}<\frac{1}{\lfloor i(\varphi)^{-r}+1\rfloor}\leq \frac{1}{\lfloor i(\varphi)^{-r}}$ for all $\varphi\in\mathcal{A}_q(1)$ with q>p for some $p\in\mathbb{N}$. Therefore, $|\frac{1}{n}-0|<\alpha_r^{\bullet}$ and thus $\frac{1}{n}\in V_r[0]$.

Let $(B_{\varphi})_{\varphi \in \mathcal{A}_0}$ be a net of subsets of \mathbb{K}^n . The internal set $B = [(B_{\varphi})_{\varphi}]$ is the subset $B \subset \widetilde{\mathbb{K}}_f^n$ whose points $p \in B$ have some representative (\hat{p}_{φ}) with $\hat{p}_{\varphi} \in B_{\varphi}$, $\forall \varphi \in \mathcal{A}_0$. Since points of $\widetilde{\mathbb{K}}_f^n$ must be thought of as germs, in the full environments, this translate into the following: there exist $q \in \mathbb{N}$ such that $\forall \varphi \in \mathcal{A}_q$, there exists $\eta_{\varphi} \in I$ such that $\hat{p}_{\varphi_{\varepsilon}} \in B_{\varphi_{\varepsilon}}$, $\forall \varepsilon < \eta_{\varphi}$.

Definition 5.4 Let $B = [(B_{\varphi})_{\varphi}]$, $C = [(C_{\varphi})_{\varphi}]$ be internal sets in \mathcal{G}_f . An application $T : B \to C$ is a map represented by a moderate net $(T_{\varphi} : B_{\varphi} \to C_{\varphi})_{\varphi \in \mathcal{A}_0}$ such that $T(f) = [(T_{\varphi}(f_{\varphi}))_{\varphi}]$ for all $f = [(f_{\varphi})_{\varphi}] \in B$.

Definition 5.5 [Contraction] Let $B = [(B_{\varphi})_{\varphi}] \subset \mathcal{G}_f$ be an internal set, and let $T: B \to B$ be an application with representative $(T_{\varphi}: B_{\varphi} \to B_{\varphi})_{\varphi \in \mathcal{A}_0}$. We say that T is a contraction if there exist $L \in \mathbb{R}^*_{+f}$ and $\lambda \in (0,1)$ such that $L < \lambda$ and $\|T(f) - T(g)\|_{l} \le L\|f - g\|_{l}$ for all $l \in \mathbb{N}$.

Given $n = [(n_{\varphi})_{\varphi}] \in \widetilde{\mathbb{N}}$, the map $T^n : B \to B$ with representative $(T_{\varphi}^{n_{\varphi}} : B_{\varphi} \to B_{\varphi})_{\varphi \in \mathcal{A}_0}$, where n_{φ} compositions of T_{φ} are understood, if well defined is a contraction since $\|T^n(f) - T^n(g)\|_l \le L^n \|f - g\|_l \le \lambda^n \|f - g\|_l$ for all $l \in \mathbb{N}$. If it is necessary to specify λ , we shall say that T is a λ -contraction. Let $\Omega \subset \mathbb{R}^n$, $l, p \in \mathbb{N}$ and consider the ultrametric in $\mathcal{G}_f(\Omega)$ $D : \mathcal{G}_f(\Omega) \times \mathcal{G}_f(\Omega) \to \mathbb{R}_+$ defined by

$$D(f,g) := \sup \left\{ \frac{2 \cdot D_{ll}(f,g)}{1 + D_{ll}(f,g)} : l \in \mathbb{N} \right\}$$

Let $r_0 < 2$ and $f \in B_{r_0}(0)$. Then it follows that $\frac{2 \cdot D_{ll}(f,g)}{1 + D_{ll}(f,g)} \le r_0$, $\forall l \in \mathbb{N}$ and hence $V_{ll}(f,g) \ge ln(\frac{2-r_0}{r_0})$, $\forall l \in \mathbb{N}$. Take d_0 such that $\ln(d_0) \le ln(\frac{2-r_0}{r_0})$, then $||f||_l \le \alpha^{\bullet}_{d_0}$, $\forall l \in \mathbb{N}$. Now let $\lambda \in]0,1[$ be fixed and suppose that we want $\lambda^{n_0} \in V_t(0)$, where $n_0 = [(n_{0\varphi})]$ and t > 0. For this to occur one must have $\lambda^{n_{0\varphi}} < i(\varphi)^t$. From this it follows that $n_{0\varphi} > (\frac{-t}{|\ln(\lambda)|}) \cdot \ln(i(\varphi))$. Hence we may take $n_{0\varphi} = 2 \cdot \left\lfloor \frac{-t}{|\ln(\lambda)|} \cdot \ln(i(\varphi)) \right\rfloor$. It is clear that $n_0 < \alpha^{\bullet}_{-1.1}$ and therefore is moderate. For any $m > n_0$ we have $\lambda^m < \lambda^{n_0} \in V_t(0)$ and thus $\lambda^m \in V_t(0)$. This proves that the hypersequence (λ^n)

converges to 0. We shall use these two facts in the proof of our next theorem.

Theorem 5.6 [Fixed Point Theorem]

Let $\Omega \subset \mathbb{R}^N$, $A = [(A_{\varphi})_{\varphi}] \subset B_{r_0}(0) \cap \mathcal{G}_f(\Omega)$, $r_0 < 2$, be an internal set, and $T : A \to A$ be given with representative $(T_{\varphi} : A_{\varphi} \to A_{\varphi})_{\varphi \in \mathcal{A}_0}$. If there exists $k = [(k_{\varphi})_{\varphi}] \in \widetilde{\mathbb{N}}$ such that $T^k = (T_{\varphi}^{k_{\varphi}})$ is a λ -contraction, then T is well defined, continuous and has a unique fixed point in A.

Proof. We start by noting that B is a complete topological space. This follows because internal sets are closed and $\mathcal{G}_f(\Omega)$ is a complete ultrametric space. The Lipschitz condition implies that T is well-defined and continuous. For $g_0 \in A$, we shall prove that the hypersequence $(T^n(g_0))_{n \in \widetilde{\mathbb{N}}}$ converges to a point $f_0 \in A$. To achieve this it is enough to prove that $(T^n(g_0))_{n \in \widetilde{\mathbb{N}}}$ is a Cauchy hypersequence. Consider the basic neighborhood $W^l_{l,r}[0] \in \mathcal{B}_f$. Since $0 < \lambda < 1$ and $A \subset B_{r_0}(0)$, and using what we mentioned in the paragraph before the theorem, we may choose $n_0 \in \widetilde{\mathbb{N}}$ and d_0 such that $\lambda^{n_0} \alpha_d^{\bullet} \in V_{4^l r}[0]$. Set $r_1 = 4^{l-1}r$ and let take $n, s > n_0$. Writing $n = n_0 + p$ and $s = n_0 + q$ we have that $||T^p(g_0) - T^q(g_0)||_l \le \alpha_{d_0}^{\bullet}$ and thus $||T^n(g_0) - T^s(g_0)||_l = ||T^{n_0}(T^p(g_0)) - T^{n_0}(T^q(g_0))||_l \le \lambda^{n_0} ||T^p(g_0) - T^q(g_0)||_l \le \lambda^{n_0} \alpha_{d_0}^{\bullet} \in V_{4r_1}[0]$. This proves that $F := T^n(g_0) - T^s(g_0) \in W^0_{l,4r_1}[0]$. Consider the embedding $\kappa : \mathcal{G}_f(\Omega) \longrightarrow \mathcal{C}^{\infty}(\widetilde{\Omega}_{cf}, \widetilde{\mathbb{K}}_f)$ and identify F with $\kappa(F)$. Without loss of generality, we may suppose that N = 1. Since F is differentiable in $\widetilde{\Omega}_{cf}$, it follows that

$$\lim_{r_1 \to +\infty} \frac{F(x + \alpha_{2r_1}^{\bullet}) - F(x)}{\alpha_{2r_1}^{\bullet}} = F'(x),$$

for all $x \in \widetilde{\Omega}_{cf}$. Therefore,

$$\lim_{r_1 \to +\infty} \frac{|F(x + \alpha_{2r_1}^{\bullet}) - F(x)|}{\alpha_{2r_1}^{\bullet}} = |F'(x)|. \tag{1}$$

Since $F \in W_{l,4r_1}^0[0]$, we have that

$$|F(x+\alpha_{2r_1}^\bullet)-F(x)|\leq |F(x+\alpha_{2r_1}^\bullet)|+|F(x)|\leq 2\alpha_{4r_1}^\bullet$$

and therefore

$$\frac{|F(x + \alpha_{2r_1}^{\bullet}) - F(x)|}{\alpha_{2r_1}^{\bullet}} \le 2\alpha_{2r_1}^{\bullet} \tag{2}$$

for all $x \in \widetilde{\Omega}_{cf}$. From 1 and 2, it follows that $|F'(x)| \leq 2\alpha_{2r_1}^{\bullet} < \alpha_{r_1}^{\bullet}$ for all $x \in \widetilde{\Omega}_{cf}$, which gives us that $F \in W^1_{l,r_1}[0]$. Since F is a C^{∞} -function, we may repeat the same process for its derivates up to order l concluding that $F \in W^l_{l,r}[0]$ and thus, $(T^n(g_0))_{n \in \widetilde{\mathbb{N}}}$ is a Cauchy hypersequence. Uniqueness of the fixed point being obvious, completes the proof.

From the induction step given in the proof of the theorem we deduce the topological tool which we coin the *Down Sequencing Argument* and shorten it to DSA. The DSA ensures that moderation and nullity at level 0 imply nullity.

Theorem 5.7 [Down Sequencing Argument] Let $f \in \mathcal{G}_f(\Omega)$, $f \in W^0_{l,4^k r}[0]$ with r > 0. Then $f \in W^k_{l,r}[0]$, i.e., $W^0_{l,4^k r}[0] \subset W^k_{l,r}[0]$.

The notion of association has long been seen and used as an algebraic notion. But here we observe that it is in fact a topological notion which implies the following.

Theorem 5.8 Let Ω be an open subset of \mathbb{R}^n . Then $\mathcal{D}'(\Omega)$ embeds as a discrete grid in $\mathcal{G}(\Omega)$. Moreover, $\mathcal{C}^{\infty}(\Omega)$ embeds as a grid of equidistant points into $\mathcal{G}_f(\Omega)$.

The theorem shows that classical solutions to differential equations are rare. This can be used as a tool to obtain classical solutions from a generalized solution. References on which this section is based are [2, 3, 4, 5, 12, 14, 18, 23, 25].

6 Fundamentals of Generalized Geometry

In this section we extend the constructions made for subset $\Omega \subset \mathbb{K}^n$ to abstract manifolds. This is done in the full milieus. Starting with a Riemannian sub-manifold $M \subset \mathbb{K}^n$ we construct a generalized sub-manifold $M^* \subset \widetilde{\mathbb{K}}^n$, whose dimensions over the respective underlying reals are equal. The construction is such that M is the shadow of M^* , or, in the notation to be introduced in this section, $ssupp(M^*) = M$. It is here that we piece the puzzle letting tools and concepts developed by several prominent researchers fall into place. In particular, Classical Space-Time can be embedded into Generalized Space-Time and thus making available tools that can be used to explain phenomena in physical reality. We begin recalling some definitions and preparing the needed full environment machinery we shall be using.

Definition 6.1 Let M be a non-empty set. A C^{∞} \mathcal{G}_f -atlas of dimension n on M is a family $\mathcal{A} = (U_{\lambda}, \varphi_{\lambda})_{\lambda \in \Lambda}$, where Λ is an index set, that satisfies the following conditions:

- 1. For every index $\lambda \in \Lambda$, the map $\varphi_{\lambda}: U_{\lambda} \to \widetilde{\mathbb{R}}_{f}^{n}$ is a bijection between the non-empty open subset $U_{\lambda} \subset M$ and the open subset $\varphi_{\lambda}(U_{\lambda}) \subset \widetilde{\mathbb{R}}_{f}^{n}$;
- 2. $M = \bigcup_{\lambda \in \Lambda} U_{\lambda};$
- 3. For every pair $\alpha, \beta \in \Lambda$ with $U_{\alpha\beta} = U_{\alpha} \cap U_{\beta} \neq \emptyset$, the subsets $\varphi_{\alpha}(U_{\alpha\beta})$ and $\varphi_{\beta}(U_{\alpha\beta})$ are open subsets contained in $\widetilde{\mathbb{R}}_{f}^{n}$ such that $\varphi_{\beta} \circ \varphi_{\alpha}^{-1} : \varphi_{\alpha}(U_{\alpha\beta}) \to \varphi_{\beta}(U_{\alpha\beta})$ is a \mathcal{C}^{∞} diffeomorphism.

The pair $(U_{\alpha}, \varphi_{\alpha})$ is called a local chart (or a coordinate system) of M. If $U \subset M$ and $\varphi : U \to \varphi(U)$ is a homeomorphism, where $\varphi(U)$ is an open subset of $\widetilde{\mathbb{R}}_f^n$, then the pair (U, φ) is said to be compatible with \mathcal{A} if for every pair $(U_{\lambda}, \varphi_{\lambda}) \in \mathcal{A}$, with $W_{\lambda} = U \cap U_{\lambda} \neq \emptyset$, we have that $\varphi \circ \varphi_{\lambda}^{-1} : \varphi_{\lambda}(W_{\lambda}) \to \varphi(W_{\lambda})$ is a \mathcal{C}^{∞} diffeomorphism, where $\varphi_{\lambda}(W_{\lambda})$ and $\varphi(W_{\lambda})$ are open subsets of $\widetilde{\mathbb{R}}_f^n$. By Zorn's Lemma, there exists a unique maximal \mathcal{C}^{∞} \mathcal{G} -atlas $\mathcal{A}^* \supset \mathcal{A}$ of dimension n on M, namely, the atlas

$$(U,\varphi) \in \mathcal{A}^*$$
, if and only if (U,φ) is compatible with \mathcal{A} .

A generalized manifold in the full environment, or a \mathcal{G}_f -manifold, is a set M equipped with a \mathcal{G}_f -atlas. A maximal \mathcal{G}_f -atlas of the \mathcal{G}_f -manifold M is called a \mathcal{G}_f -differential structure on M. The topology of M is the one that makes all local charts simultaneously homeomorphisms. If the context is clear, we shall omit the prefix \mathcal{G}_f . The proof of our next result can be found in [23].

Theorem 6.2 [Dimension Invariance] Let M be a \mathcal{G}_f -manifold. Then the dimension of a \mathcal{G}_f -atlas \mathcal{A} is constant in each connected component of M.

The notion of support of a generalized point was introduced in the context of the simplified algebra. Given a point $p = [(p_{\varepsilon})_{\varepsilon}] \in \widetilde{\mathbb{K}}^n$ one can see easily that

$$supp(p) = \bigcap_{\eta \in I} \overline{\{p_{\varepsilon} : \varepsilon \in I_{\eta}\}}$$

As we already saw, the definition can be translated in terms of idempotents as follow: $q \in supp(p)$ if and and only if these exists an idempotent $e \in \mathcal{B}(\widetilde{\mathbb{R}})$ such that $e \cdot p \approx e \cdot q$. In the full environment we have that $x \in \widetilde{\mathbb{K}}_f$ is associated with $0, x \approx 0$, if for a representative $(x_\varphi)_\varphi$ of x, there exists $p \in \mathbb{N}$ such that $x_{\varphi_\varepsilon} \to 0$ when $\varepsilon \to 0$, for all $\varphi \in \mathcal{A}_p(1)$. The identification $x_1 \approx x_2$ is equivalent to $x_1 - x_2 \approx 0$. If there exists $x_0 \in \mathbb{K}$ such that $x \approx x_0$, we say that x_0 is the associated number (or shadow) of x. This definition readily extends to $\widetilde{\mathbb{K}}_f^n$ in an obvious way.

Let $p = [(p_{\varphi})_{\varphi}] \in \widetilde{\mathbb{K}}_f^n$. The support of p is defined to be the subset

$$supp_f(p):=\{q\in\mathbb{K}^n:\ \forall\ m\in\mathbb{N},\ \exists\ \varphi\in\mathcal{A}_m\ \mathrm{and}\ \exists\ \varepsilon_l\to 0\ \mathrm{such\ that}\ p_{\varphi_{\varepsilon_l}}\to q\}$$

and the essential support of p is defined as

$$ssupp_f(p) = \bigcap_{m \in \mathbb{N}} \overline{\{p_{\varphi} : \varphi \in \mathcal{A}_m\}}$$

Clearly $supp_f(p) \subset ssupp_f(p)$. Algebraically we have: $q_0 \in supp_f(p)$ if and only if there exists $e \in \mathcal{B}(\widetilde{\mathbb{R}}_f)$ such $e \cdot p \approx e \cdot q_0$. From now on, subscripts will be omitted when referring to the support of a point. If M is a Riemannian manifold and $p \in \widetilde{M}_{cf}$ then ssupp(p) is a compact subset of M. By construction, $ssupp(\widetilde{M}_{cf}) = \overline{M}$. If r > 0, then $supp(\alpha_r^{\bullet}) = \{0\}$. More generally, if $p \in B_1(0)$ then $supp(p) = \{0\}$, since in this case we have $p \approx 0$ (see [6, Lemma 2.1]). If $p = [(\sin(\alpha_{-1}^{\bullet})]$, then supp(p) = [-1, 1]. For $x = [(x_{\varphi})_{\varphi}] \in \widetilde{\mathbb{R}}_f^n$, write $||x||_2 = [(||x_{\varphi}||_{\mathbb{R}^n})_{\varphi}]$, but note that any other norm of \mathbb{R}^n will do.

If $(B_{\varphi})_{\varphi \in \mathcal{A}_0}$ is a net of subsets of \mathbb{R}^n , then the internal set generated by $(B_{\varphi})_{\varphi \in \mathcal{A}_0}$ is $[B_{\varphi}] := \{x \in \mathbb{R}^n_f : \exists (x_{\varphi})_{\varphi} \text{ representative of } x, \exists k \in \mathbb{N}, \exists \eta_{\varphi} \in I \text{, such that } x_{\varphi_{\varepsilon}} \in B_{\varphi_{\varepsilon}} \forall \varphi \in \mathcal{A}_k \text{ and } \varepsilon \in I_{\eta_{\varphi}} \}.$ Internal sets generalize the notion of membranes and are closed in the sharp topology, τ . Strong internal sets were first introduced in the simple environments and are part of the puzzle.

Definition 6.3 [Strong Internal Sets]

- 1. Let $(B_{\varphi})_{\varphi \in \mathcal{A}_0}$ be a net of subsets of \mathbb{R}^n and let $(x_{\varphi})_{\varphi \in \mathcal{A}_0}$ be a moderated net of points in \mathbb{R}^n . We say that $(x_{\varphi})_{\varphi \in \mathcal{A}_0}$ belongs to $(B_{\varphi})_{\varphi \in \mathcal{A}_0}$ τ -strongly, and we write $x_{\varphi} \in_{\tau} B_{\varphi}$, if
 - (a) There exists $k \in \mathbb{N}$ such that $x_{\varphi_{\varepsilon}} \in B_{\varphi_{\varepsilon}}$ for all $\varphi \in A_k$ and ε sufficiently small.
 - (b) If $[(y_{\varphi})_{\varphi}] = [(x_{\varphi})_{\varphi}]$ then there exists $p \in \mathbb{N}$, $p \geq k$, such that $y_{\varphi_{\varepsilon}} \in B_{\varphi_{\varepsilon}}$, for all $\varphi \in \mathcal{A}_p$ and ε sufficiently small.
- 2. Let $(B_{\varphi})_{\varphi \in \mathcal{A}_0}$ be a net of subsets of \mathbb{R}^n . The strong internal set with respect to τ (τ -strong internal) generated by $(B_{\varphi})_{\varphi \in \mathcal{A}_0}$ is denoted and defined by: $\langle B_{\varphi} \rangle_{\tau} := \{ [(x_{\varphi})_{\varphi}] \in \widetilde{\mathbb{R}}_f^n : x_{\varphi} \in_{\tau} B_{\varphi} \}$. We shall omit τ in the notation $\langle B_{\varphi} \rangle_{\tau}$, call it a strong internal set and also replace the notation \in_{τ} by \in_{φ} .

It is straightforward to verify that $\widetilde{\mathbb{R}}_f = \langle (-e^{1/i(\varphi)}, e^{1/i(\varphi)})_{\varphi \in \mathcal{A}_0} \rangle$ and that $\langle A_{\varphi} \rangle \cap \langle B_{\varphi} \rangle = \langle A_{\varphi} \cap B_{\varphi} \rangle$.

Theorem 6.4 Let $(A_{\varphi})_{\varphi \in \mathcal{A}_0}$ be a net of subsets of \mathbb{R}^n , and let $(x_{\varphi})_{\varphi \in \mathcal{A}_0}$ be a moderated net of points in \mathbb{R}^n . Then, $x \in \langle A_{\varphi} \rangle \iff dist(x, A^c) := [(d(x_{\varphi}, A_{\varphi}^c))_{\varphi}] \in Inv(\widetilde{\mathbb{R}}_f)$.

Proof. Consider $x_{\varphi} \in_{\varphi} A_{\varphi}$ and suppose that $\forall p \in \mathbb{N} \exists \varphi \in A_p$ and $\varepsilon_k \to 0$ such that $d(x_{\varphi_{\varepsilon_k}}, A_{\varphi_{\varepsilon_k}}^c) \leq i(\varphi)^k \varepsilon_k^k$, $\forall k \in \mathbb{N}$. Then, for each $k \in \mathbb{N}$, we can choose $y_k \in A_{\varphi_{\varepsilon_k}}^c$, such that $||y_k - x_{\varphi_{\varepsilon_k}}|| < 2i(\varphi)^k \varepsilon_k^k$, $\forall k \in \mathbb{N}$.

Choose $(y_{\varphi})_{\varphi}$ equivalent to $(x_{\varphi})_{\varphi}$, so that for such $\varphi's$ that we assume exist, we have $y_{\varphi_{\varepsilon_k}} = y_k$, $\forall \ k \in \mathbb{N}$. Thus, $y_{\varphi_{\varepsilon_k}} \notin A_{\varphi_{\varepsilon_k}} \ \forall \ k \in \mathbb{N}$, a contradiction. On the other hand, if the hypothesis of the converse implication holds, then $x_{\varphi_{\varepsilon}} \in A_{\varphi_{\varepsilon}}$, $\forall \ \varphi \in \mathcal{A}_p$ and ε sufficiently small. Moreover, if $(y_{\varphi})_{\varphi}$ is equivalent to $(x_{\varphi})_{\varphi}$, then $d(y_{\varphi_{\varepsilon}}, A_{\varphi_{\varepsilon}}^c) > \frac{1}{2} \alpha_{\varphi}^{\bullet}(\varphi_{\varepsilon})$ and therefore, $y_{\varphi_{\varepsilon}} \in A_{\varphi_{\varepsilon}}$, $\forall \ \varphi \in \mathcal{A}_p$ and ε small enough. Using [6, Theorem 3.18], the result follows.

Corollary 6.5 $\langle A_{\varphi} \rangle$ is open in the sharp topology.

Let M be a connected sub-manifold of dimension n in \mathbb{R}^N , and let $(U_{\alpha}, \phi_{\alpha}), \alpha \in \Lambda$ be an atlas of M. Suppose that $\forall \alpha \in \Lambda$, $\phi_{\alpha}(U_{\alpha}) = \Omega_0 = B_r(0)$, for some r > 0. Consider $\widetilde{M}_{c,f} \subset \widetilde{\mathbb{R}}^N_{c,f} \subset \widetilde{\mathbb{R}}^N_f$ the set of compactly supported points defined by M. Define $\widetilde{\Lambda} := \{\lambda : \mathcal{A}_0 \to \Lambda\} = \mathcal{F}(\mathcal{A}_0, \Lambda)$. Each $\lambda \in \widetilde{\Lambda}$ is associated with a net $(\lambda_{\varphi})_{\varphi \in \mathcal{A}_0}$ of elements from Λ , where $\lambda_{\varphi} = \lambda(\varphi)$. For $\lambda \in \widetilde{\Lambda}$, define $U_{\lambda} := \langle U_{\lambda_{\varphi}} \rangle \subset \widetilde{\mathbb{R}}^N_{c,f}$ and define

$$\phi_{\lambda}: U_{\lambda} \longrightarrow \widetilde{\mathbb{R}}_{c,f}^{n}$$
$$[(p_{\varphi})_{\varphi}] \longmapsto [(\phi_{\lambda_{\varphi}}(p_{\varphi}))_{\varphi}].$$

Theorem 6.6 Suppose that for each $\alpha \in \Lambda$, the map ϕ_{α} and its inverse are Lipschitz. Then $\widetilde{M}_{c,f}$, with the induced topology, is a \mathcal{G}_f -manifold of $\widetilde{\mathbb{R}}_f^N$ of dimension n.

Proof. Let $p = [(p_{\varphi})_{\varphi}] \in M_{c,f}$. We shall construct a local chart containing p. In fact, since ssupp(p) is a compact subset of M, there exists a finite subset $I_0 \subset \Lambda$ such that $ssupp(p) \subset \bigcup_{\alpha \in I_0} U_{\alpha}$. Let δ be the Lebesgue number for this covering. Choose a finite number of points $q_i \in ssupp(p)$ such that $ssupp(p) \subset \bigcup_{1 \leq i \leq l} B_{\delta_1}(q_i)$, where $\delta_1 < \delta/4$. Starting with q_1 , define $\lambda \in \widetilde{\Lambda}$ by $\lambda_{\varphi} := \alpha_{q_1}$ where α_{q_1} is chosen such that $B_{\delta_1}(q_1) \subset U_{\alpha_{q_1}}$ and $p_{\varphi} \in B_{\delta_1}(q_1)$. For λ_{φ} not yet defined, continue this process defining $\lambda_{\varphi} := \alpha_{q_2}$ where α_{q_2} is chosen such that $B_{\delta_1}(q_2) \subset U_{\alpha_{q_2}}$ and $p_{\varphi} \in B_{\delta_1}(q_2)$ until completing it by defining $\lambda_{\varphi} := \alpha_{q_1}$. At the end of this process, λ will be well defined. In fact, if it were not so, then $\forall m \in \mathbb{N}$, there should exist a sequence $(\varphi_k) \in \mathcal{A}_{m_k}$, $m_k \uparrow \infty$ and a sequence $\varepsilon_k \to 0$ such that $p_{(\varphi_k)_{\varepsilon_k}} \notin B_{\delta_1}(q_i)$ for any $1 \leq i \leq l$, and for sufficiently large values of k. This is a contradiction, since the sequence $(p_{(\varphi_k)_{\varepsilon_k}})$ has an accumulation point in ssupp(p) and the latter is covert by the balls. Therefore, λ is well-defined and has finite image. Furthermore, $p \in U_{\lambda}$. To see this, we need to show that $p_{\varphi} \in_{\varphi} U_{\lambda_{\varphi}}$. But this follows directly from Theorem 6.4 and the definition of λ , since by the definition of λ , we have $p_{\varphi} \in B_{\delta_1}(q_i) \subset U_{\alpha_{q_i}}$ for some i, by Theorem 6.4 and appropriate choice of δ_1 , we have that $[(d(p_{\varphi}, U_{\lambda_{\varphi}}^c))_{\varphi}]$ is invertible in $\widetilde{\mathbb{R}}_f$.

We shall now prove that the family $\{(U_{\lambda}, \phi_{\lambda}), \lambda \in \widetilde{\Lambda}\}$, with λ of finite range, is a C^{∞} \mathcal{G}_f -atlas of dimension n for $\widetilde{M}_{c,f}$. In fact, it is clear that ϕ_{λ} is well-defined for each λ , i.e., it does not depend on representatives and $\phi_{\lambda}(p) \in \langle \Omega_0 \rangle$, $p \in U_{\lambda}$. This follows directly from the Lipschitz condition of the charts of M and their inverses, and from λ having finite range. It also follows that each ϕ_{λ} is an isometry with respect to the sharp topologies of $\widetilde{\mathbb{R}}_f^N$ and $\widehat{\mathbb{R}}_f^n$, and is therefore bijective and continuous. By Corollary 6.5, we already have that each U_{λ} and $\langle \Omega_0 \rangle$ are open in the sharp topology. It is clear that $\widetilde{M}_{c,f} = \bigcup U_{\lambda}$.

Any change of coordinates $\phi_{\beta} \circ \phi_{\lambda}^{-1}$ is a homeomorphism that has a representative consisting of finitely many \mathcal{C}^{∞} diffeomorphisms that take values in a bounded subset of \mathbb{R}^n , so $\phi_{\beta} \circ \phi_{\lambda}^{-1}$ is a \mathcal{C}^{∞} diffeomorphism from the open set $\phi_{\lambda}(U_{\lambda\beta})$ to the open set $\phi_{\beta}(U_{\lambda\beta})$. Note that we can write $\phi_{\beta} \circ \phi_{\lambda}^{-1}$ as a finite interleaving

$$\phi_{\beta} \circ \phi_{\lambda}^{-1} = \sum_{i} e_{i} f_{i}$$

where each f_i is a \mathcal{C}^{∞} diffeomorphism. Finally, for each λ_{φ} , there exists an open set $U^{\lambda_{\varphi}}$ in \mathbb{R}^N such that $U_{\lambda_{\varphi}} = M \cap U^{\lambda_{\varphi}}$. Defining $U^{\lambda} = \langle U^{\lambda_{\varphi}} \rangle$, it follows that $U_{\lambda} = \widetilde{M}_{c,f} \cap U^{\lambda}$, with U^{λ} an open subset of $\widetilde{\mathbb{R}}_f^N$. This proves that $\widetilde{M}_{c,f}$ has the induced topology of $\widetilde{\mathbb{R}}_f^N$, and concludes the proof.

Corollary 6.7 Each local chart $(U_{\lambda}, \phi_{\lambda})$ is an isometry in the sharp topologies.

Proof. In fact, by lemma A.1 of [35], for each compact subset $K \subset M$, there exists C > 0 such that $||p-q|| \le dist_M(p,q) \le C||p-q||$, for all $p,q \in K$, where $dist_M$ is the Riemannian metric of M. This implies that the local charts are isometries in the sharp topology, since, from the above proof, each one is an isometry with respect to the sharp topologies of $\widetilde{\mathbb{R}}_f^N$ and $\widetilde{\mathbb{R}}_f^n$.

If λ is constant then (U_{λ}, φ) is called a *principal chart*. For $\lambda \in \widetilde{\Lambda} \subset \widetilde{\mathbb{K}}_f$ and $x \in B_1(0)$ define $x\lambda(\varphi) = x(\varphi)\lambda(\varphi)$. In particular, if $x = e \in \mathcal{B}(\widetilde{\mathbb{R}}_f)$ then $e\lambda$ means that when $e(\varphi) = 0$ this index must be omitted. With this notation, the proof of the theorem gives us the following corollary and thus, once again, making the connection with the notion of interleaving.

Corollary 6.8 Let (U_{λ}, φ) be a local chart with λ of finite range. There exist principal charts $(U_{\alpha_i}, \varphi_i), i = 1, k$ and a complete set of mutually orthogonal idempotents $e_1, \dots, e_k \in \mathcal{B}(\widetilde{\mathbb{R}}_f)$ such that $e_i \cdot U_{\lambda} \subset e_i \cdot U_{\alpha_i}$ and $U_{\lambda} = \bigsqcup_i U_{e_i \alpha_i}$, where \bigsqcup should be understood as some kind of interleaving.

Let M be a Riemannian manifold. By Whitney's Embedding Theorem M may be embedded as sub-manifold in some Euclidean space. Applying the previous theorem, we set $M^* = \widetilde{M}_{cf}$. By construction we have that $ssupp(M^*) = M$. By construction we also have that $M \subset M^* \subset \widetilde{M}$, the latter containing the infinities and the former, M^* , containing infinitesimals.

Corollary 6.9 Let M be a Riemannian manifold. There exists a \mathcal{G}_f -manifold M^* such that M is discretely embedded in M^* and $ssupp(M^*) = M$.

Let $f \in \mathcal{C}^{\infty}(M,\mathbb{R})$ and define $\iota(f)$ on M^* such that its local expression on the local chart $(U_{\lambda},\phi_{\lambda})$ of M^* is given by $f\circ\phi_{\lambda}^{-1}([(p_{\varphi})_{\varphi}])=[(f\circ\varphi_{\lambda(\varphi)}^{-1}(p_{\varphi}))_{\varphi}]$. Since f is differentiable, the local

expressions of $\iota(f)$ are differentiable functions and hence, as defined following classical Differential Geometry, it follows that $\iota(f) \in \mathcal{C}^{\infty}(M^*, \widetilde{\mathbb{R}}_f)$. Consequently,

$$\iota: \mathcal{C}^{\infty}(M, \mathbb{R}) \longrightarrow \mathcal{C}^{\infty}(M^*, \widetilde{\mathbb{R}}_f)$$

is an \mathbb{R} -algebra monomorphism.

Let M be an orientable Riemannian manifold. We now relate our construction with the construction of [20, 21], where the authors define the basic space of generalized scaler fields on M and denote it by $\hat{\mathcal{E}}(M)$. The subset of moderate elements is denoted by $\hat{\mathcal{E}}_m(M)$ and the algebra of generalized functions on M is denoted by $\hat{\mathcal{G}}(M)$. This is done in the setting of the full algebras which happens to be our setting too (not the invariant though). We observe that $\mathcal{D}(M)$ consists of forms but this does not invalidate what comes next. Our next result extends the Embedding Theorem proved in [3] (see previous sections).

Theorem 6.10 [Embedding Theorem] Let M be an n-dimensional orientable Riemannian manifold. There exists an n-dimensional \mathcal{G}_f -manifold M^* and an algebra monomorphism

$$\kappa: \hat{\mathcal{G}}(M) \longrightarrow \mathcal{C}^{\infty}(M^*, \widetilde{\mathbb{R}}_f)$$

which commutes with derivation. Moreover, $ssupp(M^*) = M$ and equations defined on M, whose data have singularities or nonlinearities, naturally extend to equations on M^* and, on M^* , these data become C^{∞} -functions.

We skip the proof since it can be obtained from what we just saw and results of [20, 21] on $\hat{\mathcal{E}}_m(M)$. The restriction of κ to $\mathcal{C}^{\infty}(M,\mathbb{R})$ is ι . Some classical results can be readily extended: A differentiable map $g:M\to N$ between manifolds induces an algebra homomorfism $\Phi(g):\mathcal{C}^{\infty}(N^*,\widetilde{\mathbb{R}}_f)\longrightarrow \mathcal{C}^{\infty}(M^*,\widetilde{\mathbb{R}}_f)$. Another point in our approach is that the topologies of the algebras and the \mathcal{G}_f -manifolds involved have the same underlying reals, namely $\widetilde{\mathbb{R}}_f$. Note however that in our approach some infinitesimals live in M, but infinities live in \widetilde{M} . Function which are infinities do live in the algebras constructed.

We established that in the setting of Colombeau algebras on manifolds the perspective can be reduced to a setting very much alike to the settings of Classical Differential Geometry which we coined Generalized Differential Geometry. The difference being that in the former, the basic underlying structure is \mathbb{R} and in the former it is $\widetilde{\mathbb{R}}_f$. In the complex case \mathbb{R} should be changed by \mathbb{C} . The topological ring $\widetilde{\mathbb{R}}_f$ behaves much alike \mathbb{R} in the sense that an element of $\widetilde{\mathbb{R}}_f$ is either a unit or a zero divisor and its group of invertible elements, $Inv(\widetilde{\mathbb{R}}_f)$, is open and dense in $\widetilde{\mathbb{R}}_f$. The structure of \mathbb{R} is well known as is the structure of $\widetilde{\mathbb{R}}_f$. So our approach gives researchers from other field the possibility to apply the theory without having to dive into the complicated details of its construction. Basically all one needs is to get acquainted with $\widetilde{\mathbb{R}}_f$: its ideal structure, idempotents and group of units. We refer the interested reader to [6, 7, 34].

It follows from the embedding theorem above that certain problem on M involving distributions and their products or involving certain singularities can be lifted to problems on M^* involving only \mathcal{C}^{∞} -functions defined on M^* . For all the environments involved one can define the support of their elements, integration, membranes, internal sets and obtain topological results. In particular, one can prove that $\mathcal{D}'(M)$ is discretely embedded in $\hat{\mathcal{G}}(M)$ and that for $T \in \mathcal{D}'(M)$ and $\varphi \in \mathcal{D}(M)$ the following equality holds also in \mathbb{R}_f (see previous sections).

$$\langle T \mid \varphi \rangle = \int_{M} \kappa(T) \varphi$$

We now define one more notion of derivation, the notion of Gâteau and Fréchet differentiability in the full environments. Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, X and Y be topological $\widetilde{\mathbb{K}}_f$ -modules and let $\Omega \subset X$ be an open subset. A function $F: \Omega \longrightarrow Y$ is said to be $\widetilde{\mathbb{K}}_f$ -Gâteau-differentiable in $x \in int(\Omega)$, the set of interior points of Ω , if there exists a $\widetilde{\mathbb{K}}_f$ -linear map $DF(x): X \longrightarrow Y$ such that

$$\lim_{t \to 0} \frac{F(x + \alpha^{\bullet}_{-ln(t)}u) - F(x)}{\alpha^{\bullet}_{-ln(t)}} = DF(x)u, \ \forall \ u \in X$$

Such a differentiable map is said to be $\widetilde{\mathbb{K}}_f$ -Fréchet-differentiable in x if

$$\lim_{\|u\| \to 0} \frac{F(x+u) - F(x) - DF(x)u}{\alpha_{-\ln(\|u\|)}^{\bullet}} = 0, \ \forall \ u \in X$$

It is easy to see that the latter definition implies the former and also the continuity of the map F. If the map $DF: X \longrightarrow \mathcal{L}(X,Y)$, that associates each x with DF(x) is continuous then we say that F is of class \mathcal{C}^1 or continuously $\widetilde{\mathbb{K}}_f$ —Fréchet-differentiable. This extends Generalized Differential Calculus to $\widetilde{\mathbb{K}}_f$ —modules making possible applications in what we coin Generalized Variational Calculus. The development and applications of these ideas in all environments will appear elsewhere. We finish this section looking at events in Generalized Space-Time and generalized solutions of equations. We do not claim that what follows next corresponds to physical reality. Conclusion are solely based on our interpretation of our model.

Let M be four dimensional Classical Space-Time. Since M is curved, it follows by the Nash Embedding Theorem and Whitney's Embedding Theorem that there exists a smallest $n \in \{5, 6, 7, 8, 9\}$ such that M can be isometrically embedded into \mathbb{R}^n . Define Generalized Space-Time to be the \mathcal{G}_f -manifold M^* associated to M. We have that M is discretely embedded in M^* and $M \subset M^* \subset \overline{B}_1(0) \subset \widetilde{M} \subset \widetilde{\mathbb{R}}^n$, with $ssupp(M^*) = M$. The infinitesimals live in $M^* \cap B_1(\vec{0})$ and the infinities live in $\widetilde{M} \cap (\overline{B}_1(\vec{0}))^c \subset \widetilde{\mathbb{R}}^n$.

If $p \in M^*$ is an infinitesimal such that $\|p\|_2 \notin Inv(\widetilde{\mathbb{R}})$, then there exist $e, f \in \mathcal{B}(\widetilde{\mathbb{R}})$ satisfying $e \cdot p = \vec{0}$ and $f \cdot \|p\|_2 \in Inv(f \cdot \widetilde{\mathbb{R}})$. Given $p_1 \in M^*$, the interleaving $ep_1 + (1 - e)p = ep_1 + p \in M^*$. So one can interleave two events (points) without, possibly, modifying at least one of them. Once interleaved, they become a single event in M^* . In general, an interleaving is of the form $\sum_j e_j \cdot x_j$, where $x_j \in \widetilde{\mathbb{R}}^n$ and $e_j \in \mathcal{B}(\widetilde{\mathbb{R}})$, $e_i \cdot e_j = \delta_{ij} e_i$ and $\sum_j e_j = 1$, where δ_{ij} is Kronecker's delta function. A

generalized transition probability $\nu(e_j)$ is associated to each e_j and $\sum_j \nu(e_j) = 1$. If measuring on M corresponds to applying the function F, then $F(\sum_j e_j \cdot x_j) = \sum_j e_j \cdot F(x_j)$ is again an interleaving.

For example, let $T = x\delta$, $x_0, x_j \in \mathbb{R}$, $e_j \in \mathcal{B}(\widetilde{\mathbb{R}})$, $T_j(x) = T(\frac{x_j}{x_0}x)$ and consider the interleaving $F = \sum_j T_j e_j$. Then $\kappa(T)(x_0 \cdot \alpha) = x_0 \rho(x_0)$ and $\kappa(F)(x_0 \cdot \alpha) = \sum_j \kappa(T_j)(x_0 \cdot \alpha) e_j = \sum_j \kappa(T)(x_j \cdot \alpha) e_j = \sum_j x_j \rho(x_j) e_j$ showing that an infinitesimal can produce a *simultaneous interleaved effect* at points x_j

at arbitrary classical distances from $x_0\alpha \in halo(0)$ (see also [23]). The product of the infinitesimal $x_0\alpha$ and the infinity $\delta(x_0\alpha)$, can be considered the collapse of the infinity or the surge of the

infinitesimal. Summing the interleaving $x = e_1x_1 + e_2x_2$ and $y = f_1y_1 + f_2y_2$ results in the interleaving $e_1f_1(x_1 + y_1) + e_1f_2(x_1 + y_2) + e_2f_1(x_2 + y_1) + e_2f_2(x_2 + y_2)$, a superposition. So in M interleaving and their sums are perceived like waves. Motion of objects in M^* is captured by hypersequences, which can be considered to be displacement with choices, made along the displacement, leading to the same point in case of convergence.

A certain set of idempotents— which maybe are linked to a certain set of automata — in M^* seems to determine phenomenon — which are the results of interleaving — in M. That what is not interleaved cannot be observed until it is interleaved. Since there is a fixed distance between points in M — it is a grid of equidistant points — it is the time interacted interleaving between points of M, that causes the illusion of distances between classical points to emerge. Time, being partially ordered and having the possibility of taking infinitesimal values, is event dependent. It results in flows in M^* . When objects are observed, this causes the multiplication of the observed objects with an interleaving of infinities and infinitesimals, resulting in new interleaving and possibly modifying the objects. Transition probabilities, determined by observations and choices, decide how interleaving are formed. Things in M become meaningful and interact, i.e., M expands and attraction occurs, as interleaving are formed. Consequently, in M^* observations and choices are the driving forces behind spacetime expansion in M and it is interleaving that results in gravity. In case $\mathbb{K} = \mathbb{C}$ other interpretations arise, in particular, when this involves infinitesimals. For a general M, managing to control the inputs in events in M^* and to direct the outputs in M might be a tool to be used and indicate a way to glance into the unseeable, thus making possible the confirmation of its existence.

Consider a flow F in M^* . In some parts of F some of its elements can behave as infinitesimals resulting in some probability of their rendezvous with infinities in \widetilde{M} . Taking a small open generalized space-time volume Δ containing the rendezvous, there will be some probability that one observes a simultaneous interleaved effect, caused by the collapse of infinities in Δ , resulting in points contained in $\Delta \cap (F \cap M)$ and, hence, in flow patterns of $\Delta \cap F \cap M$ that exhibit chaotic and unpredictable behavior. The unpredictability depends on the generalized transition probabilities of the idempotents of the interleaving involved and the type of infinitesimals in Δ and infinities in \widetilde{M} . Instantaneousness may also occur since it is produced by the observation in M of the collapse of infinities in M^* .

Suppose that one interleaves two events F_1 and F_2 in M^* so that they are contained in a small open generalized space-time volume Δ stretch in the classical space direction. The creation of infinitesimals at $F_1 \cap M$, one end of the space-stretch, may result in the observation of an instantaneous simultaneous interleaved effect at $F_2 \cap M$, the other end of the space-stretch. No contradiction arrises in the instantaneousness, since M consists of a grid of equidistant points in M^* all resulting from the collapse of infinities. In M^* , no points of M are far away.

Other phenomena can be the result of the fact that the support of a generalized event (or generalized solution of an equation) has more than one point in its support. For certain differential equations this may lead to different numerical solutions depending on the numerical refinement or numerical grid being used in the calculations. Numerical grids correspond to idempotents and the numerical solution given by each grid corresponds to the product of the generalized solution and the corresponding grid-idempotent. For such equations, only very fine tuning on grids, which is highly improbable, will result in the same idempotent. This suggest that the support of the generalized solution of such an equation has cluster points. This may indicate a way to confirm the existence of these generalized environments. References on which this section is based are [2, 3, 4, 6, 8, 12, 14, 17, 20, 21, 22, 23, 25, 26, 27, 29].

7 Conclusion

The generalized reals were introduced and milieus were constructed resulting in a notion of Calculus which naturally extends Newton's and Schwartz's Calculus, the main difference being that classically we stop measuring at scale α , the natural gauge. Generalized Differential Geometry is introduced resulting in the construction of Generalized Space-Time, which we do not claim corresponds to physical reality. Below a certain distance in Generalized Space-Time, points in Classical Space-Time cease to exists, the latter becomes fussy, uncertainty of measurements becomes the rule and reality becomes sequential, i.e., depending on a history of (infinitely) many interleaved events. As infinities have the same nature, their rendezvous appears to cause, seemingly unpredictable, spooky interleaved events. Such encounters can be considered as the collapse of the infinities as time takes infinitesimal values. For example, the collapse of Dirac's infinity is illustrated by the function $f(t) = t\delta(t) = t_0 \rho(t_0)$, where $\delta(t)$ is the speed, at $t = t_0 \alpha \in \mathbb{R} \cdot \alpha \subset halo(0)$, a grid of equidistant points in Generalized Time. The solution $u(t,x) = \frac{x}{t+\alpha}$ of the equation $u_t + uu_x = 0, u(0,x) = x\alpha^{-1}$ satisfies $u(t_0\alpha, x_0\alpha) = \frac{x_0}{1+t_0} \in \mathbb{R}$. For $w(t) = arctan(\alpha^{-1}t)$ we have $w^2(t_0\alpha^2) \in halo(0)$, but $(w^2)'(t_0\alpha^2) \in halo(2t_0)$. Hence, in infinitesimal time classical measurements can be arbitrarily big. Consequently, phenomena in Generalized Space-Time can effect Classical Space-Time in several counterintuitive ways, being turbulence, spookiness and the illusion of distances some of the effects. The possible dependence on the grid used to find numerical solutions of certain differential equations is considered, suggesting an explanation in terms of the generalized solutions of these equations, the support of these generalized solutions and grid-idempotents. Finally, the stepping stone of generalized variational calculus is laid. Our proposal, centered around the notions of idempotents, interleaving and support, is for researchers to consider these generalized milieus and concepts as they might give insights concerning questions in classical environments and provide alternative mathematical tools. It might be however that, in terms of computation, Classical Calculus and Generalized Calculus are equivalent, which should explain the enormous success of Classical Calculus even though infinitesimals and infinities are absent in the classical realm. However, more mathematical effort may be needed and less clarity of why's might occur when using Classical Calculus to describe physical reality.

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