

Radial excitations and their potential impact on Fermi β -decay rates

L. Xayavong,¹ Y. Lim,¹ N. A. Smirnova,² and Calvin W. Johnson³

¹*Department of Physics, Yonsei University, Seoul 03722, South Korea*

²*LP2IB (CNRS/IN2P3-Université de Bordeaux), 33170 Gradignan cedex, France*

³*Department of Physics San Diego State University 5500 Campanile Drive San Diego CA 92182-1233 United States*
(Dated: May 28, 2025)

We investigate the contribution of radial excitations to Fermi β -decay matrix element. To this end, exact no-core shell model calculations are performed for the mirror β decay of tritium, where full convergence can be achieved on an ordinary computer. The differences between the isospin-mixing correction values obtained in the full and in a restricted model spaces are matched to the radial overlap correction term, analogous to that required in the shell-model approach, where the configuration space is extremely limited. We examine this complementary correction term using a nonorthogonal harmonic-oscillator basis, generated by slightly differentiating the oscillator frequencies between the initial and final nuclei, while all desirable properties, including translational invariance, are still preserved. For $N_{\max} \leq 8$, we find that the radial excitation contribution is negative, with a typical magnitude of approximately 10% to 20% of the radial diagonal contribution. This effect becomes more pronounced as the model space increases. Therefore, the δ_{C2} values obtained in the shell model approach, where radial excitations are not explicitly included, are likely overestimated. Based on experimental ft data and the corrective terms adopted in the survey by Hardy and Towner [Phys. Rev. C **102**, 045501 (2020)], we show that the incorporation of radial excitations for the superallowed $0^+ \rightarrow 0^+$ nuclear β decay tends however to worsen agreement with the Standard Model.

I. INTRODUCTION

Isospin symmetry is an efficient tool for dealing with the nuclear many-body problem, where the system's basic constituents are protons and neutrons [1]. Notably, it predicts degenerate nuclear charge states within a given isospin multiplet that possess identical spin and spatial configurations. Additionally, it implies model-independent sum rules that has been proven to be useful for both nuclear structure and reaction studies [2, 3]. This symmetry is, however, imperfect because of Coulomb repulsion between protons and nuclear charge-dependent interactions [4]. Currently, the correction due to isospin-symmetry breaking, although small, represents a limiting factor in the low-energy precision tests of the electroweak sector of the Standard Model via nuclear β decays [5–12]. For instance, experimental precision in 15 cases of the superallowed $0^+ \rightarrow 0^+$ nuclear β decay of isotriplets has reached $\mathcal{O}(10^{-4})$. These cases involve a wide variety of emitters with mass number between 10 and 74 [5, 13, 14], namely ^{10}C , ^{14}O , ^{22}Mg , ^{26m}Al , ^{26}Si , ^{34}Cl , ^{34}Ar , ^{38m}K , ^{38}Ca , ^{42}Sc , ^{42}Ti , ^{46}V , ^{50}Mn , ^{54}Co , ^{62}Ga , ^{70}Kr , and ^{74}Rb . Therefore, to preserve the meaningfulness of these studies, it is mandatory to account for all contributing effects within this level of precision [6, 7, 9, 15–24]. The so-called absolute $\mathcal{F}t$ value for the superallowed $0^+ \rightarrow 0^+$ nuclear β decay is traditionally expressed as [5]

$$\mathcal{F}t = ft(1 + \delta'_R)(1 - \delta_C + \delta_{NS}) = \frac{K}{\mathcal{M}_F^2 G_F^2 V_{ud}^2 (1 + \Delta_R^V)}, \quad (1)$$

where ft value is evaluated as the product of the statistical rate function (f) and the partial half-

life (t) [5], K is a combination of fundamental constants, namely $K/(\hbar c)^6 = 2\pi^3 \hbar \ln(2)/(m_e c^2)^5 = 8120.27648(26) \times 10^{-10} \text{ GeV}^{-4} \text{ s}$, G_F is the Fermi coupling constant, precisely known from the MuLan experiment [25, 26], and V_{ud} is the top-left element of the Cabibbo–Kobayashi–Maskawa (CKM) quark-mixing matrix [27, 28]. Note that $G_V^2 = G_F^2 V_{ud}^2$, where G_V is the vector coupling constant for weak semileptonic processes. Further, \mathcal{M}_F is the Fermi matrix element obtained within the isospin-symmetry limit and thus being model-independent, while other entries denote various corrections to the process. Radiative corrections are usually separated into three terms: Δ_R^V , which is universal [29, 30]; δ'_R , which depends on the decay Q value and atomic number [15]; and δ_{NS} , which absorbs nuclear structure effects [31–35]. Finally, δ_C is the so-called isospin-symmetry breaking correction which describes deviation of the realistic Fermi matrix element squared from its model-independent value.

An important feature of Eq. (1) is that the constancy of the corrected $\mathcal{F}t$ value across various nuclei would serve as a direct test of the conserved vector current (CVC) hypothesis, an analogy of the conservation of electric charge current. If CVC is confirmed, the average $\mathcal{F}t$ can be used further to extract V_{ud} , enabling a test of the top-row unitarity of the CKM matrix. Deviations of the average $\mathcal{F}t$ value from a constant or breaking of the CKM matrix unitarity can be attributed to new physics effects beyond the Standard Model (presence of scalar currents, right-handed currents or else [36]). The current status of these studies can be found in Ref. [5].

Radiative corrections have recently attracted a lot of attention of both particle and nuclear physicists [15, 29–35], since new calculations lead to certain tension in the

top-row CKM matrix unitarity. This in turn, encouraged nuclear theorists to put further efforts in the calculation of δ_C , whose evaluation necessitates an accurate solution of the nuclear many-body problem. Theoretical approaches to δ_C vary widely, ranging from simple schematic models that incorporate Coulomb mixing on top of harmonic oscillator functions to fully microscopic many-body treatments, such as symmetry-restored density functional theory and the shell model. This extensive body of work has resulted in more than 30 publications in high impact journals [6–9, 15, 37–64]. Among all the proposed approaches, only the one based on the phenomenological shell model with realistic radial wave functions [5, 42] shows remarkable agreement with the standard electroweak theory, particularly with the CVC and CKM top-row unitarity conditions. Within this consistent theoretical framework, δ_C is decomposed as

$$\delta_C \approx \delta_{C1} + \delta_{C2}, \quad (2)$$

where δ_{C1} corrects for isospin mixing inside the shell-model valence space, induced by the isospin-nonconserving part of the effective Hamiltonian [41, 53, 56, 57, 62], whereas δ_{C2} accounts for contributions beyond the valence space [6–9, 15, 58, 63]. Typically, δ_{C2} is evaluated using realistic Woods-Saxon radial wave functions, introducing a mismatch between protons and neutrons via the Coulomb and nuclear isovector potentials. Essentially, the radial wave functions are required to reproduce separation energies and charge radii, when experimental data are available [6, 7]. Note that several higher-order terms are excluded from Eq. (2). These terms are generally insignificant for Fermi transitions within low isospin multiplets, including isodoublets and isotriplets [9].

Despite its remarkable success, the shell-model approach was criticized by Miller and Schwenk [50] for lacking the contribution of radial excitations, which they demonstrated using a schematic model to be significant. However, no dedicated numerical calculation has yet been performed. While this effect might be partially captured through the use of an effective Hamiltonian and other operators, it cannot be explicitly treated within the shell model's valence spaces, which typically consist of a single oscillator shell. It is interesting to notice that an upper limit for the overlap $\langle \nu 1g_{9/2} | \pi 0g_{7/2} \rangle$ was experimentally determined to be 0.19% in the Gamow-Teller transition of ^{207}Hg [65]. Although this overlap could also be influenced by the spin-orbit potential, as emphasized in Ref. [66], it is primarily a manifestation of radial excitations. Additionally, radial excitations are the only mechanism considered by Damgaard [23] to derive an analytical formula for δ_C . While Damgaard's model is greatly simplified, since it incorporates only nodal mixing of harmonic oscillator functions driven by a schematic one-body Coulomb potential, it effectively captures key properties of δ_C , including Z -dependence and shell-structure effects. This result may also be regarded as evidence that radial excitations could be significant in certain scenar-

ios.

II. EXACT CALCULATIONS USING NO-CORE SHELL MODEL

To verify the importance of radial excitations in a realistic calculation, we perform exact no-core shell model (NCSM) calculations of the total isospin-symmetry breaking correction and its individual terms in Eq. (2) to extract the radial excitation contribution. For this purpose, we consider only the mirror β decay of tritium, $^3\text{H}(\beta^-)^3\text{He}$, where the computational requirements are manageable on an ordinary computer. Note that we use the exact τ_- operator for the calculations of the Fermi transition matrix element. NCSM calculations of δ_C for heavier nuclei are not expected to converge, even using advanced supercomputing resources. For reference, NCSM calculations for the superallowed $0^+ \rightarrow 0^+$ Fermi β decay in ^{10}C have been reported in Refs. [64, 67]. The reason for this undesired behavior mainly relies on the infinite-range Coulomb interaction, which is the principal driver for isospin mixing. The impact of an operator's range on the convergence of its expectation values has been discussed in Ref. [68].

In principle, the separation ansatz in Eq. (2) can be adapted to the NCSM framework, despite the lack of an appropriate strategy for ensuring asymptotic radial wave functions, as provided in the shell-model approach. By definition, the correction values deduced from the transition matrix element calculated in a finite model space specified by N_{max} correspond to δ_{C1} (see Appendix for the formalism). Only the value obtained from the fully converged matrix element (the value at $N_{\text{max}} = \infty$) corresponds to the total correction, δ_C . Ideally, the convergence issue may be addressed by evaluating δ_C through independent calculations of δ_{C1} and δ_{C2} , as conducted within the shell model framework in Refs. [6, 7, 9]. However, this approach requires a realistic single-particle basis which, besides the lack of square integrability¹ and the absence of an analytical form, can lead to significant contamination from center of mass contributions, especially for light nuclei. To avoid these technical difficulties, we perform an independent calculation of δ_{C2} by slightly differentiating the oscillator frequencies for the initial and final nuclei, while using an isospin-invariant interaction². We notice that many-body wave functions built from harmonic oscillator eigenfunctions are always factorable into intrinsic and center of mass components (provided the

¹ The overlap between states with the same quantum numbers, but opposite isospin projections is not equal to unity.

² The reader is reminded that δ_{C2} must be evaluated with an isospin-symmetry interaction, as it corrects only for the radial mismatch of basis functions, compensating for model space truncation. The isospin admixture in many-body wave functions induced by isospin nonconserving interactions is accounted for by δ_{C1} using the conventional harmonic oscillator basis.

latter is in its lowest energy state), even when the frequencies are assumed to be isospin-dependent or freely varied for each orbits. Our choice of different oscillator frequencies for a mother and a daughter nuclei meets the minimum requirement for generating radial mismatch, including radial excitations, to β decay matrix element. In general, the fact that initial and final nuclei may have minima of their ground-state energies at different $\hbar\omega$ values is natural for the NCSM approach and the converged result is independent of this parameter. In spite of these benefits, we remark that our approach does not allow to improve convergence of the calculation of δ_C and therefore is applicable only to light nuclei. Another possible solution to construct a realistic basis, would be for example the use of an isospin dependent oscillator parameter, which on the other hand, would necessitate significant effort in calculating interaction matrix elements, demanding the generalization of the well-known Talmi-Moshinsky brackets [69–71].

We employ an interaction derived from chiral effective field theory, excluding three-body forces³. The two-body matrix elements of the Hamiltonian are generated using the NuHamil code [72], incorporating up to next-to-next-to-next-to-next-to-leading order (N4LO) terms with 500 MeV regulator cutoff by Entem-Machleidt-Nosyk [73]. The Hamiltonian is evolved via the similarity renormalization group (SRG) with a flow parameter of $\lambda_{SRG} = 2 \text{ fm}^{-1}$. The isospin-symmetry interaction is obtained by switching off Coulomb repulsion and replacing the $T = 1$ channel matrix elements with the average of proton-proton, neutron-neutron, and proton-neutron matrix elements. The resulting Hamiltonian matrix is diagonalized using the BIGSTICK code [74]. It is important to emphasize that when different oscillator frequencies are used for the initial and final nuclei, we must account for the overlap integrals in the evaluation of the transition matrix element. The formalism is described in detail in Appendix. Specifically, these overlap integrals reflect differences between proton and neutron radial wave functions with both identical and different numbers of nodes. When the numbers of nodes differ, this effect is referred to as radial excitations, which is the central focus of the present study. Accordingly, δ_{C2} can be broken as

$$\delta_{C2} = \delta_{C2}^{sm} + \delta_{C2}^{re} = \delta_{C2}^{sm}(1 + \kappa), \quad (3)$$

where δ_{C2}^{sm} and δ_{C2}^{re} represent the contributions arising from the radial diagonal (obtained in the shell model approach) and radial excitation mechanisms, respectively, while κ is given by the ratio $\delta_{C2}^{re}/\delta_{C2}^{sm}$. Although the asymptotic wave functions still retain a Gaussian form

and the resulting correction terms might be unrealistic, the ratio κ is expected to be reliable, showing no substantial dependence on the mean field potential model. The results for δ_{C2} and κ as a function of the oscillator frequency difference, $\Delta\hbar\omega = \hbar\omega_i - \hbar\omega_f$, are shown in Fig. 1. We remark that, by construction, δ_{C2} is symmetric with respect to $\Delta\hbar\omega = 0$. However, since neutrons in the initial nucleus are more tightly bound than protons in the final nucleus, it is generally more natural to set $\Delta\hbar\omega$ to be slightly larger than 0. Note that we consider only small $\Delta\hbar\omega$; otherwise, Eq. (2), which will be employed in the following, would no longer be justified, as the higher-order terms would become significant. It is observed that δ_{C2} follows a quadratic trend, which can be understood from its Taylor expansion,

$$\delta_{C2}(\Delta\hbar\omega) = \delta'_{C2}(0)\Delta\hbar\omega + \frac{\delta''_{C2}(0)}{2!}\Delta\hbar\omega^2 + \mathcal{O}(\Delta\hbar\omega^3), \quad (4)$$

with $\delta_{C2}(0) = 0$. Eq. (4) is generally N_{\max} dependent.

It is interesting to emphasize that the isospin admixture spreads the model-independent Fermi matrix element [obtained under SU(2) symmetry] across all possible channels⁴, including isospin forbidden ones. This implies that δ_C applied to a particular channel must always be positive, resulting in a reduction in the magnitude of the actual Fermi matrix element. Previous shell model studies suggest that this behavior also holds for the individual correction terms, including δ_{C1} and δ_{C2}^{sm} . Only the higher-order terms, such as δ_{C2}^{re} , and some other terms discussed in Ref. [9], may be negative.

As seen in the right panels of Fig. 1, the radial excitation contribution is consistently negative. These results are in agreement with those of Miller and Schwenk [50]. We remark that the ratio κ varies very slowly with $\Delta\hbar\omega$, in contrast to δ_{C2} , whose behavior follows Eq. (4), as evident from the left panels of Fig. 1. However, κ shows a strong dependence on N_{\max} . For instance, at $\hbar\omega_i = 10 \text{ MeV}$, κ remains nearly constant for $N_{\max} \leq 6$, but jumps by almost an order of magnitude (with a negative sign) at $N_{\max} = 8$. This dependence weakens as $\hbar\omega_i$ increases. At $\hbar\omega_i = 30 \text{ MeV}$, the κ values obtained for model spaces of $2\hbar\omega$, $4\hbar\omega$, $6\hbar\omega$, and $8\hbar\omega$, are approximately -13% , -15% , -16% , and -18% , respectively. This observation is due to the fact that larger model spaces include more orbits with different nodes, leading to stronger radial excitation contribution. Both δ_{C2}^{sm} and δ_{C2}^{re} must vanish at $\Delta\hbar\omega = 0$ where the orthogonality is fully restored. Since the magnitude of δ_{C2}^{re} is always smaller than that of δ_{C2}^{sm} , κ must also be 0 at $\Delta\hbar\omega = 0$ (our results for κ do not include $\Delta\hbar\omega = 0$ because of numerical instability). In the present case, we have $\kappa = 0$ for the $0\hbar\omega$ model space because no nodal mixing occurs in the lowest energy configurations of the initial and final

³ As shown in Ref. [67], three-body forces enhance the binding energy of three-nucleon systems by more than half an MeV. Noticeably, an increase in binding energy generally causes a reduction in δ_C , as demonstrated in previous shell model studies [6–9]. However, the impact of three-body forces is unlikely to be significant in our study of radial excitations, which is a higher-order effect.

⁴ The total Fermi strength is conserved, and is given by the sum of the allowed and isospin forbidden transitions.

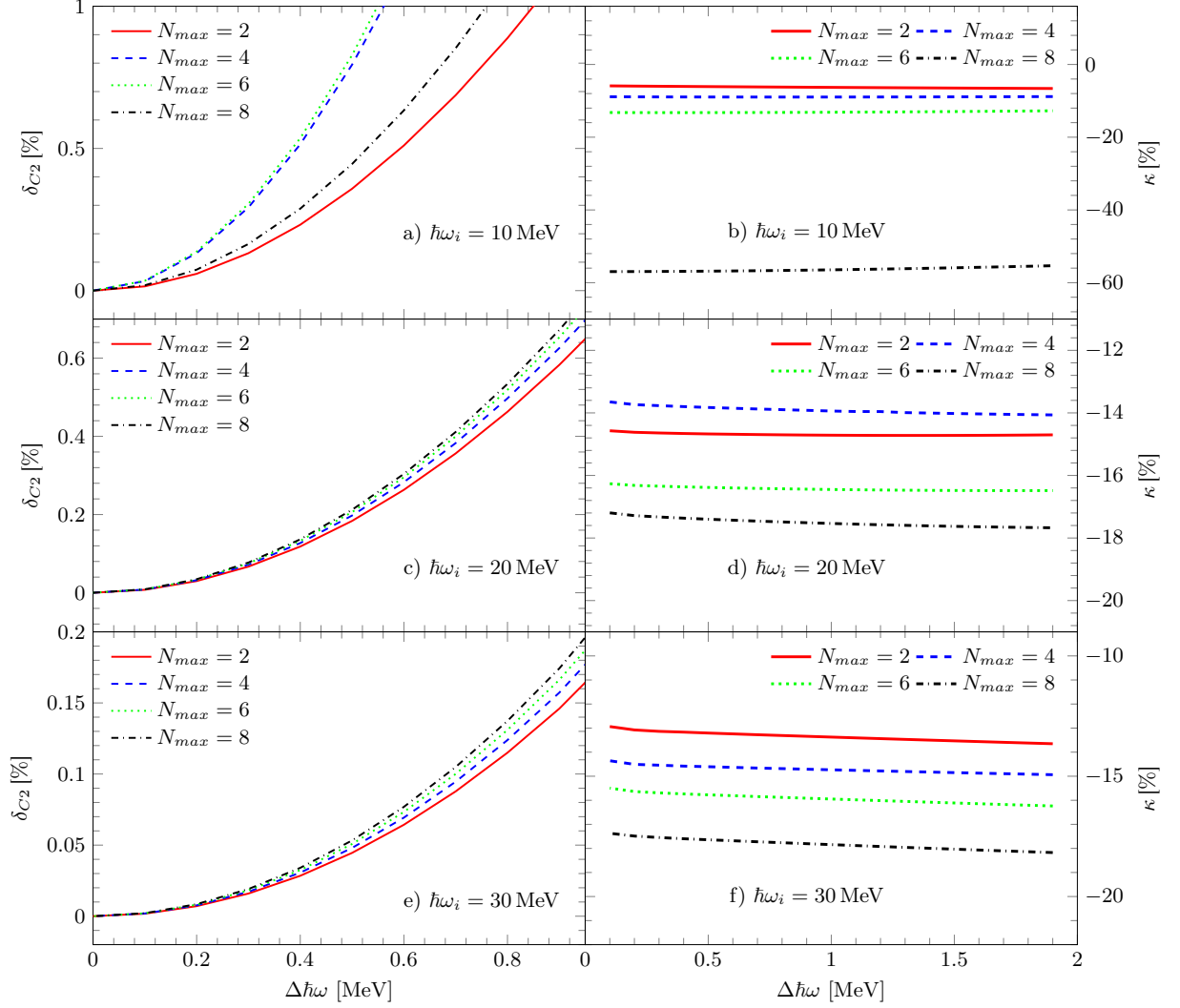


FIG. 1. (Color online) Radial mismatch correction, δ_{C2} (left panels), and ratio κ (right panels) for ${}^3\text{H}(\beta^-){}^3\text{He}$ as a function of the oscillator frequency difference, $\Delta\hbar\omega$. The oscillator frequency for the initial state, $\hbar\omega_i$, is explicitly indicated. Note that the scales differ across panels.

nuclei. Nodal mixing in the lowest energy configuration starts to appear only from *sd*-shell nuclei. Therefore, *p*- or *s*-shell nuclei, such as tritium, are free from nodal mixing with core orbits. In general, δ_{C2} decreases with increasing $\hbar\omega_i$ due to the binding energy effect⁵, while its variation with the frequency difference follows the opposite trend as described earlier. Thus so far, the key properties of δ_{C2} and its individual components, including radial excitations, are seen to be well described and are consistent with other studies. The ratio κ can already be approximately estimated, at least for its lower limit, as it is only weakly dependent on the oscillator parameters (except for small $\hbar\omega_i$, e.g., $\hbar\omega_i \leq 10$ MeV, where

the dependence is somewhat stronger). However, since full convergence is achievable for the decay of tritium, we can constrain δ_{C2} by using the values extracted as the difference between δ_C and δ_{C1} , according to Eq. (2).

Our calculation of δ_{C1} uses the conventional orthogonal harmonic oscillator basis and isospin-nonconserving interaction. This interaction is similar to the one described earlier, but without enforcing an isospin-symmetry restoration. Our calculation achieves full convergence at $N_{\text{max}} \approx 30$, where the conditions $\delta_C = \delta_{C1}$ and $\delta_{C2} = 0$ are satisfied. We obtain $\delta_C = 0.077\%$. Our results for δ_{C1} for various oscillator frequencies are displayed as a function of N_{max} in Fig. 2. The numerical values of δ_{C1} and the corresponding extracted δ_{C2} are listed in the third and fourth columns of Table I, respectively. We observe that at smaller $\hbar\omega_i$ values, δ_{C1} approaches its final (converged) value more rapidly as N_{max} increases.

⁵ As the potential depth increases, the radial mismatch becomes less sensitive to the isovector counterpart.

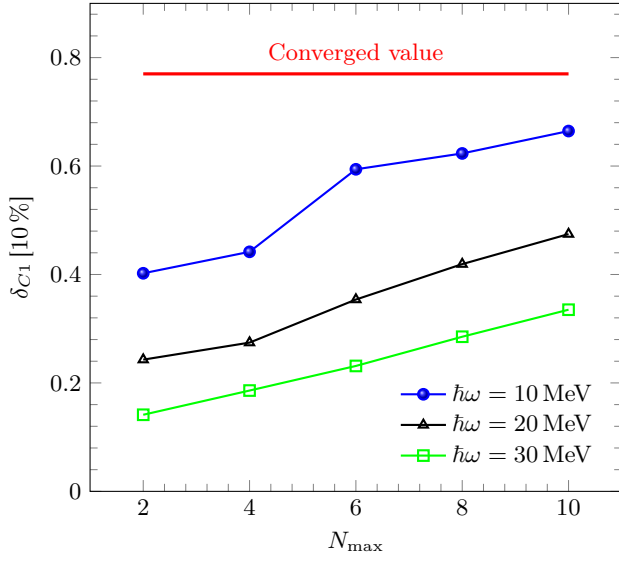


FIG. 2. (Color online) Illustration of the isospin-mixing (δ_{C1}) contribution as a function of the model space size, N_{\max} , for ${}^3\text{H}(\beta^-){}^3\text{He}$ (mirror β decay). The correction values δ_{C1} are red scaled by a factor of 10. The difference between the calculated δ_{C1} value at a finite model space and its converged value defines the radial mismatch (δ_{C2}) contributions. By definition, $\delta_{C2} = 0$ at $N_{\max} = \infty$. See text for discussion of convergence and converged value.

This trend is expected, since smaller $\hbar\omega_i$ values better simulate the weakly bound nature of light nuclei⁶. For the considered systems with 3 nucleons, the optimal absolute $\hbar\omega_i$ value is found to be around 10 to 13 MeV.

TABLE I. Results for $\Delta\hbar\omega$ and κ constrained by the extracted δ_{C2} values. The units of oscillator frequencies and their frequency differences are MeV, whereas those of δ_{C1} , δ_{C2} and κ are %. The values for δ_{C1} are obtained from exact NCSM calculations, while the other values are extracted as discussed in the main text.

$\hbar\omega_i$	N_{\max}	δ_{C1}	δ_{C2}	$\Delta\hbar\omega$	κ
10	2	0.402	0.037	0.51	-5.97
	4	0.442	0.033	0.32	-8.93
	6	0.594	0.018	0.23	-13.24
	8	0.623	0.015	0.28	-56.92
20	2	0.24	0.053	0.85	-14.70
	4	0.274	0.049	0.80	-14.00
	6	0.354	0.042	0.72	-16.42
	8	0.419	0.035	0.64	-17.45
30	2	0.141	0.064	0.60	-13.23
	4	0.186	0.059	0.55	-14.62
	6	0.231	0.055	0.52	-15.78
	8	0.285	0.049	0.48	-17.63

⁶ The Gaussian tail of harmonic oscillator functions can be well approximated with an exponential at the limit when the frequency is close to 0

By projecting the extracted δ_{C2} values onto the results illustrated in Fig. 1, we deduced the $\Delta\hbar\omega$ values for a given model space and $\hbar\omega_i$ as listed in the fifth column of Table I. Generally, different values of $\Delta\hbar\omega$ are required for different $\hbar\omega_i$ and N_{\max} to produce the correct δ_{C2} that bridges the gap between δ_{C1} and δ_C . For the considered ranges of $\hbar\omega_i$ and N_{\max} , we find the optimal $\Delta\hbar\omega$ varies between 0.23 and 0.85 MeV. At $\hbar\omega_i = 10$ MeV, the deduced κ substantially changes between the $2\hbar\omega$ and the $8\hbar\omega$ model spaces, from -5.973% to -56.921% . Whereas, at $\hbar\omega_i = 20$ and 30 MeV, it varies only from -13.234% and -17.628% . Based on these results, it is safe to conclude that, on average, the radial excitation contribution is stronger than -10% . Despite the considerable parameter and space dependence, these results suggest that δ_{C2}^{re} could be significant for the superallowed $0^+ \rightarrow 0^+$ nuclear β decay of isotriplets, especially in medium and heavy nuclei, where multiple oscillator shells are simultaneously occupied.

III. STANDARD-MODEL TESTS OF RADIAL EXCITATIONS

In order to test the potential impact of this effect on the shell model calculations of δ_C , we extract δ_{C2}^{re} by reversing the usual procedure used in the superallowed $0^+ \rightarrow 0^+$ nuclear β decay studies. Under the CKM's top-row unitarity and using the V_{us} value recommended by the Particle Data Group in 2018 [26], we deduce $|V_{ud}^{st}|^2 = 0.94969(22)$. Here the label st indicates that V_{ud}^{st} is evaluated under the Standard Model framework. In this manner, the corrected $\mathcal{F}t$ is uniquely obtained via

$$\mathcal{F}t^{st} = \frac{K}{2G_F^2(1 + \Delta_R^V)|V_{ud}^{st}|^2} = \frac{2912.95 \pm 0.54}{|V_{ud}^{st}|^2}, \quad (5)$$

where the value for Δ_R^V is taken from Ref [5]. This Standard Model constraint yields $\mathcal{F}t^{st} = 3067.26(91)$ s.

Furthermore, we decompose $\mathcal{F}t$ into two parts according to Eq. (3) as

$$\mathcal{F}t(\kappa) = \mathcal{F}t^{sm} - ft(1 + \delta'_R)\kappa\delta_{C2}^{sm}, \quad (6)$$

where $\mathcal{F}t^{sm}$ represents the conventional corrected $\mathcal{F}t$ evaluated at the limit of $\delta_{C2}^{re} = 0$. Whereas the second term in Eq. (6) is induced by δ_{C2}^{re} . For simplicity, κ is assumed to be nucleus-independent. Under these assumptions, we can effectively determine κ by adjusting it such that the χ^2/ν defined below is minimized

$$[\chi^2/\nu]^{2c} = \frac{1}{\nu} \sum_{i=1}^N \frac{[\mathcal{F}t_i(\kappa) - \mathcal{F}t^{st}]^2}{\sigma_i^2}, \quad (7)$$

where $\nu = N - 1$, standing for the number of degree of freedoms, with N being the sample size (number of transistions). As mentioned in the introduction, the uniqueness or constancy of the corrected $\mathcal{F}t$ is an implication of

CVC. Therefore, the χ^2/ν defined in Eq. (7) provides a sensitive measure of consistency with CVC. The index 2c in Eq. (7) indicates that two Standard-Model constraints are employed: CVC and CKM's top-row unitarity condition. Note that we consider only the best 15 cases selected in Ref. [5] with experimental precision of 0.1 % or better. The variant σ_i is taken as the quadratic sum of the uncertainties on $\mathcal{F}t_i^{sm}$ and $\mathcal{F}t_i^{st}$. The data for $\mathcal{F}t_i^{sm}$, $\mathcal{F}t_i$, δ'_{Ri} , and δ_{C2i}^{sm} are taken from Ref. [5].

Fig. 3 (the solid red curve) shows that the shape of the $[\chi^2/\nu]^{2c}$ curve is asymmetric with respect to $\kappa = 0$, with positive κ values producing better consistency. This is not surprising as the used set of $\mathcal{F}t_i$ values is known to underestimate the CKM's top-row unitarity condition by more than two standard deviations [5]. In this situation, one way to improve the agreement with the Standard Model is to reduce the average of $\mathcal{F}t_i$. Therefore, a negative κ as predicted by Miller and Schwenk and also indicated by our calculations in Sec. II, which results in an increase of $\mathcal{F}t_i$ [see Eq. (6)], does not align with this requirement. In this test, the Standard-Model optimized $[\chi^2/\nu]^{2c}$ corresponds to $\kappa \approx 25\%$, which is clearly opposite to the expectation from theories.

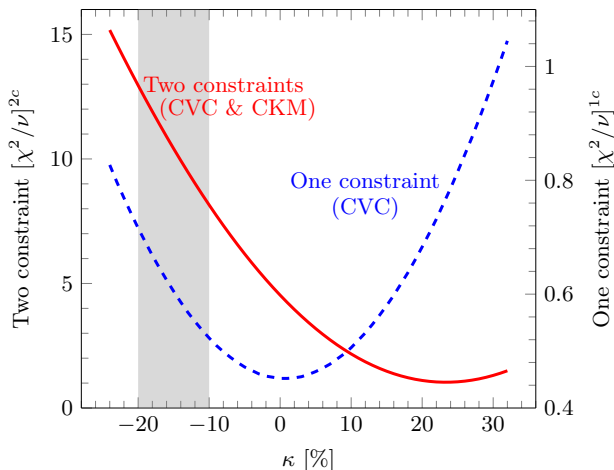


FIG. 3. (Color online) Illustration of $[\chi^2/\nu]^{1c}$ and $[\chi^2/\nu]^{2c}$ defined in Eq. (7) and (8), respectively, as function of κ . The gray band represents the typical range of κ obtained in the present NCSM calculations.

Another Standard Model test is conducted based solely on CVC, using the following formula

$$[\chi^2/\nu]^{1c} = \frac{1}{\nu} \sum_{i=1}^N \frac{[\mathcal{F}t_i(\kappa) - \overline{\mathcal{F}t}(\kappa)]^2}{\sigma_i^2}, \quad (8)$$

where the index 1c indicates that only one Standard-Model constraint is employed. Unlike the previous test, which compares the individual corrected $\mathcal{F}t_i$ values with the unique value extracted via the CKM's sum rule, Eq. (8) utilizes the sample mean (weighted average) as the reference value, evaluated according to the actual value of κ . Therefore, the goal of this test is ultimately

to find the optimal κ value by adjusting it to minimize the intrinsic scatter in the data sample, as implied by CVC. It is seen from Fig. 3 (the dashed blue curve) that the minimum value of $[\chi^2/\nu]^{1c}$ corresponds to $\kappa \approx 0$, which contradicts both the result of our first test and the theoretical expectation (as per Miller and Schwenk's criticism). Nevertheless, this CVC test is considerably less sensitive to κ compared to the first test. Specifically, $[\chi^2/\nu]^{1c}$ remains below 1 for $-30\% \leq \kappa \leq 30\%$. Additionally, we have verified that the corrected $\mathcal{F}t$ remain within the error bars despite considerable variations in κ , i.e., between -20% and 20% . This reduced sensitivity arises from allowing the reference value in the test to vary with κ , leading to a greater flexibility than using a fixed reference value. This inconsistency between these tests and our NCSM calculations may arise from inaccuracies in experimental or theoretical inputs, particularly the leading nuclear structure dependent corrections. Otherwise, it could indicate a violation of CVC or an incompleteness in the CKM sum rules.

IV. CONCLUSION

We present a comprehensive study of radial excitations and its influence on nuclear matrix elements of β decays. Using no-core shell model with an isospin-dependent harmonic oscillator basis, we show that the radial excitation contribution is negative and can be significant in first-principle calculations within a large model space, in full agreement with the prediction of Miller and Schwenk. Although it is not entirely clear whether this constitutes a significant defect of the phenomenological shell model approach, as the effect may be implicitly captured through adjustments of the model's parameters, it is explicitly excluded in the evaluation of the transition matrix elements in valence spaces. By contrast, our analysis, based on both CVC and CKM's top-row unitarity, suggests a significant positive radial excitation contribution. Conversely, this effect is disfavored under CVC when the constraint from the CKM sum rule is omitted. In addition to the relevant physics discussed, the present study also serves as a proof of principle for ab initio nuclear structure calculations using a nonorthogonal harmonic oscillator basis. More flexible scenarios, such as unequal oscillator frequencies for protons and neutrons or even state dependent frequencies, could be implemented without losing the desired properties obtained with the conventional case. Although the asymptotics of these generalized oscillator functions remain unrealistic, the relaxation could provide sufficient flexibility to capture essential effects and potentially improve computational efficiency. Extending the approach in this direction is left for future work, as it requires significant effort in transforming the interaction matrix elements from arbitrary coordinates to intrinsic system of interacting nucleons. For further investigation, the contribution of radial excitations could be addressed in a valence space by employing realistic

nuclear interactions and constructing effective transition operators.

ACKNOWLEDGMENTS

L. Xayavong would like to thank Prof. A. M. Shirokov for useful discussions on various aspects of the no-core shell model. The technical support and provision of the NuHamil code by T. Miyagi are gratefully acknowledged. L. Xayavong and Y. Lim are supported supported by the National Research Foundation of Korea(NRF) grant funded by the Korea government(MSIT)(No. RS-2024-

00457037) and by Global - Learning & Academic research institution for Master's-PhD students, and Post-docs(LAMP) Program of the National Research Foundation of Korea(NRF) grant funded by the Ministry of Education(No. RS-2024-00442483). Y. Lim is also supported by the Yonsei University Research Fund of 2024-22-0121. N. A. Smirnova acknowledges the financial support of CNRS/IN2P3, France, via ABI-CONFI Master project. The contribution of C. W. Johnson is based upon work supported by the U.S. Department of Energy, Office of Science, Office of Nuclear Physics, under Award Number DE-FG02-03ER41272.

-
- [1] D. H. Wilkinson, *Isospin in Nuclear Physics* (North-Holland, Amsterdam, 1969).
 - [2] J. B. French and M. H. MacFarlane, Nucl. Phys. **26**, 168 (1961).
 - [3] M. H. Macfarlane and J. B. French, Rev. Mod. Phys. **32**, 567 (1960).
 - [4] N. A. Smirnova, Physics **5**, 352 (2023).
 - [5] J. C. Hardy and I. S. Towner, Phys. Rev. C **102**, 045501 (2020).
 - [6] I. S. Towner and J. C. Hardy, Phys. Rev. C **77**, 025501 (2008).
 - [7] L. Xayavong and N. A. Smirnova, Phys. Rev. C **97**, 024324 (2018).
 - [8] L. Xayavong and N. A. Smirnova, Phys. Rev. C **105**, 044308 (2022).
 - [9] L. Xayavong and N. A. Smirnova, Phys. Rev. C **109**, 014317 (2024).
 - [10] N. A. Smirnova and L. Xayavong, Proceedings of the International Conference "Nuclear Theory in the Supercomputing Era-2018" (2018).
 - [11] M. Brodeur, N. Buzinsky, M. A. Caprio, V. Cirigliano, J. A. Clark, P. J. Fasano, J. A. Formaggio, A. T. Galant, A. Garcia, S. Gandolfi, S. Gardner, A. Glick-Magid, L. Hayen, H. Hergert, J. D. Holt, M. Horoi, M. Y. Huang, K. D. Launey, K. G. Leach, B. Longfellow, A. Lovato, A. E. McCoy, D. Melconian, P. Mohanmurthy, D. C. Moore, P. Mueller, E. Mereghetti, W. Mitig, P. Navratil, S. Pastore, M. Piarulli, D. Puentes, B. C. Rasco, M. Redshaw, G. H. Sargsyan, G. Savard, N. D. Scielzo, C. Y. Seng, A. Shindler, S. R. Stroberg, J. Surbrook, A. Walker-Loud, R. B. Wiringa, C. Wrede, A. R. Young, and V. Zelevinsky, Nuclear β decay as a probe for physics beyond the standard model (2023), arXiv:2301.03975 [nucl-ex].
 - [12] B. Acharya, C. Adams, A. A. Aleksandrova, K. Alfonso, P. An, S. Baekler, A. B. Balantekin, P. S. Barbeau, F. Bellini, V. Bellini, R. S. Beminiwattha, J. C. Bernauer, T. Bhattacharya, M. Bishof, A. E. Bolotnikov, P. A. Breur, M. Brodeur, J. P. Brodsky, L. J. Broussard, T. Brunner, *et al.*, Fundamental symmetries, neutrons, and neutrinos (fsmn): Whitepaper for the 2023 nsac long range plan (2023), arXiv:2304.03451 [nucl-ex].
 - [13] I. S. Towner and J. C. Hardy, Phys. Rev. C **91**, 015501 (2015).
 - [14] I. S. Towner and J. C. Hardy, Phys. Rev. C **92**, 055505 (2015).
 - [15] I. S. Towner and J. C. Hardy, Phys. Rev. C **66**, 035501 (2002).
 - [16] C.-Y. Seng, M. Gorchtein, H. H. Patel, and M. J. Ramsey-Musolf, Phys. Rev. Lett. **121**, 241804 (2018).
 - [17] C.-Y. Seng and M. Gorchtein, Phys. Rev. C **109**, 045501 (2024).
 - [18] C.-Y. Seng, V. Cirigliano, X. Feng, M. Gorchtein, L. Jin, and G. A. Miller, Phys. Lett. B **846**, 138259 (2023).
 - [19] C.-Y. Seng and M. Gorchtein, Phys. Rev. C **109**, 044302 (2024).
 - [20] C.-Y. Seng, Phys. Rev. Lett. **130**, 152501 (2023).
 - [21] C.-Y. Seng and M. Gorchtein, Phys. Lett. B **838**, 137654 (2023).
 - [22] G. F. Grinyer, C. E. Svensson, and B. A. Brown, Nucl. Instrum. Methods Phys. Res. A. **622**, 236 (2010).
 - [23] J. Damgaard, Nucl. Phys. A **130**, 233 (1969).
 - [24] L. Xayavong and Y. Lim, Phys. Rev. C **108**, 064310 (2023).
 - [25] V. Tishchenko, S. Battu, R. M. Carey, D. B. Chitwood, J. Crnkovic, P. T. Debevec, S. Dhamija, W. Earle, A. Gafarov, K. Giovanetti, T. P. Gorringer, F. E. Gray, Z. Hartwig, D. W. Hertzog, B. Johnson, P. Kammel, B. Kiburg, S. Kizilgul, J. Kunkle, B. Lauss, I. Logashenko, K. R. Lynch, R. McNabb, J. P. Miller, F. Mulhauser, C. J. G. Onderwater, Q. Peng, J. Phillips, S. Rath, B. L. Roberts, D. M. Webber, P. Winter, and B. Wolfe (MuLan Collaboration), Phys. Rev. D **87**, 052003 (2013).
 - [26] M. Tanabashi, K. Hagiwara, K. Hikasa, K. Nakamura, Y. Sumino, F. Takahashi, J. Tanaka, K. Agashe, G. Aielli, C. Amsler, M. Antonelli, D. M. Asner, H. Baer, S. Banerjee, R. M. Barnett, T. Basaglia, C. W. Bauer, J. J. Beatty, V. I. Belousov, J. Beringer, *et al.* (Particle Data Group), Phys. Rev. D **98**, 030001 (2018).
 - [27] N. Cabibbo, Phys. Rev. Lett. **10**, 531 (1963).
 - [28] M. Kobayashi and T. Maskawa, Prog. Theor. Phys. **49**, 652 (1973), <https://academic.oup.com/ptp/article-pdf/49/2/652/5257692/49-2-652.pdf>.
 - [29] C.-Y. Seng, M. Gorchtein, and M. J. Ramsey-Musolf, Phys. Rev. D **100**, 013001 (2019).
 - [30] C.-Y. Seng, M. Gorchtein, H. H. Patel, and M. J. Ramsey-Musolf, Phys. Rev. Lett. **121**, 241804 (2018).
 - [31] I. Towner, Nucl. Phys. A **540**, 478 (1992).

- [32] M. Gennari, M. Drissi, M. Gorchtein, P. Navrátil, and C.-Y. Seng, *Phys. Rev. Lett.* **134**, 012501 (2025).
- [33] C.-Y. Seng and M. Gorchtein, *Phys. Rev. C* **107**, 035503 (2023).
- [34] C.-Y. Seng, *Particles* **4**, 397 (2021).
- [35] V. Cirigliano, W. Dekens, J. de Vries, S. Gandolfi, M. Hoferichter, and E. Mereghetti, Ab initio electroweak corrections to superallowed β decays and their impact on V_{ud} (2024).
- [36] I. S. Towner and J. C. Hardy, *Rep. Prog. Phys.* **73**, 046301 (2010).
- [37] N. Auerbach and M. L. Bui, *Nucl. Phys. A* **1027**, 122521 (2022).
- [38] W. Satuła, J. Dobaczewski, W. Nazarewicz, and T. R. Werner, *Phys. Rev. C* **86**, 054316 (2012).
- [39] M. Konieczka, P. Bączyk, and W. Satuła, *Phys. Rev. C* **105**, 065505 (2022).
- [40] H. Liang, N. V. Giai, and J. Meng, *Phys. Rev. C* **79**, 064316 (2009).
- [41] W. E. Ormand and B. A. Brown, *Phys. Rev. Lett.* **62**, 866 (1989).
- [42] I. S. Towner and J. C. Hardy, *Phys. Rev. C* **82**, 065501 (2010).
- [43] A. E. Çalik, M. Gerçeklioğlu, and C. Selam, *Sci. China Phys. Mech. Astron.* **56**, 718 (2013).
- [44] A. E. Çalik, M. Gerçeklioğlu, and D. I. SALAMOV, *Pramana* **79**, 417 (2012).
- [45] H. Liang, N. V. Giai, and J. Meng, *J. Phys.: Conf. Ser.* **205**, 012028 (2010).
- [46] B. R. Barrett, P. Navrátil, and W. E. Ormand, *Czech. J. Phys.* **48**, 691 (1998).
- [47] P. Navrátil, B. R. Barrett, and W. E. Ormand, *Phys. Rev. C* **56**, 2542 (1997).
- [48] S. R. Stroberg, *Particles* **4**, 521 (2021).
- [49] G. A. Miller and A. Schwenk, *Phys. Rev. C* **78**, 035501 (2008).
- [50] G. A. Miller and A. Schwenk, *Phys. Rev. C* **80**, 064319 (2009).
- [51] L. Condren and G. A. Miller, *Phys. Rev. C* **106**, L062501 (2022).
- [52] G. A. Miller, *Universe* **9**, 10.3390/universe9050209 (2023).
- [53] Y. H. Lam, N. A. Smirnova, and E. Caurier, *Phys. Rev. C* **87**, 054304 (2013).
- [54] J. Le Bloas, L. Bonneau, P. Quentin, J. Bartel, and D. D. Strottman, *Phys. Rev. C* **86**, 034332 (2012).
- [55] H. Sagawa, Nguyen Van Giai, and T. Suzuki, *Phys. Lett. B* **353**, 7 (1995).
- [56] W. E. Ormand and B. A. Brown, *Nucl. Phys. A* **491**, 1 (1989).
- [57] K. Kaneko, Y. Sun, T. Mizusaki, S. Tazaki, and S. Ghori, *Phys. Lett. B* **773**, 521 (2017).
- [58] L. Xayavong, N. A. Smirnova, M. Bender, and K. Ben-naceur, *Act. Phys. Pol. B. Supp.* **10**, 285 (2017).
- [59] D. Wilkinson, *Phys. Lett. B* **65**, 9 (1976).
- [60] D. Wilkinson, *Nucl. Phys. A* **587**, 421 (1995).
- [61] N. Auerbach, *Phys. Rev. C* **79**, 035502 (2009).
- [62] W. E. Ormand and B. A. Brown, *Phys. Rev. C* **52**, 2455 (1995).
- [63] J. C. Hardy and I. S. Towner, *Phys. Rev. C* **79**, 055502 (2009).
- [64] E. Caurier, P. Navrátil, W. E. Ormand, and J. P. Vary, *Phys. Rev. C* **66**, 024314 (2002).
- [65] T. Berry, Z. Podolyák, R. Carroll, R. Lică, H. Grawe, N. Timofeyuk, T. Alexander, A. Andreyev, S. Ansari, M. Borge, J. Creswell, C. Fahlander, L. Fraile, H. Fynbo, W. Gelletly, R.-B. Gerst, M. Górska, A. Gredley, P. Greenlees, L. Harkness-Brennan, M. Huyse, S. Judge, D. Judson, J. Konki, J. Kurcewicz, I. Kuti, S. Lalkovski, I. Lazarus, M. Lund, M. Madurga, N. Mărginean, R. Mărginean, I. Marroquin, C. Mihai, R. Mihai, E. Nácher, S. Nae, A. Negret, C. Niță, R. Page, S. Pascu, Z. Patel, A. Perea, V. Pucknell, P. Rahkila, E. Rapisarda, P. Regan, F. Rotaru, C. Shand, E. Simpson, C. Sotly, S. Stegemann, T. Stora, O. Tengblad, A. Turturica, P. Van Duppen, V. Vedia, R. Wadsworth, P. Walker, N. Warr, F. Wearing, and H. De Witte, *Phys. Lett. B* **793**, 271 (2019).
- [66] L. Xayavong and Y. Lim, Shell-model description of the isospin-symmetry-breaking correction to gamow-teller β -decay rates and their mirror asymmetries, arXiv:2312.07900 [nucl-th].
- [67] B. R. Barrett, P. Navrátil, and J. P. Vary, *Prog. Part. Nucl. Phys.* **69**, 131 (2013).
- [68] I. Stetcu, B. R. Barrett, P. Navrátil, and J. P. Vary, *Phys. Rev. C* **71**, 044325 (2005).
- [69] V. D. Efros, *Comput. Phys. Commun.* **265**, 108005 (2021).
- [70] V. D. Efros, *Comput. Phys. Commun.* **292**, 108852 (2023).
- [71] L. Trlifaj, *Phys. Rev. C* **5**, 1534 (1972).
- [72] T. Miyagi, *Eur. Phys. J. A* **59**, 150 (2023).
- [73] D. R. Entem, R. Machleidt, and Y. Nosyk, *Phys. Rev. C* **96**, 024004 (2017).
- [74] C. W. Johnson, W. E. Ormand, K. S. McElvain, and H. Shan, Bigstick: A flexible configuration-interaction shell-model code, arXiv:1801.08432 [physics.comp-ph].
- [75] Y. Utsuno, N. Shimizu, T. Otsuka, and T. Abe, *Comput. Phys. Commun.* **184**, 102 (2013).
- [76] T. Togashi, N. Shimizu, Y. Utsuno, T. Abe, and T. Otsuka, *Procedia Comput. Sci.* **29**, 1711 (2014), 2014 International Conference on Computational Science.
- [77] J. Rodriguez-Laguna, L. M. Robledo, and J. Dukelsky, *Phys. Rev. A* **101**, 012105 (2020).
- [78] A. Scemama and E. Giner, An efficient implementation of slater-condon rules, arXiv:1311.6244 [physics.comp-ph].
- [79] D. D. Dao and F. Nowacki, *Phys. Rev. C* **105**, 054314 (2022).

Appendix: Transition matrix elements in nonorthogonal basis

In second quantization, a one-body operator such as those responsible for Fermi and Gamow-Teller transitions is written as

$$O_x = \sum_{\alpha\beta}^{N_{sp}} \langle \alpha | O_x | \beta \rangle c_\alpha^\dagger c_\beta, \quad (\text{A.1})$$

with x standing for Fermi or Gamow-Teller. N_{sp} represents the number of single-particle states available within the chosen model space. It should be noted that the basis used in Eq. (A.1) is assumed to be orthogonal such that the following conditions are fulfilled: $\{c_\alpha^\dagger, c_\beta^\dagger\} = \{c_\alpha, c_\beta\} = 0$ and $\{c_\alpha, c_\beta^\dagger\} = \delta_{\alpha\beta}$. The Greek letters α and β denote the set of spherical quantum numbers characterizing single-particle states, $nljm$.

The reduced matrix element of O_x between initial $|\Psi_i\rangle$ and final $|\Psi_f\rangle$ states is given by

$$M_x = \sum_{k_\alpha k_\beta}^{N_{sp}} \langle k_\alpha || O_x || k_\beta \rangle \langle \Psi_f || [c_{k_\alpha}^\dagger \otimes \tilde{c}_{k_\beta}]^{(\lambda_x)} || \Psi_i \rangle, \quad (\text{A.2})$$

with λ_x specifying the tensor rank of O_x which is 0 for Fermi and 1 for Gamow-Teller. Here k denotes the set of spherical quantum numbers, similar to α and β described above, excluding m . The expressions for $\langle k_\alpha || O_x || k_\beta \rangle$ is operator specific and can be found in standard nuclear structure textbooks.

The isospin-symmetry breaking correction δ_C is defined as

$$\delta_C = \left(1 - \frac{M_x^2}{\mathcal{M}_x^2}\right), \quad (\text{A.3})$$

where \mathcal{M}_x represents the transition matrix element at isospin-symmetry limit, evaluated in the same model space as M_x . For Fermi transition, the isospin-symmetry matrix element is analytically known: $\mathcal{M}_F^2 = T(T+1) - T_{zi}T_{zf}$. Contrary, \mathcal{M}_{GT} remains model-dependent.

We are interested in the scenario where the many-body states $|\Psi_i\rangle$ and $|\Psi_f\rangle$ are constructed on different harmonic oscillator bases

$$|\Psi_i\rangle = \sum_y^{N_{sd}} \xi_i^y |\Phi_i^y\rangle = \sum_y^{N_{sd}} \xi_i^y \left[\prod_\gamma^A a_\gamma^\dagger \right]_y |-\rangle, \quad (\text{A.4})$$

and

$$|\Psi_f\rangle = \sum_x^{N_{sd}} \xi_f^x |\Phi_f^x\rangle = \sum_x^{N_{sd}} \xi_f^x \left[\prod_\gamma^A b_\lambda^\dagger \right]_x |-\rangle, \quad (\text{A.5})$$

where $|-\rangle$ and N_{sd} denotes the vacuum and the number of Slater determinants spanning the many-body basis, respectively. The coefficients ξ_i^y and ξ_f^x represent the wave function amplitudes that can be obtained through the diagonalization of the Hamiltonian matrix. For simplicity, we assume that the bases of the initial and final states are intrinsically orthogonal but nucleus-dependent. These bases can be formed with harmonic oscillator eigenstates using different oscillator frequencies for the initial and final nuclei.

With this ansatz for the many-body wave functions, the one-body probability density matrix can be expressed as

$$\begin{aligned} \langle \Psi_f | c_\alpha^\dagger c_\beta | \Psi_i \rangle &= \sum_{xy}^{N_{sd}} \xi_f^{x*} \xi_i^y \langle \Phi_f^x | c_\alpha^\dagger c_\beta | \Phi_i^y \rangle \\ &= \sum_{xy}^{N_{sd}} \xi_f^{x*} \xi_i^y \langle - | b_{\lambda_A} \dots b_{\lambda_1} c_\alpha^\dagger c_\beta a_{\gamma_1}^\dagger \dots a_{\gamma_A}^\dagger | - \rangle_{xy}. \end{aligned} \quad (\text{A.6})$$

To evaluate the matrix elements in Eq. (A.6), we perform basis transformation as

$$c_\beta = \sum_\lambda^{N_{sp}} \tilde{\delta}_{\beta\lambda}^* b_\lambda, \quad (\text{A.7})$$

where the overlap matrix is given by $\tilde{\delta}_{\beta\lambda}^* = \Omega_{n_\beta n_\lambda} \delta_{l_\beta l_\lambda} \delta_{j_\beta j_\lambda} \delta_{m_\beta m_\lambda}$ with the radial component of

$$\Omega_{nn'} = \int_0^\infty R_n(r) R_{n'}(r) r^2 dr, \quad (\text{A.8})$$

it is interesting to note that for the present case, $\tilde{\delta}_{nn'}$ is invariant by exchanging n and n' . Additionally, $\Omega_{nn'}$ is slightly less than 1 when $n = n'$ and slightly greater than 0 when $n \neq n'$ given that $\Delta\hbar\omega \neq 0$.

Similarly,

$$c_\alpha^\dagger = \sum_\gamma^{N_{sp}} \tilde{\delta}_{\alpha\gamma} a_\gamma^\dagger, \quad (\text{A.9})$$

where $\tilde{\delta}_{\alpha\gamma} = \Omega_{n_\alpha n_\gamma} \delta_{l_\alpha l_\gamma} \delta_{j_\alpha j_\gamma} \delta_{m_\alpha m_\gamma}$.

Substituting Eqs. (A.7) and (A.9) into Eq. (A.6), we arrive at

$$\langle \Psi_f | c_\alpha^\dagger c_\beta | \Psi_i \rangle = \sum_{xy}^{N_{sd}} \xi_f^{x*} \xi_i^y \sum_{\lambda\gamma}^{N_{sp}} \tilde{\delta}_{\alpha\gamma} \tilde{\delta}_{\beta\lambda}^* \langle - | b_{\lambda_A} \dots b_{\lambda_1} b_\lambda^\dagger a_\gamma a_{\gamma_1}^\dagger \dots a_{\gamma_A}^\dagger | - \rangle_{xy}. \quad (\text{A.10})$$

Obviously, the choice of the common basis $\{c_\alpha^\dagger\}$ is trivial. It can be chosen to be equal to $\{a_\gamma^\dagger\}$ or $\{b_\lambda^\dagger\}$ such that one summation in Eq. (A.10) disappears.

Eq. (A.10) can be written in a compact form as

$$\langle -|b_{\lambda_A} \dots b_{\lambda_1} b_\lambda^\dagger a_\gamma a_{\gamma_1}^\dagger \dots a_{\gamma_A}^\dagger | - \rangle = \tilde{\delta}_{\lambda\gamma} \prod_k^{\lambda_k \gamma_k \neq \lambda\gamma} \tilde{\delta}_{\lambda_k \gamma_k}, \quad (\text{A.11})$$

if $a_\gamma |\Phi_i^x\rangle = 1$ and $\langle \Phi_f^y | b_\lambda^\dagger = 1$. Otherwise, it vanishes.

Finally, the reduced matrix element $\langle \Psi_f | [c_\alpha^\dagger \otimes \tilde{c}_\beta]^{(\lambda_x)} | \Psi_i \rangle$ in Eq. (A.2) can be easily obtained from Eq. (A.10) by using the Wigner–Eckart theorem. In this study, our numerical implementation of nonorthogonal basis is based on the so-called odometer method. A more efficient method would be required for heavier nuclei. More details on this computational aspect can be found in Refs. [75–79].

The matrix element on the right-hand side of