MODULI SPACES OF HOM-LIE ALGEBROID CONNECTIONS

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Abstract. We have studied irreducible Hom-Lie algebroid connections for Hom-bundle and prove that the H-gauge theoretic moduli space has a Hausdorff Hilbert manifold structure. This work generalizes some known results about simple semi-connections and Lie algebroid connections for complex vector bundles on compact complex manifold.

1. Introduction

In [5], Hartwig, Larson and Silverstov have introduced Hom-Lie algebra, while studying deformations of Witt and Virasoro algebras . A Hom-Lie algebra is a generalization of a Lie algebra. Similar to the case of Lie algebra, there is a Hom-Jacobi identity, which is also known as Jacobi identity, twisted by a linear map. There is a growing interest in Hom-Lie algebra as well as Hom-lie algebroids; some articles in this direction are [8, 1, 10].

Camille Laurent-Gengouxa and Joana Teles [8] introduced the notions of Hom-Lie algebroid, Hom-Gerstenhaber algebra, and Hom-Poisson structure. In [1], Liqiang Cai, Jiefeng Liu, and Yunhe Sheng modified the definition of Hom-Lie algebroid and gave its dual description. In section 2, we have described space of (irreducible) Hom-Lie algebroid connections $\big(\widehat{A}(E,\mathbb{L})\big)$ $A(E,\mathbb{L})$ and its affine space structure for a given Hom-bundle E and a Hom-Lie algebroid $\mathbb{L}$ on a smooth manifold X, following the definition of Cai, Liu and Sheng.

Gauge theory is useful in describing differential geometric invariants of manifolds like Seiberg-Witten and Donaldson invariants (see [3]), which are solutions of some gauge invariant equations.

In Section 3, we have described Hom-gauge group H-Gau(E) which has a Lie group structure, associated moduli space of irreducible Hom-Lie algebroid connections

$$\widehat{B}(E,\mathbb{L}) = \widehat{A}(E,\mathbb{L}) \Big/ \text{H-Gau}(E)$$

and their Sobolev completions in Section 4. Furthermore, in Section 4, we have described Hilbert manifold structure on the Sobolev space $\widehat{B}(E,\mathbb{L})_l$ and proved

$$\widehat{p} : \widehat{A}(E,\mathbb{L})_l \to \widehat{B}(E,\mathbb{L})_l$$

is a principal H-Gau$(E)^r_{l+1}$-bundle. Libor Krizka has proved similar results for Lie algebroid connections [7].

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2. Preliminaries

This section discusses the basic definitions of Hom-Lie algebra, Hom-Lie algebroid, and associated connections. Also, we verify the space of Hom-Lie algebroid connections is an affine space. We will use the notation $\mathbb{K}$ for the field $\mathbb{C}$ as well as $\mathbb{R}$. For more details about Hom-Lie algebras and Hom-Lie algebroids, see [1].

2.1. **Hom-Lie-algeba.** For a given smooth manifold X, with a smooth morphism $\phi : X \to X$, there is a canonical morphism (pullback) of the ring of smooth functions

$$\phi^\star : C^\infty(X) \to C^\infty(X)$$

that is $\phi^\star(fg) = \phi^\star(f)\phi^\star(g)$ for all $f, g \in C^\infty(X)$.

Let $E \to X$ be a smooth complex (resp. real) vector bundle over X and $\phi : X \to X$ be a smooth map. The pullback bundle of E along ϕ, which we will denoted by $\phi^! E$ and for any $s \in \Gamma(X, E)$, we use $s^! \in \Gamma(X, \phi^! E)$ to denote the corresponding pullback section, i.e. $s^!(x) = \phi^\star(s)(x) = s\big(\phi(x)\big)$ for $x \in M$.

Definition 2.1. For a smooth manifold X, an algebra morphism $D : C^\infty(X) \to C^\infty(X)$ is called (σ, τ)-derivation, if

$$D(fg) = \sigma(f)D(g) + \tau(g)D(f)$$

for $f, g \in C^\infty(X); \sigma, \tau : C^\infty(X) \to C^\infty(X)$ ring morphisms.

The collection of $(\phi^\star, \phi^\star)$-derivations for the commutative $\mathbb{K}$-algebra $C^\infty(X)$ will be denoted by $\mathrm{Der}_{\phi^\star,\phi^\star}\big(C^\infty(X)\big)$. For a given $D \in \mathrm{Der}_{\phi^\star,\phi^\star}\big(C^\infty(X)\big)$, $\phi^\star \circ D \circ (\phi^\star)^{-1} \in \mathrm{Der}_{\phi^\star,\phi^\star}\big(C^\infty(X)\big)$ because

$$\begin{aligned}
\phi^\star \circ D \circ (\phi^\star)^{-1}(fg) &= \phi^\star \circ D\big((\phi^\star)^{-1}(f)(\phi^\star)^{-1}(g)\big) \\
&= \phi^\star\Big(f\big(D \circ (\phi^\star)^{-1}\big)(g) + g\big(D \circ (\phi^\star)^{-1}\big)(f)\Big) \\
&= \phi^\star(f)\big(\phi^\star \circ D \circ (\phi^\star)^{-1}\big)(g) + \phi^\star(g)\big(\phi^\star \circ D \circ (\phi^\star)^{-1}\big)(f)
\end{aligned}$$

We have a canonical map

$$\mathrm{Ad}_{\phi^\star} : \mathrm{Der}_{\phi^\star,\phi^\star}\big(C^\infty(X)\big) \to \mathrm{Der}_{\phi^\star,\phi^\star}\big(C^\infty(X)\big) \quad \big(D \mapsto \phi^\star \circ D \circ (\phi^\star)^{-1}\big)$$

Remark 2.2. A pulled back section $D^! \in \Gamma(X, \phi^! T_X)$ acts on the ring of smooth functions given by

$$\begin{aligned}
D^!(gf) = \big(D(gf)\big)^! = \big(gD(f) + fD(g)\big)^! &= g^!\big(D(f)\big)^! + f^!\big(D(g)\big)^! \\
&= g^! D^!(f) + f^! D^!(g)
\end{aligned}$$

and can be considered as a $(\phi^\star, \phi^\star)$-derivation.

Definition 2.3. A tuple $(V, [\cdot,\cdot]_V, \psi_V)$ containing a $\mathbb{K}$-vector space V, a skew-symmetric bilinear map $[\cdot,\cdot]_V : V \times V \to V$ and a morphism $\psi_V : V \to V$ satisfying the following properties

(1) $\psi_V([x, y]_V) = [\psi_V(x), \psi_V(y)]_V$,

(2) $[\psi_V(x), [y,z]_V]_V + [\psi_V(y), [z,x]_V]_V + [\psi_V(z), [x,y]_V]_V = 0$

for all $x, y, z \in V$, is called a **Hom-Lie algebra**.

The tuple $\big(V = \mathrm{Der}_{\phi^\star,\phi^\star}\big(C^\infty(X)\big), [\cdot,\cdot]_V, \psi_V = \mathrm{Ad}_{\phi^\star}\big)$ is a Hom-Lie algebra with the Hom-Lie bracket, given by

$$[D_1, D_2]_V = \phi^\star \circ D_1 \circ (\phi^\star)^{-1} \circ D_2 \circ (\phi^\star)^{-1} - \phi^\star \circ D_2 \circ (\phi^\star)^{-1} \circ D_1 \circ (\phi^\star)^{-1}$$

for $D_i \in \mathrm{Der}_{\phi^\star,\phi^\star}\big(C^\infty(X)\big)$ $(i = 1, 2)$.

Definition 2.4. A representation of a Hom-Lie algebra $(V, [\cdot,\cdot]_V, \psi_V)$ on a vector space V' with respect to $\beta \in \mathrm{End}(V')$ is a linear map $\rho : V \to \mathrm{End}(V')$, such that for all $x, y \in V$, the following equalities are satisfied:

$$\begin{aligned} \rho\big(\psi(x)\big) \circ \beta &= \beta \circ \rho(x); \\ \rho([x,y]_V) \circ \beta &= \rho\big(\psi(x)\big) \circ \rho(y) - \rho\big(\psi(y)\big) \circ \rho(x). \end{aligned}$$

A representation of a Hom-Lie algebra $(V, [\cdot,\cdot]_V, \psi_V)$ on a vector space V' w.r.t $\beta \in \mathrm{End}(V')$ is denoted by (V', β, ρ).

2.2. **Hom-Lie algebroid.** A given vector bundle E on a smooth manifold X, is said to have a Hom-bundle structure if there is an algebra morphism

$$\phi_E : \Gamma(X, E) \to \Gamma(X, E)$$

such that $\phi_E(fs) = \phi^\star(f)\phi_E(s)$ for all $f \in C^\infty(X)$ and $s \in \Gamma(X, E)$.

A Hom-bundle is said to be an invertible Hom-bundle, if the morphism ϕ is diffeomorphism and ϕ_E is an invertible linear map. From now on, we will assume a Hom-bundle is invertible.

For given two Hom-bundle $E = (E, \phi, \phi_E)$ and $F = (F, \phi, \phi_F)$, a vector bundle morphism $\psi : E \to F$ with same base is said to respect the Hom-structure, if

$$\psi \circ \phi_E = \phi_F \circ \psi.$$

Such a vector bundle morphism is called Hom-bundle morphism.

Example. Some examples of Hom-bundles are following.

(1) For a given Hom-bundle $E = (E, \phi, \phi_E)$ with an open subset $U \subset X$, satisfying $\phi(U) \subset U$, there is a Hom-bundle $(E_U, \phi_U, \phi_{E_U})$, where

$$\phi_{E_U} : \Gamma(U, E) \to \Gamma(U, E)$$

is the restriction map $\phi_{E}{}_{|_{\Gamma(U,E)}}$. The Hom-bundle $(E_U, \phi_U, \phi_{E_U})$ is called Hom-subbundle restricted to U.

(2) For a Hom-bundle $E = (E, \phi, \phi_E)$, there is a natural bundle $\mathrm{End}(E) \cong \phi^!(E) \otimes \mathrm{Hom}\big(E, \phi^!(\underline{\mathbb{K}})\big)$, where $\underline{\mathbb{K}}$ is trivial line bundle with fiber $\mathbb{K}$. The space of global sections is

$$\Gamma\big(X, \mathrm{End}(E)\big) = \{\psi : \Gamma(X, E) \to \Gamma(X, E) \mid \psi(fs) = \phi^\star(f)\psi(s)\}$$

with a canonical morphism $\mathrm{Ad}_{\phi_E} : \Gamma\big(X, \mathrm{End}(E)\big) \to \Gamma\big(X, \mathrm{End}(E)\big)$, given by

$$\mathrm{Ad}_{\phi_E}(\psi) = \phi_E \circ \psi \circ \phi_E^{-1}$$

The tuple $(\mathrm{End}(E), \phi, \mathrm{Ad}_{\phi_E})$ is a Hom-bundle. The collection of global $\phi^\star$-linear morphisms respecting the Hom-bundle structure is given by

$$\Gamma\big(X, \mathrm{End}_{\phi_E}(E)\big) = \{\psi \in \Gamma(X, \mathrm{End}(E) \mid \psi \circ \phi_E = \phi_E \circ \psi\}$$

(3) For a smooth manifold X with smooth morphism $\phi : X \to X$, there is a smooth map $Ad_{\phi^\star} : \Gamma(X, \phi^! T_X) \to \Gamma(X, \phi^! T_X)$ given by,

$$Ad_{\phi^\star}(D) = \phi^\star \circ D \circ (\phi^\star)^{-1}$$

and the tuple $(\phi^! T_X, \phi, Ad_{\phi^\star})$ is a Hom-bundle.

(4) For a given Hom-bundle $E = (E, \phi, \phi_E)$, there is a canonical map

$$\Gamma(X, \Lambda^p E) \to \Gamma(X, \Lambda^p E) \qquad \big(\wedge_{i=1}^p D_i \mapsto \wedge_{i=1}^p \phi_E(D_i)\big) \text{ for any } p \in \mathbb{Z}^+$$

The induced map will also be denoted by the same symbol $\phi_E : \Lambda^p E \to \Lambda^p E$ and the tuple $(\Lambda^p E, \phi, \phi_E)$ is a Hom-bundle.

(5) For a Hom-bundle $E = (E, \phi, \phi_E)$, there is a canonical map $\phi_E^\dagger : \Gamma(X, \Lambda^p E^\star) \to \Gamma(X, \Lambda^p E^\star)$ given by,

$$\phi_E^\dagger(\omega)(X) = \phi^\star\Big(\omega\big(\phi_E^{-1}(X)\big)\Big) \quad \big(X \in \Gamma(X, \Lambda^p E)\big)$$

The tuple $(\Lambda^p E^\star, \phi, \phi_E^\dagger)$ is a Hom-bundle.

Definition 2.5. A **Hom-Lie algebroid** structure on a Hom-bundle $(E \to X, \phi, \phi_E)$ is a pair that consists of a Hom-Lie algebra structure $(\Gamma(X, E), [\cdot,\cdot]_E, \phi_E)$ on the space of global sections, $\Gamma(X, E)$ and a bundle map $\mathfrak{a}_E : E \to \phi^! T_X$, called the anchor map, such that the following conditions are satisfied:

(1) For all $x, y \in \Gamma(X, E)$ and $f \in C^\infty(X)$,

$$[x, fy]_E = \phi^*(f)[x, y]_E + \big[\mathfrak{a}_{E,X}\big(\phi_E(x)\big)(f)\big]\, \phi_E(y)$$

where $\mathfrak{a}_{E,X} : \Gamma(X, E) \to \Gamma(X, \phi^! T_X)$ is the induced map and $\mathfrak{a}_{E,X}\big(\phi_E(x)\big) \in \Gamma(X, \phi^! T_X)$ acts on f as in Remark 2.2;

(2) $(C^\infty(X), \phi^*, \mathfrak{a}_{E,X})$ is a representation of the Hom-Lie algebra $(\Gamma(X, E), [\cdot,\cdot]_E, \phi_E)$ on vector space $C^\infty(X)$ w.r.t $\phi^\star \in \mathrm{End}\big(C^\infty(X)\big)$.

A Hom-Lie algebroid is denoted by a quintuple $(E, \phi, \phi_E, [\cdot,\cdot]_E, \mathfrak{a}_E)$.

Example. Some examples of Hom-Lie algebroids are following.

(1) For a given manifold X, there is a canonical Hom-bundle $(\phi^! T_X, \phi, \mathrm{Ad}_{\phi^\star})$. Define a skew-symmetric bilinear map $[\cdot,\cdot]_{\phi^! T_X} : \phi^! TX \times \phi^! TX \to \phi^! TX$ given by

$$[D_1, D_2]_{\phi^! TX} = \phi^\star \circ D_1 \circ (\phi^\star)^{-1} \circ D_2 \circ (\phi^\star)^{-1} - \phi^\star \circ D_2 \circ (\phi^\star)^{-1} \circ D_1 \circ (\phi^\star)^{-1}$$

The tuple $(\phi^! TX, \phi, \mathrm{Ad}_{\phi^\star}, [\cdot,\cdot]_{\phi^! T_X}, \mathrm{id}_{\phi^! T_X})$ is a Hom-Lie algebroid.

(2) According to above example, pullback of tangent bundle T_X along a smooth morphism $\phi : X \to X$ has a Hom-Lie algebroid structure. In general, for a given Lie algebroid $(\mathcal{A}, [\cdot,\cdot]_{\mathcal{A}}, \mathfrak{a}_{\mathcal{A}})$ with additional data there is an induced Hom-Lie algebroid. Let $(\mathcal{A}, \phi, \alpha)$ be a Hom-bundle such that
   (a) $\alpha[x,y]_{\mathcal{A}} = [\alpha(x), \alpha(y)]_{\mathcal{A}}$
   (b) $\mathfrak{a}_{\mathcal{A}}\big(\alpha(x)\big) \circ \phi^\star(f) = \phi^\star \circ \mathfrak{a}_{\mathcal{A}}(x)(f)$

   then there is a canonical Hom-Lie algebroid $(A = \phi^! \mathcal{A}, \phi, \phi_A = \alpha^!, [\cdot,\cdot]_A, \mathfrak{a}_A)$, where the maps $\phi_A : \Gamma(X, A) \to \Gamma(X, A)$, $[\cdot,\cdot]_A : \Gamma(X, A) \times \Gamma(X, A) \to \Gamma(X, A)$ and $\mathfrak{a}_A : A \to \phi ! T_X$ are given by
$$\phi_A(x^!) = \alpha(x)^!, \quad [x^!, y^!]_A = [x, y]^!_{\mathcal{A}} \text{ and } \quad \mathfrak{a}_A(x^!) = \mathfrak{a}_{\mathcal{A}}(x)^!$$

From now on, a Hom-Lie algebroid $(\mathbb{L}, \phi, \phi_{\mathbb{L}}, [\cdot,\cdot]_{\mathbb{L}}, \mathfrak{a}_{\mathbb{L}})$ will be denoted by $\mathbb{L}$, a Hom-bundle (E, ϕ, ϕ_E) will be denoted by E and the induced map $\mathfrak{a}_{\mathbb{L},U}$ for any open subset $U \subset X$ will be denoted by $\mathfrak{a}_{\mathbb{L}}$.

Using the dual anchor map $\mathfrak{a}^\star_{\mathbb{L}} : \Gamma(X, \phi^! T^\star_X) \to \Gamma(X, \mathbb{L}^\star)$, define a degree-1 differential operator on the ring of smooth functions
$$d_{\mathbb{L}} = \mathfrak{a}^\star_{\mathbb{L}} \circ d : C^\infty(X) \to \Gamma(X, \mathbb{L}^\star)$$
where $d : C^\infty(X) \to \Gamma(X, \phi^! T^\star_X)$ is the canonical operator induced from the de Rham operator. The differential operator $d_{\mathbb{L}}$ can be extended to higher exterior powers of $\mathbb{L}^\star$
$$d^p_{\mathbb{L}} : \Gamma(X, \Lambda^p \mathbb{L}^\star) \to \Gamma(X, \Lambda^{p+1} \mathbb{L}^\star)$$
given by
$$\begin{aligned} d^p_{\mathbb{L}}(\omega)(\xi_0, \xi_1, \dots, \xi_p) = &\sum_{i=0}^{p} (-1)^i \mathfrak{a}_{\mathbb{L}}(\xi_i)\Big(\omega\big({\phi_{\mathbb{L}}}^{-1}(\xi_0), {\phi_{\mathbb{L}}}^{-1}(\xi_1), \dots, \widehat{\xi_i}, \dots, {\phi_{\mathbb{L}}}^{-1}(\xi_p)\big)\Big) \\ &+ \sum_{i<j, i=0}^{p} (-1)^{i+j} \phi^\dagger_{\mathbb{L}}(\omega)\big([{\phi_{\mathbb{L}}}^{-1}(\xi_i), {\phi_{\mathbb{L}}}^{-1}(\xi_j)]_{\mathbb{L}}, \xi_0, \xi_1, \dots, \xi_p\big) \qquad \big(\xi_i \in \Gamma(X, \mathbb{L})\big) \end{aligned}$$
For each $p \geq 0$ with $d^0_{\mathbb{L}} = d_{\mathbb{L}}$, it is easy to observe that $d^p_{\mathbb{L}} \circ d^{p-1}_{\mathbb{L}} = 0$ (see [1, Theorem 3.7]). The graded differential complex $(\Lambda^\bullet \mathbb{L}^\star, d^\bullet_{\mathbb{L}})$ is called **Hom-Chevalley-Eilenberg-de Rham** complex or $(\phi^\dagger_{\mathbb{L}}, \phi^\dagger_{\mathbb{L}})$-differential graded complex.

Using abuse of notation, the operator $d^\bullet_{\mathbb{L}} : \Lambda^\bullet \mathbb{L}^\star \to \Lambda^{\bullet+1} \mathbb{L}^\star$ will be denoted by $d_{\mathbb{L}}$, is called **Hom-Chevalley-Eilenberg-de Rham operator**, which is a generalization of the de Rham as well as the Chevalley-Eilenberg-de Rham operator.

For $\xi, \xi_i \in \Gamma(X, \mathbb{L})$ $(i \in I)$, the Hom-Lie derivative operator $\mathcal{L}^{\mathbb{L}}_\xi : \Lambda^p \mathbb{L}^\star \to \Lambda^p \mathbb{L}^\star$ is given by
$$\begin{aligned} (\mathcal{L}^{\mathbb{L}}_\xi \omega)(\xi_1, \xi_2, \dots, \xi_p) = &\,\mathfrak{a}_{\mathbb{L}}\big(\phi_{\mathbb{L}}(\xi)\big)\Big(\omega\big({\phi_{\mathbb{L}}}^{-1}(\xi_1), {\phi_{\mathbb{L}}}^{-1}(\xi_2), \dots, {\phi_{\mathbb{L}}}^{-1}(\xi_p)\big)\Big) \\ &- \sum_{i=1}^{p} (\phi^\dagger_{\mathbb{L}} \omega)(\xi_1, \dots, [\xi, {\phi_{\mathbb{L}}}^{-1}(\xi_i)]_{\mathbb{L}}, \dots, \xi_p) \end{aligned}$$

and the Hom-Insertion operator $i^{\mathbb{L}}_{\xi} : \Gamma(X, \Lambda^p \mathbb{L}^\star) \to \Gamma(X, \Lambda^{p-1}\mathbb{L}^\star)$ is given by

$$(i^{\mathbb{L}}_{\xi}\omega)(\xi_1, \xi_2, \ldots, \xi_{p-1}) = \phi^{\dagger}_{\mathbb{L}}\omega\big(\phi_{\mathbb{L}}(\xi), \xi_1, \xi_2, \ldots, \xi_p\big)$$

Remark 2.6. For a smooth Hom-Lie algebroid $\mathbb{L}$ and a smooth Hom-bundle E on a smooth manifold X, the space of global sections of the bundle $\Lambda^k\mathbb{L}^\star$ (resp. $\Lambda^k\mathbb{L}^\star \otimes E$) will be denoted by $\mathcal{A}^k_{\mathbb{L}}(X)$ (resp. $\mathcal{A}^k_{\mathbb{L}}(X, E))(0 \leq k \in \mathbb{Z})$, whose elements are called smooth $\mathbb{L}$-forms (resp. $\mathbb{L}$-forms taking values in E) of type k.

Remark 2.7. For a given Hom-bundle (E, ϕ, ϕ_E) and a Hom-Lie algebroid $\mathbb{L}$, the bundle $\Lambda^k\mathbb{L}^\star \otimes \mathrm{End}(E)$ has induced Hom structure with the algebra morphism

$$\phi^{\dagger}_{\mathbb{L}} \otimes \mathrm{Ad}_{\phi_E} : \Gamma\big(X, \Lambda^k\mathbb{L}^\star \otimes \mathrm{End}(E)\big) \to \Gamma\big(X, \Lambda^k\mathbb{L}^\star \otimes \mathrm{End}(E)\big) \qquad \text{given by,}$$
$$\phi^{\dagger}_{\mathbb{L}} \otimes \mathrm{Ad}_{\phi_E}(D) = (\phi^{\dagger}_{\mathbb{L}} \otimes \phi_E) \circ D \circ \big(\phi_E^{-1}\big)$$

The global sections of bundle $\Lambda^k\mathbb{L}^\star \otimes \mathrm{End}(E)$ respecting the Hom-structure is given by

$$\mathcal{A}^k_{\mathbb{L}}\big(X, \mathrm{End}_{\phi_E}(E)\big) = \{D \in \mathcal{A}^k_{\mathbb{L}}\big(X, \mathrm{End}(E)\big) \mid \phi^{\dagger}_{\mathbb{L}} \otimes \phi_E \circ D = D \circ \phi_E\}$$

2.3. **Hom-Lie algebroid connection.**

Definition 2.8. Let $\mathbb{L}, E$ be a Hom-Lie algebroid and a Hom-bundle, respectively over a smooth manifold X. A $\mathbb{K}$-linear map $\nabla : \mathcal{A}^0_{\mathbb{L}}(X, E) \to \mathcal{A}^1_{\mathbb{L}}(X, E)$ such that for $f \in C^\infty(X); s \in \Gamma(X, E)$

1. $\nabla(fs) = d_{\mathbb{L}}(f)\phi_E(s) + \phi^\star(f)\nabla(s)$
2. $\phi^{\dagger}_{\mathbb{L}} \otimes \phi_E \circ \nabla = \nabla \circ \phi_E$

is called a Hom-Lie algebroid connection or, an $\mathbb{L}$-connection.

A Hom-Lie algebroid connection is a generalization of a Lie algebroid as well as an affine connection. For a given $\xi \in \Gamma(X, \mathbb{L})$, we have a linear morphism $\nabla_\xi \in \mathcal{A}^0_{\mathbb{L}}\big(X, \mathrm{End}_{\phi_E}(E)\big)$ such that

$$\nabla_\xi(fs) = \mathcal{L}^{\mathbb{L}}_\xi(f)\phi_E(s) + \nabla_\xi(s)\phi^\star(f)$$
$$\nabla_{f\xi}(s) = \phi^\star(f)\nabla_\xi(s)$$

for $f \in \mathcal{C}^\infty(X); s \in \Gamma(X, E)$.

The element $\nabla_\xi(fs)$ is called co-variant $\mathbb{L}$-derivative and the morphism ∇_ξ is called co-variant $\mathbb{L}$-differential operator in the direction of $\xi \in \Gamma(X, \mathbb{L})$.

Theorem 2.9. *For a given $\mathbb{L}$-connection ∇, there is a degree 1 differential operator $d^\nabla : \mathcal{A}^\bullet_{\mathbb{L}}(X, E) \to \mathcal{A}^\bullet_{\mathbb{L}}(X, E)$, uniquely determined such that*

(1) for $\alpha \in \mathcal{A}^l_{\mathbb{L}}(X), \beta \in \mathcal{A}^\bullet_{\mathbb{L}}(X, E)$

$$d^\nabla(\alpha \wedge \beta) = d_{\mathbb{L}}(\alpha) \wedge (\phi^{\dagger}_{\mathbb{L}} \otimes \phi_E)(\beta) + (-1)^l \phi^{\dagger}_{\mathbb{L}}(\alpha) \wedge d^\nabla(\beta) \tag{2.1}$$

$$\phi^{\dagger}_{\mathbb{L}} \otimes \phi_E \circ d^\nabla = d^\nabla \circ \phi^{\dagger}_{\mathbb{L}} \otimes \phi_E \tag{2.2}$$

(2) for $s \in \mathcal{A}^0_{\mathbb{L}}(X,E), \xi \in \Gamma(X,\mathbb{L})$

$$d^{\nabla}_{|\mathcal{A}^0_{\mathbb{L}}(X,E)} \equiv \nabla \text{ and } \big(d^{\nabla}(s)\big)(\xi) = \nabla_{\phi_{\mathbb{L}}^{-1}(\xi)}(s)$$

The operator d^{∇} *is given by*

$$d^{\nabla}(\omega)(\xi_0,\xi_1,\dots,\xi_p) = \sum_{i=1}^{p}(-1)^i\nabla_{\phi_{\mathbb{L}}^{-1}(\xi_i)}\Big(\omega\big(\phi_{\mathbb{L}}^{-1}(\xi_0),\phi_{\mathbb{L}}^{-1}(\xi_1),\dots,\widehat{\xi_i},\dots,\phi_{\mathbb{L}}^{-1}(\xi_p)\big)\Big)+ \sum_{i<j=0}^{k}(-1)^{i+j}\phi_{\mathbb{L}}^{\dagger}\otimes\phi_E(\omega)\big([\phi_{\mathbb{L}}^{-1}(\xi_i),\phi_{\mathbb{L}}^{-1}(\xi_j)]_{\mathbb{L}},\xi_0,\xi_1,\dots,\widehat{\xi_i},\dots,\widehat{\xi_j},\dots,\xi_p\big) \tag{2.3}$$

for $\xi_i (0 \le i \le p) \in \Gamma(X,\mathbb{L})$.

Proof. The uniqueness of the operator d^{∇} satisfying properties (1) and (2) follows in the same line of arguements as in the case of affine connections only thing we need to prove is, the operator d^{∇} described in the Equation 2.3 satisfies the properties (1) and (2). We have verified the property (1), verification of (2) is easy.

$$\begin{aligned}
d^{\nabla}(f\beta)(\xi_0,\xi_1,\dots,\xi_p) &= \sum_{i=0}^{p}(-1)^i\nabla_{\phi_{\mathbb{L}}^{-1}(\xi_i)}\Big(f\beta\big(\phi_{\mathbb{L}}^{-1}(\xi_0),\dots,\widehat{\xi_i},\dots,\phi_{\mathbb{L}}^{-1}(\xi_p)\big)\Big)+\\
&\quad \sum_{i<j}(-1)^{i+j}\phi_{\mathbb{L}}^{\dagger}\otimes\phi_E(f\beta)\big([\phi_{\mathbb{L}}^{-1}(\xi_i),\phi_{\mathbb{L}}^{-1}(\xi_j)]_{\mathbb{L}},\xi_0,\dots,\widehat{\xi_i},\dots,\widehat{\xi_j},\dots,\xi_p\big)\\
&= \sum_{i=0}^{p}(-1)^i\mathcal{L}^{\mathbb{L}}_{\phi_{\mathbb{L}}^{-1}(\xi_i)}(f)\Big(\phi_{\mathbb{L}}^{\dagger}\otimes\phi_E(\beta)\big(\phi_{\mathbb{L}}^{-1}(\xi_0),\dots,\widehat{\xi_i},\dots,\phi_{\mathbb{L}}^{-1}(\xi_p)\big)\Big)+\\
&\quad \sum_{i=0}^{p}(-1)^i\phi^{\star}(f)\nabla_{\phi_{\mathbb{L}}^{-1}(\xi_i)}\big(\beta(\phi_{\mathbb{L}}^{-1}(\xi_0),\dots,\widehat{\xi_i},\dots,\phi_{\mathbb{L}}^{-1}(\xi_p)\big)+\\
&\quad\quad \sum_{i<j}(-1)^{i+j}\phi_{\mathbb{L}}^{\dagger}\otimes\phi_E(f\beta)\big([\phi_{\mathbb{L}}^{-1}(\xi_i),\phi_{\mathbb{L}}^{-1}(\xi_j)]_{\mathbb{L}},\xi_0,\dots,\widehat{\xi_i},\dots,\widehat{\xi_j},\dots,\xi_p\big)\\
&= \big(d_{\mathbb{L}}(f)\wedge\phi_{\mathbb{L}}^{\dagger}\otimes\phi_E(\beta)+\phi^{\star}(f)d^{\nabla}(\beta)\big)(\xi_0,\xi_1,\dots,\xi_p)
\end{aligned}$$

Assume for $k \le n$, the Equation (2.1) holds, we need to prove for $k = n+1$. Let $\alpha \in \Gamma(X,\wedge^n\mathbb{L}^{\star}), \beta \in \Gamma(X,\Lambda^n\mathbb{L}^{\star}\otimes E)$ and $\xi \in \Gamma(X,\mathbb{L}^{\star})$,

$$\begin{aligned}
d^{\nabla}(\alpha\wedge\xi\wedge\beta) &= d_{\mathbb{L}}(\alpha)\wedge\phi_{\mathbb{L}}^{\dagger}\otimes\phi_E(\xi\wedge\beta)+(-1)^n\phi_{\mathbb{L}}^{\dagger}(\alpha)\wedge d^{\nabla}(\xi\wedge\beta)\\
&= \quad d_{\mathbb{L}}(\alpha)\wedge\Big(\phi_{\mathbb{L}}^{\dagger}(\xi)\wedge\big(\phi_{\mathbb{L}}^{\dagger}\otimes\phi_E(\beta)\big)\Big)+\\
&\quad (-1)^n\phi_{\mathbb{L}}^{\dagger}(\alpha)\wedge\Big[d_{\mathbb{L}}(\xi)\wedge\phi_{\mathbb{L}}^{\dagger}\otimes\phi_E(\beta)-\phi_{\mathbb{L}}^{\dagger}(\xi)\wedge d^{\nabla}(\beta)\Big]\\
&= \quad \big(d_{\mathbb{L}}(\alpha)\wedge\phi_{\mathbb{L}}^{\dagger}(\xi)+(-1)^n\phi_{\mathbb{L}}^{\dagger}(\alpha)\wedge d_{\mathbb{L}}(\xi)\big)\wedge\big(\phi_{\mathbb{L}}^{\dagger}\otimes\phi_E(\beta)\big)+\\
&\quad (-1)^{n+1}\phi_{\mathbb{L}}^{\dagger}(\alpha\wedge\xi)\wedge d^{\nabla}(\beta)\\
&= d_{\mathbb{L}}(\alpha\wedge\xi)\wedge\phi_{\mathbb{L}}^{\dagger}\otimes\phi_E(\beta)+(-1)^{n+1}\phi_{\mathbb{L}}^{\dagger}(\alpha\wedge\xi)d^{\nabla}(\beta)
\end{aligned}$$

Hence, for $n = k+1$, d^∇ described in the Equation (2.3) satifies property (1). Now we will verify the Equation (2.2).

$$\begin{aligned}
d^\nabla \circ \phi_{\mathbb{L}}^\dagger \otimes \phi_E(\omega)(\xi_0, \dots, \xi_p) &= \sum_{i=0}^{p} (-1)^i \nabla_{\phi_{\mathbb{L}}^{-1}(\xi_i)} \Big[\big(\phi_{\mathbb{L}}^\dagger \otimes \phi_E(\omega)\big)\big(\phi_{\mathbb{L}}^{-1}(\xi_0), \dots, \widehat{\xi_i}, \dots \phi_{\mathbb{L}}^{-1}(\xi_p)\big)\Big] + \\
&\sum_{i<j=0}^{k} (-1)^{i+j} \big((\phi_{\mathbb{L}}^\dagger \otimes \phi_E)^2(\omega)\big)\big([\phi_{\mathbb{L}}^{-1}(\xi_i), \phi_{\mathbb{L}}^{-1}(\xi_j)]_{\mathbb{L}}, \xi_0, \dots, \widehat{\xi_i}, \dots, \widehat{\xi_j}, \dots, \xi_p\big) \\
&= \sum_{i=0}^{p} (-1)^i \Big(\nabla \Big[\big(\phi_{\mathbb{L}}^\dagger \otimes \phi_E(\omega)\big)\big(\phi_{\mathbb{L}}^{-1}(\xi_0), \dots, \widehat{\xi_i}, \dots \phi_{\mathbb{L}}^{-1}(\xi_p)\big)\Big]\Big)(\xi_i) + \\
&\sum_{i<j=0}^{k} (-1)^{i+j} \big((\phi_{\mathbb{L}}^\dagger \otimes \phi_E)^2(\omega)\big)\big([\phi_{\mathbb{L}}^{-1}(\xi_i), \phi_{\mathbb{L}}^{-1}(\xi_j)]_{\mathbb{L}}, \xi_0, \dots, \widehat{\xi_i}, \dots, \widehat{\xi_j}, \dots, \xi_p\big) \\
&= \sum_{i=0}^{p} (-1)^i \phi_E \circ \Big(\nabla \Big[\omega\big(\phi_{\mathbb{L}}^{-2}(\xi_0), \dots, \widehat{\xi_i}, \dots \phi_{\mathbb{L}}^{-2}(\xi_p)\big)\Big] \big(\phi_{\mathbb{L}}^{-1}(\xi_i)\big)\Big) + \\
&\sum_{i<j=0}^{k} (-1)^{i+j} \phi_E \circ \Big(\big((\phi_{\mathbb{L}}^\dagger \otimes \phi_E)(\omega)\big)\big([\phi_{\mathbb{L}}^{-2}(\xi_i), \phi_{\mathbb{L}}^{-2}(\xi_j)]_{\mathbb{L}}, \phi_{\mathbb{L}}^{-1}(\xi_0), \dots, \widehat{\xi_i}, \dots \\
&\qquad , \widehat{\xi_j}, \dots, \phi_{\mathbb{L}}^{-1}(\xi_p)\big)\Big) \\
&= \phi_E \circ \Big[d^\nabla(\omega)\big(\phi_{\mathbb{L}}^{-1}(\xi_0), \dots, \phi_{\mathbb{L}}^{-1}(\xi_p)\big)\Big] \\
&= \Big[\phi_{\mathbb{L}}^\dagger \otimes \phi_E \circ d^\nabla(\omega)\Big](\xi_0, \dots, \xi_p)
\end{aligned}$$

□

Lemma 2.10. *The space of $\mathbb{L}$-connections $A(E, \mathbb{L})$, is an affine space modeled on the vector space $\mathcal{A}^1_{\mathbb{L}}\big(X, End_{\phi_E}(E)\big)$*

Proof. The proof follows in two steps. First we prove, the space of $\mathbb{L}$-connections $A(E, \mathbb{L})$ is non-empty while in second step, we prove it is an affine space modelled on $\mathbb{K}$-vector space $\mathcal{A}^1_{\mathbb{L}}\big(X, \mathrm{End}_{\phi_E}(E)\big)$.

(1) Let $\{(U_\alpha, \psi_\alpha)\}_{\alpha \in I}$ be a smooth trivialization for the Hom-bundle E with $\phi(U_\alpha) \subset U_\alpha$, take $(g_\alpha)_{\alpha \in \mathcal{A}}$ be a smooth partition of unity sub-ordinate to covering $(U_\alpha)_{\alpha \in I}$. Let $(\underline{V} = X \times V, \phi, \phi_{\underline{V}})$ be a trivial Hom-bundle, where $\phi_{\underline{V}}|_{\Gamma(U_\alpha, \underline{V})} : \Gamma(U_\alpha, \underline{V}) \to \Gamma(U_\alpha, \underline{V})$ is given by

$$\phi_{\underline{V}}|_{\Gamma(U_\alpha, \underline{V})} = \psi_\alpha \circ \phi_E|_{\Gamma(U_\alpha, E)} \circ \psi_\alpha^{-1} \tag{2.4}$$

Let $\widehat{\nabla}$ be the trivial $\mathbb{L}$-connection on trivial Hom-bundle $(\underline{V}, \phi, \phi_{\underline{V}})$ that is

$$\widehat{\nabla}(f \otimes v) = d_{\mathbb{L}}(f) \otimes \phi_{\underline{V}}(v) \quad \big(f \otimes v \in \Gamma(X, \underline{V})\big)$$

For $s \in \Gamma(X, E)$, $\operatorname{supp}(g_\alpha s) \subset U_\alpha$ implies $\operatorname{supp}\Big(\psi_\alpha\big((g_\alpha s)_{|_{U_\alpha}}\big)\Big) \subset U_\alpha$, extending $\psi_\alpha\big((g_\alpha s)_{|_{U_\alpha}}\big)$ to X and applying the trivial connection $\widehat{\nabla}$, we have

$$\operatorname{supp}\bigg(\widehat{\nabla}\Big(\psi_\alpha\big((g_\alpha s)_{|_{U_\alpha}}\big)\Big)\bigg) \subset U_\alpha.$$

Furthermore, $\operatorname{supp}\left(\mathrm{id}_{\mathbb{L}^\star} \otimes \psi_\alpha^{-1} \circ \bigg(\widehat{\nabla}\Big(\psi_\alpha\big((g_\alpha s)_{|_{U_\alpha}}\big)\Big)\bigg)_{|_{U_\alpha}}\right) \subset U_\alpha$, which can be extended to a global section, $\mathrm{id}_{\mathbb{L}^\star} \otimes \psi_\alpha^{-1} \circ \bigg(\widehat{\nabla}\Big(\psi_\alpha\big((g_\alpha s)_{|_{U_\alpha}}\big)\Big)\bigg)_{|_{U_\alpha}} \in \Gamma(X, \mathbb{L}^\star \otimes E)$. Define, a natural map $\nabla : \Gamma(X, E) \to \Gamma(X, \mathbb{L}^\star \otimes E)$ given by

$$\nabla(s) = \sum_{\alpha \in I} \mathrm{id}_{\mathbb{L}^\star} \otimes \psi_\alpha^{-1} \circ \bigg(\widehat{\nabla}\Big(\psi_\alpha\big((g_\alpha s)_{|_{U_\alpha}}\big)\Big)\bigg)_{|_{U_\alpha}}$$

The map ∇ is an $\mathbb{L}$-connection because for $f \in C^\infty(X), s \in \Gamma(X, E)$, we have

$$\begin{aligned}
\nabla(fs) &= \sum_{\alpha \in I} \mathrm{id}_{\mathbb{L}^\star} \otimes \psi_\alpha^{-1} \circ \bigg(\widehat{\nabla}\Big(\psi_\alpha\big((g_\alpha f s)_{|_{U_\alpha}}\big)\Big)\bigg)_{|_{U_\alpha}} \\
&= \sum_{\alpha \in I} \mathrm{id}_{\mathbb{L}^\star} \otimes \psi_\alpha^{-1} \circ \bigg(\widehat{\nabla}\Big(f\psi_\alpha\big((g_\alpha s)_{|_{U_\alpha}}\big)\Big)\bigg)_{|_{U_\alpha}} \\
&= \sum_{\alpha \in I} \mathrm{id}_{\mathbb{L}^\star} \otimes \psi_\alpha^{-1} \circ \bigg(d_{\mathbb{L}}(f) \otimes \phi_{\underline{V}}\Big(\psi_\alpha\big((g_\alpha s)_{|_{U_\alpha}}\big)\Big) + \phi^\star(f)\widehat{\nabla}\Big(\psi_\alpha\big((g_\alpha s)_{|_{U_\alpha}}\big)\Big)\bigg)_{|_{U_\alpha}} \\
&= \sum_{\alpha \in I} \left[d_{\mathbb{L}}(f) \otimes \phi_{E_{|\Gamma(U_\alpha, E)}}\big((g_\alpha s)_{|_{U_\alpha}}\big) + \phi^\star(f)\left(\mathrm{id}_{\mathbb{L}^\star} \otimes \psi_\alpha^{-1} \circ \bigg(\widehat{\nabla}\Big(\psi_\alpha\big((g_\alpha s)_{|_{U_\alpha}}\big)\Big)\bigg)_{|_{U_\alpha}}\right)\right] \\
&= d_{\mathbb{L}}(f) \otimes \phi_E(s) + \phi^\star(f)\nabla(s)
\end{aligned}$$

Because trivial $\mathbb{L}$-connection $\widehat{\nabla}$ satisfies $\phi_{\mathbb{L}}^\dagger \otimes \phi_V \circ \widehat{\nabla} = \widehat{\nabla} \circ \phi_V$ and using the Equation (2.4), we have $\phi_{\mathbb{L}}^\dagger \otimes \phi_E \circ \nabla = \nabla \circ \phi_E$. Hence, the space of $\mathbb{L}$-connections, $A(E, \mathbb{L})$ is non-empty.

(2) Let $\nabla, \nabla' \in A(E, \mathbb{L})$, then

$$\begin{aligned}
(\nabla - \nabla')(fs) &= \big(d_{\mathbb{L}}(f)\phi_E(s) + \phi^\star(f)\nabla(s)\big) - \big(d_{\mathbb{L}}(f)\phi_E(s) + \phi^\star(f)\nabla'(s)\big) \\
&= \phi^\star(f)(\nabla - \nabla')(s)
\end{aligned}$$

It proves, the space $A(E, \mathbb{L})$ is an affine space modelled on the vector space $\mathcal{A}^1_{\mathbb{L}}\big(X, \mathrm{End}_{\phi_E}(E)\big)$. □

Remark 2.11. Using Lemma 2.10, any Hom-Lie algebroid connection $\nabla \in A(E, \mathbb{L})$ can be expressed as $\nabla = \nabla_0 + \alpha$ for some $\alpha \in \mathcal{A}^1_{\mathbb{L}}\big(X, \mathrm{End}_{\phi_E}(E)\big)$, where $\nabla_0 : \mathcal{A}^0_{\mathbb{L}}(X, E) \to \mathcal{A}^1_{\mathbb{L}}(X, E)$ is the trivial $\mathbb{L}$-connection. The space of $\mathbb{L}$-connections is given by

$$A(E, \mathbb{L})(X) = \big\{\nabla_0 + \alpha \mid \alpha \in \mathcal{A}^1_{\mathbb{L}}\big(X, \mathrm{End}_{\phi_E}(E)\big)\big\} \tag{2.5}$$

Remark 2.12. Any $\mathbb{L}$-connection ∇^E on a Hom-bundle E induces an $\mathbb{L}$-connection $\nabla^{\mathrm{End}(E)}$ for the bundle $\mathrm{End}(E)$, given by

$$\begin{aligned}(\nabla^{\mathrm{End}(E)}T)(s) &= \phi_{\mathbb{L}}^{\dagger} \otimes \phi_E \circ \nabla^E \circ \phi_E^{-1} \circ T \circ \phi_E^{-1}(s) - \phi_{\mathbb{L}}^{\dagger} \otimes \phi_E \circ \mathrm{id}_{\mathbb{L}^\star} \otimes (T \circ \phi_E^{-1}) \circ \nabla^E \circ \phi_E^{-1}(s)\\ &= [\nabla^E, T]_{\mathbb{L}}(s)\end{aligned}$$

for $T \in \mathcal{A}^0_{\mathbb{L}}\big(X, \mathrm{End}(E)\big); s \in \Gamma(X, E)$.

Remark 2.13. For $T_i \in \mathcal{A}^0_{\mathbb{L}}\big(X, \mathrm{End}_{\phi_E}(E)\big)$ $(i = 1, 2)$ and a Hom-Lie algebroid connection ∇^E, we have

$$\begin{aligned}\big(\nabla^{\mathrm{End}(E)}(T_1 \circ \phi_E^{-1} \circ T_2)\big) &= [\nabla^E, T_1 \circ \phi_E^{-1} \circ T_2]_{\mathbb{L}}\\ &= \phi_{\mathbb{L}}^{\dagger} \otimes \phi_E \circ [\nabla^E, T_1 \circ \phi_E^{-1} \circ T_2] \circ \phi_E^{-1}\\ &= \phi_{\mathbb{L}}^{\dagger} \otimes \phi_E \circ \big([\nabla^E, T_1] \circ \phi_E^{-1} \circ T_2 + \mathrm{id}_{\mathbb{L}^\star} \otimes (T_1 \circ \phi_E^{-1}) \circ [\nabla^E, T_2]\big) \circ \phi_E^{-1}\\ &= [\nabla^E, T_1]_{\mathbb{L}} \circ \phi_E^{-1} \circ T_2 + \mathrm{id}_{\mathbb{L}^\star} \otimes (T_1 \circ \phi_E^{-1}) \circ [\nabla^E, T_2]_{\mathbb{L}}\end{aligned}$$

The elements of the vector space $\mathcal{A}^\bullet_{\mathbb{L}}\big(X, \mathrm{End}(E)\big)$ has a canonical associative algebra structure with wedge product as multiplication operation given by

$$\omega \wedge \tau(\xi_1, \xi_2, \dots, \xi_{p+q}) = \frac{1}{p!q!} \sum_{\sigma \in S(p+q)} Sign(\sigma) \omega(\xi_{\sigma(1)}, \dots, \xi_{\sigma(p)}) \circ \phi_E^{-1} \circ \tau(\xi_{\sigma(p+1}, \dots, \xi_{\sigma(p+q)}) \tag{2.6}$$

for $\omega \in \mathcal{A}^p_{\mathbb{L}}\big(X, \mathrm{End}(E)\big)$ and $\tau \in \mathcal{A}^q_{\mathbb{L}}\big(X, \mathrm{End}(E)\big)$, $\omega \wedge \tau \in \mathcal{A}^{p+q}_{\mathbb{L}}\big(X, \mathrm{End}(E)\big)$.

The space $\mathcal{A}^\bullet_{\mathbb{L}}\big(X, \mathrm{End}(E)\big)$ also has, Hom-Lie algebra structure with the Hom-Lie bracket $[\cdot, \cdot]_{\mathbb{L}}$ can be described as follows.

For $\omega, \tau \in \mathcal{A}^0_{\mathbb{L}}\big(X, \mathrm{End}(E)\big)$

$$[\omega, \tau]_{\mathbb{L}} = \phi_E \circ \omega \circ \phi_E^{-1} \circ \tau \circ \phi_E^{-1} - \phi_E \circ \tau \circ \phi_E^{-1} \circ \omega \circ \phi_E^{-1} \tag{2.7}$$

and for $\Omega \in \mathcal{A}^p_{\mathbb{L}}\big(X, \mathrm{End}(E)\big), \mathrm{T} \in \mathcal{A}^q_{\mathbb{L}}\big(X, \mathrm{End}(E)\big)$ $(p, q > 0)$

$$\begin{aligned}[\Omega, \mathrm{T}]_{\mathbb{L}}(\xi_1, \dots, \xi_{p+q}) = \frac{1}{p!q!} \sum_{\sigma \in S(p+q)} Sign(\sigma) \Big[&\Omega\big({\phi_E}^{-1}(\xi_{\sigma(1)}), \dots, {\phi_E}^{-1}(\xi_{\sigma(p)})\big),\\ &\mathrm{T}\big({\phi_E}^{-1}(\xi_{\sigma(p+1)}), \dots, {\phi_E}^{-1}(\xi_{\sigma_{(p+q)}})\big)\Big]_{\mathbb{L}}\end{aligned} \tag{2.8}$$

for $\xi_1, \dots, \xi_{p+q} \in \Gamma(X, \mathbb{L})$.

3. H-Gauge group action

In this section, we have described Hom-gauge group (H-gauge group) for a given Hom-bundle along with the moduli space of irreducible $\mathbb{L}$-connections as H-gauge equivalence classes, which we have further studied in Section 4.

For a given Hom-bundle E, the collection of invertible $\phi^\star$-linear global morphisms has a group structure with the group operation $\psi \bullet \psi_2 = \psi_1 \circ \phi_E^{-1} \circ \psi_2$ and identity ϕ_E, which we will call the Hom-Gauge group, given by

$$\text{H-Gau}(E) = \left\{ \psi \in \mathcal{A}^0_{\mathbb{L}}\big(X, \operatorname{End}_{\phi_E}(E)\big) \;\middle|\; \begin{array}{l} \psi \bullet \psi' = \psi' \bullet \psi = \phi_E \\ \text{for some } \psi' \in \mathcal{A}^0_{\mathbb{L}}\big(X, \operatorname{End}_{\phi_E}(E)\big) \end{array} \right\}$$

Note that there is a unique element $[\psi]^{-1} = \phi_E \circ \psi^{-1} \circ \phi_E \in \text{H-Gau}(E)$ for each $\psi \in \text{H-Gau}(E)$ such that

$$\psi \bullet [\psi]^{-1} = [\psi]^{-1} \bullet \psi = \phi_E$$

For a given smooth Hom-bundle (E, ϕ, ϕ_E) on a smooth manifold X, define a map

$$\begin{aligned} \odot : \text{H-Gau}(E) \otimes A(E, \mathbb{L}) &\to A(E, \mathbb{L}) \text{ given by,} \\ \nabla^\psi = \odot(\psi, \nabla) &= \mathrm{id}_{\mathbb{L}^\star} \otimes \big([\psi]^{-1} \circ \phi_E^{-1}\big) \circ \nabla \circ \big(\phi_E^{-1} \circ \psi\big) \\ &= \mathrm{id}_{\mathbb{L}^\star} \otimes [\psi]^{-1} \bullet \nabla \bullet \psi \end{aligned} \tag{3.1}$$

It is easy to observe that

(1) for $f \in C^\infty(X); s \in \Gamma(X, E)$,

$$\begin{aligned} \nabla^\psi(fs) &= \mathrm{id}_{\mathbb{L}^\star} \otimes \big([\psi]^{-1} \circ \phi_E^{-1}\big) \circ \nabla\big(\phi_E^{-1} \circ \psi(fs)\big) \\ &= \mathrm{id}_{\mathbb{L}^\star} \otimes \big([\psi]^{-1} \circ \phi_E^{-1}\big) \circ \nabla\Big(f\big(\phi_E^{-1} \circ \psi\big)(s)\Big) \\ &= \mathrm{id}_{\mathbb{L}^\star} \otimes \big([\psi]^{-1} \circ \phi_E^{-1}\big)\Big[d_{\mathbb{L}}(f) \otimes \psi(s) + \phi^\star(f)\nabla\big(\phi_E^{-1} \circ \psi(s)\big)\Big] \\ &= d_{\mathbb{L}}(f)\phi_E(s) + \phi^\star(f)\nabla^\psi(s), \end{aligned}$$

also $\phi_{\mathbb{L}}^\dagger \otimes \phi_E \circ \nabla^\psi = \nabla^\psi \circ \phi$ for $\psi \in \text{H-Gau}(E)$ implies the map $\odot$ is well defined;

(2) the map $\odot$ is a left action of $\text{H-Gau}(E)$ on $A(E, \mathbb{L})$.

Remark 3.1. [9, cf.Theorem 6] The group $\text{H-Gau}(E)$ has a Lie group as well as a Hom-Lie group structure, denoted by $(\text{H-Gau}(E), \bullet, \phi_E, \mathrm{Ad}_{\phi_E})$, because

$$\mathrm{Ad}_{\phi_E} \circ \bullet(x, y) = \bullet(x, y) \qquad \big(x, y \in \text{H-Gau}(E)\big)$$

with Lie bracket structure on the Lie algebra $\text{H-}\mathfrak{g}\text{au}(E) = \mathcal{A}^0_{\mathbb{L}}\big(X, \operatorname{End}_{\phi_E}(E)\big)$ can be described using the Equation (2.7)

$$\begin{aligned} [x, y] &= x \bullet y - y \bullet x \\ &= \mathrm{Ad}_{\phi_E}(x \bullet y - y \bullet x) \\ &= [x, y]_{\text{H-Gau}(E)} \end{aligned}$$

For a given $\mathbb{L}$-connection, ∇ the isotropy subgroup $\text{H-Gau}_\nabla(E) \subset \text{H-Gau}(E)$ is given by

$$\text{H-Gau}_\nabla(E) = \{\psi \in \text{H-Gau}(E) \mid \nabla^\psi = \nabla\},$$

Definition 3.2. An $\mathbb{L}$-connection ∇ with isotropy subgroup $\text{H-Gau}_\nabla(E) = \mathbb{K}^\star \cdot \phi_E$, is called an irreducible $\mathbb{L}$-connection, which will be denoted by $\widehat{A}(E, \mathbb{L})$.

It is easy to observe

$$\text{H-Gau}_{\nabla^\psi}(E) = [\psi]^{-1} \bullet \text{H-Gau}_\nabla(E) \bullet \psi$$

Hence, the space of irreducible $\mathbb{L}$-connections $\widehat{A}(E,\mathbb{L})$ is closed under the action of H-Gau(E) and we have the quotient spaces

$$p : A(E,\mathbb{L}) \to A(E,\mathbb{L}) \Big/ \text{H-Gau}(E) = B(E,\mathbb{L})$$
and
$$\widehat{p} : \widehat{A}(E,\mathbb{L}) \to \widehat{A}(E,\mathbb{L}) \Big/ \text{H-Gau}(E) = \widehat{B}(E,\mathbb{L})$$

Remark 3.3. For a given smooth Hom-bundle E and smooth Hom-Lie algebroid $\mathbb{L}$ on a smooth manifold X, define the reduced H-Gauge group as the quotient space $\text{H-Gau}^r(E) = \text{H-Gau}(E) \Big/ \mathbb{K}^\star \cdot \phi_E$. Because for each $\mathbb{L}$-connection ∇, we have $\mathbb{K}^\star \cdot \phi_E \subset \text{H-Gau}_\nabla(E)$, the action of H-Gau$(E)$ reduces to the action of H-Gau$^r(E)$ and the moduli spaces, which we are interested in are the H-Gau$^r(E)$ equivalence classes

$$p : A(E,\mathbb{L}) \to A(E,\mathbb{L}) \Big/ \text{H-Gau}^r(E) = B(E,\mathbb{L})$$
and
$$\widehat{p} : \widehat{A}(E,\mathbb{L}) \to \widehat{A}(E,\mathbb{L}) \Big/ \text{H-Gau}^r(E) = \widehat{B}(E,\mathbb{L})$$

Remark 3.4. For a given Hom-Lie algebroid connection $\nabla \in A(E,\mathbb{L})$ and $\psi \in \text{H-Gau}(E)$, the transformed connection ∇^ψ is given by

$$\nabla^\psi = \text{id}_{\mathbb{L}^\star} \otimes ([\psi^{-1}] \circ \phi_E^{-1}) \circ \nabla_E \circ \phi_E^{-1} \circ \psi$$

Using Remark 2.12, we have

$$(\phi_{\mathbb{L}}^\dagger)^{-1} \otimes \phi_E^{-1} \circ (\nabla^{\text{End}(E)}\psi) \circ \phi_E = \nabla^E \circ \phi_E^{-1} \circ \psi - \text{id}_{\mathbb{L}^\star} \otimes (\psi^{-1} \otimes \phi_E^{-1}) \circ \nabla^E$$
$$\text{id}_{\mathbb{L}^\star} \otimes ([\psi^{-1}] \circ \phi_E^{-1}) \circ (\phi_{\mathbb{L}}^\dagger)^{-1} \otimes \phi_E^{-1} \circ (\nabla^{\text{End}(E)}\psi) \circ \phi_E = \nabla^\psi - \nabla^E \tag{3.2}$$

Using Remark 2.13, and applying $\nabla^{\text{End}(E)}$ on the equality $[\psi^{-1}] \circ \phi_E^{-1} \circ \psi = \phi_E$, we get

$$(\phi_{\mathbb{L}}^\dagger)^{-1} \otimes \phi_E^{-1} \circ (\nabla^{\text{End}(E)}[\psi^{-1}]) \circ \psi + \text{id}_{\mathbb{L}^\star} \otimes ([\psi^{-1}] \circ \phi_E^{-1}) \circ (\phi_{\mathbb{L}}^\dagger)^{-1} \otimes \phi_E^{-1} \circ (\nabla^{\text{End}(E)}\psi) \circ \phi_E = 0 \tag{3.3}$$

From Equations (3.2) and (3.3), we have

$$\nabla^\psi = \nabla - (\phi_{\mathbb{L}}^\dagger)^{-1} \otimes \phi_E^{-1} \circ (\nabla^{\text{End}(E)}[\psi^{-1}]) \circ \psi \tag{3.4}$$

More generally, for some $\mathbb{L}$-connection $\nabla = \nabla_0 + \alpha$, $\alpha \in \mathcal{A}^1_{\mathbb{L}}\big(X, \text{End}(E)\big)$, we have

$$\begin{aligned}
\nabla^\psi &= \text{id}_{\mathbb{L}^\star} \otimes ([\psi^{-1}] \circ \phi_E^{-1}) \circ (\nabla_0 + \alpha) \circ \phi_E^{-1} \circ \psi \\
&= \text{id}_{\mathbb{L}^\star} \otimes ([\psi^{-1}] \circ \phi_E^{-1}) \circ \nabla_0 \circ \phi_E^{-1} \circ \psi + \text{id}_{\mathbb{L}^\star} \otimes ([\psi^{-1}] \circ \phi_E^{-1}) \circ \alpha \circ \phi_E^{-1} \circ \psi \\
&= \nabla_0 + \text{id}_{\mathbb{L}^\star} \otimes ([\psi^{-1}] \circ \phi_E^{-1}) \circ (\phi_{\mathbb{L}}^\dagger)^{-1} \otimes \phi_E^{-1} \circ (\nabla_0^{\text{End}(E)}\psi) \circ \phi_E + \\
&\qquad \text{id}_{\mathbb{L}^\star} \otimes ([\psi^{-1}] \circ \phi_E^{-1}) \circ \alpha \circ \phi_E^{-1} \circ \psi \qquad\qquad (\text{from}(3.2)) \\
&= \nabla_0 + \text{id}_{\mathbb{L}^\star} \otimes [\psi^{-1}] \bullet (\phi_{\mathbb{L}}^\dagger)^{-1} \otimes \phi_E^{-1} \circ (\nabla_0^{\text{End}(E)}\psi) \circ \phi_E + \text{id}_{\mathbb{L}^\star} \otimes [\psi^{-1}] \bullet \alpha \bullet \psi
\end{aligned}$$

And, we can write

$$\begin{aligned}
&\nabla^\psi = \nabla_0 + \alpha^\psi \text{ such that} \\
&\alpha^\psi = \text{id}_{\mathbb{L}^\star} \otimes [\psi^{-1}] \bullet (\phi_{\mathbb{L}}^\dagger)^{-1} \otimes \phi_E^{-1} \circ (\nabla_0^{\text{End}(E)}\psi) \circ \phi_E + \text{id}_{\mathbb{L}^\star} \otimes [\psi^{-1}] \bullet \alpha \bullet \psi
\end{aligned}$$

4. Sobolve H-Gauge group action

In this section, we have discussed Sobolev completion of H-Gau(E)-moduli spaces, using Sobolev theory of function spaces. The main theorems of this section are Theorems 4.11 and 4.12, in which we have described the Hilbert manifold structure on the moduli space $\widehat{B}(E,\mathbb{L})_l$ and $\widehat{p}:\widehat{A}(E,\mathbb{L})_l\to\widehat{B}(E,\mathbb{L})_l$ has principal-H-Gau$(E)^r_{l+1}$ bundle structure.

A complex (resp. real) Hom-bundle E is said to have Hom-Hermitian (resp. Hom-Euclidean) metric if there is a Hermitian (resp. Euclidean) metric $(\cdot,\cdot)_{h^E}$ on the bundle E such that

$$\big(\phi_E(s_1),\phi_E(s_2)\big)_{h^E}=\phi^\star(s_1,s_2)_{h^E}\quad\text{ for } s_1,s_2\in\Gamma(X,E)$$

4.1. **Sobolev completion of space of $\mathbb{L}$-connections.** Let $E,\mathbb{L}$ be a complex (resp. real) Hom-bundle, Hom-Lie algebroid respectively over a compact manifold X with Hom-Hermitian (resp. Hom-Euclidean) metric corresponding to Hermitian (resp. Euclidean) metrices $h^E,h^{\mathbb{L}}$ respectively and Riemannian metric g on the manifold X. There is an induced Hom-metric on the Hom-bundle $\Lambda^k\mathbb{L}\otimes E$ $(0\leq k)$ associated to the Hermitian (resp. Euclidean) metric $h^E_{\mathbb{L}}$ induced by h^E and $h^{\mathbb{L}}$. The Riemannian metric g induces a volume form vol(g), which further induces Borel measure μ_g on the smooth manifold X.

Let $L^2_l\big(\mathcal{A}^0_{\mathbb{L}}(X,E)\big)(l\in\mathbb{Z}^+)=\mathcal{A}^0_{\mathbb{L}}(X,E)_l$ be the space of equivalence classes of Borel measurable sections with weak derivatives of order upto l are square integrable. The space $\mathcal{A}^0_{\mathbb{L}}(X,E)_l(l\in\mathbb{Z}^+)$ is called the Sobolev completion of space of global sections of bundle E, which is a Hilbert space with inner product

$$\langle s_1,s_2\rangle_{L^2_l}=\langle s_1,s_2\rangle_l=\sum_{j=0}^{l}\langle\nabla^j s_1,\nabla^j s_2\rangle_{L^2}\qquad(s_1,s_2\in\mathcal{A}^0_{\mathbb{L}}(X,E)_l)$$

where $\langle\nabla^j s_1,\nabla^j s_2\rangle_{L^2}$ can be computed using metric h^E and $h^E_{\mathbb{L}}$. The metric h^E induces a metric on the bundle $\mathrm{End}(E)\cong\phi!E\otimes Hom(E,\phi!\underline{\mathbb{K}})$. Using this induced metric and the given Hermitian metric on $\mathbb{L}$, we can describe the Sobolev completion of the space $\mathcal{A}^k_{\mathbb{L}}\big(X,\mathrm{End}(E)\big)$, which will be written as $L^2_l\Big(\mathcal{A}^k_{\mathbb{L}}\big(X,\mathrm{End}(E)\big)\Big)=\mathcal{A}^k_{\mathbb{L}}\big(X,\mathrm{End}(E)\big)_l$

Using the Lemma 2.10, define the Sobolev space of $\mathbb{L}$-connections as

$$A(E,\mathbb{L})_l=\{\nabla_0+\alpha\mid\alpha\in\mathcal{A}^1_{\mathbb{L}}\big(X,\mathrm{End}_{\phi_E}(E)\big)_l\}$$

where $\mathcal{A}^1_{\mathbb{L}}\big(X,\mathrm{End}_{\phi_E}(E)\big)_l=\{x\in\mathcal{A}^1_{\mathbb{L}}\big(X,\mathrm{End}(E)\big)_l\mid\phi^\dagger_{\mathbb{L}}\otimes\phi_E\circ x=x\circ\phi^\dagger_{\mathbb{L}}\otimes\phi_E\}$ and $\phi^\dagger_{\mathbb{L}}\otimes\phi_E$ is continuous extension of the smooth map $\phi^\dagger_{\mathbb{L}}\otimes\phi_E:\mathcal{A}^1_{\mathbb{L}}(X,E)\to\mathcal{A}^1_{\mathbb{L}}(X,E)$ to appropriate Sobolev spaces, using Sobolev multiplication theorem in the range $l>\frac{1}{2}\mathrm{dim}_{\mathbb{R}}(X)$.

The space $\mathcal{A}^1_{\mathbb{L}}\big(X,\mathrm{End}_{\phi_E}(E)\big)_l\subset\mathcal{A}^1_{\mathbb{L}}\big(X,\mathrm{End}(E)\big)_l$ is a closed subspace, furthermore a Hilbert space, which implies the Sobolev space of $\mathbb{L}$-connections is a Hilbert manifold.

4.2. **Sobolev completion of space of H-gauge transformations.** For $l > \frac{1}{2}\dim_{\mathbb{R}} X$, using Sobolev multiplication theorem, we have the continuously extended map

$$\mathcal{A}^0_{\mathbb{L}}\big(X, \mathrm{End}_{\phi_E}(E)\big)_{l+1} \times \mathcal{A}^0_{\mathbb{L}}\big(X, \mathrm{End}_{\phi_E}(E)\big)_{l+1} \to \mathcal{A}^0_{\mathbb{L}}\big(X, \mathrm{End}_{\phi_E}(E)\big)_{l+1}$$
$$\big((\psi_1, \psi_2) \mapsto \psi_1 \bullet \psi_2 = \psi_1 \circ \phi_E^{-1} \circ \psi_2\big)$$

which makes the space $\mathcal{A}^0_{\mathbb{L}}\big(X, \mathrm{End}_{\phi_E}(E)\big)_{l+1}$, a Banach $\mathbb{K}$-algebra.

Define the Sobolev space of H-gauge group

$$\text{H-Gau}(E)_{l+1} = \left\{ \psi \in \mathcal{A}^0_{\mathbb{L}}\big(X, \mathrm{End}_{\phi_E}(E)\big)_{l+1} \;\middle|\; \begin{array}{l} \psi \bullet \psi' = \psi' \bullet \psi = \phi_E \\ \text{for some } \psi' \in \mathcal{A}^0_{\mathbb{L}}\big(X, \mathrm{End}_{\phi_E}(E)\big)_{l+1} \end{array} \right\}$$

The Sobolev space of H-gauge group is the subspace of invertible elements in the Hilbert space $\mathcal{A}^0_{\mathbb{L}}\big(X, \mathrm{End}_{\phi_E}(E)\big)_{l+1}$, implies $\big(\text{H-Gau}(E)\big)_{l+1}$ is a Hilbert-Lie group. The Lie bracket structure on the Lie algebra

$$\text{H-}\mathfrak{gau}(E)_{l+1} = \mathcal{A}^0_{\mathbb{L}}\big(X, \mathrm{End}_{\phi_E}(E)\big)_{l+1}$$

can be described by continuously extending the map $[\cdot,\cdot]_{\mathbb{L}}$ described in Equation (2.7) to appropriate Sobolev space, using Sobolev multiplication theorem.

4.3. **Sobolev H-gauge action on the Sobolev space of $\mathbb{L}$-connection.** The H-Gau(E) action on the space $A(E, \mathbb{L})$ described in Equation (3.1), can be extended continuously to the Sobolev completion spaces

$$\odot : \text{H-Gau}(E)_{l+1} \times A(E, \mathbb{L})_l \to A(E, \mathbb{L})_l$$
$$(\psi, \nabla = \nabla_0 + \alpha) \mapsto \nabla^\psi = \nabla_0 + \mathrm{id}_{\mathbb{L}^\star} \otimes ([\psi^{-1}] \circ \phi_E^{-1}) \circ \big((\phi_{\mathbb{L}}^\dagger)^{-1} \otimes \phi_E^{-1}\big) \circ (d^{\nabla_0}\psi) \circ \phi_E +$$
$$\mathrm{id}_{\mathbb{L}^\star} \otimes (\psi \circ \phi_E^{-1}) \circ \alpha \circ \phi_E^{-1} \circ \psi^{-1} \qquad (\text{ see Equation (3.4)})$$

where $d^{\nabla_0} : \mathcal{A}^\bullet_{\mathbb{L}}\big(X, \mathrm{End}_{\phi_E}(E)\big)_l \to \mathcal{A}^{\bullet+1}_{\mathbb{L}}\big(X, \mathrm{End}_{\phi_E}(E)\big)_{l-1}$ is continous extension of the degree 1 operator d^{∇_0} assciated to the connection $\nabla_0^{\mathrm{End}(E)}$ (see Equation (2.1)) and ϕ_E, ψ are continuously extended locally linear maps on finite dimensional Sobolev space; hence the Sobolev Gauge action $\odot$ is a smooth map.

Similar to the smooth case discussed in previous section, for a Sobolev $\mathbb{L}$-connection ∇, the isotropy subgroup $\text{H-Gau}_\nabla(E)_{l+1} \subset \text{H-Gau}(E)_{l+1}$ is given by

$$\text{H-Gau}_\nabla(E)_{l+1} = \{\psi \in \text{H-Gau}(E)_{l+1} \mid \nabla^\psi = \nabla\} \tag{4.1}$$

and a Sobolev $\mathbb{L}$-connection is said to be an irreducible $\mathbb{L}$-connection if $\text{H-Gau}_\nabla(E) = \mathbb{K}^\star \cdot \phi_E$. The collection of irreducible Sobolev $\mathbb{L}$-connections will be denoted by $\widehat{A}(E, \mathbb{L})_l \subset A(E, \mathbb{L})_l$. Similar to the smooth case, it can be verified that,

$$\text{H-Gau}_{\nabla^\psi}(E)_{l+1} = [\psi]^{-1} \bullet \text{H-Gau}(E)_{l+1} \bullet \psi$$

and the space $\widehat{A}(E,\mathbb{L})_l$ is closed under H-gauge action. Define the H-gauge theoretic moduli spaces

$$p : A(E,\mathbb{L})_l \to A(E,\mathbb{L})_l \Big/ \text{H-Gau}(E)_{l+1} = B(E,\mathbb{L})_l$$

and

$$\widehat{p} : \widehat{A}(E,\mathbb{L})_l \to \widehat{A}(E,\mathbb{L})_l \Big/ \text{H-Gau}(E)_{l+1} = \widehat{B}(E,\mathbb{L})_l$$

For an irreducible Sobolev $\mathbb{L}$-connection $\nabla \in A(E,\mathbb{L})_l$, the isotropy subgroup $\text{H-Gau}_{\nabla}(E)_{l+1} \subset \text{H-Gau}(E)_{l+1}$ is $\mathbb{K}^\star \cdot \phi_E$, we define the reduced Sobolev H-gauge group as,

$$\text{Gau}^r(E)_{l+1} = \text{Gau}(E)_{l+1} \big/ \mathbb{K}^\star \cdot \phi_E$$

The left action of $\text{Gau}^r(E)_{l+1}$ on the space $\widehat{A}(X,\mathbb{L})_l$ is a free action.

Theorem 4.1. [4, cf. Theorem II.2] *For a given Banach Lie group G over a field $\mathbb{K}$ with a normal Banach Lie subgroup N with Lie algebras $\mathfrak{g},\mathfrak{n}$ respectively. Then G/N is a Banach Lie group with Lie algebra $\mathfrak{g}/\mathfrak{n}$, which can be described in unique way such that the map $q : G \to G/N$ is a smooth map. Moreover, for any Banach manifold X a map $f : G/N \to X$ is smooth iff $f \circ q$ is smooth.*

Using above theorem, a Lie group structure on the quotient space $\text{H-Gau}^r(E)$ can be described in a unique way with Lie algebra

$$\text{H-}\mathfrak{gau}^r(E)_{l+1} = \mathcal{A}^0_{\mathbb{L}}\big(X, \text{End}_{\phi_E}(E)\big)_{l+1} \big/ \mathbb{K}^\star \cdot \phi_E = \mathcal{A}^0_{\mathbb{L}}\big(X, \text{End}_{\phi_E}(E)\big)^0_{l+1}$$

where the space $\mathcal{A}^0_{\mathbb{L}}\big(X, \text{End}_{\phi_E}(E)\big)^0_{l+1}$ can be described using the L^2-orthogonal decomposition

$$\mathcal{A}^0_{\mathbb{L}}\big(X, \text{End}_{\phi_E}(E)\big)_{l+1} = \mathcal{A}^0_{\mathbb{L}}\big(X, \text{End}_{\phi_E}(E)\big)^0_{l+1} \oplus \mathbb{K}^\star \cdot \phi_E$$

The space $\mathcal{A}^0_{\mathbb{L}}\big(X, \text{End}_{\phi_E}(E)\big)^0_{l+1}$ is given by

$$\mathcal{A}^0_{\mathbb{L}}\big(X, \text{End}_{\phi_E}(E)\big)^0_{l+1} = \left\{ s \in \mathcal{A}^0_{\mathbb{L}}\big(X, \text{End}_{\phi_E}(E)\big)_{l+1} \mid \int_X \text{tr}(\phi_E^\star \circ s) d\mu_g = 0 \right\}$$

where $\phi_E^\star$ is adjoint operator w.r.t Hermitian metric h^E.

Remark 4.2. For a given $\alpha \in \mathcal{A}^1_{\mathbb{L}}\big(X, \text{End}(E)\big)$, we have order 0, degree 1 differential operator

$$\text{ad}(\alpha) : \mathcal{A}^0_{\mathbb{L}}\big(X, \text{End}(E)\big) \to \mathcal{A}^1_{\mathbb{L}}\big(X, \text{End}(E)\big)$$

given by $\text{ad}(\alpha)(\beta) = [\alpha,\beta]_{\mathbb{L}}$.

The adjoint operator $\text{ad}(\alpha)^\star$ w.r.t the induced metric on the space $\mathcal{A}^0_{\mathbb{L}}\big(X, \text{End}(E)\big)$ gives a ses-quilinear map

$$m : \mathcal{A}^1_{\mathbb{L}}\big(X, \text{End}(E)\big) \times \mathcal{A}^1_{\mathbb{L}}\big(X, \text{End}(E)\big) \to \mathcal{A}^0_{\mathbb{L}}\big(X, \text{End}(E)\big) \quad \big(m(\alpha,\beta) = \text{ad}(\alpha)^\star(\beta)\big)$$

Using Sobolev multiplication theorem $\big(l > \frac{1}{2}\dim_{\mathbb{R}}(X)\big)$ the map m can be extended continuously to a map m_l, given by

$$m_l : \mathcal{A}^1_{\mathbb{L}}\big(X, \text{End}(E)\big)_l \times \mathcal{A}^1_{\mathbb{L}}\big(X, \text{End}(E)\big)_l \to \mathcal{A}^0_{\mathbb{L}}\big(X, \text{End}(E)\big)_l \quad \big(m_l(\alpha,\beta) = \text{ad}(\alpha)^\star(\beta)\big)$$

and for a fixed $\alpha \in \mathcal{A}^1_{\mathbb{L}}\big(X, \mathrm{End}(E)\big)_l$, we have the continuous $(\phi^\star)^{-1}$-linear map defined on appropriate Sobolev space

$$\mathrm{ad}(\alpha)^\star : \mathcal{A}^1_{\mathbb{L}}\big(X, \mathrm{End}(E)\big)_l \to \mathcal{A}^0_{\mathbb{L}}\big(X, \mathrm{End}(E)\big)_l$$

Using Sobolev embedding theorem, we have the compact embeddings given by

$$i_l : \mathcal{A}^0_{\mathbb{L}}\big(X, \mathrm{End}(E)\big)_l \to \mathcal{A}^0_{\mathbb{L}}\big(X, \mathrm{End}(E)\big)_{l-1} \quad (l \in \mathbb{Z})$$

For a given Sobolev $\mathbb{L}$-connection $\nabla = \nabla_0 + \alpha$, we can write the differential operators

$$\begin{aligned} d^\nabla :& \mathcal{A}^0_{\mathbb{L}}\big(X, \mathrm{End}(E)\big)_{l+1} \to \mathcal{A}^1_{\mathbb{L}}\big(X, \mathrm{End}(E)\big)_l \quad \text{and} \\ (d^\nabla)^\star :& \mathcal{A}^1_{\mathbb{L}}\big(X, \mathrm{End}(E)\big)_l \to \mathcal{A}^1_{\mathbb{L}}\big(X, \mathrm{End}(E)\big)_{l-1} \end{aligned}$$

as, $d^\nabla = d^{\nabla_0} + \mathrm{ad}(\alpha) \circ i_{l+1}$ and $(d^\nabla)^\star = (d^{\nabla_0})^\star + i_l \circ \mathrm{ad}(\alpha)^\star$. The operator $(d^{\nabla_0})^\star$ is continuous extension to appropriate Sobolev spaces of the formal adjoint of d^{∇_0} w.r.t the Hermitian metric $h^{\mathrm{End}(E)}$.

From now on we will assume, the given Hom-Lie algebroid $\mathbb{L}$ is transitive, which means the anchor map $\mathfrak{a}_{\mathbb{L}} : \mathbb{L} \to \phi^! T_X$ is surjective.

Lemma 4.3. *For any $\nabla \in A(E, \mathbb{L})_l$, the composition of operators d^∇ and $(d^\nabla)^\star$*

$$(d^\nabla)^\star \circ d^\nabla : \mathcal{A}^0_{\mathbb{L}}\big(X, End(E)\big)_{l+1} \to \mathcal{A}^0_{\mathbb{L}}\big(X, End(E)\big)_{l+1}$$

is a Fredholm operator, for all $l > \frac{1}{2} dim_{\mathbb{R}}(X)$.

Proof. It is enough to prove $(d^{\nabla_0})^\star \circ d^{\nabla_0}$ is a Fredholm operator, because

$$(d^\nabla)^\star \circ d^\nabla = (d^{\nabla_0})^\star \circ d^{\nabla_0} + i \circ \mathrm{ad}(\alpha)^\star \circ d^{\nabla_0} + i \circ \mathrm{ad}(\alpha)^\star \circ \mathrm{ad}(\alpha) \circ i + (d^{\nabla_0})^\star \circ \mathrm{ad}(\alpha) \circ i$$

and $i \circ \mathrm{ad}(\alpha)^\star, \mathrm{ad}(\alpha) \circ i$ are compact implies $i \circ \mathrm{ad}(\alpha)^\star \circ d^{\nabla_0} + i \circ \mathrm{ad}(\alpha)^\star \circ \mathrm{ad}(\alpha) \circ i + (d^{\nabla_0})^\star \circ \mathrm{ad}(\alpha) \circ i$ are compact operators. And sum of a Fredholm and a compact operator is a Fredholm operator, so it is enough to prove that $(d^{\nabla_0})^\star \circ d^{\nabla_0}$ is an elliptic operator, which is true in case the Hom-Lie algebroid $\mathbb{L}$ is transitive because the principal symbol

$$\sigma_1\big((d^{\nabla_0})^\star \circ d^{\nabla_0}\big)(\xi_x) = \sigma_1\big((d^{\nabla_0})^\star\big)(\xi_x) \circ \sigma_1(d^{\nabla_0})(\xi_x) \qquad (\xi_x \in \phi^! T^\star_{X,x} \setminus \{0\})$$

is an isomorphism if $\sigma_1(d^{\nabla_0})(\xi_x)(\_) = \mathfrak{a}^\star_{\mathbb{L}}(\xi_x) \otimes \_$ is isomorphism, which is true, if $\mathfrak{a}^\star_{\mathbb{L},x}$ is injective or, the anchor map $\mathfrak{a}_{\mathbb{L}}$ is surjective. □

Lemma 4.4. *For any Sobolev $\mathbb{L}$-connection $\nabla \in A(E, \mathbb{L})$, we have the following L^2-orthogonal decomposition*

$$\mathcal{A}^1_{\mathbb{L}}\big(X, \mathrm{End}(E)\big)_l = \mathrm{im}(d^\nabla) \oplus \ker\big((d^\nabla)^\star\big) \tag{4.2}$$

for all $l > \frac{1}{2} dim_{\mathbb{R}} X$.

Proof. Using Fredholmness of the operator $\Delta = (d^\nabla)^\star \circ d^\nabla$, the dimension of $\ker(\Delta) \subset \mathcal{A}^0_{\mathbb{L}}\big(X, \mathrm{End}(E)\big)_{l+1}$ is finite and $\mathrm{im}(\Delta) \subset \mathcal{A}^0_{\mathbb{L}}\big(X, \mathrm{End}(E)\big)_{l-1}$ is a closed subspace. Furthermore, we have the L^2-orthogonal decomposition of $\mathcal{A}^0_{\mathbb{L}}\big(X, \mathrm{End}(E)\big)_{l+1}$ in closed subspaces

$$\mathcal{A}^0_{\mathbb{L}}\big(X, \mathrm{End}(E)\big)_{l+1} = \ker(\Delta) + \ker(\Delta)^\perp$$

and using the fact that $\operatorname{im}(\Delta) \subset \mathcal{A}^0_{\mathbb{L}}\big(X, \operatorname{End}(E)\big)_{l-1}$ is a closed subspace, the bijective map

$$\Delta_{|\ker(\Delta)^{\perp}} : \ker(\Delta)^{\perp} \to \operatorname{im}(\Delta)$$

is a continuous map. Using Banach open mapping theorem $\big(\Delta_{|\ker(\Delta)^{\perp}}\big)^{-1}$ is a continuous map, furthermore $\mathrm{id}_Y - d^{\nabla} \circ \big(\Delta_{|\ker(\Delta)^{\perp}}\big)^{-1} \circ (d^{\nabla})^{\star}$ is a continuous map on $Y = \big((d^{\nabla})^{\star}\big)^{-1}\big(\operatorname{im}(\Delta)\big)$. Note that $\ker(\Delta) = \ker(d^{\nabla})$ implies $\operatorname{im}(d^{\nabla}) = \operatorname{im}(d^{\nabla})_{|\ker(\Delta)^{\perp}}$ and

$$\operatorname{im}(d^{\nabla}) = \ker\big(\mathrm{id}_Y - d^{\nabla} \circ \big(\Delta_{|\ker(\Delta)^{\perp}}\big)^{-1} \circ (d^{\nabla})^{\star}\big)$$

is a closed subspace of $\mathcal{A}^1_{\mathbb{L}}\big(X, \operatorname{End}(E)\big)_l$.

Furthermore, we have the L^2-orthogonal decomposition

$$\mathcal{A}^1_{\mathbb{L}}\big(X, \operatorname{End}(E)\big)_l = \operatorname{im}(d^{\nabla}) \oplus \operatorname{im}(d^{\nabla})^{\perp}$$

Using the property of adjoint operator, we have $\operatorname{im}(d^{\nabla})^{\perp} = \ker(d^{\nabla})^{\star}$ and we have the required L^2-orthogonal decomposition. $\square$

Lemma 4.5. *Let ∇ be an $\mathbb{L}$-connection for a Hom-bundle E over a compact manifold X. The following statements are equivalent*

(1) $\text{H-Gau}_{\nabla}(E) = \mathbb{K}^{\star} \cdot \phi_E$
(2) $\ker(\nabla^{\operatorname{End}(E)}) = \mathbb{K} \cdot \phi_E$

Proof. The proof for (1) $\implies$ (2) can be proved as follows.

Let $\psi \in \mathbb{K}^{\star} \cdot \phi_E$, we have $\nabla^{\psi} = \nabla$. Form the Equation (3.4), we have $\psi \in \ker(\nabla^{\operatorname{End}(E)})$, which implies

$$\mathbb{K} \cdot \phi_E \subset \ker(\nabla^{\operatorname{End}(E)}) \tag{4.3}$$

Now let $\psi \in \ker(\nabla^{\operatorname{End}(E)})$ be a non-trivial element. Using compactness of manifold X, we can have $c \in \mathbb{K}$ with $|c|$ large enough such that $\phi_E \circ \Big(\mathrm{id}_E + (\phi_E^{-1} \circ \psi)\,/_c\Big) \in \text{H-Gau}(E)$. Also $\nabla^{\operatorname{End}(E)}\Big(\phi_E \circ \Big(\mathrm{id}_E + (\phi_E^{-1} \circ \psi)\,/_c\Big)\Big) = 0$ and using the Equation (3.4), we have $\phi_E \circ \Big(\mathrm{id}_E + (\phi_E^{-1} \circ \psi)\,/_c\Big) \in \text{H-Gau}_{\nabla}(E)$. Assuming (1) is true, we have $\psi \in \mathbb{K} \cdot \phi_E$ or,

$$\ker(\nabla^{\operatorname{End}(E)}) \subset \mathbb{K} \cdot \phi_E \tag{4.4}$$

From Equations (4.3) and (4.4), we have $\ker(\nabla^{\operatorname{End}(E)}) = \mathbb{K} \cdot \phi_E$.

To prove (2) $\implies$ (1), we need to show only $\text{H-Gau}(E) \subset K^{\star} \cdot \phi_E$ and can be proved using the Equation (3.4). $\square$

Remark 4.6. The above theorem can be proved in the same line of arguements for Sobolev space of $\mathbb{L}$-connections for Hom-bundle E on compact manifold X and we have the following lemma.

Lemma 4.7. *Let ∇ be a Sobolev $\mathbb{L}$-connection for a Hom-bundle E over a compact manifold X. The following statements are equivalent*

(1) $\text{H-Gau}_{\nabla}(E)_{l+1} = \mathbb{K}^{\star} \cdot \phi_E$
(2) $\ker(\nabla^{\mathrm{End}(E)}) = \mathbb{K} \cdot \phi_E$

where ϕ_E is continuous extension of the map $\phi_E : \Gamma(X,E) \to \Gamma(X,E)$ to appropriate Sobolev spaces, using Sobolev multiplication theorem in the range $l > \frac{1}{2} dim_{\mathbb{R}} X$.

Lemma 4.8. *The space of irreducible Sobolev $\mathbb{L}$-connections $\widehat{A}(E,\mathbb{L})_l \subset A(E,\mathbb{L})_l$ is an open subspace for all $l > \frac{1}{2} dim_{\mathbb{R}}(X)$.*

Proof. For a given Sobolev $\mathbb{L}$-connection $\nabla = \nabla_0 + \alpha$, we have a Fredholm operator Δ_α, with finite dimensional kernel space $\ker(\Delta_\alpha)$; furthermore the composition of maps

$$A(E,\mathbb{L})_l \xrightarrow{\nabla \mapsto \Delta_\alpha} \mathcal{F}(\mathcal{A}^0_{\mathbb{L},l+1}, \mathcal{A}^0_{\mathbb{L},l-1}) \xrightarrow{\Delta_\alpha \mapsto \dim(\ker \Delta_\alpha)} \mathbb{R}$$

is upper semi-continuous [2]. For an irreducible Sobolev $\mathbb{L}$-connection $\nabla = \nabla_0 + \alpha$, $\dim\big(\ker(\Delta_\alpha)\big) = 1$ (see Lemma 4.7) also we know $\ker(\Delta_\alpha) = \ker(d_\alpha)$ implies the space of irreducible Sobolev $\mathbb{L}$-connections $\widehat{A}(E,\mathbb{L})_l \subset A(E,\mathbb{L})_l$ is an open subspace, using semi-continuity. □

Since space $\widehat{A}(E,\mathbb{L})_l$ (resp. $A(E,\mathbb{L})_l$) has image $\widehat{B}(E,\mathbb{L})_l$ (resp. $B(E,\mathbb{L})_l$) under left H-Gau$(E)_{l+1}$-action and using the Lemma 4.8, the space $\widehat{B}(E,\mathbb{L})_l \subset B(E,\mathbb{L})_l$, is an open subspace under quotient toplogy.

For $\nabla \in A(E,\mathbb{L})_l$ and $\epsilon > 0$, the Hilbert submanifold $\mathcal{O}_{\nabla,\epsilon} \subset A(E,\mathbb{L})_l$

$$\mathcal{O}_{\nabla,\epsilon} = \{\nabla + \alpha \mid \alpha \in \mathcal{A}^1_{\mathbb{L}}\big(X, \mathrm{End}_{\phi_E}(E)\big)_l, (d^{\nabla})^{\star}(\alpha) = 0 \text{ and } \|\alpha\|_{L^2_l} < \epsilon\}$$

has the tangent space

$$T_{\mathcal{O}_{\nabla,\epsilon}} = \mathrm{Ker}\big((d^{\nabla})^{\star}\big) \subset \mathcal{A}^1_{\mathbb{L}}\big(X, \mathrm{End}_{\phi_E}(E)\big)$$

4.4. **Main theorem.** Considering a Hilbert submanifold $\mathcal{O}_{\nabla,\epsilon} \subset \widehat{A}(E,\mathbb{L})_l \subset A(E,\mathbb{L})_l$ for ϵ small enough,

Theorem 4.9. *The map*

$$\Psi_{\nabla} : H\text{-}Gau(E)^r_{l+1} \times \mathcal{O}_{\nabla,\epsilon} \to \widehat{A}(E,\mathbb{L})_l$$

given by

$$\Psi_{\nabla}(\psi, \nabla + \alpha) = \psi \odot (\nabla + \alpha)$$

is smooth and a local diffeomorphism.

Proof. The differential map

$$d(\Psi_{\nabla})_{(\phi_E,\nabla)} : \big[\mathcal{A}^0_{\mathbb{L}}\big(X, \mathrm{End}_{\phi_E}(E)\big)_l\big]^0 \oplus \ker\big((d^{\nabla})^{\star}\big) \to \mathcal{A}^1_{\mathbb{L}}\big(X, \mathrm{End}_{\phi_E}(E)\big)_l$$

given by

$$d(\Psi_{\nabla})_{(\phi_E,\nabla)}(A,B) = d^{\nabla}(A) + B$$

Using the L^2-orthogonal decomposition Equation (4.2), the differential map $d(\Psi_{\nabla})_{(\phi_E,\nabla)}$ is surjective, it is injective as ∇ is an irreducible $\mathbb{L}$-connection. Using Banach open mapping theorem, the differential $d(\Psi_\nabla)_{(\phi_E,\nabla)}$ is isomorphism and using inverse function theorem, the map Ψ_∇ is local diffeomorphism around the point $(\phi_E,\nabla) \in \text{H-Gau}(E)^r_{l+1} \times \mathcal{O}_{\nabla,\epsilon}$. $\square$

Theorem 4.10. *For $\epsilon > 0$ small enough, the map*

$$\widehat{p}_{\nabla,\epsilon} = \widehat{p}_{|_{\mathcal{O}_{\nabla,\epsilon}}} : \mathcal{O}_{\nabla,\epsilon} \to \widehat{B}(E,\mathbb{L})$$

is injective. If $\mathcal{U}_{\nabla,\epsilon} \subset \widehat{B}(E,\mathbb{L})$ be the image, then the map

$$\widehat{p}_{\nabla,\epsilon} : \mathcal{O}_{\nabla,\epsilon} \to \mathcal{U}_{\nabla,\epsilon}$$

is homeomorphism.

Proof. For any two $\mathbb{L}$-connections in $\mathcal{O}_{\nabla,\epsilon}$, say $\nabla+\alpha_1, \nabla+\alpha_2$, where $\alpha_i(i=1,2) \in \mathcal{A}^1_{\mathbb{L}}\big(X, \mathrm{End}_{\phi_E}(E)\big)_l$ such that there is a $\psi \in \mathcal{A}^0_{\mathbb{L}}\big(X, \mathrm{End}_{\phi_E}(E)\big)_{l+1}$ satisfying

$$\nabla + \alpha_2 = \psi \odot (\nabla + \alpha_2) \tag{4.5}$$

then we need to show $\alpha_1 = \alpha_2$.

Using the L^2-orthogonal decomposition

$$\mathcal{A}^0_{\mathbb{L}}\big(X, \mathrm{End}(E)\big)_{l+1} = \ker(d^\nabla) \oplus \ker(d^\nabla)^\perp$$

and the lemma 4.7, let $\psi = c\phi_E + \psi_0$, where $\psi_0 \in \ker(d^\nabla)^\perp \cap \mathcal{A}^0_{\mathbb{L}}\big(X, \mathrm{End}_{\phi_E}(E)\big)_{l+1}$ and $c \in K$. Using Banach open mapping theorem, the map

$$d^\nabla : \ker(d^\nabla)^\perp \to \mathrm{im}(d^\nabla)$$

is an isomorphism between Hilbert spaces; hence a lower bounded operator. For $\psi_0 \in \mathrm{Gau}(E)_{l+1}$, let $\rho \in \mathbb{R}$ be a positive real number such that

$$\rho\|\psi_0\|_{L^2_{l+1}} \le \|d^\nabla(\psi_0)\|_{L^2_l} = \|d^\nabla(\psi)\|_{L^2_l} = \|(\phi^\dagger_{\mathbb{L}})^{-1} \otimes \phi_E^{-1} \circ d^\nabla(\psi) \circ \phi_E\|_{L^2_l} \quad \big(\text{by def. of metric}\big)$$

Using Remark 3.4, for the Equation (4.5), we have

$$\rho\|\psi_0\|_{L^2_{l+1}} \le \|d^\nabla(\psi)\|_{L^2_l} = \|\mathrm{id}_{\mathbb{L}^\star} \otimes (\psi \circ \phi_E^{-1}) \circ \alpha_2 - \alpha_1 \circ \phi_E^{-1} \circ \psi\|_{L^2_l}$$

Using triangle inequality

$$\begin{aligned}
\rho\|\psi_0\|_{L^2_{l+1}} &\le \|\mathrm{id}_{\mathbb{L}^\star} \otimes (\psi \circ \phi_E^{-1}) \circ \alpha_2\|_{L^2_l} + \|\alpha_1 \circ \phi_E^{-1} \circ \psi\|_{L^2_l} \\
&\le \rho_2\|\phi_E^{-1}\|_{L^2_{l+1}}\|\psi\|_{L^2_{l+1}}\|\alpha_2\|_{L^2_l} + \rho_1\|\phi_E^{-1}\|_{L^2_{l+1}}\|\psi\|_{L^2_{l+1}}\|\alpha_1\|_{L^2_l} \\
&\le 2\tilde{\rho}\epsilon\|\phi_E^{-1}\|_{L^2_{l+1}}\|\psi\|_{L^2_{l+1}} \quad \big(\tilde{\rho} = \max.(\rho_1,\rho_2); \|\alpha_1\|, \|\alpha_2\| < \epsilon\big) \\
&\le 2\tilde{\rho}\epsilon\|\phi_E^{-1}\|_{L^2_{l+1}}\big(|c|\|\phi_E\|_{L^2_{l+1}} + \|\psi_0\|_{L^2_{l+1}}\big)
\end{aligned}$$

we have

$$\|\psi_0\|_{L^2_{l+1}} \le \frac{2|c|\tilde{\rho}\epsilon\|\phi_E\|_{L^2_{l+1}}\|\phi_E^{-1}\|_{L^2_{l+1}}}{\rho - 2\tilde{\rho}\epsilon\|\phi_E^{-1}\|_{L^2_{l+1}}}$$

for $\epsilon < \frac{\rho}{2\tilde{\rho}\|\phi_E^{-1}\|_{L^2_{l+1}}}$. Note that $c \neq 0$ (otherwise $\psi_0 = 0$ implies ψ is trivial) and

$$\|c^{-1}\psi - \phi_E\| = \frac{1}{|c|}\|\psi_0\|_{L^2_{l+1}} \le \frac{2\tilde{\rho}\epsilon\|\phi_E\|_{L^2_{l+1}}\|\phi_E^{-1}\|_{L^2_{l+1}}}{\rho - 2\tilde{\rho}\epsilon\|\phi_E^{-1}\|_{L^2_{l+1}}}$$

For $\epsilon > 0$ small enough, we can assume ψ is near ϕ_E in H-Gau$(E)^r_{l+1}$, using injectivity of Ψ_∇ in above theorem, we have $\alpha_1 = \alpha_2$.

Hence the map, $\widehat{p}_{\nabla,\epsilon} : \mathcal{O}_{\nabla,\epsilon} \to \mathcal{U}_{\nabla,\epsilon}$ is bijective continuous map, to prove homeomorphism we need to check if $(\widehat{p}_{\nabla,\epsilon})$ is an open map, which is true because action of H-Gau$(E)_{l+1}$ on $\widehat{A}(E,\mathbb{L})_l$ is continuous and for any open subset $U \subset \mathcal{O}_{\nabla,\epsilon}$, we have

$$(\widehat{p}_{\nabla,\epsilon})^{-1}\big((\widehat{p}_{\nabla,\epsilon})(U)\big) = \bigcup^{\cdot}_{g\in \text{H-Gau}(E)_{l+1}} g \odot U$$

which is open in $\widehat{A}(E,\mathbb{L})_l$, implies $(\widehat{p}_{\nabla,\epsilon})(U)$ is open in $\mathcal{U}_{\nabla,\epsilon}$ for a given open subset $U \in \mathcal{O}_{\nabla,\epsilon}$. □

Theorem 4.11 (Main theorem 1)**.** *The map*

$$\Psi_\nabla : \text{H-Gau}(E)^r_{l+1} \times \mathcal{O}_{\nabla,\epsilon} \to (\widehat{p})^{-1}(\mathcal{U}_{\nabla,\epsilon}) \subset \widehat{A}(E,\mathbb{L})_l$$
$$(g,\nabla) \mapsto g \odot \nabla$$

is diffeomorphism. In other words, the fibre bundle $\widehat{p} : \widehat{A}(E,\mathbb{L}) \to \widehat{B}(A,\mathbb{L})_l$ is a principal H-Gau$(E)^r_{l+1}$*-bundle with trivialization $\{(\mathcal{U}_{\nabla_l,\epsilon_l}, \Psi_{\nabla_l})\}_{l\in I}$.*

Proof. Note that the image of $\mathcal{O}_{\nabla,\epsilon}$ under $\widehat{p}_{\nabla,\epsilon}$ is the H-Gau$^r_{l+1}(E)$-equivalence classes of connections in $\mathcal{O}_{\nabla,\epsilon}$ implies Ψ_∇ is surjective.

The action of H-Gau$^r_{l+1}(E)$ on $\widehat{A}(E,\mathbb{L})$ is a free action and using injectivity of $\widehat{p}_{\nabla,\epsilon}$, it is easy to verify Ψ_∇ is injective, hence bijective.

The map

$$\Psi_\nabla : \textit{H-Gau}(E)^r_{l+1} \times \mathcal{O}_{\nabla,\epsilon} \to (\widehat{p})^{-1}(\mathcal{U}_{\nabla,\epsilon})$$

is local diffeomorphism and bijective implies, it is diffeomorphism. □

Theorem 4.12 (Main theorem 2)**.** *The space $\widehat{B}(E,\mathbb{L})_l$ is a smooth Hilbert manifold.*

Proof. Consider the composition of maps

$$g_\nabla : (\widehat{p})^{-1}(\mathcal{U}_{\nabla,\epsilon}) \xrightarrow{\psi_\nabla} \text{H-Gau}(E)^r_{l+1}(X) \times \mathcal{O}_{\nabla,\epsilon} \xrightarrow{pr_1} \text{H-Gau}(E)^r_{l+1}$$

where $pr_1 : \text{H-Gau}(E)^r_{l+1}(X) \times \mathcal{O}_{\nabla,\epsilon} \to \text{H-Gau}(E)^r_{l+1}(X)$ is first projection map. There is a canonical co-ordinate chart $(\mathcal{U}_{\nabla,\epsilon}, \sigma_\nabla)$ around $\nabla \in \widehat{B}(E,\mathbb{L})_l$ with co-ordinate map

$$\sigma_\nabla : \mathcal{U}_{\nabla,\epsilon} \to \mathcal{O}_{\nabla,\epsilon} \qquad (\sigma_\nabla(\nabla') = [g_\nabla(\nabla')]^{-1} \odot \nabla'$$

It is easy to verify that $\sigma_\nabla \circ \widehat{p}_{\nabla,\epsilon} = \mathrm{id}_{\mathcal{O}_{\nabla,\epsilon}}$ and $\widehat{p}_{\nabla,\epsilon} \circ \sigma_\nabla = \mathrm{id}_{\mathcal{U}_{\nabla,\epsilon}}$.

Take two charts $(\mathcal{U}_{\nabla_l,\epsilon_l}, \sigma_{\nabla_l})$ $(l = 1,2)$ and $\nabla = \nabla_1 + \alpha_1 \in \mathcal{U}_{\nabla_1,\epsilon_1}$ such that $\widehat{p}_{\nabla_1,\epsilon_1}(\nabla) \in \mathcal{U}_{\nabla_1,\epsilon_1} \cap \mathcal{U}_{\nabla_2,\epsilon_2}$, we have

$$\sigma_{\nabla_2} \circ \sigma^{-1}_{\nabla_1}(\nabla_1 + \alpha_1) = \sigma_{\nabla_2}\big(\widehat{p}_{\nabla_1,\epsilon_1}(\nabla_1 + \alpha_1)\big) = [g_{\nabla_2}(\nabla_1 + \alpha_1)]^{-1} \odot (\nabla_1 + \alpha_1)$$

Since the Sobolev gauge action is a smooth map, the transition map is smooth and the space $\widehat{B}(E,\mathbb{L})_l$ has smooth Hilbert manifold structure. □

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