Bose-Einstein Condensation, Fluctuations and Spontaneous Symmetry Breaking

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The realisation of Bose-Einstein condensation under grand-canonical conditions has provided the experimental evidence for the simultaneous occurrence of macroscopic fluctuations and phase coherence of the condensate. The observation of these two features, against a consolidated tradition which wants the fluctuations to be pathological (grand-canonical catastrophe) and incompatible with spontaneous symmetry braking, calls for a comprehensive rethinking of the approach to the problem. In this paper we consider the uniform ideal gas in a box and we present an alternative conceptual framework. We show that the usually-employed Bogoliubov quasi-average construction fails to reproduce the broken-symmetry state. The observed features are accounted for by a different pattern of spontaneous symmetry breaking, characterised by condensation of fluctuations and long-range correlations of the order parameter.

A century after its discovery, and even in the simplest setting of the ideal gas in the grand-canonical ensemble (GCE), Bose-Einstein condensation (BEC) remains not fully understood. The derivation is exact and by now standard textbook material. Yet, we are warned that what we learn should not be entirely trusted. The reason is that condensation is accompanied by macroscopic fluctuations, an outcome disturbing to the point of being tellingly dubbed as the grand-canonical catastrophe (\mathcal{GCC}) [1–5]. Some of the motivations for the refusal to accept a seemingly straightforward result are difficult to address, being grounded on vague notions like the extravagance of such large fluctuations. For a rich account of this sort of criticism see the recent review by Kruk et al. and references therein [6].

On a different level is the objection based on the alleged incompatibility of the condensate fluctuations with spontaneous symmetry breaking (SSB) [7, 8] and this needs to be addressed. The connection between BEC and SSB has been established in the strong sense of an equivalence [9, 10]. However, the chain of reasoning rests on the key assumption that the broken-symmetry state can be faithfully reproduced via the Bogoliubov quasi-average construction [11]. The problem with this is that the quasi-average scheme implies the exactness of the Bogoliubov c-number substitution which, in turn, excludes that there might be fluctuations of the condensate. Hence, BEC, SSB and \mathcal{GCC} could not coexist.

The distrust about the so-called textbook treatment

of the ideal gas has largely remained in place even after the landmark experimental realisation of BEC in a photon system has demonstrated both that the \mathcal{GCC} is a real phenomenon observable in the lab [12, 13] and that it is compatible with SSB [14]. Of course the ideal limit cannot be expected to account for all what goes on in the optical cell. Indeed, there are modifications which go from mild to severe depending on the range of parameters used, as highlighted in studies of the effects of weak interactions [15, 16]. But, if what is observed can be regarded as a tunable departure from the ideal behaviour, the ideal behaviour cannot be nonsense and it goes by itself that the understanding of the observed phenomenology requires the preliminary understanding of the ideal limit.

It is then apparent that the usual approach to the BEC-SSB connection needs some overhauling. Reconsidering the whole problem from scratch and building on ideas proposed in previous works [17, 18], in this paper we take the \mathcal{GCC} at face value and we argue that it is perfectly compatible with SSB by tracing back the source of the problem to the uncritical use of the quasiaverage procedure. The point is that, when applied in the GCE, the external-field perturbation involved in the quasi average is singular and produces an SSB pattern which is not equivalent to the unperturbed one. In the latter, which is the physically-meaningful one and that we extract from the P-representation of the density matrix, all the pieces fall into place forming a coherent picture. The main features are: i) the condensed phase is characterised by condensation of the fluctuations of the amplitude mode, ii) the \mathcal{GCC} and the non equivalence of ensembles, rather than a pathology undermining the pillars of statistical mechanics, are what are supposed to be: the manifestation of long-range correlations [19–21],

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iii) the constraint enforced by fixing the average density amounts to a mean-field approximation, whose instability to external-field perturbations is well known from other instances of the same approximation. In this respect, the photons experiment is all the more important because provides the so-far-unique experimental platform realising a theoretical scheme which in other contexts, such as magnetic systems, is elusive if not impossible to observe.

We are concerned with a system of non-interacting bosons of mass m described by the second-quantized Hamiltonian

$$\hat{\mathcal{H}} = \int d\mathbf{r} \ \hat{\psi}^{\dagger}(\mathbf{r}) \left[-\frac{\hbar^2}{2m} \nabla^2 + U(\mathbf{r}) \right] \hat{\psi}(\mathbf{r}). \tag{1}$$

We shall first work out the problem in the simpler case of the gas in a 3d box of volume $V=L^3$, setting the confining potential $U(\mathbf{r})\equiv 0$. The experimentally relevant case of the gas in the harmonic trap will be briefly discussed at the end. Imposing periodic boundary conditions, the energy eigenfunctions are the plane waves $\phi_{\mathbf{k}}(\mathbf{r}) = \frac{1}{\sqrt{V}} e^{i\mathbf{k}\cdot\mathbf{r}}$ with wave vectors $k_{x,y,z} = \frac{2\pi}{L}(0,\pm 1,..)$ and eigenvalues $\epsilon_{\mathbf{k}} = (\hbar^2 k)^2/2m$. Expanding the field operator $\hat{\psi}(\mathbf{r}) = \sum_{\mathbf{k}} \hat{a}_{\mathbf{k}} \phi_{\mathbf{k}}(\mathbf{r})$, where $\hat{a}_{\mathbf{k}}$ and $\hat{a}_{\mathbf{k}}^{\dagger}$ are the annihilation and creation operators in the one-particle state $|\mathbf{k}\rangle$, the Hamiltonian goes into the diagonal form $\hat{\mathcal{H}} = \sum_{\mathbf{k}} \hat{\mathcal{H}}_{\mathbf{k}}$, with $\hat{\mathcal{H}}_{\mathbf{k}} = \epsilon_{\mathbf{k}} \hat{a}_{\mathbf{k}}^{\dagger} \hat{a}_{\mathbf{k}}$. The Gibbs density matrix factorises $\hat{\mathcal{D}} = \prod_{\mathbf{k}} \hat{\mathcal{D}}_{\mathbf{k}}$, with

$$\hat{\mathcal{D}}_{\mathbf{k}} = \frac{1}{\mathcal{Z}_{\mathbf{k}}} \exp\{-(\beta \hat{\mathcal{H}}_{\mathbf{k}} + \alpha \hat{n}_{\mathbf{k}})\}, \quad \alpha = -\beta \mu, \quad (2)$$

where β is the inverse temperature, μ is the chemical potential, $\hat{n}_{\mathbf{k}} = \hat{a}_{\mathbf{k}}^{\dagger} \hat{a}_{\mathbf{k}}$ and $\mathcal{Z}_{\mathbf{k}}$ is obtained from $\operatorname{Tr} \hat{\mathcal{D}}_{\mathbf{k}} = 1$.

Denoting averages by angular brackets, a straightforward calculation of the **k**-mode occupation density $\rho_{\mathbf{k}} = \frac{1}{V} \langle \hat{n}_{\mathbf{k}} \rangle$ and of the mean-square fluctuations $\delta^2 \rho_{\mathbf{k}} = \frac{1}{V^2} [\langle \hat{n}_{\mathbf{k}}^2 \rangle - \langle \hat{n}_{\mathbf{k}} \rangle^2]$, yields

$$\rho_{\mathbf{k}} = \frac{1}{V[e^{\beta \epsilon_{\mathbf{k}} + \alpha} - 1]}, \quad \delta^2 \rho_{\mathbf{k}} = \rho_{\mathbf{k}}^2 + \frac{1}{V} \rho_{\mathbf{k}}. \tag{3}$$

Upon varying the control parameters (β, α, V) the behaviour of these quantities is smooth and featureless, as it should be in a free theory. So, in order to get not-trivial results some kind of interaction must be introduced. In the GCE this is covertly done by solving the equation for the total density $\rho = \sum_{\mathbf{k}} \rho_{\mathbf{k}}(\alpha)$ with respect to α and by substituting the solution $\alpha(\beta, \rho, V)$ wherever α appears (see for example Refs. [1, 22]). Trading α with ρ as a control parameter is the crucial step for the appearance of the BEC singularity. If the actual density (not the averaged one) had been kept fixed (canonical ensemble), an obvious coupling would have been introduced through the sum rule $\rho = \sum_{\mathbf{k}} \rho_{\mathbf{k}}$. In the GCE, instead, by fixing the average density the constraint is softened leaving the structure formally non-interacting. Nevertheless, a coupling is still implemented via the selfconsistency relation $\rho = \sum_{\mathbf{k}} \rho_{\mathbf{k}}(\rho)$, which is characteristic of the mean-field approximations. Particularly relevant in the present context is the connection with the mean spherical model [18, 23–28].

Separating the ground-state contribution $\rho_{\mathbf{0}}$ from the excited states $\rho' = \sum_{\mathbf{k} \neq 0} \rho_{\mathbf{k}}$ and replacing this sum with an integral, the equation for α reads

$$\rho = \rho_0(\alpha) + \lambda^{-3} g_{3/2}(\alpha), \tag{4}$$

where $g_{3/2}(\alpha)$ is the Bose function [1] and $\lambda = \sqrt{2\pi\beta\hbar^2/m}$ is the de Broglie thermal wave length. Solving, substituting the solution $\alpha(\beta, \rho, V)$ into Eq. (3) and taking the $V \to \infty$ limit, one finds BEC and, inevitably, the concomitant \mathcal{GCC}

$$\rho_{\mathbf{0}}(\rho) = \begin{cases} 0, & \text{for } \Delta \rho \le 0, \\ \Delta \rho, & \text{for } \Delta \rho > 0, \end{cases}$$
 (5)

$$\delta^{2} \rho_{\mathbf{0}}(\rho) = \begin{cases} 0, & \text{for } \Delta \rho \leq 0, \\ \Delta \rho^{2}, & \text{for } \Delta \rho > 0, \end{cases}$$
 (6)

where $\Delta \rho = \rho - \rho_c$, $\rho_c = \lambda^{-3} \zeta(3/2)$ is the critical density, and ζ is the Riemann zeta function. In the standard treatment of BEC the transition is driven by β , keeping ρ fixed. Here, by analogy with the photons' experiment, ρ is used as the driving parameter keeping β fixed. The normal phase ($\Delta \rho < 0$) and the condensed phase ($\Delta \rho > 0$) are separated by a critical point (see below) at $\Delta \rho = 0$.

Given the simplicity of the algebra involved, it is hard to see why the result for $\delta^2 \rho_0$ ought to be discarded. Nonetheless, it has been challenged invoking SSB [8]. Restricting, for simplicity, the attention to the **0**-mode, the statement is that in the condensed phase and in the thermodynamic limit the invariance of the density matrix $\hat{\mathcal{D}}_0$ under the gauge transformation $\hat{a}_0 \mapsto e^{i\theta} \hat{a}_0$ is spontaneously broken. Then, $\hat{\mathcal{D}}_0$ must be regarded as the mixture of dynamically-disjoint ergodic components [29, 30]

$$\hat{\mathcal{D}}_{\mathbf{0}} = \frac{1}{2\pi} \int_{0}^{2\pi} d\theta \, \hat{\mathcal{D}}_{\mathbf{0},\theta} \,, \tag{7}$$

each of which with a definite phase θ . Only one of these represents the physically-meaningful state, since the real system in the lab remains confined in one component [14]. Consequently, the meaningful averages are those taken with $\hat{\mathcal{D}}_{0,\theta}$ whose form, at this stage, is not known and needs to be determined.

Here, there are three distinct issues: first, establish if SSB takes place, then, if so, find out what the form of $\hat{\mathcal{D}}_{0,\theta}$ is and, finally, check if on averaging with $\hat{\mathcal{D}}_{0,\theta}$ the \mathcal{GCC} nuisance is disposed of. Actually, the last one is the easiest to deal with, since symmetrical operators, like \hat{n}_0 and \hat{n}_0^2 , are insensitive to the phase. Therefore, it doesn't matter whether the average is taken with the symmetric matrix $\hat{\mathcal{D}}_0$ or with one of the components $\hat{\mathcal{D}}_{0,\theta}$. The outcome is the same. Consequently, the results (5,6) for ρ_0

and $\delta^2 \rho_0$ hold also in the broken-symmetry state, whatever this might be, implying that the \mathcal{GCC} has nothing to do with SSB. Should it disappear, as indeed it does with the Bogoliubov procedure, it can only mean that the quasi-average construction fails to reproduce $\hat{\mathcal{D}}_{0,\theta}$.

Let us now see how this method works, since it is widely used to address the first and second issue. The symmetry is explicitly broken by applying a complex external field $\nu = |\nu|e^{i\theta_{\nu}}$, which couples to the order parameter $\hat{\psi}_{\mathbf{0}} = \hat{a}_{\mathbf{0}}\phi_{\mathbf{0}}(\mathbf{r}) = \frac{1}{\sqrt{V}}\hat{a}_{\mathbf{0}}$. There are no changes for $\mathbf{k} \neq \mathbf{0}$, while for $\mathbf{k} = \mathbf{0}$ the Hamiltonian and density matrix go into $\hat{\mathcal{H}}_{\mathbf{0}}^{(\nu)} = \hat{\mathcal{H}}_{\mathbf{0}} - V(\nu\hat{\psi}_{\mathbf{0}}^{\dagger} + \nu^*\hat{\psi}_{\mathbf{0}})$ and $\hat{\mathcal{D}}_{\mathbf{0}}^{(\nu)} = \frac{1}{Z_{\mathbf{0}}}e^{-\beta\hat{\mathcal{H}}_{\mathbf{0}}^{(\nu)} - \alpha\hat{n}_{\mathbf{0}}}$. Clearly, if $\nu \to 0$ the unbiased state $\hat{\mathcal{D}}_0$ is recovered. However, the idea is that if the removal of ν is done after taking the $V \to \infty$ limit and if SSB has occurred, then the $\nu \to 0$ limit of $\hat{\mathcal{D}}_{\mathbf{0}}^{(\nu)}$ remains locked into the particular component $\hat{\mathcal{D}}_{\mathbf{0},\theta_{\nu}}$ selected by the phase of the biasing field. The occurrence of SSB is then revealed by a non-null value of the quasi-averaged order parameter $\langle \hat{\psi}_{\mathbf{0}} \rangle_{\theta_{\nu}}^{(\text{qa})} = \lim_{\nu \to 0} \lim_{V \to \infty} \langle \hat{\psi}_{\mathbf{0}} \rangle^{(\nu)}$. The order of limits is crucial. The reverse order produces the regular average [11] $\langle \hat{\psi}_{\mathbf{0}} \rangle^{(\text{ra})} = \lim_{V \to \infty} \lim_{\nu \to 0} \langle \hat{\psi}_{\mathbf{0}} \rangle^{(\nu)}$, which vanishes by symmetry. In addition to checking on SSB, the quasi-average method is usually also tacitly extended to the more delicate task of constructing the ergodic-component $\hat{\mathcal{D}}_{\mathbf{0},\theta}$ entering the decomposition (7), by assuming that one can make the identification

$$\langle \cdot \rangle_{\theta}^{(\text{ra})} \stackrel{?}{=} \langle \cdot \rangle_{\theta}^{(\text{qa})},$$
 (8)

where the left-hand side stands for the regular average taken in the (so far unknown) ergodic component with the phase θ . The catch, here, is that the above identification holds only if the external field's perturbation is not singular, which, as we shall see, is not the case with the ideal gas in the GCE.

The issue is settled by resorting to the Glauber-Sudarshan P-representation [31, 32] of the biased density matrix $\hat{\mathcal{D}}_{\mathbf{0}}^{(\nu)} = \int d^2z \, P^{(\nu)}(z) \, |z\rangle\langle z|$, where the coherent state $|z\rangle$ is the eigenvector of the volume-rescaled annihilation operator $\hat{\psi}_{\mathbf{0}}$, with the complex eigenvalue $z = |z|e^{i\theta}$. Averages of normally-ordered products are given by [31]

$$\langle \hat{\psi}_{\mathbf{0}}^{\dagger n} \hat{\psi}_{\mathbf{0}}^{m} \rangle^{(\nu)} = \int d^2 z \, P^{(\nu)}(z) \, z^{*n} z^m. \tag{9}$$

The weight function reads [33]

$$P^{(\nu)}(z) = \frac{1}{\pi\sigma} \exp\left\{-\frac{1}{\sigma}(z^* - \gamma^*)(z - \gamma)\right\},$$
 (10)

where

$$\sigma = [V(e^{\alpha} - 1)]^{-1}, \quad \gamma = \beta \nu / \alpha, \tag{11}$$

from which follows

$$\langle \hat{\psi}_{\mathbf{0}} \rangle^{(\nu)} = \gamma, \quad \rho_{\mathbf{0}}^{(\nu)} = |\langle \hat{\psi}_{\mathbf{0}} \rangle^{(\nu)}|^2 + \sigma.$$
 (12)

Again, it is necessary to first solve for α as a function of (β, ρ, V, ν) . This is done in the Appendix obtaining a result which depends on the order of limits, thus revealing the singular nature of the perturbation. Inserting the outcome into Eq. (10) and using Eq. (11), in the quasi-average case we find

$$\lim_{\nu \to 0} \lim_{V \to \infty} P^{(\nu)}(z) = P_{\theta_{\nu}}^{(\text{qa})}(z) = \begin{cases} \delta^{2}(z), & \text{for } \Delta \rho \leq 0, \\ \delta(\theta - \theta_{\nu}) \, \delta(|z| - \sqrt{\Delta \rho}), \\ \text{for } \Delta \rho > 0. \end{cases}$$
(13)

Hence, in the condensed phase the density matrix reduces to the projector on the single coherent state selected by the phase of the vanishing external field [34]

$$\hat{\mathcal{D}}_{\mathbf{0},\theta}^{(\mathrm{qa})} = |e^{i\theta_{\nu}} \sqrt{\Delta \rho}\rangle \langle e^{i\theta_{\nu}} \sqrt{\Delta \rho}|. \tag{14}$$

Conversely, the regular-average procedure yields

$$\lim_{V \to \infty} \lim_{\nu \to 0} P^{(\nu)}(z) = P^{(ra)}(z) = \begin{cases} \delta^2(z), & \text{for } \Delta \rho \le 0, \\ \frac{1}{\pi} \frac{1}{\Delta \rho} \exp\left\{-\frac{|z|^2}{\Delta \rho}\right\}, \\ & \text{for } \Delta \rho > 0, \end{cases}$$
(15)

i.e. a distribution uniform over the phases and spreadout over the amplitudes. Therefore, the P-representation of the broken-symmetry state entering the decomposition (7) can be readily obtained by omitting the phase integration

$$\hat{\mathcal{D}}_{\mathbf{0},\theta}^{(\mathrm{ra})} = 2 \int_0^\infty d|z| \, |z| \, \frac{1}{\Delta \rho} \exp\left\{-\frac{|z|^2}{\Delta \rho}\right\} |e^{i\theta}|z| \rangle \langle e^{i\theta}|z||. \tag{16}$$

Comparing Eqs. (14) and (16), it is evident that the physical broken-symmetry state cannot be constructed by means of the quasi-average and that Eq. (8) does not hold. Carrying out the calculation of Eq. (9), in the two cases we have

$$\langle \hat{\psi}_{\mathbf{0}}^{\dagger n} \hat{\psi}_{\mathbf{0}}^{m} \rangle_{\theta}^{(\text{qa})} = e^{i(m-n)\theta} \, \Delta \rho^{(n+m)/2}, \tag{17}$$
$$\langle \hat{\psi}_{\mathbf{0}}^{\dagger n} \hat{\psi}_{\mathbf{0}}^{m} \rangle_{\theta}^{(\text{ra})} = \Gamma((n+m+2)/2) \, e^{i(m-n)\theta} \, \Delta \rho^{(n+m)/2}, \tag{18}$$

where Γ is the Euler gamma function. In particular, for (n=0,m=1) we get

$$\langle \hat{\psi}_{\mathbf{0}} \rangle_{\theta_{\nu}}^{(\text{qa})} = e^{i\theta_{\nu}} \sqrt{\Delta \rho},$$
 (19)

$$\langle \hat{\psi}_{\mathbf{0}} \rangle_{\theta}^{(\text{ra})} = \Gamma(3/2)e^{i\theta}\sqrt{\Delta\rho},$$
 (20)

with $\Gamma(3/2) = \sqrt{\pi}/2 < 1$. Hence, the c-number substitution $\hat{\psi}_{\mathbf{0}} \mapsto c = \langle \hat{\psi}_{\mathbf{0}} \rangle_{\theta_{\nu}}^{(\mathrm{qa})}$ is exact in the quasi-average case $\langle \hat{\psi}_{\mathbf{0}}^{\dagger n} \hat{\psi}_{\mathbf{0}}^{m} \rangle_{\theta_{\nu}}^{(\mathrm{qa})} = \langle \hat{\psi}_{\mathbf{0}}^{\dagger} \rangle_{\theta_{\nu}}^{(\mathrm{qa})n} \langle \hat{\psi}_{\mathbf{0}} \rangle_{\theta_{\nu}}^{(\mathrm{qa})m}$, but does not hold in the regular-average case because of the gamma functions. It is important to note that the δ -functions in Eq. (13) match the P-representation of the density matrix in the

canonical ensemble [18] and, therefore, that the quasi-average ensemble is equivalent to the canonical one implying, in turn, that the lack of equivalence between the quasi-average and the regular-average is the same as the one between the canonical and grand-canonical ensembles. Moreover, the instability under the external perturbation is a feature which further underscores the mean-field character of the formal apparatus, since it is well known to occur in the already mentioned mean-spherical model [26, 27] and in the large-N limit [35, 36], where $V \to \infty$ and $h \to 0$ do not commute, h being the magnetic field.

A major consequence of this discrepancy concerns the nature of the condensed phase, specifically the mechanism of formation of the condensate at the underlying level of the field $\hat{\psi}_{\mathbf{0}}$. Setting (n=1,m=1), in both cases we recover the same result (5) $\rho_{\mathbf{0}} = \langle \hat{\psi}_{\mathbf{0}}^{\dagger} \hat{\psi}_{\mathbf{0}} \rangle = \Delta \rho$. Since in the quasi-average case this is achieved by developing the appropriate anomalous average $\rho_{\mathbf{0}}^{(\mathrm{qa})} = \langle \hat{\psi}_{\mathbf{0}}^{\dagger} \rangle_{\mathbf{0}}^{(\mathrm{qa})} \langle \hat{\psi}_{\mathbf{0}} \rangle_{\theta}^{(\mathrm{qa})}$, BEC is an instance of ordering transition, like the ferromagnetic transition. Conversely, from Eq. (20) follows

$$\rho_{\mathbf{0}}^{(\mathrm{ra})} = \Delta \rho = \langle \hat{\psi}_{\mathbf{0}}^{\dagger} \rangle_{\theta}^{(\mathrm{ra})} \langle \hat{\psi}_{\mathbf{0}} \rangle_{\theta}^{(\mathrm{ra})} + \Phi^{(\mathrm{ra})}, \tag{21}$$

where, next to the ordering piece, there appears also the mean-square fluctuation of $\hat{\psi}_{\mathbf{0}}$

$$\Phi^{(\mathrm{ra})} = \langle \delta \hat{\psi}_{\mathbf{0}}^{\dagger} \delta \hat{\psi}_{\mathbf{0}} \rangle_{\theta}^{(\mathrm{ra})} = [1 - \pi/4] \Delta \rho, \quad \delta \hat{\psi}_{\mathbf{0}} = \hat{\psi}_{\mathbf{0}} - \langle \hat{\psi}_{\mathbf{0}} \rangle_{\theta}^{(\mathrm{ra})}.$$
(22)

Therefore, ordering is not enough to build up the condensate. Since the missing piece must be supplemented by the fluctuations of the amplitude mode, in this case BEC is a different kind of transition involving condensation of fluctuations, in the sense that fluctuations of macroscopic size are not the cumulative effect contributed by many degrees of freedom, but are produced by a single degree of freedom [17]. This is a phenomenon of general occurrence, observed in a wide range of systems [35, 37– 41]. Then, it must be stressed that BEC in the GCE involves condensation in two different ways, which must be kept distinct: next to the usual condensation in the sense of a macroscopic number of bosons accumulating in the ground state, there is also condensation of fluctuations at the underlying microscopic level of the order parameter $\hat{\psi}_{\mathbf{0}}$, which is indispensable to meet the required size of the condensate ρ_0 and whose mesoscopic manifestation is in the \mathcal{GCC} .

However, this is not yet the root cause of \mathcal{GCC} . Abiding to the rule of thumb that wants large fluctuations due to long-range correlations, we must search for these correlations. Separating the ground-state from the excited-states contribution $\hat{\psi}(\mathbf{r}) = \hat{\psi}_{\mathbf{0}}(\mathbf{r}) + \hat{\psi}'(\mathbf{r})$, the first-order connected correlation function $G_{\mathbf{c}}(\mathbf{r}-\mathbf{r}') = \langle \hat{\psi}^{\dagger}(\mathbf{r})\hat{\psi}(\mathbf{r}')\rangle - \langle \hat{\psi}^{\dagger}(\mathbf{r})\rangle \langle \hat{\psi}(\mathbf{r}')\rangle$ splits into the sum $G_{\mathbf{c}}(\mathbf{r}-\mathbf{r}') = G_{\mathbf{c},\mathbf{0}}(\mathbf{r}-\mathbf{r}') + G'_{\mathbf{c}}(\mathbf{r}-\mathbf{r}')$. The first reads $G_{\mathbf{c},\mathbf{0}}(\mathbf{r}-\mathbf{r}') = \langle \delta \hat{\psi}^{\dagger}_{\mathbf{0}} \delta \hat{\psi}_{\mathbf{0}} \rangle$, whose form in the condensed phase depends on how the

average is taken

$$G_{c,\mathbf{0}}(\mathbf{r} - \mathbf{r}') = \begin{cases} 0, & \text{quasi-average,} \\ \Phi^{(ra)}, & \text{regular-average.} \end{cases}$$
 (23)

The second does not depend on the way of averaging and for $|\mathbf{r} - \mathbf{r}'|$ and V large enough is given by [1, 42] $G'_{\mathbf{c}}(\mathbf{r} - \mathbf{r}') = \frac{\exp\{-|\mathbf{r} - \mathbf{r}'|/\xi\}}{\lambda^2|\mathbf{r} - \mathbf{r}'|}$, with $\xi = \frac{1}{\lambda^2|\Delta\rho|}$ for $\Delta\rho < 0$ and $\xi = \infty$ for $\Delta\rho \geq 0$. Hence, in the normal phase there is one fluctuating field $\hat{\psi}(\mathbf{r})$ with a finite correlation length, which diverges upon approaching the critical point at $\Delta\rho = 0$ [42, 43]. In the condensed phase both fields $\delta\hat{\psi}_{\mathbf{0}}(\mathbf{r})$ and $\hat{\psi}'(\mathbf{r})$ are critical but belong to different universality classes, as revealed by the power-law decays

$$G_{c,\mathbf{0}}(\mathbf{r} - \mathbf{r}') = \frac{\Phi^{(ra)}}{|\mathbf{r} - \mathbf{r}|^{a_0}}, \quad G'_{c}(\mathbf{r} - \mathbf{r}') = \frac{1}{\lambda^2 |\mathbf{r} - \mathbf{r}'|^{a'}},$$
(24)

with the different exponents $a_0 = 0$ and a' = 1. Correlations non-decaying with the distance seems to be a characteristic feature of condensation of fluctuations. It has been investigated in detail in the context of the Ising model with anti-periodic boundary conditions, which exhibits condensation of fluctuations below the critical temperature, in place of the usual ferromagnetic transition [44, 45]. In that case, the spin-spin correlation function remains constant and the vanishing of the exponent a has a nice geometrical interpretation in terms of the Coniglio-Klein correlated clusters [46].

As mentioned in the introductory remarks, the broad features of the scenario presented above are not just a mere theoretical possibility, but have been experimentally observed in a photonic quantum gas. By realising BEC in GCE conditions in a dye-filled cavity [12, 13], evidence has been produced for the existence of macroscopic fluctuations of the condensate, consistently with the \mathcal{GCC} . Remarkably, in the same system, phase coherence of the condensate has been detected on a time scale which grows linearly with the size of the condensate [14], indicating the occurrence of SSB in the thermodynamic limit and its compatibility with \mathcal{GCC} . The ideal limit of this experimental system is represented by a perfect gas of photons, endowed with an effective non-null mass and confined in a trap modeled by the two-dimensional harmonic potential $U(\mathbf{r}) = \frac{1}{2} m_{\text{eff}} \Omega^2 |\mathbf{r}|^2$. Carrying out the calculations in the GCE, both BEC and \mathcal{GCC} are found (see for example Ref. [47]). However, there are important formal differences with the gas in the box, the foremost one being the lack of space translational invariance. The detailed treatment of the trapped gas from the perspective of the present paper, and with particular attention to the first-order correlation function, will be presented in a future publication. Here, we just state that the ground state connected correlation function, in place of the constant behaviour of Eq. (24) obeys the Gaussian decay form

$$G_{\mathrm{c},\mathbf{0}}(\mathbf{r}, -\mathbf{r}) \sim \frac{\exp\{-(r/L)^2\}}{|\mathbf{r} - \mathbf{r}|^{a_0}},$$
 (25)

where the sector $\mathbf{r}' = -\mathbf{r}$ has been considered [47] and, again $a_0 = 0$. The characteristic length $L = \sqrt{\hbar/m_{\mathrm{eff}}\Omega}$ represents the linear size of the trap and the above result, which is in the form of finite-size scaling, indicates that the correlation length is of order L. Thus, in the thermodynamic limit - in the sense of an infinite trap $(L \to \infty)$ - a critical state is reached, which is characterised by a non-decaying correlation function, as in the uniform case. The experimental evidence for the above behaviour is in the results for the first-order correlation function reported in [48]. We stress that this kind of behaviour could not arise in the quasi-average framework, where the connected correlation function vanishes.

In conclusion, in this paper we have overturned the outlook on the \mathcal{GCC} , from culprit responsible of the failure of the GCE to most interesting manifestation of an underlying SSB pattern characterised by strong amplitude fluctuations and strong correlations. The proposed conceptual framework enhances the significance of the photon gas system as the experimental platform where to observe the distinctive features which make BEC based on the condensation of fluctuations different from BEC based on pure ordering, as realised in the cold atoms systems. It seems fitting to end up by saying that the remarkable accomplishment of BEC in GCE conditions lends support to the so-called totalitarian principle, according to which "everything not forbidden is compulsory" [49, 50].

Acknowledgements. Interesting conversations with Prof. Luca Salasnich, who participated to the early stage of this work, are gratefully acknowledged. A.S. acknowledges financial support from the MUR PRIN 2022 (project "SNO-MINK" no. 2022KWTEB7), which is funded by the European Union Next Generation EU, M4 C2 1.1 CUP B53C24006470006.

I. APPENDIX

Using the expression (12) for $\rho_{\mathbf{0}}^{(\nu)}$, keeping in mind that the excited states' contribution ρ' is not affected by the external field and using the small- α expansion of the Bose function, whereby $\lambda^{-3}g_{3/2}(\alpha)=\lambda^{-3}g_{3/2}(0)-C\sqrt{\alpha}+\dots$ with $C=2\lambda^{-3}\sqrt{\pi}$, Eq. (4) takes the form

$$\Delta \rho = \left[\left(\frac{\beta |\nu|}{\alpha} \right)^2 + \frac{1}{V\alpha} \right] - C\sqrt{\alpha}. \tag{26}$$

It is evident from the two terms in the square bracket that the order of the $V \to \infty$ and $\nu \to 0$ limits matters, revealing the singular nature of the perturbation. The solution, to leading order in ν and 1/V, in the two cases is given by

$$\alpha^{(\text{qa})} = \begin{cases} (\Delta \rho/C)^2, & \text{for } \Delta \rho < 0, \\ (\beta |\nu|/\sqrt{C})^{4/5}, & \text{for } \Delta \rho = 0, \\ \beta |\nu|/\sqrt{\Delta \rho}, & \text{for } \Delta \rho > 0, \end{cases}$$
 (27)

$$\alpha^{(\text{ra})} = \begin{cases} (\Delta \rho/C)^2, & \text{for } \Delta \rho < 0, \\ (CV)^{-2/3}, & \text{for } \Delta \rho = 0, \\ (V\Delta \rho)^{-1}, & \text{for } \Delta \rho > 0. \end{cases}$$
 (28)

Inserting the above results into Eq. (10) and using Eq. (11), after taking the limits Eqs. (13) and (15) are obtained.

- R. M. Ziff, G. E. Uhlenbeck and M. Kac, Phys. Rep. 32, 169 (1977).
- [2] M. Holthaus, E. Kalinowski, and K. Kirsten, Annals of Physics 270, 198 (1998).
- [3] I. Fujiwara, D. Ter Haar, and H. Wergeland, J. Stat. Phys. 2, 329 (1970).
- [4] S. Grossmann and M. Holthaus, Phys. Rev. Lett. 79, 3557 (1997).
- [5] C. Weiss and M. Wilkens, Optics Express 1, 272 (1997).
- [6] M. B. Kruk et al. arXiv:2502.10880
- [7] V. Yukalov, Laser Phys. Lett. **34**, 113001 (2024).
- [8] see also V. Yukalov, Major issues in the theory of Bose-Einstein condensation - arXiv:2505.16600v1 - and references therein.
- [9] A. Súto, Phys. Rev. Lett. 94, 080402 (2005).
- [10] E. H. Lieb, R. Seiringer and J. Yngvason, Phys. Rev. Lett. 94, 080401 (2005); Rep. Math. Phys. 59, 389 (2007).
- [11] N. N. Bogoliubov, Lectures on Quantum Statistics vol. 2: Quasi-Averages (Gordon and Breach, New York 1970).
- [12] J. Klaers, J. Schmitt, F. Vewinger, and M. Weitz, Nature

- **468**, 545 (2010).
- [13] J. Schmitt et al., Phys. Rev. Lett. 112, 030401 (2014).
- 14] J. Schmitt et al., Phys. Rev. Lett. 116, 033604 (2016).
- [15] C. Weiss and J. Tempere Phys. Rev. E **94**, 042124 (2016).
- [16] E. C. I. van der Wurff, A.-W. de Leeuw, R. A. Duine, and H. T. C. Stoof, Phys. Rev. Lett. 113, 135301 (2014).
- [17] M. Zannetti, Europhys. Lett. 111, 20004 (2015).
- [18] A. Crisanti, A. Sarracino and M. Zannetti, Phys. Rev. Res. 1, 023022 (2019).
- [19] A. Campa, T. Dauxois, S. Ruffo, Phys. Rep. 480, 57 (2009).
- [20] H. Touchette, J. Stat. Phys. 159, 987 (2015).
- [21] Ensemble non-equivalence and Bose-Einstein condensation has been also recently studied in weighted networks with local constraints, see Q. Zhang, D. Garlaschelli, Chaos, Solitons & Fractals 172, 113546 (2023).
- [22] K. Huang, Statistical Mechanics, 2nd Edn (Wiley 1987).
- [23] T. H. Berlin and M. Kac, Phys. Rev. 86, 821 (1952).
- [24] H. W. Lewis and G. H. Wannier, Phys. Rev. 88, 682 (1952).
- [25] H. W. Lewis and G. H. Wannier, Phys. Rev. 90, 1131

- (1953).
- [26] M. Kac and C. J. Thompson , J. Math. Phys. 18, 1650 (1977).
- [27] C. C. Yan, G. H. Wannier, J. Math. Phys. 6, 1833 (1965).
- [28] In this connection it is interesting to note that in the low temperature phase of the mean-spherical model there are macroscopic fluctuations of the order parameter, which are absent in the spherical model [23] and which are entirely analogous to the \mathcal{GCC} . The interesting aspect is that upon discovering this Yan and Wannier [27] regarded the rersult as a *freak* feature of the model.
- [29] G. Roepstorff, J. Stat. Phys. 18, 191 (1978).
- [30] W. F. Wreszinski and V. A. Zagrebnov, Theor. Math. Phys. 194, 157 (2018).
- [31] R. J. Glauber, Phys. Rev. 131, 2766 (1963).
- [32] E. C. G. Sudarshan, Phys. Rev. Lett. 10, 277 (1963).
- [33] C. L. Mehta, Phys. Rev. Lett. 18, 752 (1967).
- [34] A. Casher and M. Revzen, Am. J. Phys. 35, 1154 (1967).
- [35] C. Castellano, F. Corberi, and M. Zannetti, Phys. Rev. E 56, 4973 (1997).
- [36] N. Fusco and M. Zannetti, Phys. Rev. E 66, 066113 (2002).
- [37] F. Corberi, G. Gonnella, A. Piscitelli, and M. Zannetti, J. Phys. A: Math.Theor. 46, 042001(2013).
- [38] M. Zannetti, F. Corberi, and G. Gonnella, Phys. Rev. E

- **90**, 012143 (2014).
- [39] M. Zannetti, F. Corberi, G. Gonnella, and A. Piscitelli, Commun. Theor. Phys. 62, 555 (2014).
- [40] M. Filiasi, G. Livan, M. Marsili. M. Peressi, E. Vesselli, and E. Zarinelli, J. Stat. Mech. P09030 (2014);
- [41] L. Ferretti, M. Mamino, and G. Bianconi, Phys. Rev. E 89, 042810 (2014).
- [42] J. D. Gunton and M. J. Buckinghan, Phys. Rev. 166, 152 (1968).
- [43] I Reyes-Ayala, F J Poveda-Cuevas, V Romero-Roshín, J. Stat. Mech. 113102 (2019).
- [44] A. Fierro, A. Coniglio, and M. Zannetti, Phys. Rev. E 99, 042122 (2019).
- [45] A. Fierro, A. Coniglio, and M. Zannetti, Phys. Rev. E 102, 012144 (2020).
- [46] A. Coniglio and W. Klein, J. Phys. A 13, 2775 (1980).
- [47] A. Crisanti, L. Salasnich, A. Sarracino and M. Zannetti, Entropy 26, 367 (2024).
- [48] T. Damm, D. Dung, F. Vewinger, M. Weitz, and J. Schmitt, Nature comm. 8, 158 (2017).
- [49] Although there is no clear reference, the quotation is widely attributed to Murray Gell-Mann, see [50].
- [50] H. Kragh, arXiv:1907.04623.