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Mechanical form factors and densities of non-relativistic fermions

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The hadron physics community has been actively debating the interpretation of so-called mechanical properties of hadrons. Non-relativistic quantum-mechanical systems like the hydrogen atom have been appealed to in these debates as analogies. Since such appeals are likely to continue, it is important to have Galilei-covariant expressions for matrix elements of the energy-momentum tensor. In this work, I obtain Galilei-covariant breakdowns of such matrix elements into mechanical form factors, with a special focus on spin-half states. I additionally study the spatial densities associated with these form factors, using the pilot wave interpretation to guide their breakdown into contributions from internal structure and from quantum-mechanical effects such as wave packet dispersion. For completeness, I also obtain non-relativistic Breit frame densities.

I. INTRODUCTION

In the past decade, the hadron physics community has seen a flourishing of discussion about what are often called the mechanical properties of hadrons. These are properties described by the energy-momentum tensor (EMT), including the energy (or mass), angular momentum, and potentially internal forces. Popular topics of discussion have included the decomposition of the proton's mass [1–5] and spin [6–9], as well as spatial distributions of both [10–19].

Perhaps one of the most lively—and contentious—areas of discussion has been the hadronic stress tensor. The spacelike components of the EMT describe momentum flux densities, and in many continuum classical systems have a clear interpretation as a stress tensor. Since the seminal work of Maxim Polyakov [10]—inspired largely by analogy to liquid drops—many researchers [5, 11–18, 20–40] have utilized the traditionally classical continuum concepts of stresses, pressures, tension and shear in the interpretation of the hadronic EMT. (For reviews, see Refs. [11, 41, 42].) To be sure, a few researchers [43–45] have questioned the concept of hadronic stresses; but others [41, 42, 46] have argued to defend the use of the stress concept.

Many of these discussions use simpler systems than hadrons as test cases for claims made in the literature. After all, quantum chromodynamics is an infamously difficult theory. Systems in quantum electrodynamics such as the electron [25, 47–49] and photon [50] provide sandboxes to play with ideas often appealed to in discussions of mechanical properties, but in a better-understood theory.

Recently, the hydrogen atom has become a playground in which multiple researchers [44, 46, 51, 52] have explored the calculation and meaning of the energy-momentum tensor. As a non-relativistic quantum-mechanical system, it avoids difficulties related to relativistic effects in spatial densities and composite operator renormalization. With discussions about the meaning of the EMT far from over, it is likely that appeals to non-relativistic quantum-mechanical systems will continue.

If non-relativistic quantum mechanics is to become a playground for studying mechanical properties, it is important to have all the equipment and amenities in place. Among these are expressions for matrix elements of the EMT in terms of mechanical form factors[1]. Just as the guiding principle behind form factor breakdowns in relativistic quantum field theory is Lorentz covariance, the symmetries of non-relativistic physics—namely Galilei covariance—should guide form factor breakdowns in non-relativistic quantum mechanics. Such expressions seem to be absent in the present literature, especially for systems with non-zero spin. One of the purposes of this study is to provide a framework for constructing such expressions, and to provide form factor breakdowns for the EMT in spin-zero and spin-half systems specifically.

Another helpful amenity to have on hand is a set of formulas for relating the mechanical form factors to spatial densities of mass, energy, momentum and stresses. The other purpose of this work is to provide these density formulas. Although the difficulties of relativistic effects are absent here, there is still some ambiguity in separating the internal structure of composite systems from effects of the barycentric wave packet (such as dispersion). To this end, I use considerations from the pilot wave interpretation [53–56] of quantum mechanics to guide this separation. Nonetheless, for the benefit of readers who are skeptical of this interpretation, I provide formulas for non-relativistic densities in the standard Breit frame formalism as well.

This work is organized as follows. In Sec. II, I provide a gentle introduction to the Galilei group, using the energy-momentum tensor of spin-zero states as an example case. In particular, I exploit the fact that the Galilei group in $3+1$ dimensions is a subgroup of the $(4+1)$-dimensional Lorentz group. Next, in Sec. III, I dive more deeply into the relevant group theory, to derive non-relativistic spinors that are covariant under the Galilei group. The ultimate results of this deep dive are matrix elements of

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[1] The form factors appearing in matrix elements of the EMT are more commonly called gravitational form factors, because the EMT is the source of gravitation in general relativity. However, I have found that the phrase "gravitational form factors" confuses and misleads researchers outside the field, by giving the impression that gravitation is used to measure these form factors or that they are being used to study gravitational forces within hadrons. Since the purpose of these form factors is to characterize mechanical properties of hadrons—for the purpose of understanding the strong nuclear force, of course—it seems more apt to call them *mechanical* form factors.

the spin-half EMT, which are stated in Sec. IV. These are used to construct densities in Sec. V with the guidance of the pilot wave interpretation (but contrasting density results in the Breit frame formalism are given in Sec. IV B for completeness). I conclude and provide an outlook in Sec. VI.

II. SPIN-ZERO STATES

In non-relativistic physics, we usually consider time t and spatial coordinates $\boldsymbol{x}$ separately, only uniting them into a single quantity—the four-vector $x^\mu = (t;\boldsymbol{x})$—in relativistic physics[2]. However, there is nothing to prevent us from formally considering $(t;\boldsymbol{x})$ as a unified object in non-relativistic physics—albeit with different transformation laws than in special relativity. Under an active boost of velocity $\boldsymbol{v}$ in particular:

$$\begin{aligned} t' &= t \\ \boldsymbol{x}' &= \boldsymbol{x} + \boldsymbol{v}t \,. \end{aligned} \tag{1}$$

Another familiar four-vector is the energy-momentum four-vector $p^\mu = (E;\boldsymbol{p})$. In relativistic physics, its components transform under the same Lorentz transformation law as those of x^μ. What happens, however, in non-relativistic physics? Under the same boost of velocity $\boldsymbol{v}$, the energy and momentum of non-relativistic physics transform as:

$$\begin{aligned} E' &= E + \boldsymbol{v}\cdot\boldsymbol{p} + \frac{1}{2}m\boldsymbol{v}^2 \\ \boldsymbol{p}' &= \boldsymbol{p} + \boldsymbol{v}m \,. \end{aligned} \tag{2}$$

At first glance, Eqs. (1) and (2) seem to specify different transformation laws. However, they can be unified—the hint for this is the similar forms of the $\boldsymbol{x}$ and $\boldsymbol{p}$ transformation laws, with t and m playing analogous roles. If, rather than a four-vector, we construct a five-vector:

$$\begin{aligned} x^\mu &= (x^+;\boldsymbol{x};x^-) = (t;\boldsymbol{x};x^-) \\ p^\mu &= (p^+;\boldsymbol{p};p^-) = (m;\boldsymbol{p};E) \,, \end{aligned} \tag{3}$$

with the two non-spatial components marked by an upper + or − index, then these five-component objects transform identically under boosts:

$$\begin{aligned} x^+ &\mapsto x^+ \\ \boldsymbol{x} &\mapsto \boldsymbol{x} + \boldsymbol{v}x^+ \\ x^- &\mapsto x^- + \boldsymbol{v}\cdot\boldsymbol{x} + \frac{1}{2}x^+\boldsymbol{v}^2 \,. \end{aligned} \tag{4}$$

For the five-momentum in particular, the five components arise because the energy and mass transform differently under boosts. For the five-position on the other hand, the new component x^- does not have an apparent physical meaning, and was only introduced so that p^μ and x^μ have the same transformation laws[3]. In fact, five-vectors are widely used in studies of Galilei symmetry [57–64].

Similarly to time and position, the electric charge density $\rho(x)$ and current density $\boldsymbol{j}(x)$ are usually unified into a single electromagnetic four-current in relativistic physics; and similarly to time and position, we can construct an electromagnetic five-current:

$$j^\mu(x) = (\rho(x);\boldsymbol{j}(x);j^-(x)) \equiv (j^+(x);\boldsymbol{j}(x);j^-(x)) \,. \tag{5}$$

Just as in the time-position case as well, $j^-(x)$ does not have a clear physical meaning, and seems to be spurious. The reason for identifying the charge density as the + component rather than the − component is found in the transformation laws:

$$\begin{aligned} j^+(x) &\mapsto j^+(x') \\ \boldsymbol{j}(x) &\mapsto \boldsymbol{j}(x') + \boldsymbol{v}j^+(x') \,. \end{aligned} \tag{6}$$

The general rule for five-currents is that the + component is the density, the $(1,2,3)$ components are the current density, and the − component is spurious and unphysical.

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[2] Here, and throughout the text, I use natural units with $c = 1$. In the context of Galilei transforms, c should be understood as merely a unit conversion factor.
[3] A possible operational interpretation of the fifth coordinate has been suggested by Kapuscik [57], as a control parameter specifying the velocities of signals used to synchronize spatially-displaced clocks.

This brings us to the energy, momentum and mass currents. In principle, the energy, mass, and all three components of the momentum are locally conserved quantities which therefore each have an associated five-current. However, these five quantities are (unlike the charge) not individually invariant under boosts, being instead connected through the Galilei transformation law (4). Thus, there should be a rank-two tensor $T^{\mu\nu}(x)$ for which $T^{\mu+}(x)$ provides the mass current, $T^{\mu a}(x)$ the momentum current (given $a \in \{1,2,3\}$)[4], and $T^{\mu-}(x)$ the energy current. This tensor is in effect the non-relativistic analogue of the relativistic energy-momentum tensor; I will just call it the energy-momentum tensor through the remainder of the text.

A. Mechanical form factors

Mechanical form factors are defined by breaking the matrix element $\langle \boldsymbol{p}'|\hat{T}_q^{\mu\nu}(0)|\boldsymbol{p}\rangle$ down into the most general possible tensor structure that is compatible with Galilei covariance (and which is also invariant under parity and time reversal). Here, q specifies a particular constituent, usually in the form of a particle species, and $\hat{T}_q^{\mu\nu}$ signifies the contribution of this constituent to the EMT. To build this tensor, we have as ingredients the five-vectors $P^\mu = \frac{1}{2}\left(p^\mu + p'^\mu\right)$, $\varDelta^\mu = p'^\mu - p^\mu$ and $n^\mu = (n^+; \boldsymbol{n}; n^-) = (0; \boldsymbol{0}, 1)$, as well as the Galilei metric tensor $g^{\mu\nu}$. The vector n^μ is present in this list because it is invariant under the Galilei group, in effect representing that we could always add a constant to the energy-like component of any five-vector without altering the transformation law (4).

The Galilei metric is defined so that Galilei transforms are isometries of the metric—that is, so that $g^{\mu\nu}$ is invariant under all Galilei transforms. It must also be symmetric in its indices, and by convention I choose $g^{ab} = -\delta^{ab}$ for $a, b \in \{1,2,3\}$ (to closely mimic similar components of the Minkowski metric). These requirements impose the form:

$$g^{\mu\nu} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & g^{--} \end{bmatrix}$$

where g^{--} is an undetermined real number. Choosing $g^{--} = 0$, in line with most authors [59–61, 64, 65][5], conveniently makes the metric its own inverse:

$$g^{\mu\nu} = g_{\mu\nu} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{7}$$

Additionally, $g^{\mu\nu}$ here is the metric tensor of light front coordinates in five-dimensional spacetime. This occurs because the Poincaré group has a Galilei subgroup with one less dimension [58, 66–68], and is why I have used + and − superscripts for the null coordinates.

In light of the metric (7), scalar products between Galilei five-vectors are:

$$g_{\mu\nu} x^\mu y^\nu = x^+ y^- + x^- y^+ - \boldsymbol{x} \cdot \boldsymbol{y}, \tag{8}$$

and the invariant associated with the mass-momentum-energy five-vector is:

$$g_{\mu\nu} p^\mu p^\nu = 2mE - \boldsymbol{p}^2 = 2mE_0. \tag{9}$$

This will be useful later when constructing Galilei spinors.

In terms of the available tensors, the matrix elements of the energy-momentum tensor takes the form:

$$\langle \boldsymbol{p}'|\hat{T}_q^{\mu\nu}(0)|\boldsymbol{p}\rangle = \frac{P^\mu P^\nu}{m} A_q(\boldsymbol{\varDelta}^2) + \frac{\varDelta^\mu \varDelta^\nu + g^{\mu\nu}\boldsymbol{\varDelta}^2}{4m} D_q(\boldsymbol{\varDelta}^2) + m g^{\mu\nu} \bar{c}_q(\boldsymbol{\varDelta}^2) + P^\mu n^\nu \bar{e}_q(\boldsymbol{\varDelta}^2), \tag{10}$$

where m is the mass of the system. Strictly speaking, more tensors such as $n^\mu P^\nu$ and $n^\mu n^\nu$ could be added, but these would contribute only to the unphysical components $T^{-\nu}$ of the EMT, and can be safely discarded. I have written this breakdown to closely mirror the standard relativistic breakdown [11], with the form factors $A_q(\boldsymbol{\varDelta}^2)$ and $D_q(\boldsymbol{\varDelta}^2)$ appearing in various forms

[4] Here, and throughout the text, I use a, b, c instead of the usual i, j, k to signify spatial components of five-vectors.
[5] The early, pioneering work of Pinski [58] instead uses $g^{--} = 1$.

in works dating to the 1960s [69, 70] and $\bar{c}_q(\varDelta^2)$ first introduced by Ji in 1996 [7]. The form factor $\bar{e}_q(\varDelta^2)$ is new to the non-relativistic case, and is present because the energy density does not transform into the mass and momentum densities under boosts—-similarly to how the rest energy in Eq. (2) does not transform into the mass and momentum. Another way to think about the $\bar{e}_q(\varDelta^2)$ form factor is that it arises because Galilei symmetry does not include rotational symmetry in the unphysical fifth dimension, which makes it analogous to the "spurious" form factors appearing in form factor breakdowns in light front dynamics [35, 71].

B. Sum rules

There are several sum rules that can be imposed by the mechanical form factors in Eq. (10). First, the densities of the energy-momentum tensor $T^{+\nu}$ should integrate over all space to the five-momentum P^ν. Accordingly, once all constituents have been summed over, we should have:

$$\sum_q \langle \boldsymbol{p}|\hat{T}_q^{+\nu}(0)|\boldsymbol{p}\rangle = P^\nu\,. \tag{11}$$

This imposes the following sum rules on the form factors at $\boldsymbol{\varDelta} = 0$:

$$\begin{aligned} \sum_q A_q(0) &= 1 \\ \sum_q \left(\bar{c}_q(0) + \bar{e}_q(0)\right) &= 0\,. \end{aligned} \tag{12}$$

The first of these is analogous to the momentum sum rule for the relativistic form factors. Additionally, the local continuity equation $\partial_\mu T^{\mu\nu}(\boldsymbol{x},t) = 0$ requires, for non-zero momentum transfer:

$$\sum_q \varDelta_\mu \langle \boldsymbol{p}'|\hat{T}_q^{\mu\nu}(0)|\boldsymbol{p}\rangle\,, \tag{13}$$

and since $\varDelta_\mu P^\nu = 0$, this imposes the following sum rule:

$$\sum_q \bar{c}_q(\boldsymbol{\varDelta}^2) = 0\,, \tag{14}$$

which also holds for the relativistic $\bar{c}_q$ form factor [7]. Together with the previous sum rules, this entails:

$$\sum_q \bar{e}_q(0) = 0\,. \tag{15}$$

C. Densities of the energy-momentum tensor

With the form factor decomposition (10), we can construct its corresponding spatial densities. The starting point is the expectation value of the local operator $\hat{T}^{\mu\nu}(\boldsymbol{x})$ for physical states. This expectation value can be written[6]:

$$\langle\Psi(t)|\hat{T}_q^{\mu\nu}(\boldsymbol{x})|\Psi(t)\rangle = \int \mathrm{d}^3\boldsymbol{R} \int \frac{\mathrm{d}^3\boldsymbol{\varDelta}}{(2\pi)^3} \Psi^*(\boldsymbol{R},t)\langle\boldsymbol{p}'|\hat{T}_q^{\mu\nu}(0)|\boldsymbol{p}\rangle\Psi(\boldsymbol{R},t)\,\mathrm{e}^{-i\boldsymbol{\varDelta}\cdot(\boldsymbol{R}-\boldsymbol{x})}\Big|_{2\boldsymbol{P}\to -i\overleftrightarrow{\boldsymbol{\nabla}}}\,. \tag{16}$$

As far as specific components go: the $\mu = +$ components constitute densities, the $\mu \in \{1,2,3\}$ components constitute flux densities, and the $\mu = -$ components have no physical meaning (being akin to j^-). The index ν labels which component of the five-momentum this is a density or flux density of. For the mass density, mass flux density, and momentum density, Eq. (16) factorizes into a single internal density smeared out by different wave-packet dependent smearing functions, thus constituting "simple" densities in the nomenclature of Ref. [16]. In particular, in terms of the probability density:

$$\mathscr{P}(\boldsymbol{R},t) = \Psi^*(\boldsymbol{R},t)\Psi(\boldsymbol{R},t)\,, \tag{17}$$

[6] See Refs. [16, 72] for step-by-step derivations of similar expressions.

the Bohmian velocity [54, 56]:

$$\boldsymbol{v}_{\mathrm{B}}(\boldsymbol{R},t) = -\frac{i}{2m}\frac{\Psi^*(\boldsymbol{R},t)\overleftrightarrow{\boldsymbol{\nabla}}\Psi(\boldsymbol{R},t)}{\Psi^*(\boldsymbol{R},t)\Psi(\boldsymbol{R},t)}\,, \tag{18}$$

and the contribution of the qth particle species to the internal matter density:

$$\mathfrak{a}_q(\boldsymbol{b}) = \int \frac{\mathrm{d}^3\boldsymbol{\varDelta}}{(2\pi)^3} A_q(\boldsymbol{\varDelta}^2)\,\mathrm{e}^{-i\boldsymbol{\varDelta}\cdot\boldsymbol{b}}\,, \tag{19}$$

the expectation values in question can be written:

$$\begin{aligned}\langle\Psi(t)|\hat{T}_q^{++}(\boldsymbol{x})|\Psi(t)\rangle &= m\int \mathrm{d}^3\boldsymbol{R}\,\mathscr{P}(\boldsymbol{R},t)\mathfrak{a}_q(\boldsymbol{x}-\boldsymbol{R})\\ \langle\Psi(t)|\hat{T}_q^{a+}(\boldsymbol{x})|\Psi(t)\rangle = \langle\Psi(t)|\hat{T}_q^{+a}(\boldsymbol{x})|\Psi(t)\rangle &= m\int \mathrm{d}^3\boldsymbol{R}\,\mathscr{P}(\boldsymbol{R},t)v_{\mathrm{B}}^a(\boldsymbol{R},t)\,\mathfrak{a}_q(\boldsymbol{x}-\boldsymbol{R})\,.\end{aligned} \tag{20}$$

These expressions have a straightforward interpretation. First, the average mass density is found by smearing the internal mass density by the probability density $\mathscr{P}$. Second, the momentum density is found by smearing instead by the probability current. The second rule has a more transparent meaning in the pilot wave interpretation: $\boldsymbol{v}_{\mathrm{B}}(\boldsymbol{R},t)$ is the composite system's actual velocity, and a momentum distribution is obtained by multiplying this by the internal mass density. The average momentum density is then obtained by smearing by the probability.

Next, the stress tensor constitutes a "compound density" in the nomenclature of Ref. [16]:

$$\begin{aligned}\langle\Psi(t)|\hat{T}_q^{ab}(\boldsymbol{x})|\Psi(t)\rangle &= \int \mathrm{d}^3\boldsymbol{R}\,\left\{\left(-\Psi^*(\boldsymbol{R},t)\frac{\overleftrightarrow{\nabla}^a\overleftrightarrow{\nabla}^b}{4m}\Psi(\boldsymbol{R},t)\right)\mathfrak{a}_q(\boldsymbol{x}-\boldsymbol{R}) + \mathscr{P}(\boldsymbol{R},t)\mathfrak{t}_q^{ab}(\boldsymbol{x}-\boldsymbol{R})\right\}\\ \mathfrak{t}_q^{ab}(\boldsymbol{b}) &= \int\frac{\mathrm{d}^3\boldsymbol{\varDelta}}{(2\pi)^3}\left\{\frac{\varDelta^a\varDelta^b - \boldsymbol{\varDelta}^2\delta^{ab}}{4m}D_q(\boldsymbol{\varDelta}^2) - m\delta^{ab}\bar{c}_q(\boldsymbol{\varDelta}^2)\right\}\mathrm{e}^{-i\boldsymbol{\varDelta}\cdot\boldsymbol{b}}\,,\end{aligned} \tag{21}$$

where $\mathfrak{t}_q^{ab}(\boldsymbol{b})$ is the intrinsic stress tensor, and the $\mathfrak{a}_q(\boldsymbol{b})$ term constitutes dynamic stresses associated with wave packet motion and dispersion. I previously have explored the meaning of this breakdown in the pilot wave formulation [46].

D. Energy density

The energy density is especially important, being related to open questions and controversies regarding mass generation and the mass decomposition of hadrons in quantum chromodynamics (see e.g., Refs. [1–5]). Recalling that the energy is given by the "minus" component of the Galilei five-momentum p^-, the relevant matrix element is:

$$\langle\boldsymbol{p}'|\hat{T}_q^{+-}(0)|\boldsymbol{p}\rangle = \left(E_0 + \frac{\boldsymbol{P}^2}{2m} + \frac{\boldsymbol{\varDelta}^2}{8m}\right)A_q(\boldsymbol{\varDelta}^2) + \frac{\boldsymbol{\varDelta}^2}{4m}D_q(\boldsymbol{\varDelta}^2) + m\big(\bar{c}_q(\boldsymbol{\varDelta}^2) + \bar{e}_q(\boldsymbol{\varDelta}^2)\big)\,, \tag{22}$$

where $E_0 \approx m$ is the rest energy of the system[7]. The corresponding energy density will naturally be a compound density, with one contribution from the energy contained in barycentric motion and wave packet dispersion, and the other from the internal density.

It may be tempting to identify $\frac{\boldsymbol{P}^2}{2m}A_q(\boldsymbol{\varDelta}^2)$ as containing all of the barycentric energy, and the remaining terms as the internal energy—and this is in effect what the Breit frame formalism does [11]. Perhaps in vanilla quantum mechanics, the separation of the energy density into barycentric and internal contributions is to some extent arbitrary, and thus a matter of definition. However, in the realist pilot wave interpretation, there is a clean and unambiguous way of separating the barycentric and internal contributions to the energy density. While the truth of the pilot wave formulation is not evident—and empirically, cannot be, since it makes the same predictions as vanilla quantum theory—there is no harm in hedging our bets on the matter, and adopting a convention that *might* turn out to be the true breakdown under a realist interpretation of quantum mechanics.

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[7] Recall that in non-relativistic mechanics, mass is strictly additive: for a system of N particles, $m = m_1 + m_2 + \ldots + m_N$. However, even if we ascribe a rest energy m_n to each of these particles when they are free, the energy of the system in the center-of-mass frame still contains binding and kinetic energy contributions, and thus $E_0 \neq m$. Of course since the motion of all constituents is assumed (by the very applicability of non-relativistic physics) to be much slower than the speed of light, these deviations of E_0 from m are small, so $E_0 \approx m$.

In the pilot wave interpretation, the system as a whole has a kinetic energy K and a quantum potential energy Q, given respectively by [54, 56]:

$$\begin{aligned} K(\boldsymbol{R},t) &= \frac{(\boldsymbol{\nabla}\mathcal{S}(\boldsymbol{R},t))^2}{2m} \\ Q(\boldsymbol{R},t) &= -\frac{1}{2m}\frac{\boldsymbol{\nabla}^2\mathcal{R}(\boldsymbol{R},t)}{\mathcal{R}(\boldsymbol{R},t)}\,, \end{aligned} \tag{23}$$

where $\mathcal{R}(\boldsymbol{R},t)$ and $\mathcal{S}(\boldsymbol{R},t)$ are real-valued functions defined through the polar decomposition of the wave function:

$$\Psi(\boldsymbol{R},t) = \mathcal{R}(\boldsymbol{R},t)\,\mathrm{e}^{i\mathcal{S}(\boldsymbol{R},t)}\,. \tag{24}$$

The sum of the kinetic and quantum potential energy, weighted by the probability density, can be written:

$$\mathcal{E}_{\text{bary}}(\boldsymbol{R},t) \equiv \mathcal{P}(\boldsymbol{R},t)\big(K(\boldsymbol{R},t)+Q(\boldsymbol{R},t)\big) = -\frac{1}{8m}\Psi^*(\boldsymbol{R},t)\overleftrightarrow{\boldsymbol{\nabla}}^2\Psi(\boldsymbol{R},t) - \frac{1}{8m}\boldsymbol{\nabla}^2\big[\Psi^*(\boldsymbol{R},t)\Psi(\boldsymbol{R},t)\big]\,, \tag{25}$$

where I have identified this energy as "bary" for barycentric. The two-sided derivative in this expression translates to $-4\boldsymbol{P}^2$ in momentum space, and the total Laplacian translates to $-\boldsymbol{\Delta}^2$. Thus the barycentric energy involves not just the $\boldsymbol{P}^2$ term in the form factor breakdown (22), but also part of the $\boldsymbol{\Delta}^2$ terms. In fact, the resulting energy density breakdown is:

$$\begin{aligned} \langle\Psi(t)|\hat{T}^{+-}(\boldsymbol{x})|\Psi(t)\rangle &= \int \mathrm{d}^3\boldsymbol{R}\,\Big\{\mathcal{E}_{\text{bary}}(\boldsymbol{R},t)\mathfrak{a}_q(\boldsymbol{x}-\boldsymbol{R}) + \mathcal{P}(\boldsymbol{R},t)\mathfrak{e}_q(\boldsymbol{x}-\boldsymbol{R})\Big\} \\ \mathfrak{e}_q(\boldsymbol{b}) &= \int\frac{\mathrm{d}^3\boldsymbol{\Delta}}{(2\pi)^3}\left\{E_0A_q(\boldsymbol{\Delta}^2) + m\big(\bar{c}_q(\boldsymbol{\Delta}^2)+\bar{e}_q(\boldsymbol{\Delta}^2)\big) + \frac{\boldsymbol{\Delta}^2}{4m}D_q(\boldsymbol{\Delta}^2)\right\}\mathrm{e}^{-i\boldsymbol{\Delta}\cdot\boldsymbol{b}}\,, \end{aligned} \tag{26}$$

where $\mathfrak{e}_q(\boldsymbol{b})$ is the contribution of the qth particle species to the internal energy density.

A few remarks about this result are in order. Existing formulas for the internal energy density of composite systems (see Refs. [11, 12, 16] for examples) all contain a term where $\boldsymbol{\Delta}^2$ multiplies $A_q(\boldsymbol{\Delta}^2)$. This means, even for an elementary particle with no D-term—for which $A(\boldsymbol{\Delta}^2)=1$ and $\bar{c}(\boldsymbol{\Delta}^2)=0$—there is a non-trivial internal energy density. This is peculiar, given that the particle in question is ostensibly pointlike. By contrast, the result (26) gives a delta function times E_0 for such an elementary particle. While this by no means proves the correctness of Eq. (26) over standard formulas, it does make the result more intuitive.

It's also worth noting that the integral of Eq. (26) over all space gives:

$$\int \mathrm{d}^3\boldsymbol{b}\,\mathfrak{e}_q(\boldsymbol{b}) = E_0A_q(0) + m\big(\bar{c}_q(0)+\bar{e}_q(0)\big)\,, \tag{27}$$

making this the contribution of the qth constituent to the rest energy. This is in effect the non-relativistic analogue of Lorcé's rest energy decomposition [2], which in the relativistic case gives the constituent contribution as $m\big(A_q(0)+\bar{c}_q(0)\big)$.

E. Covariant breakdown

Before moving on to spin-half states, it is helpful to unify the densities I have written above in an apparently covariant form. To this end, it is helpful to define the following Galilei boost matrices [60]:

$$\Lambda^\mu{}_\nu(\boldsymbol{v}) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ v_x & 1 & 0 & 0 & 0 \\ v_y & 0 & 1 & 0 & 0 \\ v_z & 0 & 0 & 1 & 0 \\ \frac{1}{2}\boldsymbol{v}^2 & v_x & v_y & v_z & 1 \end{bmatrix}\,. \tag{28}$$

Using these, the expectation value of the energy-momentum tensor can be written:

$$\langle\Psi(t)|\hat{T}^{\mu\nu}_q(\boldsymbol{x})|\Psi(t)\rangle = \int\mathrm{d}^3\boldsymbol{R}\,\Lambda^\mu{}_\alpha\big(\boldsymbol{v}_\mathrm{B}(\boldsymbol{R},t)\big)\Lambda^\nu{}_\beta\big(\boldsymbol{v}_\mathrm{B}(\boldsymbol{R},t)\big)\Big\{\mathcal{P}(\boldsymbol{R},t)\mathfrak{t}^{\alpha\beta}_q(\boldsymbol{x}-\boldsymbol{R}) + T^{\alpha\beta}_Q(\boldsymbol{R},t)\mathfrak{a}_q(\boldsymbol{x}-\boldsymbol{R})\Big\}\,, \tag{29}$$

where $\mathfrak{t}^{\alpha\beta}_q(\boldsymbol{b})$ is the internal energy-momentum tensor, and $T^{\alpha\beta}_Q(\boldsymbol{R},t)$ is the quantum energy-momentum tensor. The internal energy-momentum tensor has components including:

$$\mathfrak{t}^{++}_q(\boldsymbol{b}) = \mathfrak{a}_q(\boldsymbol{b}) \qquad\qquad \mathfrak{t}^{+-}_q(\boldsymbol{b}) = \mathfrak{e}_q(\boldsymbol{b})\,, \tag{30}$$

as well as those given in Eq. (21). These can be interpreted as mass, energy and momentum currents inside the system, owing to its composite structure, in its rest frame.

The quantum energy-momentum tensor, on the other hand, has the following non-zero components:

$$\begin{aligned}
T_Q^{ab}(\boldsymbol{R},t) &= \frac{1}{2m}\Big(\big(\nabla_a\mathscr{R}(\boldsymbol{R},t)\big)\big(\nabla_b\mathscr{R}(\boldsymbol{R},t)\big) - \mathscr{R}(\boldsymbol{R},t)\big(\nabla_a\nabla_b\mathscr{R}(\boldsymbol{R},t)\big)\Big)\\
T_Q^{+-}(\boldsymbol{R},t) &= -\frac{\mathscr{R}(\boldsymbol{R},t)\boldsymbol{\nabla}^2\mathscr{R}(\boldsymbol{R},t)}{2m}\\
T_Q^{a-}(\boldsymbol{R},t) &= \frac{1}{2m}\left(\frac{\partial\mathscr{R}(\boldsymbol{R},t)}{\partial t}\big(\nabla_a\mathscr{R}(\boldsymbol{R},t)\big) - \mathscr{R}(\boldsymbol{R},t)\nabla_a\left[\frac{\partial\mathscr{R}(\boldsymbol{R},t)}{\partial t}\right]\right)
\end{aligned} \tag{31}$$

the first line of which was first given by Takabayasi in Eq. (A2) of Ref. [73]. These components effectively describe stresses and energies felt by the composite particle due to its guidance by the barycentric wave function in the particle's rest frame.

Finally, the contributions of both the intrinsic and quantum EMT in Eq. (29) are boosted by the Bohmian velocity $\boldsymbol{v}_{\mathrm{B}}(\boldsymbol{R},t)$, which effectively incorporates contributions of barycentric motion to the energy and momentum densities and fluxes. This is then in turn smeared out by the probability $\mathscr{P}(\boldsymbol{R},t)$ of the barycentric position actually being $\boldsymbol{R}$.

III. GALILEI SPINORS

With spin-zero states under our belt, the time is ripe to move on to spin-half states. To construct a spin-half analogue of Eq. (10), we need to write the matrix element $\langle \boldsymbol{p}', s'|\hat{T}^{\mu\nu}(0)|\boldsymbol{p}, s\rangle$ in the most general possible manner compatible with Galilei covariance (as well as parity and time reversal), using all the five-vectors and Galilei tensors at our disposal. Since it is a spin-half state, the available structures include Clifford algebra matrices, and everything should be sandwiched between Galilei-covariant spinors $u(\boldsymbol{p}, s)$.

There is one difficulty which will take some time to address, and which is the focus of the current section: we need the appropriate Galilei-covariant spinors to do this. The popular two-component Pauli spinors are inadequate for this purpose, because they do not transform under a matrix representation of the Galilei group [74]. There is a standard set of four-component spinors discovered by Lévy-Leblond [75] that are covariant under the part of the Galilei group connected to identity. However, they do not have the expected transformation properties under parity reversal, which limits their applicability to the construction of form factor breakdowns. (See Appendix B for details.) This necessitates the derivation of new Galilei-covariant spinors.

In this section, I will construct an eight-component spinor wave equation that is covariant under the full Galilei group, and give explicit solutions[8]. The construction will be by necessity rather mathematically involved. A reader only interested in results can skip to Sec. IV A, where I write the covariant form factor breakdown and evaluate its components in terms of standard (but non-covariant) two-component spinors.

A. Representation theory for the $(4+1)$-dimensional Lorentz group

To derive the required spinors, I will step back and consider representation theory of the $(4+1)$-dimensional Lorentz group. This group contains the connected part of the $(3+1)$-dimensional Galilei group as a subgroup, so spinors in $(4+1)$-dimensional Minkowski spacetime will be covariant under the Galilei group after restricting $p^+ = m$. A subtlety that arises along the way is that Galilean parity reversal is not present in the $(4+1)$-dimensional Lorentz group, so the Galilei subgroup contained therein will need to be extended, but we'll cross that bridge when we come to it.

The Lorentz group in $(4+1)$ dimensions is the special pseudo-orthogonal group $\mathrm{SO}(4,1,\mathbb{R})$ of matrices that preserve the metric $g = \mathrm{diag}(1,-1,-1,-1,-1)$, i.e.,

$$MgM^{\mathsf{T}} = g\,, \tag{32}$$

and have a determinant $\det(M) = 1$. The subgroup that is path-connected to identity can be written in terms of 10 generators $\lambda^{\mu\nu} = -iJ^{\mu\nu}$ (4 boosts and 6 rotations), with $\lambda_{\mu\nu}$ being antisymmetric in its indices. Given a set of 10 real-valued parameters $\omega_{\mu\nu}$ (also antisymmetric):

$$M = \exp\left\{\frac{1}{2}\omega_{\mu\nu}\lambda^{\mu\nu}\right\} = \exp\left\{-\frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}\right\}\,, \tag{33}$$

[8] It should be noted that an eight-component spinor wave equation was derived previously by Kobayashi, de Montigny and Khanna [76], by a Clifford algebra method in $5+1$ dimensions, but I will obtain simpler spinors directly in $4+1$ dimensions using representation theory of the $(4+1)$-dimensional Lorentz group.

where the factor $\frac{1}{2}$ is to counteract double-counting. It is common for mathematicians to use exponentiation without a factor i, and for physicists to use the factor i, making the conventions for the generators differ by an imaginary unit. In this section I will use the mathematicians' convention.

The algebra $\mathfrak{so}(4,1,\mathbb{R})$ of the generators is specified by the commutation rules:

$$[\lambda_{\mu\nu}, \lambda_{\rho\sigma}] = g_{\mu\sigma}\lambda_{\nu\rho} + g_{\nu\rho}\lambda_{\mu\sigma} - g_{\mu\rho}\lambda_{\nu\sigma} - g_{\nu\sigma}\lambda_{\mu\rho}\,, \tag{34}$$

or alternatively:

$$[J_{\mu\nu}, J_{\rho\sigma}] = i\Big(g_{\mu\sigma}J_{\nu\rho} + g_{\nu\rho}J_{\mu\sigma} - g_{\mu\rho}J_{\nu\sigma} - g_{\nu\sigma}J_{\mu\rho}\Big)\,, \tag{35}$$

the latter being the Lorentz algebra commutation relation familiar to most physicists [77, 78]. The components J_{0i} (or λ_{0i}) can be identified as boost generators in particular, and J_{ij} (or λ_{ij}) as generators of rotations in the (x_i, x_j) plane.

1. *Double cover of the five-dimensional Lorentz group*

Spinors are objects that transform under the double-cover of the Lorentz group. In four-dimensional spacetime, the Lorentz group $\mathrm{SO}(3,1,\mathbb{R})$ has $\mathrm{SL}(2,\mathbb{C})$ as its double cover. Left-handed and right-handed spinors then transform under two inequivalent fundamental representations of $\mathrm{SL}(2,\mathbb{C})$.

In analogy, the five-dimensional Lorentz group also has a double cover: the pseudo-unitary group $\mathrm{U}(1,1,\mathbb{H})$, of 2×2 quaternion-valued matrices that preserve the metric $\eta = \mathrm{diag}(1,-1)$ [79–81]:

$$M\eta M^\dagger = \eta\,. \tag{36}$$

Alternatively, one can avoid the use of quaternions by using the pseudo-unitary symplectic group $\mathrm{USp}(2,2,\mathbb{C})$ of complex 4×4 matrices that satisfy both [82, 83]:

$$\begin{aligned} &MHM^\dagger = H && H = \mathrm{diag}(1,1,-1,-1) \\ &M\Omega M^\intercal = \Omega && \Omega = \begin{bmatrix} i\sigma_2 & 0 \\ 0 & i\sigma_2 \end{bmatrix} \end{aligned} \tag{37}$$

which is isomorphic to $\mathrm{U}(1,1,\mathbb{H})$ [81–83]. The isomorphism maps the unit quaternions $(\mathbf{i},\mathbf{j},\mathbf{k})$ to the Pauli matrices:

$$\mathbf{i} \equiv -i\sigma_1 \qquad \mathbf{j} \equiv -i\sigma_2 \qquad \mathbf{k} \equiv -i\sigma_3\,. \tag{38}$$

Ultimately, the complex matrix representation will be necessary to construct explicit spinors, but quaternion algebra allows faster rote calculations, and the map (38) allows the result of any quaternion calculation to be promptly translated into the complex matrix representation.

The generators of $\mathrm{U}(1,1,\mathbb{H})$ can be written:

$$\begin{aligned} &\lambda_{01} = \frac{1}{2}\begin{bmatrix} 0 & -\mathbf{i} \\ \mathbf{i} & 0 \end{bmatrix} && \lambda_{02} = \frac{1}{2}\begin{bmatrix} 0 & -\mathbf{j} \\ \mathbf{j} & 0 \end{bmatrix} && \lambda_{03} = \frac{1}{2}\begin{bmatrix} 0 & -\mathbf{k} \\ \mathbf{k} & 0 \end{bmatrix} && \lambda_{04} = \frac{1}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\ &\lambda_{14} = \frac{1}{2}\begin{bmatrix} \mathbf{i} & 0 \\ 0 & -\mathbf{i} \end{bmatrix} && \lambda_{24} = \frac{1}{2}\begin{bmatrix} \mathbf{j} & 0 \\ 0 & -\mathbf{j} \end{bmatrix} && \lambda_{34} = \frac{1}{2}\begin{bmatrix} \mathbf{k} & 0 \\ 0 & -\mathbf{k} \end{bmatrix} \\ &\lambda_{23} = \frac{1}{2}\begin{bmatrix} \mathbf{i} & 0 \\ 0 & \mathbf{i} \end{bmatrix} && \lambda_{31} = \frac{1}{2}\begin{bmatrix} \mathbf{j} & 0 \\ 0 & \mathbf{j} \end{bmatrix} && \lambda_{12} = \frac{1}{2}\begin{bmatrix} \mathbf{k} & 0 \\ 0 & \mathbf{k} \end{bmatrix}\,, \end{aligned} \tag{39}$$

and obey the commutation relations (34). From these generators, one can readily confirm through exponentiation that a 2π rotation in any spatial plane produces a factor -1, as expected for fermion representations.

Unlike the four-dimensional case, there is only one fundamental representation of the covering group up to unitary equivalence. In the four-dimensional case, right- and left-handed spinors transform under $\psi_R \mapsto M\psi_R$ and $\psi_L \mapsto (M^\dagger)^{-1}\psi_L$ respectively, and there is no unitary U for which $(M^\dagger)^{-1} = UMU^\dagger$ for every $M \in \mathrm{SL}(2,\mathbb{C})$. By contrast, for any $M \in \mathrm{U}(1,1,\mathbb{H})$, $\eta = \mathrm{diag}(1,-1)$ is a unitary matrix for which:

$$\eta M \eta = (M^\dagger)^{-1}\,, \tag{40}$$

as in fact follows from the definition of the group (36). This means, for instance, if a spinor transforms as $\psi \mapsto M\Psi$, then $\eta\psi \mapsto (M^\dagger)^{-1}\eta\psi$.

2. Matrix representation of five-vectors

A helpful trick often used in four-dimensional relativistic physics is to write four-vectors as 2×2 complex-valued matrices through the construction $\hat{X} = X^\mu \sigma_\mu$, where $\sigma_\mu = (\mathbb{1}; \sigma_1, \sigma_2, \sigma_3)$ are the Infeld-van der Waerden symbols [84, 85]. This is possible because four-vectors are in a sense rank-two tensors, with indices that transform under the left-handed and right-handed defining representations of $\mathrm{SL}(2, \mathbb{C})$. (See Refs. [86–90] for further details.) A similar trick can be employed for five-vectors—which similarly behave like rank-two tensors with respect to the defining representation of $\mathrm{U}(1, 1, \mathbb{H})$, and accordingly can be represented using 2×2 quaternion-valued matrices. The appropriate construction is [80]:

$$\begin{gathered} \hat{X} = X^\mu \tau_\mu = \begin{bmatrix} X^0 & X^4 - X^1\mathbf{i} - X^2\mathbf{j} - X^3\mathbf{k} \\ X^4 + X^1\mathbf{i} + X^2\mathbf{j} + X^3\mathbf{k} & X^0 \end{bmatrix} \\ \tau_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \tau_1 = \begin{bmatrix} 0 & -\mathbf{i} \\ \mathbf{i} & 0 \end{bmatrix} \qquad \tau_2 = \begin{bmatrix} 0 & -\mathbf{j} \\ \mathbf{j} & 0 \end{bmatrix} \qquad \tau_3 = \begin{bmatrix} 0 & -\mathbf{k} \\ \mathbf{k} & 0 \end{bmatrix} \qquad \tau_4 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} . \end{gathered} \tag{41}$$

These τ_μ matrices are in effect a five-dimensional analogue of the Infeld-van der Waerden symbols σ_μ. Under a five-dimensional Lorentz transform, the five-vector transforms as:

$$\hat{X} \mapsto \hat{X}' = M \hat{X} M^\dagger . \tag{42}$$

One can verify by explicit calculation that this gives the expected transformation laws.

3. The Galilei subgroup

The matrix construction (41) for five-vectors is especially helpful for identifying the Galilei subgroup. This is the subgroup generated by rotations in the xy, yz and zx planes, along with boosts following the Galilei transformation rule (4), where $x^\pm = \frac{1}{\sqrt{2}}\left(x^0 \pm x^4\right)$. The matrix representation of five-vectors (41) can be written in terms of light front coordinates if:

$$\tau^- = \tau_+ = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \qquad \tau^+ = \tau_- = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} . \tag{43}$$

A helpful rule of thumb for identifying Galilei transforms is that they will not transform τ_- into other tau matrices, and they will not transform other tau matrices into τ_+—ensuring that Galilean time is invariant and Galilean energy does not mix into momentum or mass. One can show through explicit calculation that the Galilei subgroup can be generated by:

$$\begin{aligned} &\lambda_{23}\,, \qquad \lambda_{31}\,, \qquad \lambda_{12}\,, \\ &\beta_a = \frac{1}{\sqrt{2}}\left(\lambda_{0a} + \lambda_{4a}\right) \qquad a \in \{1, 2, 3\}\,, \end{aligned} \tag{44}$$

with the λ_{ab} constituting rotations and the β_a constituting boosts. The Galilei boosts are similar in form to the light front transverse boosts in Refs. [66–68], the three-dimensional xyz-space being the "transverse plane" in this case. Using the map (38), the complex matrix representation for the Galilei boost generators is:

$$\beta_a = \frac{i}{2\sqrt{2}} \begin{bmatrix} \sigma_a & \sigma_a \\ -\sigma_a & -\sigma_a \end{bmatrix} , \tag{45}$$

which will be indispensable for constructing the Galilei spinors.

B. The five-dimensional Dirac equation

The Dirac equation in five dimensions involves eight-component spinors, since there are are positive- and negative-energy solutions (i.e., particles and anti-particles) that can have simultaneous eigenvalues of $\pm\frac{1}{2}$ for both J_{12} and J_{34}. While this may seem counterintuitive from familiarity with three-dimensional space, rotations in the xy and zw planes commute in four spatial dimensions. The positive- and negative-energy solutions should thus each have four independent components, giving eight total.

The five-dimensional Dirac equation (with $p^\mu p_\mu = 2mE_0$ as invariant) is:

$$p_\mu \gamma^\mu u(p) \equiv p^\mu \begin{bmatrix} 0 & \tau_\mu \\ \eta\tau_\mu\eta & 0 \end{bmatrix} \begin{bmatrix} \chi_1(p) \\ \eta\chi_2(p) \end{bmatrix} = \sqrt{2mE_0} \begin{bmatrix} \chi_1(p) \\ \eta\chi_2(p) \end{bmatrix} , \tag{46}$$

which effectively defines the 8×8 gamma matrices, which are explicitly given in Appendix A. Each of $\chi_{1,2}(p)$ is a four-component column matrix in the complex defining representation of the covering group. In line with the discussion in Sec. III A 1, the top four components of $u(p)$ transform as $\chi_1(p) \mapsto M\chi_1(p)$, while the bottom four components transform as $\eta\chi_2(p) \mapsto (M^\dagger)^{-1}\eta\chi_2(p)$. The full eight-component spinor thus transforms as:

$$u(p) \mapsto \begin{bmatrix} M & 0 \\ 0 & (M^\dagger)^{-1} \end{bmatrix} u(p)\,. \tag{47}$$

Given the transformation law (42), it is straightforward to verify that the Dirac equation (46) is covariant under the full $(4+1)$-dimensional Lorentz group—and thus, accordingly, under its Galilei subgroup.

1. *Parity reversal*

Let us next consider how the new spinors transform under *Galilean* parity inversion. This will be vital when restricting ourselves to the Galilei subgroup, to ensure the physical solutions we construct transform correctly under parity.

Galilean parity is distinct from Lorentzian parity in this context, and should transform only the $(1,2,3)$ components of five-vectors, leaving the $(+,-)$ components (and thus, the $(0,4)$ components) alone:

$$\begin{aligned} P\tau^{1,2,3}(P^\dagger)^{-1} &= -\tau^{1,2,3} \\ P\tau^{0,4}(P^\dagger)^{-1} &= +\tau^{0,4}\,. \end{aligned} \tag{48}$$

A matrix that satisfies these is $M = (M^\dagger)^{-1} = \tau_4$. Given Eq. (47), the eight-component spinors transform under Galilean parity reversal as:

$$P = \begin{bmatrix} \tau_4 & 0 \\ 0 & \tau_4 \end{bmatrix} = \begin{bmatrix} 0 & \mathbb{1}_{2\times2} & 0 & 0 \\ \mathbb{1}_{2\times2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbb{1}_{2\times2} \\ 0 & 0 & \mathbb{1}_{2\times2} & 0 \end{bmatrix}. \tag{49}$$

A peculiarity of Galilean parity reversal is that it is absent from $\mathrm{U}(1,1,\mathbb{H})$—in fact, $\eta P\eta = -P = -(P^\dagger)^{-1}$. For that matter, Galilean parity reversal is also absent from $\mathrm{SO}(4,1,\mathbb{R})$, since a transformation that flips the signs of only three components of a five-vector has determinant -1. Thus, the Galilei subgroup of the five-dimensional Lorentz group needs to be extended to include parity reversal.

A consequence of the fact that $\eta P\eta \neq (P^\dagger)^{-1}$ is that the four-component spinors χ_1 and $\eta\chi_2$ are no longer equivalent if the Galilei group is extended to include parity reversal. Thus, in terms of the extended Galilei group, the eight-component spinors are the direct sum of two inequivalent representations, tightening the analogy to Dirac spinors.

2. *Solutions at five-dimensional rest*

Next next us consider solutions to the five-dimensional Dirac equation (46) that are at rest in five-dimensional spacetime. To be sure, this does not mean the system is a physical rest solution: it means $p^{1,2,3,4} = 0$, and thus $p^0 = \sqrt{2mE_0}$ and accordingly $p^+ = \sqrt{mE_0}$. The equation for positive-energy rest solutions is:

$$\begin{bmatrix} 0 & \sqrt{2mE_0} \\ \sqrt{2mE_0} & 0 \end{bmatrix} \begin{bmatrix} \chi_1(0) \\ \eta\chi_2(0) \end{bmatrix} = \sqrt{2mE_0} \begin{bmatrix} \chi_1(0) \\ \eta\chi_2(0) \end{bmatrix}, \tag{50}$$

which is solved when $\chi_1(p) = \eta\chi_2(p)$. There are thus four independent solutions. As explained above, these can be categorized by the eigenvalues of the commuting operators J_{12} and J_{34}. Since

$$J_{12} = i\lambda_{12} = \frac{1}{2}\begin{bmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{bmatrix}, \qquad\qquad J_{34} = i\lambda_{34} = \frac{1}{2}\begin{bmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{bmatrix}, \tag{51}$$

solutions with one non-zero component of $\chi_1(0)$ are the simultaneous eigenstates.

Next, we need to ensure the spinors are invariant under Galilean parity reversal, namely when the P matrix in Eq. (49) acts on them. This eliminates two of the solutions, with the remaining solutions being:

$$u_\uparrow(0) = N \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \qquad u_\downarrow(0) = N \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \tag{52}$$

which are respectively spin-up and spin-down in the xy plane[9]. Here, N is a normalization factor (to be set later).

3. *Physical solutions in momentum space*

To obtain physical solutions to the Dirac equation, we first must boost Eq. (52) in the unphysical direction by a rapidity $y = \frac{1}{2}\log\left(\frac{m}{E_0}\right)$ to set $p^+ = m$, and subsequently perform a Galilei boost of velocity $\boldsymbol{v} = \frac{\boldsymbol{p}}{m}$. The relevant boost along the unphysical direction is:

$$M_{\text{unph}}(y) = \exp\left\{\frac{y}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right\} = \begin{bmatrix} \cosh\frac{y}{2} & \sinh\frac{y}{2} \\ \sinh\frac{y}{2} & \cosh\frac{y}{2} \end{bmatrix}. \tag{53}$$

Next, using Eq. (45), the relevant Galilei boost is:

$$M_{\text{gal}}(\boldsymbol{v}) = \exp\left(\boldsymbol{\beta}\cdot\boldsymbol{v}\right) = 1 + \frac{i}{2\sqrt{2}m}\begin{bmatrix} \boldsymbol{p}\cdot\boldsymbol{\sigma} & \boldsymbol{p}\cdot\boldsymbol{\sigma} \\ -\boldsymbol{p}\cdot\boldsymbol{\sigma} & -\boldsymbol{p}\cdot\boldsymbol{\sigma} \end{bmatrix}, \tag{54}$$

where the nilpotency of the generators (i.e., $\beta_a\beta_b = 0$) makes the expansion especially simple.

Applying the sequence of boosts $M_{\text{gal}}(\boldsymbol{v})M_{\text{unph}}(y)$ to the rest spinor (52) according to the transformation law (47) gives the physical spinor solutions in momentum space:

$$u_\uparrow(\boldsymbol{p}) = \frac{1}{2^{5/4}m}\begin{bmatrix} \sqrt{2}m + ip_z \\ ip_x - p_y \\ \sqrt{2}m - ip_z \\ -ip_x + p_y \\ \sqrt{2mE_0} \\ 0 \\ \sqrt{2mE_0} \\ 0 \end{bmatrix} \qquad u_\downarrow(\boldsymbol{p}) = \frac{1}{2^{5/4}m}\begin{bmatrix} ip_x + p_y \\ \sqrt{2}m - ip_z \\ -ip_x - p_y \\ \sqrt{2}m + ip_z \\ 0 \\ \sqrt{2mE_0} \\ 0 \\ \sqrt{2mE_0} \end{bmatrix}, \tag{55}$$

which are spin-up and spin-down along the z axis, respectively, and where I have chosen the normalization factor so that $\bar{u}(\boldsymbol{p},s)\gamma^+u(\boldsymbol{p},s) = 1$. Explicit matrix elements for gamma matrices between these spinors are given in Appendix A. It is curious to note a vague resemblance between these spinors and the Kogut-Soper spinors [66], likely due to both sets of spinors being light front spinors in different spacetimes.

C. Wave equation in coordinate space

To find the Galilei spinor wave equation in coordinate space, we must define a coordinate-space wave function. I use the following definition:

$$\Psi(\boldsymbol{x},t) \equiv \sum_s \int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}\, \mathrm{e}^{i\boldsymbol{p}\cdot\boldsymbol{x}} u(\boldsymbol{p},s)\langle \boldsymbol{p},s|\Psi(t)\rangle\,. \tag{56}$$

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[9] In three spatial dimensions, this is the same as saying they're spin-up and spin-down along the z axis. In four spatial dimensions, "spin-up" means spinning from the positive x axis towards the positive y axis, and "spin-down" means spinning in the opposite direction.

From the momentum-space equation (46), one can derive the following wave equation:

$$\left(i\gamma^+\frac{\partial}{\partial t} + i\boldsymbol{\gamma}\cdot\boldsymbol{\nabla} + m\gamma^- - \sqrt{2mE_0}\right)\Psi(\boldsymbol{x},t) = 0\,. \tag{57}$$

This is the coordinate-space wave equation for Galilei spinors.

1. Two-component spinors

Since there are only two independent degrees of freedom, one may ask whether Eq. (57) and its solutions can be recast in terms of two-component objects. The answer is yes: with the aid of Eq. (55), solutions of Eq. (57) can be written in terms of two complex-valued wave functions:

$$\Psi(\boldsymbol{x},t) = \frac{1}{2^{5/4}m}\begin{bmatrix} \sqrt{2}m + \partial_z \\ \partial_x + i\partial_y \\ \sqrt{2}m - \partial_z \\ -\partial_x - i\partial_y \\ \sqrt{2mE_0} \\ 0 \\ \sqrt{2mE_0} \\ 0 \end{bmatrix}\phi_\uparrow(\boldsymbol{x},t) + \frac{1}{2^{5/4}m}\begin{bmatrix} \partial_x - i\partial_y \\ \sqrt{2}m - \partial_z \\ -\partial_x + i\partial_y \\ \sqrt{2}m + \partial_z \\ 0 \\ \sqrt{2mE_0} \\ 0 \\ \sqrt{2mE_0} \end{bmatrix}\phi_\downarrow(\boldsymbol{x},t)\,, \tag{58}$$

where $\phi_s(\boldsymbol{x},t)$ are the Fourier transforms of $\langle \boldsymbol{p},s|\Psi(t)\rangle$:

$$\phi_s(\boldsymbol{x},t) = \int\frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}\,\mathrm{e}^{i\boldsymbol{p}\cdot\boldsymbol{x}}\langle \boldsymbol{p},s|\Psi(t)\rangle\,. \tag{59}$$

The two functions $\phi_s(\boldsymbol{x},t)$ can be combined into a two-component object as such:

$$\psi(\boldsymbol{x},t) = \begin{bmatrix}\phi_\uparrow(\boldsymbol{x},t) \\ \phi_\downarrow(\boldsymbol{x},t)\end{bmatrix}\,. \tag{60}$$

This object is conveniently normalized to:

$$\int \mathrm{d}^3\boldsymbol{x}\,\psi^\dagger(\boldsymbol{x},t)\psi(\boldsymbol{x},t) = 1\,, \tag{61}$$

and in the absence of interactions, its components obey independent Schrödinger equations:

$$i\frac{\partial\phi_s(\boldsymbol{x},t)}{\partial t} = E_0\phi_s(\boldsymbol{x},t) - \frac{\boldsymbol{\nabla}^2}{2m}\phi_s(\boldsymbol{x},t)\,. \tag{62}$$

This of course seems much easier to deal with than eight-component spinors, and are accordingly preferable to work with in a fixed frame. However, the eight-component spinors and their wave equation (57) have the correct transformation properties under Galilei boosts[10]. Accordingly, the eight-component spinors are the appropriate objects to use for constructing currents and matrix elements, including those appearing in form factor breakdowns. Additionally, the eight-component spinors contain additional physical information absent in the two-component formulation, which is necessary to obtain the correct dynamics. As an example, I will show as an example that they entail the correct gyromagnetic ratio $g = 2$—a feature also of the Lévy-Leblond spinors [75].

2. Gyromagnetic ratio

One of the most celebrated successes of Lévy-Leblond spinors is that they give a gyromagnetic ratio $g = 2$, demonstrating that this is not a strictly relativistic outcome [75]. It is therefore an important sanity check that the eight-component spinors of this work also give $g = 2$.

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[10] Technically, it is $\Psi(\boldsymbol{x},t)\,\mathrm{e}^{-imx^-}$ that transforms appropriately under Galilei boosts. However, since mass is fixed in non-relativistic physics, the factor e^{-imx^-} can be safely neglected under most circumstances, and does not affect observables. The factor only needs to be tracked when analyzing transformations of the wave function under Galilei boosts, to ensure the phase correctly transforms.

Electromagnetic interactions are introduced through minimal substitution, as usual: $\partial_\mu \mapsto \mathcal{D}_\mu = \partial_\mu - ieA_\mu$. This needs to be done both in the wave equation (57) itself, and—if we're building an equation for the two-component spinors—in the derivatives of Eq. (58). To introduce a magnetic field in particular, a time-independent vector potential in particular is introduced. Performing the required substitutions and working out the rote algebra gives:

$$i\frac{\partial}{\partial t}\psi(\boldsymbol{x},t) = \left(E_0 - \frac{\left(\boldsymbol{\nabla} - ie\boldsymbol{A}\right)^2}{2m} - \frac{e\boldsymbol{B}\cdot\boldsymbol{\sigma}}{2m}\right)\psi(\boldsymbol{x},t)\,, \tag{63}$$

which is Pauli's equation with $g = 2$ [77, 91].

It is worth taking a moment to dwell on this result. If one were to perform minimal substitution in the independent Schrödinger equations (62) only, there would be no interaction with the magnetic field and we'd find $g = 0$; just as in Pauli's original formulation, we'd need to put the gyromagnetic factor in by hand. The superstructure attending the eight-component spinors contains physics that is otherwise missing from Eq. (62), related to Galilei symmetry—showing that these eight component spinors are not at all superfluous.

IV. MECHANICAL FORM FACTORS OF SPIN-HALF SYSTEMS

Now that we have Galilei-covariant spinors, we can finally construct a Galilei-covariant expression for the mechanical form factors. Like with the spin-zero case, I write the breakdown in a manner that closely mirrors existing relativistic breakdowns, using the expression from Won and Lorcé [19] in particular as a guide. The form factor breakdown is:

$$\begin{aligned}\langle \boldsymbol{p}', s'|\hat{T}^{\mu\nu}_q(0)|\boldsymbol{p}, s\rangle = \sqrt{\frac{m}{2E_0}}\bar{u}(\boldsymbol{p}',s')\bigg\{\frac{P^\mu P^\nu}{m}A_q(\varDelta^2) + \frac{\varDelta^\mu\varDelta^\nu - g^{\mu\nu}\varDelta^2}{4m}D_q(\varDelta^2) + mg^{\mu\nu}\bar{c}_q(\varDelta^2) + P^\mu n^\nu \bar{e}_q(\varDelta^2)\\ + \frac{iP^{\{\mu}\sigma^{\nu\}\varDelta}}{2m}J_q(\varDelta^2) - \frac{iP^{[\mu}\sigma^{\nu]\varDelta}}{2m}S_q(\varDelta^2) + \frac{iE_0\sigma^{\mu\varDelta}n^\nu}{2m}\bar{f}_q(\varDelta^2)\bigg\}u(\boldsymbol{p},s)\,,\end{aligned} \tag{64}$$

where $J_q(\varDelta^2)$ a total angular momentum form factor [7] and $S_q(\varDelta^2)$ is a fermion spin form factor [8, 19]. (The latter can be set to zero if we want to look at the symmetric EMT.) The form factors $\bar{e}_q(\varDelta^2)$ and $\bar{f}_q(\varDelta^2)$ here are new, and idiosyncratic to the non-relativistic case; like $\bar{e}_q(\varDelta^2)$ in the spin-zero breakdown (10), they are a consequence of the fact that energy does not mix into mass or momentum under boosts, so there can be a component of the energy density that is independent of the mass and momentum densities. The overall normalization factor $\sqrt{\frac{m}{2E_0}}$ is introduced to cancel similar factors in the explicit spinor elements of Eq. (A9).

A. Explicit expressions in terms of two-component spinors

The use of eight-component spinors in Eq. (64) is necessitated by covariance under Galilei transforms—including discrete transforms like parity reversal. However, as shown in Eq. (58), there are only two independent degrees of freedom for solutions to the wave equation, and it is often convenient in practical calculations to work with two-component spinors (60). To be sure, the latter do not transform under a matrix representation of the Galilei group [74], but within a fixed frame the two-component spinors permit easier algebraic manipulations. For this reason, it is to convenient to explicitly evaluate Eq. (64) in terms of two-component spinors. The evaluation is done with the aid of the formulas in Appendix A.

It is easiest to look at collections of components bit-by-bit. The first few components, relevant to the mass density, mass flux density, and momentum density respectively, are:

$$\begin{aligned}\langle \boldsymbol{p}', s'|\hat{T}^{++}_q(0)|\boldsymbol{p}, s\rangle &= mA_q(\varDelta^2)\delta_{s's}\\ \langle \boldsymbol{p}', s'|\hat{T}^{a+}_q(0)|\boldsymbol{p}, s\rangle &= P^a A_q(\varDelta^2)\delta_{s's} - \frac{i(\boldsymbol{\varDelta}\times\boldsymbol{\sigma}_{s's})^a}{2}\Big(J_q(\varDelta^2) + S_q(\varDelta^2)\Big)\\ \langle \boldsymbol{p}', s'|\hat{T}^{+a}_q(0)|\boldsymbol{p}, s\rangle &= P^a A_q(\varDelta^2)\delta_{s's} - \frac{i(\boldsymbol{\varDelta}\times\boldsymbol{\sigma}_{s's})^a}{2}\Big(J_q(\varDelta^2) - S_q(\varDelta^2)\Big)\,.\end{aligned} \tag{65}$$

The components relevant to the momentum flux density (i.e., the stress tensor) are:

$$\begin{aligned}\langle \boldsymbol{p}', s'|\hat{T}^{ab}_q(0)|\boldsymbol{p}, s\rangle = \frac{P^aP^b}{m}A_q(\varDelta^2)\delta_{s's} + \frac{\varDelta^a\varDelta^b - \delta^{ab}\boldsymbol{\varDelta}^2}{4m}D_q(\varDelta^2)\delta_{s's} - m\delta^{ab}\bar{c}_q(\varDelta^2)\delta_{s's}\\ - \frac{i(\boldsymbol{\varDelta}\times\boldsymbol{\sigma}_{s's})^{\{a}P^{b\}}}{2m}J_q(\varDelta^2) - \frac{i(\boldsymbol{\varDelta}\times\boldsymbol{\sigma}_{s's})^{[a}P^{b]}}{2m}S_q(\varDelta^2)\,.\end{aligned} \tag{66}$$

The component relevant to the energy density is:

$$\langle \boldsymbol{p}', s'|\hat{T}_q^{+-}(0)|\boldsymbol{p}, s\rangle = \left\{\left(E_0 + \frac{\boldsymbol{P}^2}{2m}\right) A_q(\boldsymbol{\Delta}^2) + m\Big(\bar{c}_q(\boldsymbol{\Delta}^2) + \bar{e}_q(\boldsymbol{\Delta}^2)\Big)\right\} \delta_{s's} + \frac{\boldsymbol{\Delta}^2}{4m}\left(D_q(\boldsymbol{\Delta}^2) + \frac{1}{2}A_q(\boldsymbol{\Delta}^2) - J_q(\boldsymbol{\Delta}^2) + S_q(\boldsymbol{\Delta}^2)\right)\delta_{s's} - \frac{i(\boldsymbol{\Delta}\times\boldsymbol{\sigma}_{s's})\cdot\boldsymbol{P}}{2m}\Big(J_q(\boldsymbol{\Delta}^2) - S_q(\boldsymbol{\Delta}^2)\Big) . \quad (67)$$

Lastly, the components relevant to the energy flux density are:

$$\begin{aligned}\langle \boldsymbol{p}', s'|\hat{T}_q^{a-}(0)|\boldsymbol{p}, s\rangle = {} & \frac{P^a}{m}\left\{\left(E_0 + \frac{\boldsymbol{P}^2}{2m} + \frac{\boldsymbol{\Delta}^2}{8m}\right) A_q(\boldsymbol{\Delta}^2) + m\bar{e}_q(\boldsymbol{\Delta}^2) - \frac{\boldsymbol{\Delta}^2}{4m}\Big(J_q(\boldsymbol{\Delta}^2) - S_q(\boldsymbol{\Delta}^2)\Big)\right\}\delta_{s's} \\ & - \frac{P^a}{m}\frac{i(\boldsymbol{\Delta}\times\boldsymbol{\sigma}_{s's})\cdot\boldsymbol{P}}{2m}\Big(J_q(\boldsymbol{\Delta}^2) - S_q(\boldsymbol{\Delta}^2)\Big) \\ & - \frac{i(\boldsymbol{\Delta}\times\boldsymbol{\sigma}_{s's})^a}{2m}\left\{E_0\Big(J_q(\boldsymbol{\Delta}^2) + S_q(\boldsymbol{\Delta}^2) + \bar{f}_q(\boldsymbol{\Delta}^2)\Big) + \left(\frac{\boldsymbol{P}^2}{2m} + \frac{\boldsymbol{\Delta}^2}{8m}\right)\Big(J_q(\boldsymbol{\Delta}^2) + S_q(\boldsymbol{\Delta}^2)\Big)\right\} . \end{aligned} \quad (68)$$

B. Density formulas in the Breit frame formalism

The standard formalism for defining internal densities of composite systems is the Breit frame formalism [10, 11, 92, 93]—although it is not universally accepted [13, 68, 94–96]. In Sec. V, I will provide yet another competitor to the Breit frame formalism. However, for the sake of completeness, I here provide Breit frame density formulas as well.

The relevant formulas are simple to obtain—just evaluate the matrix element (64) in the $\boldsymbol{P} = 0$ frame (i.e., the Breit frame), and perform a Fourier transform over $\boldsymbol{\Delta}$:

$$\langle T_q^{\mu\nu}(\boldsymbol{b})\rangle_{\mathrm{BF}} \equiv \sum_{ss'}\int\frac{\mathrm{d}^3\boldsymbol{\Delta}}{(2\pi)^3}\hat{\psi}^*_{s'}\hat{\psi}_s\langle\boldsymbol{p}', s'|\hat{T}_q^{\mu\nu}(0)|\boldsymbol{p}, s\rangle\,\mathrm{e}^{-i\boldsymbol{\Delta}\cdot\boldsymbol{b}}\Big|_{\boldsymbol{P}=0} , \quad (69)$$

where $\hat{\psi}$ is a two-component normal vector specifying the spin direction:

$$\hat{\psi} = \begin{bmatrix}\mathrm{e}^{-i\phi_s/2}\cos\frac{\theta_s}{2} \\ \mathrm{e}^{+i\phi_s/2}\sin\frac{\theta_s}{2}\end{bmatrix} , \quad (70)$$

so that the spin vector is given by:

$$\boldsymbol{\Sigma} = \frac{1}{2}\hat{\psi}^\dagger\boldsymbol{\sigma}\hat{\psi} . \quad (71)$$

Then, plugging the explicit matrix elements at the start of Sec. IV A into Eq. (69), the Breit frame densities are:

$$\begin{aligned}\langle T_q^{++}(\boldsymbol{b},\boldsymbol{\Sigma})\rangle_{\mathrm{BF}} &= m\int\frac{\mathrm{d}^3\boldsymbol{\Delta}}{(2\pi)^3}A_q(\boldsymbol{\Delta}^2)\,\mathrm{e}^{-i\boldsymbol{\Delta}\cdot\boldsymbol{b}} \\ \langle T_q^{a+}(\boldsymbol{b},\boldsymbol{\Sigma})\rangle_{\mathrm{BF}} &= m\int\frac{\mathrm{d}^3\boldsymbol{\Delta}}{(2\pi)^3}\frac{\boldsymbol{\Sigma}\times i\boldsymbol{\Delta}}{2m}\Big(J_q(\boldsymbol{\Delta}^2) + S_q(\boldsymbol{\Delta}^2)\Big)\mathrm{e}^{-i\boldsymbol{\Delta}\cdot\boldsymbol{b}} \\ \langle T_q^{+a}(\boldsymbol{b},\boldsymbol{\Sigma})\rangle_{\mathrm{BF}} &= m\int\frac{\mathrm{d}^3\boldsymbol{\Delta}}{(2\pi)^3}\frac{\boldsymbol{\Sigma}\times i\boldsymbol{\Delta}}{2m}\Big(J_q(\boldsymbol{\Delta}^2) - S_q(\boldsymbol{\Delta}^2)\Big)\mathrm{e}^{-i\boldsymbol{\Delta}\cdot\boldsymbol{b}} \\ \langle T_q^{ab}(\boldsymbol{b},\boldsymbol{\Sigma})\rangle_{\mathrm{BF}} &= \int\frac{\mathrm{d}^3\boldsymbol{\Delta}}{(2\pi)^3}\left\{\frac{\Delta^a\Delta^b - \delta^{ab}\boldsymbol{\Delta}^2}{4m}D_q(\boldsymbol{\Delta}^2) - m\delta^{ab}\bar{c}_q(\boldsymbol{\Delta}^2)\right\}\mathrm{e}^{-i\boldsymbol{\Delta}\cdot\boldsymbol{b}} \\ \langle T_q^{+-}(\boldsymbol{b},\boldsymbol{\Sigma})\rangle_{\mathrm{BF}} &= \int\frac{\mathrm{d}^3\boldsymbol{\Delta}}{(2\pi)^3}\left\{E_0A_q(\boldsymbol{\Delta}^2) + m\big(\bar{c}_q(\boldsymbol{\Delta}^2) + \bar{e}_q(\boldsymbol{\Delta}^2)\big) + \frac{\boldsymbol{\Delta}^2}{4m}\left(D_q(\boldsymbol{\Delta}^2) + \frac{1}{2}A_q(\boldsymbol{\Delta}^2) - J_q(\boldsymbol{\Delta}^2) + S_q(\boldsymbol{\Delta}^2)\right)\right\}\mathrm{e}^{-i\boldsymbol{\Delta}\cdot\boldsymbol{b}} \\ \langle T_q^{a-}(\boldsymbol{b},\boldsymbol{\Sigma})\rangle_{\mathrm{BF}} &= E_0\int\frac{\mathrm{d}^3\boldsymbol{\Delta}}{(2\pi)^3}\left(1 + \frac{\boldsymbol{\Delta}^2}{4mE_0}\right)\frac{\boldsymbol{\Sigma}\times i\boldsymbol{\Delta}}{2m}\left(J_q(\boldsymbol{\Delta}^2) + S_q(\boldsymbol{\Delta}^2) + \bar{f}_q(\boldsymbol{\Delta}^2) + \frac{\boldsymbol{\Delta}^2}{4mE_0}\Big(J_q(\boldsymbol{\Delta}^2) + S_q(\boldsymbol{\Delta}^2)\Big)\right)\mathrm{e}^{-i\boldsymbol{\Delta}\cdot\boldsymbol{b}} . \end{aligned} \quad (72)$$

C. Matching to relativistic form factors

An pertinent matter I have not yet addressed is the relationship between the non-relativistic form factors in Eq. (64) and the standard relativistic form factors. Non-relativistic quantum mechanics can only ever be an approximation—nature is relativistic, after all, and non-relativistic mechanics arises as an approximation when all the relevant particles move much more slowly than the speed of light. I should thus address how the form factors in Eq. (64) can be matched to relativistic mechanical form factors.

The most straightforward matching scheme is to evaluate both the relativistic and non-relativistic matrix elements of the EMT in the Breit frame (i.e., the $\boldsymbol{P} = 0$ frame) and define these to be equal. The matrix elements cannot, of course, be equal in all frames, since they transform under different groups (the Galilei and Lorentz groups). Imposing $\boldsymbol{P} = 0$ limits us to the rotational subgroup, which both groups have in common.

There are several additional impositions we must make for the matching to make sense, however. Firstly, we must set $E_0 = m$, since the rest energy is the mass in special relativity. Secondly, the matching must be performed for the symmetric EMT specifically. The reason for this is the appearance of $S_q(\boldsymbol{\varDelta}^2)$ in the matrix element of $\hat{T}^{+-}$. This happens in the Galilei-covariant framework because the energy density is given by an off-diagonal component $(+-)$ of the EMT. However, in the Lorentz-covariant framework, the energy density is given by the diagonal component $\hat{T}^{00}$. The antisymmetric form factor can thus never appear in the relativistic energy density, nor in any non-relativistic reduction thereof.

With these necessities explained, let's move on to the matching. In terms of the shorthand:

$$\langle \widetilde{T}^{\mu\nu}\rangle_{\mathrm{BF,NR}} = \left\langle \tfrac{\boldsymbol{\varDelta}}{2}, s' \middle| \hat{T}^{\mu\nu}_{\mathrm{sym.}}(0) \middle| -\tfrac{\boldsymbol{\varDelta}}{2}, s \right\rangle_{\mathrm{NR}} \Bigg|_{E_0=m} \qquad \langle \widetilde{T}^{\mu\nu}\rangle_{\mathrm{BF,rel.}} = \frac{\left\langle \tfrac{\boldsymbol{\varDelta}}{2}, s' \middle| \hat{T}^{\mu\nu}(0) \middle| -\tfrac{\boldsymbol{\varDelta}}{2}, s \right\rangle_{\mathrm{rel.}}}{\sqrt{1+\frac{\boldsymbol{\varDelta}^2}{4m^2}}}\,, \tag{73}$$

the matching dictionary is:

$$\begin{aligned} \langle \widetilde{T}^{+-}\rangle_{\mathrm{BF,NR}} &\equiv \langle \widetilde{T}^{00}\rangle_{\mathrm{BF,rel.}} \\ \langle \widetilde{T}^{+a}\rangle_{\mathrm{BF,NR}} &\equiv \langle \widetilde{T}^{0a}\rangle_{\mathrm{BF,rel.}} \\ \langle \widetilde{T}^{ab}\rangle_{\mathrm{BF,NR}} &\equiv \langle \widetilde{T}^{ab}\rangle_{\mathrm{BF,rel.}}\,. \end{aligned} \tag{74}$$

That is, matrix elements associated with the energy density, momentum density and stress tensor should match. The needed non-relativistic matrix elements are:

$$\begin{aligned} \langle \widetilde{T}^{+-}\rangle_{\mathrm{BF,NR}} &= E_0\left(A_q(\boldsymbol{\varDelta}^2) + \frac{m}{E_0}\big(\bar{c}_q(\boldsymbol{\varDelta}^2) + \bar{e}_q(\boldsymbol{\varDelta}^2)\big)\right)\delta_{s's} + \frac{\boldsymbol{\varDelta}^2}{4m}\left(D_q(\boldsymbol{\varDelta}^2) + \frac{1}{2}A_q(\boldsymbol{\varDelta}^2) - J_q(\boldsymbol{\varDelta}^2)\right)\delta_{s's} \\ \langle \widetilde{T}^{+a}\rangle_{\mathrm{BF,NR}} &= -\frac{i(\boldsymbol{\varDelta}\times\boldsymbol{\sigma}_{s's})^a}{2} J_q(\boldsymbol{\varDelta}^2) \\ \langle \widetilde{T}^{ab}\rangle_{\mathrm{BF,NR}} &= \frac{\varDelta^a\varDelta^b - \delta^{ab}\boldsymbol{\varDelta}^2}{4m} D_q(\boldsymbol{\varDelta}^2)\delta_{s's} - m\delta^{ab}\bar{c}_q(\boldsymbol{\varDelta}^2)\delta_{s's}\,, \end{aligned} \tag{75}$$

and the required relativistic matrix elements can be found in the literature [10–12]:

$$\begin{aligned} \langle \widetilde{T}^{00}\rangle_{\mathrm{BF,rel.}} &= m\left(\mathcal{A}_q(\boldsymbol{\varDelta}^2) + \bar{\mathcal{C}}_q(\boldsymbol{\varDelta}^2)\right)\delta_{s's} + \frac{\boldsymbol{\varDelta}^2}{4m}\left(\mathcal{D}_q(\boldsymbol{\varDelta}^2) + \mathcal{A}_q(\boldsymbol{\varDelta}^2) - 2\mathcal{J}_q(\boldsymbol{\varDelta}^2)\right)\delta_{s's} \\ \langle \widetilde{T}^{0a}\rangle_{\mathrm{BF,rel.}} &= -\frac{i(\boldsymbol{\varDelta}\times\boldsymbol{\sigma}_{s's})^a}{2}\mathcal{J}_q(\boldsymbol{\varDelta}^2) \\ \langle \widetilde{T}^{ab}\rangle_{\mathrm{BF,rel.}} &= \frac{\varDelta^a\varDelta^b - \delta^{ab}\boldsymbol{\varDelta}^2}{4m}\mathcal{D}_q(\boldsymbol{\varDelta}^2)\delta_{s's} - m\delta^{ab}\bar{\mathcal{C}}_q(\boldsymbol{\varDelta}^2)\delta_{s's}\,, \end{aligned} \tag{76}$$

where I have used calligraphic letters to signify the relativistic form factors. The matching dictionary (74) tells us:

$$\begin{gathered} \left(1 + \frac{\boldsymbol{\varDelta}^2}{8m^2}\right)A_q(\boldsymbol{\varDelta}^2) + \bar{e}_q(\boldsymbol{\varDelta}^2) = \mathcal{A}_q(\boldsymbol{\varDelta}^2) + \frac{\boldsymbol{\varDelta}^2}{4m^2}\left(\mathcal{A}_q(\boldsymbol{\varDelta}^2) - \mathcal{J}_q(\boldsymbol{\varDelta}^2)\right) \\ J_q(\boldsymbol{\varDelta}^2) = \mathcal{J}_q(\boldsymbol{\varDelta}^2) \qquad D_q(\boldsymbol{\varDelta}^2) = \mathcal{D}_q(\boldsymbol{\varDelta}^2) \qquad \bar{c}_q(\boldsymbol{\varDelta}^2) = \bar{\mathcal{C}}_q(\boldsymbol{\varDelta}^2)\,. \end{gathered} \tag{77}$$

Most of the form factors are apparently identical, with the exception being $A_q(\boldsymbol{\varDelta}^2) \neq \mathcal{A}_q(\boldsymbol{\varDelta}^2)$. In fact, the presence of $\bar{e}_q(\boldsymbol{\varDelta}^2)$ in the relevant expression makes the relationship to the relativistic form factors underdetermined. This underdetermination occurs because the non-relativistic theory has less symmetry than the relativistic theory—basically, the energy density does not mix into the momentum density under Galilei boosts—and $\bar{e}_q(\boldsymbol{\varDelta}^2)$ contains the additional information that symmetry alone can't provide. As an important sanity check, however, $A_q(0) + \bar{e}_q(0) = \mathcal{A}_q(0)$, meaning—in light of the sum rule (15)—that the momentum sum rule $\sum_q A_q(0) = 1$ can be satisfied simultaneously for both the relativistic and non-relativistic form factors. Additionally, the case of an elementary fermion—in which $A(\boldsymbol{\varDelta}^2) = \mathcal{A}(\boldsymbol{\varDelta}^2) = 1$, $\bar{e}(\boldsymbol{\varDelta}^2) = 0$ and $\mathcal{J}(\boldsymbol{\varDelta}^2) = \frac{1}{2}$—is consistent with this matching scheme.

V. FERMION DENSITIES IN PILOT WAVE THEORY

With the appropriate Galilei-covariant spinors (55) in hand, we can finally proceed to obtain densities of the energy-momentum tensor. The extra spin structures in the form factor breakdown complicate the identification of internal and barycentric contributions to the mechanical densities, and furthermore make any ambiguities in the separation more severe. Like in the spin-zero case, I will use considerations from the pilot wave formulation of quantum mechanics [53–56] to guide the separation. (See Refs. [56, 97, 98] for excellent introductions to the pilot wave formulation.)

A. Pilot waves for fermions

Two of the essential ingredients of the pilot wave formulation of quantum mechanics are the polar decomposition of the wave function and the continuity equation. First, for spin-half particles, the appropriate polar decomposition can be written in terms of the two-component spinors (60) as [56, 99, 100]:

$$\psi(\boldsymbol{x},t) = \begin{bmatrix} \phi_\uparrow(\boldsymbol{x},t) \\ \phi_\downarrow(\boldsymbol{x},t) \end{bmatrix} = \mathscr{R}(\boldsymbol{x},t)\,\mathrm{e}^{i\mathscr{S}(\boldsymbol{x},t)} \begin{bmatrix} \cos\left(\frac{\theta_s(\boldsymbol{x},t)}{2}\right)\mathrm{e}^{-i\phi_s(\boldsymbol{x},t)/2} \\ \sin\left(\frac{\theta_s(\boldsymbol{x},t)}{2}\right)\mathrm{e}^{+i\phi_s(\boldsymbol{x},t)/2} \end{bmatrix}. \tag{78}$$

Here, $\mathscr{R}(\boldsymbol{x},t)$ and $\mathscr{S}(\boldsymbol{x},t)$ are the real-valued magnitude and phase functions from the familiar polar decomposition of spin-zero wave functions, and $\theta_s(\boldsymbol{x},t)$ and $\phi_s(\boldsymbol{x},t)$ are angles specifying the spin direction at the location $\boldsymbol{x}$ and time t. The spin at $(\boldsymbol{x},t)$ can also be specified by the vector:

$$\boldsymbol{\Sigma}(\boldsymbol{x},t) = \frac{1}{2}\frac{\psi^\dagger(\boldsymbol{x},t)\boldsymbol{\sigma}\psi(\boldsymbol{x},t)}{\psi^\dagger(\boldsymbol{x},t)\psi(\boldsymbol{x},t)} = \frac{1}{2}\Big(\hat{z}\cos\big(\theta_s(\boldsymbol{x},t)\big) + \hat{x}\sin\big(\theta_s(\boldsymbol{x},t)\big)\cos\big(\phi_s(\boldsymbol{x},t)\big) + \hat{y}\sin\big(\theta_s(\boldsymbol{x},t)\big)\sin\big(\phi_s(\boldsymbol{x},t)\big)\Big). \tag{79}$$

Second, let us find the continuity equation entailed by the coordinate-space wave equation (57). Defining the conjugate spinor wave $\overline{\Psi}(\boldsymbol{x},t) = \Psi^\dagger(\boldsymbol{x},t)\gamma^0$, solutions to Eq. (57) obey the continuity equation:

$$\frac{\partial}{\partial t}\Big[\overline{\Psi}(\boldsymbol{x},t)\gamma^+\Psi(\boldsymbol{x},t)\Big] + \boldsymbol{\nabla}\cdot\Big[\overline{\Psi}(\boldsymbol{x},t)\boldsymbol{\gamma}\Psi(\boldsymbol{x},t)\Big] = 0\,. \tag{80}$$

This suggests the following probability density and Bohmian velocity:

$$\begin{aligned} \mathscr{P}(\boldsymbol{x},t) &= \overline{\Psi}(\boldsymbol{x},t)\gamma^+\Psi(\boldsymbol{x},t)\,, \\ \boldsymbol{v}_\mathrm{B}(\boldsymbol{x},t) &= \frac{\overline{\Psi}(\boldsymbol{x},t)\boldsymbol{\gamma}\Psi(\boldsymbol{x},t)}{\mathscr{P}(\boldsymbol{x},t)}\,. \end{aligned} \tag{81}$$

With the aid of the matrix elements in Appendix A, these can be written in terms of the two-component spinors (60) as:

$$\begin{aligned} \mathscr{P}(\boldsymbol{x},t) &= \psi^\dagger(\boldsymbol{x},t)\psi(\boldsymbol{x},t)\,, \\ \boldsymbol{v}_\mathrm{B}(\boldsymbol{x},t) &= -\frac{i}{2m}\frac{\psi^\dagger(\boldsymbol{x},t)\overleftrightarrow{\boldsymbol{\nabla}}\psi(\boldsymbol{x},t)}{\mathscr{P}(\boldsymbol{x},t)} + \frac{\boldsymbol{\nabla}\times\Big(\psi^\dagger(\boldsymbol{x},t)\boldsymbol{\sigma}\psi(\boldsymbol{x},t)\Big)}{2m\mathscr{P}(\boldsymbol{x},t)}\,, \end{aligned} \tag{82}$$

Notably, I recover Bohm and Hiley's velocity law for spinors [56]. Although it was originally derived through non-relativistic reduction of Dirac's equation, working in a fully Galilei-covariant framework is sufficient to obtain the same result. In fact, Wilkes earlier found that using Lévy-Leblond spinors also gives the same velocity law [101].

It is helpful to break the Bohmian velocity down into pieces corresponding to the convective (or irrotational) and magnetization (or solenoidal) parts of the probability current:

$$\begin{aligned} \boldsymbol{v}_\mathrm{B}(\boldsymbol{x},t) &= \boldsymbol{v}_\nabla(\boldsymbol{x},t) + \boldsymbol{v}_\sigma(\boldsymbol{x},t) \\ \boldsymbol{v}_\nabla(\boldsymbol{x},t) &= -\frac{i}{2m}\frac{\psi^\dagger(\boldsymbol{x},t)\overleftrightarrow{\boldsymbol{\nabla}}\psi(\boldsymbol{x},t)}{\mathscr{P}(\boldsymbol{x},t)} \\ \boldsymbol{v}_\sigma(\boldsymbol{x},t) &= \frac{\boldsymbol{\nabla}\times\Big(\psi^\dagger(\boldsymbol{x},t)\boldsymbol{\sigma}\psi(\boldsymbol{x},t)\Big)}{2m\mathscr{P}(\boldsymbol{x},t)}\,. \end{aligned} \tag{83}$$

Although the currents $\mathscr{P}\boldsymbol{v}_\nabla$ and $\mathscr{P}\boldsymbol{v}_\sigma$ are respectively irrotational and solenoidal (i.e., they respectively have zero curl and zero divergence), the velocities themselves may not be. In fact [100]:

$$\begin{aligned}\boldsymbol{v}_\nabla(\boldsymbol{x},t) &= \frac{\boldsymbol{\nabla}\mathscr{S}(\boldsymbol{x},t)}{m} - \frac{\cos\big(\theta_s(\boldsymbol{x},t)\big)}{2m}\boldsymbol{\nabla}\phi_s(\boldsymbol{x},t) \\ \boldsymbol{v}_\sigma(\boldsymbol{x},t) &= \frac{\boldsymbol{\nabla}\mathscr{P}(\boldsymbol{x},t)\times\boldsymbol{\Sigma}(\boldsymbol{x},t)}{m\mathscr{P}(\boldsymbol{x},t)} + \frac{\boldsymbol{\nabla}\times\boldsymbol{\Sigma}(\boldsymbol{x},t)}{m}\,. \end{aligned} \tag{84}$$

It is only when the spin direction $\boldsymbol{\Sigma}$ is fixed that $\boldsymbol{v}_\nabla$ and $\boldsymbol{v}_\sigma$ respectively have zero curl and zero divergence.

In typical circumstances, the spin direction of a wave function is not fixed. Fermions interact with their environment in a way that depends on their spin direction, especially when magnetic fields are present. Accordingly, the components $\phi_\uparrow(\boldsymbol{x},t)$ and $\phi_\downarrow(\boldsymbol{x},t)$ of the two-component spinor (60) will typically be evolve to be different functions of $(\boldsymbol{x},t)$. On the other hand, one of the underlying premises of the pilot wave formulation is that the fermion has a definite position $\boldsymbol{x}_{\text{true}}$, of which we are merely ignorant. It follows, then, that the fermion also has a definite spin given by $\boldsymbol{\Sigma}(\boldsymbol{x}_{\text{true}},t)$—similar to how $\boldsymbol{v}_{\text{B}}(\boldsymbol{x}_{\text{true}},t)$ is postulated to give the actual velocity[11]. Any point-to-point variation in the spin direction associated with the wave packet is a property of the wave packet, rather than an internal property of the fermion. Since the goal of this work is to characterize the *internal* densities of fermions, it is safe to limit our attention to wave packets with a fixed spin direction, which I will do for the remainder of the paper.

B. From form factors to densities

To relate the mechanical form factors to densities, I closely follow derivations given in Refs. [16, 72]. The expectation value of the EMT for a physical state $|\Psi(t)\rangle$ can be written with the help of completeness relations as:

$$\langle\Psi(t)|\hat{T}^{\mu\nu}(\boldsymbol{x})|\Psi(t)\rangle = \sum_{s,s'}\int\frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}\int\frac{\mathrm{d}^3\boldsymbol{p}'}{(2\pi)^3}\langle\Psi(t)|\boldsymbol{p}',s'\rangle\langle\boldsymbol{p}',s'|\hat{T}^{\mu\nu}(0)|\boldsymbol{p},s\rangle\langle\boldsymbol{p},s|\Psi(t)\rangle\,\mathrm{e}^{-i\boldsymbol{\Delta}\cdot\boldsymbol{x}}\,, \tag{85}$$

where $\boldsymbol{\Delta}=\boldsymbol{p}'-\boldsymbol{p}$, and where the matrix elements can be written in terms of mechanical form factors though Eq (64). Inverting the Fourier transform (59) allows us to write:

$$\langle\Psi(t)|\hat{T}^{\mu\nu}(\boldsymbol{x})|\Psi(t)\rangle = \sum_{s,s'}\int\frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}\int\frac{\mathrm{d}^3\boldsymbol{p}'}{(2\pi)^3}\int\mathrm{d}^3\boldsymbol{r}\int\mathrm{d}^3\boldsymbol{r}'\,\phi^*_{s'}(\boldsymbol{r}',t)\phi_s(\boldsymbol{r},t)\langle\boldsymbol{p}',s'|\hat{T}^{\mu\nu}(0)|\boldsymbol{p},s\rangle\,\mathrm{e}^{-i(\boldsymbol{\Delta}\cdot\boldsymbol{x}+\boldsymbol{p}\cdot\boldsymbol{r}-\boldsymbol{p}'\cdot\boldsymbol{r}')}\,. \tag{86}$$

Defining the variable transformations:

$$\begin{aligned}\boldsymbol{P} &= \frac{1}{2}(\boldsymbol{p}+\boldsymbol{p}') \qquad & \boldsymbol{\Delta} &= \boldsymbol{p}'-\boldsymbol{p} \\ \boldsymbol{R} &= \frac{1}{2}(\boldsymbol{r}+\boldsymbol{r}') \qquad & \boldsymbol{\rho} &= \boldsymbol{r}'-\boldsymbol{r}\,,\end{aligned}$$

we can write:

$$\langle\Psi(t)|\hat{T}^{\mu\nu}(\boldsymbol{x})|\Psi(t)\rangle = \sum_{s,s'}\int\frac{\mathrm{d}^3\boldsymbol{P}}{(2\pi)^3}\int\frac{\mathrm{d}^3\boldsymbol{\Delta}}{(2\pi)^3}\int\mathrm{d}^3\boldsymbol{R}\int\mathrm{d}^3\boldsymbol{\rho}\,\phi^*_{s'}(\boldsymbol{r}',t)\phi_s(\boldsymbol{r},t)\langle\boldsymbol{p}',s'|\hat{T}^{\mu\nu}(0)|\boldsymbol{p},s\rangle\,\mathrm{e}^{-i\boldsymbol{\Delta}\cdot(\boldsymbol{x}-\boldsymbol{R})}\,\mathrm{e}^{i\boldsymbol{P}\cdot\boldsymbol{\rho}}\,. \tag{87}$$

Just as in Refs. [16, 72], the integral over $\boldsymbol{P}$ can be performed if all instances of $\boldsymbol{P}$ appearing in the matrix element $\langle\boldsymbol{p}',s'|\hat{T}^{\mu\nu}(0)|\boldsymbol{p},s\rangle$ are replaced by the two-sided derivative $-\frac{i}{2}\overleftrightarrow{\boldsymbol{\nabla}}$, acting between $\phi^*_{s'}(\boldsymbol{r}',t)$ and $\phi_s(\boldsymbol{r},t)$. The integral over $\boldsymbol{P}$ then creates a delta function setting $\boldsymbol{\rho}=0$, giving the expression:

$$\langle\Psi(t)|\hat{T}^{\mu\nu}(\boldsymbol{x})|\Psi(t)\rangle = \sum_{s,s'}\int\mathrm{d}^3\boldsymbol{R}\int\frac{\mathrm{d}^3\boldsymbol{\Delta}}{(2\pi)^3}\phi^*_{s'}(\boldsymbol{R},t)\langle\boldsymbol{p}',s'|\hat{T}^{\mu\nu}(0)|\boldsymbol{p},s\rangle\Big|_{2i\boldsymbol{P}\to\overleftrightarrow{\boldsymbol{\nabla}}}\phi_s(\boldsymbol{R},t)\,\mathrm{e}^{-i\boldsymbol{\Delta}\cdot(\boldsymbol{x}-\boldsymbol{R})}\,. \tag{88}$$

At this point, the formulas in Sec. IV A can be used to evaluate the densities for fermions prepared in realistic wave packets.

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[11] To be sure, like the Bohmian velocity, the Bohmian spin $\boldsymbol{\Sigma}$ is not an observable, but a hidden variable. It is distinct from the spin operators $\frac{1}{2}\boldsymbol{\sigma}$. For instance, a fermion prepared in a spin-up state along the z axis has $\Sigma_x=\Sigma_y=0$, while the eigenvalues of $\frac{1}{2}\sigma_x$ and $\frac{1}{2}\sigma_y$ are always $\pm\frac{1}{2}$. When a fermion is not already in an eigenstate of a spin projection operator, a "measurement" of this spin projection disturbs the system and changes the system's spin. In fact, one of the perks of the pilot wave formulation is that it provides a detailed mechanistic picture of *how* the system is disturbed by a measurement; see Refs. [56, 98, 102–104] for more details. It should be remembered that in the pilot wave formulation, a "measurement" of any observable other than position does not passively reveal pre-existing properties. Accordingly, attributing ontological reality to what I have called the Bohmian spin $\boldsymbol{\Sigma}$ is not in contradiction with Bell-type inequalities [105–107], the Kochen-Specker theorem [108, 109] or the GHZ experiment [110, 111].

C. Densities of elementary fermions

I will now use Eq. (88) to evaluate densities of elementary fermions. Elementary fermions are a helpful intuition-building exercise, since they should have minimal internal structure, and accordingly most of the mass, momentum and energy distributions should be attributable to the wave packet. There are actually two cases to consider: (1) the asymmetric EMT, for which $S(\varDelta^2) = \frac{1}{2}$; and (2) the symmetric EMT, for which $S(\varDelta^2) = 0$. In both cases, the remaining form factors are given by $A(\varDelta^2) = 1$, $J(\varDelta^2) = \frac{1}{2}$, and $D(\varDelta^2) = \bar{c}(\varDelta^2) = \bar{e}(\varDelta^2) = \bar{f}(\varDelta^2) = 0$[12].

I will consider the mass density first. In either case, it is simply given by:

$$\langle\Psi(t)|\hat{T}^{++}(\boldsymbol{x})|\Psi(t)\rangle = m\mathscr{P}(\boldsymbol{x},t)\,, \tag{89}$$

which is compatible with the notion of the fermion being pointlike. The density in this case is entirely due to the uncertainty in the fermion's position.

Next, I consider the mass flux density. The result differs depending on whether the asymmetric or symmetric EMT is used. For the asymmetric EMT:

$$\langle\Psi(t)|\hat{T}^{a+}_{\text{asym}}(\boldsymbol{x})|\Psi(t)\rangle = m\mathscr{P}(\boldsymbol{x},t)v^a_{\text{B}}(\boldsymbol{x},t)\,, \tag{90}$$

which is compatible with the notion of a pointlike particle moving with the Bohmian velocity given by Eq. (83). On the other hand, the symmetric EMT gives a mass flux density of:

$$\langle\Psi(t)|\hat{T}^{a+}_{\text{sym}}(\boldsymbol{x})|\Psi(t)\rangle = m\mathscr{P}(\boldsymbol{x},t)\left(v^a_{\nabla}(\boldsymbol{x},t) + \frac{1}{2}v^a_{\sigma}(\boldsymbol{x},t)\right)\,, \tag{91}$$

where $\boldsymbol{v}_\nabla$ and $\boldsymbol{v}_\sigma$ are defined in Eq. (83). The interpretation of this expression is unclear, and appears at odds with the notion of a pointlike particle moving according to the Bohmian velocity law (83). The pilot wave interpretation accordingly seems to favor the asymmetric EMT.

Next, I consider the momentum density. This again differs for the asymmetric and symmetric EMT. In the first case:

$$\langle\Psi(t)|\hat{T}^{+a}_{\text{asym}}(\boldsymbol{x})|\Psi(t)\rangle = m\mathscr{P}(\boldsymbol{x},t)v^a_{\nabla}(\boldsymbol{x},t) \equiv \mathscr{P}(\boldsymbol{x},t)p^a_{\text{B}}(\boldsymbol{x},t)\,, \tag{92}$$

which in effect defines the Bohmian momentum. In fact, this expression for the Bohmian momentum agrees exactly with the result of Bohm and Hiley [56], which itself follows from the quantum Hamilton-Jacobi equation [99]. Accordingly, I also reproduce their observation that:

$$\boldsymbol{p}_{\text{B}}(\boldsymbol{x},t) \neq m\boldsymbol{v}_{\text{B}}(\boldsymbol{x},t)\,. \tag{93}$$

This again suggests that the pilot wave interpretation prefers the asymmetric EMT; the symmetric EMT instead gives equal momentum and mass flux densities, both of which involve the peculiar mixture $\boldsymbol{v}_\nabla + \frac{1}{2}\boldsymbol{v}_\sigma$.

Since the pilot wave interpretation favors it, I will focus exclusively on the asymmetric EMT for the remainder of this work. The inequivalence between momentum and velocity however raises a pertinent question: how do we define the rest frame in the pilot wave formulation? Should it be the zero-velocity frame, or the zero-momentum frame? Each choice will lead to different conclusions about the fermion's internal densities, since boosts from the rest frame mix components of the EMT—see Eq. (29).

The zero-momentum frame seems like the natural choice for the rest frame. Although the Bohmian velocity is non-zero in this frame, being given by $\boldsymbol{v}_\sigma$ (see Eq. (83)), it can be interpreted as revolutionary motion about some pivot [56], where the pivot itself moves with the convective velocity $\boldsymbol{v}_\nabla$. On the other hand, in the zero-velocity frame, there is non-zero revolutionary momentum *opposite* the spin direction, which has no sensible physical interpretation.

Internal densities of the fermion are defined relative to the rest frame of the pivot—i.e., relative to the zero-momentum frame. Accordingly, motion relative to the pivot should be considered part of the fermion's internal structure. This includes the mass flux density. For an elementary fermion with a spin vector $\boldsymbol{\varSigma}$, the internal mass flux density is thus:

$$\mathfrak{t}^{a+}_{\text{elem.}}(\boldsymbol{b},\boldsymbol{\varSigma}) = -(\boldsymbol{\varSigma}\times\boldsymbol{\nabla})^a\delta^{(3)}(\boldsymbol{b})\,. \tag{94}$$

Due to our ignorance of the pivot's exact location, we must smear this out by the probability density, meaning the smeared contribution is:

$$\int \mathrm{d}^3\boldsymbol{R}\,\psi^\dagger(\boldsymbol{R},t)\psi(\boldsymbol{R},t)\,\mathfrak{t}^{a+}\big(\boldsymbol{x}-\boldsymbol{R},\boldsymbol{\varSigma}(\boldsymbol{R},t)\big) = \frac{1}{2}\Big(\boldsymbol{\nabla}\times\big(\psi^\dagger(\boldsymbol{x},t)\boldsymbol{\sigma}\psi(\boldsymbol{x},t)\big)\Big)^a\,. \tag{95}$$

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[12] See Ref. [20] regarding the zero $D(\varDelta^2)$. The zero $\bar{c}(\varDelta^2)$ and $\bar{e}(\varDelta^2)$ are related to sum rules; cf. Sec II B. The zero $\bar{f}_q(\varDelta^2)$ is effectively a statement that mass and energy don't flow differently in an elementary fermion.

Subsequently boosting this by the pivot velocity gives Eq. (90).

The remaining components of the energy-momentum tensor can be broken down in a similar manner to Eq. (29), albeit using the pivot velocity $\boldsymbol{v}_\nabla$ in the Galilei boost matrix (28). The question now arises of how exactly to separate the rest-frame EMT into an intrinsic part $\mathfrak{t}^{\alpha\beta}$ and a quantum part $T_Q^{\alpha\beta}$. It is here that considering the case of fixed spin is helpful, for two reasons. First, any point-to-point variation of of the spin vector $\boldsymbol{\Sigma}$ is a property of the wave packet, rather than of the fermion's internal structure; thus any terms that are dropped by considering a constant spin vector are part of $T_Q^{\alpha\beta}$. Second, when the spin vector is constant, the quantum EMT is identical to the spin-zero case (31). The presence of point-to-point variations would introduce additional terms to the quantum potential [56, 99], and additional quantum stresses corresponding to torques that rotate the spin vector [99], but these matters are beyond the scope of this work and need not be considered to study internal structure.

With the spin vector now fixed, the procedure is simple. First, we evaluate Eq. (88) using the explicit matrix elements in Sec. IV A. Second, we perform the *inverse* Galilei boosts on the integrand of the result—i.e., the integrand is boosted by $-\boldsymbol{v}_\nabla$. Third, in accordance with Eq. (29), the quantum EMT (31) is subtracted off. What remains is the internal EMT of an elementary fermion, smeared by the probability density $\mathscr{P}$. The resulting non-zero components of the internal EMT are:

$$\begin{aligned} \mathfrak{t}^{++}_{\text{elem.}}(\boldsymbol{b},\boldsymbol{\Sigma}) &= m\delta^{(3)}(\boldsymbol{b}) & \qquad \mathfrak{t}^{a+}_{\text{elem.}}(\boldsymbol{b},\boldsymbol{\Sigma}) &= -(\boldsymbol{\Sigma}\times\boldsymbol{\nabla})^a\delta^{(3)}(\boldsymbol{b}) \\ \mathfrak{t}^{+-}_{\text{elem.}}(\boldsymbol{b},\boldsymbol{\Sigma}) &= E_0\delta^{(3)}(\boldsymbol{b}) & \qquad \mathfrak{t}^{a-}_{\text{elem.}}(\boldsymbol{b},\boldsymbol{\Sigma}) &= -\frac{E_0}{m}(\boldsymbol{\Sigma}\times\boldsymbol{\nabla})^a\delta^{(3)}(\boldsymbol{b})\,. \end{aligned} \tag{96}$$

In essence, the elementary fermion carries only mass and energy in the zero-momentum frame, which are rapidly revolving around a pivot that is at rest.

D. Composite fermions

Finally, let's look at densities of composite fermions. Most of the conceptual and formal elements needed to isolate the internal densities are now in place. The missing ingredient is an appropriate generalization of the breakdown (29). We could try to apply it in its current form to the special case of a constant spin vector $\boldsymbol{\Sigma}$, but we would find an extra term in the energy flux density that cannot be sorted into either of the terms present in Eq. (29). Nonetheless, I will proceed and patch up Eq. (29) afterwards.

The method is basically the same as we used to obtain Eq. (96). We first work out Eq. (88) using the matrix elements in Sec. IV A, and then apply the *inverse* of the Galilei boost matrix (28) to the integrand. Since constant $\boldsymbol{\Sigma}$ is assumed, Eq. (31) gives the quantum EMT, which is then subtracted off from the result, under the assumption that Eq. (29) is applicable. Ideally, the remaining terms should be equal to a convolution between $\mathscr{P}$ and a tensor that is independent of $\mathscr{P}$—the latter being interpretable as an internal density. This almost works out, but there is one extra term in the energy flux density that violates this expectation. Putting this extra term aside, the internal densities suggested by Eq. (29) are:

$$\begin{aligned} \mathfrak{t}^{++}_q(\boldsymbol{b},\boldsymbol{\Sigma}) &= m\int\frac{\mathrm{d}^3\boldsymbol{\Delta}}{(2\pi)^3}A_q(\boldsymbol{\Delta}^2)\,\mathrm{e}^{-i\boldsymbol{\Delta}\cdot\boldsymbol{b}} \equiv m\mathfrak{a}_q(\boldsymbol{b}) \\ \mathfrak{t}^{a+}_q(\boldsymbol{b},\boldsymbol{\Sigma}) &= m\int\frac{\mathrm{d}^3\boldsymbol{\Delta}}{(2\pi)^3}\frac{\boldsymbol{\Sigma}\times i\boldsymbol{\Delta}}{2m}\Big(J_q(\boldsymbol{\Delta}^2)+S_q(\boldsymbol{\Delta}^2)\Big)\,\mathrm{e}^{-i\boldsymbol{\Delta}\cdot\boldsymbol{b}} \\ \mathfrak{t}^{+a}_q(\boldsymbol{b},\boldsymbol{\Sigma}) &= m\int\frac{\mathrm{d}^3\boldsymbol{\Delta}}{(2\pi)^3}\frac{\boldsymbol{\Sigma}\times i\boldsymbol{\Delta}}{2m}\Big(J_q(\boldsymbol{\Delta}^2)-S_q(\boldsymbol{\Delta}^2)\Big)\,\mathrm{e}^{-i\boldsymbol{\Delta}\cdot\boldsymbol{b}} \\ \mathfrak{t}^{ab}_q(\boldsymbol{b},\boldsymbol{\Sigma}) &= \int\frac{\mathrm{d}^3\boldsymbol{\Delta}}{(2\pi)^3}\left\{\frac{\Delta^a\Delta^b-\delta^{ab}\boldsymbol{\Delta}^2}{4m}D_q(\boldsymbol{\Delta}^2)-m\delta^{ab}\bar{c}_q(\boldsymbol{\Delta}^2)\right\}\mathrm{e}^{-i\boldsymbol{\Delta}\cdot\boldsymbol{b}} \\ \mathfrak{t}^{+-}_q(\boldsymbol{b},\boldsymbol{\Sigma}) &= \int\frac{\mathrm{d}^3\boldsymbol{\Delta}}{(2\pi)^3}\left\{E_0A_q(\boldsymbol{\Delta}^2)+m\big(\bar{c}_q(\boldsymbol{\Delta}^2)+\bar{e}_q(\boldsymbol{\Delta}^2)\big)+\frac{\boldsymbol{\Delta}^2}{4m}\Big(D_q(\boldsymbol{\Delta}^2)-J_q(\boldsymbol{\Delta}^2)+S_q(\boldsymbol{\Delta}^2)\Big)\right\}\mathrm{e}^{-i\boldsymbol{\Delta}\cdot\boldsymbol{b}} \\ \mathfrak{t}^{a-}_q(\boldsymbol{b},\boldsymbol{\Sigma}) &= E_0\int\frac{\mathrm{d}^3\boldsymbol{\Delta}}{(2\pi)^3}\frac{\boldsymbol{\Sigma}\times i\boldsymbol{\Delta}}{2m}\Big(J_q(\boldsymbol{\Delta}^2)+S_q(\boldsymbol{\Delta}^2)+\bar{f}_q(\boldsymbol{\Delta}^2)\Big)\,\mathrm{e}^{-i\boldsymbol{\Delta}\cdot\boldsymbol{b}}\,. \end{aligned} \tag{97}$$

In fact, once we have properly accounted for the extra term, these turn out to be the correct internal densities. This pesky extra term in the energy flux density can be written:

$$\delta^\nu_-\int\mathrm{d}^3\boldsymbol{R}\,T_Q^{\mu b}(\boldsymbol{R},t)\mathfrak{t}^{+b}(\boldsymbol{x}-\boldsymbol{R},\boldsymbol{\Sigma})\,.$$

It is apparent that this term is *not* of the required form, i.e., it is not $\mathscr{P}$ convolved with a $\mathscr{P}$-independent function. Nonetheless, the contraction of the internal momentum density with the quantum stress tensor suggests a possible physical origin: the quantum

EMT must be boosted by the *total* convective velocity of the matter that the wave function is acting on, rather than just the convective velocity of the barycenter. This suggests the following generalization of Eq. (29):

$$\langle\Psi(t)|\hat{T}^{\mu\nu}_q(\boldsymbol{x})|\Psi(t)\rangle = \int \mathrm{d}^3\boldsymbol{R}\,\Big\{\Lambda^\mu{}_\alpha\big(\boldsymbol{v}_\nabla(\boldsymbol{R},t)\big)\Lambda^\nu{}_\beta\big(\boldsymbol{v}_\nabla(\boldsymbol{R},t)\big)\mathscr{P}(\boldsymbol{R},t)\mathfrak{t}^{\alpha\beta}_q(\boldsymbol{x}-\boldsymbol{R},\boldsymbol{\Sigma}) \\ + \Lambda^\mu{}_\alpha\Big(\boldsymbol{v}_\nabla(\boldsymbol{R},t)+\boldsymbol{u}_q(\boldsymbol{x}-\boldsymbol{R},\boldsymbol{\Sigma})\Big)\Lambda^\nu{}_\beta\Big(\boldsymbol{v}_\nabla(\boldsymbol{R},t)+\boldsymbol{u}_q(\boldsymbol{x}-\boldsymbol{R},\boldsymbol{\Sigma})\Big)T^{\alpha\beta}_Q(\boldsymbol{R},t)\mathfrak{a}_q(\boldsymbol{x}-\boldsymbol{R})\Big\}, \quad (98)$$

where $\boldsymbol{u}_q$ is the *internal* convective velocity:

$$\boldsymbol{u}_q(\boldsymbol{b},\boldsymbol{\Sigma}) = \frac{\mathfrak{t}^{+a}_q(\boldsymbol{b},\boldsymbol{\Sigma})}{\mathfrak{t}^{++}_q(\boldsymbol{b})}\,. \quad (99)$$

Now Eq. (29) is just a special case of Eq. (98), since $\boldsymbol{u}_q$ vanishes for spin-zero states. Moreover, by explicitly plugging the internal densities (97) and the quantum EMT (31) into Eq. (98), one does arrive at the same result as evaluating Eq. (88). This consistency check validates the formulas in question—and so Eq. (97) gives the internal mechanical densities of composite fermions in the pilot wave formulation.

Before concluding, it is worth taking a moment to compare and contrast the mechanical densities in the pilot wave formulation (97) and the Breit frame formalism (72). Remarkably, for nearly all components, the densities agree—the sole exceptions being the energy and energy flux densities. In both of these cases, the pilot wave expressions are simpler. While this does not prove the correctness of the pilot wave picture, it does lend some credence to the idea that Breit frame densities are contaminated by artifacts of wave function dispersion.

VI. SUMMARY AND OUTLOOK

The goals of this work were twofold: (1) obtain Galilei-covariant expressions for matrix elements of the energy-momentum tensor; and (2) obtain formulas for internal energy-momentum densities of non-relativistic composite systems. The first objective was achieved in Eq. (10) for spin-zero systems, and Eq. (64) for spin half systems. For the spin-half expression in particular, a lengthy detour into group theory was necessary to build a new set of spinors—given by Eq. (55)—that respect all the necessary symmetries (namely, the Galilei group extended by parity).

The second objective was addressed in two formalisms: the standard Breit frame formalism, where one simply sets $\boldsymbol{P} = 0$ in the matrix element and takes the Fourier transform; and the pilot wave formalism, where the composite particle is assumed to have a definite position, momentum and energy at all times, and where contributions from the barycentric wave packet can simply be subtracted off. The Breit frame densities are given in Eq. (72), and the pilot wave densities in Eq. (97). Most of these densities are identical, with the exceptions being the energy density and energy flux density. This discrepancy occurs because part of the Breit frame matrix element—namely, the $\frac{\boldsymbol{\Delta}^2}{8m}A_q(\boldsymbol{\Delta}^2)$ term in the case of the energy density—is attributed by the pilot wave formalism to wave packet dispersion.

Put another way: the Breit frame Fourier transform involves matrix elements between plane wave states with non-zero momentum—see Eq. (69)—meaning that each of these plane waves would have a barycentric kinetic energy $\frac{\boldsymbol{\Delta}^2}{8m}$. The Breit frame formalism in effect considers this barycentric energy to be part of the "internal" energy density. It thus appears that the Breit frame energy density is contaminated by what is actually part of the barycentric energy density. This contamination is absent in the pilot wave result.

This also raises the question of whether—and to what degree—expressions for relativistic densities are also contaminated by artifacts of wave packet dispersion. Most of the literature critical of relativistic Breit frame densities is concerned with relativistic effects associated with Lorentz boosts [13, 16, 68, 95, 96], while the concern I raise here is present even with Galilei symmetry. In principle, existing expressions for light front energy densities (e.g. in Ref. [16]) may be contaminated by wave packet dispersion, just as much as Breit frame densities are.

It thus seems pertinent to ask whether the pilot wave based analysis of Sec. V can be repeated for relativistic Dirac spinors, to aid in the removal of such contamination. This will be the main subject of a follow-up study. The pilot wave formulation infamously has difficulties with Lorentz covariance when applied to relativistic systems [56, 112–115], which will require careful and dedicated attention to adequately address. Such an analysis would also be more directly relevant to mechanical properties of the proton in particular, and would warrant concrete numerical and visual demonstrations in addition to formulas.

In the meantime, the most likely application of the results in this work will be to spin-zero and spin-one bound states *composed of* non-relativistic fermions. This includes calculating and interpreting mechanical properties of heavy quarkonia in non-relativistic quantum chromodynamics (NRQCD) [116, 117]. While the form factor breakdown of these systems will of course differ from Eq. (64), this formula will still be relevant: the effective EMT operator of the spin-half constituents will need to be constructed to reproduce Eq. (64) when sandwiched between one-particle states.

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Appendix A: Explicit five-dimensional gamma matrices and matrix elements

To make this Appendix more helpful and self-contained, I will reproduce several formulas present in the main text, in addition to explicit expressions for the gamma matrices and spinor elements.

The tau matrices are defined in Eq. (41), and reproduced here for convenience:

$$\tau_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \tau_1 = \begin{bmatrix} 0 & -\mathbf{i} \\ \mathbf{i} & 0 \end{bmatrix} \qquad \tau_2 = \begin{bmatrix} 0 & -\mathbf{j} \\ \mathbf{j} & 0 \end{bmatrix} \qquad \tau_3 = \begin{bmatrix} 0 & -\mathbf{k} \\ \mathbf{k} & 0 \end{bmatrix} \qquad \tau_4 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} . \tag{A1}$$

The gamma matrices are defined in terms of them via:

$$\gamma^\mu \equiv \begin{bmatrix} 0 & \tau^\mu \\ \eta\tau^\mu\eta & 0 \end{bmatrix} , \tag{A2}$$

where

$$\eta = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{A3}$$

is the "metric" for which matrices in $\mathrm{U}(1,1,\mathbb{H})$ are isometries. The quaternions can be mapped to complex matrices via:

$$\mathbf{i} \equiv -i\sigma_1 \qquad \mathbf{j} \equiv -i\sigma_2 \qquad \mathbf{k} \equiv -i\sigma_3 \, , \tag{A4}$$

allowing the tau and gamma matrices to be written entirely in terms of complex numbers. The explicit complex-valued expressions for the gamma matrices are:

$$\begin{gathered} \gamma^0 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \qquad \gamma^4 = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \qquad \gamma^+ = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \qquad \gamma^- = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \end{bmatrix} \\ \gamma^a = \begin{bmatrix} 0 & 0 & 0 & -i\sigma_a \\ 0 & 0 & i\sigma_a & 0 \\ 0 & i\sigma_a & 0 & 0 \\ -i\sigma_a & 0 & 0 & 0 \end{bmatrix} , \end{gathered} \tag{A5}$$

for $a \in \{1, 2, 3\}$. The sigma matrices, defined as $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$, are also helpful to have on hand:

$$\begin{gathered} \sigma^{0a} = \begin{bmatrix} 0 & -\sigma_a & 0 & 0 \\ \sigma_a & 0 & 0 & 0 \\ 0 & 0 & 0 & \sigma_a \\ 0 & 0 & -\sigma_a & 0 \end{bmatrix} \qquad \sigma^{4a} = \begin{bmatrix} -\sigma_a & 0 & 0 & 0 \\ 0 & \sigma_a & 0 & 0 \\ 0 & 0 & -\sigma_a & 0 \\ 0 & 0 & 0 & \sigma_a \end{bmatrix} \qquad \sigma^{04} = -\sigma^{+-} = i \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \\ \sigma^{ab} = \epsilon_{abc} \begin{bmatrix} \sigma_c & 0 & 0 & 0 \\ 0 & \sigma_c & 0 & 0 \\ 0 & 0 & \sigma_c & 0 \\ 0 & 0 & 0 & \sigma_c \end{bmatrix} \qquad \sigma^{+a} = \frac{1}{\sqrt{2}} \begin{bmatrix} -\sigma_a & -\sigma_a & 0 & 0 \\ \sigma_a & \sigma_a & 0 & 0 \\ 0 & 0 & -\sigma_a & \sigma_a \\ 0 & 0 & -\sigma_a & \sigma_a \end{bmatrix} \qquad \sigma^{-a} = \frac{1}{\sqrt{2}} \begin{bmatrix} \sigma_a & -\sigma_a & 0 & 0 \\ \sigma_a & -\sigma_a & 0 & 0 \\ 0 & 0 & \sigma_a & \sigma_a \\ 0 & 0 & -\sigma_a & -\sigma_a \end{bmatrix} . \end{gathered} \tag{A6}$$

For convenience, I reproduce the physical spinor solutions (55) here as well:

$$u_\uparrow(\boldsymbol{p}) = \frac{1}{2^{5/4}m}\begin{bmatrix} \sqrt{2}m + ip_z \\ ip_x - p_y \\ \sqrt{2}m - ip_z \\ -ip_x + p_y \\ \sqrt{2mE_0} \\ 0 \\ \sqrt{2mE_0} \\ 0 \end{bmatrix} \qquad u_\downarrow(\boldsymbol{p}) = \frac{1}{2^{5/4}m}\begin{bmatrix} ip_x + p_y \\ \sqrt{2}m - ip_z \\ -ip_x - p_y \\ \sqrt{2}m + ip_z \\ 0 \\ \sqrt{2mE_0} \\ 0 \\ \sqrt{2mE_0} \end{bmatrix}, \tag{A7}$$

which respectively have spin-up and spin-down along the z axis (i.e., within the xy plane).

Next, I will give matrix elements of the gamma matrices between the physical spin-up and spin-down solutions. The relevant matrix elements are between spinors with different initial and final five-momenta p and p', where I also use standard notation for the average $P = \frac{1}{2}(p + p')$, and momentum transfer $\varDelta = p' - p$. We require $p^+ = p'^+ = P^+$ and thus $\varDelta^+ = 0$, since $P^+ = m$ is the physical mass (which does not change in non-relativistic processes). I will use the notation:

$$[\![\gamma^\mu]\!] = \begin{bmatrix} \bar{u}_\uparrow(p')\gamma^\mu u_\uparrow(p) & \bar{u}_\uparrow(p')\gamma^\mu u_\downarrow(p) \\ \bar{u}_\downarrow(p')\gamma^\mu u_\uparrow(p) & \bar{u}_\downarrow(p')\gamma^\mu u_\downarrow(p) \end{bmatrix} \tag{A8}$$

to write the results more compactly. The resulting matrix elements are:

$$\begin{aligned} [\![\mathbb{1}]\!] &= \sqrt{\frac{2E_0}{m}}\delta_{s's} \\ [\![\gamma^+]\!] &= \delta_{s's} \\ [\![\gamma^a]\!] &= \frac{P^a}{m}\delta_{s's} - \frac{i(\boldsymbol{\varDelta}\times\boldsymbol{\sigma}_{s's})^a}{2m} \\ [\![\gamma^-]\!] &= \frac{1}{m}\left\{\left(E_0 + \frac{\boldsymbol{P}^2}{2m} - \frac{\boldsymbol{\varDelta}^2}{8m}\right)\delta_{s's} - \frac{i(\boldsymbol{\varDelta}\times\boldsymbol{\sigma}_{s's})\cdot\boldsymbol{P}}{2m}\right\} \\ [\![\sigma^{ab}]\!] &= \sqrt{\frac{2E_0}{m}}\epsilon_{abc}(\sigma_b)_{s's} \\ [\![\sigma^{+a}]\!] &= 0 \\ [\![\sigma^{-a}]\!] &= \sqrt{\frac{2E_0}{m}}\left(\frac{\epsilon_{abc}(\sigma_b)_{s's}P_c}{m} - \frac{i\varDelta_a}{2m}\right) \\ [\![\sigma^{+-}]\!] &= 0\,, \end{aligned} \tag{A9}$$

where $a, b \in \{1, 2, 3\}$.

Appendix B: Lévy-Leblond spinors

Lévy-Leblond spinors [75] are the standard set of spinors used in literature on Galilean quantum mechanics and field theory [59–63, 74, 101, 118, 119]. This appendix briefly introduces them and explains why I am not using them in this work. The main issue is that they don't have the expected transformation properties under Galilean parity reversal, which would limit the number of true tensor (as opposed to pseudotensor) structures that could be added to form factor breakdowns. For instance, this means Lévy-Leblond spinors cannot encode an anomalous magnetic moment, nor internal stresses.

The Lévy-Leblond equation can be found quickly (albeit in a non-standard form) using the tau-matrix formalism of Sec. III. The key trick is to note that if $\psi_s(p)$ transforms under the complex defining representation of $\mathrm{USp}(2, 2, \mathbb{C})$, then so does $p^\mu\tau_\mu\eta\psi_s(p)$. Accordingly, the following momentum-space equation is covariant under the connected part of the Galilei group:

$$p^\mu\tau_\mu\eta\psi_s(p) = \big(m\tau_+ + E\tau_- - \boldsymbol{p}\cdot\boldsymbol{\tau}\big)\eta\psi_s(p) = \sqrt{2mE_0}\psi_s(p)\,, \tag{B1}$$

where the τ_μ matrices are defined in Eq. (41). This equation is not, however, invariant under Galilean parity reversal unless $E_0 = 0$. To see this, parity reversal is effected by acting on on Eq. (B1) with τ_4 (see Sec. III B 1). Since $\tau_4\eta = -\eta\tau_4$:

$$p^\mu\tau_\mu\eta\psi_s(p) = \big(m\tau_+ + E\tau_- + \boldsymbol{p}\cdot\boldsymbol{\tau}\big)\eta\big(\tau_4\psi_s(p)\big) = -\sqrt{2mE_0}\big(\tau_4\psi_s(p)\big)\,, \tag{B2}$$

which is not equivalent to Eq. (B1) except when $E_0 = 0$. Setting $E_0 = 0$ gives:

$$p^\mu \tau_\mu \eta \psi_s(p) = \big(m\tau_+ + E\tau_- - \boldsymbol{p}\cdot\boldsymbol{\tau}\big)\eta\psi_s(p) = 0\,, \tag{B3}$$

which is in effect the Lévy-Leblond equation [75]. To be sure, this is not the standard form of the equation, but it can be proved that there is (up to equivalences) a unique Galilei-covariant wave equation for four-component spinors [74].

Now, even though Eq. (B3) itself is covariant under parity reversal, its solutions have unexpected parity transformation properties. Since $\tau_4\eta = -\eta\tau_4$, $\psi_s(p)$ and $\eta\psi_s(p)$ have opposite parity, and consequently the bilinear $\psi_s^\dagger(p)\eta\psi_s(p)$—the equivalent of the Dirac scalar $\bar{u}(p,s)u(p,s)$—is a pseudoscalar. This means $\bar{u}u$-like terms cannot appear in form factor breakdowns if the current in question is a true tensor.

I will presently explore this issue more concretely in terms of the standard form of the Lévy-Leblond equation.

Standard form of the Lévy-Leblond equation

The Lévy-Leblond equation can be written in its standard form using the unitary transformation matrix:

$$L = \frac{1}{2}\begin{bmatrix} 1+i & 1+i \\ i-1 & 1-i \end{bmatrix}, \tag{B4}$$

and defining the quantities:

$$\begin{aligned} u_{\mathrm{LL}}(p,s) &= L\psi_s(p) \\ \gamma^\mu_{\mathrm{LL}} &= L\tau^\mu\eta L^{-1}\,, \end{aligned} \tag{B5}$$

for which the explicit gamma matrices are:

$$\gamma^0_{\mathrm{LL}} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \qquad \gamma^a_{\mathrm{LL}} = \begin{bmatrix} i\sigma_a & 0 \\ 0 & -i\sigma_a \end{bmatrix} \qquad \gamma^4_{\mathrm{LL}} = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}. \tag{B6}$$

Note that the gamma matrices defined in this way obey the Clifford algebra relation $\{\gamma^\mu_{\mathrm{LL}}, \gamma^\nu_{\mathrm{LL}}\} = 2g^{\mu\nu}$. Plugging these definitions into Eq. (B3) gives:

$$\not{p}_{\mathrm{LL}} u_{\mathrm{LL}}(p,s) = -i\begin{bmatrix} \boldsymbol{p}\cdot\boldsymbol{\sigma} & \sqrt{2}m \\ -\sqrt{2}E & -\boldsymbol{p}\cdot\boldsymbol{\sigma} \end{bmatrix} u_{\mathrm{LL}}(p,s) = 0\,. \tag{B7}$$

This is the standard form of the Lévy-Leblond equation, given for instance (up to a difference of sign convention in the metric) by Refs. [59–61].

Defining two-component sub-spinors:

$$\phi_\uparrow = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \phi_\downarrow = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \tag{B8}$$

solutions to Eq. (B7) can be written:

$$u_{\mathrm{LL}}(p,s) = \frac{1}{\sqrt{2}}\begin{bmatrix} \phi_s \\ -\frac{\boldsymbol{p}\cdot\boldsymbol{\sigma}}{\sqrt{2}m}\phi_s \end{bmatrix}. \tag{B9}$$

The parity problem arises when building bilinear forms in u_{LL} and its Hermitian conjugate. Form factor breakdowns like Eq. (64) need to be true Galilei tensors, rather than pseudotensors. However, $\bar{u}_{\mathrm{LL}}(p',s')u_{\mathrm{LL}}(p,s)$ is a pseudoscalar rather than a true scalar, which limits the number of possible tensors that can be built. For instance, form factors akin to $D_q(\boldsymbol{\varDelta}^2)$ and $\bar{c}_q(\boldsymbol{\varDelta}^2)$ could never appear in a breakdown with Lévy-Leblond spinors.

To see this, the conjugate spinor is as usual defined via $\bar{u}_{\mathrm{LL}} = u^\dagger_{\mathrm{LL}}\gamma^0_{\mathrm{LL}}$. By the change of basis in Eq. (B4), this is equivalent to making $\bar{\psi}_s(p) = \psi_s^\dagger(p)\eta$ the conjugate of $\psi_s(p)$ in Eq. (B3). This makes sense, since under the path-connected part of the Galilei group, $\bar{\psi}_s(p) \mapsto \bar{\psi}_s(p)M^{-1}$, making $\bar{u}_{\mathrm{LL}}(p',s')u_{\mathrm{LL}}(p,s)$ transform like a scalar under this subgroup. However, this quantity is parity-odd, because $\eta\tau_4\eta = -\tau_4$. Thus $\bar{u}_{\mathrm{LL}}(p',s')u_{\mathrm{LL}}(p,s)$ is a pseudoscalar. Explicit calculation even shows:

$$\bar{u}_{\mathrm{LL}}(p',s')u_{\mathrm{LL}}(p,s) = -\frac{i\boldsymbol{\varDelta}\cdot\boldsymbol{\sigma}}{2m}\,, \tag{B10}$$

which is clearly parity-odd, and would indicate an intrinsic electric dipole moment if it appeared in an electromagnetic form factor breakdown.

In fact, for electromagnetic form factors, the most general (Galilei + parity)-covariant expression that can be built using Lévy-Leblond spinors is:

$$\langle \boldsymbol{p}', s' | \hat{J}^{\mu}(0) | \boldsymbol{p}, s \rangle = \bar{u}_{\mathrm{LL}}(p', s') \gamma^{\mu}_{\mathrm{LL}} u_{\mathrm{LL}}(p, s) F(\boldsymbol{\varDelta}^2) . \tag{B11}$$

Explicitly evaluating the $(+, 1, 2, 3)$ components for the charge density and current gives:

$$\begin{aligned} \langle \boldsymbol{p}', s' | \hat{J}^{\mu}(0) | \boldsymbol{p}, s \rangle &= \delta_{s's} F(\boldsymbol{\varDelta}^2) , \\ \langle \boldsymbol{p}', s' | \hat{J}^{a}(0) | \boldsymbol{p}, s \rangle &= \left(\frac{P^a}{m} \delta_{s's} - \frac{i(\boldsymbol{\varDelta} \times \boldsymbol{\sigma}_{s's})^a}{2m} \right) F(\boldsymbol{\varDelta}^2) , \end{aligned} \tag{B12}$$

which cannot encode an anomalous magnetic dipole moment.

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