Space- and Time-Dependent Source Identification Problem with Integral Overdetermination Condition

R.R. Ashurov[1] **and O.T. Mukhiddinova**[1,2]
ashurovr@gmail.com, oqila1992@mail.ru

[1] *V.I. Romanovskiy Institute of Mathematics, Uzbekistan Academy of Science, University str.,9, Olmazor district, Tashkent, 100174, Uzbekistan*
[2] *Tashkent University of Information Technologies named after Muhammad al-Khwarizmi, Str., 108, Amir Temur Avenue,Tashkent, 100200, Uzbekistan*

Annotation 0.1. *This paper is devoted to the study of the inverse problem of determining the right-hand side of the subdiffusion equation with the Caputo derivative with respect to time. In our case, the inverse problem consists in restoring the coefficient of the right-hand side, which depends on both the time and the spatial variable, when measured in integral form. Previously, similar inverse problems were studied for hyperbolic and parabolic equations with a different overdetermination condition, and in some works the existence and uniqueness of generalized solutions was established, while in others, the uniqueness of classical solutions was established. However, similar inverse problems for fractional equations with an integral overdetermination condition have not been considered before this work. The existence and uniqueness of a weak solution to the inverse problem under consideration is established. It is noteworthy that the results obtained are new for parabolic equations as well.*

Keywords. 0.2. *Subdiffusion equation, Caputo time derivative, inverse problem, uniqueness and existence of solution, Fourier method.*

1. Introduction

In recent decades, much attention has been paid to the study of problems for partial differential equations with fractional derivatives, which are used to model a number of phenomena in various branches of science and technology (see, e.g., [1]–[4]). It should also be noted that, according to some recent research results, it can be seen that there are some physical phenomena that cannot be modeled by partial differential equations of integer order, but are adequately described by fractional derivatives ([5] - [7]). On the other hand, partial differential equations with fractional derivatives, along with such important physical applications, in some cases generalize partial differential equations of integer order (see, e.g., [8]–[14]). Therefore, the study of direct and inverse problems associated with differential equations with fractional derivatives is of great interest from both theoretical and practical points of view.

In recent years, there has been growing interest among researchers in inverse problems of determining the source function in differential equations of both integer and fractional order (see, for example, the monograph by S. I. Kabanikhin [15] and the review article by Yamamoto [16]). This interest is due to their crucial role in applications in various fields, such as mechanics, seismology, medical tomography, and geophysics (see, for example, [1]). Mainly such problems have been studied where the source function has the form $F(x,t) = g(t)f(x)$, with either $g(t)$ or $f(x)$ being unknown. To our knowledge, the more general case $F(x,t)$ without such factorization has not been studied. In this scenario, even the choice of an appropriate overdetermination condition remains unclear. Inverse problems aimed at recovering the time-dependent component $g(t)$ are usually solved by reducing them to integral equations (see, e.g., [17]–[18] and references therein). In contrast, the inverse problem of determining the space-dependent component $f(x)$ is analyzed in two different cases: $g(t) \equiv 1$ and $g(t) \not\equiv 1$. When $g(t) \equiv 1$, such problems were studied, e.g., in [19]–[22]. The case when $g(t) \not\equiv 1$ is more complicated, and solvability depends on the sign-definiteness of the function $g(t)$ (see, for example, [25]–[30], as well as [15] and the review article [16]).

The works most closely related to our study are those by S. Z. Dzhamalov et al. [31]–[34], which address inverse problems of determining the right-hand side of equations, where the right-hand side depends on both time and a portion of the spatial variable. These studies investigate the existence of a generalized solution to the problem using the Galerkin method.

In [35], the inverse problem for a hyperbolic equation is also addressed, specifically the determination of the right-hand side, which depends on both time and a portion of the spatial variable. The authors successfully established the uniqueness of the classical solution to this inverse problem.

In recent work [36] the authors investigated the inverse problem of determining the right-hand side of a subdiffusion equation with a Caputo time derivative, where the right-hand side depends on both time and certain spatial variables. The primary objective of this paper is to investigate the inverse problem of determining the coefficient of the right-hand side, and solution of the forward problem: $\{u(t,x,y), h(t,x)\}$, $(x,\, y) \in (0,1) \times (0,\pi) \subset \mathbb{R}^2$, subject to the overdetermination condition given by:

$$u(t,x,l_0) = \psi(t,x), \quad t \in (0,T), \quad x \in (0,1), \quad l_0 \in (0,\pi), \tag{1}$$

where ψ is a known continuous function. It is proved the existence and uniqueness of the weak solution to the considered inverse problem. To solve it, the authors employed the Fourier method with respect to the variable independent of the unknown right-hand side, followed by the method of successive approximations to compute the Fourier coefficients of the solution.

Note that in all works [31] - [36] the authors considered the inverse problem with the overdetermination condition (1).

In this paper, we set the overedetermination condition in a integral form and investigate the inverse problem of determining the function $h(t,x)$ that appears in the right-hand side of a subdiffusion equation, given as $f(t,x,y)h(t,x) + g(t,x,y)$, where $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^n$.

Next, we proceed to the precise formulation of the problem.

Let $Q = (0,T) \times G_x \times \Omega_y$ and $D = G_x \times \Omega_y$, where $G := G_x \subset \mathbb{R}^m$ and $\Omega := \Omega_y \subset \mathbb{R}^n$ are a bounded domains, with a sufficiently smooth boundary. Consider the following initial - boundary value problem

$$\begin{cases} D_t^\alpha u - \Delta_x u - \Delta_y u = g(t,x,y) + f(t,x,y) \cdot h(t,x), \quad (t,x,y) \in Q, \\ u(0,x,y) = \varphi(x,y), \quad (x,y) \in D, \\ u(t,0,y) = u(t,1,y) = 0, \quad t \in [0,T], \quad y \in \Omega, \\ u(t,x,0) = u(t,x,\pi) = 0, \quad t \in [0,T], \quad x \in G. \end{cases} \tag{2}$$

Here Δ_x and Δ_y are the Laplace operators in variables x and y respectively, f, g and φ are given functions, $\alpha \in (0,1)$, and D_t^α is the fractional Caputo derivative. Recall, if $v(t)$ is an absolutely continuous function, then its Caputo derivative has the form (see, e.g., [8], p. 91)

$$D_t^\alpha v(t) \equiv J_t^{1-\alpha} D_t v(t), \quad J_t^\alpha v(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} v(\tau) d\tau,$$

where $D_t = d/dt$, $\Gamma(\alpha)$ is the gamma function, J_t^α is the Riemann-Liouville fractional integral.

If the function $h(t,x)$ is given, then problem (2) is called *the forward problem* and its solution exists and is unique under certain conditions on the problem's data (see, e.g., [13], [37]).

Now we will assume that the coefficient $h(t,x)$ of the source function is unknown and must be determined. The main goal of this paper is to study *the inverse problem* of determining a pair of functions $\{u(t,x,y), h(t,x)\}$ under the overdetermination condition specified by the following integral measurement:

$$\int_\Omega u(t,x,y)\omega(y)dy = \psi(t,x), \quad t\in[0,T], \quad x\in G, \tag{3}$$

where ω and ψ are known functions, the conditions on which we will determine later.

This paper is organized into seven sections. The next section is auxiliary, where the known results of A. Alikhanov are presented. In section 3 we recall some properties of the eigenvalues of the Laplace operator with the Dirichlet condition. In section 4 a weak solution of the inverse problem under consideration is defined and the main result of the study is presented. The solution of the problem is sought in the form of a series in eigenfunctions with unknown coefficients and in section 5 an infinite system of integro-differential equations is derived to determine these coefficients and a priori estimates are established. Sections 6 and 7 are devoted to the proof of the main result. Finally, section 8 provides a conclusion.

2. Preliminaries

In this section, we recall the definition of Mittag-Leffler functions and remind the well-known results of A. Alikhanov, which will be used in further discussions.

The solution of the subdiffusion equations is expressed through the two-parameter Mittag-Leffler functions $E_{\rho,\mu}(z)$ which are determined by the formula

$$E_{\rho,\mu}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\rho k+\mu)}, \quad \rho>0, \quad \mu, z\in\mathbb{C}.$$

If $\mu=1$, then we obtain the classical Mittag-Leffler function and is denoted by $E_\rho(z)=E_{\rho,1}(z)$. Obviously, since the Mittag-Leffler functions are an entire function, there exists a constant M such that

$$E_{\rho,\mu}(z)\le M, \quad \mu>0, \quad z\in[a,b]\subset\mathbb{R}_+. \tag{4}$$

The following connection between the fractional integral and derivative follows directly from the definitions and equality for $v\in L_1(0,T)$: $J_t^\alpha(J_t^\beta v(t)) = J_t^\beta(J_t^\alpha v(t)) = J_t^{\alpha+\beta}v(t))$, $\alpha,\beta>0$ (see [9], p. 567).

Lemma 2.1. *Let $0<\alpha<1$ and $v(t)$ be absolutely continuous on $[0,T]$: $v\in AC[0,T]$. Then*

$$J_t^\alpha D_t^\alpha v(t) = v(t) - v(0). \tag{5}$$

Proof see in [9], p. 570.

Let us now quote the following two statements, in a form convenient for us, from Alikhanov [12].

Lemma 2.2. *Let $0<\alpha<1$ and $w\in AC([0,T]:L_2(G))$. Then*

$$\frac{1}{2}D_t^\alpha\|w(t,x)\|^2_{L_2(G)} \le \int_G w(t,x)\, D_t^\alpha w(t,x)\, dx.$$

Lemma 2.3. *Let $y(t)\in AC[0,T]$ be a positive function and for all $t\in(0,T]$, the following inequality holds:*

$$D_t^\alpha y(t)\le c_1 y(t) + c_2(t), \quad 0<\alpha\le 1,$$

for almost all $t\in[0,T]$, where $c_1>0$ and $c_2(t)$ is an integrable nonnegative function on $[0,T]$. Then

$$y(t)\le y(0)\,E_\alpha(c_1t^\alpha) + \Gamma(\alpha)\,E_{\alpha,\alpha}(c_1t^\alpha)\,J_t^\alpha c_2(t). \tag{6}$$

Note also that if $v(t)$ is a positive bounded function on $[0,T]$ and $\alpha\in(0,1]$, then the following relations hold:

$$\frac{1}{T^{1-\alpha}}\int_0^t v(s)ds \le \int_0^t \frac{v(s)}{t^{1-\alpha}}ds \le \int_0^t \frac{v(s)}{(t-s)^{1-\alpha}}ds = \Gamma(\alpha)J_t^\alpha v(t) \le T^\alpha \max_{t\in[0,T]} v(t). \tag{7}$$

Next, we introduce some function spaces. Let B be a Banach space. We denote by $L_\infty(0,T;B)$ the space of functions that are essentially bounded on $(0,T)$ and take values in B. The space $L_1(0,T;B)$ is defined similarly. Let $W_2^k(D)$ denote a classical Sobolev space. Then, the symbol $\dot{W}_2^k(D)$ represents the closure of the set $C_0^\infty(\Omega)$ with respect to the norm of $W_2^k(\Omega)$.

3. Eigenvalues of the Laplace Operator with Dirichlet Boundary Conditions

Let $\Omega\subset\mathbb{R}^n$ be a bounded domain with a smooth boundary $\partial\Omega$. Consider the Laplace operator $-\Delta$, equipped with Dirichlet boundary conditions, i.e., $u=0$ on $\partial\Omega$. The eigenvalue problem for the Laplace operator is given by:

$$-\Delta v = \lambda v \quad \text{in } \Omega, \quad v=0 \quad \text{on } \partial\Omega. \tag{8}$$

This operator's spectrum consists of a sequence of positive eigenvalues $0<\lambda_1\le\lambda_2\le\lambda_3\le\ldots$, counted with multiplicity, such that $\lambda_k\to+\infty$ as $k\to\infty$. Each eigenvalue λ_k corresponds to an eigenfunction v_k, and the eigenfunctions $\{v_k\}$ form an orthonormal basis in $L^2(\Omega)$.

The asymptotic behavior of the eigenvalues is governed by Weyl's law, which provides insight into their growth rate. For large k, the eigenvalues satisfy:

$$\lambda_k \sim C_n\left(\frac{k}{|\Omega|}\right)^{2/n},$$

where $|\Omega|$ denotes the Lebesgue measure of Ω, and $C_n=4\pi^2(B_n)^{-2/n}$, with $B_n=\pi^{n/2}/\Gamma(n/2+1)$ being the volume of the unit ball in $\mathbb{R}^n$. This implies that λ_k grows approximately as $k^{2/n}$. Weyl's law is a cornerstone of spectral theory and has been extensively studied (see, e.g., [38, 39]).

Consider the series involving the eigenvalues λ_k of the Laplace operator with Dirichlet boundary conditions:

$$\sum_{k=1}^\infty \lambda_k^{-a},$$

where $a>0$ is a parameter. The convergence of this series depends on the asymptotic behavior of λ_k. Since $\lambda_k\sim k^{2/n}$, then for large k,

$$\lambda_k^{-a}\sim (k^{2/n})^{-a} = k^{-2a/n}.$$

The series $\sum_{k=1}^\infty \lambda_k^{-a}$ is therefore comparable to the p-series $\sum_{k=1}^\infty k^{-2a/n}$. A p-series $\sum k^{-p}$ converges if and only if $p>1$. Applying this criterion, we require:

$$\frac{2a}{n}>1 \implies a>\frac{n}{2}.$$

Thus, the series $\sum_{k=1}^\infty \lambda_k^{-a}$ converges if $a>n/2$ and diverges if $a\le n/2$. For any positive $\varepsilon>0$ we denote

$$C_\varepsilon := \sum_{j=1}^\infty \lambda_j^{-\frac{n}{2}-\varepsilon}. \tag{9}$$

Let us derive an estimate for the L^2-norm of the gradient of the eigenfunctions v_k of the Laplace operator with Dirichlet boundary conditions. Let λ_k be the eigenvalues and u_k the corresponding eigenfunctions satisfying:

$$-\Delta v_k = \lambda_k v_k \quad \text{in } \Omega, \quad v_k = 0 \quad \text{on } \partial\Omega,$$

with $\|v_k\|_{L^2(\Omega)} = 1$. Multiply this equation by v_k and integrate over Ω:

$$-\int_\Omega v_k \Delta v_k \, dy = \lambda_k \int_\Omega v_k^2 \, dy = \lambda_k,$$

since $\|v_k\|_{L^2(\Omega)} = 1$. Using Green's identity and the Dirichlet boundary condition ($v_k = 0$ on $\partial\Omega$), the left-hand side becomes:

$$-\int_\Omega v_k \Delta v_k \, dy = \int_\Omega |\nabla v_k|^2 \, dy,$$

as the boundary term vanishes. Thus:

$$\int_\Omega |\nabla v_k|^2 \, dy = \lambda_k. \tag{10}$$

Again, by Green's identity and since $\|v_k\|_{L^2(\Omega)} = 1$, we have

(11)
$$\int_\Omega \nabla v_k(y) \nabla v_j(y) \, dy = -\int_\Omega \Delta v_k(y) v_j(y) \, dy = \lambda_k \int_\Omega v_k(y) v_j(y) \, dy = \begin{cases} \lambda_k, & \text{if } j = k; \\ 0, & \text{if } j \neq k. \end{cases}$$

Let A be a self-adjoint extension of the Laplace operator with domain $D(-\Delta) = \{v \in C^2(\Omega) : v(y) = 0,\ y \in \partial\Omega\}$. Then A is a positive operator with domain $W_2^2(\Omega) \cap \dot{W}_2^1(\Omega)$ (see, for example, [40], Chapte 2). Therefore, we can define fractional powers of this operator using the von Neumann theorem. Namely, let τ be an arbitrary non-negative number. Then

$$A^\tau p(y) = \sum_{k=1}^\infty \lambda_k^\tau \, p_k \, v_k(y),$$

and domain of the operator A^τ has the form

$$D(A^\tau) = \{p \in L_2(\Omega) : \sum_{k=1}^\infty \lambda_k^{2\tau} \, |p_k|^2 < \infty\}.$$

Here after, the symbol p_k denotes the Fourier coefficients of a function $p(y) \in L_2(\Omega)$ with respect to the system $\{v_k(y)\}$, defined as a scalar product in $L_2(\Omega)$: $p_k = (p, v_k)$.

Note, that if $\tau_1 > \tau_2$ then $D(A^{\tau_1}) \subset D(A^{\tau_2})$. $D(A^\tau)$ is a linear space and we can introduce a scalar product in it using the formula

$$(p, q)_\tau = \sum_{k=1}^\infty \lambda_k^{2\tau} \, p_k \, q_k \ = \ (A^\tau p, A^\tau q),$$

which defines the following norm

$$||p||_\tau^2 = \sum_{k=1}^\infty \lambda_k^{2\tau} \, |p_k|^2 \ = \ \int_\Omega |A^\tau p(y)|^2 \, dy.$$

In many cases in this paper, the function p also depends on $x \in G$ and on $t \in [0, T]$, i.e. $p = p(t, x, y)$. In such cases, we introduce the notation

$$||p(t, \cdot, \cdot)||_{\tau,G}^2 := \int_G \int_\Omega |A^\tau p(t, x, y)|^2 dy \, dx = \int_G \sum_{k=1}^\infty \lambda_k^{2\tau} \, |p_k(t, x)|^2 \, dx.$$

The question naturally arises: how to check whether a given function p belongs to $D(A^\tau)$? This question has been thoroughly studied, for example, in the work of D. Fujiwara [41] (for high-order operators, see the work of Sh. Alimov [42]). If the order

τ is an integer, then the answer to this question is much simpler. For example, let us define the domain of the operator $(-\Delta)^2$. First note that $D(\Delta^2) \subset D(-\Delta)$. Therefore, if $p \in D(\Delta^2)$, then $p \in W_2^2(\Omega) \cap \dot{W}_2^1(\Omega)$ and $\Delta p \in W_2^2(\Omega) \cap \dot{W}_2^1(\Omega)$. Or in other words,

$$D((-\Delta)^2) = W_2^4(\Omega) \cap \{u \in \dot{W}_2^1(\Omega) \mid \Delta u \in \dot{W}_2^1(\Omega)\}.$$

If the equality to zero on the boundary of the domain Ω is understood as the equality to zero of the traces, then we can write

$$D((-\Delta)^2) = \{u \in W_2^4(\Omega) \mid u = 0, \Delta u = 0 \text{ on } \partial\Omega\}. \tag{12}$$

4. Definition of the Weak Solution and Formulation of the Main Result

First, we find a formal representation for the unknown function h using the overdetermination condition (3). To do this, we represent the solution to the forward problem (2) as a formal series in terms of the eigenfunctions $\{v_k(y)\}$ of the spectral problem (8):

$$u(t,x,y) = \sum_{k=1}^{\infty} u_k(t,x) v_k(y), \tag{13}$$

where $u_k(t,x)$ are the unknown functions.

In order to use the overdetermination condition (3), we multiply the equation in (2) by the function ω and integrate over the domain Ω. Then

$$D_t^\alpha \psi - \Delta_x \psi - (\Delta_y u, \omega) = (f(t,x,\cdot), \omega)\, h(t,x) + (g(t,x,\cdot), \omega).$$

Let $(f(t,x,\cdot), \omega) \neq 0$ for all $(t,x) \in [0,T] \times G$. Then the desired representation for $h(t,x)$ can be written as:

$$h(t,x) = \frac{D_t^\alpha \psi(t,x) - \Delta_x \psi(t,x) - (g(t,x,\cdot), \omega) - (\Delta_y u, \omega)}{(f(t,x,\cdot), \omega)}. \tag{14}$$

Next, we will investigate the following weak formulation of the inverse problem (2)–(3).

Definition 4.1. *Find a pair of functions $\{u(t,x,y), h(t,x)\}$, where $h(t,x)$ has the form* (14)*, and the function $u(t,x,y)$ satisfies the following conditions:*

(1) $u \in L_\infty(0,T; L_2(D))$, $\quad u \in L_1(0,T; \dot{W}_2^1(D))$;
(2) $D_t^\alpha u \in L_1(0,T; L_2(D))$;
(3) $u(0,x,y) = \varphi(x,y)$ *a.e. in* D;
(4) *For any $v \in \dot{W}_2^1(D)$ and almost every $t \in (0,T]$, the following equality holds:*

$$\int_D D_t^\alpha u v \,dx\,dy + \int_D (\nabla_x u \nabla_x v + \nabla_y u \nabla_y v)\,dx\,dy = \int_D fhv\,dx\,dy + \int_D gv\,dx\,dy. \tag{15}$$

Let $M_\alpha = MT^\alpha$, where M is from (4) and recall the constant C_ε is defined in (9). Next, it is convenient for us to introduce a notation for some positive number ε:

$$2\tau := 2\tau(\varepsilon) = \frac{n}{2} + 1 + \varepsilon. \tag{16}$$

We now state the main result of the paper.

Theorem 4.1. *Let ε be any positive number and suppose the following conditions hold:*

(1) $f(t,x,y) \in C(Q)$, $(f(t,x,\cdot),\omega) \neq 0$, *and* $f_0 = \max_{t,x} \left|\frac{1}{(f(t,x,\cdot),\omega)}\right|$;
(2) $g(t,x,y) \in C(Q)$, *and* $g_0 = \max_{t,x} |(g(t,x,\cdot),\omega)|$;
(3) $D_t^\alpha \psi$, $\Delta\psi \in C([0,T] \times G)$, *and* $\psi_0 = \max_{t,x}(|D_t^\alpha \psi| + |\Delta\psi|)$;
(4) $M_\alpha C_\varepsilon f_0^2 \,||\nabla\omega||^2_{L_2(\Omega)} \max_{t,x} ||f(t,x,\cdot)||^2_\tau \le 1$, *and* $f^* = \max_t ||f(t,\cdot,\cdot)||^2_{\tau,G} < \infty$;
(5) $g^* = \max_t ||g(t,\cdot,\cdot)||^2_{\tau,G} < \infty$;
(6) $\|\nabla_x \varphi\|^2_{L_2(D)} < \infty$, *and* $\varphi^* = ||\varphi||^2_{\tau,G} < \infty$;
(7) $\omega \in \dot{W}_2^1(\Omega)$ *and* $\int_\Omega \varphi(x,y)\omega(y)dy = \psi(0,x)$.

Then, the inverse problem has a unique weak solution. Moreover, the following estimates hold:

$$\max_{[0,T]} \|u(t,\cdot,\cdot)\|^2_{\tau,G} \le 2A_0. \tag{17}$$

$$\begin{aligned} ||\nabla_x u||^2_{L_2(Q)} + ||\nabla_y u||^2_{L_2(Q)} &\le \frac{\Gamma(\alpha)T^{1-\alpha}}{2}||\varphi||^2_{L_2(D)} + 3A_0T + \frac{T}{2}A_1 \\ &+T\,A_0C_\varepsilon\, f_0^2\,||\nabla\omega||^2_{L_2(\Omega)} \max_{x\in G}||f(t,x,\cdot)||^2_{L_2(\Omega)}, \end{aligned} \tag{18}$$

$$\begin{aligned} ||D_t^\alpha u||^2_{L_2(Q)} &\le \frac{\Gamma(\alpha)T^{1-\alpha}}{2}\Big(\|\nabla_x\varphi\|^2_{L_2(D)} + \|\nabla_y\varphi\|^2_{L_2(D)}\Big) \\ &+T\max_t\Big[f_0^2(\psi_0+g_0)^2\|f(t,\cdot,\cdot)\|^2_{L_2(D)} + \|g(t,\cdot,\cdot)\|^2_{L_2(D)}\Big] \\ &+2\,A_0C_\varepsilon\, f_0^2\,||\nabla\omega||^2_{L_2(\Omega)} \max_{x\in G}||f(t,x,\cdot)||^2_{L_2(\Omega)}, \end{aligned} \tag{19}$$

$$\begin{aligned} ||h(t,\cdot)||^2_{L_2(G)} &\le 4f_0^2\Big(||D_t^\alpha\psi(t,x)||^2_{L_2(G)} + ||\Delta_x\psi(t,x)||^2_{L_2(G)} + ||(g(t,x,\cdot),\omega)||^2_{L_2(G)}\Big) \\ &+8f_0^2\,A_0\,C_\varepsilon||\nabla\omega||^2_{L_2(\Omega)}. \end{aligned}$$

Here

$$A_1 = f_0^2(\psi_0+g_0)^2\max_t\|f(t,\cdot,\cdot)\|^2_{L_2(D)} + \max_t\|g(t,\cdot,\cdot)\|^2_{L_2(D)},$$

and

$$A_0 = M\varphi^* + M_\alpha f_0^2(\psi_0+g_0)^2 f^* + M_\alpha g^*.$$

Remark 4.2. *It is known that for $\alpha > n/4$ the eigenfunction expansion of any function $f \in D(A^\alpha)$ converges absolutely and uniformly in the closed domain $\overline{\Omega}$ (see* [43]*). And the functions φ, f and g belong to the class $D(A^\tau)$, with $\tau > \frac{n+2}{4}$, which naturally guarantees the indicated convergence.*

5. A priori estimates

Let us write the function h in a form convenient for us. First, taking into account the property of the function ω (see condition 7 of the theorem), we get $(\Delta_y u,\omega) = (\nabla_y u,\nabla\omega)$. Then, applying the formal form (13) of the function u, we will have

$$h(t,x) = \frac{D_t^\alpha\psi(t,x) - \Delta_x\psi(t,x) - (g(t,x,\cdot),\omega) - \sum_{j=1}^\infty(\nabla v_j,\nabla\omega)u_j(t,x)}{(f(t,x,\cdot),\omega)}. \tag{20}$$

We decompose the functions $f(t,x,y)$, $g(t,x,y)$, and $\varphi(x,y)$ into Fourier series (see Remark 4.2) and denote their corresponding Fourier coefficients by $f_k(t,x)$, $g_k(t,x)$, and $\varphi_k(x)$. Substitute these series and the representation (13) for $u(t,x,y)$ into equation (15), with the test function $v(x,y)$ replaced by $w(x)v_k(y)$, where $w \in \dot{W}_2^1(G)$. Next note that according to (11) one has

$$\int_D \nabla_y u\nabla_y v dxdy = \sum_{j=1}^\infty \int_G u_j(t,x)w(x)dx\int_\Omega \nabla v_j\nabla v_k dy = \lambda_k\int_G u_k w\,dx. \tag{21}$$

Thus we obtain the following equation to determine the unknown coefficients $u_k(t,x)$ (see (13)):

$$\int_G D_t^\alpha u_k w\,dx + \int_G \nabla u_k\nabla w\,dx + \lambda_k\int_G u_k w\,dx = \int_G (g_k + f_k h)w\,dx, \tag{22}$$

but here h is defined using all $u_j(t,x)$, $j = 1,2,\dots$ (see (20)). This fact creates a certain problem and in order to solve it we use the method of successive approximations, i.e. using recurrence relations we construct a sequence $\{u_k^i\}$, $i = 1,2,\dots$, and then we prove that $\{u_k^i\} \to u_k$ as $i \to \infty$ in the appropriate norm.

The corresponding recurrence relations for all $w \in \dot{W}_2^1(G)$, $k \geq 1$ and $i \geq 1$ have the form:

$$\int_G D_t^\alpha u_k^i w\, dx + \int_G \nabla u_k^i \nabla w\, dx + \lambda_k \int_G u_k^i w\, dx \tag{23}$$

$$= \int_G M_k(t,x) w\, dx - \int_G \frac{f_k(t,x)}{(f(t,x,\cdot),\omega)} \sum_{j=1}^\infty (\nabla v_j, \nabla \omega) u_j^{i-1}(t,x) w\, dx,$$

with the initial conditions:

$$u_k^i(0,x) = \varphi_k(x), \tag{24}$$

where

$$M_k(t,x) = f_k(t,x) \left[\frac{D_t^\alpha \psi(t,x) - \Delta_x \psi(t,x) - (g(t,x,\cdot),\omega)}{(f(t,x,\cdot),\omega)} \right] + g_k(t,x).$$

If the right-hand side of equation (23) is known, then the strong formulation of the problem (23)–(24) is well studied (see, e.g., [13], [37]). For instance, in [13], it is established that if the right-hand side (for all $t \in (0,T]$) and the initial function $\varphi_k(x)$ belong to $L_2(G)$ (which holds in our case), then there exists a unique strong solution to the problem satisfying $u_k^i(t,x) \in L_\infty(0,T; \dot{W}_2^2(G))$ and $D_t^\alpha u_k^i(t,x) \in L_\infty(0,T; L_2(G))$. Clearly, such a strong solution is also a weak solution.

Here's a refined version of your text:

We analyze the problem (23)–(24) as follows. We set $u_k^0 = 0$ as the initial approximation for all $k = 1, 2, \ldots$. Then, we solve the initial-boundary value problem (23)–(24) to obtain the sequence $\{u_k^1\}$, $k = 1, 2, \ldots$. Subsequently, we construct the sequence $\{u_k^i\}$ iteratively. Prior to this, assuming that all $\{u_k^i\}$ have been constructed, we first derive a priori estimates.

Lemma 5.1. *Let ε be any possitive number. If*

$$M_\alpha C_\varepsilon\, f_0^2\, ||\nabla \omega||^2_{L_2(\Omega)} \max_{t,x} ||f(t,x,\cdot)||^2_{\tau(\varepsilon)} \leq 1. \tag{25}$$

then

$$\max_{[0,T]} \sum_{k=1}^\infty \lambda_k^{2\tau(\varepsilon)} ||u_k^i||^2_{L_2(G)} \leq 2A_0, \tag{26}$$

where

$$A_0 = M\varphi^* + M_\alpha f_0^2 (\psi_0 + g_0)^2 f^* + M_\alpha g^*,\ \varphi^* = ||\varphi||^2_{\tau(\varepsilon),G},$$

and

$$f^* = \max_{[0,T]} ||f(t,\cdot,\cdot)||^2_{\tau(\varepsilon),G}\ , \quad g^* = \max_{[0,T]} ||f(t,\cdot,\cdot)||^2_{\tau(\varepsilon),G}.$$

Proof. Substitute $w = u_k^i$ into equation (23), to obtain:

$$\int_G D_t^\alpha u_k^i u_k^i\, dx + \|\nabla u_k^i\|^2_{L_2(G)} + \lambda_k \|u_k^i\|^2_{L_2(G)} \tag{27}$$

$$= \int_G M_k(t,x) u_k^i\, dx - \int_G u_k^i \frac{f_k(t,x)}{(f(t,x,\cdot),\omega)} \sum_{j=1}^\infty (\nabla v_j, \nabla \omega) u_j^{i-1}(t,x)\, dx,$$

Apply for the integral on the left-hand side Alikhanov's estimate (see Lemma 2.2). Then

$$\frac{1}{2} D_t^\alpha \|u_k^i\|^2_{L_2(G)} + \|\nabla u_k^i\|^2_{L_2(G)} + \lambda_k \|u_k^i\|^2_{L_2(G)} \tag{28}$$

$$\leq \left| \int_0^1 M_k(t,x) u_k^i\, dx \right| + \left| \int_G u_k^i \frac{f_k(t,x)}{(f(t,x,\cdot),\omega)} \sum_{j=1}^\infty (\nabla v_j, \nabla \omega) u_j^{i-1}(t,x)\, dx \right| := I_1(t) + I_2(t).$$

For the right-hand side, using the notation and conditions of Theorem 4.1, we have:

$$I_1(t) \leq f_0(\psi_0 + g_0)\int_G |f_k(t,x)u_k^i(t,x)|\,dx + \int_G |g_k(t,x)u_k^i(t,x)|\,dx.$$

Apply the inequality $2ab \leq a^2 + b^2$, to get

$$I_1(t) \leq \frac{1}{2}\left[f_0^2(\psi_0+g_0)^2\|f_k\|^2_{L_2(G)} + \|g_k\|^2_{L_2(G)}\right] + \|u_k^i\|^2_{L_2(G)}. \tag{29}$$

For I_2 it is not hard to see, that

$$I_2(t) \leq \frac{1}{2}\|u_k^i\|^2_{L_2(G)} + \frac{1}{2}f_0^2\int_G |f_k(t,x)|^2 \left|\sum_{j=1}^{\infty}(\nabla v_j, \nabla\omega)u_j^{i-1}(t,x)\right|^2 dx.$$

Let us estimate the sum separately. Due to (10) and the Cauchy–Bunyakovsky inequality we have

$$\begin{aligned} I := \left|\sum_{j=1}^{\infty}(\nabla v_j, \nabla\omega)u_j^{i-1}(t,x)\right|^2 &\leq \left[\sum_{j=1}^{\infty}||\nabla v_j||_{L_2(\Omega)}||\nabla\omega||_{L_2(\Omega)}|u_j^{i-1}(t,x)|\right]^2 \\ &= ||\nabla\omega||^2_{L_2(\Omega)}\left[\sum_{j=1}^{\infty}\lambda_j^{\frac{1}{2}}|u_j^{i-1}(t,x)|\right]^2. \end{aligned}$$

For any $\varepsilon > 0$, we express $\lambda_j^{\frac{1}{2}} = \lambda_j^{\frac{1}{2}+\frac{n+2\varepsilon}{4}}\lambda_j^{-\frac{n+2\varepsilon}{4}}$ and use the Cauchy–Schwarz inequality on the sum. Then, refer to (9):

$$I \leq C_\varepsilon||\nabla\omega||^2_{L_2(\Omega)}\sum_{j=1}^{\infty}\lambda_j^{\frac{n}{2}+1+\varepsilon}|u_j^{n-1}|^2. \tag{30}$$

By combining estimates of I_1, I_2 and (30), the expression in (27) can be reformulated as:

$$\begin{aligned} \frac{1}{2}D_t^\alpha\|u_k^i\|^2_{L_2(G)} + \|\nabla u_k^i\|^2_{L_2(G)} + \lambda_k\|u_k^i\|^2_{L_2(G)} &\leq \frac{3}{2}\|u_k^i\|^2_{L_2(G)} \\ &+\frac{1}{2}\left[f_0^2(\psi_0+g_0)^2\|f_k(t,\cdot)\|^2_{L_2(G)} + \|g_k(t,\cdot)\|^2_{L_2(G)}\right] \\ &+\frac{C_\varepsilon\, f_0^2\,||\nabla\omega||^2_{L_2(\Omega)}}{2}\int_G |f_k(t,x)|^2\sum_{j=1}^{\infty}\lambda_j^{2\tau}|u_j^{i-1}|^2\,dx. \end{aligned} \tag{31}$$

If we omit the final two terms on the left-hand side, then

$$D_t^\alpha\|u_k^i\|^2_{L_2(G)} \leq 3\|u_k^i\|^2_{L_2(G)} + c_2^k(t),$$

where $c_2^k(t) = c_{2,1}^k(t) + c_{2,2}^k(t)$ and

$$c_{2,1}^k(t) = f_0^2(\psi_0+g_0)^2\|f_k(t,\cdot)\|^2_{L_2(G)} + \|g_k(t,\cdot)\|^2_{L_2(G)},$$

$$c_{2,2}^k(t) = C_\varepsilon\, f_0^2\,||\nabla\omega||^2_{L_2(\Omega)}\int_G |f_k(t,x)|^2\sum_{j=1}^{\infty}\lambda_j^{2\tau}|u_j^{i-1}|^2\,dx.$$

By Lemma 2.3 we obtain

$$\|u_k^i\|^2_{L_2(G)} \leq \|\varphi_k\|^2_{L_2(G)}\,E_\alpha(3t^\alpha) + \Gamma(\alpha)\,E_{\alpha,\alpha}(3t^\alpha)\,J_t^\alpha c_2^k(t).$$

Relying on the boundedness of the Mittag-Leffler function (refer to (4)) and the estimate (7), we deduce that

$$\|u_k^i(t,\cdot)\|^2_{L_2(G)} \leq M\|\varphi_k\|^2_{L_2(G)} + MT^\alpha\left(\max_{[0,T]}c_{2,1}^k(t) + \max_{[0,T]}c_{2,2}^k(t)\right).$$

By multiplying this inequality by $\lambda_j^{2\tau}$ and summing over k from 1 to ∞, it follows that

$$\max_{[0,T]}\sum_{k=1}^{\infty}\lambda_k^{2\tau}\|u_k^i(t,\cdot)\|^2_{L_2(G)} \le M\varphi^* + M_\alpha f_0^2(\psi_0+g_0)^2 f^* + M_\alpha g^*$$

$$+ M_\alpha C_\varepsilon\, f_0^2\,||\nabla\omega||^2_{L_2(\Omega)}\max_{[0,T]}\int_G\sum_{k=1}^{\infty}\lambda_k^{2\tau}|f_k(t,x)|^2\sum_{j=1}^{\infty}\lambda_j^{2\tau}|u_j^{i-1}(t,x)|^2dx.$$

Employing condition (25) and the notation from Lemma 5.1, the final estimate can be reformulated as a recurrence estimate:

$$\max_{[0,T]}\sum_{k=1}^{\infty}\lambda_k^{2\tau}\|u_k^i\|^2_{L_2(G)} \le A_0 + \frac{1}{2}\max_{[0,T]}\sum_{k=1}^{\infty}\lambda_k^{2\tau}\|u_k^{i-1}\|^2_{L_2(G)}, \quad i=1,2,\dots. \tag{32}$$

As stated previously, we set $u_k^0 = 0$ as the initial approximation for all $k\ge 1$. Consequently, for $u_k^1(t,x)$, $k=1,2,\dots$, using (32), we derive:

$$\max_{[0,T]}\sum_{k=1}^{\infty}\lambda_k^{2\tau}\|u_k^1\|^2_{L_2(G)} \le A_0. \tag{33}$$

Subsequently, we insert the functions $\{u_k^1\}_{k=1}^{\infty}$ into the problem (23)–(24) to uniquely define the functions $\{u_k^2\}_{k=1}^{\infty}$. For these functions, we obtain the following estimate from (32):

$$\max_{[0,T]}\sum_{k=1}^{\infty}\lambda_k^{2\tau}\|u_k^2\|^2_{L_2(G)} \le A_0 + \frac{1}{2}A_0.$$

Proceeding with this approach, we ultimately derive:

$$\max_{[0,T]}\sum_{k=1}^{\infty}\lambda_k^{2\tau}\|u_k^n\|^2_{L_2(G)} = A_0\sum_{j=1}^{n}\left(\frac{1}{2}\right)^j = 2\left(1-\frac{1}{2^n}\right)A_0 \le 2A_0. \tag{34}$$

□

Lemma 5.2. *The following estimate is valid*

$$\sum_{k=1}^{\infty}\int_0^t\int_G\left[|\nabla\, u_k^i|^2+\lambda_k|u_k^i|^2\right]dxd\tau \le \frac{\Gamma(\alpha)T^{1-\alpha}}{2}||\varphi||^2_{L_2(D)} + 3A_0T + \frac{T}{2}\,A_1$$

$$+T\,A_0C_\varepsilon\, f_0^2\,||\nabla\omega||^2_{L_2(\Omega)}\max_{x\in G}||f(t,x,\cdot)||^2_{L_2(\Omega)}, \quad t\in[0,T]. \tag{35}$$

Proof. Applying the operator J_t^α to both sides of inequality (31) and utilizing Lemma 2.1, we obtain:

$$\|u_k^i\|^2_{L_2(G)} + 2J_t^\alpha\|\nabla u_k^i\|^2_{L_2(G)} + 2\lambda_k J_t^\alpha\|u_k^i\|^2_{L_2(G)} \tag{36}$$

$$\le \|\varphi_k\|^2_{L_2(G)} + 3J_t^\alpha\|u_k^i\|^2_{L_2(G)} + J_t^\alpha c_{2,1}^k(t) + J_t^\alpha c_{2,2}^k(t).$$

Initially, we disregard the first term on the left-hand side of the inequality. Then, by summing inequality (37) over k from 1 to ∞ and applying the estimates (7), we derive:

$$\frac{2}{\Gamma(\alpha)T^{1-\alpha}}\sum_{k=1}^{\infty}\int_0^t\int_G\left[|\nabla u_k^i|^2+\lambda_k|u_k^i|^2\right]dxd\tau \tag{37}$$

$$\le \sum_{k=1}^{\infty}\|\varphi_k\|^2_{L_2(G)} + \frac{3T^\alpha}{\Gamma(\alpha)}\max_{[0,T]}\sum_{k=1}^{\infty}\|u_k^i\|^2_{L_2(G)} + \frac{T^\alpha}{\Gamma(\alpha)}\max_{[0,T]}\sum_{k=1}^{\infty}\left(c_{2,1}^k(t)+c_{2,2}^k(t)\right).$$

Apply Parseval's equality to obtain

$$\max_{[0,T]}\sum_{k=1}^{\infty} c^k_{2,1}(t) = f_0^2(\psi_0+g_0)^2\max_{[0,T]}\|f(t,\cdot,\cdot)\|^2_{L_2(D)} + \max_{[0,T]}\|g(t,\cdot,\cdot)\|^2_{L_2(D)} := A_1.$$

Similarly,

$$\sum_{k=1}^{\infty} c^k_{2,2}(t) \le C_\varepsilon\, f_0^2\,||\nabla\omega||^2_{L_2(\Omega)}\max_{x\in G}||f(t,x,\cdot)||^2_{L_2(\Omega)}\sum_{j=1}^{\infty}\lambda_j^{2\tau}||u_j^{i-1}||^2_{L_2(G)},$$

or by (26),

$$\sum_{k=1}^{\infty} c^k_{2,2}(t) \le 2\,A_0 C_\varepsilon\, f_0^2\,||\nabla\omega||^2_{L_2(\Omega)}\max_{x\in G}||f(t,x,\cdot)||^2_{L_2(\Omega)}\,.$$

Thus, inequality (37) can be reformulated as (35). □

Lemma 5.3. *For any $t\in[0,T]$ we have*

$$\sum_{k=1}^{\infty}\int\limits_0^t\int\limits_G |D_t^\alpha u_k^i|^2 dx d\tau \le \frac{\Gamma(\alpha)T^{1-\alpha}}{2}\Big(\|\nabla_x\varphi\|^2_{L_2(D)}+\|\nabla_y\varphi\|^2_{L_2(D)}\Big) \tag{38}$$

$$+T\max_t\Big[f_0^2(\psi_0+g_0)^2\|f(t,\cdot,\cdot)\|^2_{L_2(D)}+\|g(t,\cdot,\cdot)\|^2_{L_2(D)}\Big]$$

$$+2\,A_0C_\varepsilon\, f_0^2\,||\nabla\omega||^2_{L_2(\Omega)}\max_{x\in G}||f(t,x,\cdot)||^2_{L_2(\Omega)}.$$

Proof. Substitute $w=D_t^\alpha u_k^i$ into equation (23), to get

$$\|D_t^\alpha u_k^i\|^2_{L_2(G)}+\int_G \nabla u_k^i\, D_t^\alpha\,\nabla u_k^i\,dx+\lambda_k\int_G u_k^i\, D_t^\alpha\, u_k^i\,dx \tag{39}$$

$$=\int_G M_k(t,x)D_t^\alpha u_k^i\,dx-\int_G D_t^\alpha u_k^i\frac{f_k(t,x)}{(f(t,x,\cdot),\omega)}\sum_{j=1}^{\infty}(\nabla v_j,\nabla\omega)u_j^{i-1}(t,x)\,dx.$$

We apply A. Alikhanov's estimate (refer to Lemma 2.2) to the second and third integrals on the left-hand side of the equality, yielding:

$$\|D_t^\alpha u_k^i\|^2_{L_2(G)}+\frac{1}{2}D_t^\alpha\|\nabla u_k^i\|^2_{L_2(G)}+\lambda_k\frac{1}{2}D_t^\alpha\|u_k^i\|^2_{L_2(G)} \tag{40}$$

$$\le\left|\int_G M_k(t,x)D_t^\alpha u_k^i dx\right|+\left|\int_G D_t^\alpha u_k^i\frac{f_k(t,x)}{(f(t,x,\cdot),\omega)}\sum_{j=1}^{\infty}(\nabla v_j,\nabla\omega)u_j^{i-1}(t,x)\,dx\right|:=J_1(t)+J_2(t).$$

Apply inequality $a\,b\le a^2+\frac{1}{4}\,b^2$ to $J_1(t)$ to get (see Eq. (29))

$$J_1(t)\le f_0^2(\psi_0+g_0)^2||f_k||^2_{L_2(G)}+||g_k||^2_{L_2(G)}+\frac{1}{2}\|D_t^\alpha u_k^n\|^2_{L_2(G)}.$$

The second summand in the right-hand side of (40) has the estimate (see Eq. (30))

$$J_2(t)\le\frac{1}{4}\|D_t^\alpha u_k^i\|^2_{L_2(G)}+C_\varepsilon\, f_0^2\,||\nabla\omega||^2_{L_2(\Omega)}\int_G|f_k(t,x)|^2\sum_{j=1}^{\infty}\lambda_j^{2\tau}|u_j^{i-1}|^2dx.$$

Now we apply the operator J_t^α to both parts of inequality (31) and use estimates of $J_1(t)$, $J_2(t)$ and Lemma 2.1. Then

$$\frac{1}{4}J_t^\alpha\|D_t^\alpha u_k^i\|^2_{L_2(G)}+\frac{1}{2}\|\nabla u_k^i\|^2_{L_2(G)}+\lambda_k\frac{1}{2}\|u_k^i\|^2_{L_2(G)}$$

$$\le\frac{1}{2}\|\nabla\varphi_k\|^2_{L_2(G)}+\lambda_k\frac{1}{2}\|\varphi_k\|^2_{L_2(G)}+f_0^2(\psi_0+g_0)^2J_t^\alpha\|f_k\|^2_{L_2(G)}+J_t^\alpha\|g_k\|^2_{L_2(G)}$$

$$+C_\varepsilon\, f_0^2\, ||\nabla\omega||^2_{L_2(\Omega)} J_t^\alpha \int_G |f_k(t,x)|^2 \sum_{j=1}^\infty \lambda_j^{2\tau} |u_j^{i-1}|^2 dx..$$

Initially, we exclude the second and third terms on the left-hand side of the inequality. Then, by summing over k from 1 to ∞ and utilizing the estimates (7), we derive (38). □

6. Convergence

We aim to demonstrate that the sequences u_k^i, ∇u_k^i, and $D_t^\alpha u_k^i$, for $i = 1, 2, \ldots$, are fundamental with respect to their corresponding norms for all $k \geq 1$.

Given that the initial-boundary value problem (23)–(24) is linear with respect to u_k^i, by reiterating the previous arguments, we can establish estimates (34), (35), and (41) for $u_k^{i+p} - u_k^i$, where $p = 1, 2, 3, \ldots$. For instance, consider writing equality (23) for u_k^{i+1} and subtracting (23) from it. This leads in (noting that here $M_k(t,x) \equiv 0$):

$$\int_G D_t^\alpha (u_k^{i+1} - u_k^i)\, w\, dx + \int_G \nabla (u_k^{i+1} - u_k^i)\, \nabla w\, dx + \lambda_k \int_G (u_k^{i+1} - u_k^i)\, w\, dx \tag{41}$$
$$= - \int_G \frac{f_k(t,x)}{(f(t,x,\cdot),\omega)} \sum_{j=1}^\infty (\nabla v_j, \nabla\omega) u_j^{i-1}(t,x) w\, dx,$$

for all $w \in \dot{W}_2^1(0,1)$, with the initial condition:

$$(u_k^{i+1} - u_k^i)(0,x) = 0. \tag{42}$$

By applying the same reasoning as in the proof of (32), but substituting (42) for the initial condition (24), we obtain the inequality:

$$\max_{[0,T]} \sum_{k=1}^\infty \lambda_k^{2\tau} \|u_k^{i+1} - u_k^i\|^2_{L_2(G)} \leq \frac{1}{2} \max_{[0,T]} \sum_{k=1}^\infty \lambda_k^{2\tau} \|u_k^i - u_k^{i-1}\|^2_{L_2(G)}.$$

Since $u_k^0 = 0$ for all $k \geq 1$, then from (33), we get

$$\max_{[0,T]} \sum_{k=1}^\infty \lambda_k^{2\tau} \|u_k^2 - u_k^1\|^2_{L_2(G)} \leq A_0.$$

For any $n \geq 2$ the last two estimates imply:

$$\max_{[0,T]} \sum_{k=1}^\infty \lambda_k^{2\tau} \|u_k^{i+1} - u_k^i\|^2_{L_2(G)} \leq A_0 \left(\frac{1}{2}\right)^{n-2}.$$

Hence, if we write $u_k^{i+p} - u_k^i = u_k^{i+p} - u_k^{i+p-1} + u_k^{i+p-1} - u_k^{i+p-2} + \cdots$, then

$$\max_{[0,T]} \sum_{k=1}^\infty \lambda_k^{2\tau} \|u_k^{i+p} - u_k^i\|^2_{L_2(G)} \leq A_0 \left(\frac{1}{2}\right)^{i-2} \left(1 + \frac{1}{2} + \cdots + \frac{1}{2^p}\right) \leq A_0 \left(\frac{1}{2}\right)^{i-1}. \tag{43}$$

This indicates that the sequence $\{u_k^i\}$ is fundamental in $L_2(G)$. Therefore, for each $k \geq 1$, there exist functions $u_k(t,x) \in L_2(G)$ such that, for every $0 < t \leq T$, as $i \to \infty$, the following conditions are satisfied:

$$\begin{cases} u_k^i(t,x) \to u_k(t,x) \quad \text{in } L_2(G), \quad k = 1, 2, \cdots, \\ \sum_{k=1}^\infty \lambda_k^{2\tau} \|u_k^i(t,\cdot) - u_k(t,\cdot)\|^2_{L_2(G)} \to 0. \end{cases} \tag{44}$$

By substituting $u_k^{i+p} - u_k^i$ for u_k^i in the estimate (37), it assumes the form (here, one must replicate the reasoning analogous to that provided in the proof of (43)):

$$\|u_k^{i+p} - u_k^i\|^2_{L_2(G)} + 2J_t^\alpha \|\nabla(u_k^{i+p} - u_k^i)\|^2_{L_2(G)} + 2\lambda_k J_t^\alpha \|u_k^{i+p} - u_k^i\|^2_{L_2(G)}$$
$$\leq J_t^\alpha \|u_k^{i+p} - u_k^i\|^2_{L_2(G)} + J_t^\alpha c_{2,2}^k(t).$$

Following straightforward computations and applying the estimate (7), (35) can be reformulated as:

$$\sum_{k=1}^{\infty}\int\limits_0^t\int\limits_G\left[|\nabla(u_k^{i+p}-u_k^i)|^2+\lambda_k|u_k^{i+p}-u_k^i|^2\right]dxd\tau\le\frac{3\,T}{2}\max_{t\in[0,T]}\sum_{k=1}^{\infty}\|u_k^{i+p}-u_k^i\|^2_{L_2(G)}$$

$$+\frac{T\,C_\varepsilon\,f_0^2}{2}||\nabla\omega||^2_{L_2(\Omega)}\max_{x\in G}\int\limits_\Omega|f(t,x,y)|^2dy\sum_{j=1}^{\infty}\lambda_j^{2\tau}||u_j^{i+p}-u_j^i||^2_{L_2(G)},\quad t\in[0,T].$$

The estimate (43) indicates that the sequences on the left-hand side are fundamental. Thus, for every $0<t\le T$, as $i\to\infty$, it follows that:

$$\begin{cases}\int\limits_0^t\|\nabla u_k^i(\tau,\cdot)-\nabla u_k(\tau,\cdot)\|^2_{L_2(G)}d\tau\to 0,,\quad k=1,2,\cdots,\\ \int\limits_0^t\lambda_k\|u_k^i(\tau,\cdot)-u_k(\tau,\cdot)\|^2_{L_2(G)}d\tau\to 0,,\quad k=1,2,\cdots,\\ \int\limits_0^t\sum_{k=1}^\infty\|\nabla u_k^i(\tau,\cdot)-\nabla u_k(\tau,\cdot)\|^2_{L_2(G)}d\tau\to 0,\\ \int\limits_0^t\sum_{k=1}^\infty\lambda_k\|u_k^i(\tau,\cdot)-u_k(\tau,\cdot)\|^2_{L_2(G)}d\tau\to 0.\end{cases}\tag{45}$$

Let us set $w=D_t^\alpha(u_k^{i+1}-u_k^i)$ in equality (41) and replicate the reasoning provided in the proof of (38). Then, employing arguments analogous to those in the proof of (43), we confirm that the sequence $\sum_{k=1}^\infty\int_0^t\int_G|D_t^\alpha u_k^i|^2\,dx\,d\tau$ is fundamental. Consequently, for every $0<t\le T$, as $i\to\infty$, it follows that:

$$\begin{cases}\int\limits_0^t\|D_t^\alpha u_k^i(\tau,\cdot)-D_t^\alpha u_k(\tau,\cdot)\|^2_{L_2(G)}d\tau\to 0,\quad k=1,2,\cdots,\\ \int\limits_0^t\sum_{k=1}^\infty\|D_t^\alpha u_k^i(\tau,\cdot)-D_t^\alpha u_k(\tau,\cdot)\|^2_{L_2(G)}d\tau\to 0.\end{cases}\tag{46}$$

It is easy to verify that the right-hand side of equation (23) converges in $L_2(G)$ as $i\to\infty$. Indeed (see (30)), first note

$$\left|\sum_{k=1}^{\infty}(\nabla v_k,\nabla\omega)(u_k^i(t,x)-u_k(t,x))\right|^2\le C_\varepsilon||\nabla\omega||^2_{L_2(\Omega)}\sum_{k=1}^{\infty}\lambda_k^{2\tau}|u_k^i(t,x)-u_k(t,x)|^2.\tag{47}$$

Now, we integrate (47) over the domain G and apply the relation (44). Then we obtain the required convergence.

7. Proof of Theorem 4.1

Let us pass to the limit $i\to\infty$ in (34). Then, by virtue of the convergence (44) and Parseval's equality, we see that for each $t\in[0,T]$ the function

$$u(t,x,y)=\sum_{k=1}^{\infty}u_k(t,x)v_k(y),$$

is well defined element of $L_2(D)$ and (see Lemma 5.1)

$$\max_{[0,T]}\|u(t,\cdot,\cdot)\|^2_{\tau,G}\le 2A_0.\tag{48}$$

Similarly, if we pass to the limit $i\to\infty$ in (35), then by virtue of the convergence of (45) we will obtain

$$\int\limits_0^t\left[||\nabla_x u(\tau.\cdot,\cdot)||^2_{L_2(D)}+||\nabla_y u(\tau.\cdot,\cdot)||^2_{L_2(D)}\right]d\tau\le\frac{\Gamma(\alpha)T^{1-\alpha}}{2}||\varphi||^2_{L_2(D)}+3A_0T+\frac{T}{2}A_1$$

$$+T\, A_0 C_\varepsilon\, f_0^2\, ||\nabla\omega||^2_{L_2(\Omega)} \max_{x\in G} ||f(t,x,\cdot)||^2_{L_2(\Omega)}, \quad t\in[0,T]. \tag{49}$$

For any $t\in[0,T]$ the estimate (38) implies

$$\int_0^t ||D_t^\alpha u(\tau,\cdot,\cdot)||^2_{L_2(D)} d\tau \le \frac{\Gamma(\alpha)T^{1-\alpha}}{2}\left(\|\nabla_x\varphi\|^2_{L_2(D)} + \|\nabla_y\varphi\|^2_{L_2(D)}\right) \tag{50}$$

$$+T\max_t\left[f_0^2(\psi_0+g_0)^2\|f(t,\cdot,\cdot)\|^2_{L_2(D)} + \|g(t,\cdot,\cdot)\|^2_{L_2(D)}\right]$$
$$+2\, A_0 C_\varepsilon\, f_0^2\, ||\nabla\omega||^2_{L_2(\Omega)} \max_{x\in G} ||f(t,x,\cdot)||^2_{L_2(\Omega)}.$$

Lemma 7.1. *For each $t\in[0,T]$, the function $h(t,x)$ defined by the formula* (14) *is an element of $L_2(G)$ and the estimate*

$$\int_G |h(t,x)|^2 dx \le 4f_0^2\left(|G|\,\psi_0^2 + 2\,A_0\,C_\varepsilon ||\nabla\omega||^2_{L_2(\Omega)}\right) + 2|G|\,g_0^2,\ t\in[0,T],$$

holds.

Proof. Due to inequality $(a+b+c+d)^2\le 4(a^2+b^2+c^2+d^2)$, it follows from definition (20) of $h(t,x)$

$$|h(t,x)|^2 \le 4f_0^2\left(|D_t^\alpha\psi(t,x)|^2 + |\Delta_x\psi(t,x)|^2 + |(g(t,x,\cdot),\omega)|^2 + \left|\sum_{j=1}^\infty(\nabla v_j,\nabla\omega)u_j(t,x)\right|^2\right). \tag{51}$$

For the sum we have the estimate (see (30))

$$\left|\sum_{j=1}^\infty(\nabla v_j,\nabla\omega)u_j(t,x)\right|^2 \le C_\varepsilon ||\nabla\omega||^2_{L_2(\Omega)}\sum_{j=1}^\infty \lambda_j^{2\tau}|u_j^{n-1}(t,x)|^2;, \quad x\in G, \quad t\ge 0.$$

If we integrate this inequality over $x\in G$, then by virtue of estimate (34) we will have

$$\int_G\left|\sum_{j=1}^\infty(\nabla v_j,\nabla\omega)u_j(t,x)\right|^2 dx \le 2\,A_0\,C_\varepsilon ||\nabla\omega||^2_{L_2(\Omega)}. \tag{52}$$

Now we integrate (51) with respect to $x\in G$. Then by (52), we get the required estimate. □

Now, returning to the proof of Theorem 4.1, we first show the existence of a weak solution of the inverse problem. We integrate (23) over $t\in[0,T]$, and then pass to the limit $i\to\infty$ and, taking into account (44) - (47), we obtain the following equalities, valid for an arbitrary $w\in \dot W_2^1(0,1)$ and for all k:

$$\int_0^t \left[(D_t^\alpha u_k,w) + (\nabla u_k,\nabla w) + \lambda_k(u_k,w)\right] d\tau = \int_0^t \left[(g_k,w) + (f_k h,w)\right] d\tau. \tag{53}$$

Next, we take the function $w=v(x,y)v_k(y)$, $v\in \dot W_2^1(\Omega)$, and integrate equality (53) over $y\in\Omega$. Then, summing over k from 1 to ∞, we obtain for all $t\in[0,T]$ (see (21))

$$\int_0^t\int_D\left[D_t^\alpha uv + \nabla_x u\nabla_x v + \nabla_y u\nabla_y v\right] dx\,dyd\tau = \int_0^t\int_D\left[fhv + gv\right] dx\,dyd\tau. \tag{54}$$

Note that if $s(t)$ is integrable in any subset $(0,t)$ of the interval $[0,T]$ and $\int_0^t s(\tau)d\tau = 0$, then obviously $s(t)=0$ almost everywhere on $[0,T]$. Therefore, the equality (54) coincides with (15). Consequently, the function $u(t,x,y)$ defined by formula (13), together with h, defined by formula (20), is a weak solution of the inverse problem, i.e., it satisfies all the conditions of definition 4.1. As for the estimates (17) - (19), they are proved above (see the estimates (48) - (50)). The estimate of h is established in Lemma 7.1.

Let us prove the uniqueness of the weak solution of the inverse problem.

Suppose the opposite, i.e. there are two solutions to the inverse problem: (u_1, h_1) and (u_2, h_2). Then the functions $u = u_1 - u_2$ and $h = h_1 - h_2$ satisfy all the conditions of Definition 4.1 with the functions $\varphi(x, y) \equiv 0$, $g(t, x, y) \equiv 0$. Let us denote

$$u_k(t, x) = \int\limits_{\Omega} u(t, x, y) v_k(y)\, dy.$$

Let us consider the sum that defines the function $h(t, x)$ (see (20)):

$$\sum_{j=1}^{\infty} (\nabla v_j, \nabla \omega) u_j(t, x).$$

Estimate (52) means that this function exists almost everywhere in G and for all $t \in [0, T]$. Therefore, the function

$$h(t, x) = -\frac{\sum_{j=1}^{\infty} (\nabla v_j, \nabla \omega) u_j(t, x)}{(f(t, x, \cdot), \omega)}$$

(see (20)) is correctly defined. Then for functions u_k one has (see (28)):

$$\frac{1}{2} D_t^{\alpha} \|u_k\|^2_{L_2(G)} + \|\nabla u_k\|^2_{L_2(G)} + \lambda_k \|u_k\|^2_{L_2(G)} \leq$$
$$\leq \left| \int_G u_k(t, x) \frac{f_k(t, x)}{(f(t, x, \cdot), \omega)} \sum_{j=1}^{\infty} (\nabla v_j, \nabla \omega) u_j(t, x)\, dx \right|$$

Since $M_k(t, x) \equiv 0$, then, repeating the same reasoning as in the proof (34), we get:

$$\max_{[0,T]} \sum_{k=1}^{\infty} \lambda_k^{2\tau} \|u_k\|^2_{L_2(G)} \leq 0$$

(note that $A_0 = 0$). Hence,

$$u_k(t, x) = 0 \quad \text{a.e. in } G \text{ for all } k \geq 1 \text{ and } t \in [0, T].$$

Therefore, $h(t, x) = 0$ a.e. in G for all $t \in [0, T]$. Completeness of the system $\{v_k(y)\}$ implies

$$u(t, x, y) = 0 \quad \text{for almost all } x \in G,\ y \in \Omega \text{ and } t \in [0, T].$$

Thus, the theorem is completely proved.

Remark 7.2. *It is important to emphasize that theorem 4.1 remains valid for parabolic equations as well. In this case, instead of Alikhanov's estimate, one should use the usual equality* $\frac{d}{dt} u^2 = 2u \frac{d}{dt} u$.

8. Conclusions

In this paper, we study a new inverse problem for subdiffusion equations, namely, the problem of determining the coefficient of the source function. The elliptic part of the equation is given by $\Delta_x u + \Delta_y u$, $x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$, and the unknown function depends on both time and a part of the spatial variables, denoted by $h(t, x)$. Moreover, the overdetermination condition is given in integral form. To our knowledge, such a problem has not been studied before for subdiffusion equations or even for diffusion equations. The existence and uniqueness of a weak solution to the inverse problem are established, as well as coercive estimates.

The choice of constructing a weak solution is motivated by the relative simplicity of obtaining a priori estimates in such a formulation. If we consider the classical formulation of the problem (23), (24) (for example, see [37]), then we can prove the existence of a classical solution to the inverse problem. However, this will be the subject of a future paper.

ACKNOWLEDGEMENTS

The authors acknowledge financial support from the Ministry of Innovative Development of the Republic of Uzbekistan, Grant No F-FA-2021-424.

References

[1] J.A.T. Machado, Editor: Handbook of fractional calculus with applications. Vol. 1 - 8, DeGruyter, 2019.
[2] V.V. Uchaikin, Fractional derivatives for Physicists and Engineers. Vol. 1, Background and Theory. Vol. 2, Application, Springer. 2013.
[3] R. Hilfer, aditor: Applications of fractional calculas in physics. Singapore, World Scientific. 2000.
[4] S. Umarov, M. Hahn, K. Kobayashi, Beyond the triangle: Browian motion, Ito calculas, and Fokker-Plank equation-fractional generalizations. World Scientific. 2017.
[5] R. Herrmann. Common aspects of deformed Lie algebras and fractional calculus. Phys A Stat Mech Appl 2010;389(21):4613–22. http://dx.doi.org/10.1016/j.physa.2010.07.004.
[6] Kiryakova V. Generalized fractional calculus and applications. In: Pitman research notes in mathematics, vol. 301, Harlow: Longman; 1994.
[7] Podlubny I. FractionalDiffusion equation, mathematics in science and engineer- ing. New York: Academic Press; 1999.
[8] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, Theory and applications of fractional differential equations. Elsevier, North-Holland, Mathematics studies. 2006.
[9] M. Djrbashian, *Integral transforms and representation of functions in the complex domain*, M. Nauka, (1966). (in Russian)
[10] R. Gorenflo, A.A. Kilbas, F. Mainardi, S.V.Rogozin, Mittag-Leffler functions, related topics and applications. Springer. 2014.
[11] A. V. Pskhu, Partial Differential Equations of Fractional Order, Moscow: Nauka, 2005.
[12] A. A. Alikhanov, "A Priori Estimates of Solutions to Boundary Value Problems for Equations of Fractional Order," Differential Equations, 2010, Vol. 46, No. 5, 658–664 pp.
[13] M. Sakamoto, Yamamoto, Initial value/boundary value problems for fractional diffusion-wave equations and applications to some inverse problems. Journal of Mathematical Analysis and Applications. 2011, Vol.382, No. 1, 426–447p.
[14] R. R. Ashurov, A. Cabada, B.Turmetov, Operator method for construction of solutions of linear fractional differential equations with constant coefficients. Frac. Calculus Appl. Anal. 2016, Vol. 1., 229-252p.
[15] S.I. Kabanikhin, Inverse and Ill-Posed Problems. Theory and Applications, De Gruyter, 2011.
[16] Y. Liu, Z. Li, M. Yamamoto, Inverse problems of determining sources of the fractional partial differential equations. Handbook of Fractional Calculus with Appl., J.A.T. Marchado Ed. De Gruyter., 2019, Vol.2, 411–430p.
[17] R. R. Ashurov, M. Shakarova, Time-dependent source identification problem for fractional Schrödinger type equations. Lobachevskii Journal of Mathematics, 2022, Vol.42, No.3, 517–525p.

[18] R. R. Ashurov, M. Shakarova, Time-dependent source identification problem for a fractional Schrödinger equation with the Riemann-Liouville derivative. Ukr. Mat. Journal, 2023, Vol.75, No.7, 871–887p.

[19] M. Kirane, A.S. Malik, Determination of an unknown source term and the temperature distribution for the linear heat equation involving fractional derivative in time. Applied Mathematics and Computation. 2011, Vol. 218, 163-170p.

[20] Z. Li, Y. Liu, M. Yamamoto, Initial-boundary value problem for multi-term time-fractional diffusion equation with positive constant coefficients. Applied Mathematica and Computation, 2015, Vol. 257, 381–397p.

[21] M. Ruzhansky, N. Tokmagambetov, B.T. Torebek, Inverse source problems for positive operators. I: Hypoelliptic diffusion and subdiffusion equations. J. Inverse Ill-Possed Probl. 2019. Vol. 27. 891-911p.

[22] R. R. Ashurov, A. Mukhiddinova, Inverse Problem of Determining the Heat Source Density for the Subdiffusion Equation, Differential Equations. 2020, Vol. 56, No. 12, 1550–1563p.

[23] M. Kirane, B. Samet, B.T. Torebek, Determination of an unknown source term and the temperature distribution for the subdiffusion equation at the initial and final data. Electronic Journal of Differential Equations. 2017, Vol. 217, 1-13p.

[24] B.T. Torebek, R. Tapdigoglu, Some inverse problems for the nonlocal heat equation with Caputo fractional derivative. Mathematical Methods in Applied Sciences. 2017, Vol. 40, 6468-6479p.

[25] R. R. Ashurov, M. Shakarova, Inverse problem for the subdiffusion equation with fractional Caputo derivative. UFA Mathematical Journal, 2024, Vol. 16, No.1, 111–126p.

[26] R. R. Ashurov, M. Shakarova, S. Umarov, An inverse problem for the subdiffusion equation with a non-local in time condition. Fractal and Fractional, Vol. 878, No.8, 2024.

[27] A.V. Nagumanova, V.E. Fedorov, Linear inverse problems for degenerate evolution equations with the Caputo derivative in the sectorial case. Mathematical notes of NEFU, 2020, Vol. 27, No.2, 54–76p.

[28] D.G. Orlovskii, On a problem of determining the parameter of an evolution equation. Differ. Uravn., 1990, Vol. 26, No. 9, 1614–1621p.

[29] A.I. Prilepko, A.B. Kostin, On certain inverse problems for parabolic equations with final and integral observation, Mat. Sb., 1992, Vol. 183, No.4, 49–68p.

[30] K. Van Bockstal, Uniqueness for inverse source problems of determining a space-dependent source in time-fractional equations with non-smooth solutions. Fractal and Fractional, 2021, Vol.5, 169p.

[31] S. Z. Dzhamalov, R. R. Ashurov, "On a Linear Inverse Problem for a Multidimensional Second-Order Equation of Mixed Type of the Second Kind," Differential Equations, 2019, Vol. 55, No. 1, 34–44 pp.

[32] S. Z. Dzhamalov, R. R. Ashurov, "On a Linear Inverse Problem for a Multidimensional Second-Order Equation of Mixed Type of the First Kind," Russian Mathematics (Izvestiya Vysshikh Uchebnykh Zavedenii. Matematika), 2019, No. 6, 1–12 pp.

[33] S. Z. Dzhamalov, Sh. Sh. Khudoykulov. On Linear Two-Point Inverse Problem for a Multidimensional Wave Equation with Semi-Nonlocal Boundary Conditions. Lobachevskii Journal of Mathematics, 2024, Vol. 45, No. 3, 1032–1044p.

[34] S.Z.Dzhamalov, Sh.Sh.Khudoykulov. On some linear two-point inverse problem for a multidimensional heat conduction equation with semi-nonlocal boundary conditions. Bulletin of the Karaganda University. Mathematics series, 2024, Vol. 2, No. 114, 71–85p.

[35] Fikret G¨olgeleyen and Masahiro Yamamoto, Uniqueness of solution of an inverse source problem for ultrahyperbolic equations, arXiv:

[36] R.R. Ashurov, O.T. Mukhiddinova Space-Time-Dependent Source Identification Problem for a Subdiffusion Equation. http://arxiv.org/abs/2505.04252

[37] R.R. Ashurov, O.T. Mukhiddinova, Initial-boundary value problem for a time-fractional subdiffusion equation with an arbitrary elliptic differential operator. Lobachevski Journal of Mathematics, 2021, Vol. 42, No.3, 517-525 p.

[38] M. Reed and B. Simon, *Methods of Modern Mathematical Physics, Volume IV: Analysis of Operators*, Academic Press, 1978.

[39] H. Weyl, Das asymptotische Verteilungsgesetz der Eigenwerte linearer partieller Differentialgleichungen, *Mathematische Annalen*, 71(4):441–479, 1912.

[40] M. Berezans'kii Expansions in Eigenfunctions of Selfadjoint Operators, Transl. Math. Monogr. 17, American Mathematical Society, Providence, 1968

[41] Fujiwara D. Concrete characterization of the domains of fractional powers of some elliptic differential operators of the second order. Proceedings of the Japan Academy, Ser. A, Mathematical. -1967. -Vol. 43. -P. 82-86.

[42] Sh.A. Fractional powers of elliptic operators and isomorphism of classes of differentiable functions. Differ. Uravn. -1972. -Vol. 8(9). -P. 1609-1626.

[43] V.A. Il'in, On the uniform convergence of expansions in eigenvalues functions in the entire closed region, *Sbornik Mathematics* **45** (1958), 195–232.