# SHARP ESTIMATE ON THE RESOLVENT OF A FINITE-DIMENSIONAL CONTRACTION

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ABSTRACT. We compute an asymptotic formula for the supremum of the resolvent norm  $\|(\zeta - T)^{-1}\|$  over  $|\zeta| \ge 1$  and contractions T acting on an n-dimensional Hilbert space, whose spectral radius does not exceed a given  $r \in (0, 1)$ . We prove that this supremum is achieved on the unit circle by an analytic Toeplitz matrix.

### 1. Introduction

1.1. Notation and main result. Let  $\mathcal{M}_n$  be the set of complex  $n \times n$  matrices and let ||T|| denote the operator norm of  $T \in \mathcal{M}_n$  associated with the Hilbert norm on  $\mathbb{C}^n$ . We denote by  $\sigma = \sigma(T)$  the spectrum of T and by  $\rho(T) = \max_{\lambda \in \sigma(T)} |\lambda|$  its spectral radius. In our discussion we will assume that  $||T|| \leq 1$  and call such T a contraction. Let  $\mathcal{C}_n \subset \mathcal{M}_n$  denote the set of all contractions. We denote by  $R(\zeta, T) = (\zeta - T)^{-1}$  the resolvent of T at point  $\zeta \notin \sigma$  and are interested in sharp estimates on  $||R(\zeta, T)||$  in terms of the spectral data of T. E. B. Davies and B. Simon [4] proved that the supremum

$$\sup_{|\zeta| \ge 1} \sup \left\{ d(\zeta, \, \sigma) \, \| R(\zeta, \, T) \| \, : \, T \in \mathcal{C}_n \right\}$$

is attained for  $|\zeta| = 1$  and that for such  $\zeta$  we have

(1.1) 
$$\sup \left\{ d(\zeta, \sigma) \left\| R(\zeta, T) \right\| : T \in \mathcal{C}_n \right\} = \cot(\frac{\pi}{4n}).$$

In this paper we will show – using different methods from those in [4] – that given  $r \in (0,1)$  the supremum

$$\sup_{|\zeta| \ge 1} \sup \{ \|R(\zeta, T)\| : T \in \mathcal{C}_n, \, \rho(T) \le r \}$$

is again attained for  $|\zeta| = 1$  and that for such  $\zeta$  we have

(1.2) 
$$\mathcal{R}_{n,r} := \sup \{ \| R(\zeta, T) \| : T \in \mathcal{C}_n, \, \rho(T) \le r \} \sim \frac{2}{\pi} \frac{1+r}{1-r} n$$

as  $n \to \infty$ . The paper is organized as follows: Section 1.2 motivates our study relating our question to problems previously considered by V. Pták and N. Young [11, 12]. In Section 1.3 we state our results in full detail (see Theorems 1 - 2 - 3 below) and exhibit an analytic  $n \times n$  Toeplitz matrix that achieves the supremum in (1.2). We also mention the techniques from matrix analysis on the one hand and from interpolation theory on the other hand, which we employ to prove Theorem 1, Theorem 2 and Theorem 3 respectively. In Section 2 we first lay down the required definitions and basic results from the theory of model spaces and their operators, to the approach we choose to estimate  $||R(\zeta, T)||$ . Then we relate our question to interpolation theory

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by expressing the supremum in (1.2) as an interpolation quantity in the algebra  $H^{\infty}$  of bounded holomorphic functions in the unit disc  $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ . Finally Section 3 collects the proofs of Theorems 1–3.

1.2. **Motivation.** Our motivation to consider such of a variant of Davies-Simon's question comes from problems studied by Pták and Young [11, 12]. For a finite sequence  $\sigma$  in  $\mathbb{D}$ , we denote by  $P_{\sigma}$  the monic polynomial with zero set  $\sigma$  (counted with multiplicities). Given a finite sequence  $\sigma$  in  $\mathbb{D}$  and  $f \in \mathcal{H}ol(\mathbb{D})$ ,  $\mathcal{H}ol(\mathbb{D})$  being the space of holomorphic functions in  $\mathbb{D}$ , we introduce the quantity

$$\mathcal{S}(f, \sigma) = \sup \left\{ \| f(T) \| : T \in \mathcal{C}_n, m_T = P_\sigma \right\},\,$$

where  $m_T$  stands for the minimal polynomial of  $T \in \mathcal{C}_n$ . The case  $f|_{\sigma} = z^k|_{\sigma}$  (estimates on the norm of the powers of an  $n \times n$  contraction) is considered by Pták in [11] who proved that the quantity

$$\sup \left\{ \left\| T^k \right\| : \ T \in \mathcal{C}_n, \ \rho(T) \le r \right\},\,$$

is achieved by an  $n \times n$  analytic Toeplitz matrix. In the same time Young [16] proved among other things that at fixed r the above quantity is less than

$$nr + O(r^2)$$
.

Later Pták and Young [12, Section 2] considered the more general quantity

$$\sup \big\{ \mathcal{S}(f, \, \sigma) : \, \max_{\lambda \in \sigma} |\lambda| \le r \big\},\,$$

where  $\mathcal{S}(f,\sigma)$  is defined above,  $\sigma$  is a finite sequence of  $\mathbb{D}$  and f is an analytic polynomial. They proved [12, Section 2, p. 365] that the above supremum is achieved by an  $n \times n$  analytic Toeplitz matrix. Here we study the case  $f|_{\sigma} = (\zeta - z)^{-1}|_{\sigma}$  where  $|\zeta| = 1$  (estimates on the norm of the resolvent of an  $n \times n$  contraction) and our estimate (1.2) can be reformulated using Pták and Young's notation as

$$\sup \left\{ \mathcal{S}\left( (\zeta - z)^{-1}, \, \sigma \right) : \, \max_{\lambda \in \sigma} |\lambda| \le r \right\} \sim \frac{2}{\pi} \frac{1+r}{1-r} n, \qquad n \to \infty.$$

1.3. Statement of our result and main ingredients to its proof. We prove the following theorem for which we recall that the definition of  $\mathcal{R}_{n,r}$  is given by the left-hand side of (1.2).

**Theorem 1.** Given  $n \geq 1$ ,  $r \in (0, 1)$   $\zeta \in \mathbb{C} \setminus \mathbb{D}$  and  $T \in \mathcal{C}_n$  such that  $\rho(T) \leq r$  we have

(1.3) 
$$||R(\zeta, T)|| \le ||(1 - T^*)^{-1}|| = \frac{1}{1 - r} ||X_{1+r}||,$$

where  $T^* \in \mathcal{C}_n$  is the analytic  $n \times n$  Toeplitz matrix

$$T^* = \begin{pmatrix} r & 0 & \dots & 0 \\ 1 - r^2 & r & \ddots & \vdots \\ -r(1 - r^2) & 1 - r^2 & r & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ (-r)^{n-2}(1 - r^2) & \dots & -r(1 - r^2) & 1 - r^2 & r \end{pmatrix}$$

and the analytic  $n \times n$  Toeplitz matrix  $X_{1+r}$  is entry-wise given by:

$$(X_{1+r})_{ij} = \begin{cases} 0 & \text{if } i < j \\ 1 & \text{if } i = j \\ 1+r & \text{if } i > j \end{cases}$$

In particular, we have  $\mathcal{R}_{n,r} = \left\| \left( 1 - T^* \right)^{-1} \right\|$ .

**Theorem 2.** At fixed  $r \in (0, 1)$  and in the limit of large n we have

(1.4) 
$$\mathcal{R}_{n,r} \sim \frac{2}{\pi} \frac{1+r}{1-r} n.$$

The proof of (1.3) is based on an application of the Commutant Lifting Theorem of B. Sz.-Nagy and C. Foiaş [6, 7, 13] to the rational function  $f(z) = \frac{1}{\zeta - z}$ . Our approach to compute the norm of  $||X_{1+r}||$  and prove the asymptotic formula (1.4), follows the one from [14, 15] and builds on the techniques developed in [5].

Using a purely interpolation-theoretic approach we obtain the following results, which are asymptotically weaker than formula (1.4), but hold for any  $n \ge 1$  and  $r \in [0, 1)$ .

**Theorem 3.** Let  $T \in \mathcal{C}_n$  with minimal polynomial m and spectrum  $\sigma = (\lambda_1, \lambda_2, \ldots, \lambda_{|m|}) \in \mathbb{D}^{|m|}$ , where |m| denotes the degree of m. We have

(1.5) 
$$||R(\zeta, T)|| \le \sum_{k=1}^{|m|} \frac{1 + |\lambda_k|}{1 - |\lambda_k|}, \qquad |\zeta| \ge 1.$$

In particular, the upper estimate

$$\mathcal{R}_{n,\,r} \le \frac{1+r}{1-r}n,$$

holds for any  $n \geq 1$  and  $r \in [0,1)$ . Moreover the lower estimate

(1.6) 
$$\mathcal{R}_{n,r} \ge \frac{1}{2} \frac{1+r}{1-r} n + \frac{1}{2},$$

holds also for any  $n \ge 1$  and  $r \in [0, 1)$ .

The asymptotically sharp prefactor  $\frac{2}{\pi}$  does not appear in the upper estimate on  $\mathcal{R}_{n,r}$  from Theorem 3, which is therefore asymptotically weaker than formula (1.4). The proof of (1.5) is obtained by applying Von Neumann's inequality to  $T \in \mathcal{C}_n$  and to the orthogonal projection of the function  $z \mapsto \frac{1}{\zeta - z}$  onto  $K_{B_{\sigma}}$ ,  $\sigma$  being the spectrum of T. The lower estimate (1.6) is derived by making use of  $H^{\infty}$ —interpolation techniques developed in [2,17].

## 2. Main ingredients

This section lays down the required definitions and basic results used in the approach we choose to estimate  $||R(\zeta, T)||$ . We begin with the definition of model spaces and model operators.

2.1. Model spaces and model operators. We denote by  $H^{\infty}$  the Banach algebra of bounded analytic functions in  $\mathbb{D}$ , endowed with the supremum norm on  $\mathbb{D}$ :  $||f||_{\infty} = \sup_{z \in \mathbb{D}} |f(z)|$ . We recall that the standard Hardy space  $H^2$  is defined as the subspace of  $\mathcal{H}ol(\mathbb{D})$  consisting of those functions f such that

$$||f||_{H^2}^2 := \sup_{0 \le r \le 1} \int_{\partial \mathbb{D}} |f(rz)|^2 dm(z) < \infty,$$

where m is the normalized Lebesgue measure on the unit circle  $\partial \mathbb{D} := \{z \in \mathbb{C} : |z| = 1\}$ . Endowed with  $\|\cdot\|_{H^2}$ ,  $H^2$  is a Hilbert space.

Let  $\sigma$  be a finite sequence of points in  $\mathbb{D}$ . The finite Blaschke product  $B = B_{\sigma}$  corresponding to  $\sigma$  is defined by

$$B = B_{\sigma} = \prod_{\lambda \in \sigma} b_{\lambda},$$

where  $b_{\lambda}(z) = \frac{\lambda - z}{1 - \overline{\lambda}z}$  is the Blaschke factor corresponding to  $\lambda \in \mathbb{D}$ . Then one defines the model space  $K_B$  as the finite dimensional subspace of  $H^2$  given by

$$K_B = \left(BH^2\right)^{\perp} = H^2 \ominus BH^2.$$

The reproducing kernel of the model space  $K_B$  corresponding to a point  $\zeta \in \mathbb{D}$  is of the form

$$k_{\zeta}^{B}(z) = \frac{1 - \overline{B(\zeta)}B(z)}{1 - \overline{\zeta}z} = (1 - \overline{B(\zeta)}B(z))k_{\zeta}(z),$$

where  $k_{\zeta}(z) = \frac{1}{1-\overline{\zeta}z}$  is the Cauchy kernel at  $\zeta$ . If  $\sigma = (\lambda_1, \ldots, \lambda_n) \in \mathbb{D}^n$ , then we set  $f_j(z) = k_{\lambda_j}(z) = \frac{1}{1-\overline{\lambda_j}z}$ ,  $j = 1, \ldots, n$ . Observe that  $||f_j||_{H^2} = (1-|\lambda_j|^2)^{-1/2}$ . Now one can check that the family  $(e_k)_{1\leq k\leq n}$  given by

$$e_1 = \frac{f_1}{\|f_1\|_{H^2}}$$
 and  $e_k = \prod_{i=1}^{k-1} b_{\lambda_i} \frac{f_k}{\|f_k\|_{H^2}}, \quad k = 2, \dots, n,$ 

is an orthonormal basis of  $K_B$  (known as the Takenaka-Malmquist-Walsh basis, see [10, p. 117]). In particular for any  $\zeta \in \overline{\mathbb{D}}$  we have:

(2.1) 
$$k_{\zeta}^{B} = P_{B}(k_{\zeta}) = \sum_{k=1}^{n} \overline{e_{k}(\zeta)} e_{k}.$$

Further, we define the model operator  $M_B$  acting on  $K_B$  as follows:

$$M_B: \left\{ \begin{array}{ccc} K_B & \to & K_B \\ f & \mapsto & P_B(zf), \end{array} \right.$$

where  $P_B$  denotes the orthogonal projection on  $K_B$ . We finally denote by  $\hat{M}_{\sigma}$  the matrix representation of  $M_B$  with respect to the Takenaka-Malmquist-Walsh basis  $(e_k)_{1 \leq k \leq n}$  of  $K_B$ . By [14, Proposition III.4],

$$(\hat{M}_{\sigma})_{ij} = \begin{cases} 0 & \text{if } i < j \\ \lambda_i & \text{if } i = j \\ (1 - |\lambda_i|^2)^{1/2} (1 - |\lambda_j|^2)^{1/2} \prod_{\mu=j+1}^{i-1} (-\bar{\lambda}_{\mu}) & \text{if } i > j, \end{cases}$$

where  $(\hat{M}_{\sigma})_{ij}$  stands for the i,j entry of  $\hat{M}_{\sigma}$ , and the empty product is defined to be 1.

We refer the reader to [9,10] for a thorough study of model operators and model spaces.

2.2. Link to interpolation theory. Given a Blaschke sequence  $\sigma$  in  $\mathbb{D}$  and  $f \in H^{\infty}$  it is possible to evaluate  $\mathcal{S}(f, \sigma)$  as follows:

(2.2) 
$$\mathcal{S}(f, \sigma) = \|f\|_{H^{\infty}/B_{\sigma}H^{\infty}} = \|f(M_{B_{\sigma}})\|,$$

where  $M_{B_{\sigma}}$  is the compression of the multiplication operation by z to the model space  $K_{B_{\sigma}}$ . This formula is due to N. K. Nikolski [8, Theorem 3.4] while the last equality is a well-known corollary of Commutant Lifting Theorem of B. Sz.-Nagy and C. Foiaş [6,7,13].

It is shown in [2] that the above equality on  $S(f, \sigma)$  naturally extends to any  $f \in \mathcal{H}ol(\mathbb{D})$  as follows. We reproduce the proof of this fact for completeness. There exists an analytic polynomial p interpolating f on the finite set  $\sigma$ . Therefore for any  $T \in \mathcal{C}_n$  with  $m_T = P_{\sigma}$  and  $\sigma \subset \mathbb{D}$ , we have f(T) = p(T) (since  $f = p + m_T h$  for some  $h \in \mathcal{H}ol(\mathbb{D})$ ). Hence,

$$S(f, \sigma) = S(p, \sigma) = ||p||_{H^{\infty}/B_{\sigma}H^{\infty}}$$
$$= ||p(M_{B_{\sigma}})|| = ||f(M_{B_{\sigma}})||.$$

Here we used (2.2) applied to p. Moreover

$$||p||_{H^{\infty}/B_{\sigma}H^{\infty}} = \inf\{||p + B_{\sigma}h||_{\infty} : h \in H^{\infty}\}$$

$$= \inf\{||g||_{\infty} : g|_{\sigma} = p|_{\sigma}, g \in H^{\infty}\}$$

$$= \inf\{||g||_{\infty} : g|_{\sigma} = f|_{\sigma}, g \in H^{\infty}\}.$$

We conclude that

(2.3) 
$$\mathcal{S}(f,\,\sigma) = \inf \left\{ \|g\|_{\infty} : g \in H^{\infty}, g|_{\sigma} = f|_{\sigma} \right\}.$$

In particular we will apply (2.3) with  $f(z) = (\zeta - z)^{-1}$  to prove the lower estimate (1.6) from Theorem 3.

#### 3. Proofs

## 3.1. Proof of the upper bound in Theorem 3.

Proof. Let  $T \in \mathcal{C}_n$  with minimal polynomial m and spectrum  $\sigma$ . Let  $B = B_{\sigma}$  the finite Blaschke product corresponding to  $\sigma = (\lambda_1, \lambda_2, \dots, \lambda_{|m|})$ . We suppose that the spectral radius of T is less than  $r \in (0, 1)$ . We first assume that  $|\zeta| > 1$  (the more general case  $\zeta \in \overline{\mathbb{D}}$  will follow immediately by continuously moving  $\zeta$  towards the boundary  $\partial \mathbb{D}$  of  $\mathbb{D}$ ) and consider the orthogonal projection of  $\frac{1}{\zeta - z}$  onto the model space  $K_B$ :

$$g(z) = \frac{1}{\zeta} P_B(k_{1/\bar{\zeta}})(z).$$

The function g clearly interpolates  $\frac{1}{\zeta-z}$  on  $\sigma$ . Moreover expanding g on the Takena-Malmquist-Walsh  $(e_k)_{k=1}^{|m|}$  basis of  $K_B$  we get

$$g = \frac{1}{\zeta} P_B(k_{1/\bar{\zeta}}) = \frac{1}{\zeta} \sum_{k=1}^{|m|} \langle k_{1/\bar{\zeta}}, e_k \rangle e_k$$
$$= \frac{1}{\zeta} \sum_{k=1}^{|m|} \overline{e_k(1/\bar{\zeta})} e_k.$$

That is to say that for any  $u \in \partial \mathbb{D}$ 

$$g(u) = \sum_{k=1}^{|m|} \frac{1 - |\lambda_k|^2}{\zeta - \lambda_k} \prod_{j=1}^{k-1} b_{\lambda_j} (1/\overline{\zeta}) \prod_{j=1}^{k-1} b_{\lambda_j} (u) \frac{1}{1 - \overline{\lambda_k} u}.$$

Clearly we have

$$||g||_{\infty} \le \sum_{k=1}^{|m|} \frac{1+|\lambda_k|}{1-|\lambda_k|},$$

and the proof of (1.5) follows by making use of Von Neumann's inequality:

$$||R(\zeta, T)|| = ||g(T)|| \le ||g||_{\infty}$$
.

3.2. Proof of the lower bound in Theorem 3. In this paragraph we prove (1.6) by using  $H^{\infty}$ -interpolation techniques.

*Proof.* Again we first assume that  $|\zeta| > 1$  (the more general case  $\zeta \in \overline{\mathbb{D}}$  will follow immediately by continuously moving  $\zeta$  towards  $\partial \mathbb{D}$ ). Consider  $\lambda \in \mathbb{D}$ ,  $B = b_{\lambda}^{n}$  and the function

$$g(z) = \frac{1}{\zeta} P_B(k_{1/\bar{\zeta}})(z)$$

$$= \frac{1}{\zeta} \frac{1 - \overline{B(1/\bar{\zeta})}B(z)}{1 - z/\zeta}$$

$$= \frac{1}{\zeta} \sum_{k=0}^{n-1} (\overline{b_{\lambda}(1/\bar{\zeta})})^k \frac{1}{1 - \frac{\lambda}{\zeta}} (b_{\lambda}(z))^k \frac{1 - |\lambda|^2}{1 - \bar{\lambda}z},$$

where the last equality is due to (2.1). Clearly,

$$g - \frac{1}{\zeta} k_{1/\bar{\zeta}} \in B\mathcal{H}ol(\mathbb{D}),$$

and in particular

$$||g||_{H^{\infty}/b_{\lambda}^{n}H^{\infty}} = ||g(M_{B})|| = ||(\zeta - M_{B})^{-1}||.$$

Since  $1 - \overline{\lambda}b_{\lambda}(z) = \frac{1 - \overline{\lambda}z - |\lambda|^2 + \overline{\lambda}z}{1 - \overline{\lambda}z} = \frac{1 - |\lambda|^2}{1 - \overline{\lambda}z}$  and  $b_{\lambda}$  is self-inverse, we have

$$g \circ b_{\lambda}(z) = \sum_{k=0}^{n-1} (\overline{b_{\lambda}(1/\zeta)})^{k} \frac{1}{\zeta - \lambda} z^{k} \frac{1 - |\lambda|^{2}}{1 - \overline{\lambda}b_{\lambda}(z)}$$

$$= \frac{1}{\zeta - \lambda} \sum_{k=0}^{n-1} (\overline{b_{\lambda}(1/\zeta)})^{k} z^{k} \left(1 - \overline{\lambda}z\right)$$

$$= \frac{1}{\zeta - \lambda} \left(1 - \overline{\lambda}(\overline{b_{\lambda}(1/\zeta)})^{n-1} z^{n} + \sum_{k=1}^{n-1} (\overline{b_{\lambda}(1/\zeta)})^{k} z^{k} - \overline{\lambda} \sum_{k=1}^{n-1} (\overline{b_{\lambda}(1/\zeta)})^{k-1} z^{k}\right)$$

$$= \frac{1}{\zeta - \lambda} \left(1 - \overline{\lambda}(\overline{b_{\lambda}(1/\zeta)})^{n-1} z^{n} + \left(\overline{b_{\lambda}(1/\zeta)} - \overline{\lambda}\right) \sum_{k=1}^{n-1} (\overline{b_{\lambda}(1/\zeta)})^{k-1} z^{k}\right)$$

$$= \frac{1}{\zeta - \lambda} \left(1 - \overline{\lambda}(\overline{b_{\lambda}(1/\zeta)})^{n-1} z^{n} - \frac{1 - |\lambda|^{2}}{\zeta - \lambda} \sum_{k=1}^{n-1} (\overline{b_{\lambda}(1/\zeta)})^{k-1} z^{k}\right)$$

Given  $r \in (0, 1)$  and  $-\frac{1}{r} < \zeta < -1$ , we consider  $\lambda = -r$ . The function

$$\Psi_n(z) := g \circ b_{-r}(z) = \frac{1}{\zeta + r} \left( 1 + r \left( -\frac{1 + r\zeta}{\zeta + r} \right)^{n-1} z^n - \frac{1 - r^2}{r + \zeta} \sum_{k=1}^{n-1} \left( -\frac{1 + r\zeta}{r + \zeta} \right)^{k-1} z^k \right),$$

is an analytic polynomial of degree n. We need a lower estimate for  $||g||_{H^{\infty}/b_{\lambda}^{n}H^{\infty}}$ : to this end consider G such that  $R_{n} - G \in b_{\lambda}^{n}\mathcal{H}ol(\mathbb{D})$ , i.e. such that  $R_{n} \circ b_{\lambda} - G \circ b_{\lambda} \in z^{n}\mathcal{H}ol(\mathbb{D})$ , as  $b_{\lambda}$  is self-inverse.

Due to invariance of the norm in  $H^{\infty}$  with respect to the composition by  $b_{\lambda}$  we have

$$\begin{split} \|g\|_{H^{\infty}/b_{\lambda}^{n}H^{\infty}} &= \|\Psi_{n}\|_{H^{\infty}/z^{n}H^{\infty}} \\ &= \frac{1}{|\zeta + r|} \|\Phi_{n}\|_{H^{\infty}/z^{n}H^{\infty}} \,, \end{split}$$

where

$$\Phi_n(z) = 1 + r \left( -\frac{1 + r\zeta}{\zeta + r} \right)^{n-1} z^n - \frac{1 - r^2}{r + \zeta} \sum_{k=1}^{n-1} \left( -\frac{1 + r\zeta}{r + \zeta} \right)^{k-1} z^k,$$

is an analytic polynomial of degree n with nonnegative coefficients because the condition  $-\frac{1}{r} < \zeta < -1$  implies  $-\frac{1+r\zeta}{r+\zeta} > 0$ .

Denote by  $F_n$  the (n-1)-th Fejer kernel,  $F_n(z) = \frac{1}{2\pi} \sum_{|j| \le n-1} \left(1 - \frac{|j|}{n}\right) z^j$ , and denote by \* the usual convolution operation in  $L^1(\partial \mathbb{D})$ . Then, for any  $h \in L^{\infty}(\partial \mathbb{D})$ , we have  $\|h * F_n\|_{\infty} \le \|h\|_{\infty} \|F_n\|_{H^1} = \|h\|_{\infty}$ . On the other hand, since  $\widehat{h_1 * h_2}(j) = \widehat{h_1}(j)\widehat{h_2}(j)$  and  $\widehat{F}_n(j) = 0$  for every  $j \ge n$ , we have

$$h * F_n = \Phi_n * F_n$$

for any  $h \in H^{\infty}$  such that  $\hat{h}(k) = \widehat{\Phi}_n(k)$ ,  $k = 0, 1, \dots, n-1$ . Hence, for any such h,  $||h||_{\infty} \ge ||\Phi_n * F_n||_{\infty}$  and so

$$\|\Phi_n\|_{H^{\infty}/z^n H^{\infty}} = \inf \left\{ \|h\|_{\infty} : h \in H^{\infty}, \ \hat{h}(k) = \widehat{\Phi}_n(k), \ 0 \le k \le n-1 \right\}$$
$$\ge \|\Phi_n * F_n\|_{\infty} \ge (\Phi_n * F_n)(1).$$

Note that the convolution with  $F_n$  gives us the Cesàro mean of the partial sums of the Fourier series. Denoting by  $S_j$  the j-th partial sum for  $\Phi_n$  at 1 we have

$$(\Phi_n * F_n)(1) = \frac{1}{n} \sum_{j=0}^{n-1} S_j(1).$$

For  $j = 0, \ldots, n-1$  we have

$$S_{j}(1) = 1 - \frac{1 - r^{2}}{r + \zeta} \sum_{k=1}^{j} \left( -\frac{1 + r\zeta}{r + \zeta} \right)^{k-1}$$
$$= \frac{r + \zeta + (1 - r) \left( -\frac{1 + r\zeta}{r + \zeta} \right)^{j}}{\zeta + 1},$$

and a computation shows that

$$\sum_{j=0}^{n-1} S_j(1) = \frac{\zeta + r}{(\zeta + 1)^2 (1+r)} \left( \left( \frac{-r\zeta - 1}{r + \zeta} \right)^n (r-1) + (-1 + (\zeta + 1)n)r + 1 + (\zeta + 1)n \right).$$

Passing to the limit as  $\zeta \to -1$  we obtain

$$\lim_{\zeta \to -1} \| (\zeta - M_B)^{-1} \| = \| (-1 - M_B)^{-1} \|$$

$$= \lim_{\zeta \to -1} \| g \|_{H^{\infty}/b_{-r}^n H^{\infty}}$$

$$= \lim_{\zeta \to -1} \| \Psi_n \|_{H^{\infty}/z^n H^{\infty}}$$

$$\geq \frac{1}{1 - r} \cdot \lim_{\zeta \to -1} \frac{1}{n} \sum_{j=0}^{n-1} S_j(1)$$

$$= \frac{n(1 + r) + 1 - r}{2(1 - r)},$$

which completes the proof.

3.3. **Proof of Theorem 1.** Of particular importance to our discussion in the proof of Theorem 1 will be the analytic Toeplitz matrix  $X_{\beta}$  given entry-wise by

(3.1) 
$$(X_{\beta})_{ij} = \begin{cases} 0 & \text{if } i < j \\ 1 & \text{if } i = j \\ \beta & \text{if } i > j \end{cases}$$

with  $\beta \in [0, 2]$ . The spectral norm of  $X_{\beta}$  is computed in [15, Proposition II.6].

*Proof.* We first prove (1.3). We consider  $T \in \mathcal{C}_n$  with spectral radius less than r. It follows from [14, Theorem III.2] that for any  $\zeta \in \mathbb{C} - \sigma(T)$  the resolvent of T is bounded by

where B is the Blaschke product associated with  $m_T$ . We recall [14, Theorem III.2] that given a finite Blaschke product  $B(z) = \prod_{i=1}^{|m|} \frac{\lambda_i - z}{1 - \lambda_i z}$ , of degree  $|m| \geq 1$ , with zeros  $\lambda_i \in \mathbb{D}$ , the components of the resolvent of the model operator  $M_B$  at any point  $\zeta \in \mathbb{C} - \{\lambda_1, ..., \lambda_{|m|}\}$  with respect to the Takenaka-Malmquist-Walsh basis are given by

(3.3) 
$$\left( \left( \zeta - M_B \right)^{-1} \right)_{1 \le i, j \le |m|} = \begin{cases} 0 & \text{if } i < j \\ \frac{1}{\zeta - \lambda_i} & \text{if } i = j \\ \frac{(1 - |\lambda_i|^2)^{1/2}}{\zeta - \lambda_i} \frac{(1 - |\lambda_j|^2)^{1/2}}{\zeta - \lambda_j} \prod_{k=j+1}^{i-1} \frac{1 - \bar{\lambda}_k \zeta}{\zeta - \lambda_k} & \text{if } i > j \end{cases}.$$

Recall that for any  $n \times n$  matrices  $A = (a_{ij})$  and  $A = (a'_{ij})$ , the condition  $|a_{ij}| \leq a'_{ij}$  implies that  $||A|| \leq ||A'||$ . Hence, supposing  $|\zeta| \geq 1$  we have

$$\frac{1}{|\zeta - \lambda_i|} \le \frac{1}{|\zeta| - |\lambda_i|} \le \frac{1}{1 - r}$$

on the one hand and

$$\frac{(1-|\lambda_i|^2)^{1/2}}{|\zeta-\lambda_i|} \frac{(1-|\lambda_j|^2)^{1/2}}{|\zeta-\lambda_j|} \le \frac{(1-|\lambda_i|^2)^{1/2}}{1-|\lambda_i|} \frac{(1-|\lambda_j|^2)^{1/2}}{1-|\lambda_j|} \le \frac{1+r}{1-r},$$

for if i > j on the other hand and estimate

$$\left| \left( \zeta - M_B \right)_{i,j}^{-1} \right| \le \begin{cases} 0 & \text{if } i < j \\ \frac{1}{1-r} & \text{if } i = j \\ \frac{1+r}{1-r} & \text{if } i > j \end{cases} = \frac{1}{1-r} X_{1+r},$$

where the analytic Toeplitz matrix  $X_{\beta}$  is entry-wise defined by in (3.1). Moreover choosing  $\lambda_1 = \cdots = \lambda_n = r$ ,  $T^* = M_B$  and  $\zeta = 1$  we get

$$\left( \left( 1 - T^* \right)^{-1} \right)_{1 \le i, j \le n} = \begin{cases} 0 & \text{if } i < j \\ \frac{1}{1-r} & \text{if } i = j \\ \frac{1+r}{1-r} & \text{if } i > j \end{cases}$$
$$= \frac{1}{1-r} X_{1+r}.$$

This completes the proof of (1.3) and proves in particular that for any  $|\zeta| \geq 1$ 

$$||R(\zeta, T)|| \le ||(1 - T^*)^{-1}|| = \frac{1}{1 - r} ||X_{1+r}||.$$

## 3.4. Proof of Theorem 2.

*Proof.* The spectral norm of  $X_{\beta}$  is computed in [15, Proposition II.6]. In particular for any  $\beta \in [0, 2]$ 

$$||X_{\beta}|| = \frac{1}{2} \sqrt{(\beta - 2)^2 + \frac{\beta^2}{\cot^2(\theta^*/2)}},$$

where  $\theta^*$  is the unique solution of

$$\cot(n\theta) = \frac{\beta - 2}{\beta}\cot(\theta/2)$$

in  $\left[\frac{2n-1}{2n}\pi, \pi\right)$  and it follows, see [14], that

$$||X_0|| = 1,$$
  $||X_1|| = \frac{1}{2\sin(\frac{\pi}{4n+2})},$   $||X_2|| = \cot(\frac{\pi}{4n}).$ 

In our situation  $\beta = 1 + r$  and

$$||X_{1+r}|| = \frac{1}{2}\sqrt{(1-r)^2 + \frac{(1+r)^2}{\cot^2(\theta^*/2)}},$$

where  $\theta^*$  is the unique solution of

$$\cot(n\theta) = -\frac{1-r}{1+r}\cot(\theta/2)$$

in  $\left[\frac{2n-1}{2n}\pi, \pi\right)$ . Note that if  $\beta = 1$  then  $\theta^* = \frac{2n\pi}{2n+1}$  while if  $\beta = 2$  then  $\theta^* = \frac{2n-1}{2n}\pi$ . We consider the case  $\beta = 1 + r$  and put

$$f_n(\theta) = \cot(n\theta) + \frac{1-r}{1+r}\cot(\theta/2).$$

We first observe that

$$f_n\left(\frac{2n-1}{2n}\pi\right) = \cot\left(\frac{2n-1}{2}\pi\right) + \frac{1-r}{1+r}\cot\left(\frac{2n-1}{4n}\pi\right)$$
$$= \frac{1-r}{1+r}\cot\left(\frac{2n-1}{4n}\pi\right) > 0.$$

Moreover  $\lim_{\theta \to \pi^-} \cot(n\theta) = -\infty$  and  $\cot(\pi/2) = 0$  so that

$$\lim_{\theta \to \pi^{-}} f_n(\theta) = -\infty.$$

The function  $\theta \mapsto \cot(n\theta)$  is strictly decreasing on  $[(n-1)\pi/n,\pi) \supset [\frac{2n-1}{2n}\pi,\pi)$ , and this is also the case of the function  $\theta \mapsto \frac{1-r}{1+r}\cot(\theta/2)$  on  $(0,\pi) \supset [\frac{2n-1}{2n}\pi,\pi)$ . Therefore the function  $f_n$  is also strictly decreasing on the interval  $[\frac{2n-1}{2n}\pi,\pi)$ , which shows the existence and the uniqueness of  $\theta^* \in [\frac{2n-1}{2n}\pi,\pi)$  such that

$$f_n(\theta^*) = 0.$$

Now we show that for n large enough we have

$$f_n\left(\frac{2n}{2n+1}\pi\right) < 0.$$

On one hand we have

$$\cot\left(n\frac{2n}{2n+1}\pi\right) = \cot\left(n\pi - \frac{\pi}{2} + \frac{\pi}{2(2n+1)}\right)$$

$$= -\tan\left(n\pi + \frac{\pi}{2(2n+1)}\right)$$

$$= -\frac{\sin\left(n\pi + \frac{\pi}{2(2n+1)}\right)}{\cos\left(n\pi + \frac{\pi}{2(2n+1)}\right)}$$

$$= -\frac{(-1)^n \sin\left(\frac{\pi}{2(2n+1)}\right)}{(-1)^n \cos\left(\frac{\pi}{2(2n+1)}\right)}$$

$$= -\tan\left(\frac{\pi}{2(2n+1)}\right) = -\frac{\pi}{4n} + \mathcal{O}\left(\frac{1}{n^2}\right)$$

as  $n \to \infty$ . On the other hand

$$\cot\left(\frac{n}{2n+1}\pi\right) = \frac{\pi}{4n} + \mathcal{O}\left(\frac{1}{n^2}\right).$$

Therefore

$$f_n\left(\frac{2n}{2n+1}\pi\right) = -\frac{\pi}{4n}\left(1 - \frac{1-r}{1+r}\right) + \mathcal{O}\left(\frac{1}{n^2}\right)$$
$$= -\frac{\pi r}{2(1+r)}\frac{1}{n}\left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right) < 0$$

for n large enough. Thus, for  $\beta = 1 + r$  we have

$$\frac{2n-1}{2n}\pi \le \theta^* \le \frac{2n\pi}{2n+1},$$

and

$$\cot(\theta^*/2) \sim \frac{\pi - \theta^*}{2} \sim \frac{\pi}{4n}$$

as n tends to  $+\infty$ . In particular

$$||X_{1+r}|| \sim \frac{1}{2} \frac{1+r}{\cot(\theta^*/2)},$$
  
 $\sim \frac{2n(1+r)}{\pi}$ 

as n tends to  $+\infty$ .

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