

Seeing through the light cone: Visualizing electromagnetic fields in special relativity

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Abstract

The theoretical framework of electromagnetism played a foundational role in Einstein's development of special relativity. To support conceptual understanding, we present a fully special relativistic computer simulation that visualizes electromagnetic fields from the perspective of a moving observer. In this simulation, the user observes electromagnetic phenomena through their past light cone and directly experiences the Lorentz force acting at that spacetime point. The electromagnetic field is computed from the subluminal motion of point charges at the intersection of their worldlines with the observer's past light cone, ensuring causal consistency and Lorentz covariance. This approach offers an interactive and intuitive representation of relativistic electromagnetism. It provides insight into how electric and magnetic fields transform across inertial frames, and serves as a bridge between abstract formalism and physical intuition. The simulation also lends itself to pedagogical use in courses on special relativity or electrodynamics.

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1 Introduction

Special relativity, together with quantum mechanics, forms a foundational pillar of modern physics, providing a framework for understanding spacetime and fundamental interactions [1]. Unlike quantum phenomena, which are beyond the reach of our everyday classical perception, relativistic effects such as time dilation and length contraction can be directly observed, making them particularly suitable for visualization-based learning. Interactive simulations offer an intuitive way to approach these abstract concepts.

Students often face significant conceptual difficulties in mastering special relativity, particularly when grappling with frames of reference, simultaneity, and transformation laws [2, 3]. Traditional instruction methods, which primarily rely on algebraic derivations and static spacetime diagrams, frequently fall short in promoting deep understanding. In response, educators have developed a variety of innovative tools—including interactive simulations and multimedia platforms—to dynamically visualize relativistic phenomena [4, 5, 6]; see also Ref. [7] for a collection of related work. These efforts suggest that combining interactive visualizations with guided conceptual inquiry can significantly enhance learning outcomes.

Previous computational work has primarily focused on relativistic kinematics, offering simulations of time dilation, length contraction, and frame transformations [4, 5, 6]. These efforts have significantly aided conceptual understanding, but have largely overlooked the role of electromagnetism—even though Einstein’s formulation of special relativity was originally motivated by the transformation properties of electric and magnetic fields [8].

In Ref. [9], we introduced a visualization method based on the past-light-cone (PLC) perspective, enabling a Lorentz-covariant treatment of relativistic motion. In this paper, we extend that approach to include electromagnetic fields, providing, to our knowledge, the first fully relativistic and Lorentz-covariant computer simulation of electromagnetic fields from the viewpoint of a moving observer.

In our simulation, users perceive the world through intersections with their PLC, and experience electromagnetic fields in a manifestly Lorentz-covariant manner. This builds on a theoretical framework that formulates electromagnetic fields in terms of PLCs [10], enabling a consistent and causally grounded approach to relativistic electrodynamics.

Our implementation consists of three main components:

1. At each spacetime point on the user’s PLC, the electromagnetic field tensor is evaluated and transformed into the user’s rest frame, demonstrating the frame-dependent mixing of electric and magnetic components.
2. Every charged particle, including the user, experiences the Lorentz force based on the electromagnetic field at its own spacetime location.
3. The electromagnetic field at any spacetime point is computed from the subluminal motion of point charges at the intersection of their worldlines with the PLC of that point, ensuring causal consistency and relativistic covariance.

By combining these elements into a single interactive simulation, we provide an intuitive and visually coherent representation of relativistic electromagnetism. The simulation operates in real time as the observer moves, serving as a bridge between abstract formalism and physical intuition. It also serves as a pedagogical demonstration for courses on special relativity or electrodynamics.

This paper is organized as follows. In Sec. 2, we describe the Lorentz-covariant kinematics underlying the simulation. In Sec. 3, we describe how the visible world is constructed from intersections with the PLC. In Sec. 4, we present a covariant description of charged particle motion and introduce the electromagnetic potential in a relativistic setting. In Sec. 5, we apply the Lorentz-covariant formalism to compute electromagnetic fields from moving charges. Sec. 6 describes the practical aspects of implementing these theoretical concepts, detailing the computational framework and visualization techniques. Finally, Sec. 7 summarizes the key contributions of this work.

2 Covariant kinematics for visualization

This section sets up the Lorentz-covariant framework needed to describe particle motion in special relativity, building on the formalism introduced in Ref. [9] and restructuring key elements to suit our simulation-oriented perspective. Coordinates, proper time, velocity, and acceleration are introduced, along with the role of PLCs in determining what is observable. These concepts lay the foundation for frame-independent, causally consistent simulations.

We base our formulation on the following three principles of special relativity:

- I. The constancy of the speed of light in vacuum.
- II. Lorentz invariance, implying that the laws of physics are the same for all inertial observers.
- III. The equivalence principle, adapted for specific contexts within special relativity.

These principles guide the structure of this section: principle I motivates the formulation in Sec. 2.1; principle II underpins the developments in Secs. 2.2–2.6; and the adapted equivalence principle III finds its application in Sec. 2.7.

2.1 Spacetime coordinates

We first prepare a Cartesian coordinate system for a reference frame, which we call the *world frame* hereafter:

$$\vec{x} = (x^0, \mathbf{x}) = (x^0, x^1, x^2, x^3), \quad (1)$$

where

$$x^0 := ct \quad (2)$$

denotes the time t multiplied by the (constant) speed of light c and $\mathbf{x} = (x^1, x^2, x^3)$ is the spatial coordinates, which are more frequently (but not in this paper) written as (x, y, z) . Hereafter, we call x^0 , having the dimension of length, the *time length*. If one does not intend to change the speed of light depending on the game/simulation settings, one may take the *natural units*

$$c = 1. \quad (3)$$

Throughout this paper, we keep the non-natural units for the possible reader's ease.

We also employ the matrix notation

$$\vec{x} = \begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix}, \quad \vec{x}^t = [x^0 \ x^t] = [x^0 \ x^1 \ x^2 \ x^3], \quad (4)$$

where the superscript “t” denotes the transpose.

In the actual implementation in Sec. 6, the spatial and time-length coordinates are indexed as

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \end{bmatrix} := \begin{bmatrix} x^1 \\ x^2 \\ x^3 \\ x^0 \end{bmatrix}. \quad (5)$$

The authors are sorry for the confusing notation but this is how the physics and information-technology communities translate.

For spatial vectors $\mathbf{x} = (x^1, x^2, x^3)$ and $\mathbf{y} = (y^1, y^2, y^3)$, we write their inner product

$$\mathbf{x} \cdot \mathbf{y} := \sum_{i=1}^3 x^i y^i = [x^1 \ x^2 \ x^3] \begin{bmatrix} y^1 \\ y^2 \\ y^3 \end{bmatrix} = \mathbf{x}^t \mathbf{y}, \quad (6)$$

$$\mathbf{x}^2 := \mathbf{x} \cdot \mathbf{x} = (x^1)^2 + (x^2)^2 + (x^3)^2 = \sum_{i=1}^3 (x^i)^2, \quad (7)$$

$$|\mathbf{x}| := \sqrt{\mathbf{x}^2} = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2} = \sqrt{\sum_{i=1}^3 (x^i)^2}. \quad (8)$$

Throughout this paper, the roman and greek letters i, j, \dots and μ, ν, \dots run for 1, 2, 3 and 0, ..., 3, respectively. A hat symbol denotes a unit vector:

$$\hat{\mathbf{x}} := \frac{\mathbf{x}}{|\mathbf{x}|}. \quad (9)$$

For later use, we define a matrix called *metric*:¹

$$\eta = [\eta_{\mu\nu}]_{\mu,\nu=0,\dots,3} = \begin{bmatrix} \eta_{00} & \eta_{01} & \eta_{02} & \eta_{03} \\ \eta_{10} & \eta_{11} & \eta_{12} & \eta_{13} \\ \eta_{20} & \eta_{21} & \eta_{22} & \eta_{23} \\ \eta_{30} & \eta_{31} & \eta_{32} & \eta_{33} \end{bmatrix} := \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \quad (10)$$

Since $\eta^2 = I$, with I being the identity matrix, the inverse of η becomes identical to itself: $\eta^{-1} = \eta$, where

$$\eta^{-1} = [\eta^{\mu\nu}]_{\mu,\nu=0,\dots,3} = \begin{bmatrix} \eta^{00} & \eta^{01} & \eta^{02} & \eta^{03} \\ \eta^{10} & \eta^{11} & \eta^{12} & \eta^{13} \\ \eta^{20} & \eta^{21} & \eta^{22} & \eta^{23} \\ \eta^{30} & \eta^{31} & \eta^{32} & \eta^{33} \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \quad (11)$$

¹Here and hereafter, we use the sans-serif italic font to denote matrices in the $d+1$ spacetime dimensions.

Using η , we define the Lorentzian inner product and the Lorentzian (squared-)norm:

$$(\vec{x}, \vec{y}) := \vec{x}^t \eta \vec{y} = [x^0 \ x^1 \ x^2 \ x^3] \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y^0 \\ y^1 \\ y^2 \\ y^3 \end{bmatrix}$$

$$= -x^0 y^0 + \mathbf{x} \cdot \mathbf{y} = -x^0 y^0 + x^1 y^1 + x^2 y^2 + x^3 y^3, \quad (12)$$

$$(\vec{x})^2 := (\vec{x}, \vec{x}) = -(x^0)^2 + \mathbf{x}^2 = -(x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2. \quad (13)$$

We might sometimes write

$$\vec{x} \cdot \vec{y} := (\vec{x}, \vec{y}), \quad \vec{x}^2 := (\vec{x})^2. \quad (14)$$

2.2 Lorentz transformations

Special relativity postulates the light-speed invariance. The coordinates $\mathbf{x} = (x^1, x^2, x^3)$ of a spherical light wave originating from $\mathbf{x} = 0$ at the time length $x^0 = 0$ are prescribed by the following constraint at a time length x^0 :

$$\mathbf{x}^2 = (x^0)^2. \quad (15)$$

This condition for the spherical light wave can be written as

$$(\vec{x})^2 = 0. \quad (16)$$

In the spirit of the light-speed invariance, a linear spacetime-coordinate transformation (represented by a matrix) Λ ,

$$\vec{x} \rightarrow \vec{x}' = \Lambda \vec{x}, \quad \begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix} \rightarrow \begin{bmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{bmatrix} = \begin{bmatrix} \Lambda^0_0 & \Lambda^0_1 & \Lambda^0_2 & \Lambda^0_3 \\ \Lambda^1_0 & \Lambda^1_1 & \Lambda^1_2 & \Lambda^1_3 \\ \Lambda^2_0 & \Lambda^2_1 & \Lambda^2_2 & \Lambda^2_3 \\ \Lambda^3_0 & \Lambda^3_1 & \Lambda^3_2 & \Lambda^3_3 \end{bmatrix} \begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix}, \quad (17)$$

is called the *Lorentz transformation* when it leaves the left-hand side of Eq. (16) invariant: For all \vec{x} ,

$$(\vec{x}')^2 = (\vec{x})^2 \iff (\Lambda \vec{x})^t \eta (\Lambda \vec{x}) = \vec{x}^t \eta \vec{x} \iff \vec{x}^t (\Lambda^t \eta \Lambda) \vec{x} = \vec{x}^t \eta \vec{x}. \quad (18)$$

That is, a transformation (represented by) Λ is called the Lorentz transformation when and only when²

$$\Lambda^t \eta \Lambda = \eta. \quad (19)$$

This can be equivalently written as

$$\Lambda^{-1} = \eta \Lambda^t \eta. \quad (20)$$

² This relation results in $\det \Lambda = \pm 1$ and $|\Lambda^0_0| \geq 1$. Hereafter, when we say Lorentz transformation, it denotes the proper ($\det \Lambda = 1$) orthochronous ($\Lambda^0_0 \geq 1$) Lorentz transformation, barring the parity transformation (or space inversion) $P := \text{diag}(1, -1, -1, -1)$ and the time reversal $T := \text{diag}(-1, 1, 1, 1)$, where “diag” denotes the diagonal matrix.

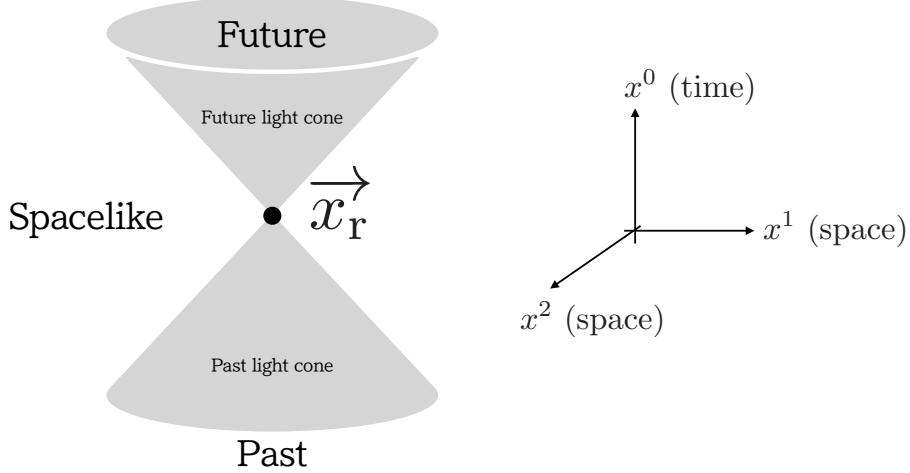


Figure 1: Schematic figure (in two spatial dimensions) for the past, spacelike, and future regions of \vec{x}_r , as well as its future and past light cones.

When a “spacetime vector” $\vec{V} = (V^0, V^1, V^2, V^3)$ transforms the same as the coordinates \vec{x} under the Lorentz transformations (17), namely $\vec{V} \rightarrow \vec{V}' = \Lambda \vec{V}$, we call \vec{V} a *covariant* vector.³ Special relativity is a theory that describes everything in terms of covariant and invariant quantities.

2.3 Light cones

The above argument sets the spacetime origin as the reference point. Now we generalize it. Given a reference point \vec{x}_r and another arbitrary point \vec{x} , let us consider the following Lorentz *invariant*:

$$(\vec{x} - \vec{x}_r)^2 = - (x^0 - x_r^0)^2 + (x^1 - x_r^1)^2 + (x^2 - x_r^2)^2 + (x^3 - x_r^3)^2. \quad (21)$$

With it, we may divide the whole spacetime into three regions (see Fig. 1):

- \vec{x} belongs to the *past* region of \vec{x}_r if the separation is *timelike* $(\vec{x} - \vec{x}_r)^2 < 0$ and if $x^0 < x_r^0$.
- \vec{x} belongs to the *spacelike* region of \vec{x}_r if the separation is *spacelike* $(\vec{x} - \vec{x}_r)^2 > 0$.
- \vec{x} belongs to the *future* region of \vec{x}_r if the separation is timelike $(\vec{x} - \vec{x}_r)^2 < 0$ and if $x^0 > x_r^0$.

These three regions are separated by the *light cones*:

- \vec{x} belongs to the *past light cone* (PLC) of \vec{x}_r if the separation is *lightlike* $(\vec{x} - \vec{x}_r)^2 = 0$ and if $x^0 < x_r^0$.

³In this paper, we call both the contravariant and covariant vectors the covariant vectors in the spirit that both are Lorentz covariant.

- \vec{x} belongs to the *future light cone* of \vec{x}_r if the separation is *lightlike* $(\vec{x} - \vec{x}_r)^2 = 0$ and if $x^0 > x_r^0$.

We emphasize that the three regions and the light cones are defined independently of the chosen coordinate system.⁴

2.4 Proper time and covariant velocity

Let us consider objects O_1, O_2, \dots . We write the O_n 's spacetime position \vec{x}_n . In the next iteration of the computer program, it moves to \vec{x}'_n .⁵ From the displacement vector

$$\vec{\Delta x}_n := \vec{x}'_n - \vec{x}_n, \quad \begin{bmatrix} \Delta x_n^0 \\ \Delta \mathbf{x}_n \end{bmatrix} = \begin{bmatrix} x_n'^0 - x_n^0 \\ \vec{x}'_n - \vec{x}_n \end{bmatrix}, \quad (22)$$

we may define a Lorentz invariant: $(\vec{\Delta x}_n)^2 = -(\Delta x_n^0)^2 + (\Delta \mathbf{x}_n)^2$. A particle is called *massive*, *massless*, and *tachyon* when it moves with this Lorentz-invariant quantity being always negative, zero, and positive, respectively:

$$(\vec{\Delta x}_n)^2 \begin{cases} < 0 & \text{massive (moving timelike),} \\ = 0 & \text{massless (moving lightlike),} \\ > 0 & \text{tachyon (moving spacelike).} \end{cases} \quad (23)$$

The light is massless. Hereafter, we assume all the objects (other than light rays) are massive unless otherwise stated. We also assume that an object moves forward in the future direction: $\Delta x_n^0 > 0$.⁶

Note that by the definition of massiveness,

$$(\vec{\Delta x}_n)^2 < 0 \iff (\Delta \mathbf{x}_n)^2 < (\Delta x_n^0)^2, \quad (24)$$

the (non-covariant) velocity of the massive particle⁷

$$\mathbf{v}_n := c \frac{d\mathbf{x}_n}{dx_n^0} = \lim_{\Delta x_n^0 \rightarrow 0} c \frac{\Delta \mathbf{x}_n}{\Delta x_n^0} \quad (25)$$

always satisfies

$$|\mathbf{v}_n| < c. \quad (26)$$

That is, all the massive objects move slower than the speed of light.

For each iteration, we define what we call a *proper time distance* for O_n by

$$\Delta s_n := \sqrt{-(\vec{\Delta x}_n)^2} = \Delta x_n^0 \sqrt{1 - \left(\frac{\Delta \mathbf{x}_n}{\Delta x_n^0}\right)^2}. \quad (27)$$

⁴ The order of times is Lorentz invariant when the separation is either timelike or lightlike; see also footnote 2.

⁵ It should be understood that the prime symbol here, “*r*”, has nothing to do with the Lorentz transformation (17).

⁶ The order of time is well-defined both for massive and massless particles that move into the future and the future-light cone, respectively; see footnote 4. We also note that there is no need to consider an object moving backward in time because, in quantum field theory, a particle moving backward in time is equivalent to an anti-particle moving forward in time, and vice versa.

⁷ The expression $\mathbf{v}_n = \frac{d\mathbf{x}_n}{dt}$ in the world frame might be more familiar for some readers; recall Eq. (2).

By adding the proper time distance of all the iterations (up to the spacetime point of interest), we obtain what we call the *proper time length* s_n that parametrizes each world line of O_n ; it becomes the same as the *proper time* $\tau_n := s_n/c$ in natural units $c = 1$.⁸

In the limit of infinitesimal iterations, the proper time length s_n becomes a continuous parameter. It is important that the proper time length can parametrize the O_n 's *worldline* \mathcal{W}_n in a Lorentz-*invariant* fashion, where \mathcal{W}_n is the trajectory of O_n in the spacetime:

$$\mathcal{W}_n := \left\{ \vec{x}_n(s_n) \mid s_n^{\text{start}} \leq s_n \leq s_n^{\text{cease}} \right\}, \quad (28)$$

with s_n^{start} and s_n^{cease} being the proper time lengths at which O_n starts and ceases to exist, respectively.

In the limit of infinitesimal lapse of the proper time length $\Delta s_n \rightarrow 0$, we define the dimensionless *covariant velocity*:

$$\vec{u}_n := \lim_{\Delta s_n \rightarrow 0} \frac{\overrightarrow{\Delta x}_n}{\Delta s_n} = \frac{dx_n}{ds_n}, \quad (29)$$

where the particle position (on its worldline) is regarded as a function of its proper time length: $\vec{x}_n(s_n)$. Hereafter, when we simply write *velocity*, it denotes the dimensionless covariant velocity, and we always call \mathbf{v}_n the non-covariant velocity.

By definition (27), we see that

$$(\vec{u}_n)^2 = -1, \quad u_n^0 = \sqrt{1 + \mathbf{u}_n^2}. \quad (30)$$

It is important that \vec{u}_n has only 3 independent components $\mathbf{u}_n = (u_n^1, u_n^2, u_n^3)$ and u^0 is always given in terms of them.

In the limit of infinitesimal time-lapse, the proper time distance becomes

$$ds_n = dx_n^0 \sqrt{1 - \frac{\mathbf{v}_n^2}{c^2}}. \quad (31)$$

Therefore, we obtain,

$$\frac{\mathbf{v}_n}{c} = \frac{\mathbf{u}_n}{\sqrt{1 + \mathbf{u}_n^2}} = \frac{\mathbf{u}_n}{u_n^0}, \quad \mathbf{u}_n = \frac{\frac{\mathbf{v}_n}{c}}{\sqrt{1 - \frac{\mathbf{v}_n^2}{c^2}}}. \quad (32)$$

The range of velocity is $0 \leq |\mathbf{u}_n| < \infty$ for $0 \leq |\mathbf{v}_n| < c$.

For any velocity \vec{u} , its time component

$$u^0 = \sqrt{1 + \mathbf{u}^2} \left(= \frac{1}{\sqrt{1 - \frac{\mathbf{v}^2}{c^2}}} \right) \quad (33)$$

takes the values $1 \leq u^0 < \infty$ for the above range of velocities. It governs the time dilation in the sense that

$$dx_n^0 = u_n^0 ds_n, \quad (34)$$

namely $dt_n = u_n^0 d\tau_n$; see the next subsection.⁹

⁸As we are in the non-natural units, we unconventionally call the proper time (multiplied by c) the proper time length.

⁹The time component $u^0 = \sqrt{1 + \mathbf{u}^2}$ is sometimes denoted by $\gamma(\mathbf{u})$ and is called the gamma factor.

2.5 Covariant acceleration

Along with the covariant velocity, we define the *covariant acceleration* of O_n as

$$\overrightarrow{\alpha}_n(s_n) := \frac{d\overrightarrow{u}_n(s_n)}{ds_n}. \quad (35)$$

Note that this quantity has the dimension of inverse length.

Taking the derivative of the normalization condition $(\overrightarrow{u}_n)^2 = -1$ with respect to s_n , we obtain

$$(\overrightarrow{\alpha}_n(s_n), \overrightarrow{u}_n(s_n)) = 0, \quad (36)$$

which shows that the time component of the covariant acceleration is not independent of the spatial ones:

$$\alpha_n^0 = \frac{\boldsymbol{\alpha}_n \cdot \mathbf{u}_n}{\sqrt{1 + \mathbf{u}_n^2}}. \quad (37)$$

This relation holds at any proper time s_n in any inertial frame. In particular, $\alpha_n^0 = 0$ whenever $\mathbf{u}_n = 0$.

2.6 Transformation to instantaneous rest frame

For each object O_n at any proper time length s_n , there exists a unique reference frame in which it is momentarily at rest. This frame, known as the *instantaneous rest frame*, corresponds to the viewpoint of O_n itself at that moment—it is the frame with respect to which O_n observes the world around it.

This frame is related to the world frame by a Lorentz transformation (17) determined by O_n 's covariant velocity $\overrightarrow{u}_n(s_n)$, and is explicitly given by $\Lambda = L(\mathbf{u}_n(s_n))$, where

$$L(\mathbf{u}) := \begin{bmatrix} u^0 & -u^1 & -u^2 & -u^3 \\ -u^1 & 1 + (u^0 - 1) \frac{u^1 u^1}{|\mathbf{u}|^2} & (u^0 - 1) \frac{u^1 u^2}{|\mathbf{u}|^2} & (u^0 - 1) \frac{u^1 u^3}{|\mathbf{u}|^2} \\ -u^2 & (u^0 - 1) \frac{u^2 u^1}{|\mathbf{u}|^2} & 1 + (u^0 - 1) \frac{u^2 u^2}{|\mathbf{u}|^2} & (u^0 - 1) \frac{u^2 u^3}{|\mathbf{u}|^2} \\ -u^3 & (u^0 - 1) \frac{u^3 u^1}{|\mathbf{u}|^2} & (u^0 - 1) \frac{u^3 u^2}{|\mathbf{u}|^2} & 1 + (u^0 - 1) \frac{u^3 u^3}{|\mathbf{u}|^2} \end{bmatrix}, \quad (38)$$

with u^0 defined in Eq. (33). It is straightforward to check that L satisfies the Lorentz condition (19) and obeys

$$L(\mathbf{u}) \overrightarrow{u} = \begin{bmatrix} 1 \\ \mathbf{0} \end{bmatrix}, \quad (L(\mathbf{u}))^{-1} = L(-\mathbf{u}). \quad (39)$$

Hereafter, we use uppercase letters for quantities in the instantaneous rest frame of an object:

$$\overrightarrow{X} = L(\mathbf{u}_n(s_n)) \overrightarrow{x}, \quad \overrightarrow{U}_m(s_m) = L(\mathbf{u}_n(s_n)) \overrightarrow{u}_m(s_m). \quad (40)$$

In particular, O_n itself is at rest in this frame: $\vec{U}_n(s_n) = (1, \mathbf{0})$ at $\vec{X}_n(s_n)$.

Furthermore, the infinitesimal time lapse in the rest frame coincides with the proper time interval:

$$ds_n = dX_n^0 \sqrt{1 - \left(\frac{d\mathbf{X}_n}{dX_n^0} \right)^2} = dX_n^0. \quad (41)$$

Thus, the Lorentz-invariant proper time s_n matches the time flow experienced by O_n .

Since $\alpha_n^0 = 0$ whenever $\mathbf{u}_n = 0$ (as shown in Sec. 2.5), the rest-frame acceleration satisfies

$$A_n^0(s_n) = 0, \quad (42)$$

where

$$\vec{A}_n(s_n) := L(\mathbf{u}_n(s_n)) \vec{\alpha}_n(s_n) \quad (43)$$

is the covariant acceleration transformed to the instantaneous rest frame.

2.7 Equation of motion

Let m_n be the mass of O_n . In the instantaneous rest frame (40), the object O_n is at rest and hence its motion is governed by a non-relativistic equation of motion:

$$m_n c^2 \frac{d\mathbf{U}_n(s_n)}{dX_n^0} = \mathbf{F}_n(s_n), \quad (44)$$

where $\mathbf{F}_n(s_n)$ is the sum of all the non-relativistic forces felt by O_n at s_n ; recall that \mathbf{U}_n is dimensionless and that the time-length X_n^0 has the dimension of length, hence the extra factor c^2 . Since dX_n^0 here is equal to the Lorentz-*invariant* ds_n (recall Eq. (41)), we may obtain its equation of motion in the world frame by

$$m_n c^2 \frac{d\vec{u}_n(s_n)}{ds_n} = \vec{f}_n(s_n), \quad (45)$$

where

$$\vec{f}_n(s_n) := L(-\mathbf{u}_n(s_n)) \begin{bmatrix} 0 \\ \mathbf{F}_n(s_n) \end{bmatrix}, \quad (46)$$

in which we used the inverse expression (39). We stress that the time component of the equation of motion does not give independent information and can be totally neglected.¹⁰

In electromagnetism, conversely, we can directly derive \vec{f}_n in fully Lorentz-*covariant* fashion, and then derive the expression of the rest-frame force \mathbf{F}_n if necessary. The derivation of \vec{f}_n is one of the main subjects in the following.

This covariant equation of motion provides the theoretical foundation for updating particle trajectories in our simulation, as we detail in the following implementation sections.

¹⁰Indeed, Eq. (45) leads to

$$m_n c^2 \begin{bmatrix} \frac{\alpha_n \cdot \mathbf{u}_n}{u_n^0} \\ \boldsymbol{\alpha}_n \end{bmatrix} = \begin{bmatrix} u_n^0 & \mathbf{u}_n^t \\ \mathbf{u}_n & I + (u_n^0 - 1) \hat{\mathbf{u}}_n \hat{\mathbf{u}}_n^t \end{bmatrix} \begin{bmatrix} 0 \\ \mathbf{F}_n \end{bmatrix},$$

where we used Eq. (37). The temporal and spatial components read

$$m_n c^2 \frac{\boldsymbol{\alpha}_n \cdot \mathbf{u}_n}{u_n^0} = \mathbf{u}_n \cdot \mathbf{F}_n, \quad m_n c^2 \boldsymbol{\alpha}_n = u_n^0 \mathbf{F}_{n\parallel} + \mathbf{F}_{n\perp},$$

where $\mathbf{F}_{n\parallel} := (\mathbf{F}_n \cdot \hat{\mathbf{u}}_n) \hat{\mathbf{u}}_n$ and $\mathbf{F}_{n\perp} := \mathbf{F}_n - \mathbf{F}_{n\parallel}$. Noting that $\hat{\mathbf{u}}_n \cdot \mathbf{F}_{n\perp} = 0$ and hence $\mathbf{u}_n \cdot \mathbf{F}_n = \mathbf{u}_n \cdot \mathbf{F}_{n\parallel}$, we see that the temporal component is nothing but the inner product of the spatial component with \mathbf{u}_n .

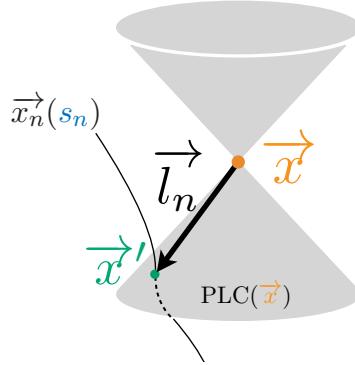


Figure 2: Schematic illustration: \vec{x} denotes the spacetime point at which the electromagnetic field is evaluated. $\text{PLC}(\vec{x})$ is its PLC. $\vec{x}_n(s_n)$ parametrizes the worldline of the n th charge. \vec{x}' indicates the spacetime location of the charge that influences the field at the observation point \vec{x} , and \vec{l}_n is the chargeward vector connecting the two points.

3 Drawing the world on PLC

This section explains how to simulate what a player sees by reconstructing the world on their PLC, using stored worldline data and Lorentz transformations.

3.1 Definition and visualization of PLC

When the player P is at a spacetime point \vec{x}_P (in the world frame), the world seen by P is the collection of points from which a light ray can reach \vec{x}_P —the PLC:

$$\text{PLC}(\vec{x}_P) := \left\{ \vec{x} \mid (\vec{x} - \vec{x}_P)^2 = 0, x^0 < x_P^0 \right\}. \quad (47)$$

Note that $\text{PLC}(\vec{x}_P)$ is defined Lorentz invariantly.¹¹ An object O_n is observed by the player at the intersection between its worldline \mathcal{W}_n and $\text{PLC}(\vec{x}_P)$; see Fig. 2, where \vec{x}_P is denoted by \vec{x} . As said above, the player at its proper time length s_P sees the world in its instantaneous rest frame:

$$\vec{X} = L(u_P(s_P)) \vec{x}. \quad (48)$$

3.2 Intersection with worldlines

Here we explain how to implement the worldline in a computer game. We let the program store data of the worldlines in the world frame \vec{x} . After N iterations, O_n 's worldline \mathcal{W}_n becomes a discrete set of its past spacetime positions:

$$\mathcal{W}_n = \{ \vec{x}_{n(i)} \}_{i=0,\dots,N} = \{ \vec{x}_{n(0)}, \vec{x}_{n(1)}, \dots, \vec{x}_{n(N)} \}, \quad (49)$$

where we order from the past to the future.

¹¹The condition $x^0 < x_P^0$ is also invariant under the (orthochronous) Lorentz transformation; see footnote 2.

Now we illustrate how to obtain an intersection between a particle's worldline \mathcal{W}_n and $\text{PLC}(\vec{x}_P)$. First, we check, from the past to the future, whether $\vec{x}_{n(i)} \in \mathcal{W}_n$ ($i = 0, \dots, N$) fits in the past-side of $\text{PLC}(\vec{x}_P)$:

$$[\vec{x}_{n(i)} - \vec{x}_P]^2 < 0 \quad \text{and} \quad x_{n(i)}^0 < x_P^0. \quad (50)$$

Let j ($\in \{0, \dots, N\}$) be the first iteration that violates this check. That is, $\vec{x}_{n(j-1)}$ and $\vec{x}_{n(j)}$ are the last and first spacetime points inside and outside of $\text{PLC}(\vec{x}_P)$, respectively. By linear-interpolating a point \vec{x} in between $\vec{x}_{n(j-1)}$ and $\vec{x}_{n(j)}$ as

$$\vec{x} = (1 - \lambda) \vec{x}_{n(j-1)} + \lambda \vec{x}_{n(j)}, \quad (0 \leq \lambda \leq 1), \quad (51)$$

the value of λ that gives the intersection point can be determined by the condition $(\vec{x} - \vec{x}_P)^2 = 0$:

$$\lambda = \frac{B - \sqrt{B^2 - AC}}{A}, \quad (52)$$

where

$$A := -[\vec{x}_{n(j)} - \vec{x}_{n(j-1)}]^2 > 0, \quad (53)$$

$$B := -(\vec{x}_{n(j)} - \vec{x}_{n(j-1)}, \vec{x}_P - \vec{x}_{n(j-1)}) > 0, \quad (54)$$

$$C := -(\vec{x}_P - \vec{x}_{n(j-1)})^2 > 0. \quad (55)$$

We have discarded the intersection with the future light cone $(B + \sqrt{B^2 - AC})/A$.

If necessary, the O_n 's velocity on $\text{PLC}(\vec{x}_P)$, which is nothing but the velocity from $\vec{x}_{n(j-1)}$ to $\vec{x}_{n(j)}$, can be computed as

$$\vec{u}_{n(j-1)} := \frac{\vec{\Delta x}_{n(j-1)}}{\Delta s_{n(j-1)}}, \quad (56)$$

where

$$\vec{\Delta x}_{n(j-1)} := \vec{x}_{n(j)} - \vec{x}_{n(j-1)}, \quad (57)$$

$$\Delta s_{n(j-1)} := \sqrt{-\left(\vec{\Delta x}_{n(j-1)}\right)^2} = \sqrt{-\left(\vec{x}_{n(j)} - \vec{x}_{n(j-1)}\right)^2}. \quad (58)$$

Accordingly, one may also store the information of the proper time length of O_n given by

$$s_{n(j)} := s_{n(j-1)} + \Delta s_{n(j-1)} \quad (59)$$

in \mathcal{W}_n .

Given $\vec{x}_{n(j)}$, $\vec{x}_{n(j-1)}$, and $\vec{x}_{n(j-2)}$, the covariant acceleration can be approximated by

$$\begin{aligned} \vec{\alpha}_{n(j-1)} &:= \frac{\vec{u}_{n(j-1)} - \vec{u}_{n(j-2)}}{\frac{\Delta s_{n(j-1)} + \Delta s_{n(j-2)}}{2}} \\ &= \frac{2}{\Delta s_{n(j-1)} + \Delta s_{n(j-2)}} \left(\frac{\vec{x}_{n(j)} - \vec{x}_{n(j-1)}}{\Delta s_{n(j-1)}} - \frac{\vec{x}_{n(j-1)} - \vec{x}_{n(j-2)}}{\Delta s_{n(j-2)}} \right). \end{aligned} \quad (60)$$

3.3 Drawing the observed world

Once the intersecting points $\{\vec{x}_n\}_{n=1,\dots,N}$ with $\text{PLC}(\vec{x}_P)$ are obtained for N point charges, they are transformed to the player's instantaneous rest frame. As in Eq. (48), this transformation is given by

$$\vec{X}_n = L(\mathbf{u}_P)(\vec{x}_n - \vec{x}_P), \quad (61)$$

where \mathbf{u}_P is the player's (dimensionless covariant) velocity at \vec{x}_P in the world frame, and we have subtracted \vec{x}_P to place the player at the origin of the new coordinate system.

3.4 Time evolution of player

We present a schematic time evolution for the player to account for where c appears.

In our implementation, the real-world time is synchronized with the player's proper time τ_P , or the corresponding proper time length $s_P = c\tau_P$. At the real-world time t , we have $s_P = ct$. Let Δt be the real-world time-lapse in the next iteration. Then the player's proper-time-length lapse is $\Delta s_P = c\Delta t$. In the next iteration, the player's position and velocities change into, in the world frame,¹²

$$\vec{x}_P(s_P + \Delta s_P) = \vec{x}_P(s_P) + \vec{u}_P(s_P) \Delta s_P, \quad (62)$$

$$\mathbf{u}_P(s_P + \Delta s_P) = \mathbf{u}_P(s_P) + \boldsymbol{\alpha}_P(s_P) \Delta s_P, \quad (63)$$

where the acceleration is given as

$$\boldsymbol{\alpha}_P(s_P) = \frac{\vec{f}_P(s_P)}{m_P c^2} \quad (64)$$

from the equation of motion (45).¹³

From the dimensionalities

$$[\Delta s_P] = [c] [\text{Time}], \quad (= [\text{Length}]) \quad (65)$$

$$[\mathbf{u}_P] = [\vec{u}_P] = \left[\frac{\text{Non-relativistic velocity}}{c} \right], \quad (= [\text{Dimensionless}]) \quad (66)$$

$$[\boldsymbol{\alpha}_P] = [\vec{\alpha}_P] = \left[\frac{\text{Non-relativistic acceleration}}{c^2} \right], \quad (= \left[\frac{1}{\text{Length}} \right]) \quad (67)$$

we see that the time evolution in Eqs. (62) and (63) has the same dimensionality as the ordinary non-relativistic one.

In actual implementation, the Euler method in Eqs. (62) and (63) is known to increase the total energy of the system exponentially. Consequently, this method results in pathological behaviors such as perpetually rotating opposite charges despite the emission of electromagnetic waves. This issue can be mitigated by employing the symplectic (semi-implicit) Euler method:

$$\mathbf{u}_P(s_P + \Delta s_P) = \mathbf{u}_P(s_P) + \boldsymbol{\alpha}_P(s_P) \Delta s_P, \quad (68)$$

$$\vec{x}_P(s_P + \Delta s_P) = \vec{x}_P(s_P) + \vec{u}_P(s_P + \Delta s_P) \Delta s_P. \quad (69)$$

We utilize this symplectic Euler method in the sample code explained in Sec. 6.

¹²The time component of the velocity can be fixed as $u_P^0(s_P + \Delta s_P) = \sqrt{1 + \mathbf{u}_P^2(s_P + \Delta s_P)}$.

¹³If the player is charged, the player will feel the electromagnetic force (93) below, which should also be added in the force in Eq. (64).

3.5 Time evolution of others

At s_P , the program draws the world on PLC($\vec{x}_P(s_P)$). In the next iteration, the program reads out the real-world time-lapse, identified as $\Delta\tau_P$, or $\Delta s_P = c \Delta\tau_P$. Then, we need to time-evolve all the point charges up to PLC($\vec{x}_P(s_P + \Delta s_P)$). For a consistent time evolution in accordance with causality, we employ the following algorithm, which is newly explained here but has already been implemented in Ref. [9].

Let $\{\vec{x}_n(s_n)\}_{n=1,\dots,N}$ be the last intersecting points of the N worldlines with PLC($\vec{x}_P(s_P)$); see Fig. 2. We want to evolve the N charges consistently with causality. We first move a past-most charge with a pre-fixed small time step Δx^0 in the world frame.¹⁴ We then evolve the past-most one after each iteration until all the charged particles reach PLC($\vec{x}_P(s_P + \Delta s_P)$).

At each of the iterations, the position and velocity of the charge are determined by the Lorentz force described below.

4 Relativistic electromagnetism

We briefly review the relativistic electromagnetism. Once again, we ensure to include details that, although might be elementary for experts in physics and mathematics, are crucial for making the content accessible to a broader audience.

4.1 Equation of motion

In the electromagnetic dynamics, we promote the equation of motion (44) (of O_n at a space-time point \vec{x}_n with velocity \mathbf{u}_n) to¹⁵

$$m_n c^2 \frac{d\mathbf{u}_n}{dx_n^0} = \mathbf{f}_n^{\text{Lorentz}}(\vec{x}_n), \quad (70)$$

where $\mathbf{f}_n^{\text{Lorentz}}$ is the Lorentz force felt by O_n :

$$\mathbf{f}_n^{\text{Lorentz}}(\vec{x}_n) := q_n [\mathbf{E}(\vec{x}_n) + \mathbf{v}_n \times \mathbf{B}(\vec{x}_n)], \quad (71)$$

in which q_n is the charge of O_n ; \mathbf{E} and \mathbf{B} are the electric and magnetic fields at the location of the charge, respectively; \mathbf{v}_n is the non-covariant velocity (25); and the vector product is defined by, for $i = 1, 2, 3$,

$$(\mathbf{A} \times \mathbf{B})_i = \sum_{j,k=1}^3 \epsilon_{ijk} A_j B_k, \quad \begin{bmatrix} (\mathbf{A} \times \mathbf{B})_1 \\ (\mathbf{A} \times \mathbf{B})_2 \\ (\mathbf{A} \times \mathbf{B})_3 \end{bmatrix} = \begin{bmatrix} A_2 B_3 - A_3 B_2 \\ A_3 B_1 - A_1 B_3 \\ A_1 B_2 - A_2 B_1 \end{bmatrix}, \quad (72)$$

¹⁴In the implementation of Ref. [9], the interplay among the player and non-player characters was important. Therefore, a fixed proper-time length Δs_n was used. In the current implementation, a charge obeys only the Lorentz force and does not change its move by some extra acceleration of its own. So it suffices to consider fixed world-frame time lapse.

¹⁵In a more familiar-looking expression, $c \frac{d}{dx_n^0} = \frac{d}{dt}$. In the non-relativistic regime, $c\mathbf{u}_n \approx \mathbf{v}_n$, Eq. (70) reduces to the non-relativistic equation of motion $m_n \frac{d\mathbf{v}_n}{dt} \approx \mathbf{f}_n(\vec{x}_n)$. Here, we write the left-hand side of the equation of motion (70) in the form that is correct in the relativistic theory from the beginning. On the other hand, the Lorentz force (71) does not need such a modification to be relativistic.

where

$$\epsilon_{ijk} = \begin{cases} 1 & (\text{when } i, j, k \text{ is an even permutation of } 1, 2, 3), \\ -1 & (\text{when } i, j, k \text{ is an odd permutation of } 1, 2, 3), \\ 0 & (\text{otherwise}), \end{cases} \quad (73)$$

is the Levi-Civita symbol, namely, the totally anti-symmetric tensor for the (spatial) rotational group $SO(3)$. With our metric convention (10), the upper and lower spatial indices are not distinguished:

$$A^i = A_i, \quad \epsilon^{ijk} = \epsilon_{ijk}, \quad (74)$$

etc.; see Eq. (87) below.

It is crucial to note that once the charge q_n is specified, the motion of O_n is governed by the electromagnetic fields. These fields are determined at any spacetime position of the charge, denoted by \vec{x}_n .

Given the motion of the objects $\{O_n\}_{n=1,2,\dots}$, namely their world lines $\{\mathcal{W}_n\}_{n=1,2,\dots}$, the charge and (3D) current densities $\rho(\vec{x})$ and $\mathbf{j}(\vec{x})$ at a spacetime point \vec{x} are obtained as

$$\rho(\vec{x}) = \sum_n q_n \int d\mathbf{s}_n \delta^4(\vec{x} - \vec{x}_n(\mathbf{s}_n)), \quad (75)$$

$$\mathbf{j}(\vec{x}) = \sum_n q_n \int d\mathbf{s}_n \delta^4(\vec{x} - \vec{x}_n(\mathbf{s}_n)) c \mathbf{u}_n(\mathbf{s}_n). \quad (76)$$

Recall that $\mathbf{u}_n(\mathbf{s}_n)$ is dimensionless, and hence the extra factor c . Here and hereafter, we sometimes put colors for ease of eyes.

These charge and current densities determine the electromagnetic fields via Maxwell's equations at each spacetime point \vec{x} :

$$\nabla \cdot \mathbf{E}(\vec{x}) = \frac{\rho(\vec{x})}{\epsilon_0}, \quad \nabla \times \mathbf{B}(\vec{x}) = \frac{\mathbf{j}(\vec{x})}{\epsilon_0 c^2} + \frac{1}{c^2} c \partial_0 \mathbf{E}(\vec{x}), \quad (77)$$

$$\nabla \cdot \mathbf{B}(\vec{x}) = 0, \quad \nabla \times \mathbf{E}(\vec{x}) = -c \partial_0 \mathbf{B}(\vec{x}). \quad (78)$$

where, for $\mu = 0, \dots, 3$,

$$\partial_\mu := \frac{\partial}{\partial x^\mu}, \quad c \partial_0 = c \frac{\partial}{\partial x^0} = \frac{\partial}{\partial t}, \quad (79)$$

and ϵ_0 is the electric constant (vacuum permittivity).

The physical insight is most simply obtained by taking the natural units

$$\epsilon_0 = c = 1, \quad (80)$$

which also yields the natural unit for the magnetic constant (vacuum permeability) $\mu_0 := \frac{1}{\epsilon_0 c^2} = 1$. Although these constants can be easily recovered when necessary by dimensional analysis, we take the non-natural units and leave them as they are for readers unfamiliar with the dimensional analysis. Finally, even if one varies c , as in our sample program that will be described in Sec. 6, one can still safely take

$$\epsilon_0 = 1 \quad (81)$$

(yielding $\mu_0 = 1/c^2$), unless one further varies ϵ_0 .

Note that the tensor ∂_μ transforms under the inverse representation of the Lorentz group. That is, when the coordinates transforms according to Eq. (17), we obtain

$$\partial_\mu = \frac{\partial}{\partial x^\mu} \rightarrow \partial'_\nu = \frac{\partial}{\partial x'^\mu} = \sum_{\nu=1}^3 \frac{\partial x^\nu}{\partial x'^\mu} \frac{\partial}{\partial x^\nu} = \sum_{\nu=1}^3 [\Lambda^{-1}]^\nu_\mu \partial_\nu, \quad (82)$$

where we used

$$\frac{\partial x'^\mu}{\partial x^\nu} = \Lambda^\mu_\nu, \quad \frac{\partial x^\nu}{\partial x'^\mu} = [\Lambda^{-1}]^\nu_\mu. \quad (83)$$

We always write an inverse representation of the Lorentz group, such as ∂_μ , with the lower indices so that the contraction of all the lower and upper indices gives a Lorentz invariant.

The purely electromagnetic part (78) is automatically solved when we rewrite the magnetic and electric fields in terms of the scalar and (3D) vector potentials $\phi(\vec{x})$ and $\mathbf{A}(\vec{x})$:

$$\mathbf{B}(\vec{x}) = \nabla \times \mathbf{A}(\vec{x}), \quad \mathbf{E}(\vec{x}) = -\nabla \phi(\vec{x}) - c\partial_0 \mathbf{A}(\vec{x}). \quad (84)$$

From (ϕ, \mathbf{A}) and (ρ, \mathbf{j}) , we respectively define a (covariant) vector potential \vec{A} , which we call the *gauge field* hereafter, and a (covariant) current density \vec{j} :

$$\vec{A}(\vec{x}) = (A^\mu(\vec{x}))_{\mu=0,\dots,3} := \left(\frac{\phi(\vec{x})}{c}, \mathbf{A}(\vec{x}) \right), \quad (85)$$

$$\vec{j}(\vec{x}) = (j^\mu(\vec{x}))_{\mu=0,\dots,3} := (c\rho(\vec{x}), \mathbf{j}(\vec{x})). \quad (86)$$

In the relativistic electrodynamics, we assume that they transform covariantly under the Lorentz transformation.

For any tensor such as A^μ and ∂_μ , we define

$$A_\mu := \sum_{\nu=0}^3 \eta_{\mu\nu} A^\nu, \quad \partial^\mu := \sum_{\nu=0}^3 \eta^{\mu\nu} \partial_\nu, \quad (87)$$

etc. That is, $A_0 := -A^0$, $\partial^0 := -\partial_0$ and $A_i := A^i$, $\partial^i = \partial_i$ ($i = 1, 2, 3$). Since η is the invariant tensor of the Lorentz group in the sense of Eq. (19), we see that A_μ transforms under the inverse representation of the Lorentz group, whereas ∂^μ the same as x^μ . In this notation, the inverse transformation (20), or (83), reads $[\Lambda^{-1}]^\nu_\mu = A_\mu^\nu$.

We define a rank-two anti-symmetric tensor, the *field strength*,

$$F_{\mu\nu}(\vec{x}) := \partial_\mu A_\nu(\vec{x}) - \partial_\nu A_\mu(\vec{x}). \quad (88)$$

We see that the electromagnetic fields can be written in terms of the field strength: For $i = 1, 2, 3$, Eq. (84) reads

$$\begin{aligned} E_i(\vec{x}) &= cF_{i0}(\vec{x}) = cF^{0i}(\vec{x}), \\ B_i(\vec{x}) &= \frac{1}{2} \sum_{j,k=1}^3 \epsilon_{ijk} F_{jk}(\vec{x}) = \frac{1}{2} \sum_{j,k=1}^3 \epsilon_{ijk} F^{jk}(\vec{x}). \end{aligned} \quad (89)$$

More explicitly, the magnetic reads

$$B_1(\vec{x}) = F_{23}(\vec{x}), \quad B_2(\vec{x}) = F_{31}(\vec{x}), \quad B_3(\vec{x}) = F_{12}(\vec{x}). \quad (90)$$

Now the equation of motion under the Lorentz force (70) can be written in a manifestly Lorentz-covariant (and invariant) fashion: For $\mu = 0, \dots, 3$ and for each n ,

$$m_n c \frac{du_n^\mu(\vec{s}_n)}{ds_n} = q_n \sum_{\rho, \sigma=0}^3 F^{\mu\rho}(\vec{x}_n(\vec{s}_n)) \eta_{\rho\sigma} u^\sigma(\vec{s}_n), \quad (91)$$

or more concisely,

$$m_n c \frac{du_n^\mu(\vec{s}_n)}{ds_n} = q_n \sum_{\nu=0}^3 F^{\mu\nu}(\vec{x}_n(\vec{s}_n)) u_\nu(\vec{s}_n), \quad (92)$$

where we have recovered the dependence on the proper time length that was implicit in Eq. (70).¹⁶

In the matrix notation (see Eqs. (104) and (105) below), the above equation of motion reads

$$m_n c \frac{d\vec{u}_n(\vec{s}_n)}{ds_n} = q_n F(\vec{x}_n(\vec{s}_n)) \eta \vec{u}(\vec{s}_n). \quad (93)$$

This form may be more usable in practical implementation.

4.2 General solution to relativistic Maxwell's equation

The general form of the gauge field is obtained as follows [10]:

$$A^\mu(\vec{x}) = \frac{1}{4\pi\epsilon_0 c^2} \int d^4 \vec{x}' \delta(\vec{x}^0 - \vec{x}'^0 - |\vec{x} - \vec{x}'|) \frac{j^\mu(\vec{x}')}{|\vec{x} - \vec{x}'|}. \quad (94)$$

Hereafter, we sometimes write the integral more concisely

$$\int_{\text{PLC}(\vec{x})} d^3 \vec{x}' [\dots] := \int d^4 \vec{x}' \delta(\vec{x}^0 - \vec{x}'^0 - |\vec{x} - \vec{x}'|) [\dots] \quad (95)$$

such that

$$A^\mu(\vec{x}) = \frac{1}{4\pi\epsilon_0 c^2} \int_{\text{PLC}(\vec{x})} d^3 \vec{x}' \frac{j^\mu(\vec{x}')}{|\vec{x} - \vec{x}'|}. \quad (96)$$

Physically, \vec{x}' is each location of the charge in the past that affects the electromagnetic field at \vec{x} in the future; see Fig. 2. Here, the restriction of \vec{x}' onto $\text{PLC}(\vec{x})$ is achieved by imposing the time difference $\vec{x}^0 - \vec{x}'^0$ to be equal to the distance $|\vec{x} - \vec{x}'| \geq 0$, by the delta function.

¹⁶We have used $\frac{d}{dx_n^0} = \frac{1}{u_n^0} \frac{d}{ds_n}$ in Eq. (70):

$$m_n c^2 \underbrace{\frac{du_n^i}{dx_n^0}}_{\frac{1}{u_n^0} \frac{du_n^i}{ds_n}} = q_n \left[c F^{0i} + \sum_{j,k=1}^3 \epsilon_{ijk} \underbrace{c \frac{dx_n^j}{ds_n}}_{c \frac{u_n^j}{u_n^0}} \left(\sum_{l,m=1}^3 \frac{1}{2} \epsilon_{klm} F_{lm} \right) \right].$$

5 Covariant formalism for field strength from point charges

We compute how the relativistic motion of a charged particle affects the electromagnetic field in the future. In other words, we compute a fully relativistic expression of the Liénard-Wiechert potential in terms of covariant quantities only. Our main goal is to find out the field strength at \vec{x} in terms of the positions and velocities of point charges on PLC(\vec{x}). A reader who is interested only in its final form may skip to Eqs. (102) and (103).

5.1 Continuous worldlines and covariant current density

The electromagnetic field at a spacetime point \vec{x} is determined by summing over the influence from each object O_n on PLC(\vec{x}), namely at the intersection between \mathcal{W}_n and PLC(\vec{x}). Each influence is solely determined by the position \vec{x}_n and velocity \vec{u}_n on PLC(\vec{x}). In the actual implementation on computers, we may always obtain them from the discrete worldline as in Sec. 3.2. Therefore, we take \vec{x}_n and \vec{u}_n for granted, and compute its electromagnetic influence on \vec{x} . This way, we treat each worldline (49) as if it were continuously parametrized by its proper time length s_n :

$$\mathcal{W}_n = \{ \vec{x}_n(s_n) \mid -\infty < s_n < \infty \}. \quad (97)$$

Given the worldlines $\{ \mathcal{W}_n \}_{n=1,2,\dots}$ of the charged particles, the covariant current density at a spacetime point \vec{x}' is given by

$$\vec{j}(\vec{x}') = \sum_n c q_n \int d s_n \delta^4(\vec{x}' - \vec{x}_n(s_n)) \vec{u}_n(s_n), \quad (98)$$

where q_n is the charge of O_n . Here, c is supplied to make the dimension of the current density (charge) / (length)² (time).

5.2 Modified gamma factor and chargeward vector

To write down the field strength below, we define a *modified gamma factor*:

$$\begin{aligned} \gamma_n(s_n, \vec{x}) &:= \frac{dx_n^0(s_n)}{ds_n} + \frac{\partial |\vec{x} - \vec{x}_n(s_n)|}{\partial s_n} \\ &= u_n^0(s_n) + \hat{l}_n(s_n, \vec{x}) \cdot \vec{u}_n(s_n), \end{aligned} \quad (99)$$

where we define the *chargeward vector* as the covariant position vector of the charge relative to the reference point:

$$\vec{l}_n(s_n, \vec{x}) := \vec{x}_n(s_n) - \vec{x}, \quad \text{namely,} \quad \begin{bmatrix} l_n^0(s_n, \vec{x}^0) \\ l_n(s_n, \vec{x}) \end{bmatrix} := \begin{bmatrix} x_n^0(s_n) - \vec{x}^0 \\ \vec{x}_n(s_n) - \vec{x} \end{bmatrix}; \quad (100)$$

see Fig. 2. Recall also the notation (9):

$$\hat{l}_n(s_n, \vec{x}) = \frac{\vec{l}_n(s_n, \vec{x})}{|\vec{l}_n(s_n, \vec{x})|}. \quad (101)$$

Note that the modified gamma factor is written in terms of the covariant quantities \vec{u}_n and \vec{l}_n .

5.3 Field strength

The components of field strength $F_{i0} = -F_{0i} = F^{0i} = -F^{i0}$ and $F_{ij} = -F_{ji} = -F^{ji} = F^{ij}$ are obtained as follows [10]:

$$\begin{aligned} F^{0i}(\vec{x}) &= -\partial_0 A^i(\vec{x}) - \partial_i A^0(\vec{x}) \\ &= \sum_n \frac{q_n}{4\pi\epsilon_0 c |\mathbf{l}_n|} \left[\hat{l}_n^i \frac{u_n^0 (\hat{\mathbf{l}}_n \cdot \boldsymbol{\alpha}_n - \frac{1}{|\mathbf{l}_n|}) - \alpha_n^0 (\hat{\mathbf{l}}_n \cdot \mathbf{u}_n)}{\gamma_n^3} + u_n^i \frac{\alpha_n^0 + \hat{\mathbf{l}}_n \cdot \boldsymbol{\alpha}_n - \frac{1}{|\mathbf{l}_n|}}{\gamma_n^3} - \frac{\alpha_n^i}{\gamma_n^2} \right], \end{aligned} \quad (102)$$

$$\begin{aligned} F^{ij}(\vec{x}) &= \partial_i A^j(\vec{x}) - \partial_j A^i(\vec{x}) \\ &= \sum_n \frac{q_n}{4\pi\epsilon_0 c |\mathbf{l}_n|} \left[\frac{\hat{l}_n^i \alpha_n^j - \hat{l}_n^j \alpha_n^i}{\gamma_n^2} - \frac{\hat{l}_n^i u_n^j - \hat{l}_n^j u_n^i}{\gamma_n^3} \left(\alpha_n^0 + \hat{\mathbf{l}}_n \cdot \boldsymbol{\alpha}_n - \frac{1}{|\mathbf{l}_n|} \right) \right], \end{aligned} \quad (103)$$

where \mathbf{l}_n and $\hat{\mathbf{l}}_n$ are given in Eqs. (100) and (101), respectively, and \vec{u}_n , $\vec{\alpha}_n$, and γ_n are all evaluated at the intersection between \mathcal{W}_n and PLC(\vec{x}). This expression for the field strength can be used directly in computer programs to obtain the Lorentz transformation of the electromagnetic fields below.

Given the field strength, the electric and magnetic fields can be derived from Eq. (89). For an actual implementation in a computer program, one may write the field strength as an anti-symmetric matrix F whose μ, ν components are given by the upper-indexed counterparts:

$$[F(\vec{x})]^{\mu,\nu} = F^{\mu\nu}(\vec{x}). \quad (104)$$

Then its Lorentz transformation law under the coordinate transformation $\vec{x} \rightarrow \vec{x}' = \Lambda \vec{x}$ is

$$F \rightarrow F' = \Lambda F \Lambda^t. \quad (105)$$

Accordingly, the Lorentz transformation for the electromagnetic fields are

$$E^i \rightarrow E'^i = \textcolor{red}{c} F'^{0i} = \textcolor{red}{c} [\Lambda F \Lambda^t]^{0,i}, \quad (106)$$

$$B^i \rightarrow B'^i = \frac{1}{2} \sum_{j,k=1}^3 \epsilon_{ijk} F'^{jk} = \frac{1}{2} \sum_{j,k=1}^3 \epsilon_{ijk} [\Lambda F \Lambda^t]^{j,k}; \quad (107)$$

recall Eqs. (74) and (90).

6 Overview of concrete implementation

We have developed a sample code to demonstrate a practical implementation [11]. Below, we describe its functionality.

In our visualization of electric and magnetic fields, they are represented by green and yellow arrows, respectively. The electromagnetic fields are evaluated at lattice points, initially located on two planes perpendicular to each other, on the player's PLC. The speed of light c is expressed in units of this lattice grid as grid/s.

Positive and negative charges are represented by red and blue regular icosahedrons, respectively. We may define an arbitrary unit charge, say the elementary charge e , to be

dimensionless: In the sample program, we choose to set $e/4\pi\epsilon_0 = 1$ for simplicity of the code, while retaining the dimensionality of c .¹⁷ In the sample program, we choose q_n to be $q_n = \pm e$ for the positive and negative charges; the values of the charges can be modified in the code if necessary.

6.1 Speed of Light

We have implemented a slider that adjusts the value of c by factors of 2^n , where the integer n ranges from -3 to 10 . When c is increased, the simulation continues unchanged, while when c is decreased, the simulation is reset to its initial configuration so that the speed of massive charges never exceed c .

We have also implemented a button that allows users to disable the Lorentz transformation to the player's rest frame (61). Note that even with the Lorentz transformation disabled, the world is still viewed from the player's PLC.

6.2 Arrow

The length ℓ of each arrow is defined such that the electric field at a distance of 1 grid unit from the point charge is represented by an arrow with length 1 grid unit when the Log Reduction Count $N = 0$ and the 10-Exponent $n = 0$, both of which are explained below. The length of the arrows for the magnetic field is defined to match that for the electric field when $c = 1$ grid/s and when $|\mathbf{B}| = |\mathbf{E}|/c$, such as when \mathbf{E} is perpendicular to the line of sight $\hat{\mathbf{l}}$.

We also provide an option for displaying the Poynting vector

$$\mathbf{S} := \epsilon_0 c^2 \mathbf{E} \times \mathbf{B} \quad (108)$$

with magenta arrows; see also Ref. [10]. The length of the arrows for the Poynting vector is defined such that it becomes the same as that for the electric field when $c = 1$ grid/s and when \mathbf{E} and \mathbf{B} are perpendicular to each other, namely when $|\mathbf{S}| = \epsilon_0 c^2 |\mathbf{B}| |\mathbf{E}|$.

Given that ℓ can vary significantly across different lattice points, we have implemented a “Log Reduction” scaling option to manage this variability. This scaling applies the transformation

$$\ell \mapsto \ln(1 + \ell) \quad (109)$$

iteratively, N times, where \ln represents the natural logarithm. We refer to N as the “Log Reduction Count.” Additionally, we implement scaling by an extra factor of 10^n before applying the Log Reduction, where n is referred to as the “10-Exponent.”

6.3 Preset

Regarding the sources of electromagnetic fields, we offer multiple preset configurations:

1. Static Neg & Rotating Pos: A fixed negative charge, paired with a dynamical positive charge.

¹⁷The fine structure constant is dimensionless and is represented as $\alpha = e^2/4\pi\epsilon_0\hbar c \simeq 1/137$., where e is the elementary charge and $\hbar c \simeq 0.20$ GeV fm (in SI units, $e \simeq 1.6 \times 10^{-19}$ C and $\hbar c \simeq 3.2 \times 10^{-26}$ J m). Therefore, in natural units $\hbar = c = \epsilon_0 = 1$, the elementary charge e becomes dimensionless and has a fixed value $e \simeq 0.30$. On the other hand, in the sample program, we take $\epsilon_0 = 1$ (see Eq. (81)) and leave c dimensionful, while setting $e = 4\pi\epsilon_0 (= 4\pi)$.

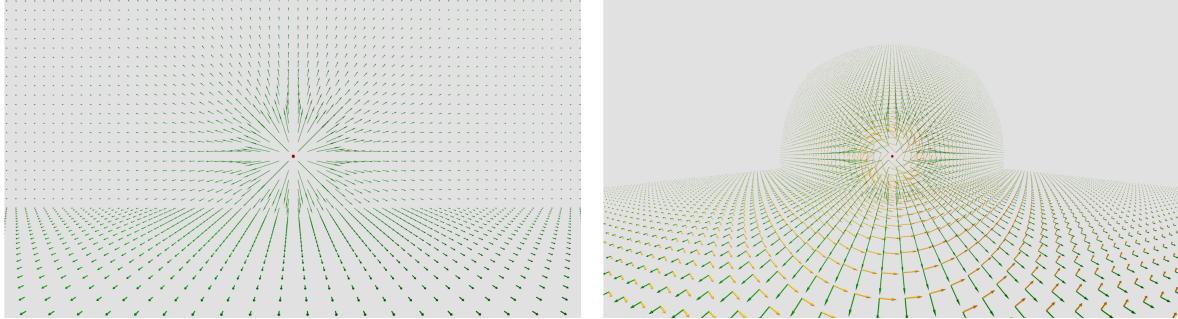


Figure 3: Scene of the sample program with the Static Charge preset. Electromagnetic fields are depicted on vertical and horizontal planes. The setup is $c = 1$ grid/s, Log Reduction Count $N = 1$, and 10-Exponent $n = 2$. **Left:** Static Coulomb electric fields (green arrows) emanate from the positive point charge at rest (red icosahedron). **Right:** After forward acceleration of the player, the point charge appears to move toward the player, generating an electric current; this produces magnetic fields (yellow arrows) circulating around the charge's velocity vector toward the player. The planes are Lorentz contracted due to the player acceleration, actually elongated along the direction of acceleration; see Appendix B in Ref. [9] for a detailed explanation.

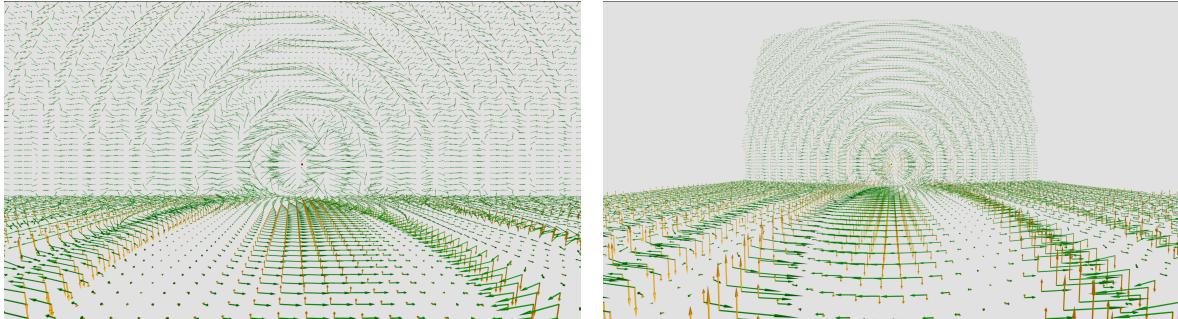


Figure 4: Scene of the sample program with the Harmonic Oscillator preset, with $c = 4$ grid/s, Log Reduction Count $N = 1$, and 10-Exponent $n = 2$; other explanations of Fig. 3 apply here too unless otherwise stated. **Left:** Electromagnetic fields (green and yellow arrows) emitted from the positive point charge (red icosahedron) under forced harmonic oscillation. The emission of electromagnetic waves is apparent both in the vertical and horizontal planes. **Right:** With the forward acceleration of the player, the planes are Lorentz contracted/elongated as in the Right of Fig. 3. On the real-time simulator, one can observe an increase in the oscillation frequency, where the time “contraction” of the incoming charge is observed; see Appendix C in Ref. [9] for a detailed explanation. In both Left and Right, the horizontal plane exhibits apparently non-spherical waves because it does not correspond to a plane on a fixed time slice of the world frame but on the player’s PLC, such that the more distant region corresponds to the further past.

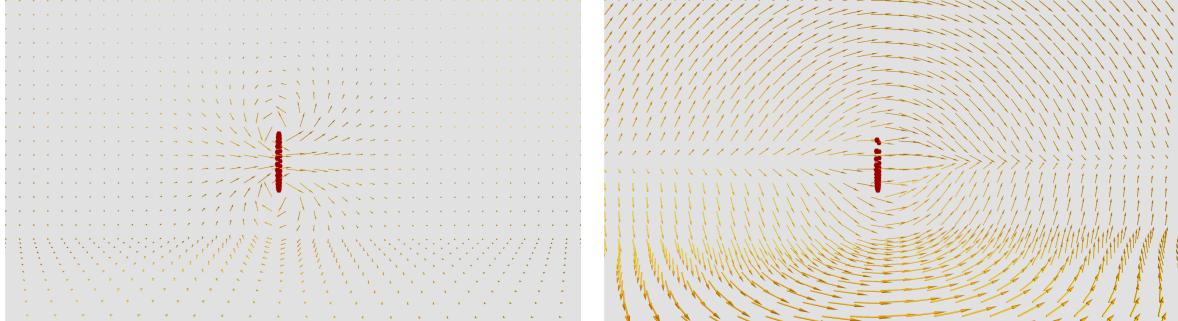


Figure 5: Scene of the sample program with the Current Loop preset, with $c = 4$ grid/s, Log Reduction Count $N = 2$, and 10-Exponent $n = 2$; other explanations from Fig. 3 apply here as well, unless otherwise stated. The electric field is not displayed here. **Left:** Magnetic field winding around the electric current generated by the rotating loop of positive charges at a non-relativistic speed. The magnetic flux through the loop appears as expected, with the direction of the flux being from left to right inside the loop and from right to left outside the loop. **Right:** At an ultra-relativistic speed of the charges, a vortex-like magnetic field is produced along the rotation axis. Both inside and outside the loop, the direction of the magnetic flux becomes from left to right. The charges on the upper (lower) side move toward (away from) the viewer, becoming sparse (dense) due to Lorentz stretching (contraction), as explained at the end of the caption for Fig. 3. In the real-time simulation, one can observe the dramatic transition from Left to Right.

2. Dynamic Opposite Charges: A pair of positive and negative dynamical charges.
3. Static Charge: A static charge; see Fig. 3.
4. Oscillating Charge: A positive charge undergoing predetermined harmonic oscillation; see Fig. 4.
5. Current Loop: A loop of positive charges is initially at rest. Its rotation accelerates towards the speed of light.¹⁸ The electric field is not displayed in the default setup to focus on the magnetic field. At non-relativistic speeds of the charges, the magnetic field exhibits the typical winding around the current, resulting in flux from left to right inside the loop and from right to left outside it. As the rotation of the charges accelerates, one can observe a dramatic transition to a vortex-like magnetic flux, with the flux direction being from left to right both inside and outside the loop.

In the first two presets, the dynamical charges adhere to the fully relativistic equation of motion as given in Eq.(93), with their time evolution outlined in Sec. 3.5.

6.4 Grid

For visualizing electromagnetic fields, two rendering options are provided:

¹⁸Specifically, we set the angular velocity $\frac{d\theta}{dt} = \frac{c}{r} \frac{1+\tanh(b(t-t_0))}{2}$, where r is the radius of the loop, t_0 shifts the initial time, and b is an arbitrary constant. The resultant angle behaves as $\theta(t) = \theta_0 + \frac{c}{2r} \left[t - t_0 + \frac{\ln \cosh(b(t-t_0))}{b} \right]$.

- Visualization only at lattice points located on two mutually perpendicular planes.
- Visualization at all lattice points within a three-dimensional grid.

6.5 Controls

On a tablet device, such as a smartphone, acceleration in the direction of sight is achieved by pinching out the screen, while pinching in results in acceleration in the opposite direction. Braking is accomplished by double-tapping the screen. The line of sight can be rotated by swiping.

When using a keyboard, constant acceleration towards the forward, backward, left, and right directions is achieved by pressing the W, S, A, and D keys, respectively. Similarly, upward and downward accelerations are controlled by the X and Z keys, respectively. Rolling is performed using the Q and E keys, while braking is achieved by pressing the R key.

7 Summary and discussion

We have presented a fully relativistic and Lorentz-covariant simulation framework for visualizing electromagnetic fields in special relativity. By incorporating the Lorentz transformation, light-cone structure, and the dynamics of charged particles, our approach enables users to directly observe the relativistic intermixing of electric and magnetic fields from the viewpoint of a moving observer.

In our simulation, the electromagnetic field at each spacetime point is determined by the motion of point charges at the intersection of their worldlines with the PLC of that point, ensuring causal consistency and Lorentz covariance. Each charged particle—including the observer—then experiences the Lorentz force from the field evaluated at its own spacetime location. Finally, the observer perceives both the fields and other particles as they appear on their PLC, constructing a consistent visual world from relativistic principles.

A key feature of our simulation is the visualization of observer-dependent electromagnetic fields. Users can witness how electric and magnetic components transform under changes in velocity, thereby gaining intuition for time dilation, length contraction, and the unity of electric and magnetic phenomena. The simulation also illustrates the Lorentz stretching effect—an often-overlooked consequence of field transformations—offering a more comprehensive view of relativistic dynamics.

Beyond its technical contributions, this simulation serves as a pedagogical tool to make abstract concepts in relativity and electromagnetism more accessible. Its interactive nature promotes intuitive understanding and offers potential for use in both classroom and outreach settings.

Future extensions may include incorporating gravitational effects or enhancing the visualization for broader engagement. We hope this work will contribute to both scientific communication and deeper conceptual insight into the relativistic structure of our physical world.

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