

ON THE MOTIVIC HOMOTOPY TYPE OF ALGEBRAIC STACKS

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ABSTRACT. We construct smooth presentations of algebraic stacks that are local epimorphisms in the Morel-Voevodsky \mathbb{A}^1 -homotopy category. As a consequence, we show that the motive of a smooth stack (in Voevodsky's triangulated category of motives) has many of the same properties as the motive of a smooth scheme.

1. INTRODUCTION

An important technique in motivic homotopy theory of algebraic stacks is reduction to the scheme case by means of homotopical descent. This is possible, for instance, when the stacks in question are Nisnevich locally quotient stacks. The results in [AHR19] (and further generalisation in [AHHLR24]) show that stacks with linearly reductive stabilisers are Nisnevich locally quotient stacks.

In this note, we establish a certain homotopy descent result for any algebraic stack. This allows us to conclude that the various formalisms of motives and motivic homotopy theory¹ produce the correct results for algebraic stacks. To wit, we will show that the motive of a smooth algebraic stack has similar properties as the motive of a smooth scheme. This generalises the results in [CDH20] that were established for Nisnevich locally quotient stacks. Another improvement, albeit minor, that we can make is to show that for algebraic stacks with separated diagonal the existing notions of stable homotopy category coincide: in [Cho21], it is shown that the stable motivic homotopy category of [Cho21] and the lisse-extended category of [KR21] are equivalent when the stack admits a smooth presentation with a Nisnevich local section (i.e., the *smooth-Nisnevich coverings* of [Pir18]). In Appendix A, we will show that all algebraic stacks admit such a presentation (Theorem A.1).

Definition 1.1. A *smooth-Nisnevich covering* of algebraic stacks is a morphism $f : \mathcal{Y} \rightarrow \mathcal{X}$ such that f is smooth and every morphism $\mathrm{Spec} K \rightarrow \mathcal{X}$ from the spectrum of a field K lifts to \mathcal{Y} .

When f is étale and \mathcal{Y} and \mathcal{X} are algebraic spaces, these are just the standard Nisnevich coverings.

The existence of smooth-Nisnevich coverings for algebraic stacks goes back to [LMB00, §6] (quasi-compact and separated diagonal) and [Čes15, Appendix B] (diagonal has relatively separated fibers). If the stack is of finite type over an infinite field with affine stabilizers, then it was shown in [Pir18] that there exists a smooth-Nisnevich covering of finite type. In Appendix A, we generalize all of these results by eliminating separation hypotheses for the existence result and noetherian hypotheses in the boundedness result; we also prove an analogous result for Deligne–Mumford stacks (Theorem A.1). It was brought to our attention during the preparation of our results that the existence result was recently proved in a similar way in [CD24].

The existence of such covers has implications for motivic homotopy theory of algebraic stacks. More specifically, it shows that the Nisnevich homotopy type of an algebraic stack can be described by a simplicial scheme (or simplicial algebraic space). This makes many homotopical descent arguments accessible for algebraic stacks.

Theorem 1.2. *Let $p : X \rightarrow \mathcal{X}$ be a smooth-Nisnevich covering over a field k . Let X_\bullet denote the Čech nerve of p . Then the morphism $p_\bullet : X_\bullet \rightarrow \mathcal{X}$ induces an equivalence in the Morel-Voevodsky \mathbb{A}^1 -homotopy category, $\mathcal{H}(k)$.*

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¹In the case of stacks one has to distinguish between *genuine* vs *Kan-extended* motivic spectra. For instance K -theory is *genuine* but Chow groups are *Kan-extended*. In this note we will be concerned with the latter kind of objects.

A consequence of the above result is that the motive of a smooth algebraic stack continues to enjoy the same properties as the motive of a smooth scheme (see Section 2). So far this was only known for stacks which satisfied local structure theorems in the sense of [AHR19, AHHLR24] or in a different direction for motives of smooth stacks in the étale topology. We describe various results of this kind in the text.

Remark 1.3. Another application of Theorem A.1 is the following: In [KM22], the authors use the boundedness result in Theorem A.1 to apply Elkik’s approximation technique to the Picard stack and study the following question of Grothendieck: when is the map

$$H^2(X, \mathbb{G}_m) \rightarrow \varprojlim H^2(X_n, \mathbb{G}_m)$$

injective for a proper morphism $X \rightarrow \mathrm{Spec} A$, where (A, \mathfrak{m}) be a complete Noetherian ring and $X_n := X \times_A \mathrm{Spec} A/\mathfrak{m}^{n+1}$ is the n -th infinitesimal thickening.

Conventions. We work with algebraic stacks in the sense of [Sta18]; that is, without separation assumptions unless noted otherwise.

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2. THE MOTIVE OF AN ALGEBRAIC STACK

Let k be a field. Let Sm/k denote the category of smooth schemes over k . Let $\Delta^op PSh(Sm/k)$ be the category of simplicial presheaves on Sm/k . Then $\Delta^op PSh(Sm/k)$ has a local model structure with respect to the Nisnevich topology (see [Jar87]). A morphism $f : X \rightarrow Y$ in $\Delta^op PSh(Sm/k)$ is a weak equivalence if the induced morphisms on stalks (for the Nisnevich topology) are weak equivalences of simplicial sets. Cofibrations are monomorphisms, and fibrations are characterised by the right lifting property.

We Bousfield localise this model structure with respect to the class of maps $X \times \mathbb{A}^1 \rightarrow X$ (see [MV99, 3.2]). The resulting model structure is called the Nisnevich motivic model structure. Denote by $\mathcal{H}(k)$ the resulting homotopy category. This is the (unstable) \mathbb{A}^1 -homotopy category for smooth schemes over k .

Let us also rapidly recall the definition of Voevodsky’s triangulated category of motive $\mathbf{DM}^{eff}(k, \mathbb{Z})$. Let Cor_k denote the category of finite correspondences whose objects are smooth separated schemes over k . For any two X, Y , the morphisms of Cor_k are given by $Cor(X, Y)$ which is the free abelian group generated by irreducible closed subschemes $W \subset X \times Y$ that are finite and surjective over X . An additive functor $F : Cor_k^{op} \rightarrow \mathbf{Ab}$ is called a presheaf with transfers. Let $PST(k, \mathbb{Z})$ denote the category presheaves with transfers. For any smooth scheme X , let $\mathbb{Z}_{tr}(X)$ be the presheaf with transfers which on any smooth scheme Y is defined as

$$\mathbb{Z}_{tr}(X)(Y) := Cor(X, Y)$$

Let $K(PST(k, \mathbb{Z}))$ denote the category of complexes of presheaves with transfers. The category $K(PST(k, \mathbb{Z}))$ also has a Nisnevich motivic model structure which is defined analogously as in the case of $\Delta^op PSh(Sm/k)$. We denote the associated homotopy category by $\mathbf{DM}^{eff}(k, \mathbb{Z})$. This is Voevodsky’s triangulated category of mixed motives in the Nisnevich topology (for details, see [MVW06]).

We now have the following trivial observation.

Lemma 2.1. *Let $p: X \rightarrow \mathcal{X}$ be a smooth-Nisnevich covering of algebraic stacks. If $\mathrm{Spec} \mathcal{O}$ is the spectrum of a Henselian local ring \mathcal{O} , then every morphism $\mathrm{Spec} \mathcal{O} \rightarrow \mathcal{X}$ factors through p .*

Proof. Let K be the residue field of \mathcal{O} and $X_{\mathcal{O}} := X \times_{\mathcal{X}} \mathrm{Spec} \mathcal{O}$ denote the base change of X to $\mathrm{Spec} \mathcal{O}$. Then the composite $\mathrm{Spec} K \rightarrow \mathrm{Spec} \mathcal{O} \rightarrow \mathcal{X}$ lifts to a morphism $\mathrm{Spec} K \rightarrow X$. This also gives us a morphism $\mathrm{Spec} K \rightarrow X_{\mathcal{O}}$. Now, as $X_{\mathcal{O}} \rightarrow \mathrm{Spec} \mathcal{O}$ is smooth, we have a surjection of sets $X_{\mathcal{O}}(\mathcal{O}) \twoheadrightarrow X_{\mathcal{O}}(K)$. Thus, we have a section $\mathrm{Spec} \mathcal{O} \rightarrow X_{\mathcal{O}}$. Composing with the projection map $X_{\mathcal{O}} \rightarrow X$ gives us the desired lift. \square

Proposition 2.2. *Let \mathcal{X} be an algebraic stack that is locally of finite type over a field k . Let $X \rightarrow \mathcal{X}$ be a smooth-Nisnevich covering. Then the associated Čech hypercover X_{\bullet} is weakly equivalent to \mathcal{X} in the Nisnevich local model structure on the category $\Delta^{\mathrm{op}} \mathrm{PSh}(\mathrm{Sm}/k)$ of simplicial presheaves.*

Proof. By Lemma 2.1, we know that hensel local points lift along smooth-Nisnevich coverings. Adapting the proof of [CDH20, Theorem 1.2] gives the result. \square

The proof of Theorem 1.2 follows from \mathbb{A}^1 -localising the above proposition.

Proof of Theorem 1.2. As \mathbb{A}^1 -localisation preserves simplicial equivalences, the result follows from the previous proposition. \square

As a consequence, we have the following result.

Corollary 2.3. *The weak equivalence above induces an equivalence of motives $M(X_{\bullet}) \simeq M(\mathcal{X})$ in $\mathbf{DM}^{\mathrm{eff}}(k, \mathbb{Z})$.*

Proof. Use the functor $M: \mathcal{H}_{\bullet}(k) \rightarrow \mathbf{DM}^{\mathrm{eff}}(k, \mathbb{Z})$ (see [CDH20] for details). \square

Combining these results with Theorem A.1, we obtain motivic results for smooth algebraic stacks. The proofs are exactly as in [CDH20] after replacing GL_n -presentations with smooth-Nisnevich presentations.

Theorem 2.4. *Let \mathcal{X} be an algebraic stack that is smooth over a field k . Then its motive $M(\mathcal{X}) \in \mathbf{DM}^{\mathrm{eff}}(k, \mathbb{Z})$ satisfies the following properties:*

- (1) $M(\mathcal{X})$ satisfies Nisnevich descent.
- (2) (Projective bundle formula) For any vector bundle \mathcal{E} of rank $n+1$ over \mathcal{X} , we have a canonical isomorphism

$$M(\mathbb{P}\mathrm{roj}(\mathcal{E})) \simeq \bigoplus_{i=0}^n M(\mathcal{X})(i)[2i]$$

- (3) (Blow-up formula) For $\mathcal{Z} \subset \mathcal{X}$ a smooth closed substack of pure codimension c we have

$$M(\mathrm{Bl}_{\mathcal{Z}}(\mathcal{X})) \simeq M(\mathcal{X}) \oplus_{i=0}^{c-1} M(\mathcal{Z})(i)[2i]$$

- (4) (Gysin triangle) For $\mathcal{Z} \subset \mathcal{X}$ a smooth closed substack of codimension c , we have a Gysin triangle:

$$M(\mathcal{X} \setminus \mathcal{Z}) \rightarrow M(\mathcal{X}) \rightarrow M(\mathcal{Z})(c)[2c] \rightarrow M(\mathcal{X} \setminus \mathcal{Z})[1].$$

Proof. The proofs in [CDH20, §4] go through using a smooth-Nisnevich covering $X \rightarrow \mathcal{X}$. \square

Remark 2.5. Note that the transfer functor $M: \mathcal{H}_{\bullet}(k) \rightarrow \mathbf{DM}^{\mathrm{eff}}(k, \mathbb{Z})$ allows us to define the motive of *any* smooth stack. What is apriori unclear is whether this motive has any properties (good or bad). This is why in [CDH20], the authors work with stacks that are Nisnevich locally quotient stacks in order to use homotopical descent to prove the above formulae for the motive. With Theorem 1.2, those arguments now work for all smooth stacks.

3. MOTIVIC COHOMOLOGY

3.1. Hypercohomology. We briefly recall the definitions of smooth-Nisnevich and smooth-Zariski sites of an algebraic stack.

Definition 3.1. Let \mathcal{X} be an algebraic stack. The smooth-Nisnevich (resp. smooth-Zariski) site of \mathcal{X} , denoted by $\mathcal{X}_{\text{lis-nis}}$ (resp. $\mathcal{X}_{\text{lis-zar}}$) is the category whose objects consist of pairs (U, p) where p is a smooth morphism $p : U \rightarrow \mathcal{X}$ from an algebraic space U and coverings in $\mathcal{X}_{\text{lis-nis}}$ are given by Nisnevich coverings (resp. Zariski coverings) of algebraic spaces.

The following result states that motivic cohomology can be computed as hypercohomology of the motivic complexes $\mathbb{Z}(i)$ on the smooth-Nisnevich or smooth-Zariski (in the Deligne–Mumford case) site of the stack.

Proposition 3.2. *Let \mathcal{X} be an algebraic stack that is smooth over a field k . The motivic cohomology of \mathcal{X} agrees with the hypercohomology of the motivic complexes $\mathbb{Z}(j)$ on $\mathcal{X}_{\text{lis-nis}}$. That is,*

$$\text{Ext}_{D(\mathcal{X})}^i(\mathbb{Z}, \mathbb{Z}(j)|_{\mathcal{X}}) \simeq \text{Ext}_{DA(\mathcal{X})}^i(\mathbb{Z}, \mathbb{Z}(j)|_{\mathcal{X}}) \simeq \text{Hom}_{\mathbf{DM}^{\text{eff}}(k, \mathbb{Z})}(M(\mathcal{X}), \mathbb{Z}(j)[i]),$$

where \mathbb{Z} denotes the constant sheaf \mathbb{Z} on the $\mathcal{X}_{\text{lis-nis}}$. In addition, if \mathcal{X} is separated and Deligne–Mumford with schematic coarse space, then the motivic cohomology of \mathcal{X} agrees with the hypercohomology of the motivic complexes $\mathbb{Z}(j)$ on $\mathcal{X}_{\text{lis-zar}}$.

Proof. See [CDH20, Theorem 5.2] for the smooth-Nisnevich case. In the Deligne–Mumford case, it follows from [Kre09], where it is shown that such stacks are Zariski-locally quotient stacks. \square

3.2. Motivic cohomology with finite coefficients. We also have the following comparison theorem relating motivic cohomology with $\mathbb{Z}/n\mathbb{Z}$ to étale cohomology with μ_n -coefficients.

Corollary 3.3. *Let \mathcal{X} be an algebraic stack that is smooth over a field k . Then the homomorphisms*

$$H_M^{p,q}(\mathcal{X}, \mathbb{Z}/n\mathbb{Z}) \rightarrow H_{\text{ét}}^p(\mathcal{X}, \mu_n^{\otimes q}),$$

are isomorphisms for $p \leq q$ and monomorphisms for $p = q + 1$.

Proof. See [CDH20, Corollary 5.3]. \square

3.3. Motives with compact support. In this subsection, we will define the notion of a motive with compact support for smooth algebraic stacks with affine stabilizers over a field k . The definition is very similar to Totaro’s construction for quotient stacks using approximation by vector bundles [Tot99], except that since we cannot approximate non-quotient stacks by algebraic spaces, we are forced to work with homotopy limits. We will use the definition of dimension for algebraic stacks as in [Sta18, Tag 0AFL]

Definition 3.4. Let \mathcal{X} be a connected algebraic stack of finite type over a field k with affine stabilizers. Consider a smooth-Nisnevich covering $p : X \rightarrow \mathcal{X}$, where X is of finite type over k (Theorem A.1) and let n denote the relative dimension of $p : X \rightarrow \mathcal{X}$. Let $p_{\bullet} : X_{\bullet} \rightarrow \mathcal{X}$ be the Čech nerve of p . Then we define the *compactly supported motive* of \mathcal{X} as

$$M^c(\mathcal{X}) := \text{holim}_i M^c(X_i)(-(i+1)n^2)[-2n^2i].$$

This definition is a bit weird looking, but it has the following pleasant feature, which also shows that the definition is independent of choices for smooth algebraic stacks.

Theorem 3.5 (Poincaré Duality). *Let \mathcal{X} be a smooth algebraic stack of dimension d . Then we have an isomorphism,*

$$M^c(\mathcal{X}) = M(\mathcal{X})^{\vee}(d)[2d].$$

Proof. The proof is a straightforward manipulation of the definition. Tensoring by the dualizable object $\mathbb{Z}(-d)[-2d]$ we get:

$$\begin{aligned} M^c(\mathcal{X}) \otimes \mathbb{Z}(-d)[-2d] &= (\operatorname{holim}_i M^c(X_i)(-(i+1)n^2)[-2n^2i]) \otimes \mathbb{Z}(-d)[-2d] \\ &= \operatorname{holim}_i M^c(X_i)(-(i+1)n^2 - d)[-2n^2i - 2d] \\ &= \operatorname{holim}_i M(X_i)^\vee. \end{aligned}$$

where the last step follows from the fact that each X_i is smooth. Further, since $M(\mathcal{X}) \simeq \operatorname{hocolim}_i M(X_i)$, taking dual we get that $M(\mathcal{X})^\vee = \operatorname{holim}_i M(X_i)^\vee$. Thus, we have proved that

$$M^c(\mathcal{X})(-d)[2d] \simeq M(\mathcal{X})^\vee.$$

Tensoring by $\otimes \mathbb{Z}(d)[2d]$, gives the required expression. \square

In the following subsection, we note an application to stable motivic homotopy theory of stacks. We will not explain any details to prevent drowning the reader in ∞ -categories. Our intention is to simply indicate how the geometric content of Theorem A.1 can be applied in the world of motivic homotopy theory. The interested reader may consult [Cho21] or [KR21] for further details.

4. THE STABLE MOTIVIC HOMOTOPY CATEGORY OF A STACK

Recently, two different definitions of the stable motivic homotopy category for stacks have appeared in literature. The limit-extended category $SH_{\triangleleft}(\mathcal{X})$ in [KR21] and the category $SH_{ext}^{\otimes}(\mathcal{X})$ in [Cho21] that is extended from schemes to stacks having Nisnevich local sections. In [Cho21] it is proved that the two definition are equivalent whenever the stack admits a smooth-Nisnevich covering. As coverings such always exist for algebraic stacks by Theorem A.1, we have a strengthening of [Cho21, Corollary 2.5.4].

Definition 4.1. Let \mathcal{X} be an algebraic stack. A smooth presentation $X \rightarrow \mathcal{X}$ is said to admit *Nisnevich local sections* if, for any scheme Y and a morphism $Y \rightarrow \mathcal{X}$, there exists a Nisnevich covering $Y' \rightarrow Y$ with a section $Y' \rightarrow Y \times_{\mathcal{X}} X$. Thus, we have the following commutative diagram,

$$\begin{array}{ccccc} & & Y \times_{\mathcal{X}} X & \longrightarrow & X \\ & \nearrow & \downarrow & & \downarrow \\ Y' & \longrightarrow & Y & \longrightarrow & \mathcal{X} \end{array}$$

The following lemma shows that the above definition is equivalent to the notion of a smooth-Nisnevich covering.

Lemma 4.2. *Let \mathcal{X} be an algebraic stack. A smooth presentation $f : X \rightarrow \mathcal{X}$ admits Nisnevich local sections if and only if it is a smooth-Nisnevich covering.*

Proof. The only if part is clear. To see the if part, observe that if $X \rightarrow \mathcal{X}$ is a smooth-Nisnevich covering, then by Cor 2.7, for any Henselian local ring \mathcal{O} with a map $\operatorname{Spec} \mathcal{O} \rightarrow \mathcal{X}$ we have a lift $\operatorname{Spec} \mathcal{O} \rightarrow X$. Thus, if $V \rightarrow \mathcal{X}$ is a morphism and $\mathcal{O}_{v,V}$ is the Henselian local ring at a point $v \in V$, we have a lift $\operatorname{Spec} \mathcal{O}_{v,V} \rightarrow V \times_{\mathcal{X}} X$. By standard limit arguments [EGA], there exists a Nisnevich neighbourhood V' of $v \in V$, and a section $V' \rightarrow V \times_{\mathcal{X}} X$. \square

Corollary 4.3. *Let \mathcal{X} be a quasi-separated algebraic stack. Then $SH_{\triangleleft}(\mathcal{X}) \simeq SH_{ext}^{\otimes}(\mathcal{X})$, whenever they are defined.*

Proof. The proof in [Cho21, Corollary 2.5.4] works verbatim using a smooth-Nisnevich covering $X \rightarrow \mathcal{X}$. \square

Remark 4.4. In fact, Theorem A.1 implies several other interesting facts about $SH(\mathcal{X})$. In forthcoming work (with Felix Sefzig), we will show that the Morel-Voevodsky construction on the smooth-Nisnevich site of \mathcal{X} recovers $SH_{ext}^{\otimes}(\mathcal{X})$. A similar result for the Nis-loc topology on \mathcal{X} has been obtained in [CD24] and as observed in *loc. cit.*, the smooth-Nisnevich and the Nis-loc constructions

coincide (essentially due to Lemma 4.2). In the same forthcoming work, we will also present a model for framed motivic spectra $SH^{fr}(\mathcal{X})$, which implies that the motivic cohomology spectrum over \mathcal{X} can still be described as the framed suspension of the constant sheaf \mathbb{Z} with transfers as described in [Hoy21].

APPENDIX A. SMOOTH-NISNEVICH COVERS (JOINT WITH JACK HALL)

The main result of this appendix is the following.

Theorem A.1. *If \mathcal{X} is an algebraic stack, then there exists a smooth-Nisnevich covering $p: X \rightarrow \mathcal{X}$, where X is a scheme. If \mathcal{X} is quasi-compact and quasi-separated with affine stabilizers, then we may take X to be affine. Moreover, if \mathcal{X} is Deligne–Mumford, then we may take p to be étale.*

Nisnevich coverings for algebraic stacks were discussed in [HR18, §3]. The difference between smooth-Nisnevich and Nisnevich coverings is that the latter morphisms are étale and require a lift of the residual gerbe—not just a rational point. Like their Nisnevich counterparts, however, smooth-Nisnevich coverings are clearly stable under composition and base change. We begin with some examples.

Example A.2. Let $f: X \rightarrow S$ be a smooth morphism of stacks, where S is a quasi-separated algebraic space. Then every point $s \in |S|$ has a residue field $\kappa(s)$ [Sta18, Tag 03JV]. Hence, f is a smooth-Nisnevich covering if and only if for every $s \in |S|$, the morphism $f_s: X \times_S \text{Spec } \kappa(s) \rightarrow \text{Spec } \kappa(s)$ admits a section. In particular, if X is also an algebraic space, then étale smooth-Nisnevich coverings are equivalent to Nisnevich covers in the sense of [HR18, §3]. Thus, if \mathcal{X} is a quasi-separated algebraic space, then Theorem A.1 holds for \mathcal{X} [RG71, Prop. 5.7.6].

Example A.3. $\text{Spec } \mathbb{Z} \rightarrow B\text{GL}_{n,\mathbb{Z}}$ is a smooth-Nisnevich covering. More generally, if $Y \rightarrow S$ is a morphism of quasi-separated algebraic spaces such that Y admits an S -action of $\text{GL}_{n,S}$, then Theorem A.1 holds for the quotient stack $[Y/\text{GL}_{n,S}]$. Indeed, passing to a Nisnevich covering of S , we may assume that S is an affine scheme. Then $Y \rightarrow [Y/\text{GL}_{n,S}]$ is a smooth-Nisnevich morphism (it is the base change of $\text{Spec } \mathbb{Z} \rightarrow B\text{GL}_n$). Passing to a Nisnevich étale cover of Y finishes the argument. This holds more generally for $[Y/G]$, where $G \rightarrow S$ is a smooth group algebraic space such that $H^1(\text{Spec } K)_{\text{et}}, G_{\text{Spec } K} = 0$ for all $\text{Spec } K \rightarrow S$.

Example A.4. If \mathcal{X} is an algebraic stack with quasi-finite diagonal, then Theorem A.1 holds for \mathcal{X} . We may of course assume that \mathcal{X} is quasi-compact. By [HR18, Thm. 4.1] there exist morphisms of algebraic stacks $V \xrightarrow{v} \mathcal{W} \xrightarrow{w} \mathcal{X}$ such that V is an affine scheme, v is finite and faithfully flat of finite presentation, and w is a Nisnevich étale covering. Hence, we may replace \mathcal{X} by \mathcal{W} and assume that \mathcal{X} admits a finite and faithfully flat cover of finite presentation by an affine scheme V . By [Gro17, Prop. 4.3(vii)], \mathcal{X} has the resolution property and so the Totaro–Gross Theorem [Gro17, Thm. 5.4] implies that $\mathcal{X} \simeq [X/\text{GL}_n]$, where X is quasi-affine. Now apply Example A.3.

Let $f: X \rightarrow S$ be a morphism of algebraic stacks. A *splitting cover* for f is a sequence of quasi-compact immersions $Z_j \hookrightarrow S$ for $j = 1, \dots, r$ with $|S| = \bigcup_{j=1}^r |Z_j|$ such that $f_{Z_j}: X \times_S Z_j \rightarrow Z_j$ admits a section for each j .

Remark A.5. If S is quasi-compact, then the existence of a splitting cover for f is equivalent to the existence of a *splitting sequence* for f ; that is, there is a filtration $\emptyset = S_0 \subseteq S_1 \subseteq \dots \subseteq S_t = S$ by quasi-compact open subsets such that $f_{S_i \setminus S_{i-1}}: X \times_S (S_i \setminus S_{i-1}) \rightarrow S_i \setminus S_{i-1}$ admits a section for each $i = 1, \dots, t$. The equivalence is due to the close relationship between partitions and filtrations in the constructible topology [Sta18, Tag 09XY].

The following lemma is a smooth analog of [HR18, Prop. 3.3].

Lemma A.6. *Let $f: X \rightarrow S$ be a smooth morphism of algebraic stacks. If S is a quasi-compact and quasi-separated algebraic space, then f is a smooth-Nisnevich covering if and only if f admits a splitting cover.*

Proof. Clearly, if f has a splitting cover, then it is a smooth-Nisnevich covering. For the other direction: let $s \in |S|$ be a point. By Example A.2, $f_s: X \times_S \text{Spec } \kappa(s) \rightarrow \text{Spec } \kappa(s)$ admits a section. By [HR18, Lem. 2.1] and [Ryd15, Prop. B.2 & B.3], there exists an immersion $V_s \hookrightarrow S$ of finite presentation such that $s \in |V_s|$ and a section to $f_{V_s}: X \times_S V_s \rightarrow V_s$. Since the V_s are constructible, S is covered by finitely many and so there are $s_j \in |S|$, $j = 1, \dots, r$ such that $\cup_{j=1}^r |V_{s_j}| = |S|$. \square

The following Proposition leverages splitting coverings to obtain a boundedness result.

Proposition A.7. *Consider a cartesian square of algebraic stacks:*

$$\begin{array}{ccc} X' & \xrightarrow{x} & X \\ f' \downarrow & & \downarrow f \\ S' & \xrightarrow{s} & S. \end{array}$$

Assume that

- (1) *f and s are smooth-Nisnevich coverings; and*
- (2) *X is a quasi-compact and quasi-separated algebraic space.*

Then there exists a quasi-compact open $S'' \subseteq S'$ such that $S'' \rightarrow S$ is a smooth-Nisnevich covering.

Proof. Since X is a quasi-compact algebraic space and x is a smooth-Nisnevich covering, it follows from Lemma A.6 that x admits a splitting cover $|X| = \cup_{j=1}^r |W_j|$ with sections $t_j: W_j \rightarrow X \times_S W_j$ to $f_{W_j}: X \times_S W_j \rightarrow W_j$. Since x is quasi-separated and locally of finite presentation, the sections t_j are of finite presentation. But the W_j are quasi-compact, so the subsets $t_j(W_j) \subseteq X'$ are all constructible (Chevalley's Theorem). Write $S' = \cup_{i \in I} S_i$ as an increasing union of quasi-compact open subsets. Since $t_j(W_j)$ is constructible and f' is quasi-compact, for each j there is an i_j such that $t_j(W_j) = f'^{-1}(S') \cap t_j(W_j) = f'^{-1}(S_{i_j}) \cap t_j(W_j)$; in other words, $t_j(W_j) \subseteq f'^{-1}(S_{i_j})$. Take $v = \max\{i_j\}$ and set $S'' = S_v$ and $X'' = f'^{-1}(S'') \subseteq X$. Then the induced morphism $X'' \rightarrow X$ is a smooth-Nisnevich covering (Lemma A.6). Thus, $X'' \rightarrow S$ is a smooth-Nisnevich covering and so $S'' \rightarrow S$ is a smooth-Nisnevich covering. \square

Let $f: X \rightarrow S$ be a morphism of algebraic stacks. Let $\underline{\text{HS}}_f \rightarrow S$ be the *Hilbert stack* of f , which parameterizes quasi-finite and representable morphisms to X that are proper over the base. If f has affine stabilizers with quasi-compact and separated diagonal, then $\underline{\text{HS}}_f \rightarrow S$ is a morphism of algebraic stacks with quasi-affine diagonal, which is locally of finite presentation if f is so [HR14, HR15b]. As noted in [HR18, Thm. 5.1], if we let $\underline{\text{HS}}_f^{\text{qfb}} \subseteq \underline{\text{HS}}_f$ be the open substack parameterizing those families that are quasi-finite over S , then these existence results are much simpler to prove. Let $\underline{\text{HS}}_f^{\text{étb}} \subseteq \underline{\text{HS}}_f^{\text{qfb}}$ be the substack parameterizing those families that are étale over S . If f is a morphism of algebraic spaces, then $\underline{\text{HS}}_f^{\text{étb}} \simeq [(X/S)^d/S_d]$ [Ryd11, Thm. 5.1], where $(X/S)^d$ denotes the d -fold fiber product of X over S and S_d acts on this product via permutation. If f is a separated morphism of algebraic spaces, then there is an open subspace $\underline{\text{Hilb}}_f^{\text{étb}} \subseteq \underline{\text{HS}}_f^{\text{étb}}$ parameterizing closed immersions into X and $\underline{\text{Hilb}}_f^{\text{étb}} \subseteq \underline{\text{HS}}_f^{\text{étb}}$ corresponds to the open subset of $[(X/S)^d/S_d]$ that is the complement of the diagonals [Ryd11, Thm. 5.1]. The following result summarizes the relevant results of [Ryd11] and can be viewed as a smooth variant of [HR18, §§5-6].

Proposition A.8. *Let $f: X \rightarrow S$ be a representable smooth covering of algebraic stacks.*

- (1) *$\underline{\text{HS}}_f^{\text{étb}} \rightarrow S$ is relatively Deligne–Mumford with separated diagonal, and a smooth-Nisnevich covering.*
 - (a) *If f is étale, then $\underline{\text{HS}}_f^{\text{étb}} \rightarrow S$ is étale.²*
 - (b) *If X is a Deligne–Mumford stack (with separated diagonal), then so is $\underline{\text{HS}}_f^{\text{étb}}$.*
- (2) *If f is separated, then $\underline{\text{Hilb}}_f^{\text{étb}} \rightarrow S$ is representable, separated, and a smooth-Nisnevich covering.*

²If X and S are quasi-separated, then f is even a Nisnevich covering [HR18, Proof of Thm. 4.1].

- (a) If X is a (separated) algebraic space, then so is $\underline{\mathrm{Hilb}}_f^{\text{étb}}$.
- (b) If X is a (separated) scheme and f is étale, then so is $\underline{\mathrm{Hilb}}_f^{\text{étb}}$.

Proof. The first parts of (1) and (2) are smooth-local on S and follow trivially from the explicit descriptions above when f is a morphism of algebraic spaces. To see that $\underline{\mathrm{HS}}_f^{\text{étb}} \rightarrow S$ is a smooth-Nisnevich covering, let $s: \mathrm{Spec} K \rightarrow S$ be a morphism, where K is a field. Since $f: X \rightarrow S$ is a smooth covering, there is a finite separable field extension $K \subseteq K'$ together with a lift $x: \mathrm{Spec} K' \rightarrow X$ of s . That is, we have $\mathrm{Spec} K' \rightarrow X \times_S \mathrm{Spec} K \rightarrow \mathrm{Spec} K$, which corresponds to a lift of x to $\underline{\mathrm{HS}}_f^{\text{étb}}$. If $X \rightarrow S$ is separated, then $\underline{\mathrm{Hilb}}_f^{\text{étb}} \rightarrow S$ is a smooth-Nisnevich covering as we may replace $\mathrm{Spec} K'$ to be the residue field of the closed point in its image in $X \times_S \mathrm{Spec} K$. Next, the universal family gives us a diagram:

$$\begin{array}{ccccc} \mathcal{Z} & \longrightarrow & X \times_S \underline{\mathrm{HS}}_f^{\text{étb}} & \longrightarrow & X \\ & \searrow & \downarrow & & \downarrow \\ & & \underline{\mathrm{HS}}_f^{\text{étb}} & \longrightarrow & S. \end{array}$$

If X is Deligne–Mumford (with separated diagonal), then $X \times_S \underline{\mathrm{HS}}_f^{\text{étb}}$ is Deligne–Mumford (with separated diagonal). The map $\mathcal{Z} \rightarrow X \times_S \underline{\mathrm{HS}}_f^{\text{étb}}$ is quasi-finite, separated and representable and so \mathcal{Z} is Deligne–Mumford (with separated diagonal). But $\mathcal{Z} \rightarrow \underline{\mathrm{HS}}_f^{\text{étb}}$ is finite étale and surjective and so $\underline{\mathrm{HS}}_f^{\text{étb}}$ is Deligne–Mumford (with separated diagonal). Finally, (2a) follows from [Ryd11, Thm. 4.1] and (2b) follows from [Ryd11, Rem. 2.3 & Thm. 2.4]. \square

Proof of Theorem A.1. For the existence: let $v: V \rightarrow \mathcal{X}$ be a smooth cover, where V is a scheme (take v to be étale if \mathcal{X} is Deligne–Mumford). Then v is representable and let $w: W = \underline{\mathrm{HS}}_v^{\text{étb}} \rightarrow \mathcal{X}$ be the induced morphism. Proposition A.8(1) implies that w is a smooth-Nisnevich covering and W is a Deligne–Mumford stack with separated diagonal (if \mathcal{X} is a Deligne–Mumford stack, w is even étale). Replacing \mathcal{X} by W we are reduced to the situation where \mathcal{X} is Deligne–Mumford with separated diagonal. In this case, v is separated, representable, and étale. Then Proposition A.8(2) implies that $p: X = \underline{\mathrm{Hilb}}_v^{\text{étb}} \rightarrow \mathcal{X}$ is an étale smooth-Nisnevich covering and X is a separated scheme.

For the boundedness: by [HR15a, Prop. 2.6(i)], $|\mathcal{X}|$ admits a finite partition $\coprod_{j=1}^r |\mathcal{W}_j|$, where $\mathcal{W}_j = [W_j/\mathrm{GL}_{n_j}]$ and W_j is a quasi-affine scheme. For each j form the following cartesian diagram:

$$\begin{array}{ccc} W'_j & \xrightarrow{p'_j} & W_j \\ q'_j \downarrow & & \downarrow q_j \\ X \times_{\mathcal{X}} \mathcal{W}_j & \xrightarrow{p_j} & \mathcal{W}_j. \end{array}$$

Then p_j is a smooth-Nisnevich covering and so too is q_j (Example A.3). By Proposition A.7, it follows that there is a quasi-compact open $W''_j \subseteq X \times_{\mathcal{X}} \mathcal{W}_j$, which is also a scheme, such that $W''_j \rightarrow \mathcal{W}_j$ is a smooth-Nisnevich cover. Now take a quasi-compact open $X'' \subseteq X$ such that $W''_j \subseteq X'' \times_{\mathcal{X}} \mathcal{W}_j$ for all j . Then $X'' \rightarrow \mathcal{X}$ is a smooth-Nisnevich cover, which proves the claim. \square

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