# A note on nonlocal discrete problems involving sign-changing Kirchhoff functions

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**Abstract.** In this note, we establish a multiplicity theorem for a nonlocal discrete problem of the type

$$\begin{cases} -\left(a\sum_{m=1}^{n+1}|x_m - x_{m-1}|^2 + b\right)(x_{k+1} - 2x_k + x_{k-1}) = h_k(x_k) & k = 1, ..., n, \\ x_0 = x_{n+1} = 0 & \end{cases}$$

assuming a > 0 and (for the first time) b < 0.

**Keywords.** Minimax theorems; Kirchhoff functions; difference equations; variational methods; multiplicity of solutions.

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#### 1. Introduction

If  $\Omega$  is a bounded domain of  $\mathbf{R}^n$  and  $K:[0,+\infty[\to\mathbf{R},\,\varphi:\Omega\times\mathbf{R}\to\mathbf{R}]$  are two given functions, the nonlocal problem

$$\left\{ \begin{aligned} -K(\int_{\Omega}|\nabla u(x)|^2)\Delta u &= \varphi(x,u) &&\text{in } \Omega\\ u &= 0 &&\text{on } \partial\Omega \end{aligned} \right.$$

is certainly among the most studied ones in today's nonlinear analysis (we refer to [7] for an introduction to the subject).

In checking the relevant literature, one can realize that, in the majority of the papers, one assumes K(t) = at + b with a > 0 and  $b \ge 0$  and, in any case, that the Kirchhoff function K is assumed to be, in particular, continuous and non-negative in  $[0, +\infty[$ .

However, it is natural to ask what happens when at least one of these properties fails.

The case where K can be discontinuous in  $[0, +\infty[$  has been considered for the first time in [4], for n = 1, and then in [5] for the general case (see also [1], [2], [3]). In these papers, however, K is non-negative.

The papers dealing with a sign-changing function K are more numerous, but in each of them it is assumed that K(t) = at + b with a < 0 and  $b \ge 0$ . The first of these papers was [8].

In the present very short note, we are interested in the discrete counterpart of the above problem. That is to say, given n continuous functions  $f_k : \mathbf{R} \to \mathbf{R}$  (k = 1, ..., n), we deal with the problem

$$\begin{cases}
-K\left(\sum_{h=1}^{n+1}|x_h-x_{h-1}|^2\right)(x_{k+1}-2x_k+x_{k-1}) = f_k(x_k) & k=1,...,n, \\
x_0 = x_{n+1} = 0.
\end{cases}$$

Also for this discrete problem, we can repeat what we said before, even in a stronger way: it seems that in each paper on the subject, the function K is continuous and non-negative in  $[0, +\infty[$ .

Our aim is to establish a multiplicity result for this problem where (for the first time) the Kirchhoff function K changes sign.

## 2. Results

Before stating our result, we recall the following two theorems which will be key tools used in our proof.

THEOREM 2.A.([6]) - Let X be a topological space, let Y be a convex set in a topological vector space and let  $h: X \times Y \to \mathbf{R}$  be lower semicontinuous and inf-compact in X, and continuous and quasi-concave in Y. Also, assume that

$$\sup_{V} \inf_{X} h < \inf_{X} \sup_{V} h .$$

Moreover, let  $\varphi: X \to \mathbf{R}$  be a lower semicontinuous function such that

$$\sup_X \varphi - \inf_X \varphi < \inf_X \sup_Y h - \sup_Y \inf_X h.$$

Then, for each convex set  $S \subseteq Y$ , dense in Y, there exists  $\tilde{y} \in S$  such that the function  $h(\cdot, \tilde{y}) + \varphi(\cdot)$  has at least two global minima.

THEOREM 2.B.([6]) - Let X be a topological space, let H be a real Hilbert, let Y be a closed ball in H centered at 0, and let  $Q: X \to \mathbf{R}, \ \psi: X \to H$ . Assume that the functional  $x \to Q(x) - \langle \psi(x), y \rangle$  is lower semicontinuous for each  $y \in Y$ , while the functional  $x \to Q(x) - \langle \psi(x), y_0 \rangle$  is inf-compact for some  $y_0 \in Y$ . Moreover, assume that, for each  $x \in X$ , there exists  $u \in X$  such that

$$Q(x) = Q(u)$$

and

$$\psi(x) = -\psi(u) .$$

Finally, assume that there is no global minimum of Q at which  $\psi$  vanishes.

Then, we have

$$\sup_{y \in Y} \inf_{x \in X} (Q(x) - \langle \psi(x), y \rangle) < \inf_{x \in X} \sup_{y \in Y} (Q(x) - \langle \psi(x), y \rangle) .$$

Our main result is as follows:

THEOREM 2.1. - Let  $K:[0,+\infty[\to \mathbf{R},\ f_1,...,f_n:\mathbf{R}\to\mathbf{R}\ be\ n+1\ continuous\ functions\ satisfying\ the$ following conditions:

- $\begin{array}{l} (a) \ \inf_{t>0} \int_{0}^{t} K(s) ds < 0 \ and \ \liminf_{t\to +\infty} \frac{\int_{0}^{t} K(s) ds}{t} > 0; \\ (b) \ \limsup_{|t|\to +\infty} \frac{\left|\int_{0}^{t} f_{k}(s) ds\right|}{t^{2}} < +\infty \ for \ each \ k=1,...,n; \end{array}$
- (c) for each k = 1, ..., n, the function  $t \to \int_0^t f_k(s) ds$  is odd and vanishes only at 0.

Then, for each r > 0, there exists a number  $\delta > 0$  with the following property: for every n-uple of continuous functions  $g_1, ..., g_n : \mathbf{R} \to \mathbf{R}$  satisfying

$$\max_{1 \leq k \leq n} \left( \sup_{t \in \mathbf{R}} \int_0^t g_k(s) ds - \inf_{t \in \mathbf{R}} \int_0^t g_k(s) ds \right) < \delta,$$

there exists  $(\tilde{\mu}_1,...\tilde{\mu}_n) \in \mathbf{R}^n$ , with  $\sum_{k=1}^n |\tilde{\mu}_k|^2 < r^2$ , such that the problem

$$\begin{cases}
-K\left(\sum_{h=1}^{n+1}|x_h - x_{h-1}|^2\right)(x_{k+1} - 2x_k + x_{k-1}) = g_k(x_k) + \tilde{\mu}_k f_k(x_k) & k = 1, ..., n, \\
x_0 = x_{n+1} = 0
\end{cases}$$

has at least three solutions.

PROOF. Fix r > 0. First, we are going to apply Theorem 2.B. In this connection, take

$$X = \{(x_0, x_1, ..., x_n, x_{n+1}) \in \mathbf{R}^{n+2} : x_0 = x_{n+1} = 0\},\$$

with the scalar product

$$\langle x, y \rangle_1 = \sum_{k=1}^{n+1} (x_k - x_{k-1})(y_k - y_{k-1}).$$

We denote by  $\langle \cdot, \cdot \rangle_2$  the usual scalar product on  $\mathbf{R}^n$ . That is

$$\langle x, y \rangle_2 = \sum_{k=1}^n x_k y_k.$$

Fix  $\gamma > 0$  so that

$$||x||_2 \le \gamma ||x||_1 \tag{2.1}$$

for all  $x \in X$ . Consider the functions  $Q: X \to \mathbf{R}$  and  $\psi: X \to \mathbf{R}^n$  defined by

$$Q(x) = \frac{1}{2} \int_0^{\|x\|_1^2} K(s) ds$$

and

$$\psi(x) = \left( \int_0^{x_1} f_1(s) ds, ..., \int_0^{x_n} f_n(s) ds \right)$$

for all  $x \in X$ . Fix  $\mu \in \mathbf{R}^n$ . In view of (a) and (b), there exists  $\eta_1, \eta_2, \eta_3 > 0$  such that

$$\int_0^t K(s)ds \ge \eta_1 t - \eta_2 \tag{2.2}$$

for all  $t \geq 0$  and

$$\left| \int_0^t f_k(s)ds \right| \le \eta_3(t^2 + 1) \tag{2.3}$$

for all  $t \in \mathbb{R}$ , k = 1, ..., n. Fix  $x \in X$ . Using (2.2) and the Cauchy-Schwarz inequality, we obtain

$$Q(x) - \langle \psi(x), \mu \rangle_2 \ge \frac{1}{2} \eta_1 \|x\|_1^2 - \frac{1}{2} \eta_2 - |\langle \psi(x), \mu \rangle_2| \ge \frac{1}{2} \eta_1 \|x\|_1^2 - \frac{1}{2} \eta_2 - \|\mu\|_2 \|\psi(x)\|_2. \tag{2.4}$$

On the other hand, in view of (2.3), for each k = 1, ..., n, we have

$$\left| \int_0^{x_k} f_k(s) ds \right| \le \eta_3(|x_k|^2 + 1)$$

and hence

$$\|\psi(x)\|_{2} \le \eta_{3} \sqrt{\sum_{k=1}^{n} (|x_{k}|^{2} + 1)^{2}} \le \eta_{3} \left(\sum_{k=1}^{n} |x_{k}|^{2} + n\right).$$
 (2.5)

Putting (2.1), (2.4) and (2.5) together, we get

$$Q(x) - \langle \psi(x), \mu \rangle_2 \ge \frac{1}{2} \eta_1 \|x\|_1^2 - \|\mu\|_2 \eta_3(\gamma^2 \|x\|_1^2 + n) - \frac{1}{2} \eta_2.$$
 (2.6)

Now, fix  $\sigma > 0$  so that

$$\sigma < \min \left\{ \frac{\eta_1}{2\eta_3 \gamma^2}, r \right\}.$$

Let Y be the closed ball in  $\mathbb{R}^n$  centered at 0, of radius  $\sigma$ . If  $\mu \in Y$ , in view of (2.6), we have

$$\lim_{\|x\|_1 \to +\infty} (Q(x) - \langle \psi(x), \mu \rangle_2) = +\infty$$

and so the function  $x \to Q(x) - \langle \psi(x), \mu \rangle_2$  is inf-compact. Further, observe that, by (c), the function  $\psi$  vanishes only at 0, while, by (a), 0 is not a global minimum of Q. Clearly, Q is even and  $\psi$  is odd, in view of (c). In other words, each assumption of Theorem 2.B is satisfied. Consequently, the number

$$\delta := \frac{1}{n} \left( \inf_{x \in X} \sup_{\mu \in Y} (Q(x) - \langle \psi(x), \mu \rangle_2) - \sup_{\mu \in Y} \inf_{x \in X} (Q(x) - \langle \psi(x), \mu \rangle_2) \right)$$
(2.7)

is positive. At this point, we apply Theorem 2.A taking

$$h(x,\mu) = Q(x) - \langle \psi(x), \mu \rangle_2$$

for all  $(x, \mu) \in X \times Y$ . Fix n continuous functions  $g_1, ..., g_n : \mathbf{R} \to \mathbf{R}$  satisfying

$$\max_{1 \le k \le n} \left( \sup_{t \in \mathbf{R}} \int_0^t g_k(s) ds - \inf_{t \in \mathbf{R}} \int_0^t g_k(s) ds \right) < \delta \tag{2.8}$$

and consider the function  $\varphi: X \to \mathbf{R}$  defined by

$$\varphi(x) = -\sum_{k=1}^{n} \int_{0}^{x_k} g_k(s) ds$$

for all  $x \in X$ . Clearly, in view of (2.7) and (2.8), we have

$$\sup_{X} \varphi - \inf_{X} \varphi \le \sum_{k=1}^{n} \left( \sup_{t \in \mathbf{R}} \int_{0}^{t} g_{k}(s) ds - \inf_{t \in \mathbf{R}} \int_{0}^{t} g_{k}(s) ds \right)$$

$$\leq n \max_{1 \leq k \leq n} \left( \sup_{t \in \mathbf{R}} \int_0^t g_k(s) ds - \inf_{t \in \mathbf{R}} \int_0^t g_k(s) ds \right) < \inf_X \sup_Y h - \sup_Y \inf_X h.$$

So, each assumption of Theorem 2.A is satisfied. As a consequence, there exists  $\tilde{\mu} \in Y$  such that the function

$$J_{\tilde{\mu}}(\cdot) := h(\cdot, \tilde{\mu}) + \varphi(\cdot)$$

has at least two global minima in X. It is clear that this function  $J_{\tilde{\mu}}$  is  $C^1$ , with derivative given by

$$J'_{\tilde{\mu}}(x)(y) = K\left(\sum_{h=1}^{n+1} |x_h - x_{h-1}|^2\right) \langle x, y \rangle_1 - \sum_{k=1}^n g_k(x_k) y_k - \sum_{k=1}^n \tilde{\mu}_k f_k(x_k) y_k$$

for all  $x, y \in X$ . So, taking into account that

$$\langle x, y \rangle_1 = -\sum_{k=1}^n (x_{k+1} - 2x_k + x_{k-1})y_k$$
,

we have

$$J'_{\tilde{\mu}}(x)(y) = -K\left(\sum_{h=1}^{n+1} |x_h - x_{h-1}|^2\right) \sum_{k=1}^{n} (x_{k+1} - 2x_k + x_{k-1})y_k - \sum_{k=1}^{n} g_k(x_k)y_k - \sum_{k=1}^{n} \tilde{\mu}_k f_k(x_k)y_k$$
(2.9)

for all  $x, y \in X$ . Since  $J_{\tilde{\mu}}$  is coercive and has at least two global minima, by a classical theorem of Courant, it possesses at least three critical points which, by (2.9), are three solutions of the problem.

Here is a remarkable corollary of Theorem 2.1:

COROLLARY 2.1. - Let  $f_1, ... f_n : \mathbf{R} \to \mathbf{R}$  be n continuous functions satisfying conditions (b) and (c) of Theorem 2.1.

Then, for each a, r > 0 and b < 0, there exists a number  $\delta > 0$  with the following property: for every n-uple of continuous functions  $g_1, ..., g_n : \mathbf{R} \to \mathbf{R}$  satisfying

$$\max_{1 \leq k \leq n} \left( \sup_{t \in \mathbf{R}} \int_0^t g_k(s) ds - \inf_{t \in \mathbf{R}} \int_0^t g_k(s) ds \right) < \delta,$$

there exists  $(\tilde{\mu}_1,...\tilde{\mu}_n) \in \mathbf{R}^n$ , with  $\sum_{k=1}^n |\tilde{\mu}_k|^2 < r^2$ , such that the problem

$$\begin{cases} -\left(a\sum_{h=1}^{n+1}|x_h-x_{h-1}|^2+b\right)(x_{k+1}-2x_k+x_{k-1})=g_k(x_k)+\tilde{\mu}_kf_k(x_k) & k=1,...,n, \\ x_0=x_{n+1}=0 \end{cases}$$

has at least three solutions.

PROOF. It is enough to observe that the function K(t) = at + b satisfies condition (a) of Theorem 2.1.  $\triangle$ 

REMARK 1.1. - It is important to remark that the technique adopted in the proof Theorem 2.1 cannot be used to treat the non-discrete problem, keeping condition (a). This is due to the fact that, under condition (a), the functional

$$u \to \int_0^{\int_{\Omega} |\nabla u(x)|^2 dx} K(s) ds$$

is not weakly lower semicontinuous in  $H_0^1(\Omega)$ .

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