Generalized spectral characterization of signed bipartite graphs

Songlin Guo[a] Wei Wang[b*] Lele Li[a]

[a] School of General Education, Wanjiang University of Technology, Ma'anshan 243031, P. R. China

[b]School of Mathematics, Physics and Finance, Anhui Polytechnic University, Wuhu 241000, P. R. China

Abstract

Let Σ be an n-vertex controllable or almost controllable signed bipartite graph, and let Δ_Σ denote the discriminant of its characteristic polynomial $\chi(\Sigma; x)$. We prove that if (i) the integer $2^{-\lfloor n/2 \rfloor}\sqrt{\Delta_\Sigma}$ is squarefree, and (ii) the constant term (even n) or linear coefficient (odd n) of $\chi(\Sigma; x)$ is ± 1, then Σ is determined by its generalized spectrum. This result extends a recent theorem of Ji, Wang, and Zhang [Electron. J. Combin. 32 (2025), #P2.18], which established a similar criterion for signed trees with irreducible characteristic polynomials.

Keywords: signed bipartite graph; generalized spectrum; discriminant; determined by spectrum.

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1 Introduction

A signed graph Σ is a pair (G, σ), where G is a simple graph and σ is a mapping from its edge set $E(G)$ to $\{+1, -1\}$. The adjacency matrix $A(\Sigma) = (a_{ij})$ of a signed graph $\Sigma = (G, \sigma)$ is defined by:

$$a_{ij} = \begin{cases} \sigma(ij) & \text{if } i \text{ and } j \text{ are adjacent;} \\ 0 & \text{otherwise.} \end{cases}$$

Two signed graphs Σ_1 and Σ_2 are said to be generalized cospectral [7] if their adjacency matrices $A(\Sigma_1)$ and $A(\Sigma_2)$ satisfy:

1. $\det(xI - A(\Sigma_1)) = \det(xI - A(\Sigma_2))$ and
2. $\det(xI - (J - I - A(\Sigma_1))) = \det(xI - (J - I - A(\Sigma_2)))$,

*Corresponding author. Email address: wangwei.math@gmail.com

where I is the identity matrix, J is the all-ones matrix, and $(J-I-A(\Sigma_1))$ formally denotes the 'complement' of Σ_1. A signed graph Σ is determined by its generalized spectrum (DGS) [7] if every signed graph generalized cospectral with Σ is isomorphic to it.

For convenience, we say a signed graph Σ is *reducible* (resp. *irreducible*) if its characteristic polynomial $\chi(\Sigma;x)$, which is $\det(xI-A(\Sigma))$, is reducible (resp. irreducible) over $\mathbb{Q}$. We use Δ_Σ to denote the discriminant of $\chi(\Sigma;x)$. Recently, Ji, Wang and Zhang [7] obtained a simple criterion for a signed tree to be DGS. We state it in a slightly different but essentially equivalent form.

Theorem 1 ([7])**.** *Let Σ be an n-vertex ($n\geq 2$) irreducible signed tree. If $2^{-n/2}\sqrt{\Delta_\Sigma}$ is an odd squarefree integer, then Σ is DGS.*

We note that the order n is necessarily even in Theorem 1 by the irreducibility assumption. Indeed, if Σ is a signed tree (or more generally, a signed bipartite graph) whose order is odd, then it is not difficult to see that $A(\Sigma)$ is singular and hence $\chi(\Sigma;x)$ has x a factor. Furthermore, since for any signed tree Σ, the constant term of its characteristic polynomial $\chi(\Sigma;x)$ belongs to $\{0,1,-1\}$, the irreducibility assumption in Theorem 1 clearly implies that the constant term of $\chi(\Sigma;x)$ is ±1.

At the end of [7], Ji et al. proposed the following question for further study.

Question 1 ([7])**.** *How can Theorem 1 be generalized to signed bipartite graphs?*

In the same paper, Ji et al. realized that some essential difficulties will inevitably appear when we try to generalize Theorem 1 to signed bipartite graphs. They also reported a 'counterexample' indicating that Theorem 7 in that paper, which is the main tool to prove Theorem 1, does not hold without the irreducibility assumption of Σ, even if Σ is controllable. Thus it is not clear how to generalize Theorem 1 to reducible signed trees (which include all signed trees with $2k+1$ vertices).

In this paper, we employ a new approach to give an answer to Question 1. The main result of this paper is the following theorem.

Theorem 2. *Let Σ be an n-vertex controllable or almost controllable signed bipartite graph. Assume that the coefficient of the constant term (n even) or linear term (n odd) of $\chi(\Sigma;x)$ is ±1. If $2^{-n/2}\sqrt{\Delta_\Sigma}$ is squarefree, then Σ is DGS.*

Compared with Theorem 1, Theorem 2 has broader applicability and subsumes Theorem 1 as a special case. Note that Theorem 2 no longer requires the oddness condition explicitly, since we will prove that the value $2^{-n/2}\sqrt{\Delta_\Sigma}$ must be odd when squarefree.

2 Preliminaries

We first recall some basic notations. Let Σ be an n-vertex signed graph. The walk-matrix of Σ is defined as:

$$W(\Sigma):=[e,Ae,\ldots,A^{n-1}e],$$

where e is the all-ones vector and A is the adjacency matrix of Σ. We say Σ is controllable (resp. almost controllable) if $\operatorname{rank}(W(\Sigma))=n$ (resp. $\operatorname{rank}(W(\Sigma))=n-1$).

2.1 Totally isotropic subspace in generalized cospectrality

An orthogonal matrix Q is called regular if it satisfies $Qe = e$. The following theorem states that for controllable or almost controllable signed graphs, generalized cospectrality can be characterized by a regular orthogonal matrix with rational entries. The result was usually stated and proved in the setting of unsigned graphs; nevertheless, the original proofs are still valid for signed graphs.

Theorem 3 ([2, 3, 11]). *Let Σ and Γ be two signed graphs with n vertices. Then Σ and Γ are generalized cospectral if and only if there exists a regular orthogonal matrix Q such that*

$$Q^{\mathrm{T}} A(\Sigma) Q = A(\Gamma). \tag{1}$$

Moreover,

(i) *if Σ is controllable then $Q^{\mathrm{T}} = W(\Gamma)(W(\Sigma))^{-1}$ and hence Q is unique and rational.*

(ii) *if Σ is almost controllable then Eq.* (1) *has exactly two solutions for Q, both of which are rational.*

Let $\mathrm{RO}_n(\mathbb{Q})$ and $\mathrm{S}_n(\mathbb{Z})$ denote the sets of all $n \times n$ rational regular orthogonal matrices and all $n \times n$ symmetric integer matrices, respectively. For a signed graph Σ, we define

$$\mathcal{Q}(\Sigma) = \{Q \in \mathrm{RO}_n(\mathbb{Q}) \colon\ Q^{\mathrm{T}} A(\Sigma) Q \in \mathrm{S}_n(\mathbb{Z})\}.$$

For a matrix $Q \in O_n(\mathbb{Q})$, its level, denoted by $\ell(Q)$ (or simply write it as ℓ), is defined as the smallest positive integer k such that kQ is an integer matrix. Clearly, matrices in $\mathrm{RO}_n(\mathbb{Q})$ with level 1 are precisely the permutation matrices. The following observation is standard and frequently used to show a graph (or a signed graph) to be DGS.

Lemma 1. *Let Σ be a controllable or almost controllable signed graph. If each matrix $Q \in \mathcal{Q}(\Sigma)$ has level 1 then Σ is DGS.*

Remark 1. For almost controllable (signed) graphs, Lemma 1 is only applicable for those that have a nontrivial automorphism. Indeed, if Σ is almost controllable but has no nontrivial automorphism, then $\mathcal{Q}(\Sigma)$ contains a non-permutation matrix and hence the condition of Lemma 1 fails.

Let p be a prime. We consider the n-dimensional vector space $\mathbb{F}_p^n$ over the finite field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$, consisting of all column vectors $(x_1, x_2, ..., x_n)^{\mathrm{T}}$ with components $x_i \in \mathbb{F}_p$. This space is endowed with the standard inner product defined by $\langle u, v\rangle = u^{\mathrm{T}} v$. Two vectors $u, v \in \mathbb{F}_p^n$ are called *orthogonal*, denoted by $u \perp v$, if $u^{\mathrm{T}} v = 0$. Similarly, two subspaces U and V are *orthogonal*, denoted by $U \perp V$, if $u \perp v$ for any $u \in U$ and $v \in V$. A nonzero vector $u \in \mathbb{F}_p^n$ is *isotropic* if $u \perp u$, i.e., $u^{\mathrm{T}} u = 0$. For a subspace V of $\mathbb{F}_p^n$, the *orthogonal space* of V is

$$V^{\perp} = \{u \in \mathbb{F}_p^n \colon\ v^{\mathrm{T}} u = 0 \text{ for every } v \in V\}.$$

A subspace V of $\mathbb{F}_p^n$ is *totally isotropic* [8] if $V \subset V^{\perp}$, i.e., every pair of vectors in V are orthogonal.

Let M be an $n \times n$ matrix over $\mathbb{F}_p$. We identify M with the linear transformation $x \mapsto Mx$ for $x \in \mathbb{F}_p^n$. A subspace V of $\mathbb{F}_p^n$ is *M-invariant* if $Mx \in V$ for any $x \in V$.

Lemma 2 ([9])**.** *Let $\chi(M;x) \in \mathbb{F}_p[x]$ be the characteristic polynomial of M and*

$$\chi(M;x) = \phi_1^{r_1}(x)\phi_2^{r_2}(x)\cdots\phi_k^{r_k}(x),$$

be the standard factorization of $\chi(M;x)$. Let U be an M-invariant subspace and denote $V_i = \ker\phi_i^{r_i}(M), i = 1,\ldots,k$. Then
(i) if $\phi_i(x)$ is a simple factor (i.e., $r_i = 1$), then $U \cap V_i = 0$;
(ii) if $U \cap \ker\phi_i^{r_i}(M)$ is nonzero, then $U \cap \ker\phi_i(M)$ is also nonzero;
(iii) $U = \oplus(U \cap V_i)$, where the summation is taken over all subscripts i satisfying $r_i \geq 2$.

Suppose that $Q \in \mathcal{Q}(\Sigma)$ such that $\ell := \ell(Q) > 1$. We define

$$\hat{Q} = \ell \cdot Q \in \mathbb{Z}^{n\times n}.$$

Let $M \in \mathbb{Z}^{n\times n}$ be an integer matrix and p be a fixed prime. We use $\mathrm{col}_p(M)$ and $\ker_p(M)$ to denote the column space and null space (or kernel), which are subspaces of $\mathbb{F}_p^n$.

Lemma 3 ([9])**.** *For any prime factor p of $\ell(Q)$, the space $\mathrm{col}_p(\hat{Q})$ is nonzero, totally isotropic and A-invariant.*

The following proposition is immediate from Lemmas 2 and 3.

Proposition 1. *For any prime factor p of $\ell(Q)$, the characteristic polynomial $\chi(A;x)$ has a multiple factor $\phi(x)$ such that*

$$\mathrm{col}(\hat{Q}) \cap \ker\phi(A) \neq 0, \text{ over } \mathbb{F}_p.$$

Proposition 2 ([9])**.** *Let p be an odd prime factor of $\ell(Q)$ and $\phi(x) \in \mathbb{Z}[x]$ satisfy the conclusion of Proposition 1. Then we have*

$$p^{\deg\phi(x)+1} \mid \det\phi(A).$$

2.2 Resultant and discriminant

For a monic polynomial $f(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0 \in Z[x]$, the discriminant of $f(x)$ is defined as:

$$\Delta(f) = \prod_{1\leq i<j\leq n} (\alpha_j - \alpha_i)^2,$$

where $\alpha_1,\ldots,\alpha_n$ are the roots of $f(x)$ in $\mathbb{C}$. The resultant of f and its derivative f', denoted by $\mathrm{Res}(f,f')$, is the determinant of the $(2n-1)\times(2n-1)$ Sylvester matrix

$$S(f,f') = \begin{pmatrix} 1 & a_{n-1} & & \cdots & \cdots & \cdots & a_0 & & & & \\ & 1 & a_{n-1} & \cdots & \cdots & & \cdots & a_0 & & & \\ & & \cdots & \cdots & \cdots & & \cdots & & & & \\ & & & 1 & a_{n-1} & & \cdots & \cdots & \cdots & a_0 \\ n & (n-1)a_{n-1} & \cdots & \cdots & a_1 & & & & & \\ & n & (n-1)a_{n-1} & \cdots & \cdots & & a_1 & & & \\ & & \cdots & \cdots & \cdots & & \cdots & & & \\ & & & & n & & (n-1)a_{n-1} & \cdots & \cdots & a_1 \end{pmatrix}. \quad (2)$$

It is known that $\Delta(f) = \pm\mathrm{Res}(f,f')$ and hence $\Delta(f)$ is an integer.

Lemma 4 ([10])**.** *Let p be a prime and $f(x) \in \mathbb{Z}[x]$ be a monic polynomial. Then $f(x)$ has a multiple factor over $\mathbb{F}_p$ if and only if $p|\Delta(f)$.*

Let $M \in \mathrm{S}_n(\mathbb{Z})$, we use Δ_M to denote $\Delta(\chi(M;x))$, the discriminant of $\chi(M;x)$. For two polynomial $f(x), g(x) \in \mathbb{Z}[x]$ and an integer q, we denote $f(x) \equiv g(x) \pmod q$ if all corresponding coefficients of f and g are congruent modulo q. The following three lemmas were obtained by Wang and Yu [4, Theorem 3.3, Lemma 4.3, and Lemma 4.4]):

Lemma 5 ([4])**.** *Let $M \in \mathrm{S}_n(\mathbb{Z})$ and Q be a rational orthogonal matrix such that $Q^{\mathrm{T}}MQ \in \mathrm{S}_n(\mathbb{Z})$. Then any prime factor of $\ell(Q)$ is a factor of Δ_M.*

Lemma 6 ([4])**.** *Let $M \in \mathrm{S}_n(\mathbb{Z})$ and p be any odd prime. If $p \mid \Delta_M$ but $p^2 \nmid \Delta_M$, then there exists an integer λ_0 and a polynomial $\varphi(x)$ with integer coefficients such that $\chi(M;x) \equiv (x-\lambda_0)^2\varphi(x) \pmod p$, where $\varphi(x)$ is squarefree over $\mathbb{Z}_p$ and $\varphi(\lambda_0) \not\equiv 0 \pmod p$.*

Lemma 7 ([4])**.** *Let $M \in \mathrm{S}_n(\mathbb{Z})$ and p be any odd prime. Suppose that $\chi(M;x) \equiv (x-\lambda_0)^2\varphi(x) \pmod p$ for some $\lambda_0 \in \mathbb{Z}$ and $\varphi(x) \in \mathbb{Z}[x]$, where $\varphi(x)$ is squarefree over $\mathbb{F}_p$ and $\varphi(\lambda_0) \not\equiv 0 \pmod p$. Then the equation*

$$\chi(M;x)u(x) \equiv \chi'(M;x)v(x) \pmod{p^2}$$

has a solution $(u(x), v(x)) \in (\mathbb{Z}[x])^2$ with:
(i) $u(x), v(x) \not\equiv 0 \pmod p$;
(ii) $\deg(u(x)) < n-1 = \deg(\chi'(M;x))$;
(iii) $\deg(v(x)) < n = \deg(\chi(M;x))$,
if and only if $p^2 \mid \det(M-\lambda_0 I)$.

Remark 2. Under the assumption of this lemma, the truth of $p^2 \mid \det(M-\lambda_0 I)$ does not depend on the choice of λ_0 in its residue class, i.e., for any λ_1 and λ_2 such that $\lambda_1 \equiv \lambda_2 \equiv \lambda_0 \pmod p$, $p^2 \mid \det(M-\lambda_1 I)$ if and only if $p^2 \mid \det(M-\lambda_2 I)$. This can be easily seen from the conclusion of Lemma 7 since the existence of a solution (u,v) satisfying the given conditions clearly does not depend on the choice of λ_0 in its residue class.

Let M be a nonsingular $n \times n$ integer matrix. It is well-known that there exist two unimodular matrices U and V such that UMV is a diagonal matrix $S = \mathrm{diag}(d_1, d_2, \ldots, d_n)$, where the diagonal entries $d_1, \ldots, d_n$ are positive integers with $d_i \mid d_{i+1}$ for $i = 1, 2 \ldots, n$. The matrix S is unique and is called the Smith normal form of M; the diagonal entries are called the invariant factor of M. We summarize some basic properties of a matrix using its invariant factors.

Lemma 8. *Let p be any prime. For an integral matrix M with invariant factors $d_1, d_2, \ldots, d_n$. We have*
(i) $p^{n-\mathrm{rank}_p(M)} \mid \det M$;
(ii) $\det(M) = \pm d_1 d_2 \ldots d_n$;
(iii) $\mathrm{rank}_p M = \max\{i : p \nmid d_i\}$;
(iv) $Mx \equiv 0 \pmod{p^2}$ has a solution $x \not\equiv 0 \pmod p$ if and only if $p^2 \mid d_n$.

Lemma 9. *Let p be a prime and $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a monic polynomial over $\mathbb{F}_p$. Then $\deg\gcd(f, f') = \dim\ker S(f, f') = (2n-1) - \mathrm{rank}_p S(f, f')$.*

Proof. Let

$$u(x) = u_0 x^{n-2} + u_1 x^{n-3} + \cdots + u_0, \text{ and } v(x) = v_{n-1}x^{n-1} + v_{n-2}x^{n-2} + \cdots + v_0.$$

We consider the equation

$$f(x)u(x) - f'(x)v(x) = 0 \text{ with variables } (u, v). \tag{3}$$

Let $g(x) = \frac{f(x)}{d(x)}$ and $h(x) = \frac{f'(x)}{d(x)}$ where $d(x) = \gcd(f(x), f'(x))$. Then the equation $f(x)u(x) - f'(x)v(x) = 0$ reduces to

$$g(x)u(x) = h(x)v(x). \tag{4}$$

Noting that $g(x)$ and $h(x)$ are coprime, the solution (u, v) of Eq. (4) has the form $(u, v) = (h(x)r(x), g(x)r(x))$ for some $r(x) \in \mathbb{F}_p[x]$. Let $k = \deg d(x)$. To satisfy the restrictions of degrees of $u(x)$ and $v(x)$, we need (and only need) $\deg r(x) \le k - 1$. This means that the solution space of Eq. (4) (or equivalently Eq. (3)) has dimension k.

Write $\eta = (u_{n-2}, u_{n-3}, \ldots, u_0, v_{n-1}, v_{n-2}, \ldots, v_0)$ and $S = S(f, f')$. Then Eq. (3) is equivalent to $S^{\mathrm{T}}\eta = 0$. This means that the solution subspace of Eq. (3) is isomorphic to $\ker S(f, f')$. It follows that $\dim\ker S(f, f') = k = \deg\gcd(f, f')$. This completes the proof. □

Proposition 3. *Let $f(x)$ be a monic polynomial with integer coefficients. Then $\Delta(f) \not\equiv 2 \pmod 4$.*

Proof. Let $f(x) = \phi_1^{r_1}(x)\phi_2^{r_2}(x)\cdots\phi_k^{r_k}(x)$ be the standard factorization of $f(x)$ over $\mathbb{F}_2$. We claim that, over $\mathbb{F}_2$,

$$\gcd(f, f') = \phi_1^{s_1}(x)\phi_2^{s_2}(x)\cdots\phi_k^{s_k}(x),$$

where

$$s_i = \begin{cases} r_i & r_i \text{ even;} \\ r_i - 1 & r_i \text{ odd.} \end{cases}$$

Indeed, if some r_i, say r_1 is odd, then we have $r_1 \equiv 1 \pmod 2$ and hence

$$f'(x) = \phi_1^{r_1-1}(x)\left(\prod_{j=2}^{k}\phi_j^{r_j}(x)\right) + \phi_1^{r_1}(x)\left(\prod_{j=2}^{k}\phi_j^{r_j}(x)\right)' \text{ over } \mathbb{F}_2,$$

which implies that the multiplicity of the factor $\phi_1(x)$ in $f'(x)$ is exactly $r_1 - 1$. Thus, in this case, the multiplicity of $\phi_1(x)$ in $\gcd(f, f')$ is $r_1 - 1$. However, if r_1 is even, i.e., $r_1 \equiv 0 \pmod 2$, then over $\mathbb{F}_2$, we have $(\phi_1^{r_1}(x))' = 0$ and hence

$$f'(x) = \phi_1^{r_1}(x)\left(\prod_{j=2}^{k}\phi_j^{r_j}(x)\right)' \text{ over } \mathbb{F}_2,$$

which implies that the multiplicity of $\phi_1(x)$ in $\gcd(f, f')$ is r_1. This proves the claim. It follows that the degree of $\gcd(f, f')$ must be even as each multiplicity s_i is even.

Let $S \in \mathbb{Z}^{(2n-1)\times(2n-1)}$ be the Sylvester matrix of f and f'. Suppose to the contrary that $\Delta(f) \equiv 2 \pmod 4$, or equivalently, $\det S \equiv 2 \pmod 4$. Then, by Lemma (ii), we see that S has exactly one invariant factor that is even, that is, $\mathrm{rank}_2 S = 2n - 2$. But this implies $\deg\gcd(f, f') = 1$ by Lemma 9, contradicting the established fact that the degree of $\gcd(f, f')$ is always even. This completes the proof of Proposition 3. □

3 Proof of Theorem 2

For convenience, we define

$$\delta = \delta(n) = \lceil n/2 \rceil - \lfloor n/2 \rfloor = \begin{cases} 0 & n \text{ even;} \\ 1 & n \text{ odd.} \end{cases}$$

Let Σ be an n-vertex controllable or almost controllable signed graph such that $c_\delta = \pm 1$, where c_δ is the coefficient of the term x^δ in $\chi(\Sigma; x)$. As Σ is bipartite, we may write the adjacency matrix $A = A(\Sigma)$ in the form

$$A = \begin{pmatrix} 0 & B \\ B^{\mathrm{T}} & 0 \end{pmatrix},$$

where B is an $s \times (n-s)$ matrix with $s \le \lfloor n/2 \rfloor$. We claim that the equality must hold. Indeed, if $s < \lfloor n/2 \rfloor$ then we have $\operatorname{rank}(A) = 2\operatorname{rank}(B) \le 2(\lfloor n/2 \rfloor - 1) \le n-2$, which implies that 0 is a multiple eigenvalue of A. This contradicts the requirement that $c_\delta = \pm 1$.

Lemma 10. $\chi(A; x) = x^\delta \chi(BB^{\mathrm{T}}; x^2) = x^{-\delta}\chi(B^{\mathrm{T}}B; x^2)$.

Proof. From the identity

$$\begin{pmatrix} xI_{\lfloor n/2 \rfloor} & -B \\ -B^{\mathrm{T}} & xI_{\lceil n/2 \rceil} \end{pmatrix} \begin{pmatrix} I_{\lfloor n/2 \rfloor} & 0 \\ \frac{1}{x}B^{\mathrm{T}} & I_{\lceil n/2 \rceil} \end{pmatrix} = \begin{pmatrix} xI_{\lfloor n/2 \rfloor} - \frac{1}{x}BB^{\mathrm{T}} & -B \\ 0 & xI_{\lceil n/2 \rceil} \end{pmatrix},$$

we obtain

$$\det \begin{pmatrix} xI_{\lfloor n/2 \rfloor} & -B \\ -B^{\mathrm{T}} & xI_{\lceil n/2 \rceil} \end{pmatrix} = \det \begin{pmatrix} xI_{\lfloor n/2 \rfloor} - \frac{1}{x}BB^{\mathrm{T}} & -B \\ 0 & xI_{\lceil n/2 \rceil} \end{pmatrix} = x^{-\lfloor n/2 \rfloor} \det(x^2 I - BB^{\mathrm{T}}) x^{\lceil n/2 \rceil},$$

i.e., $\chi(A; x) = x^\delta \chi(BB^{\mathrm{T}}; x^2)$. Similarly, by the identity

$$\begin{pmatrix} xI_{\lfloor n/2 \rfloor} & -B \\ -B^{\mathrm{T}} & xI_{\lceil n/2 \rceil} \end{pmatrix} \begin{pmatrix} I_{\lfloor n/2 \rfloor} & \frac{1}{x}B \\ 0 & I_{\lceil n/2 \rceil} \end{pmatrix} = \begin{pmatrix} xI_{\lfloor n/2 \rfloor} & 0 \\ -B^{\mathrm{T}} & xI_{\lceil n/2 \rceil} - \frac{1}{x}B^{\mathrm{T}}B \end{pmatrix},$$

we obtain $\chi(A; x) = x^{-\delta}\chi(B^{\mathrm{T}}B; x^2)$. This completes the proof. $\square$

The following basic connection between Δ_A and $\Delta_{BB^{\mathrm{T}}}$ was obtained by Ji et al. [7] for the case that n is even. We include the short proof for completeness.

Lemma 11 ([7])**.** $\Delta_A = 4^{\lfloor n/2 \rfloor} \Delta^2_{BB^{\mathrm{T}}}$ *and hence* $2^{-\lfloor n/2 \rfloor}\sqrt{\Delta_A} = \Delta_{BB^{\mathrm{T}}}$.

Proof. By Lemma 10, we have $\chi(A; x) = x^\delta \chi(BB^{\mathrm{T}}; x^2)$. Denote $m = \lfloor n/2 \rfloor$ and let the spectrum of BB^{T} be $\operatorname{spec}(BB^{\mathrm{T}}) = \{\lambda_1^2, \ldots, \lambda_m^2\}$. Then we have

$$\operatorname{spec}(A) = \begin{cases} \{\lambda_1, \ldots, \lambda_m\} \cup \{-\lambda_1, \ldots, -\lambda_m\} & n \text{ even;} \\ \{\lambda_1, \ldots, \lambda_m\} \cup \{-\lambda_1, \ldots, -\lambda_m\} \cup \{0\} & n \text{ odd.} \end{cases} \tag{5}$$

First consider the case that n is even. Then, we have

$$\begin{aligned}
\Delta_A &= \prod_{1\le i<j\le m}(\lambda_j-\lambda_i)^2\prod_{1\le i<j\le m}(-\lambda_j+\lambda_i)^2\prod_{\substack{1\le i\le m\\1\le j\le m}}(-\lambda_j-\lambda_i)^2\\
&= \left(\prod_{1\le i<j\le m}(\lambda_j-\lambda_i)^2\right)^2\left(\prod_{1\le i\le m}(2\lambda_i)^2\right)\left(\prod_{1\le i<j\le m}(\lambda_j+\lambda_i)\right)^2\\
&= 4^m\prod_{1\le i\le m}\lambda_i^2\left(\prod_{1\le i<j\le m}\left(\lambda_j^2-\lambda_i^2\right)^2\right)^2\\
&= 4^m\det(BB^{\mathrm{T}})\Delta^2_{BB^{\mathrm{T}}}. \qquad (6)
\end{aligned}$$

Noting that $\det(BB^{\mathrm{T}})=\pm\det A=\pm c_0=\pm1$ and $\det(BB^{\mathrm{T}})=(\det(B))^2\ge0$, we must have $\det(BB^{\mathrm{T}})=1$ and hence Eq. (6) reduces to $\Delta_A=4^m\Delta^2_{BB^{\mathrm{T}}}$. This proves the lemma for the even case.

Now we consider the case that n is odd. Denote $R=4^m\det(BB^{\mathrm{T}})\Delta^2_{BB^{\mathrm{T}}}$, which is the result of Δ_A for the even case. From Eq. (5), it is not difficult to see that for odd n,

$$\Delta_A=R\times\left(\prod_{1\le i\le m}\lambda_i^2\right)^2=4^m\left(\det(BB^{\mathrm{T}})\right)^3\Delta^2_{BB^{\mathrm{T}}}. \qquad (7)$$

Recall that the coefficient of the linear term in $\chi(A;x)$ is ±1. Differentiating both sides of the equation $\chi(A;x)=x\cdot\chi(BB^{\mathrm{T}};x^2)$ and evaluating at $x=0$ gives $\chi(BB^{\mathrm{T}};0)=\pm1$, i.e., $\det(BB^{\mathrm{T}})=\pm1$. As BB^{T} is positive semidefinite, we must have $\det(BB)^{\mathrm{T}}\ge0$ and hence $\det(BB^{\mathrm{T}})=1$. Thus, Eq. (7) also reduces to $\Delta_A=4^m\Delta^2_{BB^{\mathrm{T}}}$. This completes the proof. □

Let Q be any matrix in $\mathcal{Q}(\Sigma)$. We shall prove Theorem 2 by establishing the following two propositions.

Proposition 4. *If $\ell(Q)$ is even then so is $\Delta_{BB^{\mathrm{T}}}$.*

Proposition 5. *If p is an odd prime factor of $\ell(Q)$ then $p^2\mid\Delta_{BB^{\mathrm{T}}}$.*

The proofs of these Propositions 4 and 5 will be presented in the following two subsections. It turns out that Theorem 2 is an easy consequence of these two propositions.

Proof of Theorem 2 Suppose that $2^{-\lfloor n/2\rfloor}\sqrt{\Delta_\Sigma}$ is squarefree. Let Q be any matrix in $\mathcal{Q}(\Sigma)$. By Lemma 11, $\Delta_{BB^{\mathrm{T}}}=2^{-\lfloor n/2\rfloor}\sqrt{\Delta_\Sigma}$ and hence is squarefree. In particular, $4\nmid\Delta_{BB^{\mathrm{T}}}$, i.e., $\Delta_{BB^{\mathrm{T}}}\not\equiv0\pmod 4$. By Proposition 3, $\Delta_{BB^{\mathrm{T}}}\not\equiv2\pmod 4$. It follows that $\Delta_{BB^{\mathrm{T}}}\equiv0,1\pmod 4$, i.e., $\Delta_{BB^{\mathrm{T}}}$ is odd. Proposition 4 implies that $\ell(Q)$ must be odd. Moreover, as $\Delta_{BB^{\mathrm{T}}}$ is squarefree, Proposition 5 implies that $\ell(Q)$ has no odd prime factor. Thus, $\ell(Q)=1$ and hence Σ is DGS by Lemma 1. This completes the proof of Theorem 2. □

Remark 3. As a byproduct of the proof of Theorem 2, we note that if Σ is almost controllable and satisfies the conditions of Theorem 2, then $\mathcal{Q}(\Sigma)$ contains only permutation matrix and hence Σ must have a nontrivial automorphism.

3.1 The case $p = 2$

The main aim of this subsection is to prove Proposition 4. We fix $p = 2$ here, and for simplicity, we omit the subscript p for some notations. For example, $\mathrm{col}(M)$ means $\mathrm{col}_2(M)$, which is a subspace of $\mathbb{F}_2^n$. Let $v \in \mathbb{Z}^n$ be an integer vector and V be a subspace of $\mathbb{F}_2^n$. By slight abuse of notation, we write $v \in V$ to mean that the reduction of v modulo 2 lies in V.

Lemma 12 ([6])**.** *Suppose $Q \in \mathcal{Q}(\Sigma)$ with even level. Then for any integer vector $q \in \mathrm{col}(\hat{Q}) \subset \mathbb{F}_2^n$ and any nonnegative integer k, we have $q^{\mathrm{T}} A^k q \equiv 0 \pmod 4$.*

In the following, we always assume $\ell(Q)$ is even. Thus, by Lemma 3 for $p = 2$, we know that $\mathrm{col}(\hat{Q})$ is a nonzero and totally isotropic A-invariant subspace. It follows from Lemma 2 that the characteristic polynomial $\chi(A; x) \in \mathbb{F}_2[x]$ has a multiple factor $\phi(x)$ such that

$$\mathrm{col}(\hat{Q}) \cap \ker \phi(A) \neq 0.$$

We shall show that $\phi(x)$ is also a multiple factor of $\chi(BB^{\mathrm{T}}; x)$, which clearly completes the Proposition 4 by Lemma 15. We first show that $\phi(x)$ is indeed a factor of $\chi(BB^{\mathrm{T}}; x)$.

Lemma 13. *$\phi(x) \mid \chi(BB^{\mathrm{T}}; x)$ and $\phi(0) = 1$ over $\mathbb{F}_2$.*

Proof. Since we are working over $\mathbb{F}_2$, we have $f(x^2) = f^2(x)$ when f is a polynomial. Thus, the first equality in Lemma 10 becomes $\chi(A; x) = x^{\delta}(\chi(BB^{\mathrm{T}}; x))^2$. Noting that $\delta \leq 1$ and $\phi(x)$ is a multiple factor of $\chi(A; x)$, we must have $\phi(x) \mid \chi(BB^{\mathrm{T}}; x)$. It remains to show that $\phi(0) = 1$. Suppose to the contrary that $\phi(0) = 0$. Since $\phi(x)$ is a multiple factor of $\chi(A; x)$ (over $\mathbb{F}_2$), we see that the coefficients of the constant term and the linear term in $\chi(A; x)$ are both 0 over $\mathbb{F}_2$, i.e., both are even as ordinary integers. But this contradicts the requirement that $c_\delta = \pm 1$. Thus, $\phi(0) = 1$ and the proof is complete. $\square$

Lemma 14. *There exist two vectors $u \in \mathbb{F}_2^{\lfloor n/2 \rfloor}$ and $v \in \mathbb{F}_2^{\lceil n/2 \rceil}$ such that $Bv \neq 0$ and $\begin{pmatrix} u \\ v \end{pmatrix} \in \mathrm{col}(\hat{Q}) \cap \ker \phi(A)$.*

Proof. Let $q = (q_1, \ldots, q_n)^{\mathrm{T}} \in \mathbb{F}_2^n$ be any nonzero vector in $\mathrm{col}(\hat{Q}) \cap \ker \phi(A)$. Denote $u = (q_1, q_2, \ldots, q_{\lfloor n/2 \rfloor})^{\mathrm{T}}$ and $v = (q_{\lfloor n/2 \rfloor + 1}, \ldots, q_{n-1}, q_n)^{\mathrm{T}}$.

Claim 1: $Aq \in \mathrm{col}(\hat{Q}) \cap \ker \phi(A)$ and $Aq \neq 0$.

The first assertion should be clear as both $\mathrm{col}(\hat{Q})$ and $\ker \phi(A)$ are A-invariant. It remains to show $Aq \neq 0$. Suppose to the contrary that $Aq = 0$. Then we have $\phi(A)q = \phi(0)q$. By Lemma 13, we have $\phi(0)q = q$ and hence is nonzero. But clearly, $\phi(A)q = 0$. This is a contradiction and hence Claim 1 follows.

Claim 2: If $u = 0$ then $Bv \neq 0$.

As $A = \begin{pmatrix} 0 & B \\ B^{\mathrm{T}} & 0 \end{pmatrix}$ and $q = \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ v \end{pmatrix}$, we see that $Aq = \begin{pmatrix} Bv \\ 0 \end{pmatrix}$. Thus, by Claim 1, $Bv \neq 0$ and hence Claim 2 follows.

By Claim 2, to complete the proof of Lemma 14, it suffices to consider the case that $u \neq 0$ and $Bv = 0$ hold simultaneously. Now let $\tilde{q} = Aq$ and let $\tilde{u} \in \mathbb{F}_2^{\lfloor n/2 \rfloor}$ and $\tilde{v} \in \mathbb{F}_2^{\lceil n/2 \rceil}$

such that $\tilde{q} = \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix}$. Then we have

$$\begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} = \begin{pmatrix} 0 & B \\ B^{\mathrm{T}} & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} Bv \\ B^{\mathrm{T}}u \end{pmatrix} = \begin{pmatrix} 0 \\ B^{\mathrm{T}}u \end{pmatrix}.$$

Now we show that $\tilde{u}$ (which is 0) and $\tilde{v}$ satisfy all requirements of Lemma 14. By Claim 1, we know that $\tilde{q}$ is a nonzero vector in $\mathrm{col}(\hat{Q}) \cap \ker \phi(A)$. Consequently, using Claim 2 for the vector $\tilde{q} = \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} = \begin{pmatrix} 0 \\ \tilde{v} \end{pmatrix}$, we obtain that $B\tilde{v} \neq 0$. This completes the proof of Lemma 14. □

Lemma 15. *$\phi^2(x) \mid \chi(BB^{\mathrm{T}}; x)$ over $\mathbb{F}_2$.*

Proof. Suppose to the contrary that $\phi^2(x) \nmid \chi(BB^{\mathrm{T}}; x)$. Then Lemma 13 implies that $\phi(x)$ is a simple factor of $\chi(BB^{\mathrm{T}}; x)$. According to Lemma 14, there exists a vector $q = \begin{pmatrix} u \\ v \end{pmatrix} \in \mathbb{Z}^n$ such that $Bv \not\equiv 0 \pmod 2$ and the reduction of q over $\mathbb{F}_2$ lies in $\mathrm{col}(\hat{Q}) \cap \ker \phi(A)$.

Claim 1: $u, Bv \in \ker \phi(BB^{\mathrm{T}})$, i.e., $\phi(BB^{\mathrm{T}})u \equiv \phi(BB^{\mathrm{T}})Bv \equiv 0 \pmod 2$.

Since $f(x^2) \equiv f^2(x) \pmod 2$ for any $f \in \mathbb{Z}[x]$, we have

$$\phi(A^2) \begin{pmatrix} u \\ v \end{pmatrix} \equiv \phi(A)\phi(A) \begin{pmatrix} u \\ v \end{pmatrix} \equiv 0 \pmod 2.$$

On the other hand, noting that $A^2 = \begin{pmatrix} 0 & B \\ B^{\mathrm{T}} & 0 \end{pmatrix}^2 = \begin{pmatrix} BB^{\mathrm{T}} & 0 \\ 0 & B^{\mathrm{T}}B \end{pmatrix}$, we obtain

$$\phi(A^2) \begin{pmatrix} u \\ v \end{pmatrix} \equiv \begin{pmatrix} \phi(BB^{\mathrm{T}}) & 0 \\ 0 & \phi(B^{\mathrm{T}}B) \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \equiv \begin{pmatrix} \phi(BB^{\mathrm{T}})u \\ \phi(B^{\mathrm{T}}B)v \end{pmatrix} \pmod 2.$$

It follows that $\phi(BB^{\mathrm{T}})u \equiv 0 \pmod 2$.

Note that $\begin{pmatrix} Bv \\ B^{\mathrm{T}}u \end{pmatrix} = \begin{pmatrix} 0 & B \\ B^{\mathrm{T}} & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = Aq \in \ker \phi(A)$. The same argument indicates that $\phi(BB^{\mathrm{T}})(Bv) \equiv 0 \pmod 2$. Claim 1 follows.

Since the irreducible polynomial $\phi(x)$ is a simple factor of $\chi(BB^T; x)$, we see that $\dim \ker \phi(A) = \deg \phi(x)$ and $\ker \phi(A)$ is a cyclic subspace generated by any nonzero vector in it. Let $d = \deg \phi(x)$. Note that $Bv \neq 0$. Then it follows from Claim 1 that (over $\mathbb{F}_2$)

$$\ker \phi(BB^{\mathrm{T}}) = \mathrm{span}\{Bv, (BB^{\mathrm{T}})Bv, \ldots, (BB^{\mathrm{T}})^{d-1}Bv\} \tag{8}$$

and

$$\ker \phi(BB^{\mathrm{T}}) = \mathrm{span}\{u, (BB^{\mathrm{T}})u, \ldots, (BB^{\mathrm{T}})^{d-1}u\} \text{ when } u \neq 0. \tag{9}$$

Claim 2: $\ker \phi(BB^{\mathrm{T}})$ is totally isotropic.

We first consider the case that $u \equiv 0 \pmod 2$. As $\begin{pmatrix} 0 \\ v \end{pmatrix} \in \mathrm{col}(\hat{Q})$, Lemma 12 implies that, for any $k \geq 0$,

$$(0^{\mathrm{T}}, v^{\mathrm{T}}) \begin{pmatrix} 0 & B \\ B^{\mathrm{T}} & 0 \end{pmatrix}^{2k} \begin{pmatrix} 0 \\ v \end{pmatrix} \equiv 0 \pmod 4, \text{ i.e., } v^{\mathrm{T}}(B^{\mathrm{T}}B)^k v \equiv 0 \pmod 4. \tag{10}$$

Note that $Bv, (BB^{\mathrm{T}})Bv, \ldots, (BB^{\mathrm{T}})$ constitute a basis of $\ker\phi(BB^{\mathrm{T}})$. To show that $\ker\phi(BB^{\mathrm{T}})$ is totally isotropic, it suffices to show that any two vectors α and β in the basis, say $\alpha = (BB^{\mathrm{T}})^i Bv$ and $\beta = (BB^{\mathrm{T}})^j Bv$, are orthogonal over $\mathbb{F}_2$. Direct computation shows that

$$\alpha^{\mathrm{T}}\beta = ((BB^{\mathrm{T}})^i Bv)^{\mathrm{T}}(BB^{\mathrm{T}})^j Bv) = v^{\mathrm{T}}(B^{\mathrm{T}}B)^{i+j+1}v.$$

Thus, by Eq. (10), we have $\alpha^{\mathrm{T}}\beta \equiv 0 \pmod 4$, which clearly implies $\alpha^{\mathrm{T}}\beta \equiv 0 \pmod 2$, or equivalently, $\alpha \perp \beta$ over $\mathbb{F}_2$. This proves Claim 2 for the case that $u \equiv 0 \pmod 2$.

Now assume $u \not\equiv 0 \pmod 2$. By Eqs. (8) and (9), it suffice to show that

$$(BB^{\mathrm{T}})^i u \perp (BB^{\mathrm{T}})^j Bv, \text{ over } \mathbb{F}_2, \text{ for any } i, j \geq 0. \tag{11}$$

As $\begin{pmatrix} u \\ v \end{pmatrix} \in \mathrm{col}(\hat{Q})$, Lemma 12 implies that, for any $k \geq 0$,

$$(u^{\mathrm{T}}, v^{\mathrm{T}}) \begin{pmatrix} 0 & B \\ B^{\mathrm{T}} & 0 \end{pmatrix}^{2k+1} \begin{pmatrix} u \\ v \end{pmatrix} = (u^{\mathrm{T}}, v^{\mathrm{T}}) \begin{pmatrix} (BB^{\mathrm{T}})^k & 0 \\ 0 & (B^{\mathrm{T}}B)^k \end{pmatrix} \begin{pmatrix} Bv \\ B^{\mathrm{T}}u \end{pmatrix} \equiv 0 \pmod 4,$$

i.e., $u^{\mathrm{T}}(BB^{\mathrm{T}})^k Bv + v^{\mathrm{T}}(B^{\mathrm{T}}B)^k B^{\mathrm{T}}u \equiv 0 \pmod 4$. Since the two additive terms on the left-hand side are equal (as can be seen by taking the transpose of one term), we obtain $u^{\mathrm{T}}(BB^{\mathrm{T}})^k Bv \equiv 0 \pmod 2$. Taking $k = i + j$, it follows that

$$((BB^{\mathrm{T}})^i u)^{\mathrm{T}}(BB^{\mathrm{T}})^j Bv = u^{\mathrm{T}}(BB^{\mathrm{T}})^{i+j} Bv \equiv 0 \pmod 2,$$

that is, Eq. (11) holds. This completes the proof of Claim 2.

Let $M = BB^{\mathrm{T}}$ and $U = \ker\phi(M)$. Clearly U is M-invariant. By Claim 2, U is totally isotropic. As $\phi(x)$ is a simple factor of $\chi(M; x)$, we find that $U \cap \ker\phi(M) = 0$ by Lemma 2. This is a contradiction as $U = \ker\phi(M)$ and is nonzero. The proof of Lemma 15 is complete. $\square$

Proof of Proposition 4 Let $f = \chi(BB^{\mathrm{T}}; x) \in \mathbb{F}_2[x]$. We know from Lemma 15, that $\phi(x)$ is a multiple factor of $f(x)$ over $\mathbb{F}_2$. Thus, Lemma 4 implies that $2 \mid \Delta(f)$, completing the proof. $\square$

3.2 The case p is odd

The main aim of this subsection is to prove Proposition 5 by contradiction.

Proof of Proposition 5 Suppose to the contrary that there exists an odd prime p such that $p \mid \ell(Q)$ but $p^2 \nmid \Delta_{BB^{\mathrm{T}}}$. Then by Lemma 5, we have $p \mid \Delta_A$. On the other hand, we know from Lemma 11 that $\Delta_A = 4^{\lfloor n/2 \rfloor}\Delta^2_{BB^{\mathrm{T}}}$. Thus, $p \mid \Delta_{BB^{\mathrm{T}}}$ as p is an odd prime. It follows from Lemma 6 that there exists an integer λ_0 and a polynomial $\phi(x)$ such that $\chi(BB^{\mathrm{T}}; x) = (x - \lambda_0)^2\varphi(x)$ over $\mathbb{F}_p$. Noting that $\chi(A; x) = x^\delta\chi(BB^{\mathrm{T}}; x^2)$ by Lemma 10, we obtain $\chi(A; x) = x^\delta(x^2 - \lambda_0)^2\varphi(x^2)$. Denote $\psi(x) = x^\delta\varphi(x^2)$. Then we can write

$$\chi(A; x) = (x^2 - \lambda_0)^2\psi(x) \text{ over } \mathbb{F}_p. \tag{12}$$

Since the constant term or linear coefficient of $\chi(A; x)$ is ± 1 (and hence nonzero modulo p), we find from Eq. (12) that $\lambda_0 \not\equiv 0 \pmod p$. Let S be the Sylvester matrix of $\chi(A; x)$ and its derivative $\chi'(A;x)$. Noting that $p^2 \nmid \Delta_{BB^{\mathrm{T}}}$ and $\Delta_A = 4^{\lfloor n/2 \rfloor}\Delta^2_{BB^{\mathrm{T}}}$, we obtain $p^3 \nmid \Delta_A$, or equivalently, $p^3 \nmid \det S$. It follows from Lemma 8 that $\dim\ker S \le 2$ over $\mathbb{F}_p$. Consequently, by Lemma 9, we obtain

$$\deg\gcd(\chi(A;x),\chi'(A;x)) = \dim\ker S \le 2.$$

Therefore, we see from Eq. (12) that $\psi(x)$ is squarefree and coprime to $(x^2-\lambda_0)$, since otherwise the polynomial $\gcd(\chi(A;x),\chi'(A;x)$ would have degree at least 3, a contradiction.

Claim: $p^2 \mid \det(\lambda_0 I - BB^{\mathrm{T}})$.

We prove the Claim by considering two cases:

Case 1: $x^2-\lambda_0$ is irreducible over $\mathbb{F}_p$.

As $A^2 = \mathrm{diag}(BB^{\mathrm{T}}, B^{\mathrm{T}}B)$, we find that $\chi(A^2;x) = \chi(BB^{\mathrm{T}};x)\chi(B^{\mathrm{T}}B;x)$ and hence

$$\chi(A^2;\lambda_0) = \chi(BB^{\mathrm{T}};\lambda_0)\chi(B^{\mathrm{T}}B;\lambda_0). \tag{13}$$

By Lemma 10, we have $\chi(B^{\mathrm{T}}B;x^2) = (x^2)^\delta\chi(BB^{\mathrm{T}};x^2)$, i.e., $\chi(B^{\mathrm{T}}B;x) = x^\delta\chi(BB^{\mathrm{T}};x)$. Taking $x=\lambda_0$, we obtain $\chi(B^{\mathrm{T}}B;\lambda_0) = \lambda_0^\delta\chi(BB^{\mathrm{T}};\lambda_0)$. Noting that $\lambda_0 \ne 0 \pmod p$, we find that, for any $k \ge 1$,

$$p^k \mid \chi(BB^{\mathrm{T}};\lambda_0) \text{ if and only if } p^k \mid \chi(B^{\mathrm{T}}B;\lambda_0). \tag{14}$$

On the other hand, since $\psi(x)$ is squarefree coprime to $x^2-\lambda_0$, we see that $\phi(x) = (x^2-\lambda_0)$ is the only multiple factor of $\chi(A;x)$. It follows from Proposition 2 that $p^3 \mid \det\phi(A)$, i.e., $p^3 \mid \chi(A^2;\lambda_0)$. This, combining with Eq. (13) and Eq. (14) for $k=2$, leads to $p^2 \mid \chi(BB^{\mathrm{T}};\lambda_0)$.

Case 2: $x^2-\lambda_0$ is reducible over $\mathbb{F}_p$, i.e., $x^2-\lambda_0 \equiv (x-\lambda_1)(x+\lambda_1) \pmod p$ for some $\lambda_1 \in \mathbb{Z}$.

According to Remark 2, the truth of the Claim does not dependent on the choice of λ_0 in its residue class modulo p. Consequently, since $\lambda_0 \equiv \lambda_1^2 \pmod p$, we may safely assume $\lambda_0 = \lambda_1^2$. It follows from Lemma 10 that

$$\chi(A;\lambda_1) = \lambda_1^\delta\chi(BB^{\mathrm{T}};\lambda_0) \text{ and } \chi(A;-\lambda_1) = (-\lambda_1)^\delta\chi(BB^{\mathrm{T}};\lambda_0). \tag{15}$$

As $\lambda_0 \not\equiv 0 \pmod p$, we see that $\lambda_1 \not\equiv 0 \pmod p$. Note that $\chi(A;x)$ has exactly two multiple factors $\phi_1(x) = x-\lambda_1$ and $\phi_2(x) = x+\lambda_1$. It follows from Proposition 2 that $p^2 \mid \det\phi_1(A)$ or $p^2 \mid \det\phi_2(A)$. Nevertheless, either implies $p^2 \mid \chi(BB^{\mathrm{T}};\lambda_0)$ according to Eq. (15).

This proves the Claim. By Lemma 7, the equation

$$\chi(BB^{\mathrm{T}};x)u(x) \equiv \chi(BB^{\mathrm{T}};x)'v(x) \pmod{p^2}$$

has a solution $(u(x),v(x)) \in (\mathbb{Z}[x])^2$ satisfying:

$$u(x), v(x) \not\equiv 0 \pmod p,\ \deg(u(x)) < \lfloor n/2 \rfloor - 1, \text{ and } \deg(v) < \lfloor n/2 \rfloor.$$

Let S be the Sylvester matrix of $\chi(BB^{\mathrm{T}};x)$ and its derivative $\chi'(BB^{\mathrm{T}};x)$. Using a similar argument as in the proof of Lemma 9, the existence of a solution (u,v) described above means that the linear equations $S^{\mathrm{T}}\eta \equiv 0 \pmod{p^2}$ has a solution $\eta \not\equiv 0 \pmod p$.

On the other hand, as $p^2 \nmid \Delta_{BB^{\mathrm{T}}}$ and $\Delta_{BB^{\mathrm{T}}} = \pm\det S^{\mathrm{T}}$, we have $p^2 \nmid \det S^{\mathrm{T}}$. Consequently, $p^2 \nmid d_n$, where d_n denotes the last invariant of S^{T}. This contradicts Lemma 8 (iv) and hence completes the proof of Proposition 5.

4 Examples

In this section, we present some examples to illustrate Theorem 2. All computations were performed using Wolfram Mathematica.

Example 1: Let $n = 13$ and Σ be the signed bipartite graph with adjacency matrix as follows:

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & -1 & 0 & -1 & -1 \\ -1 & 1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

We obtain the factorization of $\chi(A; x)$ over $\mathbb{Q}$:

$$-x\left(x^{12} - 24x^{10} + 194x^8 - 679x^6 + 1022x^4 - 496x^2 + 1\right),$$

along with

$$\det W(\Sigma) = 2^6 \times 11^3 \times 3413 \times 697913,$$

and

$$2^{-n/2}\sqrt{\Delta_\Sigma} = 107 \times 15259 \times 12978894869.$$

By Theorem 2, we conclude that Σ is DGS.

Example 2: Let $n = 14$ and Σ be the controllable signed bipartite graph with adjacency matrix as follows:

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The characteristic polynomial $\chi(A; x)$ factors over $\mathbb{Q}$ as:

$$\left(x^7 - x^6 - 6x^5 + 4x^4 + 9x^3 - 4x^2 - 3x + 1\right)\left(x^7 + x^6 - 6x^5 - 4x^4 + 9x^3 + 4x^2 - 3x - 1\right),$$

with $\det W(\Sigma) = 2^{14}$ and $2^{-n/2}\sqrt{\Delta_\Sigma} = 17 \times 23 \times 64879$. We observe that this case does not satisfy the conditions of Theorem 1. However, by Theorem 2, we conclude that Σ is DGS.

Example 3: Let $n = 14$ and Σ be the almost controllable signed bipartite graph with adjacency matrix as follows:

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & -1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The characteristic polynomial $\chi(A;x)$ factors over $\mathbb{Q}$ as:

$$(x-1)(x+1)\left(x^{12}-15x^{10}+75x^{8}-151x^{6}+111x^{4}-23x^{2}+1\right),$$

with rank $W(\Sigma)=13$ and $2^{-n/2}\sqrt{\Delta_\Sigma}=13\times 45953\times 106501$. By Theorem 2, we conclude that Σ is DGS.

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