scientific reports

OPEN

The giant increase of stiffness of inhomogeneous rods and beams

A. A. Kolpakov[1]✉ & A. G. Kolpakov[2]

We demonstrate that stiffnesses of an inhomogeneous beam of coaxial structure coincide with the ones predicted by classical Bernoulli–Euler and Saint-Venant theories if and only if the indicated below conditions on the local Poisson's ratio are satisfied. If the conditions are not satisfied, the stiffnesses of the inhomogeneous beam exceed the stiffnesses predicted by classical theories. The difference in Poisson's ratios of the components of the rod/beam can result in a giant increase in stiffness when using materials possessing a negative Poisson's ratio.

The classical Bernoulli–Euler and Saint-Venant beam theories were presented in[1–3], see also[4]. While the local deformations of the beam are very different in the classical theories (Bernoulli–Euler theory ignores and the Saint-Venant theory accounts the transverse deformations of the beam), the stiffnesses predicted by both the theories turn out to be the same. Following the classical theories, advanced rod/beam theories were developed, see[5–8] and references herein. At the last time, the inhomogeneous beams (in particular, laminated beams, which are a special case of the coaxial beams) attract attention of the researchers, see, e.g.,[9,10]. The general theory of inhomogeneous beams was developed by using the ideas of the asymptotic homogenization theory[11], see, e.g.,[12–14] and references herein. However, the relationship of the stiffnesses calculated in various ways is not fully clarified until now.

We consider a beam which axis coincides with the Ox_1-axis of the standard orthonormal coordinate system $Ox_1x_2x_3$ ($\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are the unit basis vectors). In accordance with[12–14], the displacement of the beam $\mathbf{u} \approx u_1(x_1)\mathbf{e}_1 + w_A(x_1)\mathbf{e}_A + \varepsilon \mathbf{N}^0(\mathbf{x}/\varepsilon)u_1'(x_1) + \varepsilon \mathbf{N}^{1A}(\mathbf{x}/\varepsilon)w_A''(x_1)$, where $u_1(x_1)$ is the overall axial displacement and $w_A(x_1)$ $(A = 2, 3)$ are the overall normal deflections of the beam. The functions $\mathbf{N}^0$ and $\mathbf{N}^{1A}$ are solutions to the so-called periodicity cell problem[12]:

$$\begin{cases} (a_{ijkl}(y_2, y_3)N^{\lambda A}_{k,l} + (-1)^{\lambda} a_{ij11}(y_2, y_3) y_A^{\lambda})_{,jy} = 0 \text{ in } \mathbf{P}, \\ (a_{ijkl}(y_2, y_3)N^{\lambda A}_{k,l} + (-1)^{\lambda} a_{ij11}(y_2, y_3) y_A^{\lambda}) n_{\alpha} = 0 \text{ on } \Gamma, \\ \mathbf{N}^{\lambda A}(\mathbf{y}) \text{ periodic in } y_1 \end{cases} \tag{1}$$

which describes the local deformations of the periodicity cell $\mathbf{P}$ of the beam (Γ is the lateral surface of $\mathbf{P}$). Problem (1) with $\lambda = 0$ corresponding to the overall axial deformations, with $\lambda = 1$—the overall bending in the Oy_1y_A-plane, see Fig. 1c. Following[11], we use the original $\mathbf{x}$ and the local $\mathbf{y} = \mathbf{x}/\varepsilon$ coordinates.

Stiffnesses $d^{\lambda+\mu}_{AB}$ of the beam ($\lambda, \mu = 0, 1, A, B = 2, 3$) are calculated according to the formula[12–14]

$$d^{\lambda+\mu}_{AB} = (-1)^{\mu} L^{-1} \int_{\mathbf{P}} (a_{11kl}(\mathbf{y}) N^{\lambda A}_{k,ly} + (-1)^{\lambda} a_{1111}(\mathbf{y}) y_A^{\lambda}) y_B^{\mu} \, d\mathbf{y}. \tag{2}$$

Formula (2) determines the tensile stiffness d^0 for $\lambda = \mu = 0$, (it does not have the index AB) and bending stiffnesses for $\lambda = \mu = 1$.

We will analyze problem (1) for a beam of coaxial structure made of isotropic material(s) and demonstrate that the stiffnesses (2) coincide with the classical ones if and only if the local Poisson's ratio satisfies a condition derived below. If the condition is not satisfied, the stiffnesses (2) are greater than the stiffnesses of the corresponding homogeneous beam. Our numerical calculations show that the difference of the stiffnesses can be very large.

[1]University of Neuchatel, Rue Emile-Argand 11, 2000 Neuchâtel, Switzerland. [2]SysAn (System Analysis in Engineering), Ul. Aleksandra Nevskogo, 12a, Novosibirsk, Russia 630075. ✉email: kolpakov.alexander@gmail.com

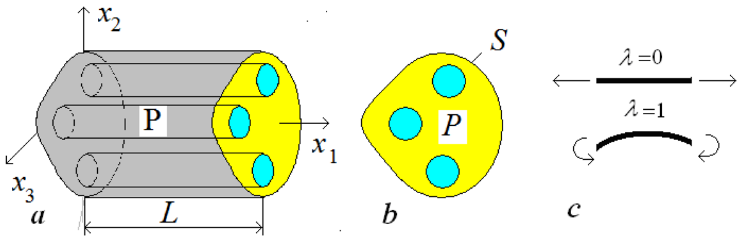

Figure 1. The periodicity cell **P** of the cylindrical rod/beam—(**a**); cross section P—(**b**); deformations of the rod ($\lambda = 0$) and beam ($\lambda = 1$)—(**c**).

Rods/beams of coaxial structure. Transition to two-dimensional problems

The beam under consideration is a thin inhomogeneous cylinder of characteristic diameter ε made of isotropic material, Fig. 1a,b. Its elastic constants dependent on y_2, y_3 only (do not depend on y_1): $a_{ijkl} = a_{ijkl}(y_2, y_3)$ and the periodicity cell (representative fragment) of the beam can be chosen as $\mathbf{P} = [0, L] \times P$, Fig. 1a, where L is an arbitrary positive number, P is the cross section of the rod/beam, Fig. 1b.

For the beam under consideration, solution to problem (1) has the form $\mathbf{N}^{\lambda A}(\mathbf{y}) = \mathbf{N}^{\lambda A}(y_2, y_3)$. Since the material is isotropic, we obtain from the first ($i = 1$) equation in (1) that $N_1^{\lambda A}(y_2, y_3) = 0$. For $i = 2, 3$, we have from (1) the following planar problem ($\alpha, \beta, \gamma, \delta = 2, 3$):

$$\begin{cases} (a_{\alpha\beta\gamma\delta}(y_2, y_3) N^{\lambda A}_{\gamma,\delta} + (-1)^{\lambda} a_{\alpha\beta 11}(y_2, y_3) y_A^{\lambda})_{,\beta y} = 0 \text{ in } P, \\ (a_{\alpha\beta\gamma\delta}(y_2, y_3) N^{\lambda A}_{\gamma,\delta} + (-1)^{\lambda} a_{\alpha\beta 11}(y_2, y_3) y_A^{\lambda}) n_\beta = 0 \text{ on } S. \end{cases} \tag{3}$$

The equalities in (3) are satisfied if the expression in brackets in (3) is equal to zero. Let us ask ourselves: are there any strains $e^A_{\gamma\delta}$ (possibly inconsistent) that deliver a solution to the following algebraic system of equations?

$$a_{\alpha\beta\gamma\delta}(y_2, y_3) e^A_{\gamma\delta} + (-1)^{\lambda} a_{\alpha\beta 11}(y_2, y_3) y_A^{\lambda} = 0. \tag{4}$$

Equation (4) in coordinate form is the following:

$$a_{2222} e^A_{22} + a_{2233} e^A_{33} = -(-1)^{\lambda} a_{2211}(y_2, y_3) y_A^{\lambda},\ a_{3322} e^A_{22} + a_{3333} e^A_{33} = -(-1)^{\lambda} a_{3311}(y_2, y_3) y_A^{\lambda},\ a_{2323} e^A_{23} = 0. \tag{5}$$

For isotropic materials, elastic constants

$$a_{2222} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)},\ a_{1122} = \frac{E\nu}{(1+\nu)(1-2\nu)},\ a_{2222} + a_{2233} = \frac{E}{(1+\nu)(1-2\nu)}, \tag{6}$$

where $E = E(y_2, y_3)$ is the Young's modulus and $\nu = \nu(y_2, y_3)$ is the Poisson's ratio[4]. Solution to (5) with the coefficients (6) is the following ($\delta_{\alpha\beta}$ is Knonecker's delta):

$$e^{\lambda A}_{\alpha\beta} = -(-1)^{\lambda} \delta_{\alpha\beta} \nu(y_2, y_3) y_A^{\lambda} \ (\lambda, \mu = 0, 1). \tag{7}$$

Solution (7) for homogeneous beams has been known since the time of Saint-Venant[3]. Also, solution (7) remains valid for inhomogeneous beams.

Taking into account (4) and (7), problem (3) can be rewritten in the form

$$\begin{cases} (a_{\alpha\beta\gamma\delta}(y_2, y_3)(N^{\lambda A}_{\gamma,\delta} - e^{\lambda A}_{\gamma\delta}))_{,\beta} = 0 \text{ in } P, \\ a_{\alpha\beta\gamma\delta}(y_2, y_3)(N^{\lambda A}_{\gamma,\delta} - e^{\lambda A}_{\gamma\delta}) n_\beta = 0 \text{ on } S. \end{cases} \tag{8}$$

For the cylindrical periodicity (representative) cell $L^{-1} \int_{\mathbf{P}} d\mathbf{y} = \int_P dy_2 dy_3$, and thus the general formula (2) takes the form

$$d^{\lambda+\mu}_{AB} = (-1)^{\mu} \int_P (a_{11\alpha\beta}(\mathbf{y}) N^{\lambda A}_{\alpha,\beta} + (-1)^{\lambda} a_{1111}(\mathbf{y}) y_A^{\lambda}) y_B^{\mu} dy_2 dy_3. \tag{9}$$

If the strains (7) are compatible, i.e., there exists displacement $\mathbf{N}^{\lambda A}(\mathbf{y})$, such that

$$N^{\lambda A}_{\alpha,\beta} = -(-1)^{\lambda} \delta_{\alpha\beta} \nu(y_2, y_3) y_A^{\lambda}, \tag{10}$$

then this $\mathbf{N}^{\lambda A}(\mathbf{y})$ is the solution to problem (3). Substituting (10) into (9) leads to the equality

$$d^{\lambda+\mu}_{AB} = (-1)^{\lambda+\mu} \int_P (a_{1111}(y_2, y_3) - a_{1122}(y_2, y_3)\nu(y_2, y_3) - a_{1133}(y_2, y_3)\nu(y_2, y_3)) y_2^{\lambda+\mu} dy_2 dy_3.$$

From (6) it follows that $a_{1111} - a_{1122}\nu - a_{1133}\nu = E$, then the equation above takes the form (hereafter "$\triangleq$" means "equal by definition")

$$d_{AB}^{\lambda+\mu} = D_{AB}^{\lambda+\mu} \triangleq (-1)^{\lambda+\mu} \int\limits_P E(y_2, y_3) y_A^{\lambda} y_B^{\mu} dy_2 dy_3. \tag{11}$$

Hereafter, $D_{AB}^{\nu+\mu}$ mean the classical stiffnesses, in particular, $D^0 = \int\limits_P E(y_2, y_3) dy_2 dy_3$ is the tensile stiffness, $D_{AA}^2 = \int\limits_P E(y_2, y_3) y_A^2 dy_A dy_3$ $(A = 2, 3)$ are the bending stiffnesses.

Equality (11) means that if the strains (7) are compatible, the stiffnesses in the classical and the asymptotic theories coincide. So, it can be concluded that the compatibility of strains $e_{\alpha\beta}^{\lambda A}$ (7) is a key point in the analysis of rod/beam stiffnesses.

The case of compatible strains (7)

Let's denote $Inc \triangleq \frac{\partial^2}{\partial y_3^2} + \frac{\partial^2}{\partial y_{22}^2} - 2\frac{\partial^2}{\partial y_2 \partial y_3}$ the operator characterizing the incompatibility of planar strains[4]. The strains are compatible if and only if $Inc(e_{\alpha\beta}) = 0$[4]. For the strains $e_{\alpha\beta}^{\lambda A} = -(-1)^{\lambda}\delta_{\alpha\beta}\nu(y_2, y_3) y_A^{\lambda}$ (7), the equality $Inc(e_{\alpha\beta}) = 0$ takes the form

$$\Delta(\nu(y_2, y_3) y_A^{\lambda}) = 0 \ (\lambda = 0, 1), \tag{12}$$

Equality (12) is a sufficient condition for the stiffness of the considered composite beam to coincide with the classical ones. Below, we will show that condition (12) is also necessary for the stiffness of the rod/beam to coincide with the classical ones.

The case of incompatible strains (7). Non-classical stiffnesses

Let us consider the problem (8), (9) for incompatible $e_{\alpha\beta}^{\lambda A}$ (7). If (12) fails, then $N_{\alpha,\beta}^{\lambda A} \neq e_{\alpha\beta}^{\lambda A}$ for any displacements $\mathbf{N}^{\lambda A}(y_2, y_3)$. Let's write the problem under consideration with respect to the function $N_{\alpha,\beta y}^{\lambda A} - e_{\alpha\beta}^{\lambda A}$.

We write (9) as follows:

$$d_{AB}^{\lambda+\mu} = (-1)^{\mu} \int\limits_P (a_{11\alpha\beta}(y_2, y_3)(N_{\alpha,\beta y}^{\lambda A} - e_{\alpha\beta}^{\lambda A}) + a_{11\alpha\beta}(y_2, y_3) e_{\alpha\beta}^{\lambda A} + (-1)^{\lambda} a_{1111}(y_2, y_3) y_A^{\lambda}) y_B^{\mu} dy_2 dy_3.$$

The integral of the last two terms is equal to the classical stiffness $D_{AB}^{\lambda+\mu}$ (11). Then

$$d_{AB}^{\lambda+\mu} = D_{AB}^{\lambda+\mu} + (-1)^{\mu} \int\limits_P a_{11\alpha\beta}(y_2, y_3)(N_{\alpha,\beta}^{\lambda A} - e_{\alpha\beta}^{\lambda A}) y_B^{\mu} dy_2 dy_3. \tag{13}$$

The problem (8) is yet written in the term of the function $N_{\alpha,\beta y}^{\lambda A} - e_{\alpha\beta}^{\lambda A}$.
Stresses $\sigma_{\alpha\beta}$ corresponding to the problem (8) are the following:

$$\sigma_{\alpha\beta} = a_{\alpha\beta\gamma\delta}(y_2, y_3)(N_{\gamma,\delta}^{\lambda A} - e_{\gamma\delta}^{\lambda A}) = \sigma_{\alpha\beta}^0 + \sigma_{\alpha\beta}^i, \tag{14}$$

where

$$\sigma_{\alpha\beta}^0 = a_{\alpha\beta\gamma\delta}(y_2, y_3) N_{\gamma,\delta}, \ \sigma_{\alpha\beta}^i = -a_{\alpha\beta\gamma\delta}(y_2, y_3) e_{\gamma\delta}. \tag{15}$$

Let us introduce a function $\Phi(y_2, y_3)$ $(\Phi = \Phi^{\lambda A})$ (Airy-type function) by the equalities

$$\sigma_{22} = \frac{\partial^2 \Phi}{\partial y_3^2}, \ \sigma_{23} = -\frac{\partial^2 \Phi}{\partial y_2 \partial y_3}, \ \sigma_{33} = \frac{\partial^2 \Phi}{\partial y_2^2}. \tag{16}$$

The quantities $\sigma_{\alpha\beta}$ (16) satisfy the equilibrium equations $\sigma_{\alpha\beta,\beta} = 0$ corresponding to (8).
The compatibility condition may be written in the terms of stresses. For isotropic material it has the form[4]

$$Inc(\sigma_{\alpha\beta}) \triangleq \frac{\partial^2}{\partial y_3^2}\left[\frac{1+\nu}{E}(\sigma_{22} - \nu(\sigma_{22} + \sigma_{33}))\right] + \frac{\partial^2}{\partial y_{22}^2}\left[\frac{1+\nu}{E}(\sigma_{33} - \nu(\sigma_{22} + \sigma_{33}))\right] - 2\frac{\partial^2}{\partial y_2 \partial y_3}\left[\frac{1+\nu}{E}\sigma_{23}\right] = 0. \tag{17}$$

Substituting into (17) σ_{22}, σ_{23}, σ_{33} in accordance with (16), we obtain

$$Inc(\sigma_{\alpha\beta}) = \frac{\partial^2}{\partial y_3^2}\left[\frac{1+\nu}{E}\left(\frac{\partial^2 \Phi}{\partial y_3^2} - \nu\Delta\Phi\right)\right] + \frac{\partial^2}{\partial y_2^2}\left[\frac{1+\nu}{E}\left(\frac{\partial^2 \Phi}{\partial y_2^2} - \nu\Delta\Phi\right)\right] + 2\frac{\partial^2}{\partial y_2 \partial y_3}\left[\frac{1+\nu}{E}\frac{\partial^2 \Phi}{\partial y_2 \partial y_3}\right] \triangleq L\Phi. \tag{18}$$

By virtue of (14), $Inc(\sigma_{\alpha\beta}) = Inc(\sigma_{\alpha\beta}^0 + \sigma_{\alpha\beta}^i)$. Since operator Inc is linear and stresses $\sigma_{\alpha\beta}^0$ are compatible, then $Inc(\sigma_{\alpha\beta}) = Inc(\sigma_{\alpha\beta}^i)$. Let's calculate $Inc(\sigma_{\alpha\beta}^i)$ (17) for $\sigma_{\gamma\delta}^i$ (15). Substituting $e_{\gamma\delta} = e_{\alpha\beta}^{\lambda A} = -(-1)^{\lambda}\delta_{\alpha\beta}\nu(y_2, y_3) y_A^{\lambda}$ (7) into (15), we get $\sigma_{\gamma\delta}^i = a_{\gamma\delta\alpha\alpha}(y_2, y_3)(-1)^{\lambda}\nu(y_2, y_3) y_A^{\lambda}$. From these equations, we have

$$\begin{aligned}\sigma^i_{22} &= (a_{2222}(y_2, y_3) + a_{2233}(y_2, y_3))(-1)^\lambda \nu(y_2, y_3) y^\lambda_A,\\ \sigma^i_{33} &= (a_{3322}(y_2, y_3) + a_{3333}(y_2, y_3))(-1)^\lambda \nu(y_2, y_3) y^\lambda_A.\end{aligned} \tag{19}$$

For isotropic material $a_{2222} + a_{2233} = a_{3333} + a_{3322} = \frac{E}{(1+\nu)(1-2\nu)}$ and we obtain from (19) that $\sigma^i_{22} = \sigma^i_{33} = (-1)^\lambda \frac{E\nu}{(1+\nu)(1-2\nu)} y^\lambda_A$. Also, $\sigma^i_{23} = \sigma^i_{32} = 0$. Substituting these $\sigma^i_{\alpha\beta}$ into (17), we have $Inc(\sigma^i_{\alpha\beta}) = (-1)^\lambda \Delta[\nu y^\lambda_A]$. Finally, we obtain from (18) that $L\Phi^{\lambda A} = (-1)^\lambda \Delta[\nu y^\lambda_A]$.

Let's write the boundary condition $\sigma_{\alpha\beta} n_\beta = 0$ on S (8) in the term of function Φ. Substituting $\sigma_{22}, \sigma_{23}, \sigma_{33}$ according to (16), we obtain $\frac{\partial^2\Phi}{\partial y_3^2} n_2 - \frac{\partial^2\Phi}{\partial y_2 \partial y_3} n_3 = 0$ and $-\frac{\partial^2\Phi}{\partial y_2 \partial y_3} n_2 + \frac{\partial^2\Phi}{\partial y_2^2} n_3 = 0$ on S. Since $\frac{\partial}{\partial y_3} n_2 - \frac{\partial}{\partial y_2} n_3 = \frac{d}{ds}$ is the operator of differentiation along the boundary S, we obtain $\frac{d}{ds}\frac{\partial\Phi}{\partial y_3} = 0, \frac{d}{ds}\frac{\partial^2\Phi}{\partial y_2} = 0$ on S. From here we obtain after some computations $\frac{\partial\Phi}{\partial \mathbf{n}} = \Phi = 0$ on S.

Non-classical corrections to the stiffnesses

Equality (13) means that $d^{\lambda+\mu}_{AB} = D^{\lambda+\mu}_{AB} + \delta D^{\lambda+\mu}_{AB}$, where $D^{\lambda+\mu}_{AB}$ are the classical stiffnesses (11), and $\delta D^{\lambda+\mu}_{AB} = \int_P a_{11\alpha\beta}(y_2, y_3)(N^{\lambda A}_{\alpha,\beta} - e^{\lambda 2}_{\alpha\beta}) y^\mu_B dy_2 dy_3$ are the (non-classical) corrections. Let's investigate them further. With regard to (6), we have

$$\delta D^{\lambda+\mu}_{AB} = (-1)^\mu \int_S \frac{E\nu}{(1+\nu)(1-2\nu)} \left[(N^{\lambda A}_{2,2} - e^{\lambda A}_{22}) + (N^{\lambda A}_{3,3} - e^{\lambda A}_{33})\right] y^\mu_B dy_2 dy_3. \tag{20}$$

Using (17), we express $N^{\lambda A}_{2,2} - e^{\lambda A}_{22}$ and $N^{\lambda A}_{3,3} - e^{\lambda A}_{33}$ from (14) as follows:

$$N^{\lambda A}_{2,2} - e^{\lambda A}_{22} = \frac{[\sigma_{22}(1-\nu) - \sigma_{33}\nu](1+\nu)}{E},\ N^{\lambda A}_{3,3} - e^{\lambda A}_{33} = \frac{[(1-\nu)\sigma_{33} - \sigma_{22}\nu](1+\nu)}{E}.$$

Substituting here σ_{22} and σ_{33} according to (16), we arrive at the equalities ($\Phi = \Phi^{\lambda A}$)

$$N^{\lambda A}_{2,2} - e^{\lambda A}_{22} = \frac{\left[\frac{\partial^2\Phi}{\partial y_3^2}(1-\nu) - \frac{\partial^2\Phi}{\partial y_{22}^2}\nu\right](1+\nu)}{E},\ N^{\lambda A}_{3,3} - e^{\lambda A}_{33} = \frac{\left[(1-\nu)\frac{\partial^2\Phi}{\partial y_{22}^2} - \frac{\partial^2\Phi}{\partial y_3^2}\nu\right](1+\nu)}{E}. \tag{21}$$

Substituting (21) into (20), we find that the non-classical correction to the stiffness $D^{\nu+\mu}_{AB}$ is

$$\delta B^{\lambda+\mu}_{AB} = (-1)^\mu \int_P \nu(y_2, y_3) \Delta\Phi^{\lambda A} y^\mu_B dy_2 dy_3. \tag{22}$$

It is seen that the non-classical correction takes place both for the tensile stiffness D^0 (the case $\lambda = \mu = 0$) and for bending stiffnesses D^2_{22}, D^2_{33} (the case $\lambda = \mu = 1$).

The problem of non-classical corrections to stiffness

Finally, we arrive at the following boundary-value problem ($\Phi = \Phi^{\lambda A}$, $L\Phi$ is defined in (18)):

$$\begin{cases} L\Phi = (-1)^\lambda \Delta[\nu(y_2, y_3) y^\lambda_A] \text{ in } D,\\ \Phi = \dfrac{\partial\Phi}{\partial\mathbf{n}} = 0 \text{ on } S,\end{cases} \tag{23}$$

and the following Main question: What are the properties of the non-classical corrector $\delta B^{\lambda+\mu}_{AB}$ (22), where $\Phi = \Phi^{\lambda A}$ is determined from the boundary-value problem (23)?

Energy identity and inequalities for the stiffnesses

Multiplying the LHP equation in (23) by Φ ($\Phi = \Phi^{\lambda A}$, $L\Phi$ is defined in (18)) and then integrating twice by parts, taking into account $\Phi = \frac{\partial\Phi}{\partial\mathbf{n}} = 0$ on S, we obtain as a result

$$\int_P \left\{\left[\frac{1+\nu}{E}\left(\frac{\partial^2\Phi}{\partial y_3^2} - \nu\Delta\Phi\right)\right]\frac{\partial^2\Phi}{\partial y_3^2} + \left[\frac{1+\nu}{E}\left(\frac{\partial^2\Phi}{\partial y_2^2} - \nu\Delta\Phi\right)\right]\frac{\partial^2\Phi}{\partial y_2^2} + 2\left[\frac{1+\nu}{E}\frac{\partial^2\Phi}{\partial y_2 \partial y_3}\right]\frac{\partial^2\Phi}{\partial y_2 \partial y_3}\right\} dy_2 dy_3. \tag{24}$$

Denote (24) $E(\Phi)$ (it has the meaning of the energy).

Multiplying the RHP equation in (23) by Φ ($\Phi = \Phi^{\lambda A}$) and then integrating twice by parts, we obtain $(-1)^\lambda \int_P \Delta[\nu y^\lambda_A]\Phi dy_2 dy_3 = (-1)^\lambda \int_P \nu y^\lambda_A \Delta\Phi dy_2 dy_3$. Finally, we obtain the energy identity

$$E(\Phi^{\lambda A}) = (-1)^\lambda \int_P \nu(y_2, y_3) y^\lambda_A \Delta\Phi^{\lambda A} dy_2 dy_3. \tag{25}$$

Energy $E(\Phi) \geq 0$ for any Φ, and $E(\Phi) > 0$ if the second derivatives of Φ are nonzero. To verify this, compute the eigen values of the quadratic form in the integrand in (24). They are positive: $\lambda_1 = 1$, $\lambda_2 = 1 - 2\nu > 0$ since Poisson's ratio of an isotropic material $\nu \leq 0.5$[4].

From (22) and (25), we obtain $\delta D_{AB}^{\lambda+\mu} = (-1)^{\mu+\nu} E(\Phi^{\lambda A})$.

Inequalities for stiffnesses
If condition (12) is not met, the tensile stiffness d^0 and bending stiffnesses d_{22}^2 and d_{33}^2 are greater the classical stiffness: $d^0 > D^0$, $d_{22}^2 > D_{22}^2$, $d_{33}^2 > D_{33}^2$.

This follows from the equality $d_{AB}^{\lambda+\mu} = D_{AB}^{\lambda+\mu} + \delta D_{AB}^{\lambda+\mu}$ and the equality of correctors δD^0 ($\lambda = \mu = 0$) and δD_{AA}^2 ($\lambda = \mu = 1$) to the energy $E(\Phi) > 0$ (with proper $\Phi = \Phi^{\lambda A}$).

Numerical experiments to determine the stiffnesses of a rod/beam of a coaxial structure
Let us present the results of numerical calculations confirming our theoretical conclusions and showing that the corrector $\delta D^{\lambda+\mu}$ may take very large values.

Note that the numerical solutions of the problems of elasticity theory, performed on modern FEM software, can be considered equivalent, in terms of accuracy and reliability of the results, to both analytical solutions and full-scale experiments.

Tension
The tensile stiffness corrector δD^0 is non-zero, only if the Poisson's ratio is not a harmonic function. An example of a non-harmonic function is a piecewise-constant function. The piecewise-constant function corresponds to a rod/beam made of several homogeneous materials widely used in practice. Present our numerical solutions to the problem of tension of a cylindrical rod made of two coaxial circular cylinders, Fig. 2a. Although we reduced the original three-dimensional periodicity problem (1) to two-dimensional problem (8), solution to the problem (1) is more illustrative. In our computations $R_1 = 1$, $R_2 = 2$, $L = 1$, Fig. 2a. Young's moduli of both materials are taken equal to 1 (the problem is linear, and we may take Young's modules equal to 1 without loss of generality). The boundary conditions corresponding to the axial tension: displacement $u_1 = 0$ on one side of the disk, $u_1 = 1$ on another side. Due to the linearity of the problem, such non-physical values (calculated strains are of the order of 100%) does not lead to any problems, but makes the calculation results more visual.

Figure 2b shows the numerically computed deformed cell and the von Misess stress value. Figure 2c shows cross-sections of the rod before (left) and after (right) deformation. Note that we consider the tension of the rod and see that the diameter of the rod increases under tension. This is not very surprising since one of the components of the rod has negative Poisson's ratio. What is surprising is the giant increase in the tensile rigidity of the rod. The numerically computed tensile stiffness d^0 and ratio $\delta D^0/D^0$ are presented in Table 1 for various ν_1 and ν_2. The classical stiffness $D^0 = 1$.

The theoretical possible value of the Poisson's ratio of isotropic materials ranges from −1 to 0.5[4]. The values of Poisson's ratio of widely used elastic isotropic materials range from 0 (cork) to 0.3 (metals) and 0.4 (plastics). The existence of materials with negative Poisson's ratios has been confirmed both experimentally and theoretically[15,16]. Poisson's ratio of isotropic material can achieve the value −1[17]. See also reviews[18–24] and references therein.

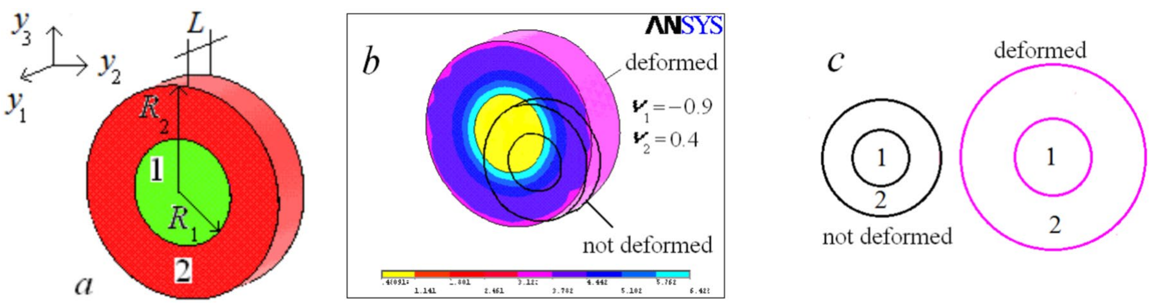

Figure 2. Fragment of rod of coaxial structure—(**a**), numerical solution to the problem about deformation of this fragment (tension along Oy_1-axis)—(**b**), deformation of the cross-section—(**c**).

ν_1	0.3	0.1	0.4	0.3	0.45	−0.9	−0.99
ν_2	0.3	0.4	0	−0.9	−0.95	0.4	0.49
d^0	1	1.02	1.03	1.33	1.47	2.67	21.68
$\delta D^0/D^0\%$	0%	2%	3%	33%	47%	167%	2068%

Table 1. Tensile stiffnesses of rod.

ν_1	0.4	− 0.9	− 0.95
ν_2	0.4	0.4	0.45
$\delta D_{22}^2/D_{22}^2\%$	0	30%	49%

Table 2. Bending stiffnesses of beam.

Bending

The problem (1) corresponding to the bending was solved for the three-dimensional cell shown in Fig. 2a. For the bending, the displacement is linearly dependent on the transverse coordinate: $u_z = x_2$ at the side $x_1 = 2$ (here $L = 2$), and $u_z = 0$ on the side $x_1 = 0$. Due to the symmetry of the beam in Fig. 2a, these boundary conditions correspond to the bending. The ratio $\delta D_{22}^2/D_{22}^2$ for various ν_1 and ν_2 are given in Table 2.

Conclusion

The stiffnesses of rods/beams of coaxial structure coincide with the stiffnesses predicted by Bernoulli–Euler and Saint-Venant theories if and only if the condition $\Delta(\nu(y_2, y_3)y_A^\nu) = 0$ is satisfied.

When the condition above is not satisfied, the stiffnesses of composite rod/beam are higher than those given by the classical theories.

Differences in value of the stiffnesses given by the asymptotic theory from those given by classical theories can be significant for rods/beams formed from materials with a Poisson's ratio close to -1 and close to 0.5. Although such materials are currently "exotic", their existence is theoretically justified and materials of this kind are being manufactured.

Prospectives

Our results indicate one more potential way to produce high stiffness fibers, which have wide applications ranging from aircraft to sport equipment[25]. In the example above, the diameter of the rod decreases under the axial compression and the stiffnesses are large. Such a combination looks promising for the design of tools possessing high penetration properties. The strength of the rods/beams made of unusual materials is still an open problem. Until the strength properties are clarified, the discussed rods/beams may be recommend for design of tools penetrating into soft materials, for example, the biological matter ([26,27] present examples of the use the materials with negative Poisson's ratio in biomedical applications).

Data availability

All data is available from the corresponding author (A.A.) upon request.

Received: 21 September 2023; Accepted: 5 August 2024
Published online: 30 May 2025

References

1. Bernoulli D. De vibrationibus et sono laminarum elasticarum commentationes physico-geometricae *Commentari Academiae Scientiarum Imperialis Petropolitanae*. 1751. T.13 ad annum 1741, **43**, 105–120.
2. Euler, L. *Methodus Inveniendi Lineas Curvas Maximi Minimive Proprietate Gaudentes, Sive Solutio Problematis Isoperimetrici Lattissimo Sensu Accepti* 1–322 (Marcum-Michaelem Bousquet, 1744).
3. Saint-Venant B. Memoire sur la flexion des prismes, sur les glissements transversaux et longitudinaux qui l'accompanent lorsqu'elle ne s'opere pas uniformerrient on en arc de cercle, et sur la forme courbe affectee alors par leurs section transversalles primitivement planes. *J. Math. Pures Appl.* (J. Liouville), 1856, 2 serie, T.1, 89–189
4. Love, A. E. H. *A Treatise on the Mathematical Theory of Elasticity* (Cambridge University Press, 1929).
5. Gere, J. M. & Timoshenko, S. P. *Mechanics of Materials* (PWS-KENT Publ. Comp, 1984).
6. Reddy, J. N. Nonlocal nonlinear formulations for bending of classical and shear deformation theories of beams and plates. *Int. J. Eng. Sci.* **48**(11), 1507–1518 (2010).
7. Carrera, E., Giunta, G. & Petrolo, M. *Beam Structures* (Wiley, 2011).
8. Polizzotto, C. From the Euler–Bernoulli beam to the Timoshenko one through a sequence of Reddy-type shear deformable beam models of increasing order. *Eur. J. Mech. A/Solids* **53**, 62–74 (2015).
9. Magnucki, K., Lewinski, J. & Magnucka-Blandzi, E. Bending of two-layer beams under uniformly distributed load: Analytical and numerical FEM studies. *Comp. Struct.* **235**(1), 111777 (2020).
10. Bîrsan, M., Pietras, D. & Sadowski, T. Determination of effective stiffness properties of multilayered composite beams. *Cont. Mech. Thermodyn.* **33**, 1781–1803 (2021).
11. Sanchez-Palencia, E. *Non Homogeneous Media and Vibration Theory* (Springer, 1980).
12. Kolpakov, A. G. Calculation of the characteristics of thin elastic rods of periodic structure. *J. Appl. Math. Mech.* **57**, 440–448 (1991).
13. Trabucho, L. & Viaño, J. M. Mathematical modeling of rods. In *Handbook of Numerical Analysis* (eds Ciarlet, P. G. & Lions, J. L.) (Elsevier, 1996).
14. Kalamkarov, A. L. & Kolpakov, A. G. *Analysis, Design and Optimization of Composite Structures* (Wiley, 1997).
15. Kolpakov, A. G. Determining of the averaged characteristics of elastic frameworks. *J. Appl. Math. Mech.* **49**, 969–977 (1985).
16. Lakes, R. Foam structures with negative Poisson's ratio. *Science* **235**, 1038 (1987).
17. Almgren, R. An isotropic three-dimensional structure with Poisson's ratio = – 1. *J. Elasticity* **15**, 427–430 (1985).
18. Lakes, R. Advances in negative Poisson's ratio materials. *Adv. Mater.* **5**, 393–296 (1993).
19. Liu, Q. *Literature Review: Materials with Negative Poisson's Ratio and Potential Applications to Aerospace and Defence* (DSTO, 2006).
20. Critchley, R. *et al.* A review of the manufacture, mechanical properties and potential applications of auxetic foams. *Phys. Status Solidi (B)* **250**(10), 1963–1982 (2013).
21. Bhullar, S. K. Three decades of auxetic polymers: A review. *e-Polymers* **15**(5), 205–215 (2015).
22. Wojciechowski, K. W. Auxetics and other systems with unusual characteristics. *Basic Solid State Phys.* **259**(12), 2200324 (2022).

23. Shukls, Sh. & Behera, B. K. Auxetic fibrous structures and their composites: A review. *Comp. Struct.* **290**, 115530 (2022).
24. Xue, X., Lin, C., Wu, F., Li, Z. & Liao, J. Lattice structures with negative Poisson's ratio: A review. *Materialstoday Comm.* **34**, 105132 (2023).
25. Agarwal, B. D., Broutman, L. J. & Chandrashekhara, K. *Analysis and Performance of Fiber Composites* 4th edn. (Wiley, 2017).
26. Zamani, A. M., Etemadi, E., Bodaghi, M. & Hu, H. Conceptual design and analysis of novel hybrid auxetic stents with superior expansion. *Mech. Mater.* **187**, 104813 (2023).
27. Rose, S., Siu, D., Zhu, J. D. & Roufail, R. Auxetics in biomedical applications: A review. *J. Miner. Mater. Character. Eng.* **11**(3), 27–35 (2023).

Author contributions

A.A. and A.G. wrote the main text of the paper, developed the methodology, carried out formal analysis, and numerical computations. A.A. provided the necessary hard- and software. A.G. prepared the figures. All authors reviewed the manuscript.

Funding

This research was supported by Swiss National Science Foundation project number PP00P2-202667.

Competing interests

The authors declare no competing interests.

Additional information

Correspondence and requests for materials should be addressed to A.A.K.