ARITHMETIC PROPERTIES OF MACMAHON-TYPE SUMS OF DIVISORS: THE ODD CASE

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Abstract. A century ago, P. A. MacMahon introduced two families of generating functions,

$$\sum_{1\le n_1<n_2<\cdots<n_t}\prod_{k=1}^{t}\frac{q^{n_k}}{(1-q^{n_k})^2}\quad\text{and}\quad\sum_{\substack{1\le n_1<n_2<\cdots<n_t\\ n_1,n_2,\ldots,n_t\ \text{odd}}}\prod_{k=1}^{t}\frac{q^{n_k}}{(1-q^{n_k})^2},$$

which connect sum-of-divisors functions and integer partitions. These have recently drawn renewed attention. In particular, Amdeberhan, Andrews, and Tauraso extended the first family above by defining

$$U_t(a,q):=\sum_{1\le n_1<n_2<\cdots<n_t}\prod_{k=1}^{t}\frac{q^{n_k}}{1+aq^{n_k}+q^{2n_k}}$$

for $a=0,\pm1,\pm2$ and investigated various properties, including some congruences satisfied by the coefficients of the power series representations for $U_t(a,q)$. These arithmetic aspects were subsequently expanded upon by the authors of the present work. Our goal here is to generalize the second family of generating functions, where the sums run over odd integers, and then apply similar techniques to show new infinite families of Ramanujan–like congruences for the associated power series coefficients.

1. Introduction

The two families of generating functions

$$\sum_{1\le n_1<n_2<\cdots<n_t}\prod_{k=1}^{t}\frac{q^{n_k}}{(1-q^{n_k})^2}\quad\text{and}\quad\sum_{\substack{1\le n_1<n_2<\cdots<n_t\\ n_1,n_2,\ldots,n_t\ \text{odd}}}\prod_{k=1}^{t}\frac{q^{n_k}}{(1-q^{n_k})^2},\tag{1}$$

introduced by P. A. MacMahon in [19], relate integer partitions and sum-of-divisors functions. Note that the coefficient of q^n of the first function in (1) is the sum of products of the multiplicities $m_1, m_2, \ldots, m_t$ such that n can be partitioned as

$$n = m_1\cdot n_1 + m_2\cdot n_2+\cdots+m_t\cdot n_t$$

with $1\le n_1<n_2<\cdots<n_t$. Restricting to the case of odd multiplicities, we find the coefficient of q^n of the second function in (1).

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Such functions have attracted renewed interest as demonstrated by the considerable number of recent articles [1, 3, 4, 6, 7, 11, 20, 22, 23]. Furthermore, in [2], Amdeberhan, Andrews, and Tauraso extended the first family in (1) by defining

$$U_t(a,q) := \sum_{1\le n_1<n_2<\cdots<n_t} \prod_{k=1}^{t} \frac{q^{n_k}}{1+aq^{n_k}+q^{2n_k}}$$

for $a = 0, \pm 1, \pm 2$ (so that the roots of $1 + ax + x^2 = 0$ are roots of unity). They investigated various properties including several congruences for their coefficients. These arithmetic aspects were later expanded upon in [24].

In this work, we take a similar approach for the second family in (1) by defining the generating functions

$$\widetilde{U}_t(a,q) := \sum_{\substack{1\le n_1<n_2<\cdots<n_t\\ n_1,n_2,\ldots,n_t \text{ odd}}} \prod_{k=1}^{t} \frac{q^{n_k}}{1+aq^{n_k}+q^{2n_k}} = \sum_{n\ge 0} \mathfrak{m}_{\text{odd}}(a,t;n)q^n, \tag{2}$$

where once again $a = 0, \pm 1, \pm 2$. Since $\widetilde{U}_t$ satisfies the identity

$$\widetilde{U}_t(-a,q) = (-1)^t \widetilde{U}_t(a,-q),$$

we will limit ourselves to the values $-2, 0, 1$.

When $t = 1$, the functions $\widetilde{U}_1(-2,q)$, $\widetilde{U}_1(0,q)$ and $\widetilde{U}_1(1,q)$ exhibit some known and easily identifiable features. For example,

$$\begin{aligned}\widetilde{U}_1(-2,q) &= \sum_{n\ge 1} \frac{q^{2n-1}}{(1-q^{2n-1})^2} = \sum_{n,m\ge 1} mq^{m(2n-1)} = \sum_{n\ge 1}\Big(\sum_{d\mid n, n/d \text{ is odd}} d\Big)q^n \\ &= q + 2q^2 + 4q^3 + 4q^4 + 6q^5 + 8q^6 + 8q^7 + 8q^8 + 13q^9 + \ldots\end{aligned}$$

or equivalently,

$$\widetilde{U}_1(-2,q) = \sum_{n\ge 1} \frac{q^n}{(1-q^n)^2} - \sum_{n\ge 1} \frac{q^{2n}}{(1-q^{2n})^2} = \sum_{n\ge 1} \sigma(n)(q^n - q^{2n})$$

which implies

$$\mathfrak{m}_{\text{odd}}(-2,1;n) = \begin{cases} \sigma(n) - \sigma(n/2) & \text{if } n \text{ is even,} \\ \sigma(n) & \text{if } n \text{ is odd,} \end{cases}$$

where $\sigma(n)$ is the sum of the positive divisors of n. See [21, A002131].
For $a = 0$, we have

$$\begin{aligned}\widetilde{U}_1(0,q) &= \sum_{n\ge 1} \frac{q^{2n-1}}{1+q^{2(2n-1)}} = \sum_{n\ge 1} \frac{q^n(1-q^n)(1-q^{2n})(1-q^{3n})}{1-q^{8n}} \\ &= q\frac{f_8^4}{f_4^2} = q + 2q^5 + q^9 + 2q^{13} + 2q^{17} + 3q^{25} + 2q^{29} + 2q^{37} + \ldots\end{aligned}$$

where $f_r = (q^r;q^r)_\infty = \prod_{k\geq 1}(1-q^{rk})$. Note that $\mathfrak{m}_{\mathrm{odd}}(0,1;4n+1)$ is the number of ways of writing n as the sum of two triangular numbers.

$$\mathfrak{m}_{\mathrm{odd}}(0,1;4n+1) = [q^n]\frac{f_2^4}{f_1^2} = [q^n]\Big(\sum_{n\geq 0} q^{T_n}\Big)^2$$

where $T_n = n(n+1)/2$ is the nth triangular number. See [21, A008441]. Moreover, it can be verified that, for all $n \geq 0$,

$$\mathfrak{m}_{\mathrm{odd}}(0,1;4(9n+5)+1) = \mathfrak{m}_{\mathrm{odd}}(0,1;4(9n+8)+1) = 0.$$

On the other hand, for $a = 1$, we have

$$\begin{aligned}
\widetilde{U}_1(1,q) &= \sum_{n\geq 1} \frac{q^{2n-1}}{(1+q^{2n-1}+q^{2(2n-1)})} = \sum_{n\geq 1}\frac{q^{2n-1}(1-q^{2n-1})}{(1-q^{3(2n-1)})} \\
&= \sum_{n\geq 1}\frac{q^n(1-q^n)}{1-q^{3n}} - \sum_{n\geq 1}\frac{q^{2n}(1-q^{2n})}{1-q^{6n}} \\
&= \sum_{n\geq 1}\frac{q^n - 2q^{2n} + 2q^{4n} - q^{5n}}{1-q^{6n}} \\
&= q - q^2 + q^3 + q^4 - q^6 + 2q^7 - q^8 + q^9 + \dots.
\end{aligned}$$

that is

$$\mathfrak{m}_{\mathrm{odd}}(1,1;n) = \tau_1(n) - 2\tau_2(n) + 2\tau_4(n) - \tau_5(n)$$

where $\tau_j(n)$ is the number of positive divisors d of n such that $d \equiv j \pmod 6$. See [21, A093829]. Notice that

$$\mathfrak{m}_{\mathrm{odd}}(1,1;6n+5) = \tau_1(6n+5) - \tau_5(6n+5) = 0 \tag{3}$$

because if d divides $6n+5$ then $d \equiv 1 \pmod 6$ if and only if $n/d \equiv 5 \pmod 6$.

With the above in mind, we see that the generating functions $\widetilde{U}_t(a,q)$ naturally generalize various well–studied arithmetic functions. For $t > 1$, our first goal is to obtain a more convenient formula for $\widetilde{U}_t(a,q)$ that avoids multiple summations. This will be discussed in Section 2, where we find that $\widetilde{U}_t(a,q)$ can be written as

$$\sum_{k\geq 0} b_k(a)q^k \cdot \sum_{n\geq 0} c_n(a,t)q^{r(n)} \tag{4}$$

where $r(n)$ is a quadratic polynomial. (Similar work was completed in [24] for the related functions $U_t(a,q)$.)

After laying this groundwork, we turn our attention to congruences. Some of these congruences are straightforward. For example, thanks to (2), we immediately see that, for any t,

$$\begin{aligned}
\widetilde{U}_t(-2,q) &\equiv \widetilde{U}_t(0,q) \pmod 2, \\
\widetilde{U}_t(-2,q) &\equiv \widetilde{U}_t(1,q) \pmod 3.
\end{aligned} \tag{5}$$

In order to address less trivial congruences, we take advantage of the product in (4) and study the arithmetic properties of the coefficients $b_k(a)$ and $c_n(a,t)$. From there, starting in Section 5 and beyond, we will derive our results for $\mathfrak{m}_{\mathrm{odd}}(a,t;n)$, grouped according to the value of a.

It is worth pointing out that, for $a=1$, we will come across the prefactor

$$\frac{f_1 f_6}{f_2^2 f_3} = 1 - q + q^2 - q^4 + 2q^5 - 3q^6 + 4q^7 - 5q^8 + 7q^9 + \dots$$

which appears to be novel in the literature and exhibits intriguing arithmetic properties which are of independent interest. These will be discussed in Section 3.

2. Explicit form of $\widetilde{U}_t(a,q)$

In [6, Corollary 2], we have an explicit form for the case $a=-2$, namely

$$\widetilde{U}_t(-2,q) = \frac{f_2}{f_1^2} \cdot \sum_{n=1}^{\infty} c(-2,t) q^{n^2} \tag{6}$$

where

$$c(-2,t) = (-1)^{n+t} \frac{2n}{n+t} \binom{n+t}{2t}$$

and the prefactor is the generating function of number of overpartitions of n:

$$\frac{f_2}{f_1^2} = \frac{(-q;q)_\infty}{(q;q)_\infty} = \sum_{n=0}^{\infty} \overline{p}(n) q^n = 1 + 2q + 4q^2 + 8q^3 + 14q^4 + 24q^5 + \dots$$

An overpartition of n is a non-increasing sequence of natural numbers whose sum is n in which the first occurrence of a number may be overlined.

The formula (6) was derived by making use of the properties of the Chebychev polynomials of the first kind, defined by $T_n(\cos\theta) = \cos(n\theta)$, and of their representation in terms of the sum

$$T_n(x) = \frac{n}{2} \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k}{n-k} \binom{n-k}{k} (2x)^{n-k}. \tag{7}$$

As we will see below, the same method is effective for $a=0$ and $a=1$ as well. Noting that the Chebychev polynomial $T_n(x)$ is alternately an even and odd function, depending on n, we define

$$to_n(x) = \frac{T_{2n+1}(\sqrt{x})}{\sqrt{x}} \quad \text{and} \quad te_n(x) = T_{2n}(\sqrt{x}).$$

Then, by [6, Theorem 1],

$$\sum_{n \geq 0} to_n\left(\frac{x}{4}\right) q^{\binom{n+1}{2}} = (q;q)_\infty^3 \prod_{n=1}^{\infty} \left(1 + \frac{xq^n}{(1-q^n)^2}\right)$$

and

$$1+2\sum_{n>0} te_n\left(\frac{x}{4}\right)q^{n^2}=\frac{(q;q)_\infty}{(-q;q)_\infty}\prod_{n=1}^{\infty}\left(1+\frac{xq^{2n-1}}{(1-q^{2n-1})^2}\right). \tag{8}$$

In [2, Theorem 2.1], the authors established

$$\sum_{t\geq 0} U_t(a,q)x^t=\prod_{n=1}^{\infty}\frac{1}{(1+aq^n+q^{2n})(1-q^n)}\cdot\sum_{n\geq 0} to_n\left(\frac{x+a+2}{4}\right)q^{\binom{n+1}{2}}.$$

Along the same lines we show the following identity.

Theorem 2.1. *We have*

$$\sum_{t\geq 0}\widetilde{U}_t(a,q)x^t=\prod_{n=1}^{\infty}\frac{1}{(1+aq^{2n-1}+q^{2(2n-1)})(1-q^{2n})}\cdot\left(1+2\sum_{n\geq 1} te_n\left(\frac{x+a+2}{4}\right)q^{n^2}\right). \tag{9}$$

Proof. We have that

$$\begin{aligned}\sum_{t\geq 0}\widetilde{U}_t(a,q)x^t&=\prod_{n=1}^{\infty}\left(1+\frac{xq^{2n-1}}{1+aq^{2n-1}+q^{2(2n-1)}}\right)\\&=\prod_{n=1}^{\infty}\frac{1}{1+aq^{2n-1}+q^{2(2n-1)}}\cdot\prod_{n=1}^{\infty}\left(1+(x+a)q^{2n-1}+q^{2(2n-1)}\right)\\&=\prod_{n=1}^{\infty}\frac{(1-q^{2n-1})^2}{1+aq^{2n-1}+q^{2(2n-1)}}\cdot\prod_{n=1}^{\infty}\left(1+\frac{(x+a+2)q^{2n-1}}{(1-q^{2n-1})^2}\right)\\&=\prod_{n=1}^{\infty}\frac{1}{(1+aq^{2n-1}+q^{2(2n-1)})(1-q^{2n})}\\&\quad\cdot\left(1+2\sum_{n\geq 1} te_n\left(\frac{x+a+2}{4}\right)q^{n^2}\right)\end{aligned}$$

where at the last step we applied (8). □

We are now in a position to present explicit formulas for $a=0$ and $a=1$. By (7), we obtain

$$\begin{aligned}[x^t]\,2te_n\left(\frac{x+a+2}{4}\right)&=[x^t]\,2n\sum_{k=0}^{n}\frac{(-1)^{n-k}}{n+k}\binom{n+k}{2k}(x+a+2)^k\\&=\sum_{k=t}^{n}(-1)^{n-k}\frac{2n}{n+k}\binom{n+k}{2k}\binom{k}{t}(a+2)^{k-t}.\end{aligned} \tag{10}$$

Moreover, the binomial sum on the right-hand side of (10) admits the following useful representation, which is derived by using Riordan arrays (see [25] and also [2, Appendix 2]).

Lemma 2.1. *For $n \geq t \geq 1$, we have that*

$$\sum_{k=t}^{n}(-1)^{n-k}\frac{2n}{n+k}\binom{n+k}{2k}\binom{k}{t}(a+2)^{k-t} = [z^n]\frac{z^t - z^{t+2}}{(1-az+z^2)^{t+1}}. \tag{11}$$

Proof. If $F(z) = \sum_{k\geq 0} c_k z^k$ then

$$\sum_{k\geq 0}(-1)^{n-k}\frac{2n}{n+k}\binom{n+k}{2k}c_k = [z^n]\,\frac{1-z}{1+z}\cdot F\left(\frac{z}{(1+z)^2}\right).$$

Hence, by letting

$$F(z) = \sum_{k\geq t}\binom{k}{t}(a+2)^{k-t}z^k = \frac{z^t}{(1-(a+2)z)^{t+1}},$$

we find that the left-hand side of (11) is equal to

$$\begin{aligned}[z^n]\,\frac{1-z}{1+z}\cdot F\left(\frac{z}{(1+z)^2}\right) &= [z^n]\,\frac{1-z}{1+z}\cdot\frac{\left(\frac{z}{(1+z)^2}\right)^t}{\left(1-(a+2)\frac{z}{(1+z)^2}\right)^{t+1}} \\ &= [z^n]\frac{z^t - z^{t+2}}{(1-az+z^2)^{t+1}}.\end{aligned}$$

□

Letting $a = 0$ in (11), we find

$$2n\sum_{k=t}^{n}\frac{(-1)^{n-k}}{n+k}\binom{n+k}{2k}\binom{k}{t}2^{k-t} = \begin{cases}(-1)^{\frac{n-t}{2}}\frac{2n}{n+t}\binom{\frac{n+t}{2}}{t} & \text{if } n+t \text{ is even,}\\ 0 & \text{if } n+t \text{ is odd.}\end{cases}$$

Hence, from (9), by separately considering the even and odd cases, we obtain

$$\widetilde{U}_{2t}(0,q) = \frac{f_8}{f_4^2}\cdot\sum_{n=1}^{\infty}(-1)^{n+t}\frac{2n}{n+t}\binom{n+t}{2t}q^{4n^2} = \widetilde{U}_t(-2,q^4) \tag{12}$$

and

$$\widetilde{U}_{2t+1}(0,q) = \frac{f_8}{f_4^2}\cdot\sum_{n=1}^{\infty}(-1)^{n-t-1}\frac{2n-1}{n+t}\binom{n+t}{2t+1}q^{(2n-1)^2} = qW_t(q^4) \tag{13}$$

with

$$W_t(q) = \frac{f_2}{f_1^2}\cdot\sum_{n=1}^{\infty}c_n(0,t)q^{n(n-1)} \tag{14}$$

and

$$c_n(0,t) = (-1)^{n-t-1}\frac{2n-1}{n+t}\binom{n+t}{2t+1}.$$

Finally, for $a = 1$, from (9) and (10), we get

$$\widetilde{U}_t(1,q) = \frac{f_1 f_6}{f_2^2 f_3} \cdot \sum_{n=1}^{\infty} c_n(1,t) q^{n^2} \tag{15}$$

where

$$c_n(1,t) = \sum_{k=t}^{n} (-1)^{n-k} \frac{2n}{n+k}\binom{n+k}{2k}\binom{k}{t} 3^{k-t}.$$

For $a = 1$, equation (11) does not yield a closed-form expression for $c_n(1,t)$, but it will prove useful later for studying its arithmetic properties.

3. Information about $\frac{f_1 f_6}{f_2^2 f_3}$

Let $a(n)$ be defined by the generating function identity

$$A(q) := \sum_{n=0}^{\infty} a(n) q^n = \frac{f_1 f_6}{f_2^2 f_3}. \tag{16}$$

This is the function which appears in (15) above. Our goal in this section is to prove a number of Ramanujan–like congruences satisfied by $a(n)$. These will then be used in Section 7 to prove several arithmetic properties satisfied by $\mathfrak{m}_{\text{odd}}(1,t;N)$ for various values of t and N. To begin this analysis, we list several q–series identities that will be used in our proofs below.

Lemma 3.1. *We have*

$$\frac{f_1}{f_3} = \frac{f_2 f_{16} f_{24}^2}{f_6^2 f_8 f_{48}} - q\frac{f_2 f_8^2 f_{12} f_{48}}{f_4 f_6^2 f_{16} f_{24}}.$$

Proof. See Hirschhorn [15, (30.10.1)]. □

Lemma 3.2. *We have*

$$\frac{f_3}{f_1} = \frac{f_4 f_6 f_{16} f_{24}^2}{f_2^2 f_8 f_{12} f_{48}} + q\frac{f_6 f_8^2 f_{48}}{f_2^2 f_{16} f_{24}}.$$

Proof. See Hirschhorn [15, (30.10.3)]. □

Lemma 3.3. *We have*

$$f_1 f_3 = \frac{f_2 f_8^2 f_{12}^4}{f_4^2 f_6 f_{24}^2} - q\frac{f_4^4 f_6 f_{24}^2}{f_2 f_8^2 f_{12}^2}.$$

Proof. See Hirschhorn [15, (30.12.1)]. □

Lemma 3.4. *We have*

$$\frac{1}{f_1 f_3} = \frac{f_8^2 f_{12}^5}{f_2^2 f_4 f_6^4 f_{24}^2} + q\frac{f_4^5 f_{24}^2}{f_2^4 f_6^2 f_8^2 f_{12}}.$$

Proof. See Hirschhorn [15, (30.12.3)]. □

Lemma 3.5. *We have*

$$\frac{1}{f_1^2 f_3^2} = \frac{f_8^5 f_{24}^5}{f_2^5 f_6^5 f_{16}^2 f_{48}^2} + 2q\frac{f_4^4 f_{12}^4}{f_2^6 f_6^6} + 4q^4\frac{f_4^2 f_{12}^2 f_{16}^4 f_{48}^2}{f_2^5 f_6^5 f_8 f_{24}}.$$

Proof. See Baruah and Ojah [8, Lemma 2.2]. □

Lemma 3.6. *We have*

$$\frac{1}{f_1^2} = \frac{f_8^5}{f_2^5 f_{16}^2} + 2q\frac{f_4^2 f_{16}^2}{f_2^5 f_8}.$$

Proof. See Berndt [9, p. 40]. □

We will also require a few 3–dissection results in our work below.

Lemma 3.7. *We have*

$$\frac{f_1^2}{f_2} = \varphi(-q) = \varphi(-q^3) - 2q\frac{f_3 f_{18}^2}{f_6 f_9}$$

where

$$\varphi(q) = 1 + 2\sum_{k=1}^{\infty} q^{k^2}$$

is one of Ramanujan's famous theta functions.

Proof. See Hirschhorn [15, (1.5.8)] and Andrews, Hirschhorn, and Sellers [5, Lemma 2.6]. □

Lemma 3.8. *We have*

$$\psi(q) = P(q^3) + q\psi(q^9)$$

where

$$\psi(q) = \sum_{k=0}^{\infty} q^{k(k+1)/2}$$

is one of Ramanujan's famous theta functions and

$$P(q) = 1 + q + q^2 + q^5 + q^7 + q^{12} + q^{15} + \dots$$

whose exponents are the generalized pentagonal numbers.

Proof. See Hirschhorn and Sellers [18, p. 274]. □

Lemma 3.9. *We have*

$$\frac{f_2}{f_1^2} = \frac{f_6^4 f_9^6}{f_3^8 f_{18}^3} + 2q\frac{f_6^3 f_9^3}{f_3^7} + 4q^2\frac{f_6^2 f_{18}^3}{f_3^6}, \quad \textit{and}$$
$$\frac{f_2^3}{f_1^3} = \frac{f_6}{f_3} + 3q\frac{f_6^4 f_9^5}{f_3^8 f_{18}} + 6q^2\frac{f_6^3 f_9^2 f_{18}^2}{f_3^7} + 12q^3\frac{f_6^2 f_{18}^5}{f_3^6 f_9}.$$

Proof. See da Silva and Sellers [13, Lemma 2]. □

Lemma 3.10. *We have*

$$\frac{f_4}{f_1} = \frac{f_{12}f_{18}^4}{f_3^3 f_{36}^2} + q\frac{f_6^2 f_9^3 f_{36}}{f_3^4 f_{18}^2} + 2q^2\frac{f_6 f_{18} f_{36}}{f_3^3}$$

Proof. See Andrews, Hirschhorn, and Sellers [5, Theorem 3.1]. ☐

Next, define the functions $a(q)$, $b(q)$, and $c(q)$ by

$$\begin{aligned} a(q) &:= \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2}, \\ b(q) &:= \sum_{m,n=-\infty}^{\infty} \omega^{m-n} q^{m^2+mn+n^2}, \quad \text{and} \\ c(q) &:= q^{1/3} \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2+m+n} \end{aligned}$$

where ω is a cube root of unity other than 1. These functions were introduced by Borwein, Borwein, and Garvan [10].

Several properties of the function $b(q)$ are known.

Lemma 3.11. *We have*

$$b(q) = \frac{f_1^3}{f_3}.$$

Proof. See Hirschhorn [15, (22.1.6)]. ☐

Lemma 3.12. *We have*

$$b(q) = 1 - 3\sum_{n=-\infty}^{\infty} \frac{q^{3n+1}\left(1-2q^{6n+2}\right)}{1-q^{9n+3}}.$$

Proof. See Hirschhorn [15, (27.1.9)]. ☐

Lemma 3.13. *We have*

$$b(q) = b(q^4) - 3q\psi(q^6)\left(\psi(q^2) - 3q^2\psi(q^{18})\right).$$

Proof. See Hirschhorn [15, (22.6.10)]. ☐

Lemma 3.14. *We have*

$$b(q) = a(q^3) - 3q\frac{f_9^3}{f_3}.$$

Proof. See Hirschhorn [15, (22.1.5) and (22.1.7)]. ☐

We are now in a position to find the 2–dissection of the generating function for $a(n)$.

Theorem 3.1. *We have*

$$\sum_{n=0}^{\infty} a(2n)q^n = \frac{f_8 f_{12}^2}{f_1 f_3 f_4 f_{24}}, \quad \textit{and} \tag{17}$$

$$\sum_{n=0}^{\infty} a(2n+1)q^n = -\frac{f_4^2 f_6 f_{24}}{f_1 f_2 f_3 f_8 f_{12}}. \tag{18}$$

Proof. From (16) and Lemma 3.1, we know

$$\begin{aligned}\sum_{n=0}^{\infty} a(2n)q^{2n} &= \frac{f_6}{f_2^2}\left(\frac{f_2 f_{16} f_{24}^2}{f_6^2 f_8 f_{48}}\right)\\ &= \frac{f_{16} f_{24}^2}{f_2 f_6 f_8 f_{48}}.\end{aligned}$$

Replacing q^2 by q throughout yields (17). Again thanks to (16) and Lemma 3.1, we know

$$\begin{aligned}\sum_{n=0}^{\infty} a(2n+1)q^{2n+1} &= \frac{f_6}{f_2^2}\left(-q\frac{f_2 f_8^2 f_{12} f_{48}}{f_4 f_6^2 f_{16} f_{24}}\right)\\ &= -q\frac{f_8^2 f_{12} f_{48}}{f_2 f_4 f_6 f_{16} f_{24}}.\end{aligned}$$

Dividing both sides by q, and then replacing q^2 by q throughout yields (18). $\square$

With the above in hand, we can now state a characterization modulo 2 satisfied by $a(2n)$ for all $n \geq 0$.

Theorem 3.2. *We have*

$$a(2n) \equiv \begin{cases} 1 \pmod{2}, & \textit{if } n = 0 \textit{ or } n \textit{ is a square not divisible by } 3\\ 0 \pmod{2}, & \textit{otherwise.}\end{cases}$$

Proof. From Theorem 3.1, we know

$$\begin{aligned}
\sum_{n=0}^{\infty} a(2n)q^n &= \frac{f_8 f_{12}^2}{f_1 f_3 f_4 f_{24}} \\
&\equiv \frac{f_1^8 f_{12}^2}{f_1 f_3 f_1^4 f_{12}^2} \pmod{2} \\
&= \frac{f_1^3}{f_3} \\
&= b(q) \quad \text{thanks to Lemma 3.11} \\
&\equiv 1 + \sum_{n=-\infty}^{\infty} \frac{q^{3n+1}}{1-q^{9n+3}} \pmod{2} \quad \text{thanks to Lemma 3.12} \\
&\equiv 1 + \sum_{n\geq 1,\ 3\nmid n} q^{n^2} \pmod{2}
\end{aligned}$$

and this yields our result. At the last step we applied

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} \frac{q^{3n+1}}{1-q^{9n+3}} &= \sum_{n=0}^{\infty} \frac{q^{3n+1}}{1-q^{9n+3}} + \sum_{n=0}^{\infty} \frac{q^{-3n-2}}{1-q^{-9n-6}} \\
&= \sum_{m,n\geq 0} q^{3n+1} q^{m(9n+3)} - \sum_{n=0}^{\infty} \frac{q^{6n+4}}{1-q^{9n+6}} \\
&= \sum_{m,n\geq 0} q^{(3n+1)(3m+1)} - \sum_{m,n\geq 0} q^{(3n+2)(3m+2)} \\
&= \sum_{n=0}^{\infty} q^{(3n+1)^2} - \sum_{n=0}^{\infty} q^{(3n+2)^2} \\
&\quad + 2\sum_{0\leq n<m} q^{(3n+1)(3m+1)} - 2\sum_{0\leq n<m} q^{(3n+2)(3m+2)}.
\end{aligned}$$

□

This now leads to numerous corollaries modulo 2, including the following congruences.

Corollary 3.1. *For all* $n \geq 0$,

$$\begin{aligned}
a(6n+4) &\equiv 0 \pmod{2}, \\
a(6n+6) &\equiv 0 \pmod{2}.
\end{aligned}$$

Proof. Note that there are no squares of the form $3n+2$ since 2 is a quadratic nonresidue modulo 3. This yields the result for $2(3n+2) = 6n+4$. Moreover, for $n \geq 0$, no number of the form $3n+3$ appears as an exponent on the right–hand side of Theorem 3.2. This completes the proof. □

Corollary 3.2. *For all $n \geq 0$,*

$$a(8n+4) \equiv 0 \pmod{2},$$
$$a(8n+6) \equiv 0 \pmod{2}.$$

Proof. Note that there are no squares of the form $4n+2$ nor $4n+3$ because all squares are congruent to either 0 or 1 modulo 4. The result follows from Theorem 3.2. □

Corollary 3.3. *For all $n \geq 0$, and for all primes $p \geq 5$,*

$$a(2(pn+r)) \equiv 0 \pmod{2}$$

where r is a quadratic nonresidue modulo p with $1 \leq r \leq p-1$.

Proof. This follows immediately from Theorem 3.2. □

We require one additional parity result for our work below, which is the following.

Theorem 3.3. *For all $n \geq 0$, $a(24n+13) \equiv 0 \pmod{2}$.*

Proof. From Theorem 3.1, we know

$$\begin{aligned}
\sum_{n=0}^{\infty} a(2n+1)q^n &= -\frac{f_4^2 f_6 f_{24}}{f_1 f_2 f_3 f_8 f_{12}} \\
&\equiv \frac{f_1^8 f_3^2 f_{12}^2}{f_1 f_2 f_3 f_1^8 f_{12}} \pmod{2} \\
&= \frac{f_{12}}{f_2} \cdot \frac{f_3}{f_1}.
\end{aligned}$$

Using Lemma 3.2, we then see that

$$\begin{aligned}
\sum_{n=0}^{\infty} a(4n+1)q^n &\equiv \frac{f_6}{f_1} \cdot \frac{f_2 f_3 f_8 f_{12}^2}{f_1^2 f_4 f_6 f_{24}} \pmod{2} \\
&\equiv \frac{f_3}{f_1} \cdot \frac{f_6 f_2 f_4^2 f_{12}^2}{f_2 f_4 f_6 f_{12}^2} \pmod{2} \\
&= \frac{f_3}{f_1} \cdot f_4.
\end{aligned}$$

Thanks again to Lemma 3.2, we then see that

$$\begin{aligned}
\sum_{n=0}^{\infty} a(8n+5)q^n &\equiv f_2 \cdot \frac{f_3 f_4^2 f_{24}}{f_1^2 f_8 f_{12}} \pmod{2} \\
&\equiv \frac{f_2 f_3 f_2^4 f_{12}^2}{f_2 f_2^4 f_{12}} \pmod{2} \\
&= f_3 f_{12}
\end{aligned}$$

which is a function of q^3. Thus, for all $n \geq 0$, $a(8(3n+1)+5) = a(24n+13) \equiv 0 \pmod{2}$. □

Next, we wish to prove the following congruences modulo 4 that are satisfied by $a(n)$.

Theorem 3.4. *For all $n \geq 0$,*

$$a(12n+6) \equiv 0 \pmod{4}, \tag{19}$$
$$a(16n+6) \equiv 0 \pmod{4}, \tag{20}$$
$$a(24n+16) \equiv 0 \pmod{4}, \quad \textit{and} \tag{21}$$
$$a(24n+22) \equiv 0 \pmod{4}. \tag{22}$$

Proof. We begin the proof of this theorem by returning to Theorem 3.1 and performing another 2–dissection. Namely, from Lemma 3.4, we have

$$\begin{aligned}\sum_{n=0}^{\infty} a(2n)q^n &= \frac{f_8 f_{12}^2}{f_4 f_{24}}\left(\frac{1}{f_1 f_3}\right)\\ &= \frac{f_8 f_{12}^2}{f_4 f_{24}}\left(\frac{f_8^2 f_{12}^5}{f_2^2 f_4 f_6^4 f_{24}^2} + q\frac{f_4^5 f_{24}^2}{f_2^4 f_6^2 f_8^2 f_{12}}\right).\end{aligned}$$

We then immediately see that

$$\sum_{n=0}^{\infty} a(4n)q^n = \frac{f_4^3 f_6^7}{f_1^2 f_2^2 f_3^4 f_{12}^3} \tag{23}$$

and

$$\sum_{n=0}^{\infty} a(4n+2)q^n = \frac{f_2^4 f_6 f_{12}}{f_1^4 f_3^2 f_4} \tag{24}$$

after elementary simplifications.

We begin by rewriting (23) as

$$\sum_{n=0}^{\infty} a(4n)q^n = \frac{f_6^7}{f_3^4 f_{12}^3}\cdot\frac{f_2}{f_1^2}\cdot\frac{f_4^3}{f_2^3}$$

which allows us to perform a 3–dissection thanks to Lemma 3.9. In this vein, we have

$$\begin{aligned}\sum_{n=0}^{\infty} a(4(3n+1))q^{3n+1} &\equiv \frac{f_6^7}{f_3^4 f_{12}^3}\left(\frac{f_6^4 f_9^6}{f_3^8 f_{18}^3}\cdot 6q^4\frac{f_{12}^3 f_{18}^2 f_{36}^2}{f_6^7} + 2q\frac{f_6^3 f_9^3}{f_3^7}\cdot\frac{f_{12}}{f_6}\right) \pmod{4}\\ &= 6q^4\frac{f_6^4 f_9^6 f_{36}^2}{f_3^{12} f_{18}} + 2q\frac{f_6^9 f_9^3}{f_3^{11} f_{12}^2}\end{aligned}$$

which implies

$$\begin{aligned}\sum_{n=0}^{\infty} a(12n+4)q^n &\equiv 2q\frac{f_2^4f_3^6f_{12}^2}{f_1^{12}f_6} + 2\frac{f_2^9f_3^3}{f_1^{11}f_4^2} \pmod 4\\ &\equiv 2q\frac{f_2^4f_6^3f_{12}^2}{f_2^6f_6} + 2\frac{f_2^9f_6}{f_2^5f_4^2}\cdot\frac{f_3}{f_1} \pmod 4\\ &= 2q\frac{f_2^4f_6^3f_{12}^2}{f_2^6f_6} + 2\frac{f_2^9f_6}{f_2^5f_4^2}\left(\frac{f_4f_6f_{16}f_{24}^2}{f_2^2f_8f_{12}f_{48}} + q\frac{f_6f_8^2f_{48}}{f_2^2f_{16}f_{24}}\right)\end{aligned}$$

using Lemma 3.2. This means that

$$\begin{aligned}\sum_{n=0}^{\infty} a(12(2n+1)+4)q^{2n+1} &\equiv 2q\frac{f_2^4f_6^3f_{12}^2}{f_2^6f_6} + 2q\frac{f_2^9f_6}{f_2^5f_4^2}\cdot\frac{f_6f_8^2f_{48}}{f_2^2f_{16}f_{24}} \pmod 4\\ &= 2q\frac{f_6^2f_{12}^2}{f_2^2} + 2q\frac{f_2^2f_6^2f_8^2f_{48}}{f_4^2f_{16}f_{24}}\\ &\equiv 2q\frac{f_3^4f_3^8}{f_1^4} + 2q\frac{f_1^4f_3^4f_1^{16}f_3^{16}}{f_1^8f_1^{16}f_3^8} \pmod 4\\ &\equiv 2q\frac{f_3^{12}}{f_1^4} + 2q\frac{f_3^{12}}{f_1^4} \pmod 4\\ &\equiv 0 \pmod 4.\end{aligned}$$

This proves (21).

Next, from (24) and the fact that $f_1^4 \equiv f_2^2 \pmod 4$, we have

$$\begin{aligned}\sum_{n=0}^{\infty} a(4n+2)q^n &\equiv \frac{f_2^2f_6f_{12}}{f_3^2f_4} \pmod 4 \qquad (25)\\ &= \frac{f_2^2f_6f_{12}}{f_4}\left(\frac{f_{24}^5}{f_6^5f_{48}^2} + 2q^3\frac{f_{12}^2f_{48}^2}{f_6^5f_{24}}\right)\end{aligned}$$

where we also applied Lemma 3.6. From the above, we see that

$$\sum_{n=0}^{\infty} a(4(2n+1)+2)q^{2n+1} \equiv 2q^3\frac{f_2^2f_6f_{12}}{f_4}\cdot\frac{f_{12}^2f_{48}^2}{f_6^5f_{24}} \pmod 4$$

which means

$$\begin{aligned}\sum_{n=0}^{\infty} a(8n+6)q^n &\equiv 2q\frac{f_1^2f_6^3f_{24}^2}{f_2f_3^4f_{12}} \pmod 4\\ &\equiv 2q\frac{f_2f_6^3f_{12}^4}{f_2f_6^2f_{12}} \pmod 4\\ &\equiv 2qf_6f_{12}^3 \pmod 4.\end{aligned}$$

Note that this last expression, when written as a power series in q, contains only odd powers of q. Therefore, for all $n \geq 0$, $a(8(2n)+6) = a(16n+6) \equiv 0 \pmod 4$, which is (20). In addition, the expression $2qf_6f_{12}^3$ can also be viewed as a function of q^{3n+1}, and this means that, for all $n \geq 0$, $a(8(3n+2)+6) = a(24n+22) \equiv 0 \pmod 4$. This is (22).

Returning to (25), by Lemma 3.7 we see that

$$\begin{aligned}\sum_{n=0}^{\infty} a(4n+2)q^n &\equiv \frac{f_2^2 f_6 f_{12}}{f_3^2 f_4} \pmod 4 \\ &\equiv \frac{f_6 f_{12}}{f_3^2}\left(\varphi(-q^6) - 2q^2\frac{f_6 f_{36}^2}{f_{12}f_{18}}\right) \pmod 4\end{aligned}$$

and this last expression, when written as a power series in q, contains no powers of the form q^{3n+1}. Therefore, for all $n \geq 0$,

$$a(4(3n+1)+2) = a(12n+6) \equiv 0 \pmod 4.$$

This is (19), and this completes the proof of the theorem. □

We now transition to proving a pair of congruences modulo 8 satisfied by $a(n)$.

Theorem 3.5. *For all $n \geq 0$,*

$$\begin{aligned}a(12n+9) &\equiv 0 \pmod 8, \quad \textit{and} \\ a(24n+19) &\equiv 0 \pmod 8.\end{aligned}$$

Proof. From (18), we know

$$\begin{aligned}\sum_{n=0}^{\infty} a(2n+1)q^n &= -\frac{f_4^2 f_6 f_{24}}{f_2 f_8 f_{12}} \cdot \frac{1}{f_1 f_3} \\ &= -\frac{f_4^2 f_6 f_{24}}{f_2 f_8 f_{12}}\left(\frac{f_8^2 f_{12}^5}{f_2^2 f_4 f_6^4 f_{24}^2} + q\frac{f_4^5 f_{24}^2}{f_2^4 f_6^2 f_8^2 f_{12}}\right) \qquad (26)\end{aligned}$$

using Lemma 3.4. Thus, we have

$$\begin{aligned}\sum_{n=0}^{\infty} a(4n+1)q^n &= -\frac{f_2^2 f_3 f_{12} f_4^2 f_6^5}{f_1 f_4 f_6 f_1^2 f_2 f_3^4 f_{12}^2} \\ &= -\frac{f_6^4}{f_3^3 f_{12}} \cdot \frac{f_2 f_4}{f_1^3} \\ &= -\frac{f_6^4}{f_3^3 f_{12}} \cdot \frac{f_2}{f_1^2} \cdot \frac{f_4}{f_1}.\end{aligned}$$

Using Lemma 3.9 and Lemma 3.10, we know

$$\begin{aligned}&\sum_{n=0}^{\infty} a(4(3n+2)+1)q^{3n+2} \\ &= -\frac{f_6^4}{f_3^3 f_{12}}\left(2q^2\frac{f_6^4 f_9^6}{f_3^8 f_{18}^3}\frac{f_6 f_{18} f_{36}}{f_3^3} + 2q^2\frac{f_6^3 f_9^3}{f_3^7}\frac{f_6^2 f_9^3 f_{36}}{f_3^4 f_{18}^2} + 4q^2\frac{f_6^2 f_{18}^3}{f_3^6}\frac{f_{12} f_{18}^4}{f_3^3 f_{36}^2}\right).\end{aligned}$$

This yields

$$\begin{aligned}
\sum_{n=0}^{\infty} a(12n+9)q^n &= -\frac{f_2^4}{f_1^3 f_4}\left(2\frac{f_2^5 f_3^6 f_{12}}{f_1^{11} f_6^2} + 2\frac{f_2^5 f_3^6 f_{12}}{f_1^{11} f_6^2} + 4\frac{f_2^2 f_4 f_6^7}{f_1^9 f_{12}^2}\right)\\
&= -\frac{f_2^4}{f_1^3 f_4}\left(4\frac{f_2^5 f_3^6 f_{12}}{f_1^{11} f_6^2} + 4\frac{f_2^2 f_4 f_6^7}{f_1^9 f_{12}^2}\right)\\
&\equiv -\frac{f_2^4}{f_1^3 f_4}\left(4\frac{f_3^6}{f_1} + 4\frac{f_3^6}{f_1}\right) \pmod 8\\
&\equiv 0 \pmod 8.
\end{aligned}$$

This proves the first congruence in this theorem. Returning to (26), we can also see that

$$\begin{aligned}
\sum_{n=0}^{\infty} a(4n+3)q^n &= -\frac{f_2^2 f_3 f_{12}}{f_1 f_4 f_6} \cdot \frac{f_2^5 f_{12}^2}{f_1^4 f_3^2 f_4^2 f_6}\\
&= -\frac{f_2^7 f_{12}^3}{f_1^5 f_3 f_4^3 f_6^2}\\
&= -\frac{f_2^7 f_{12}^3}{f_1^8 f_4^3 f_6^2} b(q)\\
&\equiv -\frac{f_2^7 f_{12}^3}{f_2^4 f_4^3 f_6^2} b(q) \pmod 8\\
&= -\frac{f_2^3 f_{12}^3}{f_4^3 f_6^2}\left(b(q^4) - 3q\psi(q^6)\left(\psi(q^2) - 3q^2\psi(q^{18})\right)\right)
\end{aligned}$$

from Lemma 3.13. Thus, we have

$$\sum_{n=0}^{\infty} a(4(2n)+3)q^{2n} \equiv -\frac{f_2^3 f_{12}^3}{f_4^3 f_6^2} b(q^4) \pmod 8$$

which means

$$\begin{aligned}
\sum_{n=0}^{\infty} a(8n+3)q^n &\equiv -\frac{f_1^3 f_6^3}{f_2^3 f_3^2} b(q^2) \pmod 8\\
&= -\frac{f_1^3 f_6^3}{f_2^3 f_3^2} \cdot \frac{f_2^3}{f_6}\\
&= -\frac{f_1^3 f_6^2}{f_3^2}\\
&= -\frac{f_6^2}{f_3} b(q)\\
&= -\frac{f_6^2}{f_3}\left(a(q^3) - 3q\frac{f_9^3}{f_3}\right) \quad \text{from Lemma 3.14.}
\end{aligned}$$

Given the definition of $a(q)$ above, we know that $a(q^3)$ is a function of q^3. Therefore, when written as a power series in q, the expression

$$-\frac{f_6^2}{f_3}\left(a(q^3)-3q\frac{f_9^3}{f_3}\right)$$

contains no terms of the form q^{3n+2}. From the string of congruences and equalities above, we then know that, for all $n\geq 0$, $a(8(3n+2)+3)=a(24n+19)\equiv 0 \pmod 8$. This completes the proof of the theorem. $\square$

We can prove one additional congruence modulo 8 satisfied by the function $a(n)$.

Theorem 3.6. *For all $n\geq 0$, $a(32n+28)\equiv 0 \pmod 8$.*

Proof. From (23), we know

$$\sum_{n=0}^{\infty}a(4n)q^n=\frac{f_4^3f_6^7}{f_2^2f_{12}^3}\cdot\frac{1}{f_1^2f_3^4}.$$

Using Lemma 3.5 and Lemma 3.6, we then have

$$\begin{aligned}\sum_{n=0}^{\infty}a(8n+4)q^{2n+1}&=\frac{f_4^3f_6^7}{f_2^2f_{12}^3}\cdot\left(2q^3\frac{f_8^5f_{24}^5}{f_2^5f_6^5f_{16}^2f_{48}^2}\cdot\frac{f_{12}^2f_{48}^2}{f_6^5f_{24}}+2q\frac{f_4^4f_{12}^4}{f_2^6f_6^6}\cdot\frac{f_{24}^5}{f_6^5f_{48}^2}\right.\\&\qquad\left.+8q^7\frac{f_4^2f_{12}^2f_{16}^4f_{48}^2}{f_2^5f_6^5f_8f_{24}}\cdot\frac{f_{12}^2f_{48}^2}{f_6^5f_{24}}\right)\\&\equiv 2q^3\frac{f_4^3f_8^5f_{24}^4}{f_2^7f_6^3f_{12}f_{16}^2}+2q\frac{f_4^7f_{12}f_{24}^5}{f_2^8f_6^4f_{48}^2}\pmod 8\end{aligned}$$

after simplification. Dividing both sides of the above by q and then replacing q^2 by q throughout yields

$$\begin{aligned}\sum_{n=0}^{\infty}a(8n+4)q^n&\equiv 2q\frac{f_2^3f_4^5f_{12}^4}{f_1^7f_3^3f_6f_8^2}+2\frac{f_2^7f_6f_{12}^5}{f_1^8f_3^4f_{24}^2}\pmod 8\\&\equiv 2q\frac{f_2f_4f_{12}^4}{f_1^3f_3^3f_6}+2\frac{f_1^4f_2f_{12}}{f_6}\pmod 8\\&\equiv 2q\frac{f_2f_4f_{12}^4}{f_1^3f_3^3f_6}+2\frac{f_2^3f_{12}}{f_6}\pmod 8\qquad(27)\\&\equiv 2q\frac{f_4f_{12}^4}{f_2f_6^3}(f_1f_3)+2\frac{f_2^3f_{12}}{f_6}\pmod 8\end{aligned}$$

utilizing the fact that $f_2^2\equiv f_1^4 \pmod 4$ several times. We now apply Lemma 3.3 to obtain

$$\sum_{n=0}^{\infty}a(8n+4)q^n\equiv 2q\frac{f_4f_{12}^4}{f_2f_6^3}\left(\frac{f_2f_8^2f_{12}^4}{f_4^2f_6f_{24}^2}-q\frac{f_4^4f_6f_{24}^2}{f_2f_8^2f_{12}^2}\right)+2\frac{f_2^3f_{12}}{f_6}\pmod 8.$$

From the above congruence, we see that

$$\begin{aligned}\sum_{n=0}^{\infty} a(8(2n+1)+4)q^{2n+1} &\equiv 2q\frac{f_4 f_{12}^4}{f_2 f_6^3}\cdot\frac{f_2 f_8^2 f_{12}^4}{f_4^2 f_6 f_{24}^2} \pmod{8}\\ &\equiv 2q\frac{f_8^2 f_{12}^6}{f_4 f_{24}^2} \pmod{8}\end{aligned}$$

after simplification. Dividing both sides of the above by q and replacing q^2 by q yields

$$\sum_{n=0}^{\infty} a(16n+12)q^n \equiv 2\frac{f_4^2 f_6^6}{f_2 f_{12}^2} \pmod{8}$$

and this last expression is an even function in q. Thus, for all $n \geq 0$,

$$a(16(2n+1)+12) = a(32n+28) \equiv 0 \pmod{8}.$$

□

We share one last congruence satisfied by $a(n)$, again modulo 4.

Theorem 3.7. *For all $n \geq 0$, $a(32n+20) \equiv 0 \pmod{4}$.*

Proof. Returning to (27), and recognizing that divisibility by 8 immediately implies divisibility by 4, we have

$$\begin{aligned}\sum_{n=0}^{\infty} a(8n+4)q^n &\equiv 2q\frac{f_2 f_4 f_{12}^4}{f_1^3 f_3^3 f_6} + 2\frac{f_2^3 f_{12}}{f_6} \pmod{4}\\ &\equiv 2q\frac{f_1^2 f_4 f_3^4 f_{12}^3}{f_1^3 f_3^5} + 2\frac{f_2^3 f_6^2}{f_6} \pmod{4}\\ &\equiv 2q f_4 f_{12}^3\cdot\frac{1}{f_1 f_3} + 2f_2^3 f_6 \pmod{4}\\ &= 2q f_4 f_{12}^3\left(\frac{f_8^2 f_{12}^5}{f_2^2 f_4 f_6^4 f_{24}^2} + q\frac{f_4^5 f_{24}^2}{f_2^4 f_6^2 f_8^2 f_{12}}\right) + 2f_2^3 f_6\end{aligned}$$

using Lemma 3.4. Simplifying further modulo 4, we have

$$\sum_{n=0}^{\infty} a(8n+4)q^n \equiv 2q\frac{f_4^3 f_{12}^6}{f_{24}^2} + 2q^2\frac{f_4^4 f_{12} f_{24}^2}{f_8^2} + 2f_2^3 f_6 \pmod{4}$$

which implies that

$$\sum_{n=0}^{\infty} a(16n+4)q^{2n} \equiv 2q^2\frac{f_4^4 f_{12} f_{24}^2}{f_8^2} + 2f_2^3 f_6 \pmod{4}.$$

Replacing q^2 by q and then continuing to simplify modulo 4, we have

$$\begin{aligned}
\sum_{n=0}^{\infty} a(16n+4)q^n &\equiv 2q\frac{f_2^4 f_6 f_{12}^2}{f_4^2} + 2f_1^3 f_3 \pmod 4 \\
&\equiv 2q\frac{f_2^4 f_6 f_{12}^2}{f_4^2} + 2f_2 f_6 \cdot \frac{f_1}{f_3} \pmod 4 \\
&= 2q\frac{f_2^4 f_6 f_{12}^2}{f_4^2} + 2f_2 f_6 \left(\frac{f_2 f_{16} f_{24}^2}{f_6^2 f_8 f_{48}} - q\frac{f_2 f_8^2 f_{12} f_{48}}{f_4 f_6^2 f_{16} f_{24}} \right)
\end{aligned}$$

from Lemma 3.1. We then see that

$$\begin{aligned}
\sum_{n=0}^{\infty} a(16(2n+1)+4)q^{2n+1} &\equiv 2q\frac{f_2^4 f_6 f_{12}^2}{f_4^2} - 2q\frac{f_2^2 f_8^2 f_{12} f_{48}}{f_4 f_6 f_{16} f_{24}} \pmod 4 \\
&\equiv 2q f_6 f_{12}^2 - 2q\frac{f_{12} f_{48}}{f_6 f_{24}} \pmod 4 \\
&\equiv 2q f_6 f_{12}^2 - 2q\frac{f_6^2 f_{12}^4}{f_6 f_{12}^2} \pmod 4 \\
&\equiv 2q f_6 f_{12}^2 - 2q f_6 f_{12}^2 \pmod 4 \\
&\equiv 0 \pmod 4.
\end{aligned}$$

Thus, for all $n \geq 0$,

$$a(16(2n+1)+4) = a(32n+20) \equiv 0 \pmod 4.$$

□

4. Congruences for coefficients $c_n(a,t)$

As our focus turns to the divisibility of binomial coefficients by powers of primes (mainly 2 and 3), we recall a fundamental result, namely Kummer's Theorem (see, for instance, [14]): if p is a prime, then the largest power of p dividing $\binom{n}{m}$ is equal to

$$\nu_p\left(\binom{n}{m}\right) = \frac{S_p(m) + S_p(n-m) - S_p(n)}{p-1}.$$

where $\nu_p(k)$ is the largest power of p dividing an integer k and $S_p(k)$ is the sum of its base-p digits.

We also require the following Lemma, which plays a key role in studying congruences modulo powers of two.

Lemma 4.1. *For any integer $s \geq 1$,*

$$(1+z)^{2^s} \equiv (1+z^2)^{2^{s-1}} \pmod{2^s}. \tag{28}$$

Proof. We show the claim by induction with respect to the exponent s. For $s = 1$, it holds:

$$(1+z)^2 = 1 + 2z + z^2 \equiv 1 + z^2 \pmod 2.$$

Let us now proceed to the inductive step. If the statement holds for s then

$$(1+z)^{2^s} = (1+z^2)^{2^{s-1}} + 2^s P(z)$$

for some polynomial P with integer coefficients. This implies that

$$\begin{aligned}(1+z)^{2^{s+1}} &= \left((1+z)^{2^s}\right)^2 \\ &= \left((1+z^2)^{2^{s-1}} + 2^s P(z)\right)^2 \\ &= (1+z^2)^{2^s} + 2(1+z^2)^{2^{s-1}} \cdot 2^s P(z) + 2^{2s} P(z)^2 \\ &\equiv (1+z^2)^{2^s} \pmod{2^{s+1}}\end{aligned}$$

and we are done. □

We now present three theorems, each establishing congruences for the coefficients $c_n(a,t)$ for $a=-2,0,1$, in that order.

Theorem 4.1. *Let*

$$c_n(-2,t) = (-1)^{n+t}\frac{2n}{n+t}\binom{n+t}{2t}.$$

Then

1) if $s \geq 1$, $J \geq 1$, and $n \geq 0$ is even, then

$$c_n(-2, 2^s J - 1) \equiv 0 \pmod{2^{s+1}},$$

2) if $J \geq 0$ and $n \not\equiv 13, 14 \pmod{27}$, then

$$c_n(-2, 27J+13) \equiv 0 \pmod{3},$$

3) if $J \geq 1$ and $n \not\equiv \pm 1 \pmod{27}$, then

$$c_n(-2, 27J-1) \equiv 0 \pmod{3}.$$

Proof. 1) Let $t = 2^s J - 1$, then by (11) and (28) we have that

$$\begin{aligned}c_n(-2, 2^s J - 1) &= [z^n]\frac{z^t - z^{t+2}}{(1+2z+z^2)^{t+1}} \\ &= [z^n]\frac{z^{2^s J-1} - z^{2^s J+1}}{(1+z)^{2^{s+1}J}} \\ &\equiv [z^n]\frac{z^{2^s J-1} - z^{2^s J+1}}{(1+z^2)^{2^s J}} \pmod{2^{s+1}}\end{aligned}$$

Since $\frac{z^{2^s J-1} - z^{2^s J+1}}{(1+z^2)^{2^s J}}$ is an odd function, the right-hand side is zero when n is even.
2) Let $t = 27J+13$ and $n = 27N + r$ with $0 \leq r < 27$. Then

$$\nu_3(c_n(-2,t)) = \nu_3\left(\frac{n}{t}\binom{n+t-1}{2t-1}\right) = \nu_3(n) + \nu_3\left(\binom{n+t-1}{2t-1}\right).$$

If n is a multiple of 3, then the congruence trivially holds. Assume that $r \in \{1, 2, 4, 5, 7, 8, 10, 11\}$. Then

$$\begin{aligned}
\nu_3(c_n(-2,t)) &= \nu_3\left(\binom{n+t-1}{2t-1}\right) \\
&= \frac{1}{2}\Big(S_3(2t-1) + S_3(n-t) - S_3(n+t-1)\Big) \\
&= \frac{1}{2}\Big(S_3(27\cdot 2J+25) + S_3(27(N-J-1)+r+14) \\
&\qquad\qquad - S_3(27(N+J)+r+12)\Big) \\
&= \frac{1}{2}\Big(S_3(2J) + 5 + S_3(N-J-1) + S_3(r+14) \\
&\qquad\qquad - S_3(N+J) - S_3(r+12)\Big) \\
&= 3 - \nu_3\left(\binom{r+14}{2}\right) + \nu_3(N-J) + \nu_3\left(\binom{N+J}{2J}\right) \\
&\geq 3 - \nu_3(r+14) - \nu_3(r+13) \geq 1
\end{aligned}$$

where, in the last step, we made use of the inequality

$$\nu_3(r+14) + \nu_3(r+13) \leq \max(\nu_3(r+14), \nu_3(r+13)) < 3$$

which holds because $\gcd(r+14, r+13) = 1$ and $r + 14 < 3^3$.
Similarly, for $r \in \{16, 17, 19, 20, 22, 23, 25, 26\}$,

$$\begin{aligned}
\nu_3(c_n(-2,t)) &= 3 - \nu_3\left(\binom{r-13}{2}\right) + \nu_3(N+1-J) + \nu_3\left(\binom{N+1+J}{2J}\right) \\
&\geq 3 - \nu_3(r-14) - \nu_3(r-13) \geq 1.
\end{aligned}$$

The proof of 3) is similar. $\square$

Theorem 4.2. *Let*

$$c_n(0,t) = (-1)^{n+t-1}\frac{2n-1}{n+t}\binom{n+t}{2t+1}.$$

Then

1) if $J \geq 1$ and $n \not\equiv 0, 1 \pmod 4$, then

$$\begin{aligned}
c_n(0, 4J-1) &\equiv 0 \pmod 4 \\
c_n(0, 8J-1) &\equiv 0 \pmod 8,
\end{aligned}$$

2) if $J \geq 1$ and $n \not\equiv 0, 1 \pmod 8$, then

$$\begin{aligned}
c_n(0, 32J-1) &\equiv 0 \pmod{16} \\
c_n(0, 64J-1) &\equiv 0 \pmod{32},
\end{aligned}$$

3) if $J \geq 0$ *and* $n \not\equiv 13, 15 \pmod{27}$, *then*

$$c_n(0, 27J + 12) \equiv 0 \pmod 3,$$

4) if $J \geq 1$ *and* $n \not\equiv \pm 1 \pmod{27}$, *then*

$$c_n(0, 27J - 1) \equiv 0 \pmod 3.$$

Proof. We note that

$$\frac{2n-1}{n+t}\binom{n+t}{2t+1} = \frac{2n-1}{2t+1}\binom{n+t-1}{2t}.$$

For 1) and 2), let $t = 2^s J - 1$ and $n = 2^s N + r$ with $1 < r < 2^s$. Then

$$\begin{aligned}
\nu_2(c_n(0,t)) &= \nu_2\left(\binom{n+t-1}{2t}\right) \\
&= S_2(2t) + S_2(n-t-1) - S_2(n+t-1) \\
&= S_2(2^s(J-1) + 2^s - 1) + S_2(2^s(N-J) + r) \\
&\qquad - S_2(2^s(N+J) + r - 2) \\
&= S_2(J-1) + s + S_2(N-J) + S_2(r) - S_2(N+J) - S_2(r-2) \\
&= s - \nu_2\left(\binom{r}{2}\right) + \nu_2\left(J\binom{N+J}{2J}\right) \\
&\geq s + 1 - \nu_2(r) - \nu_2(r-1) \geq s.
\end{aligned}$$

Hence, if $n \not\equiv 0, 1 \pmod 4$, then

$$\nu_2(c_n(0,t)) \geq s$$

which implies 1). On the other hand, if $n \not\equiv 0, 1$ modulo 8 then

$$\nu_2(c_n(0,t)) \geq s - 1$$

which implies 2).

The proofs of 3) and 4) can be carried out in a similar way. □

Theorem 4.3. *Let*

$$c_n(1,t) = \sum_{k=t}^{n} (-1)^{n-k} \frac{2n}{n+k}\binom{n+k}{2k}\binom{k}{t} 3^{k-t}.$$

Then

1) if $J \geq 1$ *and* n *is even, then*

$$c_n(1, 2J - 1) \equiv 0 \pmod 2,$$

2) if $J \geq 1$ *and* $n \not\equiv \pm 1 \pmod{2^{s-1}}$ *with* $s \geq 2$, *then*

$$c_n(1, 2^s J - 1) \equiv 0 \pmod 4,$$

3) if $J \geq 1$ *and* $n \not\equiv \pm 1 \pmod{32}$, *then*

$$c_n(1, 64J - 1) \equiv 0 \pmod 8.$$

Proof. If $k \geq t \geq 1$, then

$$\frac{2n}{n+k}\binom{n+k}{2k}\binom{k}{t} = \frac{n}{t}\binom{n+k-1}{2k-1}\binom{k-1}{t-1},$$

which implies that if $t = 2J-1$ and n is even then 1) trivially holds.
2) Let $t = 2^s J - 1$. By (28) for $s = 2$,

$$(1+z^m)^4 \equiv (1+z^{2m})^2 \pmod 4,$$

and from (11) we obtain

$$\begin{aligned} c_n(1,t) &= [z^n]\frac{z^t - z^{t+2}}{(1-z+z^2)^{t+1}} \\ &= [z^n]\frac{(z^{2^sJ-1} - z^{2^sJ-1})(1+z)^{2^sJ}}{(1+z^3)^{2^sJ}} \\ &\equiv [z^n]\frac{(z^{2^sJ-1} - z^{2^sJ+1})(1+z^{2^{s-1}})^{2J}}{(1+z^{3\cdot 2^{s-1}})^{2J}} \pmod 4 \end{aligned}$$

where the right-hand side in 0 as soon as $n \not\equiv \pm 1 \pmod{2^{s-1}}$.
The argument for 3) proceeds in a similar manner after noting that, by (28) for $s = 3$,

$$(1+z^m)^8 \equiv (1+z^{2m})^4 \pmod 8.$$

□

We now have all of the tools needed to prove congruences satisfied by the functions $\mathfrak{m}_{\mathrm{odd}}(a,t;N)$ for various values of a, t, and N. In the next three sections, we provide such results, categorized by the value of a.

5. Congruences for $a = -2$

We begin by noting that

$$\mathfrak{m}_{\mathrm{odd}}(-2,1;N) \equiv \begin{cases} 1 \pmod 2 & N \text{ is an odd square,} \\ 0 \pmod 2 & \text{otherwise,} \end{cases}$$

and

$$\begin{aligned} \mathfrak{m}_{\mathrm{odd}}(-2,1;6N+5) &\equiv \mathfrak{m}_{\mathrm{odd}}(1,1;6N+5) \pmod 3 \\ &= 0 \end{aligned}$$

using (3) and (5). Therefore, for all $N \geq 0$,

$$\mathfrak{m}_{\mathrm{odd}}(-2,1;6N+5) = 0 \pmod 6.$$

Next, we recall from [16] that, if $8n+r>0$, then

$$\nu_2(\overline{p}(8n+r)) \geq \begin{cases} 1 & \text{if } r=0,1,4, \\ 2 & \text{if } r=2, \\ 3 & \text{if } r=3,5,6, \\ 6 & \text{if } r=7. \end{cases} \tag{29}$$

and, if $9n+r>0$, then

$$\nu_2(\overline{p}(9n+r)) \geq \begin{cases} 1 & \text{if } r=0,1,4,7, \\ 2 & \text{if } r=2,5,8, \\ 3 & \text{if } r=3,6. \end{cases}$$

Moreover, from [12, (1.8)] and [17, Theorem 2.1] we have, respectively,

$$\begin{aligned} \overline{p}(16n+10) &\equiv 0 \pmod{8}, \\ \overline{p}(27n+18) &\equiv 0 \pmod{3} \end{aligned} \tag{30}$$

for all $n \geq 0$. We now use these congruence results for $\overline{p}(n)$ to prove divisibilities for $\mathfrak{m}_{\text{odd}}(-2,t;N)$ since, by (6), we have

$$\mathfrak{m}_{\text{odd}}(-2,t;N) = \sum_{k+n^2=N} \overline{p}(k)\cdot c_n(-2,t).$$

For this first group of congruences we just need the divisibility properties of $\overline{p}(k)$.

Theorem 5.1. *For all $t \geq 1$ and all $N \geq 0$,*

$$\mathfrak{m}_{\text{odd}}(-2,t;8N+r) \equiv 0 \pmod{4} \quad \textit{with } r \in \{3,6\}, \tag{31}$$
$$\mathfrak{m}_{\text{odd}}(-2,t;9N+r) \equiv 0 \pmod{4} \quad \textit{with } r \in \{3,6\}, \tag{32}$$
$$\mathfrak{m}_{\text{odd}}(-2,t;8N+7) \equiv 0 \pmod{8}. \tag{33}$$

Proof. We only show the proof of (33). The congruences (31) and (32) can be verified in the same way. Since $n^2 \equiv 0,1,4 \pmod{8}$, we then know that $k+n^2 = 8N+7$ implies that $k \equiv 7,6,3 \pmod{8}$. For such values of k, by (29), $\nu_2(\overline{p}(k)) \geq 3$ and therefore $\mathfrak{m}_{\text{odd}}(-2,t;8N+7) \equiv 0 \pmod{8}$. □

For the next group of congruences we have some restrictions on the parameter t because the proofs depend also on the divisibility properties of the coefficients $c_n(-2,t)$ given in Section 4.

Theorem 5.2. *For all $J \geq 0$ and all $N \geq 0$,*

$$\mathfrak{m}_{\mathrm{odd}}(-2, 2J+1; 8N+r) \equiv 0 \pmod 4 \quad \textit{with } r \in \{0,4\}, \tag{34}$$
$$\mathfrak{m}_{\mathrm{odd}}(-2, 2J; 8N+2) \equiv 0 \pmod 4, \tag{35}$$
$$\mathfrak{m}_{\mathrm{odd}}(-2, 2J+1; 8N+6) \equiv 0 \pmod 8, \tag{36}$$
$$\mathfrak{m}_{\mathrm{odd}}(-2, 2J; 8N+3) \equiv 0 \pmod 8, \tag{37}$$
$$\mathfrak{m}_{\mathrm{odd}}(-2, 4J+3; 8N+r) \equiv 0 \pmod 8 \quad \textit{with } r \in \{0,4\}, \tag{38}$$
$$\mathfrak{m}_{\mathrm{odd}}(-2, 4J+2; 16N+14) \equiv 0 \pmod 8, \tag{39}$$
$$\mathfrak{m}_{\mathrm{odd}}(-2, 4J; 8N+7) \equiv 0 \pmod{16}, \tag{40}$$
$$\mathfrak{m}_{\mathrm{odd}}(-2, 8J+7; 8N) \equiv 0 \pmod{16}, \tag{41}$$
$$\mathfrak{m}_{\mathrm{odd}}(-2, 16J+15; 8N) \equiv 0 \pmod{32}, \tag{42}$$
$$\mathfrak{m}_{\mathrm{odd}}(-2, 32J+31; 8N) \equiv 0 \pmod{64}. \tag{43}$$

Proof. For (34), if $t = 2J+1$ then, by 1) of Theorem 4.1 with $s = 1$, we have that if $c_n(-2, 2J+1) \not\equiv 0 \pmod 4$, then n is odd. Hence $n^2 \equiv 1 \pmod 8$ and if $k+1 \equiv r \pmod 8$ with $r = 0, 4$ then, by (29), $\overline{p}(k) \equiv 0 \pmod 4$.
For (35), let $t = 2J$. Since $n^2 \equiv 0, 1, 4 \pmod 8$, we know that $k + n^2 = 8N + 2$ implies that $k \equiv 2, 1, 6 \pmod 8$. By (29), if $k \equiv 2, 6 \pmod 8$, then $\nu_2(\overline{p}(k)) \geq 2$ and we are done, whereas if $k \equiv 1 \pmod 8$, then we only have $\nu_2(\overline{p}(k)) \geq 1$. In this case it suffices to observe that n is odd and t is even which implies that $c_n(-2, t) \equiv 2 \pmod 2$. The proofs of (36), (37) and (40) are similar.
For (38), by 1) of Theorem 4.1 with $s = 2$, we have that if $c_n(-2, 4J+3) \not\equiv 0 \pmod 8$ then n is odd. Hence $n^2 \equiv 1 \pmod 8$ and, therefore,

$$\mathfrak{m}_{\mathrm{odd}}(-2, 4J+3; 8N+r) \equiv \sum_{k+n^2=8N+r} \overline{p}(k) \cdot c_n(-2, 4J+3) \equiv 0 \pmod 8$$

because, by (29), if $k+1 \equiv 0, 4 \pmod 8$ then $\overline{p}(k) \equiv 0 \pmod 8$. The proof of (39) can be done similarly.
For (41), by 1) of Theorem 4.1 with $s = 3$, we have that if $c_n(-2, 8J+7) \not\equiv 0 \pmod{16}$ then $n^2 \equiv 1 \pmod 8$ and, therefore,

$$\mathfrak{m}_{\mathrm{odd}}(-2, 8J+7; 8N) \equiv \sum_{k+n^2=8N} \overline{p}(k) \cdot c_n(-2, 8J+7) \equiv 0 \pmod{16}$$

because, by (29), if $k+1 \equiv 0 \pmod 8$ then $\overline{p}(k) \equiv 0 \pmod{16}$. The proofs of (42) and (43) are similar. □

Theorem 5.3. *For all $J \geq 0$ and all $N \geq 0$,*

$$\mathfrak{m}_{\mathrm{odd}}(-2, 27J+13; 27N+25) \equiv 0 \pmod 3, \tag{44}$$
$$\mathfrak{m}_{\mathrm{odd}}(-2, 27J+26; 27N+19) \equiv 0 \pmod 3. \tag{45}$$

Proof. For (44), by 2) of Theorem 4.1 we have that, if $c_n(-2,27J+13)\not\equiv 0 \pmod 3$, then $n^2\equiv 7 \pmod{27}$ and, therefore,

$$\mathfrak{m}_{\rm odd}(-2,27J+13;27N+25)\equiv \sum_{k+n^2=27N+25}\overline{p}(k)\cdot c_n(-2,27J+13)\equiv 0 \pmod 3$$

because, by (30), if $k+7\equiv 25 \pmod{27}$, then $\overline{p}(k)\equiv 0 \pmod 3$.
For (45), by 3) of Theorem 4.1 we have that, if $c_n(-2,27J+26)\not\equiv 0 \pmod 3$, then $n^2\equiv 1 \pmod{27}$ and, therefore,

$$\mathfrak{m}_{\rm odd}(-2,27J+26;27N+19)\equiv \sum_{k+n^2=27N+19}\overline{p}(k)\cdot c_n(-2,27J+26)\equiv 0 \pmod 3$$

because, by (30), if $k+1\equiv 19 \pmod{27}$, then $\overline{p}(k)\equiv 0 \pmod 3$. □

6. Congruences for $a=0$

From (12), we know that, for all $t\geq 1$ and all $N\geq 0$, $\mathfrak{m}_{\rm odd}(0,2t;4N+r)=0$ for $r=1,2,3$ and

$$\mathfrak{m}_{\rm odd}(0,2t;4N)=\mathfrak{m}_{\rm odd}(-2,t,N).$$

Therefore, the congruences of the previous section can be reinterpreted in this setting.
Next, by (13), it follows that, for all $t\geq 0$ and all $N\geq 0$, $\mathfrak{m}_{\rm odd}(0,2t+1;4N+r)=0$ for $r=0,2,3$ and, together with (14),

$$\mathfrak{m}_{\rm odd}(0,2t+1;4N+1)=\sum_{k+n(n-1)=N}\overline{p}(k)\cdot c_n(0,t).$$

We now prove additional congruences satisfied in the case $a=0$.

Theorem 6.1. *For all $J\geq 0$ and all $N\geq 0$,*

$$\mathfrak{m}_{\rm odd}(0,2J+1;36N+r)\equiv 0 \pmod 4 \quad \textit{with } r\in\{21,33\}.$$

Proof. Let $r=21$. Since $36N+21=4(9N+5)+1$ and $n(n-1)\equiv 0,2,3,6 \pmod 9$, we know that $k+n(n-1)=9N+5$ implies that $k\equiv 5,3,2,8 \pmod 9$. For such values of k, $\overline{p}(k)\equiv 0 \pmod 4$. A similar proof works also for the case $r=33$. □

Theorem 6.2. *For all $J\geq 0$ and all $N\geq 0$,*

$$\mathfrak{m}_{\rm odd}(0,8J+7;16N+r)\equiv 0 \pmod 4 \quad \textit{with } r\in\{9,13\}, \tag{46}$$
$$\mathfrak{m}_{\rm odd}(0,16J+15;16N+13)\equiv 0 \pmod 8, \tag{47}$$
$$\mathfrak{m}_{\rm odd}(0,64J+63;32N+29)\equiv 0 \pmod{16} \tag{48}$$
$$\mathfrak{m}_{\rm odd}(0,128J+127;32N+29)\equiv 0 \pmod{32}. \tag{49}$$

Proof. For (46), by 1) of Theorem 4.2 we know that, if $c_n(0,4J+3)\not\equiv 0 \pmod 4$, then $n(n-1)\equiv 0 \pmod 4$ and, therefore,

$$\mathfrak{m}_{\rm odd}(0,8J+7;16N+4r+1)\equiv \sum_{k+n(n-1)=4N+r}\overline{p}(k)\cdot c_n(0,4J+3)\equiv 0 \pmod 4$$

because, by (29), if $k \equiv 2, 3 \pmod 4$, then $\overline{p}(k) \equiv 0 \pmod 4$.
For (47), by 1) of Theorem 4.2 we have that, if $c_n(0, 8J+7) \not\equiv 0 \pmod 8$, then $n(n-1) \equiv 0 \pmod 4$ and, therefore,

$$\mathfrak{m}_{\text{odd}}(0, 16J+15; 16N+13) \equiv \sum_{k+n(n-1)=4N+3} \overline{p}(k) \cdot c_n(0, 8J+7) \equiv 0 \pmod 8$$

because, by (29), if $k \equiv 3 \pmod 4$, then $\overline{p}(k) \equiv 0 \pmod 8$.
For (48), by 1) of Theorem 4.2 we have that, if $c_n(0, 32J+31) \not\equiv 0 \pmod 8$, then $n(n-1) \equiv 0 \pmod 8$ and, therefore,

$$\mathfrak{m}_{\text{odd}}(0, 64J+63; 32N+29) \equiv \sum_{k+n(n-1)=8N+7} \overline{p}(k) \cdot c_n(0, 32J+31) \equiv 0 \pmod{16}$$

because, by (29), if $k \equiv 7 \pmod 8$, then $\overline{p}(k) \equiv 0 \pmod{16}$. The remaining congruence (49) can be done in a similar way. $\square$

Theorem 6.3. *For all $J \geq 0$ and all $N \geq 0$,*

$$\mathfrak{m}_{\text{odd}}(0, 54J+25; 108N+49) \equiv 0 \pmod 3, \tag{50}$$
$$\mathfrak{m}_{\text{odd}}(0, 54J+53; 108N+73) \equiv 0 \pmod 3. \tag{51}$$

Proof. For (50), by 3) of Theorem 4.2 we have that, if $c_n(0, 27J+12) \not\equiv 0 \pmod 3$, then $n(n-1) \equiv 21 \pmod{27}$ and, therefore,

$$\mathfrak{m}_{\text{odd}}(0, 54J+25; 108N+49) \equiv \sum_{k+n(n-1)=27N+12} \overline{p}(k) \cdot c_n(0, 27J+12) \equiv 0 \pmod 3$$

because, by (30), if $k+21 \equiv 12 \pmod{27}$, then $\overline{p}(k) \equiv 0 \pmod 3$.
For (51), by 4) of Theorem 4.2 we have that, if $c_n(0, 27J+26) \not\equiv 0 \pmod 3$, then $n(n-1) \equiv 0 \pmod{27}$ and, therefore,

$$\mathfrak{m}_{\text{odd}}(0, 54J+53; 108N+73) \equiv \sum_{k+n(n-1)=27N+18} \overline{p}(k) \cdot c_n(0, 27J+26) \equiv 0 \pmod 3$$

because, by (30), if $k \equiv 18 \pmod{27}$, then $\overline{p}(k) \equiv 0 \pmod 3$. $\square$

7. Congruences for $a = 1$

In Section 4, we gathered results that can now be summarized in the following two convenient forms:
If $24n + r > 0$, then

$$\nu_2(a(24n+r)) \geq \begin{cases} 1 & \text{if } r = 0, 4, 10, 12, 13, 14, 20, \\ 2 & \text{if } r = 6, 16, 18, 22, \\ 3 & \text{if } r = 9, 19, 21. \end{cases} \tag{52}$$

If $32n + r > 0$, then

$$\nu_2(a(32n+r)) \geq \begin{cases} 1 & \text{if } r = 4, 10, 12, 14, 16, 24, 26, 30, \\ 2 & \text{if } r = 6, 20, 22, \\ 3 & \text{if } r = 28. \end{cases} \tag{53}$$

By (15), we also know that

$$\mathfrak{m}_{\text{odd}}(1, t; N) = \sum_{k+n^2=N} a(k) \cdot c_n(1, t).$$

We now use the above to prove congruences satisfied by $\mathfrak{m}_{\text{odd}}(1, t; N)$ for various values of t and N.

Theorem 7.1. *For all $J \geq 0$ and $N \geq 0$,*

$$\mathfrak{m}_{\text{odd}}(1, J; 24N + 22) \equiv 0 \pmod{2}, \tag{54}$$

$$\mathfrak{m}_{\text{odd}}(1, 2J + 1; 12N + 7) \equiv 0 \pmod{2}, \tag{55}$$

$$\mathfrak{m}_{\text{odd}}(1, 4J + 3; 24N + 7) \equiv 0 \pmod{4}. \tag{56}$$

Proof. We only show the proof of (56); the proofs of (54) and (55) follow in similar fashion.

Note that, for any n, $n^2 \equiv 0, 1, 4, 9, 12, 16 \pmod{24}$, which implies that $k \equiv 7 - n^2 \pmod{24}$ must satisfy $k \equiv 7, 6, 3, 22, 15 \pmod{24}$. By (52), if $k \equiv 6, 22 \pmod{24}$, then $a(k) \equiv 0 \pmod{4}$. In the other cases, when $k \equiv 7, 3, 15 \pmod{24}$, then n has to be even, and by 2) of Theorem 4.3 with $s = 2$, $c_n(1, 4J + 3) \equiv 0 \pmod{4}$. Thus, we may conclude that $\mathfrak{m}_{\text{odd}}(1, 4J + 3; 24N + 7) \equiv 0 \pmod{4}$. $\square$

Theorem 7.2. *For all $J \geq 0$ and all $N \geq 0$,*

$$\mathfrak{m}_{\text{odd}}(1, 2J + 1; 8N + r) \equiv 0 \pmod{2} \quad \textit{with } r \in \{5, 7\}, \tag{57}$$

$$\mathfrak{m}_{\text{odd}}(1, 16J + 15; 16N + 7) \equiv 0 \pmod{4} \tag{58}$$

$$\mathfrak{m}_{\text{odd}}(1, 32J + 31; 32N + r) \equiv 0 \pmod{4} \quad \textit{with } r \in \{21, 29\}, \tag{59}$$

$$\mathfrak{m}_{\text{odd}}(1, 64J + 63; 32N + 29) \equiv 0 \pmod{8}. \tag{60}$$

Proof. For (57), by 1) of Theorem 4.3 we know that, if $c_n(1, 2J + 1) \not\equiv 0 \pmod{2}$, then n is odd. Therefore, $n^2 \equiv 1 \pmod{8}$ and

$$\mathfrak{m}_{\text{odd}}(1, 2J + 1; 8N + r) \equiv \sum_{k+n^2=8N+r} \overline{p}(k) \cdot c_n(1, 2J + 1) \equiv 0 \pmod{2}$$

because, by (53), if $k + 1 \equiv 5, 7 \pmod{8}$, $a(k) \equiv 0 \pmod{2}$.
For (58), by 2) of Theorem 4.3 with $s = 2$ we know that, if $c_n(1, 16J + 15) \not\equiv 0 \pmod{4}$, then $n^2 \equiv 1 \pmod{8}$ and, therefore,

$$\mathfrak{m}_{\text{odd}}(1, 16J + 15; 16N + 7) \equiv \sum_{k+n^2=16N+7} \overline{p}(k) \cdot c_n(-2, 16J + 15) \equiv 0 \pmod{4}$$

because, by (53), if $k + 1 \equiv 7 \pmod{16}$, then $a(k) \equiv 0 \pmod{4}$.

The proofs of (59) and (60) are similar. □

We close by presenting the following pair of congruences modulo 3 which are a direct consequence of (5) and Theorem 5.3. They can also be proved by using (30).

Theorem 7.3. *For all $J \geq 0$ and all $N \geq 0$,*

$$\begin{aligned} \mathfrak{m}_{\mathrm{odd}}(1, 27J+13; 27N+25) &\equiv 0 \pmod{3}, \\ \mathfrak{m}_{\mathrm{odd}}(1, 27J+26; 27N+19) &\equiv 0 \pmod{3}. \end{aligned}$$

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