SUPERCONNECTION AND ORBIFOLD CHERN CHARACTER

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ABSTRACT. We use flat antiholomorphic superconnections to study orbifold Chern character following the method introduced by Bismut, Shen, and Wei. We show the uniqueness of orbifold Chern character by proving a Riemann-Roch-Grothendieck theorem for orbifold embeddings.

CONTENTS

1. INTRODUCTION

1.1. **Overview.**

1.1.1. *Orbifolds.* Orbifolds are geometric objects introduced by I. Satake [Sat56] as a natural generalization of manifolds. In analogy to manifolds, which are locally isomorphic to Euclidean spaces, an orbifold X is a topological space equipped with the data of *orbifold charts* $\{(U, G)\}$, which exhibit this orbifold locally as a quotient $[U/G]$ of an Euclidean space U by a finite group G.

Following subsequent developments, it became clear that orbifolds provide a natural geometric context to consider group actions on manifolds: for instance, orbifolds arise naturally in symplectic reduction (or geometric invariant theory quotients). Parameter spaces for interesting mathematical structures, also known as moduli spaces, often carry natural orbifold structures.

Orbifolds may be studied via *groupoids*. This approach is well-documented in the literature, see for example [MP97], [Moe02], [ALR07]. From this point of view, an orbifold X is represented by a groupoid $\mathcal{G} = (G_0, G_1)$. Here the topological space underlying X is given by the quotient space G_0/G_1. Geometric objects on the orbifold X are given by geometric objects on its groupoid representation $\mathcal{G}$ which are equivariant with respect to suitably defined actions of the groupoid $\mathcal{G}$. Different groupoids representing the same orbifold are related by *Morita equivalences*.

In this paper, we study *complex orbifolds* via groupoids. For this purpose, we present in Section 2 some basic notions and properties of groupoids and orbifolds, most of which are taken from existing literature. Nevertheless, to develop the Riemann-Roch-Grothendieck theorem for orbifold embeddings, we introduce a concept (Definition 2.35) of embedding generalized morphism for Lie groupoids. In our definition, an embedding of orbifolds only requires the associated morphisms on stabilizer groups to be injective, not necessarily isomorphisms. Such a flexibility is needed to study graph embeddings of orbifolds, c.f. 2.50.

1.1.2. *Coherent sheaves.* We consider the notion of sheaves on a complex orbifold X, via groupoids. For a complex groupoid $\mathcal{G}$, we formulate in Definition 4.5 the notion of coherent $\mathcal{G}$-sheaves on $\mathcal{G}$. The derived category $D^b_{\text{coh}}(\mathcal{G})$ of coherent $\mathcal{G}$-sheaves on $\mathcal{G}$ is defined in Definition 4.15. For a complex orbifold X, different groupoid presentations of X are Morita equivalent. As a consequence of Proposition 4.11, different groupoid presenting the same X have equivalent derived categories of coherent sheaves, which we define to be the derived category

$$D^b_{\text{coh}}(X)$$

of coherent sheaves on X.

Our study of coherent sheaves on X closely follows the recent development [BSW23] on complex manifolds. Inspired by [BD10], we generalize the approach [BSW23] to coherent sheaves on complex orbifolds via *antiholomorphic flat superconnections*. Roughly speaking, in a groupoid representation $\mathcal{G} = (G_0, G_1)$ of X, an antiholomorphic flat superconnection on $\mathcal{G}$ is a bounded, finite rank, $\mathbb{Z}$-graded, left, $\mathcal{G}$-equivariant, C^∞-vector bundle $E^\bullet$ on G_0 together with a $\mathcal{G}$-equivariant superconnection with total degree 1,

$$A^{E^\bullet\prime\prime}\colon\ \wedge^\bullet\,\overline{T^*G_0}\times_{\mathcal{G}} E^\bullet \to \wedge^\bullet\overline{T^*G_0}\times_{\mathcal{G}} E^\bullet,$$

such that $A^{E^\bullet\prime\prime}\circ A^{E^\bullet\prime\prime} = 0$. Details can be found in Definition 5.2.

We show in Proposition 5.5 that the above notion is independent of the choice of groupoid representation and is intrinsic to X, in the sense that the dg-category $B(\mathcal{G})$ of antiholomorphic flat superconnections on $\mathcal{G}$ remains unchanged when $\mathcal{G}$ undergoes Morita equivalences. This leads to the definition of

$$B(X),$$

the dg-category of antiholomorphic flat superconnections on the orbifold X. We then develop the basics of antiholomorphic flat superconnections on complex orbifolds. And we establish the foundational result:

$$D^b_{\text{coh}}(X) \simeq \underline{B}(X), \tag{1.1}$$

which is an equivalence between the category of coherent sheaves on X and the homotopy category associated to the dg-category $B(X)$ of antiholomorphic flat superconnections on X. Details are given in Corollary 6.8.

The equivalence (1.1) provides a way to study coherent sheaves on complex orbifolds X by working with antiholomorphic flat superconnections. In particular, for $\mathcal{F} \in D^b_{\text{coh}}(X)$ and an embedding map

$$i_{X,Y} : X \to Y,$$

it is in general not clear if $i_{X,Y,*}\mathcal{F} \in D^b_{\text{coh}}(Y)$ admits a resolution by holomorphic vector bundles even when $\mathcal{F}$ does. However, as an element in $\underline{B}(X)$, $i_{X,Y,*}\mathcal{F}$ does admit a representation by an antiholomorphic flat superconnection. This observation, going back to [Blo06, BSW23], is the key to our study of coherent sheaves on orbifolds in this paper.

1.1.3. *Chern character.* The main object of study in this paper is the *orbifold Chern character* on complex orbifolds. The study of orbifold Chern character goes back to the exploration of equivariant index theory, see for example [AB67], [Kaw79], [Seg68]. Below is an incomplete list of works on Chern character of holomorphic vector bundles.

In [ALR07, Section 2.3], one can find a discussion on defining Chern classes of certain holomorphic vector bundles on orbifolds using Chern-Weil theory, resulting in classes in de Rham cohomology. For a complex orbifold X, this leads to the definition of Chern characters of (complexes of) holomorphic vector bundles on X, taking values in the de Rham cohomology of X.

X. Ma [Ma05, Section 1.2] incorporates isotropy group actions to give a definition of the *orbifold Chern character* of holomorphic vector bundles on a complex orbifold X, taking values in the Bott-Chern cohomology of the *inertia orbifold* IX associated to X. Basic definitions and properties of Bott-Chern cohomology of a complex orbifold from the groupoid perspective are presented in Section 3.

For an orbifold X, its inertia orbifold IX is the (disconnected) orbifold that can be viewed as parametrizing pairs (x, g) where $x \in X$ and g is an element of the isotropy group of x. Locally on X, the inertia orbifold can be understood in terms of the local orbifold chart (U, G):

$$IX|_{[U/G]} = \coprod_{(g)\text{: conjugacy class of } G} [U^g / Z_G(g)],$$

where $Z_G(g) \subset G$ is the centralizer of $g \in G$. See Remark 2.53 for more details.

Given a groupoid presentation $\mathcal{G}$ of X, the inertia orbifold IX can be represented by the inertia groupoid $I\mathcal{G}$ associated to $\mathcal{G}$. This is defined in detail in Section 2.7. Therefore, we can study IX using the groupoid $I\mathcal{G}$.

Inertia orbifolds arise in some natural contexts in the geometry of orbifolds, such as loop spaces [LU02] and Riemann-Roch theorem [Kaw79].

In this article, we are interested in the Chern character of coherent sheaves X. Our main strategy is to invoke the equivalence (1.1) and consider antiholomorphic flat superconnections on X. Together with a generalized metric h, we use curvatures of superconnections and supertrace to define the Chern character of an antiholomorphic flat superconnection $(E^\bullet, A^{E^\bullet\prime\prime})$,

$$\text{ch}(A^{E^\bullet\prime\prime}, h),$$

see Definition 8.1 for details. We show in Section 8 that this yields a well-defined Bott-Chern cohomology class on IX associated to the object in $D^b_{\text{coh}}(X)$ represented by $(E^\bullet, A^{E^\bullet\prime\prime})$.

This gives the *orbifold Chern character*, which is a group homomorphism

$$\mathrm{ch}_{\mathrm{BC}} : K(X) \to H^{(=)}_{\mathrm{BC}}(IX, \mathbb{C}), \tag{1.2}$$

from the K-group of coherent sheaves on X to the Bott-Chern cohomology of the inertia orbifold IX. The detailed definition is given in Definition 8.11.

Our orbifold Chern character $\mathrm{ch}_{\mathrm{BC}}$ in (1.2) is easily seen to satisfy the following properties:

- ($\star$1) For complex vector bundles E on complex orbifolds, our definition of $\mathrm{ch}_{\mathrm{BC}}(E)$ agrees with the one in [Ma05, Section 1.2].
- ($\star$2) $\mathrm{ch}_{\mathrm{BC}}$ is functorial under pullbacks.

We show that our $\mathrm{ch}_{\mathrm{BC}}$ satisfies the following crucial property:

Theorem 1.1. *Let $i_{X,Y} \colon X \hookrightarrow Y$ be an embedding of compact complex orbifold groupoid. Let $\mathcal{F} \in D^b_{coh}(X)$ and $i_{X,Y,*}\mathcal{F} \in D^b_{coh}(Y)$ be its direct image. We have*

$$ch_{BC}(i_{X,Y,*}\mathcal{F}) = Ii_{X,Y,*}\left(\frac{ch_{BC}(\mathcal{F})}{Td_{BC}(N_{X/Y})}\right) \text{ in } H^{(=)}_{BC}(IY, \mathbb{C}), \tag{1.3}$$

where $Ii_{X,Y}$ is the induced morphism between inertia groupoids.

We show that properties ($\star$1), ($\star$2), together with Theorem 1.1 characterize $\mathrm{ch}_{\mathrm{BC}}$:

Theorem 1.2. *The orbifold Chern character $ch_{BC} \colon K(X) \to H^{(=)}_{BC}(IX, \mathbb{C})$ in (1.2) is the unique group homomorphism satisfying ($\star$1), ($\star$2) and*

- *($\star$3) ch_{BC} satisfies the Riemann-Roch-Grothendieck formula for orbifold embeddings, Equation (1.3).*

We view Theorem 1.2, which is an orbifold version of the uniqueness result [BSW23, Theorem 9.4.1], as a demonstration that our definition of $\mathrm{ch}_{\mathrm{BC}}$ is the correct one.

Theorem 1.1 is established in Section 9. For this, we need a structure result on orbifold embeddings. We introduce two kinds of embeddings of orbifolds: *stabilizer-preserving* embedding (Definition 2.38) and *iso-spatial* embedding (Definition 2.40). We show that any orbifold embedding can be decomposed into the composition of an iso-spatial embedding followed by a stabilizer-preserving embedding, see Proposition 2.42.

Our proof of Theorem 1.1 is divided into three parts. We first treat the iso-spatial case by direct computations, see Theorem 9.4. This part is intrinsically associated with the geometry of stabilizer groups. To the best of our knowledge, all prior works did not consider embeddings of this type. Thus Theorem 9.4 is a genuine new result in the study of the Riemann-Roch formula for embeddings of orbifolds. In contrast, the stabilizer-preserving case, Theorem 9.8, is established by a deformation to the normal cone argument following closely the method developed in [BSW23]. When $\mathcal{F}$ is a vector bundle[1], Theorem 9.8 was proved in [Ma05]. Our Theorem 9.8 covers all $\mathcal{F} \in D^b_{\mathrm{coh}}(X)$. In Section 9.5, we put everything together to prove Theorem 1.1.

Theorem 1.1 can be used to calculate the Chern character of pushforwards $i_{X,Y,*}\mathcal{F}$ of coherent sheaves under an embedding $i_{X,Y} \colon X \hookrightarrow Y$. This is very useful in practice.

[1] [Ma05] assumed that the pushforward of $\mathcal{F}$ admits a locally free resolution.

1.2. **Outlook.** Riemann-Roch type results for orbifolds start with the work of T. Kawasaki [Kaw79], who proved a formula for holomorphic Euler characteristics of vector bundles on compact complex orbifolds.

After Grothendieck (see [BGI66]), Riemann-Roch type results refer to transformations from K-theory to suitable cohomology theories that commute with pushforwards of proper morphisms.

For algebraic orbifolds, more precisely Deligne-Mumford stacks, a Riemann-Roch theorem was proved by B. Toen [Toe99].

We aim to establish a Riemann-Roch-Grothendieck theorem for complex orbifolds, which will calculate the orbifold Chern character $\mathrm{ch}_{\mathrm{BC}}(f_*\mathcal{F})$ of the pushforward of a coherent sheaf $\mathcal{F}$ under a holomorphic map f.

A holomorphic map

$$f\colon X \to Y$$

between complex orbifolds can be decomposed as the composition of the embedding

$$i_f\colon X \to X \times Y$$

and the projection

$$p\colon X \times Y \to Y.$$

Our plan to establish the Riemann-Roch-Grothendieck theorem for f is by proving the Riemann-Roch-Grothendieck theorems for i_f and p separately. In this paper we prove the case of embeddings (Theorem 1.1), which covers i_f. In the sequel, we will prove the case that covers p and thus completing the proof of the Riemann-Roch-Grothendieck for f. We will also apply our results to obtain a Riemann-Roch theorem for coherent sheaves on complex orbifolds *twisted* by flat $\mathbb{C}^*$-gerbes.

1.3. **Outline.** The rest of this paper is organized as follows. We present a treatment of groupoids and complex orbifolds in Section 2, which contains the definitions and results that we need. In Section 3 we define Bott-Chern cohomology for complex orbifolds and develop its basic properties. We discuss coherent sheaves on complex orbifolds in Section 4. In Section 5 we introduce and study antiholomorphic superconnections on complex orbifolds. In Section 6 we establish an equivalence between coherent sheaves and antiholomorphic superconnections on a complex orbifold. Section 7 contains a discussion about generalized metrics and their curvatures. The metric description of Chern character for complex orbifolds is given in Section 8. In Section 9, we prove the Riemann-Roch-Grothendieck Theorem for embeddings of orbifolds (Theorem 1.1) and apply it to establish the uniqueness property of the orbifold Chern character (Theorem 1.2).

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2. A REVIEW OF GROUPOIDS AND ORBIFOLDS

This section consists of two parts. In the first part, we review the related materials for groupoids and orbifolds; in the second part, we introduce the groupoid presentation of an orbifold embedding, which is crucial for our development of an orbifold Riemann-Roch-Grothendieck theorem.

2.1. **Lie groupoids.** We briefly review the groupoid approach to orbifolds, mostly following [Moe02]

Definition 2.1. *A groupoid is a (small) category in which each morphism is invertible. Alternatively, a groupoid $\mathcal{G}$ consists of a set G_0 of objects and a set G_1 of arrows. There are maps s and $t\colon G_1 \rightrightarrows G_0$ which are called the source map and the target map, respectively. Moreover we have the unit map $u\colon G_0 \to G_1$, the inverse map $i\colon G_1 \to G_1$, and the composition map $m\colon G_1 \times_{G_0} G_1 \to G_1$.*

Definition 2.2. *A homomorphism $\phi : \mathcal{H} \to \mathcal{G}$ is by definition a functor. In more details, a homomorphism consists of two maps (both) denoted by $\phi : H_0 \to G_0$ and $\phi : H_1 \to G_1$, which together commute with all structure maps.*

We need to consider groupoids in the category of smooth manifolds.

Definition 2.3. *A Lie groupoid is a groupoid $\mathcal{G}$ for which G_0 and G_1 are smooth manifolds and the structure maps s, t, u, i, and m are smooth. Furthermore, s and $t\colon G_1 \rightrightarrows G_0$ are required to be submersive.*

Definition 2.4. *A homomorphism $\phi\colon \mathcal{H} \to \mathcal{G}$ between Lie groupoids is by definition a smooth functor. In more details, a homomorphism consists of two smooth maps (both) denoted by $\phi\colon H_0 \to G_0$ and $\phi\colon H_1 \to G_1$, which together commute with all structure maps.*

Definition 2.5. *A homomorphism $\phi\colon \mathcal{H} \to \mathcal{G}$ between Lie groupoids is called an equivalence if*

(1) the map

$$t \circ \pi_1 \colon G_1 \times_{G_0} H_0 \to G_0$$

is a surjective submersion;

(2) the square

$$\begin{array}{ccc} H_1 & \xrightarrow{\phi} & G_1 \\ {\scriptstyle (s,t)}\Big\downarrow & & \Big\downarrow{\scriptstyle (s,t)} \\ H_0 \times H_0 & \xrightarrow{\phi\times\phi} & G_0 \times G_0 \end{array}$$

is cartesian.

Definition 2.6. *Two Lie groupoids $\mathcal{G}$ and $\mathcal{G}'$ are called Morita equivalent if there exists a third groupoid $\mathcal{H}$ and equivalences*

$$\mathcal{G} \overset{\phi}{\leftarrow} \mathcal{H} \overset{\phi'}{\to} \mathcal{G}'.$$

It can be shown that this defines an equivalence relation on Lie groupoids. To define an orbifold, we will work with proper étale Lie groupoid.

Definition 2.7. *A Lie groupoid $\mathcal{G}$ is called a proper groupoid if the map $(s,t)\colon G_1 \to G_0 \times G_0$ is proper.*

Definition 2.8. *A Lie groupoid $\mathcal{G}$ is called an étale groupoid if s and t are local diffeomorphisms.*

Definition 2.9. *Let $\mathcal{G}$ be an étale groupoid. Then any arrow $g\colon x \to y$ in $\mathcal{G}$ induces a well-defined germ of a diffeomorphism $\tilde{g}\colon (U_x, x) \xrightarrow{\sim} (V_y, y)$ as $\tilde{g} = t \circ \hat{g}$, where $\hat{g}\colon U_x \to G_1$ is a section of the source map $s\colon G_1 \to G_0$ defined on a sufficiently small neighborhood U_x of x and with $\hat{g}(x) = g$.*

We call $\mathcal{G}$ effective (or reduced) if the assignment $g \mapsto \tilde{g}$ is faithful, or equivalently, if for each $x \in G_0$ the map $g \mapsto \tilde{g}$ is an injective group homomorphism $G_x \to \mathrm{Diff}_x(G_0)$.

Now we can define orbifolds.

Definition 2.10. *An orbifold groupoid is a proper étale groupoid $\mathcal{G}$, and an orbifold X has its underlying space given by the quotient space G_0/G_1 of an orbifold groupoid.*

Two orbifolds $X = [G_0/G_1]$, $Y = [H_0/H_1]$ are said to be equivalent if the corresponding proper étale groupoid presentations $\mathcal{G}$ and $\mathcal{H}$ are Morita equivalent.

Next we describe the notion of groupoid actions on spaces.

Definition 2.11. *Let $\mathcal{G}$ be a Lie groupoid. A (right) $\mathcal{G}$-space is a manifold M equipped with an action by $\mathcal{G}$. This action is given by two smooth maps*[2]

$$\pi\colon M \to G_0, \quad \mu\colon M \times_{G_0} G_1 \to M,$$

which satisfy the usual identities for an action.

A vector bundle over $\mathcal{G}$ is a $\mathcal{G}$-space E for which $\pi\colon E \to G_0$ is a vector bundle and the action of $\mathcal{G}$ on E is fiberwise linear.

We can define a left $\mathcal{G}$-space and a left vector bundle over $\mathcal{G}$ in a similar manner.

We need the following construction.

Definition 2.12. *Let $\mathcal{G}$ be a Lie groupoid. For a right $\mathcal{G}$-space E and a left $\mathcal{G}$-space F, we define their fiber product $E \times_{\mathcal{G}} F$ to be*

$$E \times_{\mathcal{G}} F := E \times_{G_0} F/\sim,$$

where $\sim$ is the relation $(eg, f) \sim (e, gf)$ for $g \in G_1$.

Definition 2.13. *Let $\mathcal{G}$ be a Lie groupoid. A (right) $\mathcal{G}$-space P is called free if $pg_1 = pg_2$ implies $g_1 = g_2$. It is called proper if the map $P \times_{G_0} G_1 \to P$ is proper.*

A (right) $\mathcal{G}$-space P is called a principal $\mathcal{G}$-bundle if it is both free and proper.

We can define a left principal $\mathcal{G}$-bundle in the same way.

Definition 2.14. *Let $\mathcal{G}$ and $\mathcal{H}$ be two Lie groupoids. A $\mathcal{G}$-$\mathcal{H}$ principal bibundle is a manifold P with commuting left $\mathcal{G}$-space and right $\mathcal{H}$-space structures that are both principal, and satisfy the following extra conditions:*

[2]The map π is usually assumed to be submersive.

(1) The quotient space $\mathcal{G}\backslash P$ (with its quotient topology) is diffeomorphic to H_0 in a way that identifies the map $P \to H_0$ with the quotient map $P \to \mathcal{G}\backslash P$.
(2) The quotient space $P/\mathcal{H}$ (with its quotient topology) is diffeomorphic to G_0 in a way that identifies the map $P \to G_0$ with the quotient map $P \to P/\mathcal{H}$.

The following proposition gives an equivalent definition of Morita equivalence.

Proposition 2.15. *Two Lie groupoids $\mathcal{G}$ and $\mathcal{H}$ are Morita equivalent in the sense of Definition 2.6 if and only if there exists a $\mathcal{G}$-$\mathcal{H}$ principal bibundle P.*

Proof. See [BX11, Theorem 2.2]. □

2.2. **Generalized morphisms.** In this section, we introduce the notion of generalized morphisms between groupoids.

Definition 2.16. *Let $\mathcal{G}$ and $\mathcal{H}$ be two Lie groupoids. A generalized morphism from $\mathcal{G}$ to $\mathcal{H}$ is a triple (Z, ρ, σ) where*

$$G_0 \overset{\rho}{\leftarrow} Z \overset{\sigma}{\rightarrow} H_0.$$

The manifold Z is endowed with a left $\mathcal{G}$-action and a right $\mathcal{H}$-action which commute, such that

(1) the action of $\mathcal{H}$ is free and proper,
(2) ρ induces a diffeomorphism $Z/\mathcal{H} \simeq G_0$.

Remark 2.17. *We can define a groupoid $\mathcal{Z} = (Z_0, Z_1)$ from a generalized morphism (Z, ρ, σ) where $Z_0 = Z$ and*

$$Z_1 = \{(z, g, h) \in Z \times G_1 \times H_1 | s(g) = \rho(z),\ t(h) = \sigma(z)\}.$$

We define $s(z, g, h) = z$ and $t(z, g, h) = gzh$. In addition, we define

$$(z_1, g_1, h_1) \cdot (z_2, g_2, h_2) = (z_2, g_1g_2, h_2h_1)$$

when $z_1 = g_2z_2h_2$.

We have natural groupoid homomorphisms $\phi_\rho \colon \mathcal{Z} \to \mathcal{G}$ and $\phi_\sigma \colon \mathcal{Z} \to \mathcal{H}$. Actually, on the Z_0 level, we define

$$\phi_\rho(z) = \rho(z) \text{ and } \phi_\sigma(z) = \sigma(z).$$

On the Z_1 level, we define

$$\phi_\rho(z, g, h) = g \text{ and } \phi_\sigma(z, g, h) = h^{-1}.$$

It is easy to check that ϕ_ρ and ϕ_σ are groupoid homomorphisms and ϕ_ρ is a Morita equivalence morphism. If (Z, ρ, σ) is a principal bibundle, then ϕ_σ is also a Morita equivalence morphism.

Example 2.18. *Let $f \colon \mathcal{G} \to \mathcal{H}$ be a morphism between étale Lie groupoids. We consider the* comma generalized morphism *(C, ρ, σ) as follows.*

Define

$$C := \{(g_0, h) \in G_0 \times H_1 | f_0(g_0) = t(h)\}.$$

Since the map t is étale, the space C is a smooth manifold.

Define the left $\mathcal{G}$-action on C as

$$g(g_0, h) := (t(g), f_1(g)h),$$

and the right $\mathcal{H}$-action on C as

$$(g_0, h)\tilde{h} = (g_0, h\tilde{h}).$$

The above left $\mathcal{G}$-action can be depicted in the diagram below:

$$\begin{tikzcd}
g_0 \arrow[d, "f_0"'] & & {} \arrow[r, mapsto] & t(g) \arrow[d, "f_0"] & & \\
f_0(g_0) = t(h) & s(h) \arrow[l, bend right, "h"'] & & t(f_1(g)) = f_0(t(g)) & f_0(g_0) = t(h) \arrow[l, dashed, bend right, "f_1(g)"'] & s(h). \arrow[l, dashed, bend right, "h"'] \arrow[ll, bend right=40, "f_1(g)h"']
\end{tikzcd}$$

The above right $\mathcal{H}$-action can be depicted in the diagram below:

$$\begin{tikzcd}
g_0 \arrow[d, "f_0"'] & & {} \arrow[r, mapsto] & g_0 \arrow[d, "f_0"] & & \\
f_0(g_0) = t(h) & s(h) \arrow[l, bend right, "h"'] & & f_0(g_0) = t(h) & s(h) = t(\tilde{h}) \arrow[l, dashed, bend right, "h"'] & s(\tilde{h}). \arrow[l, dashed, bend right, "\tilde{h}"'] \arrow[ll, bend right=40, "h\tilde{h}"']
\end{tikzcd}$$

We define the map $\rho\colon C \to G_0$ as

$$\rho(g_0, h) = g_0,$$

and $\sigma\colon C \to H_0$ as

$$\sigma(g_0, h) = s(h).$$

It is straightforward to check that (C, ρ, σ) gives a generalized morphism from $\mathcal{G}$ to $\mathcal{H}$ in the sense of Definition 2.16.

For another generalized morphism (W, τ, χ) from $\mathcal{H}$ to $\mathcal{K}$, we define the *composition* of (Z, ρ, σ) and (W, τ, χ) as follows. We first consider the submanifold

$$\tilde{Y} = \{(z, w) \in Z \times W | \sigma(z) = \tau(w) \in H_0\}.$$

There is a right $\mathcal{H}$-action on $\tilde{Y}$ given by

$$h(z, w) = (zh, h^{-1}w).$$

Since the right $\mathcal{H}$-action on Z is free and proper, so is the right $\mathcal{H}$-action on $\tilde{Y}$. Therefore the quotient $Y := \tilde{Y}/\mathcal{H}$ is a manifold.

Define

$$\theta\colon Y \to G_0, \quad \theta(z, w) = \rho(z)$$

and

$$\eta : Y \to K_0, \quad \eta(z, w) = \chi(w).$$

Moreover, we define a left $\mathcal{G}$-action on Y as

$$g(z, w) = (gz, w)$$

and a right $\mathcal{K}$-action on Y as

$$(z, w)k = (z, wk).$$

It is clear that the triple (Y, θ, η) gives a generalized morphism from $\mathcal{G}$ to $\mathcal{K}$, which we define to be the composition of (Z, ρ, σ) and (W, τ, χ).

Moreover, if $f\colon \mathcal{G} \to \mathcal{H}$ and $g\colon \mathcal{H} \to \mathcal{K}$ are two ordinary morphisms, then it is easy to check that the comma generalized morphism of f composed with the comma generalized morphism of g is naturally isomorphic to the comma generalized morphism of $g \circ f$.

Remark 2.19. *Locally, an orbifold groupoid is a transformation groupoid associated with a finite group G action on a manifold X. In this way, a generalized morphism can be represented as a groupoid homomorphism $G \ltimes X \to H \ltimes Y$ defined by a map $\underline{f}\colon X \to Y$ equivariant with respect to the group homomorphism $\bar{f} : G \to H$.*

2.3. **Proper generalized morphisms.** We introduce the notion of proper morphisms between complex orbifold groupoids following [Tu04, Section 7]. We begin with some basic constructions.

For a generalized morphism (Z, ρ, σ) from $\mathcal{G}$ to $\mathcal{H}$. We consider the quotient $\mathcal{G}\backslash Z$. Let $\tilde{\rho}$ be the map

$$\tilde{\rho}\colon \mathcal{G}\backslash Z \to \mathcal{G}\backslash G_0$$

induced by $\rho\colon Z \to G_0$. We have the following commutative diagram,

$$\begin{array}{ccc} G_0 & \xleftarrow{\rho} & Z \\ \Big\downarrow{\scriptstyle \pi_{\mathcal{G}}} & & \Big\downarrow{\scriptstyle \pi_{\mathcal{G}\backslash Z}} \\ \mathcal{G}\backslash G_0 & \xleftarrow{\tilde{\rho}} & \mathcal{G}\backslash Z \end{array} \tag{2.1}$$

Meanwhile, the map $\sigma\colon Z \to H_0$ induces a map

$$\tilde{\sigma}\colon \mathcal{G}\backslash Z \to H_0 \tag{2.2}$$

since σ satisfies $\sigma(gz) = \sigma(z)$. Furthermore, σ and $\tilde{\sigma}$ are both $\mathcal{H}$-invariant maps, hence they induce maps

$$\theta\colon Z/\mathcal{H} \cong G_0 \to H_0/\mathcal{H}, \quad \tilde{\theta}\colon \mathcal{G}\backslash Z/\mathcal{H} \to H_0/\mathcal{H}$$

which make the following diagram commute:

$$\begin{array}{ccc} Z & \xrightarrow{\sigma} & H_0 \\ \Big\downarrow{\scriptstyle \pi_{\mathcal{H}\backslash Z}} & & \Big\downarrow{\scriptstyle \pi_{\mathcal{H}}} \\ G_0 \cong Z/\mathcal{H} & \xrightarrow{\theta} & H_0/\mathcal{H} \\ \Big\downarrow & & \Big\downarrow{\scriptstyle \mathrm{id}} \\ \mathcal{G}\backslash Z/\mathcal{H} & \xrightarrow{\tilde{\theta}} & H_0/\mathcal{H}. \end{array} \tag{2.3}$$

We know that ρ induces a diffeomorphism $Z/\mathcal{H} \simeq G_0$, hence $\mathcal{G}\backslash Z/\mathcal{H} \simeq \mathcal{G}\backslash G_0$ and the map $Z/\mathcal{H} \to \mathcal{G}\backslash Z/\mathcal{H}$ agrees with the quotient map $G_0 \to \mathcal{G}\backslash G_0$.

In summary, we have the following commutative diagram,

$$
\begin{tikzcd}
Z \arrow[d, "\pi_{\mathcal{G}\backslash Z}"'] \arrow[dr, "\sigma"] & \\
\mathcal{G}\backslash Z \arrow[r, "\tilde{\sigma}"] \arrow[d, "\tilde{\rho}"'] & H_0 \arrow[d, "\pi_{\mathcal{H}}"] \\
\mathcal{G}\backslash G_0 \arrow[r, "\tilde{\theta}"] & H_0/\mathcal{H}.
\end{tikzcd}
\tag{2.4}
$$

The following definitions are taken from [Tu04, Introduction].

Definition 2.20. *Let Z be a space with $\mathcal{G}$ action. A subspace Y of Z is called $\mathcal{G}$-compact if Y is $\mathcal{G}$-invariant and $Y/\mathcal{G}$ is compact.*

Definition 2.21. *A generalized morphism $(Z, \rho, \sigma)\colon \mathcal{G} \to \mathcal{H}$ between Lie groupoids is called* locally proper *if the action of $\mathcal{G}$ on Z is proper.*

Remark 2.22. *Let $f\colon \mathcal{G} \to \mathcal{H}$ be a groupoid morphism. Then by* [Tu04, Proposition 7.4] *the associated generalized groupoid morphism is locally proper if and only if the map $(f, r, s)\colon G_1 \to H_1 \times G_0 \times G_0$ is proper.*

Proposition 2.23. *If $\mathcal{G}$ is a proper Lie groupoid, then any generalized morphism $(Z, \rho, \sigma)\colon \mathcal{G} \to \mathcal{H}$ is locally proper.*

Proof. Consider the map $\phi\colon G_1 \times_{G_0} Z \to Z \times Z$ defined by $\phi(g, z) = (gz, z)$. Then we have the following commutative diagram

$$
\begin{tikzcd}
G_1 \times_{G_0} Z \arrow[r, "\phi"] \arrow[dr, "(t,s)"'] & Z \times Z \arrow[d, "(\rho,\rho)"] \\
& G_0 \times G_0.
\end{tikzcd}
\tag{2.5}
$$

For any compact subset $K \subset Z$, $K \times K$ is a compact subset of $Z \times Z$. Hence $(\rho, \rho)(K \times K)$ is a compact subset in $G_0 \times G_0$. Since $\mathcal{G}$ is a proper Lie groupoid, the preimage $(t, s)^{-1}((\rho, \rho)(K \times K))$ is a compact subset in $G_1 \times_{G_0} Z$. Since the diagram commutes, we know that $\phi^{-1}(K \times K) \subset (t, s)^{-1}((\rho, \rho)(K \times K))$ is also compact. So the map $G_1 \times_{G_0} Z \to Z$ is proper. □

Now we are ready to present the main definition of this section.

Definition 2.24. *A generalized morphism $(Z, \rho, \sigma)\colon \mathcal{G} \to \mathcal{H}$ between Lie groupoids is called* proper *if it is locally proper and, in addition, for any compact subset K in H_0, $\sigma^{-1}(K)$ is $\mathcal{G}$-compact.*

Remark 2.25. *It is clear that (Z, ρ, σ) is proper if and only if it is locally proper and the map $\tilde{\sigma}$ in (2.2) is proper.*

Proposition 2.26. *Let $\mathcal{G}$ and $\mathcal{H}$ be proper Lie groupoids. A generalized morphism (Z, ρ, σ) from $\mathcal{G}$ to $\mathcal{H}$ is proper if and only if the induced map $\tilde{\theta}\colon \mathcal{G}\backslash G_0 \to H_0/\mathcal{H}$ is a proper map between topological spaces.*

Proof of Proposition 2.26. Only the "if" part is non-trivial.

We assume $\tilde{\theta}$ is a proper map. Suppose C is a compact subset of H_0, then $\pi_{\mathcal{H}}(C)$ is a compact subset of $H_0/\mathcal{H}$. Since $\tilde{\theta}$ is a proper map, we know that $\tilde{\theta}^{-1}(\pi_{\mathcal{H}}(C))$ is a proper subset of $\mathcal{G}\backslash G_0$.

Recall that we have the commutative diagram:

$$\begin{tikzcd} Z/\mathcal{H} \cong G_0 \arrow[r, "\theta"] \arrow[d, "\pi_{\mathcal{G}}"'] & H_0/\mathcal{H} \\ \mathcal{G}\backslash Z/\mathcal{H} \cong \mathcal{G}\backslash G_0. \arrow[ur, "\tilde{\theta}"'] & \end{tikzcd} \tag{2.6}$$

We need the following

Lemma 2.27. *There exists a compact subset D in G_0 such that $\tilde{\theta}^{-1}(\pi_{\mathcal{H}}(C))$ is contained in $\pi_{\mathcal{G}}(D)$, and a compact subset E in Z such that D is contained in $\rho(E)$.*

Proof of Lemma 2.27. We can choose such a D by covering $\tilde{\theta}^{-1}(\pi_{\mathcal{H}}(C))$ by sufficiently small open subsets whose closures are also compact. Since the groupoid $\mathcal{G}$ is proper, we can find local sections of the map $\pi_{\mathcal{G}}\colon G_0 \to \mathcal{G}\backslash G_0$ and choose D to be the union of the local sections.

Similarly, we can cover D by sufficiently small open subsets whose closures are also compact. Since $\mathcal{H}$ acts on Z freely and properly, we can find local sections of the map $Z \to Z/\mathcal{H}$ and choose E to be the union of the local sections. ☐

We resume the proof of Proposition 2.26. Let D and E be as in Lemma 2.27. We consider $\pi_{\mathcal{G}\backslash Z}(E)$, which is a compact subset of $\mathcal{G}\backslash Z$.

We can also consider $\tilde{\sigma}^{-1}(C) \subset \mathcal{G}\backslash Z$. It is clear that $\tilde{\rho}$ maps $\tilde{\sigma}^{-1}(C)$ to $\tilde{\theta}^{-1}(\pi_{\mathcal{H}}(C))$. In particular, for any $x \in \tilde{\sigma}^{-1}(C)$, we know that $\tilde{\rho}(x) \in \tilde{\theta}^{-1}(\pi_{\mathcal{H}}(C))$. By Lemma 2.27, we can find $y \in D$ and $z \in E$ such that

$$\pi_{\mathcal{G}}(y) = \tilde{\rho}(x) \text{ and } \pi_{Z/\mathcal{H}}(z) = y.$$

Since we have the commutative diagram

$$\begin{tikzcd} Z \arrow[r, "\pi_{Z/\mathcal{H}}"] \arrow[d, "\pi_{\mathcal{G}\backslash Z}"'] & G_0 \cong Z/\mathcal{H} \arrow[d, "\pi_{\mathcal{G}}"] \\ \mathcal{G}\backslash Z \arrow[r, "\tilde{\rho}"'] & \mathcal{G}\backslash G_0 \cong \mathcal{G}\backslash Z/\mathcal{H}, \end{tikzcd} \tag{2.7}$$

we get

$$\tilde{\rho}(\pi_{\mathcal{G}\backslash Z}(z)) = \tilde{\rho}(x).$$

In other words, $\pi_{\mathcal{G}\backslash Z}(z)$ and x are in the same orbit of the $\mathcal{H}$-action on $\mathcal{G}\backslash Z$. Hence there exists an h such that

$$\pi_{\mathcal{G}\backslash Z}(z)h = x.$$

As $x \in \tilde{\sigma}^{-1}(C)$, we get

$$t(h) = \tilde{\sigma}(\pi_{\mathcal{G}\backslash Z}(z)h) = \tilde{\sigma}(x) \in C.$$

On the other hand,

$$s(h) = \tilde{\sigma}(\pi_{\mathcal{G}\backslash Z}(z)) = \sigma(z) \in \sigma(E).$$

In conclusion, for any $x \in \tilde{\sigma}^{-1}(C)$, we have

$$x \in \pi_{\mathcal{G}\backslash Z}(E)h,\ s(h) \in \sigma(E) \text{ and } t(h) \in C. \tag{2.8}$$

Let

$$H_1' = \{h \in H_1 | s(h) \in \sigma(E) \text{ and } t(h) \in C\}.$$

Since $\mathcal{H}$ is proper and both E and C are compact sets, we know that H_1' is compact. Moreover (2.8) implies that $\tilde{\sigma}^{-1}(C) \subset \pi_{\mathcal{G}\backslash Z}(E)H_1'$, hence $\tilde{\sigma}^{-1}(C)$ is also compact. □

2.4. **Complex groupoids.** Here we consider groupoids in the category of complex manifolds.

Definition 2.28. *A Lie groupoid $\mathcal{G} = (G_0, G_1)$ is called a complex Lie groupoid if G_0 and G_1 are complex manifolds and all structure maps are holomorphic.*

Definition 2.29. *A* complex orbifold *is an orbifold X, which has a representation by a complex proper étale Lie groupoid $\mathcal{G}$.*

We call a complex proper étale Lie groupoid a complex orbifold groupoid.

A morphism between complex orbifold groupoids is a generalized morphism in the sense of Definition 2.16 where diffeomorphisms are replaced by holomorphic maps.

Two complex orbifold groupoids are Morita equivalent if there is a principal bibundle that carries a bi-invariant complex structure such that the associated structure maps are holomorphic.

See [Tom12] for relations between Definition 2.29 and other definitions of complex orbifolds.

In the remainder of this section, we discuss a few basic facts about complex orbifolds.

Definition 2.30. *A complex orbifold represented by a groupoid $\mathcal{G}$ is* compact *if its underlying topological space G_0/G_1 is compact.*

Remark 2.31. *By Proposition 2.26, a complex orbifold is compact if and only if for one (and hence any) groupoid representative $\mathcal{G}$, the only generalized morphism from $\mathcal{G}$ to the trivial groupoid is proper.*

Definition 2.32. *Let $\mathcal{G} = (G_0, G_1)$ be a complex orbifold groupoid and $x \in G_0$ be a point. Then there exists a neighborhood U of x in G_0 such that $\mathcal{G}|_U$ is the action groupoid of a finite group G acting holomorphically on U. We call (U, G) a neighborhood of x in $\mathcal{G}$.*

Proposition 2.33. *For any point $x \in G_0$, a neighborhood of x in $\mathcal{G}$ exists.*

Proof. This is a special case of the linearization theorem [Wei02]. □

For compact complex orbifolds (in the sense of Definition 2.30), we have the following further result.

Proposition 2.34. *For a compact complex orbifold, we can choose a representing orbifold groupoid $\mathcal{G} = (G_0, G_1)$ such that for every $x \in G_0$ and U an invariant open neighborhood of x in G_0 there is an arbitrarily small $\mathcal{G}$-invariant open subset V on G_0 satisfying the following properties:*

- *V is a disjoint union of open subsets $\{V_i\}$ such that each point in the $\mathcal{G}$-orbit of x belongs to one and only one V_i;*
- *the restriction of $\mathcal{G}$ to each V_i is isomorphic to the transformation groupoid $\Gamma_i \rtimes V_i \rightrightarrows V_i$ with Γ_i a finite group.*

Proof. Since the orbifold is compact, we can choose to work with a special orbifold groupoid $\mathcal{G} = (G_0, G_1)$ such that every $\mathcal{G}$-orbit on G_0 is a finite set. This property, together with the linearization theorem for proper étale groupoids [Wei02], gives the claim. □

2.5. **Orbifold embeddings.** A morphism $f\colon \mathcal{G} \to \mathcal{H}$ between Lie groupoids is called an *embedding* if each $f_i\colon G_i \to H_i$, $i = 0, 1$ is an embedding of manifolds. The definition of embedding of orbifolds, which we develop in Definition 2.35 below, is more involved.

Let $\mathcal{G}$ and $\mathcal{H}$ be complex orbifold groupoids and (Z, ρ, σ) be a generalized morphism from $\mathcal{G}$ to $\mathcal{H}$. Let $\tilde{\mathcal{G}}$ be the pullback of $\mathcal{G}$ to Z under $\rho\colon Z \to G_0$. More explicitly, we have $\tilde{\mathcal{G}} = (\widetilde{G}_0, \widetilde{G}_1)$ where $\widetilde{G}_0 = Z$ and

$$\widetilde{G}_1 = \{(z_1, g, z_2) \in Z \times G_1 \times Z | \rho(z_1) = t(g),\ \rho(z_2) = s(g)\}. \tag{2.9}$$

The source and target maps of $\tilde{\mathcal{G}}$ are the obvious projections to Z. Similarly, we can define $\tilde{\mathcal{H}}$ to be the pullback of $\mathcal{H}$ to Z under $\sigma\colon Z \to H_0$. More explicitly, we have $\tilde{\mathcal{H}} = (\widetilde{H}_0, \widetilde{H}_1)$ where $\widetilde{H}_0 = Z$ and

$$\widetilde{H}_1 = \{(z_1, h, z_2) \in Z \times H_1 \times Z | \sigma(z_1) = t(h),\ \sigma(z_2) = s(h)\}. \tag{2.10}$$

We can define a morphism $\phi\colon \tilde{\mathcal{G}} \to \tilde{\mathcal{H}}$ as follows: $\phi_0\colon \widetilde{G}_0 \to \widetilde{H}_0$ is the identity map on Z. As for $\phi_1\colon \widetilde{G}_1 \to \widetilde{H}_1$, consider an element $(z_1, g, z_2) \in \widetilde{G}_1$. Notice that we have

$$\rho(gz_2) = t(g) = \rho(z_1) \in G_0.$$

Since the $\mathcal{H}$-action on Z is free and ρ induces an isomorphism $Z/\mathcal{H} \xrightarrow{\simeq} G_0$, there exists a unique $h \in H_1$ such that

$$gz_2 = z_1 h.$$

We define $\phi_1(z_1, g, z_2)$ to be

$$\phi_1(z_1, g, z_2) := (z_1, h, z_2). \tag{2.11}$$

Definition 2.35. *A generalized morphism (Z, ρ, σ) from $\mathcal{G}$ to $\mathcal{H}$ is an* embedding *if it satisfies the following conditions.*

1. *The image of $\sigma\colon Z \to H_0$ is an embedded submanifold of H_0.*
2. *The map $\sigma\colon Z \to H_0$ is an étale submersion to its image.*
3. *The morphism $\phi\colon \tilde{\mathcal{G}} \to \tilde{\mathcal{H}}$ is a* saturated embedding *of groupoids.*

Here, "saturated" means that for any $h \in \widetilde{H}_1$, there exists a $g \in \widetilde{G}_1$ such that $s(h) = s(\phi(g))$ and $t(h) = t(\phi(g))$.

Remark 2.36. *It follows from the embedding condition that for each point $x \in \sigma(Z) \subset H_0$, there exists an open neighborhood U of x such that $\sigma(Z) \cap U$ has only finitely many branches.*

Proposition 2.37. *If a generalized morphism (Z, ρ, σ) from $\mathcal{G}$ to $\mathcal{H}$ is an embedding, then it induces an embedding of the quotient space $G_0/\mathcal{G}$ into $H_0/\mathcal{H}$.*

Proof. Let (Z, ρ, σ) be an embedding generalized morphism. We can pass to consider the groupoid morphism $\tilde{\mathcal{G}} \to \mathcal{H}$.

We observe that for any $z \in Z$ and $n = \sigma(z) \in H_0$, and any $h \in H_1$ such that $t(h) = n$, we have $s(h) = \sigma(z \cdot h)$. Therefore, $L := \sigma(Z)$ is an $\mathcal{H}$-saturated submanifold of H_0.

Consider the diagram

$$\begin{tikzcd}
H_1|_L \arrow[r, hook] \arrow[d, shift left] \arrow[d, shift right] & H_1 \arrow[d, shift left] \arrow[d, shift right] \\
L \arrow[r, hook, "i"] \arrow[d, "\pi_L"'] & H_0 \arrow[d, "\pi_{\mathcal{H}}"] \\
L/(H_1|_L) \arrow[r, "\bar{i}"] & H_0/H_1.
\end{tikzcd}$$

The map $\bar{i}\colon L/(H_1|_L) \to H_0/H_1$ is obviously continuous. Since L is $\mathcal{H}$-saturated, $\bar{i}$ is injective. We can prove that $\bar{i}$ is a closed map.

Let $[l_k] \in L/(H_1|_L)$ and $[h] \in H_0/H_1$ be such that $[l_k] \to [h]$ in H_0/H_1. We choose a lift $h \in H_0$ of $[h]$ and a precompact neighborhood U of h in H_0 which is sufficiently small so that $U \to [U] \subset H_0/H_1$ is a quotient of U by a finite group. Choose a lift $l_k \in U$ of $[l_k]$.

Since U is precompact and the restriction $\mathcal{H}|_U$ is a transformation groupoid associated with a finite group action, $\{l_k\}$ has a subsequence converging to $l_0 \in U$. Without loss of generality, we can assume $l_k \to l_0$. Since L is an embedded submanifold of H_0, we know $l_0 \in L$. Hence $[l_0] \in L/(H_1|_L)$. We prove that $\bar{i}\colon L/(H_1|_L) \to H_0/H_1$ is a closed map and an embedding.

We consider the groupoid morphism

$$\begin{tikzcd}
\widetilde{H}_1 \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & H_1|_L \arrow[d, shift left] \arrow[d, shift right] \\
Z \arrow[r] & L.
\end{tikzcd}$$

As $\widetilde{H}_1$ is the pullback of H_1 to Z via $\sigma\colon Z \to L$, the above morphism is a Morita equivalence. Hence the quotient $Z/\widetilde{H}_1$ is homeomorphic to $L/(H_1|_L)$.

We also consider

$$\begin{tikzcd}
\widetilde{G}_1 \arrow[r, "\phi"] \arrow[d, shift left] \arrow[d, shift right] & \widetilde{H}_1 \arrow[d, shift left] \arrow[d, shift right] \\
Z \arrow[r, equal] & Z.
\end{tikzcd}$$

Since ϕ is saturated, the induced map $Z/\widetilde{G}_1 \to Z/\widetilde{H}_1$ is the identity map.

Combining the above results, we conclude that the map $G_0/G_1 \to H_0/H_1$ is an embedding. $\square$

For later purposes, we need to distinguish the following two types of groupoid embeddings.

Definition 2.38. *A generalized morphism (Z, ρ, σ) from $\mathcal{G}$ to $\mathcal{H}$ is a* stabilizer-preserving *embedding if it is an embedding in the sense of Definition 2.35 and, in addition, the morphism $\phi\colon \tilde{\mathcal{G}} \to \tilde{\mathcal{H}}$ is an isomorphism of groupoids.*

Remark 2.39. *The notion of stabilizer-preserving morphism is also considered in a different context in* [CDH24, Definition 5.2].

Definition 2.40. *A generalized morphism (Z, ρ, σ) from $\mathcal{G}$ to $\mathcal{H}$ is an* iso-spatial *embedding if it is an embedding in the sense of Definition 2.35 and, in addition, the map $\sigma\colon Z \to H_0$ is a surjective submersion.*

Remark 2.41. *Locally, the groupoid $\mathcal{G}$ is given by a manifold X with an action of a finite group G. A similar description is valid for the groupoid $\mathcal{H}$. When we restrict to the local case, an iso-spatial embedding is given by an inclusion of the finite group $\bar{f}\colon G \hookrightarrow H$ and the identity map on the base manifold X; and a stabilizer-preserving embedding is given by the identity map of the finite group and an embedding of complex manifolds $\underline{f}\colon X \hookrightarrow Y$.*

We have the following factorization result for embeddings.

Proposition 2.42. *Any groupoid embedding can be written as the composition of an iso-spatial embedding followed by a stabilizer-preserving embedding.*

Proof. Let (Z, ρ, σ) be an embedding from $\mathcal{G}$ to $\mathcal{H}$. We consider the groupoid $\mathcal{H}|_{\mathrm{Im}\sigma}$. Since (Z, ρ, σ) is an embedding, the generalized morphism (Z, ρ, σ) from $\mathcal{G}$ to $\mathcal{H}|_{\mathrm{Im}\sigma}$ is an iso-spatial embedding, and the morphism $\sigma\colon \mathcal{H}|_{\mathrm{Im}\sigma} \hookrightarrow \mathcal{H}$ is a stabilizer-preserving embedding. It is easy to check that their composition is exactly (Z, ρ, σ). $\square$

2.6. **Graph generalized morphisms.** For a generalized morphism (Z, ρ, σ) from $\mathcal{G}$ to $\mathcal{H}$, we can define its *graph* generalized morphism from $\mathcal{G}$ to $\mathcal{G} \times \mathcal{H}$ as the triple

$$(\mathrm{Gr}(Z), \mathrm{Gr}(\rho), \mathrm{Gr}(\sigma)),$$

where

$$\mathrm{Gr}(Z) := \{(g, z) \in G_1 \times Z | s(g) = \rho(z)\}. \tag{2.12}$$

The map $\mathrm{Gr}(\rho)\colon \mathrm{Gr}(Z) \to G_0$ is defined to be

$$\mathrm{Gr}(\rho)(g, z) := t(g),$$

and the map $\mathrm{Gr}(\sigma)\colon \mathrm{Gr}(Z) \to G_0 \times H_0$ is defined to be

$$\mathrm{Gr}(\sigma)(g, z) := (s(g), \sigma(z)).$$

The left $\mathcal{G}$-action on $\mathrm{Gr}(Z)$ is defined to be

$$\tilde{g}(g, z) := (\tilde{g}g, z), \tag{2.13}$$

and the right $\mathcal{G} \times \mathcal{H}$-action on $\mathrm{Gr}(Z)$ is defined to be

$$(g, z)(\tilde{g}, h) := (g\tilde{g}, \tilde{g}^{-1}zh). \tag{2.14}$$

It is easy to check that $(\mathrm{Gr}(Z), \mathrm{Gr}(\rho), \mathrm{Gr}(\sigma))$ is indeed a generalized morphism from $\mathcal{G}$ to $\mathcal{G} \times \mathcal{H}$.

Moreover, we consider the natural projection $\mathcal{G} \times \mathcal{H} \to \mathcal{H}$. It is easy to see that the associated comma generalized morphism is given by the triple $(G_0 \times H_1, \mathrm{pr}_1 \times t, s \circ \mathrm{pr}_2)$. It is easy to check that the composition of $(\mathrm{Gr}(Z), \mathrm{Gr}(\rho), \mathrm{Gr}(\sigma))$ with $(G_0 \times H_1, \mathrm{pr}_1 \times t, s \circ \mathrm{pr}_2)$ is canonically isomorphic to (Z, ρ, σ). This justifies the term "graph generalized morphism".

Proposition 2.43. *For a generalized morphism (Z, ρ, σ) from $\mathcal{G}$ to $\mathcal{H}$, the graph generalized morphism $(Gr(Z), Gr(\rho), Gr(\sigma))$ is an embedding from $\mathcal{G}$ to $\mathcal{G} \times \mathcal{H}$.*

Proof. The proof consists of the following lemmas.

First, we show that the image of $\mathrm{Gr}(\sigma)\colon \mathrm{Gr}(Z) \to G_0 \times H_0$ is an embedded submanifold.

Lemma 2.44. *The map $Gr(\sigma)$ is an immersion.*

Proof of Lemma 2.44. Let (g, z) be a point in $\mathrm{Gr}(Z)$. We know $s(g) = \rho(z)$. Notice that $\rho\colon Z \to G_0$ is a surjective submersion and $\mathcal{H}$ acts on Z properly and freely. Since $\mathcal{H}$ is a proper étale groupoid, we know that ρ is a local diffeomorphism.

Let U be a neighborhood of $s(g)$, V be a neighborhood of g, and W be a neighborhood of z, such that $s\colon V \to U$ and $\rho\colon W \to U$ are diffeomorphisms. Therefore, $V \times_{G_0} W \cong U$ is a neighborhood of (g, z) in $\mathrm{Gr}(Z)$, and the map

$$\mathrm{Gr}(\sigma)\colon V \times_{G_0} W \to G_0 \times H_0$$

is given by

$$(g', w) \mapsto (g', \sigma \circ \rho^{-1} \circ s(g')).$$

On $V \times_{G_0} W$, $\mathrm{Gr}(\sigma)$ is the graph of the map $\sigma \circ \rho^{-1} \circ s\colon V \to W$. Hence $\mathrm{Gr}(\sigma)$ is an immersion. ☐

Lemma 2.45. *The map $Gr(\sigma)$ is a closed map.*

Proof of Lemma 2.45. Let

$$\{(\tilde{g}_k, z_k)\} \in \mathrm{Gr}(Z) = G_1 \times_{G_0} Z$$

be a sequence in $G_1 \times_{G_0} Z$ such that

$$\lim_{k\to\infty} \mathrm{Gr}(\sigma)(\tilde{g}_k, z_k) = (g, h) \in G_0 \times H_0.$$

We will show that there exists a $(\tilde{g}, z) \in G_1 \times_{G_0} Z$ such that $\mathrm{Gr}(\sigma)(\tilde{g}, z) = (g, h)$.

Recall that

$$\mathrm{Gr}(\sigma)(\tilde{g}_k, z_k) := (s(\tilde{g}_k), \sigma(z_k)) = (\rho(z_k), \sigma(z_k)).$$

We denote $s(\tilde{g}_k)$ by g_k. It is clear that $\lim_{k\to\infty} g_k = g$. We choose $z_0 \in Z$ such that $\rho(z_0) = g$. Since the $\mathcal{H}$-action on Z is free and proper and $\mathcal{H}$ is an étale groupoid, the map $\rho\colon Z \to Z/\mathcal{H} \cong G_0$ is étale. Hence we can choose neighborhoods W and U of z_0 and g respectively such that $\rho|_W\colon W \to U$ is a diffeomorphism.

Since the $\mathcal{H}$-action on Z is free and proper, for every z_k, there exists a unique $\tilde{h}_k \in H_1$ such that

$$z_k = (\rho|_W)^{-1}(\rho(z_k)) \cdot \tilde{h}_k.$$

Hence $\sigma(z_k) = s(\tilde{h}_k) \to h \in H_0$.

Let V be a neighborhood of h in H_0. Then for sufficiently large k we have $s(\tilde{h}_k) \in V$ and
$$t(\tilde{h}_k) = \sigma((\rho|_W)^{-1}(\rho(z_k))) \in \sigma(W).$$

We can choose V and W to be precompact. By the properness of $\mathcal{H}$, we know that there is a subsequence of $\tilde{h}_k$ which has a limit $\tilde{h}_0 \in H_1$. Without loss of generality, we can work with this subsequence. Hence
$$(u(\rho(z_k)), (\rho|_W)^{-1}(\rho(z_k)) \cdot \tilde{h}_k) \in G_1 \times_{G_0} Z$$
converge to
$$(u(\rho(z_0)), (\rho|_W)^{-1}(\rho(z_0)) \cdot \tilde{h}_0) = (u(g), (\rho|_W)^{-1}(\rho(z_0)) \cdot \tilde{h}_0)$$
as k goes to ∞. Here, $u\colon G_0 \to G_1$ is the unit map. Let
$$\tilde{g} = u(g) \text{ and } z = (\rho|_W)^{-1}(\rho(z_0)) \cdot \tilde{h}_0.$$
We have $\mathrm{Gr}(\sigma)(\tilde{g}, z) = (g, h)$ as expected. So $\mathrm{Gr}(\sigma)$ is a closed map. □

Lemma 2.46. *The image of Gr(σ) is an embedded submanifold of $G_0 \times H_0$.*

Proof of Lemma 2.46. By Lemma 2.44 and Lemma 2.45, $\mathrm{Gr}(\sigma)\colon G_1 \times_{G_0} Z \to G_0 \times H_0$ is a closed immersion, hence its image is an embedded submanifold of $G_0 \times H_0$. □

Lemma 2.47. *For any point x in the image of Gr(σ), there exists an open neighborhood U of x such that Image(Gr(σ)) $\cap\, U$ has only finitely many branches.*

Proof of Lemma 2.47. Let (g, z) and $(\tilde{g}, \tilde{z})$ be another point in $\mathrm{Gr}(Z)$ such that $\mathrm{Gr}(\sigma)(g, z) = \mathrm{Gr}(\sigma)(\tilde{g}, \tilde{z})$, then $s(g) = s(\tilde{g}) = \rho(z) = \rho(\tilde{z})$. So there exist $g_1 \in G_1$ and $h_1 \in H_1$ such that $g_1 V$ and $W h_1$ are neighborhoods of $\tilde{g} \in G_1$ and $\tilde{z} \in Z$, respectively. It is clear that $s(h_1) = t(h_1) = \sigma(z) = \sigma(\tilde{z})$, i.e. h_1 is in the isotropy group of $\sigma(z)$, which is a finite group, since the groupoid $\mathcal{H}$ is proper.

The image of $g_1 V \times_{G_0} W h_1$ will be the same as that of $V \times_{G_0} W$ on the G_0 component but twisted by h_1 on the H_0 component. Hence Image($\mathrm{Gr}(\sigma)$) has only finitely many branches if restricted to a sufficiently small open neighborhood of $\mathrm{Gr}(\sigma)(g, z)$. □

We can define the pullback groupoids $\tilde{\mathcal{G}}$, $\widetilde{\mathcal{G} \times \mathcal{H}}$ and the map $\phi\colon \tilde{\mathcal{G}} \to \widetilde{\mathcal{G} \times \mathcal{H}}$ as in Section 2.5. In more detail, $\tilde{\mathcal{G}} = (\widetilde{G}_0, \widetilde{G}_1)$ where $\widetilde{G}_0 = \mathrm{Gr}(Z)$ as in (2.12), and
$$\widetilde{G}_1 = \{((g_1, z_1), g, (g_2, z_2)) \in \mathrm{Gr}(Z) \times G_1 \times \mathrm{Gr}(Z) | s(g) = t(g_2),\ t(g) = t(g_1)\}.$$
Similarly, $\widetilde{\mathcal{G} \times \mathcal{H}} = (\widetilde{K}_0, \widetilde{K}_1)$ where $\widetilde{K}_0 = \mathrm{Gr}(Z)$ and
$$\begin{aligned}\widetilde{K}_1 = \{&((g_1, z_1), g, h, (g_2, z_2)) \in \mathrm{Gr}(Z) \times G_1 \times H_1 \times \mathrm{Gr}(Z)| \\ &s(g_1) = \rho(z_1) = t(g),\ \sigma(z_1) = t(h), s(g_2) = \rho(z_2) = s(g),\ \sigma(z_2) = s(h)\}.\end{aligned}$$
Moreover, the map ϕ is defined by $\phi_0\colon \widetilde{G}_0 \to \widetilde{K}_0$ being the identity map and $\phi_1\colon \widetilde{G}_1 \to \widetilde{K}_1$ being
$$\phi_1((g_1, z_1), g, (g_2, z_2)) = ((g_1, z_1), g_1^{-1} g g_2^{-1}, h(z_1, g_1^{-1} g g_2^{-1}, z_2), (g_2, z_2)),$$
where $h(z_1, g_1^{-1} g g_2^{-1}, z_2)$ is the unique element in H_1 such that
$$g_1^{-1} g g_2^{-1} z_2 = z_1 h(z_1, g_1^{-1} g g_2^{-1}, z_2).$$

By definition, it is clear that ϕ is saturated.

Lemma 2.48. *ϕ_1 is an injective immersion.*

Proof of Lemma 2.48. By definition, it is clear that ϕ is injective.

We show that ϕ is an immersion. Let U be an open subset of G_1 such that s and t are diffeomorphisms when restricted to U. Let U_1 and U_2 be open neighborhoods of g_1 and g_2, respectively, so that

$$s\colon U_1 \to s(U_1) \subset s(U), \quad t\colon U_2 \to t(U_2) \subset t(U)$$

are diffeomorphisms.

Since $\rho\colon Z \to G_0$ is a local diffeomorphism, we can choose open neighborhoods V_1 and V_2 of z_1 and z_2, respectively, so that

$$\rho\colon V_1 \to t(U_1), \quad \rho\colon V_2 \to t(U_2)$$

are also diffeomorphisms. Therefore we have the diffeomorphism

$$(U_1 \times_{G_0} V_1) \times_{G_0} U \times_{G_0} (U_2 \times_{G_0} V_2) \xrightarrow{\simeq} U. \tag{2.15}$$

On the other hand, we can choose an open neighborhood W of $h(z_1, g_1^{-1} g g_2^{-1}, z_2)$ in H_1 such that s and t map W onto $\sigma(V_1)$ and $\sigma(V_2)$, respectively. Therefore we have another diffeomorphism

$$(U_1 \times_{G_0} V_1) \times_{G_0 \times H_0} (U \times W) \times_{G_0 \times H_0} (U_2 \times_{G_0} V_2) \xrightarrow{\simeq} U \times W. \tag{2.16}$$

We consider the restriction of $\phi_1\colon \widetilde{G}_1 \to \widetilde{G \times H}_1$ to $(U_1 \times_{G_0} V_1) \times_{G_0} U \times_{G_0} (U_2 \times_{G_0} V_2)$. Under the diffeomorphisms (2.15) and (2.16), $\phi_1|_{(U_1 \times_{G_0} V_1) \times_{G_0} U \times_{G_0} (U_2 \times_{G_0} V_2)}$ is nothing but the embedding U to $U \times W$. Therefore ϕ_1 is an injective immersion. $\square$

It remains to show that ϕ_1 is an embedding, which reduces to showing the following

Lemma 2.49. *ϕ_1 is a closed map.*

Proof of Lemma 2.49. Suppose we have a sequence $((g_1^k, z_1^k), g^k, (g_2^k, z_2^k))$ such that

$$((g_1^k, z_1^k), (g_1^k)^{-1} g^k (g_2^k)^{-1}, h(z_1^k, (g_1^k)^{-1} g^k (g_2^k)^{-1}, z_2^k), (g_2^k, z_2^k))$$

approaches to $((g_1^0, z_1^0), g^0, h^0, (g_2^0, z_2^0))$ as $k \to \infty$. This implies that as $k \to \infty$

$$h(z_1^k, (g_1^k)^{-1} g^k (g_2^k)^{-1}, z_2^k) \to h^0, \tag{2.17}$$

and $g_i^k \to g_i^0$, $z_i^k \to z_i^0$ for $i = 1$ and 2, and

$$(g_1^k)^{-1} g (g_2^k)^{-1} \to g^0.$$

Hence $g^k \to g_1^0 g^0 g_2^0$ as $k \to \infty$. In conclusion

$$((g_1^k, z_1^k), g^k, (g_2^k, z_2^k)) \to ((g_1^0, z_1^0), g_1^0 g^0 g_2^0, (g_2^0, z_2^0)),$$

as $k \to \infty$. We need to show that

$$\phi_1((g_1^0, z_1^0), g_1^0 g^0 g_2^0, (g_2^0, z_2^0)) = ((g_1^0, z_1^0), (g^0, h^0), (g_2^0, z_2^0)).$$

By definition, we know that

$$(g_1^k)^{-1} g^k (g_2^k)^{-1} z_2^k = z_1^k h(z_1^k, (g_1^k)^{-1} g^k (g_2^k)^{-1}, z_2^k).$$

Let $k \to \infty$, we get

$$g^0 z_2^0 = z_1^0 h(z_1^0, (g_1^0)^{-1} g^0 (g_2^0)^{-1}, z_2^0).$$

Since we assume that the $\mathcal{H}$-action on Z is free, we can show that for each k, the element $h(z_1^k, (g_1^k)^{-1} g^k (g_2^k)^{-1}, z_2^k) \in H_1$ is contained in a precompact neighborhood of $h(z_1^0, (g_1^0)^{-1} g^0 (g_2^0)^{-1}, z_2^0)$. Therefore

$$h(z_1^k, (g_1^k)^{-1} g^k (g_2^k)^{-1}, z_2^k) \to h(z_1^0, (g_1^0)^{-1} g^0 (g_2^0)^{-1}, z_2^0) \tag{2.18}$$

as $k \to \infty$. Combining (2.17) and (2.18) we know that

$$h(z_1^0, (g_1^0)^{-1} g^0 (g_2^0)^{-1}, z_2^0) = h^0,$$

Hence

$$\phi_1((g_1^0, z_1^0), g_1^0 g^0 g_2^0, (g_2^0, z_2^0)) = ((g_1^0, z_1^0), (g^0, h^0), (g_2^0, z_2^0)),$$

as desired. □

We have now completed the proof of Proposition 2.43 with Lemmas 2.44-2.49. □

Remark 2.50. *In general, the embedding $(Gr(Z), Gr(\rho), Gr(\sigma))$ is neither iso-spatial nor stabilizer-preserving. However, by Proposition 2.42, $(Gr(Z), Gr(\rho), Gr(\sigma))$ is the composition of an iso-spatial embedding and a stabilizer-preserving embedding.*

2.7. Inertia groupoids.

Following [Moe02, Section 6.4], we can define the inertia groupoid of a complex orbifold groupoid $\mathcal{G} = (G_0, G_1)$. We consider

$$B_{\mathcal{G}} = \{g \in G_1 | s(g) = t(g)\}. \tag{2.19}$$

It is easy to see that $B_{\mathcal{G}}$ is a disjoint union of complex manifolds whose dimensions may vary with different components even when G_0 is connected. We also notice that $\mathcal{G}$ acts on the space $B_{\mathcal{G}}$ by conjugation. Then the *inertia groupoid* $I\mathcal{G}$ is defined to be the semi-direct product $B_{\mathcal{G}} \ltimes \mathcal{G}$. More precisely,

$$(I\mathcal{G})_0 = B_{\mathcal{G}}, \tag{2.20}$$

and for g_1 and $g_2 \in S_{\mathcal{G}}$, an arrow $g_1 \to g_2$ consists of $g \in G_1$ such that $g g_1 g^{-1} = g_2$. In other words

$$(I\mathcal{G})_1 = \{(g_1, g_2, g) | g_1, g_2 \in B_{\mathcal{G}} \text{ and } g g_1 g^{-1} = g_2\}, \tag{2.21}$$

and $s(g_1, g_2, g) = g_1$, $t(g_1, g_2, g) = g_2$. It is clear that $I\mathcal{G}$ is also a proper, étale complex Lie groupoid.

Definition 2.51. *There is a proper holomorphic groupoid map $\beta_{\mathcal{G}} \colon I\mathcal{G} \to \mathcal{G}$ which maps $g \in (I\mathcal{G})_0$ to $s(g) = t(g) \in G_0$ and maps $(g_1, g_2, g) \in (I\mathcal{G})_1$ to $g \in G_1$.*

Definition 2.52. *On $I\mathcal{G}$ we have the tautological section $\tau_{\mathcal{G}}$ with value in G_1 given by*

$$\tau_{\mathcal{G}}(g) = (g, g, g), \text{ for } g \in (I\mathcal{G})_0. \tag{2.22}$$

Remark 2.53. *Locally an orbifold groupoid $\mathcal{G}$ is a finite group G acting on a complex manifold X. In this case, the inertia groupoid $I\mathcal{G}$ is*

$$G \ltimes \coprod_{g \in G} X^g, \tag{2.23}$$

where X^g is the fixed point set of g, and the G-action is given as follows: for any $g' \in G$,

$$g' \colon X^g \to X^{g' g (g')^{-1}}, \quad x \mapsto g' x. \tag{2.24}$$

It is easy to see that (2.23) is Morita equivalent to

$$\coprod_{(g)\in \mathit{Conj}(G)} Z_G(g) \ltimes X^g, \tag{2.25}$$

where $\mathit{Conj}(G)$ *is the set of conjugacy classes of* G *and* $Z_G(g)$ *is the centralizer of* g *in* G.

From the viewpoint of (2.25), the map $\beta_{\mathcal{G}}$ *is the natural map*

$$Z_G(g) \ltimes X^g \to G \ltimes X,$$

which are inclusions on both $Z_G(g)$ *and* X^g. *The map* $\tau_{\mathcal{G}}$ *is given on each component by*

$$\tau_{\mathcal{G}}(x) = (g, x), \text{ for } x \in X^g. \tag{2.26}$$

Next, we consider induced (generalized) morphisms between inertia groupoids.

Definition 2.54. *Let* $(Z, \rho, \sigma)\colon \mathcal{G} \to \mathcal{H}$ *be a generalized morphism. We define a generalized morphism*

$$(IZ, I\rho, I\sigma)\colon I\mathcal{G} \to I\mathcal{H}$$

as follows: let

$$IZ := \{(g, z, h) \in G_1 \times Z \times H_1 | gzh = z\}. \tag{2.27}$$

This implies

$$s(g) = t(g) = \rho(z), \text{ and } s(h) = t(h) = \sigma(z), \tag{2.28}$$

hence we define $I\rho\colon IZ \to (I\mathcal{G})_0$ *and* $I\sigma\colon IZ \to (I\mathcal{H})_0$ *as*

$$I\rho(g, z, h) := g, \text{ and } I\sigma(g, z, h) := h. \tag{2.29}$$

The groupoid $I\mathcal{G}$ *acts on* IZ *from the left by*

$$(g_1, g_2, g) \cdot (g_1, z, h) := (g_2, gz, h) = (gg_1g^{-1}, gz, h), \tag{2.30}$$

and $I\mathcal{H}$ *acts on* IZ *from the right by*

$$(g, z, h_2) \cdot (h_1, h_2, h) := (g, zh, h_1) = (g, zh, h^{-1}h_2h). \tag{2.31}$$

Remark 2.55. *Since the* $\mathcal{H}$*-action on* Z *is free, the* h *component of* IZ *as in (2.27) is uniquely determined by* (g, z) *and the condition* $gzh = z$. *Hence* IZ *is isomorphic to*

$$\{(g, z) \in G_1 \times Z | s(g) = t(g) = \rho(z)\}. \tag{2.32}$$

Remark 2.56. *In the local case, let* $f\colon G \ltimes X \to H \ltimes Y$ *be a morphism which consists of* $\underline{f}\colon X \to Y$ *and* $\bar{f}\colon G \to H$. *Then the induced morphism*

$$If\colon \coprod_{(g)\in \mathit{Conj}(G)} Z_G(g) \ltimes X^g \to \coprod_{(h)\in \mathit{Conj}(H)} Z_H(h) \ltimes Y^h$$

consists of $\overline{If} = \bar{f}\colon Z_G(g) \to Z_H(\bar{f}(g))$ *and* $\underline{If}$ *is given on each component by* $\underline{f}|_{X^g}\colon X^g \to Y^{\bar{f}(g)}$.

Let $(Z,\rho,\sigma)\colon \mathcal{G}\to\mathcal{H}$ be a Morita equivalence. It is easy to see that

$$(IZ, I\rho, I\sigma)\colon I\mathcal{G}\to I\mathcal{H}$$

is also a Morita equivalence. In other words, the inertia groupoid is invariant under Morita equivalence as in Definition 2.6, and hence we can consider the inertia orbifold of a complex orbifold.

The following proposition, which will be used later, shows that being stabilizer-preserving is preserved under taking induced (generalized) morphisms between inertia groupoids.

Proposition 2.57. *Let $(Z,\rho,\sigma)\colon \mathcal{G}\to\mathcal{H}$ be a generalized morphism between complex orbifold groupoids. If (Z,ρ,σ) is a stabilizer-preserving embedding, then $(IZ,I\rho,I\sigma)$ is also a stabilizer-preserving embedding.*

Proof of Proposition 2.57. First we need to check that $(IZ,I\rho,I\sigma)$ satisfies the conditions in Definition 2.35.

Lemma 2.58. *The image of $I\sigma$ is an embedded submanifold of $(I\mathcal{H})_0$.*

Proof of Lemma 2.58. First, we show that $I\sigma$ is a local embedding. For $(g,z)\in IZ$, choose a neighborhood U of z in Z such that $\rho|_U\colon U\to G_0$ is a diffeomorphism and $\sigma|_U\colon U\to H_0$ is an embedding. Choose a neighborhood W of g in $(IG)_0$ and V of $I\sigma(g,z)$ in $(I\mathcal{H})_0$ such that W and V are embedded in $\rho(U)$ and $\sigma(U)$, respectively.

We observe that $U\ {}_\rho{\times}_t W\ {}_s{\times}_\rho U$ forms a neighborhood of (z,g,z') in $Z\ {}_\rho{\times}_t (I\mathcal{G})_1\ {}_s{\times}_\rho Z$, and $U\ {}_\sigma{\times}_t V\ {}_s{\times}_\sigma U$ forms a neighborhood of $(z, I\sigma(g,z), z')$ in $Z\ {}_\rho{\times}_t (I\mathcal{H})_1\ {}_s{\times}_\rho Z$. Since

$$\phi\colon Z\ {}_\rho{\times}_t G_1\ {}_s{\times}_\rho Z\to Z\ {}_\rho{\times}_t H_1\ {}_s{\times}_\rho Z$$

is an embedding, the restriction

$$\phi\colon U\ {}_\rho{\times}_t W\ {}_s{\times}_\rho U\to U\ {}_\sigma{\times}_t V\ {}_s{\times}_\sigma U$$

is also an embedding. Notice that $\rho|_U$ is a diffeomorphism, hence $U\ {}_\rho{\times}_t W\ {}_s{\times}_\rho U$ is diffeomorphic to $W\ {}_s{\times}_\rho U$. On the other hand, $U\ {}_\sigma{\times}_t V\ {}_s{\times}_\sigma U$ is naturally embedded into V. This shows that the map $W\ {}_s{\times}_\rho U\to V$ is an embedding. Hence $IZ\to(I\mathcal{H})_0$ is a local embedding.

We need to show that $I\sigma$ is a closed map. Let $h_n\in(I\mathcal{H})_0$ be in the image of $I\sigma$. In other words, there exist $g_n\in G_1$ and $z_n\in Z$ such that

$$g_n z_n h_n = z_n.$$

If $h_n\to h_0\in(I\mathcal{H})_0$, then we need to show that h_0 is also in the image of $I\sigma$.

Notice that $h_0\in H_1$ and $s(h_0)=t(h_0)\in H_0$. Since (Z,ρ,σ) is an embedding, by Condition (2) of Definition 2.35 there exists an open neighborhood $\tilde{W}$ of $s(h_0)=t(h_0)$ such that $\sigma(Z)\cap\tilde{W}$ has finitely many branches. Since $\sigma(z_n)\in H_0$ converges to $s(h_0)=t(h_0)$, there exists a subsequence of $\{z_n\}$ whose images under σ are contained in one branch. By abuse of notation, we still denote this subsequence by $\{z_n\}$.

We choose an open subset $U\subset Z$ such that $\sigma|_U$ is a diffeomorphism to the branch mentioned above and $\rho|_U$ is a diffeomorphism to an open subset of G_0.

We can also choose an open neighborhood W of h_0 in H_1 such that $s|_W$ and $t|_W$ are both diffeomorphisms to $\tilde{W}$. Similarly, we choose an open neighborhood $V \subset G_1$ such that $s|_V$ is a diffeomorphism to $\rho(U) \subset G_0$ and $t|_V$ is a diffeomorphism to its image.

Since $\sigma|_U$ is a diffeomorphism to a branch containing $\{\sigma(z_n)\}$, there exists a unique $z'_n \in U$ such that $\sigma(z'_n) = \sigma(z_n)$ for each $n \geq 1$. Therefore

$$(z'_n, \mathrm{id}, z_n) \in \widetilde{H}_1$$

in the sense of (2.10). Since $(Z, \rho, \sigma) \colon \mathcal{G} \to \mathcal{H}$ is an embedding, the induced map $\phi \colon \tilde{\mathcal{G}} \to \tilde{\mathcal{H}}$ is saturated. Hence there exists an $g'_n \in G_1$ such that

$$(z'_n, g'_n, z_n) \in \widetilde{G}_1.$$

Let $\phi(z'_n, g'_n, z_n) = (z'_n, h'_n, z_n)$. Then for each $n \geq 1$ we found $g'_n \in G_1$ and $h'_n \in H_1$ such that

$$g'_n z_n h'_n = z'_n. \tag{2.33}$$

Since $g_n z_n h_n = z_n$, we get

$$g'_n g_n (g'_n)^{-1} z'_n (h'_n)^{-1} h_n h'_n = z'_n. \tag{2.34}$$

We notice that

$$s(h'_n) = t(h'_n) = \sigma(z'_n). \tag{2.35}$$

Since there are only finitely many isotropy elements for every point in H_0, (2.35) implies that there exists a subsequence of $\{h'_n\}$ that converges to some $h'_0 \in H_1$. Notice that both h'_0 and h_0 are in the isotropy group of $\sigma(z_0)$. On the other hand, we know that h_n converges to $h_0 \in H_1$, therefore $(h'_n)^{-1} h_n h'_n$ converges to $(h'_0)^{-1} h_0 h'_0$. Again, since $\phi \colon \tilde{\mathcal{G}} \to \tilde{\mathcal{H}}$ is an embedding, $g'_n g_n (g'_n)^{-1}$ also converges to some $g'_0 \in G_1$. Since $z'_n \in U$, there is also a subsequence of z'_n which converges to $z'_0 \in Z$.

Therefore, we can assume that $(g'_n g_n (g'_n)^{-1}, z'_n)$ converges to $(g'_0, z'_0) \in IZ$. Hence, $(h'_0)^{-1} h_0 h'_0 = I\sigma(g'_0, z'_0)$ is in the image of $I\sigma$. Therefore, h_0 is also in the image of $I\sigma$ as $h_0 = I\sigma(g'_0, z'_0 (h'_0)^{-1})$. □

Lemma 2.59. *For each point in the image of $I\sigma$ in $(I\mathcal{H})_0$, there exists a neighborhood such that it contains finitely many branches.*

Proof of Lemma 2.59. It follows from the fact that every point in H_0 has only finitely many isotropy elements. □

Conditions (3) and (4) of Definition 2.35 are clear, as they are local properties and (Z, ρ, σ) is a stabilizer-preserving embedding. This completes the proof of Proposition 2.57. □

Remark 2.60. *In general $(IZ, I\rho, I\sigma)$ is not an iso-spatial embedding even if (Z, ρ, σ) is.*

3. Bott-Chern cohomology of complex orbifolds

In this section, we introduce the Bott-Chern cohomology for complex orbifolds.

3.1. **Definition.** Let X be an n-dimensional complex orbifold with a groupoid representation $\mathcal{G} = (G_0, G_1)$. Let

$$\Omega^{p,q}_{\mathcal{G}}(G_0, \mathbb{C})$$

be the space of $\mathcal{G}$-equivariant (p, q)-forms on G_0. $\Omega^{\bullet,\bullet}_{\mathcal{G}}(G_0, \mathbb{C})$ is a bigraded algebra with the ordinary wedge product.

It is clear that the operators ∂ and $\overline{\partial}$ map $\Omega^{p,q}_{\mathcal{G}}(G_0, \mathbb{C})$ to $\Omega^{p+1,q}_{\mathcal{G}}(G_0, \mathbb{C})$ and $\Omega^{p,q+1}_{\mathcal{G}}(G_0, \mathbb{C})$, respectively.

Remark 3.1. *In the local case, $\mathcal{G} = G \ltimes X$ and $\Omega^{p,q}_{\mathcal{G}}(G_0, \mathbb{C})$ is nothing but $\Omega^{p,q}_{G}(X, \mathbb{C})$, the space of G-invariant (p, q)-forms on X.*

Definition 3.2. *The* Bott-Chern cohomology group $H^{p,q}_{BC}(\mathcal{G}, \mathbb{C})$ *is defined as*

$$H^{p,q}_{BC}(\mathcal{G}, \mathbb{C}) := \big(\Omega^{p,q}_{\mathcal{G}}(G_0, \mathbb{C}) \cap \ker d\big)/\overline{\partial}\partial\Omega^{p-1,q-1}_{\mathcal{G}}(G_0, \mathbb{C}), \tag{3.1}$$

where $d = \partial + \overline{\partial}$ is the de Rham differential.

$H^{\bullet,\bullet}_{\mathrm{BC}}(\mathcal{G}, \mathbb{C})$ inherits the structure of a bigraded algebra from $\Omega^{\bullet,\bullet}_{\mathcal{G}}(G_0, \mathbb{C})$. As in [BSW23, Section 2.1], we put

$$\Omega^{(=)}_{\mathcal{G}}(G_0, \mathbb{C}) = \bigoplus_{p=0}^{n} \Omega^{p,p}_{\mathcal{G}}(G_0, \mathbb{C}),\ \ H^{(=)}_{\mathrm{BC}}(\mathcal{G}, \mathbb{C}) = \bigoplus_{p=0}^{n} H^{p,p}_{\mathrm{BC}}(\mathcal{G}, \mathbb{C}). \tag{3.2}$$

We can similarly define the real vector space $\Omega^{(=)}_{\mathcal{G}}(G_0, \mathbb{R})$ and $H^{(=)}_{\mathrm{BC}}(\mathcal{G}, \mathbb{R})$, which are algebras as well.

Remark 3.3. *For a general complex groupoid we need to use the Bott-Schumann double complexes to define its Bott-Chern cohomology. However, in the special case of proper étale groupoid, Definition 3.2 is sufficient.*

3.2. **Current and Bott-Chen cohomology.** A review of the theory of currents in complex geometry can be found in [GH94, Section 3.1 and Section 3.2].

Let $\mathcal{D}^{p,q}_{\mathcal{G}}(G_0, \mathbb{C})$ denote the space of $\mathcal{G}$-equivariant (p, q)-currents on G_0. We can also define the operators ∂ and $\overline{\partial}$ which map $\mathcal{D}^{p,q}_{\mathcal{G}}(G_0, \mathbb{C})$ to $\mathcal{D}^{p+1,q}_{\mathcal{G}}(G_0, \mathbb{C})$ and $\mathcal{D}^{p,q+1}_{\mathcal{G}}(G_0, \mathbb{C})$, respectively. Again, let $\mathrm{d} = \partial + \overline{\partial}$.

By an argument similar to that in [Ang13], we can prove that

$$H^{p,q}_{\mathrm{BC}}(\mathcal{G}, \mathbb{C}) \cong \big(\mathcal{D}^{p,q}_{\mathcal{G}}(G_0, \mathbb{C}) \cap \ker \mathrm{d}\big)/\overline{\partial}\partial\mathcal{D}^{p-1,q-1}_{\mathcal{G}}(G_0, \mathbb{C}). \tag{3.3}$$

3.3. **Functorial properties of Bott-Chern cohomology.** Let $\mathcal{G}$ and $\mathcal{H}$ be complex orbifold groupoids of dimensions n and n' respectively. Let $f = (Z, \rho, \sigma) \colon \mathcal{G} \to \mathcal{H}$ be a generalized morphism. Then we have morphisms of bigraded algebras

$$\sigma^* \colon \Omega^{\bullet,\bullet}_{\mathcal{H}}(H_0, \mathbb{C}) \to \Omega^{\bullet,\bullet}_{\mathcal{G}\times\mathcal{H}}(Z, \mathbb{C}), \quad \rho^* \colon \Omega^{\bullet,\bullet}_{\mathcal{G}}(G_0, \mathbb{C}) \to \Omega^{\bullet,\bullet}_{\mathcal{G}\times\mathcal{H}}(Z, \mathbb{C}).$$

Moreover $\rho^* \colon \Omega^{\bullet,\bullet}_{\mathcal{G}}(G_0, \mathbb{C}) \to \Omega^{\bullet,\bullet}_{\mathcal{G}\times\mathcal{H}}(Z, \mathbb{C})$ is an isomorphism. Thus we can define $f^* \colon = (\rho^*)^{-1} \circ \sigma^*$, and it induces a morphism of bigraded algebras $H^{\bullet,\bullet}_{\mathrm{BC}}(\mathcal{H}, \mathbb{C})$ to $H^{\bullet,\bullet}_{\mathrm{BC}}(\mathcal{G}, \mathbb{C})$. Also, f^* preserves the corresponding real vector spaces.

By duality, we can define the pushforward map $f_* = \sigma_* \circ (\rho_*)^{-1}$ which maps $\mathcal{D}_{\mathcal{G}}^{p,q}(G_0, \mathbb{C})$ to $\mathcal{D}_{\mathcal{H}}^{p-n+n',q-n+n'}(H_0, \mathbb{C})$. Moreover, by Definition 2.54 we have a generalized morphism $If\colon I\mathcal{G} \to I\mathcal{H}$ between inertia groupoids, hence a morphism

$$If_*\colon H_{\mathrm{BC}}^{\bullet,\bullet}(I\mathcal{G}, \mathbb{C}) \to H_{\mathrm{BC}}^{\bullet-n+n',\bullet-n+n'}(I\mathcal{H}, \mathbb{C}). \tag{3.4}$$

Proposition 3.4. *The Bott-Chern cohomology is independent of the choice of the groupoid representation. In particular, if two complex orbifold groupoids $\mathcal{G}$ and $\mathcal{H}$ are Morita equivalent, then we have*

$$H_{BC}^{p,q}(\mathcal{G}, \mathbb{C}) \cong H_{BC}^{p,q}(\mathcal{H}, \mathbb{C}). \tag{3.5}$$

Proof. By Definition 2.6, we only need to consider the Bott-Chern cohomology under an equivalence $\psi\colon \mathcal{H} \to \mathcal{G}$ as in Definition 2.5. By the same argument as in the proof of [ALR07, Lemma 2.2], we obtain that

$$\Omega_{\mathcal{G}}^{p,q}(G_0, \mathbb{C}) \cong \Omega_{\mathcal{H}}^{p,q}(H_0, \mathbb{C}). \tag{3.6}$$

Hence $H_{\mathrm{BC}}^{p,q}(\mathcal{G}, \mathbb{C}) \cong H_{\mathrm{BC}}^{p,q}(\mathcal{H}, \mathbb{C})$. □

3.4. **Pushforward map If_* for iso-spatial embeddings.** In this subsection we consider the case that $f\colon \mathcal{G} \to \mathcal{H}$ is an iso-spatial embedding. We give an explicit description of the pushforward map

$$If_*\colon H_{\mathrm{BC}}^{\bullet,\bullet}(I\mathcal{G}, \mathbb{C}) \to H_{\mathrm{BC}}^{\bullet-n+n',\bullet-n+n'}(I\mathcal{H}, \mathbb{C}),$$

at the level of differential forms when we restrict to the local case.

As in Remark 2.41, the iso-spatial embedding is locally given by the inclusion of finite groups $G \hookrightarrow H$ and the identity map on the complex manifold X. Hence, by Remark 2.56 the induced morphism between inertia groupoids,

$$If\colon \coprod_{(g)\in C(G)} Z_G(g) \ltimes X^g \to \coprod_{(h)\in C(H)} Z_H(h) \ltimes X^h, \tag{3.7}$$

consists of the inclusion of finite groups $G \hookrightarrow H$ together with the identity map $X^g \to X^g$. For each $\omega \in \Omega_G^{p,q}(X^g, \mathbb{C})$, we define

$$If_*(\omega)\colon = \frac{1}{|Z_G(g)|} \sum_{h\in Z_H(g)} h^*\omega, \tag{3.8}$$

where $Z_G(g)$ and $Z_H(g)$ are the centralizer of g in G and H, respectively.

Proposition 3.5. *Let $f\colon G \ltimes X \to H \ltimes X$ be an iso-spatial embedding and If be the induced morphism between inertia groupoids as in (3.7). Then the pushforward map If_* in (3.8) gives the pushforward map If_* in (3.4).*

Proof. Let $\omega \in \Omega_{Z_G(g)}^{\bullet}(X^g, \mathbb{C})$, we can consider ω as a $Z_G(g)$-invariant current on X^g by

$$\omega(\theta) := \frac{1}{|Z_G(g)|} \int_{X^g} \omega \wedge \theta, \tag{3.9}$$

where θ is a testing form $\theta \in \Omega_{Z_G(g)}^{\bullet}(X^g, \mathbb{C})$.

It is clear that for any form $\theta \in \Omega^{\bullet}_{Z_H(g)}(X^g, \mathbb{C})$, we have $If^*(\theta) = \theta$ considered as a form in $\Omega^{\bullet}_{Z_G(g)}(X^g, \mathbb{C})$. Therefore we have

$$\omega(If^*(\theta)) = \frac{1}{|Z_G(g)|} \int_{X^g} \omega \wedge \theta. \tag{3.10}$$

On the other hand, by (3.8) we have

$$If_*(\omega)(\theta) = \frac{1}{|Z_H(g)|} \int_{X^g} If_*(\omega) \wedge \theta = \frac{1}{|Z_H(g)||Z_G(g)|} \sum_{h \in Z_H(g)} \int_{X^g} h^*\omega \wedge \theta. \tag{3.11}$$

Since θ is $Z_H(g)$-invariant, we have

$$h^*\omega \wedge \theta = h^*(\omega \wedge \theta), \text{ for any } h \in Z_H(g), \tag{3.12}$$

hence

$$If_*(\omega)(\theta) = \frac{1}{|Z_H(g)||Z_G(g)|} \sum_{h \in Z_H(g)} \int_{X^g} h^*(\omega \wedge \theta) = \frac{|Z_H(g)|}{|Z_H(g)||Z_G(g)|} \int_{X^g} \omega \wedge \theta, \tag{3.13}$$

which is exactly (3.10). Hence (3.8) indeed gives the pushforward map If_* in (3.4). □

4. Coherent sheaves on complex orbifolds

In this section, we discuss the notion of coherent sheaves on complex orbifolds from the viewpoint of groupoids. We follow the construction in [Moe01, Section 2.1].

Definition 4.1. *Let $\mathcal{G} = (G_0, G_1)$ be a complex orbifold groupoid. A right $\mathcal{G}$-sheaf is a sheaf $\mathcal{F}$ on G_0 together with a right action of $\mathcal{G}$. In more detail, each $g\colon x \to y$ in G_1 gives a map*

$$\mathcal{F}_x \to \mathcal{F}_y,\ a \mapsto a \cdot g,$$

where $\mathcal{F}_x$ is the stalk of $\mathcal{F}$ at x. It satisfies the conditions that $(a \cdot g_1) \cdot g_2 = a \cdot g_1 g_2$ and $a \cdot 1_x = a$. If $\mathcal{F}$ admits additional structures, say vector space, algebra, etc., then we require the $\mathcal{G}$-action to preserve such structures.

We define left $\mathcal{G}$-sheaves in the same way.

The following definitions can be found in [MP97, Section 3].

Definition 4.2. *Let $\mathcal{G} = (G_0, G_1)$ be a complex orbifold groupoid. We can define the structure sheaf $\mathcal{O}_{\mathcal{G}}$ on $\mathcal{G}$ as the sheaf of germs of holomorphic functions on G_0. This sheaf carries natural left and right $\mathcal{G}$-actions.*

Definition 4.3. *Let $\mathcal{G} = (G_0, G_1)$ be a complex orbifold groupoid. A sheaf of $\mathcal{O}_{\mathcal{G}}$ modules $\mathcal{F}$ is a sheaf on G_0 equipped with a left $\mathcal{G}$ action and an $\mathcal{O}_{\mathcal{G}}$ action which are compatible. Compatibility means that we have*

$$g \cdot (fs) = (g \cdot f)(g \cdot s), \tag{4.1}$$

where $g \in G_1$, $f \in \mathcal{O}_{\mathcal{G}}$, and $s \in \mathcal{F}$.

We denote the category of sheaves of $\mathcal{O}_{\mathcal{G}}$ modules by $\mathcal{O}_{\mathcal{G}}$-Mod.

The following result is essential in the constructions of derived functors.

Proposition 4.4. *The category $\mathcal{O}_{\mathcal{G}}$-Mod is an abelian category with enough injectives.*

Proof. The claim is local, so we can reduce to the case that $\mathcal{G}$ is the action groupoid of a finite group acting on a complex manifolds. The claim then follows from [Gro57, Proposition 5.1.1 and Proposition 5.1.2]. □

Let $f\colon \mathcal{G} \to \mathcal{H}$ be a holomorphic morphism between complex orbifold groupoids. We can define the pull back functor

$$f^*\colon \mathcal{O}_{\mathcal{H}}\text{-Mod} \to \mathcal{O}_{\mathcal{G}}\text{-Mod}$$

as in [Moe01, Section 3.1]. In more detail, for a sheaf $\mathcal{F} \in \mathcal{O}_{\mathcal{H}}$-Mod, the map $f_0\colon G_0 \to H_0$ gives a sheaf $f^{-1}\mathcal{F}$ on G_0, which has a natural left $\mathcal{G}$-action: for each $g\colon x \to y$ in G_1, the action of g on $a \in (f^{-1}\mathcal{F})_x$ is simply given by

$$g \cdot a := f(g) \cdot a.$$

Then we define $f^*\mathcal{F}$ as

$$f^*\mathcal{F} := \mathcal{O}_{\mathcal{G}} \otimes_{f^{-1}\mathcal{O}_{\mathcal{H}}} f^{-1}\mathcal{F}, \tag{4.2}$$

and the $\mathcal{G}$-action on $f^*\mathcal{F}$ by

$$g(s \otimes a) := (g \cdot s) \otimes (f(g) \cdot a).$$

It is clear that $f^*\mathcal{F}$ is a sheaf of $\mathcal{O}_{\mathcal{G}}$ modules. We define pullbacks of generalized morphisms in the same way as in Corollary 5.8.

Definition 4.5. *Let $\mathcal{G}$ be a complex orbifold groupoid. A sheaf $\mathcal{F}$ of $\mathcal{O}_{\mathcal{G}}$-modules is called a* coherent *sheaf if it satisfies the following conditions.*

(1) $\mathcal{F}$ is finite type, i.e. for every $x \in G_0$ there exists an open neighborhood (U, G) of x (Definition 2.32) such that there exists a $\mathcal{G}$-equivariant surjective map $\mathcal{O}_U^n \twoheadrightarrow \mathcal{F}|_U$;

(2) For every (U, G) and any $\mathcal{G}$-equivariant map $\phi\colon \mathcal{O}_U^n \to \mathcal{F}|_U$, the kernel of ϕ is also finite type.

We denote the category of coherent sheaves on $\mathcal{G}$ by coh$(\mathcal{G})$.

Proposition 4.6. *A sheaf $\mathcal{F}$ of $\mathcal{O}_{\mathcal{G}}$-modules is coherent if it satisfies Condition (2) in Definition 4.5 and the following weakened version of Condition (1):*

$(1')$. *For every $x \in G_0$, there exists an open neighborhood (U, G) of x such that there exists a (not necessarily $\mathcal{G}$-equivariant) surjective map $\mathcal{O}_U^n \twoheadrightarrow \mathcal{F}|_U$.*

Proof. Let $\phi\colon \mathcal{O}_U^n \twoheadrightarrow \mathcal{F}|_U$ be a surjective map, which is not necessarily $\mathcal{G}$-equivariant. Consider

$$\bigoplus_{g \in G} (\mathcal{O}_U^n)_g,$$

where $(\mathcal{O}_U^n)_g$ is a copy of $\mathcal{O}_U^n$ indexed by g. Since G is a finite group, $\oplus_{g\in G}(\mathcal{O}_U^n)_g$ is still a finite-rank free sheaf on U. Define the G-action on $\oplus_{g\in G}(\mathcal{O}_U^n)_g$ as follows: for a section $s \in \Gamma(\mathcal{O}_U^n)_g$ and $h \in G$, we have $h \cdot s = hs \in (\mathcal{O}_U^n)_{hg}$, where the hs on the right-hand side is the action of h on the original $\mathcal{O}_U^n$.

Now we define $\tilde{\phi}\colon \oplus_{g\in G} (\mathcal{O}_U^n)_g \to \mathcal{F}$. For $s \in \Gamma(\mathcal{O}_U^n)_g$, define

$$\tilde{\phi}(s) := g\phi(g^{-1}s)$$

where $g^{-1}s$ comes from the original G-action on $\mathcal{O}_U^n$. We can check that for any $h \in G$,

$$\tilde{\phi}(h \cdot s) = hg\phi((hg)^{-1}hs) = hg\phi(g^{-1}s) = h(\tilde{\phi}(s)),$$

hence $\tilde{\phi}$ is $\mathcal{G}$-equivariant. It is clear that $\tilde{\phi}$ is still surjective. So $\mathcal{F}$ satisfies Condition (1) in Definition 4.5. □

The following result is a version of Oka's coherence theorem for complex orbifolds.

Proposition 4.7. *Let $\mathcal{G}$ be a complex orbifold groupoid. Then the sheaf $\mathcal{O}_{\mathcal{G}}$ is coherent.*

Proof. We only need to prove that $\mathcal{O}_{\mathcal{G}}$ satisfies Condition (2) in Definition 4.5. For any $\mathcal{G}$-equivariant map $\phi\colon \mathcal{O}_U^n \to \mathcal{O}_U$. Forget the G-action, by the ordinary Oka's coherence theorem, $\ker\phi$ is finite type, i.e. there exists a finite set S in $\ker\phi$ that generate $\ker\phi$. Let GS be the G-orbit of S. Since G and S are both finite, GS is still a finite subset of $\ker\phi$. Take the free $\mathcal{O}_U$-module with a basis GS. Then it has an obvious G-action and an $\mathcal{G}$-equivariant surjection to $\ker\phi$. □

The following result is a version of the Syzygy theorem for complex orbifold groupoids.

Proposition 4.8. *Let $\mathcal{G}$ be a complex orbifold groupoid. Then for any coherent sheaf $\mathcal{F}$ and every $x \in G_0$ there exists an open neighborhood (U, G) of x such that there exists a $\mathcal{G}$-equivariant finite length resolution*

$$0 \to \mathcal{O}_U^{n_k} \to \ldots \to \mathcal{O}_U^{n_0} \to \mathcal{F}.$$

Proof. We use the Syzygy theorem [Kob14, Theorem 5.3.11] in the non-equivariant case and argue as in the proof of Proposition 4.7 repeatedly. □

Let $\mathcal{O}_{\mathcal{G}}^\infty$ denote the sheaf of C^∞-functions on G_0. For a sheaf of $\mathcal{O}_{\mathcal{G}}$-modules $\mathcal{F}$, let $\mathcal{F}^\infty$ denote the tensor product $\mathcal{F} \otimes_{\mathcal{O}_{\mathcal{G}}} \mathcal{O}_{\mathcal{G}}^\infty$. For a C^∞-vector bundle V on G_0, let $\mathcal{O}_{\mathcal{G}}^\infty V$ denote the associated sheaf of C^∞-sections of V.

The following lemma is important for what we need later.

Lemma 4.9. *For a coherent sheaf $\mathcal{F}$, there exists a sufficiently large integer n and a surjective morphism*

$$\phi\colon (\mathcal{O}_{\mathcal{G}}^\infty)^{\oplus n} \twoheadrightarrow \mathcal{F}^\infty. \tag{4.3}$$

Proof. By Proposition 4.8, for V sufficiently small we have a surjective morphism

$$\phi_V\colon \mathcal{O}_V^{n_V} \to \mathcal{F}|_V. \tag{4.4}$$

By Proposition 2.34, we then choose V_i so that they form a finite open cover of G_0 and use the partition of unity s_i on $\mathcal{G}$. For each V_i, $s_i\phi_{V_i}$ is a morphism from $(\mathcal{O}_{\mathcal{G}}^\infty)^{\oplus n_{V_i}}$ to $\mathcal{F}^\infty$. Then we define

$$(\mathcal{O}_{\mathcal{G}}^\infty)^{\oplus n} := \bigoplus_i (\mathcal{O}_{\mathcal{G}}^\infty)^{\oplus n_{V_i}}, \tag{4.5}$$

and

$$\phi := \sum_i s_i \phi_{V_i}. \tag{4.6}$$

It is clear that $\phi\colon (\mathcal{O}_{\mathcal{G}}^\infty)^{\oplus n} \to \mathcal{F}^\infty$ is surjective. □

The following result plays a key role in finding the relation between coherent sheaves and anti-holomorphic flat superconnections.

Proposition 4.10. *Let $\mathcal{G}$ be a compact complex orbifold groupoid. For a bounded complex of sheaves $\mathcal{F}^\bullet$ in $\mathcal{O}_\mathcal{G}$-Mod with coherent cohomologies, there exists a bounded complex of finite-dimensional C^∞-, left $\mathcal{G}$-vector bundles $V^\bullet$ on $\mathcal{G}$ and a quasi-isomorphism*

$$\mathcal{O}^\infty_\mathcal{G} V^\bullet \xrightarrow{\sim} \mathcal{F}^{\bullet,\infty}.$$

Proof. With Lemma 4.9, the proof of this result is similar to [BSW23, Proposition 6.3]. □

It is clear that for a morphism $f\colon \mathcal{G} \to \mathcal{H}$ between complex orbifold groupoids, the induced functor $f^*\colon \mathcal{O}_\mathcal{H}$-Mod $\to \mathcal{O}_\mathcal{G}$-Mod restricts to a functor

$$f^*\colon \mathrm{coh}(\mathcal{H}) \to \mathrm{coh}(\mathcal{G}).$$

Proposition 4.11. *If $f\colon \mathcal{G} \to \mathcal{H}$ is an equivalence between complex orbifold groupoids in the sense of Definition 2.5, then the induced functor $f^*\colon coh(\mathcal{H}) \to coh(\mathcal{G})$ is an equivalence of categories.*

Proof. This follows from the works of Moerdijk and Pronk [MP97, Defintion 3.1, Remark (3)] by replacing the sheave of smooth functions with the sheave of holomorphic functions. □

We take the following definition from [MM05, Section 3.2].

Let $f = (Z, \rho, \sigma)\colon \mathcal{G} \to \mathcal{H}$ be a generalized morphism between complex orbifold groupoids. The pullback functor $f^*\colon \mathcal{O}_\mathcal{H}$-Mod $\to \mathcal{O}_\mathcal{G}$-Mod has a right adjoint

$$f_*\colon \mathcal{O}_\mathcal{G}\text{-Mod} \to \mathcal{O}_\mathcal{H}\text{-Mod}.$$

More explicitly, given an $\mathcal{O}_\mathcal{G}$-module E, we pull it back to Z to get an $\mathcal{O}_\mathcal{Z}$-module $\rho^*(E)$. Then we push $\rho^*(E)$ to $\mathcal{H}$ by taking the $\mathcal{G}$-invariant sections of the usual pushforward of sheaves between complex manifolds under σ. The pushforward inherits a natural $\mathcal{H}$-action as $\rho^*(E)$ is an $\mathcal{G} \times \mathcal{H}$ module. The $\mathcal{G}$-action on Z commutes with the $\mathcal{H}$-action and $\mathcal{O}_\mathcal{Z}$-module structure. The pushforward $\sigma_*\rho^*(E)$ is an $\mathcal{O}_\mathcal{H}$-module. We define

$$f_*(E) := \sigma_*\rho^*(E). \tag{4.7}$$

Proposition 4.12. *The push forward functor $f_*\colon \mathcal{O}_\mathcal{G}$-Mod $\to \mathcal{O}_\mathcal{H}$-Mod defined in (4.7) is right adjoint to the pullback functor $f^*\colon \mathcal{O}_\mathcal{H}$-Mod $\to \mathcal{O}_\mathcal{G}$-Mod.*

Proof. It is clear that σ_* is right adjoint to σ^*. Since ρ^* is an equivalence, we get the desired result. □

Corollary 4.13. *The pushforward functor defined in (4.7) is functorial. More precisely, for $f\colon \mathcal{G} \to \mathcal{H}$ and $p\colon \mathcal{H} \to \mathcal{K}$, we have*

$$(pf)_* = p_* \circ f_*\colon \mathcal{O}_\mathcal{G}\text{-Mod} \to \mathcal{O}_\mathcal{K}\text{-Mod}. \tag{4.8}$$

Proof. It is immediate from Proposition 4.12. □

Proposition 4.14. *For a proper generalized morphism $f\colon \mathcal{G} \to \mathcal{H}$, the induced functor $f_*\colon \mathcal{O}_\mathcal{G}$-Mod $\to \mathcal{O}_\mathcal{H}$-Mod restricts to a functor $f_*\colon coh(\mathcal{G}) \to coh(\mathcal{H})$.*

Proof. It suffices to prove that $f_*(\mathcal{O}_\mathcal{G})$ is coherent. By (4.7), it reduces to proving that $\sigma_*(\mathcal{O}_\mathcal{Z})$ is coherent, which is clear by definition. □

Definition 4.15. *Let*

$$D^b_{coh}(\mathcal{G})$$

be the derived category of bounded complexes of objects in $\mathcal{O}_\mathcal{G}$-Mod with coherent cohomologies.

Definition 4.16. *For $\mathcal{F}_1$ and $\mathcal{F}_2 \in D^b_{coh}(\mathcal{G})$, we define their derived tensor product*

$$\mathcal{F}_1 \otimes^L \mathcal{F}_2$$

via flat resolutions as in [Sta22, Tag 06YH].

Definition 4.17. *For a generalized morphism $f\colon \mathcal{G} \to \mathcal{H}$, we define the* derived pull back

$$Lf^*\colon D^b_{coh}(\mathcal{H}) \to D^b_{coh}(\mathcal{G})$$

as in [Sta22, Tag 07BD].

Definition 4.18. *For a proper generalized morphism $f\colon \mathcal{G} \to \mathcal{H}$, we define the* derived push-forward

$$Rf_*\colon D^b_{coh}(\mathcal{G}) \to D^b_{coh}(\mathcal{H})$$

by injective resolutions as in [GM03, Section III.6]. *By Proposition 4.4, this definition makes sense.*

By [GM03, Theorem III.7.1], we have a natural isomorphism

$$R(f \circ g)_* \xrightarrow{\sim} Rf_* \circ Rg_*. \tag{4.9}$$

According to [GM03, Proposition III.8.12], we can compute Rf_* via soft resolutions instead of injective resolutions. Notice that [GM03, Proposition III.8.12] is about $Rf_!$, the right derived functor of the proper pushforward functor $f_!$. Nevertheless, in our case f_* and $f_!$ coincide, as both $\mathcal{G}$ and $\mathcal{H}$ are required to be compact.

Let $K(D^b_{\mathrm{coh}}(\mathcal{G}))$ denote the Grothendieck group of $D^b_{\mathrm{coh}}(\mathcal{G})$. Using the same argument as in [Sta22, Tag 0FCP], we get

$$K(D^b_{\mathrm{coh}}(\mathcal{G})) \cong K(\mathcal{G}).$$

For a holomorphic map $f\colon \mathcal{G} \to \mathcal{H}$, the derived pushforward map Rf_* induces a homomorphism

$$f_!\colon K(\mathcal{G}) \to K(\mathcal{H}). \tag{4.10}$$

5. Antiholomorphic flat superconnection on complex orbifolds

In this section, we introduce the theory of antiholomorphic flat superconnections on complex orbifolds.

5.1. **Basic constructions.** Our presentation of this section follows that of [BD10].

Lemma 5.1. *On a complex orbifold groupoid $\mathcal{G}$, the bundle of $(0,1)$-cotangent vectors $\overline{T^*G_0}$ is a $\mathcal{G}$-$\mathcal{G}$ bibundle.*

Proof. This follows from the fact that the groupoid $\mathcal{G}$-action on G_0 preserves the complex structure. □

Definition 5.2. *Let X be a complex orbifold with a groupoid representation $\mathcal{G} = (G_0, G_1)$. An* antiholomorphic flat superconnection *on $\mathcal{G}$ is a bounded, finite rank, $\mathbb{Z}$-graded, left $\mathcal{G}$-equivariant, C^∞-vector bundle $E^\bullet$ on G_0 together with a $\mathcal{G}$-equivariant superconnection of total degree 1,*

$$A^{E^\bullet\prime\prime}\colon \wedge^\bullet \overline{T^*G_0} \times_{\mathcal{G}} E^\bullet \to \wedge^\bullet \overline{T^*G_0} \times_{\mathcal{G}} E^\bullet,$$

such that $A^{E^\bullet\prime\prime} \circ A^{E^\bullet\prime\prime} = 0$.

In more detail, $A^{E^\bullet\prime\prime}$ decomposes into

$$A^{E^\bullet\prime\prime} = v_0 + \nabla^{E^\bullet\prime\prime} + v_2 + \dots, \tag{5.1}$$

where

$$\nabla^{E^\bullet\prime\prime}\colon E^\bullet \to \overline{T^*G_0} \times_{\mathcal{G}} E^\bullet$$

is a $\mathcal{G}$-equivariant $\overline{\partial}$-connection and

$$v_i\colon E^\bullet \to \wedge^i \overline{T^*G_0} \times_{\mathcal{G}} E^{\bullet+1-i} \tag{5.2}$$

is $\mathcal{G}$-equivariant and $C^\infty(G_0)$-linear.

Antiholomorphic flat superconnections on $\mathcal{G}$ form a dg-category denoted by

$$B(\mathcal{G}).$$

In more detail, let $(E^\bullet, A^{E^\bullet\prime\prime})$ and $(F^\bullet, A^{F^\bullet\prime\prime})$ be two flat antiholomorphic superconnections on X where

$$A^{E^\bullet\prime\prime} = v_0 + \nabla^{E^\bullet\prime\prime} + v_2 + \dots$$

and

$$A^{F^\bullet\prime\prime} = u_0 + \nabla^{F^\bullet\prime\prime} + u_2 + \dots.$$

A morphism $\phi\colon (E^\bullet, A^{E^\bullet\prime\prime}) \to (F^\bullet, A^{F^\bullet\prime\prime})$ of degree k is given by

$$\phi = \phi_0 + \phi_1 + \dots$$

where

$$\phi_i\colon E^\bullet \to \wedge^i \overline{T^*G_0} \times_{\mathcal{G}} F^{\bullet+k-i}$$

is $\mathcal{G}$-equivariant and $C^\infty(G_0)$-linear.

The differential of ϕ is given by

$$d\phi = A^{F^\bullet\prime\prime}\phi - (-1)^k \phi A^{E^\bullet\prime\prime}.$$

More explicitly, the l-th component $(d\phi)_l\colon E^\bullet \to \wedge^l \overline{T^*G_0} \times_{\mathcal{G}} F^{\bullet+k+1-l}$ is given by

$$(d\phi)_l = \sum_{i\neq 1} \left(u_i \phi_{l-i} - (-1)^k \phi_{l-i} v_i\right) + \nabla^{F^\bullet\prime\prime}\phi_{l-1} - (-1)^k \phi_{l-1} \nabla^{E^\bullet\prime\prime}. \tag{5.3}$$

5.2. **Pullback and equivalence.** We want to study the functoriality property of $B(\mathcal{G})$. First, we define the pullbacks under holomorphic morphisms.

Proposition 5.3. *Let $\psi\colon \mathcal{G} \to \mathcal{H}$ be a holomorphic morphism between complex orbifold groupoids. Then ψ induces a pullback dg-functor*

$$\psi_b^*\colon B(\mathcal{H}) \to B(\mathcal{G}). \tag{5.4}$$

Proof. We simply have pullback bundles, pullback connections, and pullback bundle maps. □

Remark 5.4. *Here we use the notation ψ_b^* to emphasize that we pull back objects of the dg-category $B(\mathcal{H})$.*

The following result tells us that the dg-category is independent of the representative groupoid.

Proposition 5.5. *If $\mathcal{G}$ and $\mathcal{H}$ are Morita equivalent, then the dg-categories of flat antiholomorphic superconnections on $\mathcal{G}$ and $\mathcal{H}$ are quasi-equivalent.*

Proof of Proposition 5.5. The proof is essentially the same as that of [BD10, Theorem 7.2, Corollary 7.3]. We give a proof here to show the geometric picture.

By Proposition 2.15, there exists a $\mathcal{G}$-$\mathcal{H}$ principal bibundle P. We will construct a dg-functor $P_*\colon B(\mathcal{H}) \to B(\mathcal{G})$ using P. First, for a left $\mathcal{H}$-vector bundle E on $\mathcal{H}$, we construct the fiber product $P \times_{\mathcal{H}} E$ as in Definition 2.12. From the definition of the principal bibundle P, it is easy to show that $P \times_{\mathcal{H}} E$ is a vector bundle over $\mathcal{G}$.

We can also consider P as a left $\mathcal{H}$-space with $h \cdot p := ph^{-1}$. Hence we can define the fiber product $P \times_{\mathcal{H}} P$. Similarly, we can consider P as a right $\mathcal{G}$-space and define $P \times_{\mathcal{G}} P$.

Lemma 5.6. *There are canonical isomorphisms $\mathcal{H} \simeq P \times_{\mathcal{G}} P$ and $\mathcal{G} \cong P \times_{\mathcal{H}} P$, which are compatible with the actions.*

Proof of Lemma 5.6. See [dH13, Proposition 4.6.2]. □

Lemma 5.7. *For complex orbifold groupoids $\mathcal{G}$ and $\mathcal{H}$ and a $\mathcal{G}$-$\mathcal{H}$ principal bibundle P, we have a canonical isomorphism of $\mathcal{G}$-$\mathcal{H}$ bimodules,*

$$\wedge^i \overline{T^*G_0} \times_{\mathcal{G}} P \cong P \times_{\mathcal{H}} \wedge^i \overline{T^*H_0}. \tag{5.5}$$

Proof of Lemma 5.7. Since $\mathcal{G}$ and $\mathcal{H}$ are étale, both sides are canonically isomorphic to $\wedge^i \overline{T^*P}$. □

Now for $(E^\bullet, A^{E^\bullet\prime\prime}) \in B(\mathcal{H})$, we define $P_*(E^\bullet, A^{E^\bullet\prime\prime})$ as follows: the graded bundles are given by $P \times_{\mathcal{H}} E^\bullet$. As for the superconnection, we know that

$$v_i\colon P \times_{\mathcal{H}} E^\bullet \to P \times_{\mathcal{H}} \wedge^i \overline{T^*H_0} \times_{\mathcal{H}} E^{\bullet+1-i}.$$

Using the isomorphism (5.5), we can consider v_i as

$$P_*(v_i)\colon P \times_{\mathcal{H}} E^\bullet \to \wedge^i \overline{T^*G_0} \times_{\mathcal{G}} P \times_{\mathcal{H}} E^{\bullet+1-i}. \tag{5.6}$$

The $P_*(\nabla^{E^\bullet\prime\prime})$ is defined in a similar way.

Lemma 5.6 ensures that the map $E \mapsto P \times_{\mathcal{H}} E$ gives an equivalence of categories $\text{Vect}(\mathcal{H}) \to \text{Vect}(\mathcal{G})$ with inverse $F \mapsto P \times_{\mathcal{G}} F$, where we consider P as a right $\mathcal{G}$-space in the second functor. Hence we define P_*^{-1} on bundles as

$$F \mapsto P \times_{\mathcal{G}} F$$

and define P_*^{-1} on superconnections in the same way as in (5.6). By Lemma 5.6 and Lemma 5.7, it is clear that P_*^{-1} gives an inverse of P_*. We finish the proof of Proposition 5.3. $\square$

Corollary 5.8. *Let $f = (Z, \rho, \sigma)\colon \mathcal{G} \to \mathcal{H}$ be a generalized morphism between complex orbifold groupoids. Then f induces a dg-functor*

$$f_b^*\colon B(\mathcal{H}) \to B(\mathcal{G}). \tag{5.7}$$

Proof. As shown in Remark 2.17, from (Z, ρ, σ) we can define a groupoid $\mathcal{Z}$ together with morphisms $\phi_\rho\colon \mathcal{Z} \to \mathcal{G}$ and $\phi_\sigma\colon \mathcal{Z} \to \mathcal{H}$ where ϕ_ρ is a Morita equivalence. Hence by Proposition 5.5, $(\phi_\rho)_b^*$ is a dg-equivalence. We define f_b^* as $f_b^* := ((\phi_\rho)_b^*)^{-1} \circ (\phi_\sigma)_b^*$. $\square$

We denote by

$$B(X)$$

the dg-category of flat antiholomorphic superconnections on the complex orbifold X.

We also need the following local version of antiholomorphic flat superconnections.

Lemma 5.9. *When we restrict $\mathcal{G}$ to a small open set U, $\mathcal{G}|_U$ becomes the transformation groupoid associated with a finite group G acting on a complex manifold U. Then an antiholomorphic flat superconnection restricts to a G-equivariant antiholomorphic flat superconnection on U.*

Proof. Observe that $U \rtimes G$ is an open subgroupoid of $\mathcal{G}$. The Lemma follows from the property that the restriction of an equivariant antiholomorphic flat superconnection to an open subgroupoid is again an equivariant antiholomorphic flat superconnection. $\square$

We define mapping cones and shifts in $B(X)$. For a degree zero closed map $\phi\colon (E^\bullet, A^{E^\bullet\prime\prime}) \to (F^\bullet, A^{F^\bullet\prime\prime})$, its mapping cone $(C^\bullet, A_\phi^{C^\bullet\prime\prime})$ is defined by

$$C^n = E^{n+1} \oplus F^n, \tag{5.8}$$

and

$$A^{C^\bullet\prime\prime} = \begin{bmatrix} A^{E^\bullet\prime\prime} & 0 \\ \phi(-1)^{\deg(\cdot)} & A^{F^\bullet\prime\prime} \end{bmatrix}. \tag{5.9}$$

The shift $(E^\bullet, A^{E^\bullet\prime\prime})[1]$ is defined by

$$E[1]^n = E^{n+1}, \tag{5.10}$$

and

$$A^{E^\bullet\prime\prime}[1] = A^{E^\bullet\prime\prime}(-1)^{\deg(\cdot)}.$$

It is clear that they give $B(X)$ a pre-triangulated structure, hence its homotopy category

$$\underline{B}(X)$$

is a triangulated category.

5.3. **Tensor product.** Let $\mathcal{G}$ be a complex orbifold groupoid and $\mathcal{E} = (E^\bullet, A^{E^\bullet\prime\prime})$ and $\mathcal{F} = (F^\bullet, A^{F^\bullet\prime\prime})$ be two objects in $B(\mathcal{G})$.

Definition 5.10. *We define the tensor product of $\mathcal{E}$ and $\mathcal{F}$, denoted by $\mathcal{E}\widehat{\otimes}_b\mathcal{F}$, to be*

$$(E^\bullet\widehat{\otimes}F^\bullet, A^{E^\bullet\prime\prime}\widehat{\otimes}_b A^{F^\bullet\prime\prime}),$$

where $E^\bullet\widehat{\otimes}F^\bullet$ is the graded tensor product of graded vector bundles over G_0, and $A^{E^\bullet\prime\prime}\widehat{\otimes}_b A^{F^\bullet\prime\prime}$ is the graded tensor product of superconnections.

In particular, the $\mathcal{G}$-action on $E^\bullet\widehat{\otimes}F^\bullet$ and $A^{E^\bullet\prime\prime}\widehat{\otimes}_b A^{F^\bullet\prime\prime}$ is the diagonal action.

6. An equivalence of categories

The goal of this section is to explain the equivalence between coherent sheaves and antiholomorphic superconnections on a complex orbifold. This is an extension of the case for complex manifolds treated in [BSW23].

6.1. **The functor.** For a flat antiholomorphic superconnection $(E^\bullet, A^{E^\bullet\prime\prime})$, let $\mathcal{E}^n$ be the sheafification of

$$\bigoplus_{p+q=n} \wedge^p\overline{T^*G_0} \times_G E^q.$$

Note that $\mathcal{E}^n$ is a $\mathcal{G}$-sheaf of C^∞-modules hence a sheaf of $\mathcal{O}_\mathcal{G}$-modules.

It is clear that $A^{E^\bullet\prime\prime}$ gives a map of $\mathcal{O}_\mathcal{G}$-modules $\mathcal{E}^n \to \mathcal{E}^{n+1}$.

Proposition 6.1. *The cochain complex of sheaves of $\mathcal{O}_\mathcal{G}$-modules $(\mathcal{E}^\bullet, A^{E^\bullet\prime\prime})$ has coherent cohomologies.*

Proof. By [BSW23, Theorem 5.3], for each x, there exists a local chart (U, H) such that $(\mathcal{E}^\bullet|_U, A^{E^\bullet\prime\prime})$ is (not necessarily H-equivariantly) quasi-isomorphic to a bounded complex of locally free sheaves. By Proposition 4.6, we can make this quasi-isomorphism H-equivariant. □

Definition 6.2. *The assignment $(E^\bullet, A^{E^\bullet\prime\prime}) \mapsto (\mathcal{E}^\bullet, A^{E^\bullet\prime\prime})$ defines a functor*

$$F_X\colon \underline{B}(X) \to D^b_{coh}(X) \tag{6.1}$$

where $\underline{B}(X)$ is the homotopy category of the dg-category of flat antiholomorphic superconnections on X.

6.2. **Equivalence.** For an $\mathcal{F} \in D^b_{\text{coh}}(X)$ considered as a complex of $\mathcal{G}$-sheaves on a groupoid representative $\mathcal{G} = (G_0, G_1)$. Let $\mathcal{F}^\infty$ be the tensor product $\mathcal{O}^\infty_{G_0} \otimes_{\mathcal{O}_\mathcal{G}} \mathcal{F}$ and put

$$\overline{\mathcal{F}}^\infty = \mathcal{O}^\infty_{G_0}(\wedge\overline{T^*G_0}) \otimes_{\mathcal{O}_\mathcal{G}} \mathcal{F}. \tag{6.2}$$

We equip $\overline{\mathcal{F}}^\infty$ with the Dolbeault differential $\overline{\partial}$ and it is clear that the canonical morphism $\mathcal{F} \to \overline{\mathcal{F}}^\infty$ is a quasi-isomorphism of complexes of $\mathcal{O}_\mathcal{G}$-modules.

Theorem 6.3. *The functor F_X is essentially surjective when X is compact. More precisely, for any object $\mathcal{F} \in D^b_{coh}(X)$ there exists an object $\mathcal{E} = (E^\bullet, A^{E^\bullet\prime\prime}) \in B(X)$ together with a $\mathcal{G}$-morphism of $\mathcal{O}^\infty_{G_0}(\wedge\overline{T^*G_0})$-modules $\phi\colon F_X(\mathcal{E}) \to \overline{\mathcal{F}}^\infty$ which is a quasi-isomorphism of complexes of $\mathcal{O}_{G_0}$-modules. Hence $F_X(\mathcal{E})$ and $\mathcal{F}$ are isomorphic in $D^b_{coh}(X)$.*

Proof of Theorem 6.3. By Proposition 4.10, there exists a bounded complex of finite-dimensional complex C^∞-vector bundles $E^\bullet$ on X and a quasi-isomorphism

$$\mathcal{O}_X^\infty E^\bullet \xrightarrow{\sim} \mathcal{F}^\infty. \tag{6.3}$$

Notice that $\mathcal{F}^\infty$ has a flat antiholomorphic superconnection

$$A^{\mathcal{F}} = d^{\mathcal{F}} + \overline{\partial}.$$

We want to use the quasi-isomorphism (6.3) to lift the superconnection $A^{\mathcal{F}}$ to $E^\bullet$. For this purpose, we need the following lemma.

Lemma 6.4. *Let X be a complex orbifold and $E^\bullet$ be a bounded complex of finite dimensional C^∞-vector bundles on X. Then for any acyclic complex of sheaves of $\mathcal{O}_X^\infty$-modules $N^\bullet$, the complex of morphisms Hom$(E^\bullet, N^\bullet)$ is still acyclic.*

Proof of Lemma 6.4. The claim is standard and we prove by induction on the amplitude of $E^\bullet$, which is due to [Sta22, Tag 013R]. First, if the amplitude is zero, then $E^\bullet$ is a single C^∞-vector bundle E on X. Using a similar argument as in [Seg68, Proposition 2.4], we can find a finite-dimensional trivial vector bundle T on X such that E is a direct summand of T. Then the claim is obvious.

Suppose that the claim holds if the amplitude of $E^\bullet$ is no more than l. Now consider $E^\bullet$ with amplitude $l+1$. Let $\phi\colon E^\bullet \to N^\bullet$ be a closed morphism of degree k with components $\phi^s\colon E^s \to N^{s+k}$. Closedness means

$$d_{N^\bullet} \circ \phi^s - (-1)^k \phi^{s+1} \circ d_{E^\bullet} = 0.$$

Let E^m be the highest non-zero component of $E^\bullet$. Then we have

$$d_{N^\bullet} \circ \phi^m = 0.$$

Since $N^\bullet$ is acyclic, there exists a map $\psi^m\colon E^m \to N^{m+k-1}$ such that

$$d_{N^\bullet} \circ \psi^m = \phi^m.$$

Now we consider the "naive" truncation $\sigma_{\leq m-1} E^\bullet$, which is of amplitude l. Define

$$\tilde{\phi}\colon \sigma_{\leq m-1} E^\bullet \to N^\bullet$$

to be

$$\tilde{\phi}^s = \begin{cases} \phi^s, & \text{if } s \leq m-1, \\ \phi^m - (-1)^k \psi^m \circ d_{E^\bullet}, & \text{if } s = m. \end{cases}$$

Then it is easy to see that $\tilde{\phi}$ is closed, hence there exists a map $\tilde{\theta}\colon \sigma_{\leq m-1} E^\bullet \to N^\bullet$ of degree $k-1$ such that

$$d_{N^\bullet} \circ \tilde{\theta}^s - (-1)^{k-1} \tilde{\theta}^{s+1} \circ d_{E^\bullet} = \tilde{\phi}.$$

In particular, we have

$$d_{N^\bullet} \circ \tilde{\theta}^{m-1} = \tilde{\phi}^{m-1} = \phi^m - (-1)^k \psi^m \circ d_{E^\bullet}.$$

We now define $\theta\colon E^\bullet \to N^\bullet$ as

$$\theta^s = \begin{cases} \tilde{\theta}^s, & \text{if } s \leq m-1, \\ \psi^m, & \text{if } s = m. \end{cases}$$

It is clear that
$$d_{N^\bullet} \circ \theta^s - (-1)^{k-1}\theta^{s+1} \circ d_{E^\bullet} = \phi.$$
□

The rest of the proof of Theorem 6.3 is exactly the same as the proof of [BSW23, Theorem 6.3.6], by Lemma 6.4. □

The following proposition can be proved in the same way as the proofs of [BSW23, Theorem 5.7, Theorem 5.10, and Proposition 6.8].

Proposition 6.5. *Let X be a complex orbifold, $(E^\bullet, A^{E^\bullet\prime\prime})$ and $(F^\bullet, A^{F^\bullet\prime\prime})$ be two flat anti-holomorphic superconnection on X. More explicitly we have*
$$A^{E^\bullet\prime\prime} = v_0 + \nabla^{E^\bullet\prime\prime} + v_2 + \dots$$
and
$$A^{F^\bullet\prime\prime} = u_0 + \nabla^{F^\bullet\prime\prime} + u_2 + \dots .$$

Let $\phi = \phi_0 + \phi_1 + \dots$ be a closed degree 0 morphism from $(E^\bullet, A^{E^\bullet\prime\prime})$ to $(F^\bullet, A^{F^\bullet\prime\prime})$. Then the following are equivalent.

(1) ϕ_0 gives a homotopy equivalence between complexes of C^∞-vector bundles $(E^\bullet, v_0)$ and $(F^\bullet, u_0)$;
(2) ϕ is a homotopy equivalence, i.e. ϕ has an inverse in the homotopy category $\underline{B}(X)$;
(3) ϕ induces a quasi-isomorphism between $F_X(E^\bullet, A^{E^\bullet\prime\prime})$ and $F_X(F^\bullet, A^{F^\bullet\prime\prime})$, where F_X is the functor introduced in Definition 6.2.

Definition 6.6. *We call a closed degree 0 morphism ϕ from $(E^\bullet, A^{E^\bullet\prime\prime})$ to $(F^\bullet, A^{F^\bullet\prime\prime})$ a quasi-isomorphism if it satisfies the equivalent conditions in Proposition 6.5.*

Theorem 6.7. *The functor $F_X \colon \underline{B}(X) \to D^b_{coh}(X)$ is fully faithful.*

Proof. The proof is the same as the proof of [BSW23, Theorem 6.5.1]. □

We arrive at the main result of this section.

Corollary 6.8. *The functor $F_X \colon \underline{B}(X) \to D^b_{coh}(X)$ is an equivalence of triangulated categories.*

Proof. It is clear that F_X preserves the triangulated structure. Then we conclude by Theorem 6.3 and Theorem 6.7. □

Next, we discuss some compatibility results of the equivalence F_X.

Proposition 6.9 (Compatibility with pull-backs)**.** *Let $f\colon X \to Y$ be a morphism. Let $f_b^*\colon B(Y) \to B(X)$ be as in Proposition 5.3 and $Lf^*\colon D^b_{coh}(Y) \to D^b_{coh}(X)$ be as in Definition 4.17. Then f_b^* and Lf^* are compatible with F_X.*

Proof. It is a direct check with definitions. And we leave the details to the reader. □

Proposition 6.10 (Compatibility with tensor products)**.** *The tensor product in $B(X)$ as in Definition 5.10 and the derived tensor product in Definition 4.16 are compatible under F_X.*

Proof. For an antiholomorphic flat superconnection $(E^\bullet, A^{E^\bullet\prime\prime})$, its sheafification $F_X(E^\bullet, A^{E^\bullet\prime\prime})$ is a complex of flat sheaves, where F_X is given in Definition 6.2. The claim is then clear. □

7. Generalized Metrics and Curvature

In this section, let X be an n-dimensional complex orbifold with a proper complex étale effective groupoid representation $\mathcal{G} = (G_0, G_1)$. We introduce the concepts of generalized metric and curvature on a complex orbifold.

7.1. **Generalized Metrics.** For $\alpha \in \wedge^p T^*_{\mathbb{C}} G_0$, let

$$\tilde{\alpha} = (-1)^{\frac{p(p+1)}{2}} \alpha, \quad \alpha^* = \bar{\tilde{\alpha}}. \tag{7.1}$$

Let $E^\bullet$ be a graded complex vector bundle over G_0. For $A \in \mathrm{Hom}(E^\bullet, (\bar{E}^\bullet)^*)$, let $A^* \in \mathrm{Hom}(E^\bullet, (\bar{E}^\bullet)^*)$ be its adjoint. For

$$h = \alpha \widehat{\otimes} A \in \wedge^\bullet T^*_{\mathbb{C}} G_0 \,\widehat{\otimes}\, \mathrm{Hom}(E^\bullet, (\bar{E}^\bullet)^*),$$

we define its adjoint

$$h^* : = \alpha^* \widehat{\otimes} A^*. \tag{7.2}$$

If $h \in \wedge^\bullet T^*_{\mathbb{C}} G_0 \,\widehat{\otimes}\, \mathrm{Hom}(E^\bullet, (\bar{E}^\bullet)^*)$, we can write h as

$$h = \sum_{i=0}^{2n} h_i,$$

where

$$h_i \in \wedge^i T^*_{\mathbb{C}} G_0 \,\widehat{\otimes}\, \mathrm{Hom}(E^\bullet, (\bar{E}^\bullet)^*). \tag{7.3}$$

We will often count the degrees in $\wedge^p T^* G_0$ and $\wedge^q \overline{T^* G_0}$ as $-p$ and q respectively, and we introduce the corresponding degree $\deg\_$. Therefore, for $h \in \wedge^{p,q} T^* G_0 \,\widehat{\otimes}\, \mathrm{Hom}^r(E^\bullet, (\bar{E}^\bullet)^*)$, we define

$$\deg\_\, h = q - p + r. \tag{7.4}$$

Now we are ready to define generalized metrics.

Definition 7.1. *Let X and $\mathcal{G} = (G_0, G_1)$ be as above. Let $E^\bullet$ be a $\mathcal{G}$-equivariant graded complex vector bundle on G_0. An element $h \in \wedge^i T^*_{\mathbb{C}} G_0 \,\widehat{\otimes}\, \mathrm{Hom}(E^\bullet, (\bar{E}^\bullet)^*)$ is called a generalized metric on $E^\bullet$ if it satisfies the following conditions:*

1. *h is $\mathcal{G}$-equivariant;*
2. *h is of degree 0, i.e $\deg\_\, h = 0$;*
3. *h is self-adjoint, i.e. $h^* = h$;*
4. *the 0-th component h_0 defines a graded Hermitian metric on $E^\bullet$.*

Let $\mathscr{M}^E$ denote the set of generalized metrics on $E^\bullet$.

A generalized metric is said to be pure if $h = h_0$.

7.2. **Superconnections and generalized metrics.** Recall that a cut-off function $\chi\colon G_0 \to \mathbb{R}_{\geq 0}$ is a smooth function on G_0 such that $\forall x \in G_0$,

$$\int_{t(g)=x} \chi(s(g))dg = 1.$$

According to [Tu99], a cut-off function exists on all proper étale groupoid $\mathcal{G}$. Furthermore, when G_0/G_1 is compact, we can choose a cut-off function χ that has a compact support.

Put $\Omega_{\mathcal{G}}(G_0, \mathbb{C}) = C^\infty(G_0, \wedge^\bullet T^*_{\mathbb{C}} G_0)^{\mathcal{G}}$. If α, $\alpha' \in \Omega_{\mathcal{G}}(G_0, \mathbb{C})$, then we put

$$\theta(\alpha, \alpha') = \Big(\frac{\sqrt{-1}}{2\pi}\Big)^n \int_{G_0} \chi\tilde{\alpha} \wedge \overline{\alpha'}, \tag{7.5}$$

where χ is the cut-off function.

Let $E^\bullet$ be a $\mathcal{G}$-equivariant graded complex vector bundle on G_0. Put

$$\Omega_{\mathcal{G}}(G_0, E^\bullet) = C^\infty(G_0, \wedge^\bullet T^*_{\mathbb{C}} G_0 \,\widehat{\otimes}\, E^\bullet)^{\mathcal{G}}.$$

We again equip $\Omega_{\mathcal{G}}(G_0, E^\bullet)$ with the total degree associated with $\deg_-$ on $\Omega_{\mathcal{G}}(G_0, \mathbb{C})$ and the degree on $E^\bullet$.

Definition 7.2. *Let h be a generalized metric. For s, $s' \in \Omega_c(G_0, E^\bullet)$, put*

$$\theta_h(s, s') = \theta(s, hs'). \tag{7.6}$$

Definition 7.3. *Let $\mathcal{E} = (E^\bullet, A^{E^\bullet\prime\prime})$ be a flat antiholomorphic superconnection on $\mathcal{G} = (G_0, G_1)$ equipped with a generalized metric h. Let*

$$A^{E^\bullet\prime}$$

be the formal adjoint of $A^{E^\bullet\prime\prime}$ with respect to θ_h.

Proposition 7.4. *Under the above condition, $A^{E^\bullet\prime}$ exists, is unique, of degree -1, and is $\mathcal{G}$-equivariant.*

Proof. The existence, uniqueness, and degree condition follow from [BSW23, Section 7.1] by forgetting the $\mathcal{G}$-action. Since $A^{E^\bullet\prime\prime}$ and h are $\mathcal{G}$-equivariant, the adjoint $A^{E^\bullet\prime}$ must be $\mathcal{G}$-equivariant by uniqueness. $\square$

Let $\nabla^{E^\bullet\prime}$ be the adjoint of $\nabla^{E^\bullet\prime\prime}$ with respect to θ_h and v_i^* be the adjoint of v_i with respect to θ_h, $i = 0$ or $i \geq 2$. We have an analogy of (5.1)

$$A^{E^\bullet\prime} = v_0^* + \nabla^{E^\bullet\prime} + v_2^* + \dots. \tag{7.7}$$

Remark 7.5. *If h is a pure metric, then we have*

$$v_i^*\colon E^\bullet \to \wedge^i T^* G_0 \times_G E^{\bullet+i-1},\ i = 0,\ 2, \dots$$

and $\nabla^{E^\bullet\prime}$ is an ordinary ∂-connection. This is not the case if h is not pure.

7.3. **Curvature.** Let $\mathcal{E} = (E^\bullet, A^{E^\bullet\prime\prime})$ be a flat antiholomorphic superconnection on $\mathcal{G} = (G_0, G_1)$. Recall that $A^{E^\bullet\prime}$ is the formal adjoint of $A^{E^\bullet\prime\prime}$ in Definition 7.3. It is clear that

$$(A^{E^\bullet\prime})^2 = 0. \tag{7.8}$$

Let

$$A^{E^\bullet} = A^{E^\bullet\prime\prime} + A^{E^\bullet\prime} \tag{7.9}$$

be the superconnection on $E^\bullet$. The curvature of $A^{E^\bullet}$ is given by

$$A^{E^\bullet,2} = [A^{E^\bullet\prime\prime}, A^{E^\bullet\prime}]. \tag{7.10}$$

Then $A^{E^\bullet,2}$ is a smooth section of $\wedge^\bullet T^*_{\mathbb{C}} G_0 \times_G \mathrm{End}(E^\bullet)$ with total degree 0 with respect to $\deg_-$.

We also have the Bianchi identities

$$[A^{E^\bullet\prime\prime}, A^{E^\bullet,2}] = 0, \ [A^{E^\bullet\prime}, A^{E^\bullet,2}] = 0. \tag{7.11}$$

In addition, it is easy to see that

$$(A^{E^\bullet,2})^* = A^{E^\bullet,2}. \tag{7.12}$$

8. Chern Character

In this section, we define the Chern character for a flat antiholomorphic superconnection on an orbifold.

8.1. **Supertrace and Chern character.** Let $\mathcal{E} = (E^\bullet, A^{E^\bullet\prime\prime})$ be a flat antiholomorphic superconnection on $\mathcal{G} = (G_0, G_1)$ equipped with a generalized metric h. In Definition 2.51 we have the morphism $\beta_{\mathcal{G}} \colon I\mathcal{G} \to \mathcal{G}$ mapping the inertia groupoid $I\mathcal{G}$ to $\mathcal{G}$. Consider the pullback of $\mathcal{E}$ via $\beta_{\mathcal{G}}$, as in Proposition 5.3,

$$\beta^*_{\mathcal{G},b}\mathcal{E} = (\beta^*_{\mathcal{G}} E^\bullet, \beta^*_{\mathcal{G},b} A^{E^\bullet\prime\prime}).$$

In addition, we can also pull back the metric h to $\beta^*_{\mathcal{G},b} h$, which is a generalized metric on $\beta^*_{\mathcal{G},b}\mathcal{E}$. We denote the adjoint of $\beta^*_{\mathcal{G},b} A^{E^\bullet\prime\prime}$ under $\beta^*_{\mathcal{G},b} h$ by $\beta^*_{\mathcal{G},b} A^{E^\bullet\prime}$. Let

$$\beta^*_{\mathcal{G},b} A^{E^\bullet} = \beta^*_{\mathcal{G},b} A^{E^\bullet\prime} + \beta^*_{\mathcal{G},b} A^{E^\bullet\prime\prime}. \tag{8.1}$$

We can therefore form the curvature

$$\beta^*_{\mathcal{G},b} A^{E^\bullet,2} = [\beta^*_{\mathcal{G},b} A^{E^\bullet\prime\prime}, \beta^*_{\mathcal{G},b} A^{E^\bullet\prime}], \tag{8.2}$$

which is a section of $\wedge^\bullet T^*_{\mathbb{C}}(I\mathcal{G})_0 \times_{I\mathcal{G}} \mathrm{End}(E^\bullet)$ with total degree 0 with respect to $\deg_-$

On $I\mathcal{G}$ we also have the tautological section $\tau_{\mathcal{G}}$ with value in G_1 as in Definition 2.52. For any $g \in (I\mathcal{G})_0$, let x be the point $s(g) = t(g) \in G_0$. By definition we know that $\beta^* E^\bullet_g = E^\bullet_x$. Since $\mathcal{E} = (E^\bullet, A^{E^\bullet\prime\prime})$ is $\mathcal{G}$-equivariant, we know that $\tau_{\mathcal{G}}(g)$ acts on $\beta^*_{\mathcal{G}} E^\bullet_g$.

We have a $\mathcal{G}$-equivariant supertrace map

$$\mathrm{Tr_s} \colon \ \wedge^\bullet T^*_{\mathbb{C}}(I\mathcal{G})_0 \times \mathrm{End}(E^\bullet) \to \wedge^\bullet T^*_{\mathbb{C}}(I\mathcal{G})_0 \tag{8.3}$$

that vanishes on supercommutators.

Let φ be the following morphism of $\wedge^\bullet T^*_{\mathbb{C}}(I\mathcal{G})_0$,

$$\varphi \colon \ \wedge^\bullet T^*_{\mathbb{C}}(I\mathcal{G})_0 \to \wedge^\bullet T^*_{\mathbb{C}}(I\mathcal{G})_0, \quad \alpha \mapsto (2\pi i)^{\frac{\deg \alpha}{2}} \alpha.$$

Here we use the ordinary degree instead of $\deg_-$.

Definition 8.1. *We set*

$$ch(A^{E^\bullet\prime\prime}, h) = \varphi Tr_s[\tau_{\mathcal{G}} \exp(-\beta^*_{\mathcal{G},b} A^{E^\bullet,2})] \in \Omega^{(=)}_{I\mathcal{G}}((I\mathcal{G})_0, \mathbb{C}). \tag{8.4}$$

Remark 8.2. *In the local case, we have $\mathcal{G} = G \ltimes X$ and $I\mathcal{G} = \coprod_{(g)\in Conj(G)} Z_G(g) \ltimes X^g$ as in* (2.25). *Then we have*

$$\varphi Tr_s[\tau_{\mathcal{G}} \exp(-\beta^*_{\mathcal{G},b} A^{E^\bullet,2})] = \varphi Tr_s[g \exp(-A^{E^\bullet,2})|_{X^g}], \tag{8.5}$$

which is the definition of equivariant Chern character in [Ma05, Equation (1.6)] *and* [Bis13, Equation (4.3.10)].

Proposition 8.3. *The form $ch(A^{E^\bullet\prime\prime}, h)$ is de Rham closed.*

Proof. Since $\beta^*_{\mathcal{G},b} A^{E^\bullet\prime\prime}$, $\beta^*_{\mathcal{G},b} A^{E^\bullet\prime}$, and the de Rham differential operator are $I\mathcal{G}$-equivariant, the claim is a simple consequence of the Bianchi identities (7.11). □

Recall that $\mathscr{M}^E$ is the set of generalized metrics on $E^\bullet$. Let $\mathrm{d}^{\mathscr{M}^E}$ denote the de Rham operator on $\mathscr{M}^E$. Then $h^{-1}\mathrm{d}^{\mathscr{M}^E} h$ is a 1-form on $\mathscr{M}^E$ with values in degree 0 morphisms in $\wedge^\bullet T^*_{\mathbb{C}} I\mathcal{G}_0 \times_{I\mathcal{G}} \mathrm{End}(E^\bullet)$ that are self-adjoint with respect to h.

Theorem 8.4. *The Bott-Chern cohomology class of $ch(A^{E^\bullet\prime\prime}, h)$ is independent of the metric h. More precisely we have*

$$\mathrm{d}^{\mathscr{M}^E} ch(A^{E^\bullet\prime\prime}, h) = -\frac{\bar{\partial}\partial}{2\pi i} \varphi Tr_s[h^{-1}\mathrm{d}^{\mathscr{M}^E} h \exp(-A^{E^\bullet,2})]. \tag{8.6}$$

Proof. The proof is essentially the same as [Bis13, Theorem 4.7.1]. □

Theorem 8.4 implies that the following definition is independent of the choice of the generalized metric h.

Definition 8.5. *We denote by $ch_{BC}(A^{E^\bullet\prime\prime})$ the Bott-Chern cohomology class of $ch(A^{E^\bullet\prime\prime}, h)$ in $H^{(=)}_{BC}(I\mathcal{G}, \mathbb{C})$.*

8.2. **Chern Character Form and Scaling of the Metric.** In this subsection, we assume that the metric h is a pure metric. Let N^E be the number operator on $E^\bullet$, i.e. for $e \in E^k$ we have $N^E(e) = ke$.

Definition 8.6. *For a parameter $T > 0$, let h_T be the deformed pure metric defined by*

$$h_T = hT^{N^E}. \tag{8.7}$$

In other words, for $e_1, e_2 \in E^k$, we have

$$h_T(e_1, e_2) = T^k h(e_1, e_2).$$

Let $(E^\bullet, A^{E^\bullet\prime\prime})$ be a flat antiholomorphic superconnection with a pure metric h. Now we assume that the cohomologies $HE^\bullet$ of the complex $(E^\bullet, v_0)$ are locally of constant ranks, hence they are C^∞-vector bundles. In this case, the $\bar{\partial}$-connection $\nabla^{E^\bullet\prime\prime}$ induces a flat $\bar{\partial}$-connection $\nabla^{HE^\bullet\prime\prime}$ on the graded C^∞-vector bundle $HE^\bullet$.

Put

$$\mathcal{H}E^\bullet = \{e \in E^\bullet | v_0 e = 0 \text{ and } v_0^* e = 0\}. \tag{8.8}$$

By finite-dimensional Hodge theory, we know that $HE^\bullet \cong \mathcal{H}E^\bullet$. Hence $\mathcal{H}E^\bullet$ is a graded C^∞-vector subbundle of $E^\bullet$. As a subbundle, $\mathcal{H}E^\bullet$ inherits from $E^\bullet$ a graded metric which we denote by $h^{\mathcal{H}E}$. Let $\nabla^{\mathcal{H}E}$ be the corresponding Chern connection on $\mathcal{H}E^\bullet$.

Lemma 8.7. *We have*

$$\nabla^{\mathcal{H}E} = P\nabla^E, \tag{8.9}$$

where $\nabla^E = \nabla^{E^\bullet\prime} + \nabla^{E^\bullet\prime\prime}$ as in (5.1) and (7.7), and $P\colon E^\bullet \to \mathcal{H}E^\bullet$ is the orthogonal projection.

Proof. The proof is the same as that of [Bis13, Theorem 4.10.4]. □

Under the isomorphism $HE^\bullet \cong \mathcal{H}E^\bullet$, we obtain the corresponding graded metric h^{HE} and connection ∇^{HE} on $HE^\bullet$.

Theorem 8.8. *If the cohomologies $HE^\bullet$ of the complex $(E^\bullet, v_0)$ are locally of constant ranks, then we have*

$$ch(A^{E^\bullet\prime\prime}, h_T) = ch(\nabla^{HE}, h^{HE}) + \mathcal{O}(1/\sqrt{T}) \text{ as } T \to \infty. \tag{8.10}$$

Hence, under the same condition we have

$$ch_{BC}(A^{E^\bullet\prime\prime}) = ch_{BC}(HE) \text{ in } H_{BC}^{(=)}(\mathcal{IG}, \mathbb{C}). \tag{8.11}$$

In particular, if the complex $(E^\bullet, v_0)$ is acyclic, then

$$ch_{BC}(A^{E^\bullet\prime\prime}) = 0 \text{ in } H_{BC}^{(=)}(\mathcal{IG}, \mathbb{C}). \tag{8.12}$$

Proof. This is a finite-dimensional analogue of [Bis13, Theorem 4.10.4]. □

8.3. Chern character of mapping cones.

Recall the mapping cone $(C^\bullet, A_\phi^{C^\bullet\prime\prime})$ of a degree 0 closed map $\phi\colon (E^\bullet, A^{E^\bullet\prime\prime}) \to (F^\bullet, A^{F^\bullet\prime\prime})$, defined in Section 5.

Theorem 8.9. *The following identity holds:*

$$ch_{BC}(A_\phi^{C^\bullet\prime\prime}) = ch_{BC}(A^{E^\bullet\prime\prime}) - ch_{BC}(A^{F^\bullet\prime\prime}) \text{ in } H_{BC}^{(=)}(I\mathcal{G}, \mathbb{C}). \tag{8.13}$$

In particular, if ϕ is a quasi-isomorphism as in Definition 6.6, then we have

$$ch_{BC}(A^{E^\bullet\prime\prime}) = ch_{BC}(A^{F^\bullet\prime\prime}) \text{ in } H_{BC}^{(=)}(I\mathcal{G}, \mathbb{C}). \tag{8.14}$$

Proof. This is an equivariant version of [BSW23, Proposition 8.7.1]. The proof is the same and is left to the reader. □

The following is an immediate consequence of Theorem 8.9.

Corollary 8.10. *Let $\mathcal{F} \in D^b_{coh}(\mathcal{G})$ and $(E^\bullet, A^{E^\bullet\prime\prime})$ be as in Theorem 6.3. Then the Bott-Chern class of $ch_{BC}(A^{E^\bullet\prime\prime})$ depends only on the isomorphism class of $\mathcal{F}$ in $D^b_{coh}(\mathcal{G})$.*

The following definition thus makes sense.

Definition 8.11. *We denote by $ch_{BC}(\mathcal{F})$ the Bott-Chern cohomology class of $ch_{BC}(A^{E^\bullet\prime\prime})$ in Corollary 8.10 .*

The following is a direct consequence of Corollary 6.8 and Theorem 8.9.

Corollary 8.12. *The Chern character ch_{BC} can be viewed as a homomorphism from $K(\mathcal{G})$ into $H^{(=)}_{BC}(I\mathcal{G},\mathbb{C})$.*

Next, we consider two compatibility properties of Chern character.

Proposition 8.13. *For $f\colon \mathcal{G}\to\mathcal{H}$ and $\mathcal{F}\in D^b_{coh}(\mathcal{H})$, we have*

$$ch_{BC}(Lf^*\mathcal{F}) = If^*ch_{BC}(\mathcal{F}) \text{ in } H^{(=)}_{BC}(I\mathcal{G},\mathbb{C}). \tag{8.15}$$

Proof. Let $\mathcal{E}$ be an object in $B(X)$ which represents $\mathcal{F}$. The claim is a direct consequence of Proposition 6.9 and the construction of Chern characters. □

Proposition 8.14. *For $\mathcal{F}_1$ and $\mathcal{F}_2\in D^b_{coh}(\mathcal{G})$, we have*

$$ch_{BC}(\mathcal{F}_1\widehat{\otimes}^L\mathcal{F}_2) = ch_{BC}(\mathcal{F}_1)ch_{BC}(\mathcal{F}_2) \text{ in } H^{(=)}_{BC}(I\mathcal{G},\mathbb{C}). \tag{8.16}$$

Proof. Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be objects in $B(X)$ which represent $\mathcal{F}_1$ and F_2 respectively. By Proposition 6.10, $\mathcal{E}_1\widehat{\otimes}_b\mathcal{E}_2$ represents $\mathcal{F}_1\widehat{\otimes}^L\mathcal{F}_2$. Recall Definition 5.10, the $\mathcal{G}$-action on $\mathcal{E}_1\widehat{\otimes}_b\mathcal{E}_2$ is given by the diagonal action. The claim then follows from the construction of orbifold Chern characters. We leave the details to the reader. □

9. Riemann-Roch-Grothendieck Theorem for embeddings

The goal of this section is to prove a Riemann-Roch-Grothendieck theorem for embeddings between complex orbifolds.

In Proposition 2.42, we show that an orbifold embedding can be decomposed into a composition of an iso-spatial embedding followed by a stabilizer-preserving embedding. This splits our proof into the corresponding two cases.

9.1. Iso-spatial case.

Recall that (4.7) defines the pushforward of sheaves of $\mathcal{O}_X$-modules under a generalized morphism. We define the pushforward of sheaves of C^∞-modules in the same way.

The following proposition is important in subsequent constructions.

Proposition 9.1. *Let $f=(Z,\rho,\sigma)\colon\mathcal{G}\to\mathcal{H}$ be an iso-spatial embedding of complex orbifold groupoids. For any finite-dimensional C^∞-vector bundle E on $\mathcal{G}$, its pushforward $f_*(E)$ is a C^∞-vector bundle on $\mathcal{H}$.*

Proof of Proposition 9.1. By (4.7), we know $f_*(E)=\sigma_*\rho^*(E)$. Recall the definition of $\tilde{\mathcal{G}}$ in Section 2.5. Since $f=(Z,\rho,\sigma)$ is iso-spatial (in the sense of Definition 2.40), $\sigma\colon Z\to H_0$ is a submersion. So σ_* is simply taking $\tilde{\mathcal{G}}$-invariant sections. Notice that $\tilde{\mathcal{H}}$ only acts on Z while $\tilde{\mathcal{G}}$ acts on Z as well as on the fibers of E.

Since $\rho\colon Z\to G_0$ and $\sigma\colon Z\to H_0$ are local diffeomorphisms, for a point $x\in H_0$, we can pick a neighborhood U of x in H_0 such that $\sigma^{-1}(U)$ and $\rho(\sigma^{-1}(U))$ are finite disjoint unions of open subsets which are diffeomorphic to U. More precisely, we know $\sigma^{-1}\{x\}$

is a discrete subset of Z and $\rho(\sigma^{-1}\{x\})$ is a discrete subset of G_0. Furthermore, we have diffeomorphisms

$$\sigma^{-1}(U) \cong U \times \sigma^{-1}\{x\}, \tag{9.1}$$

and

$$\rho(\sigma^{-1}(U)) \cong U \times \rho(\sigma^{-1}\{x\}). \tag{9.2}$$

Since the map $\tilde{\mathcal{G}} \hookrightarrow \tilde{\mathcal{H}}$ is saturated, it is easy to see that $\mathcal{G}$ acts transitively on $\rho(\sigma^{-1}\{x\})$. Pick a point $\tilde{x} \in \rho(\sigma^{-1}\{x\}) \subset G_0$, we have the following result.

Lemma 9.2. *We have the following bijection between sets:*

$$\rho^{-1}(\tilde{x}) \cap \sigma^{-1}(x) \cong \mathcal{H}_x, \tag{9.3}$$

hence a diffeomorphism

$$\rho^{-1}(\tilde{U}) \cap \sigma^{-1}(U) \cong U \times \mathcal{H}_x, \tag{9.4}$$

where $\mathcal{H}_x$ is the isotropy group of x.

Proof of Lemma 9.2. (9.4) follows from (9.3) so it suffices to prove (9.3).

We pick and fix a $z \in \rho^{-1}(\tilde{x}) \cap \sigma^{-1}(x)$. Since the $\mathcal{H}$-action on Z is free, it is clear that $z\mathcal{H}_x$, which is a subset of $\rho^{-1}(\tilde{x}) \cap \sigma^{-1}(x)$, is bijective to $\mathcal{H}_x$.

Moreover, since $Z/\mathcal{H} \cong G_0$, for any $z' \in \rho^{-1}(\tilde{x}) \cap \sigma^{-1}(x)$, there exists $h \in H_1$ such that $z'h = z$. Since $\sigma(z') = \sigma(z) = x$, we get $xh = x$, i.e. $h \in \mathcal{H}_x$. Therefore, we know that $z\mathcal{H}_x$ is the whole set $\rho^{-1}(\tilde{x}) \cap \sigma^{-1}(x)$, hence the latter is bijective to $\mathcal{H}_x$. This completes the proof of (9.3) and of Lemma 9.2. □

By definition we know that

$$\Gamma(U, \sigma_* \rho^*(E)) = \Gamma(\sigma^{-1}(U), \rho^*(E))^{\mathcal{G}}. \tag{9.5}$$

Let $\mathcal{G}_{\tilde{x}}$ be the isotopy group at $\tilde{x} \in G_0$. Since the map $\tilde{\mathcal{G}} \hookrightarrow \tilde{\mathcal{H}}$ is saturated, $\mathcal{G}$ acts transitively on $\rho(\sigma^{-1}\{x\})$. Hence we have

$$\Gamma(\sigma^{-1}(U), \rho^*(E))^{\mathcal{G}} \cong \Gamma(\rho^{-1}(\tilde{U}) \cap \sigma^{-1}(U), \rho^*(E))^{\mathcal{G}_{\tilde{x}}}. \tag{9.6}$$

Since the map $\tilde{\mathcal{G}} \hookrightarrow \tilde{\mathcal{H}}$ is an embedding, it is easy to see that $\mathcal{G}_{\tilde{x}}$ acts freely on $\rho^{-1}(\tilde{x}) \cap \sigma^{-1}(x)$. Hence $\mathcal{G}_{\tilde{x}}$ can be considered as a subgroup of $\mathcal{H}_x$. Therefore (9.4) and (9.6) imply

$$\Gamma(\sigma^{-1}(U), \rho^*(E))^{\mathcal{G}} \cong \Gamma(U \times (\mathcal{H}_x/\mathcal{G}_{\tilde{x}}), E). \tag{9.7}$$

This completes the proof of Proposition 9.1. □

Recall that the definition of antiholomorphic flat superconnections (Definition 5.2) requires the superconnection be $\mathcal{G}$-equivariant, hence it induces an antiholomorphic flat superconnection on the $\tilde{\mathcal{G}}$-invariant sections. Hence for an antiholomorphic flat superconnection $\mathcal{E} = (E^\bullet, A^{E^\bullet \prime\prime})$ on $\mathcal{G}$, we can define its pushforward $f_{b,*}(\mathcal{E})$ as an antiholomorphic flat superconnection on $\mathcal{H}$ under an iso-spatial embedding $f = (Z, \rho, \sigma)\colon \mathcal{G} \to \mathcal{H}$.

Proposition 9.3. *Let $f = (Z, \rho, \sigma) \colon \mathcal{G} \to \mathcal{H}$ be an iso-spatial embedding of complex orbifold groupoids. For any $\mathcal{F} \in D^b_{coh}(\mathcal{G})$ which corresponds to $\mathcal{E} = (E^\bullet, A^{E^\bullet\prime\prime})$ under the equivalence $F_\mathcal{G}$ in Definition 6.2, the derived pushforward $Rf_*\mathcal{F}$ is given by $f_{b,*}(\mathcal{E})$ under the equivalence $F_\mathcal{H}$.*

Proof. According to [GM03, Proposition III.8.12], we can choose the soft resolution in the computation of $Rf_*\mathcal{F}$. □

Now we come to the Riemann-Roch-Grothendieck theorem for iso-spatial embeddings.

Theorem 9.4. *Let $f = (Z, \rho, \sigma) \colon \mathcal{G} \to \mathcal{H}$ be an iso-spatial embedding of complex orbifold groupoids. Let $\mathcal{F} \in K(\mathcal{G})$ and $f_!\mathcal{F} \in K(\mathcal{H})$ be its direct image. Then we have*

$$ch_{BC}(f_!\mathcal{F}) = If_* ch_{BC}(\mathcal{F}) \text{ in } H^{(=)}_{BC}(I\mathcal{H}, \mathbb{C}), \tag{9.8}$$

where $If \colon I\mathcal{G} \to I\mathcal{H}$ is the induced generalized morphism as in Definition 2.54.

Proof. Recall that, for $\mathcal{G}$ and $\mathcal{H}$, Definition 2.51 gives natural morphisms $\beta_\mathcal{G} \colon I\mathcal{G} \to \mathcal{G}$ and $\beta_\mathcal{H} \colon I\mathcal{H} \to \mathcal{H}$. Let $\mathcal{E}$ be an antiholomorphic flat superconnection on $\mathcal{G}$ which corresponds to $\mathcal{F}$. By Definition 8.1, we have

$$\mathrm{ch}_{\mathrm{BC}}(\mathcal{F}) = \varphi \mathrm{Tr_s}[\tau_\mathcal{G} \exp(-(\beta^*_{\mathcal{G},b} A^{E^\bullet})^2)], \tag{9.9}$$

and

$$\mathrm{ch}_{\mathrm{BC}}(f_!\mathcal{F}) = \varphi \mathrm{Tr_s}[\tau_\mathcal{H} \exp(-(\beta^*_{\mathcal{H},b} f_{b,*} A^{E^\bullet})^2)], \tag{9.10}$$

where $\tau_\mathcal{G}$ and $\tau_\mathcal{H}$ are the tautological sections on $I\mathcal{G}$ and $I\mathcal{H}$ respectively, and φ is the normalizing operator.

We can prove that, locally, the two sides in (9.8) are equal at the level of differential forms. Recall that, as in Remark 2.41, the iso-spatial embedding is locally given by (up to Morita equivalence) the inclusion of finite groups $G \hookrightarrow H$ and the identity map on the complex manifold X. Moreover, by Remark 2.56, the induced morphism between the inertia groupoids

$$If \colon \coprod_{(g) \in C(G)} Z_G(g) \ltimes X^g \to \coprod_{(h) \in C(H)} Z_H(h) \ltimes X^h$$

consists of the inclusion of finite groups $G \hookrightarrow H$ together with the identity map $X^g \to X^g$.

Locally we have $\mathcal{E} = (E^\bullet, A^{E^\bullet\prime\prime})$ where $E^\bullet$ is a $\mathbb{Z}$-graded, G-equivariant vector bundle on X and $A^{E^\bullet\prime\prime}$ is a G-equivariant anti-holomorphic flat superconnection on $E^\bullet$. Hence (9.9) becomes

$$\mathrm{ch}_{\mathrm{BC}}(\mathcal{F}) = \varphi \mathrm{Tr_s}[g \exp(-(A^{E^\bullet})^2)] \text{ on } X^g. \tag{9.11}$$

In Section 3.3 we use current to define the pushforward of Bott-Chern cohomology as $If_* = (I\sigma_*) \circ (I\rho_*)^{-1}$. Since f is an iso-spatial embedding, locally we can define If_* using differential forms as shown in Proposition 3.5. For $h \in H$, let $g_1, \ldots, g_k$ be representatives of G-conjugate classes of elements in G such that g_i and h are conjugate in H. On X^h we have

$$If_* \mathrm{ch}_{\mathrm{BC}}(\mathcal{F}) = \sum_{i=1}^k \frac{1}{Z_G(g_i)} \sum_{\tilde{h} \in Z_H(g_i)} (\tilde{h})^* \Big(\varphi \mathrm{Tr_s}[g_i \exp(-(A^{E^\bullet})^2)]\Big). \tag{9.12}$$

Moreover, in the local case we have

$$f_{b,*}(\mathcal{E}) = (H \times_G E^\bullet, 1 \times A^{E^\bullet\prime\prime}). \tag{9.13}$$

Therefore (9.10) locally becomes

$$\mathrm{ch}_{\mathrm{BC}}(f_!\mathcal{F}) = \varphi \mathrm{Tr}_{\mathrm{s}}[h \exp(-1 \times (A^{E^\bullet})^2)] \text{ on } X^h. \tag{9.14}$$

The proof then follows from [FH91, Exercise 3.19]. □

9.2. **Transversality and direct image.** In this section, we consider two stabilizer-preserving embeddings of complex orbifold groupoids

$$i_{X,Z} \colon X \hookrightarrow Z, \quad i_{Y,Z} \colon Y \hookrightarrow Z. \tag{9.15}$$

Definition 9.5. *Let $X \hookrightarrow Z$ and $Y \hookrightarrow Z$ be as in (9.15). Let $\mathcal{G}_1$, $\mathcal{G}_2$, and $\mathcal{H}$ be the groupoid representations of X, Y, and Z, respectively, and let $(T_1, \rho_1, \sigma_1) \colon \mathcal{G}_1 \to \mathcal{H}$ and $(T_2, \rho_2, \sigma_2) \colon \mathcal{G}_2 \to \mathcal{H}$ be two generalized morphisms representing the embeddings. We say that X and Y* intersect transversely *if the images of $\sigma_1 \colon T_1 \to H_0$ and $\sigma_2 \colon T_2 \to H_0$ intersect transversely.*

Definition 9.6. *Let $X \hookrightarrow Z$ and $Y \hookrightarrow Z$ be as in (9.15) which intersect transversely. We define $X \cap Y$ to be the fiber product $X \times_Z Y$.*

The following proposition is an orbifold version of [BSW23, Proposition 9.1.1].

Proposition 9.7. *Let $i_{X,Z} \colon X \hookrightarrow Z$ and $i_{Y,Z} \colon Y \hookrightarrow Z$ be as in (9.15) which intersect transversely. Let $U = X \times_Z Y$ be the fiber product and $i_{U,X} \colon U \to X$, $i_{U,Y} \colon U \to Y$ be the natural maps, summarized in the following diagram:*

$$\begin{tikzcd} X \times_Z Y \arrow[r, equal] & U \arrow[r, hook, "i_{U,X}"] \arrow[d, hook, "i_{U,Y}"'] & X \arrow[d, hook, "i_{X,Z}"] \\ & Y \arrow[r, hook, "i_{Y,Z}"'] & Z. \end{tikzcd}$$

Then for any $\mathcal{F} \in D^b_{coh}(X)$ there exists an isomorphism in $D^b_{coh}(Y)$

$$Li^*_{Y,Z} i_{X,Z,*}\mathcal{F} \simeq i_{U,Y,*} Li^*_{U,X}\mathcal{F}. \tag{9.16}$$

Notice that for closed embeddings, the derived pushforward coincides with the pushforward.

Proof. Since the statement is local, for any $x \in U$, we can focus on a small neighborhood of x in Z. Therefore, we can assume that X, Y, Z, and U are global quotient orbifolds and furthermore, they are suborbifolds of a quotient orbifold associated to $\mathbb{C}^m$. More precisely, we can assume that there exists a finite group G acting holomorphically on a complex manifold M such that

$$Z = [M/G].$$

Moreover we can assume M is an open subset of $\mathbb{C}^m$ and x corresponds to $0 \in \mathbb{C}^m$.

Then we can further assume that there exist L_1 and L_2 which are open neighborhood of $\mathbb{C}^{l_1} \subset \mathbb{C}^m$ and $\mathbb{C}^{l_2} \subset \mathbb{C}^m$, respectively, and subgroups H_1, H_2 of G acting on L_1 and L_2, respectively, such that

$$X = [L_1/H_1] \text{ and } Y = [L_2/H_2.]$$

In this case, the embeddings of X in Z and Y in Z are given by

$$X \cong [(L_1 \times_{H_1} G)/G] \hookrightarrow [M/G] \tag{9.17}$$

and

$$Y \cong [(L_2 \times_{H_2} G)/G] \hookrightarrow [M/G], \tag{9.18}$$

respectively. Then U is given by

$$U = \left[\left(\coprod_{[g_1]\in G/H_1,\ [g_2]\in G/H_2} g_1 L_1 \cap g_2 L_2 \right) \Big/ G \right]. \tag{9.19}$$

By Corollary 6.8, there exists a flat antiholomorphic superconnection $(E^\bullet, A^{E^\bullet\prime\prime})$ on $X = [L_1/H_1]$ which represents $\mathcal{F}$. By definition, $(E^\bullet, A^{E^\bullet\prime\prime})$ is actually an H_1-equivariant flat superconnection on L_1.

The normal bundle $N_{X/Z}$ is then a trivial bundle of dimension $m - l_1$ on X. Since X and Y intersect transversely, the normal bundle $N_{U/Y}$ is also a trivial bundle of dimension $m-l_1$ on U. We then use the same Koszul resolution as in the proof of [BSW23, Proposition 9.1.1] to prove the proposition. □

9.3. **Deformation to the normal cone.** In this subsection we focus on stabilizer-preserving embeddings as in Definition 2.38.

Let $i_{X,Y}\colon X \hookrightarrow Y$ be a stabilizer-preserving embedding of compact complex orbifold groupoids. Let $N_{X/Y}$ be the normal bundle to X in Y. We construct the deformation to the normal cone of X to Y, which generalizes the construction in [BGS90, Section 4] and [BSW23, Section 9.2].

Let W be the blow-up[3] of $Y\times\mathbb{P}^1$ along $X\times\infty$. Then we have the embedding $i_{X\times\mathbb{P}^1,W}\colon X\times\mathbb{P}^1 \hookrightarrow W$. Let P be the exceptional divisor of the blow-up, i.e.

$$P\colon = \mathbb{P}(N_{X\times\infty/Y\times\mathbb{P}^1}). \tag{9.20}$$

So we have the natural embedding $i_{X\times\infty,P}\colon X\times\infty \hookrightarrow P$ as the 0-section.

Let $p_X\colon X\times\infty \to X$ and $p_\infty\colon X\times\infty\to\infty$ be the projections. Let

$$A := p_X^* N_{X/Y} \otimes p_\infty^* N^{-1}_{\infty/\mathbb{P}^1}. \tag{9.21}$$

Then we have

$$P = \mathbb{P}(A \oplus \underline{\mathbb{C}}), \tag{9.22}$$

and $i_{\mathbb{P}(N_{X/Y}),P}\colon \mathbb{P}(N_{X/Y}) \cong \mathbb{P}(A) \hookrightarrow P$ as the ∞-section.

Let $\widetilde{Y}$ be the blow-up of Y along X. The exceptional divisor in $\widetilde{Y}$ of this blow-up may be identified with $\mathbb{P}(N_{X/Y})$. Let $q_{W,Y}\colon W\to Y$ and $q_{W,\mathbb{P}^1}\colon W\to\mathbb{P}^1$ be the obvious maps. For $z\in\mathbb{P}^1$, put

$$Y_z := q^{-1}_{W,\mathbb{P}^1} z \subset W. \tag{9.23}$$

[3]In the case of orbifold groupoids, we can apply the blow up construction introduced by [DS21, PTW21].

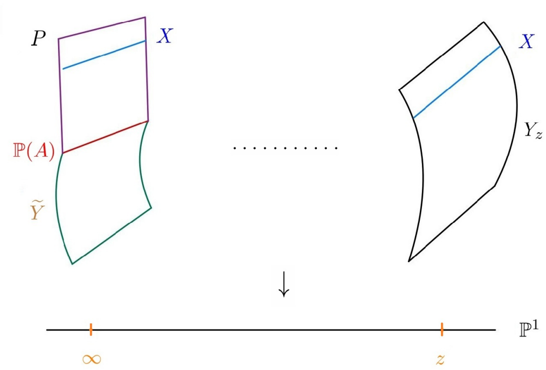

FIGURE 1. The total space W

Then

$$Y_z \cong \begin{cases} Y, & \text{if } z \neq \infty, \\ P \cup \widetilde{Y}, & \text{if } z = \infty. \end{cases} \tag{9.24}$$

For $z = \infty$, P and $\widetilde{Y}$ meet transversely along $\mathbb{P}(N_{X/Y})$. The map $q_{W,\mathbb{P}^1}$ is a submersion except on $\mathbb{P}(N_{X/Y})$, where it has ordinary double points as singularities.

Let $U = \mathcal{O}_P(-1)$ be the universal line bundle on P. We have the following exact sequence of holomorphic vector bundles on P:

$$0 \to U \to A \oplus \underline{\mathbb{C}} \to (A \oplus \underline{\mathbb{C}})/U \to 0. \tag{9.25}$$

Let σ be the image of $1 \in \underline{\mathbb{C}}$ in $(A \oplus \underline{\mathbb{C}})/U$. Then σ is a holomorphic section of $(A \oplus \underline{\mathbb{C}})/U$ which vanishes exactly on $X \times \infty$. On $\mathbb{P}(A)$, U restricts to the corresponding universal line bundle, the exact sequence (9.25) restricts to

$$0 \to U \to A \oplus \underline{\mathbb{C}} \to (A/U) \oplus \underline{\mathbb{C}} \to 0, \tag{9.26}$$

and σ restricts to the section 1 of $\underline{\mathbb{C}}$ in $(A/U) \oplus \underline{\mathbb{C}}$.

We consider the Koszul complex

$$(\wedge^\bullet((A \oplus \underline{\mathbb{C}})/U)^*, i_\sigma),$$

where i_σ is the contraction by σ. This complex provides a resolution of $i_{X\times\infty,P,*}\mathcal{O}_{X\times\infty}$. The restriction of the Koszul complex to $\mathbb{P}(A) \simeq \mathbb{P}(N_{X/Y})$ is just the split complex

$$\wedge^\bullet(A/U)^* \widehat{\otimes} (\wedge^\bullet(\underline{\mathbb{C}}), i_1).$$

9.4. **Stabilizer-preserving case.** As in [Ma05, Equation (1.7) and (1.8)], we define the Bott-Chern Todd class $\mathrm{Td}_{\mathrm{BC}}(N_{X/Y}) \in H^{(=)}_{\mathrm{BC}}(IX, \mathbb{C})$ where IX is the inertia groupoid of X.

The following is the Riemann-Roch-Grothendieck theorem for stabilizer-preserving embeddings.

Theorem 9.8. *Let $i_{X,Y} \colon X \hookrightarrow Y$ be a stabilizer-preserving embedding of compact complex orbifold groupoids. Let $\mathcal{F} \in D^b_{coh}(X)$ and $i_{X,Y,*}\mathcal{F} \in D^b_{coh}(Y)$ be its direct image. We have*

$$ch_{BC}(i_{X,Y,*}\mathcal{F}) = Ii_{X,Y,*}\left(\frac{ch_{BC}(\mathcal{F})}{Td_{BC}(N_{X/Y})}\right) \text{ in } H^{(=)}_{BC}(IY, \mathbb{C}). \tag{9.27}$$

Proof. Let $p_{X\times\mathbb{P}^1,X}\colon X\times\mathbb{P}^1\to X$ be the natural projection. Since $p_{X\times\mathbb{P}^1,X}$ is flat, we have $p^*_{X\times\mathbb{P}^1,X}=Lp^*_{X\times\mathbb{P}^1,X}$.

Recall that we have the natural embedding $i_{X\times\mathbb{P}^1,W}\colon X\times\mathbb{P}^1\hookrightarrow W$. For $\mathcal{F}\in D^b_{\mathrm{coh}}(X)$ we have

$$i_{X\times\mathbb{P}^1,W,*}p^*_{X\times\mathbb{P}^1,X}\mathcal{F}\in D^b_{\mathrm{coh}}(W). \tag{9.28}$$

Moreover, let

$$\begin{aligned} &i_{Y,W}\colon Y\hookrightarrow W,\ i_{P,W}\colon P\hookrightarrow W,\\ &i_{\widetilde{Y},W}\colon \widetilde{Y}\hookrightarrow W,\ i_{X,X\times\mathbb{P}^1}\colon X\times 0=X\hookrightarrow X\times\mathbb{P}^1 \end{aligned} \tag{9.29}$$

be the natural embeddings, where the last embedding is the inclusion of $0\in\mathbb{P}^1$ into $\mathbb{P}^1$. We summarize the various maps in the following diagram:

$$\begin{tikzcd}
 & Y \arrow[r, hook, "i_{Y,W}"'] & W & Y \arrow[l, hook', "i_{Y,W}"] & P \arrow[ll, hook', bend right, "i_{P,W}"'] \\
X & X\times\infty \arrow[l, equal] \arrow[u, hook, "i_{X\times\infty,Y}"] \arrow[r, hook, "i_{X\times\infty,X\times\mathbb{P}^1}"] & X\times\mathbb{P}^1 \arrow[u, hook, "i_{X\times\mathbb{P}^1,W}"'] \arrow[ll, bend left, "p_{X\times\mathbb{P}^1,X}"] \arrow[rr, bend right, "p_{X\times\mathbb{P}^1,X}"'] & X\times 0 \arrow[l, hook', "i_{X,X\times\mathbb{P}^1}"'] \arrow[u, hook, "i_{X,Y}"'] \arrow[r, equal] & X. \arrow[u, hook, "i_{X,P}"']
\end{tikzcd}$$

Notice that the composition $p_{X\times\mathbb{P}^1,X}\circ i_{X,X\times\mathbb{P}^1}$ is the identity map on X, so we have

$$Li^*_{X,X\times\mathbb{P}^1}\,p^*_{X\times\mathbb{P}^1,X}\mathcal{F}\simeq\mathcal{F}\in D^b_{\mathrm{coh}}(X). \tag{9.30}$$

By the same reason, if we identify $X\times\infty$ with X, then we get

$$Li^*_{X\times\infty,X\times\mathbb{P}^1}\,p^*_{X\times\mathbb{P}^1,X}\mathcal{F}\simeq\mathcal{F}\in D^b_{\mathrm{coh}}(X). \tag{9.31}$$

Recall that W is the blow-up of $Y\times\mathbb{P}^1$ along $X\times\infty$ with P the exceptional divisor. Then Y and P are both transverse to $X\times\mathbb{P}^1$ in W. By Proposition 9.7 we have

$$Li^*_{Y,W}\circ i_{X\times\mathbb{P}^1,W,*}\circ p^*_{X\times\mathbb{P}^1,X}\mathcal{F}\simeq i_{X,Y,*}\mathcal{F}\text{ in }D^b_{\mathrm{coh}}(Y), \tag{9.32}$$

and

$$Li^*_{P,W}\circ i_{X\times\mathbb{P}^1,W,*}\circ p^*_{X\times\mathbb{P}^1,X}\mathcal{F}\simeq i_{X,P,*}\mathcal{F}\text{ in }D^b_{\mathrm{coh}}(P). \tag{9.33}$$

Then by Proposition 8.13 we have

$$Ii^*_{Y,W}\mathrm{ch}_{\mathrm{BC}}(i_{X\times\mathbb{P}^1,W,*}\circ p^*_{X\times\mathbb{P}^1,X}\mathcal{F})=\mathrm{ch}_{\mathrm{BC}}(i_{X,Y,*}\mathcal{F})\text{ in }H^{(=)}_{\mathrm{BC}}(IY,\mathbb{C}) \tag{9.34}$$

and

$$Ii^*_{P,W}\mathrm{ch}_{\mathrm{BC}}(i_{X\times\mathbb{P}^1,W,*}\circ p^*_{X\times\mathbb{P}^1,X}\mathcal{F})=\mathrm{ch}_{\mathrm{BC}}(i_{X,P,*}\mathcal{F})\text{ in }H^{(=)}_{\mathrm{BC}}(IP,\mathbb{C}). \tag{9.35}$$

Let z be the canonical meromorphic function on $\mathbb{P}^1$ that vanishes at 0 and with a pole at ∞. We have the Poincaré-Lelong equation

$$\frac{\overline{\partial}^{\mathbb{P}^1}\partial^{\mathbb{P}^1}}{2\pi i}\log(|z|^2)=\delta_0-\delta_\infty. \tag{9.36}$$

Recall (9.23) that $Y_z=q^{-1}_{W,\mathbb{P}^1}z\subset W$ and IY_z is the inertia groupoid of Y_z.

Let δ_{IY_0} and δ_{IY_∞} be the currents on IW defined by integration along IY_0 and IY_∞ respectively. Since $q_{W,\mathbb{P}^1}$ has ordinary double points as singularities near $\mathbb{P}(N_{X/Y})$ in the sense of orbifolds, we have an integrable current $q^*_{IW,\mathbb{P}^1}\log(|z|^2)$ in IW. Then (9.36) gives

$$\frac{\overline{\partial}^{IW}\partial^{IW}}{2\pi i}q^*_{IW,\mathbb{P}^1}\log(|z|^2)=\delta_{IY_0}-\delta_{IY_\infty}. \tag{9.37}$$

Let $\alpha\in\Omega^{(=)}(IW,\mathbb{C})$ denote the smooth form representing $\mathrm{ch}_{\mathrm{BC}}(i_{X\times\mathbb{P}^1,W,*}\circ p^*_{X\times\mathbb{P}^1,X}\mathcal{F})$ in $H^{(=)}_{\mathrm{BC}}(IW,\mathbb{C})$. Then (9.37) gives

$$\frac{\overline{\partial}^{IW}\partial^{IW}}{2\pi i}\left(\alpha q^*_{IW,\mathbb{P}^1}\log(|z|^2)\right)=\alpha\delta_{IY_0}-\alpha\delta_{IY_\infty}. \tag{9.38}$$

Let $q_{Y_\infty,Y}$ be the restriction of $q_{W,Y}$ to Y_∞. For $z\in\mathbb{P}^1$, let $Ii_z\colon IY_z\hookrightarrow IW$ be the embedding. It is clear that $Ii_0=Ii_{Y,W}$ which is the induced morphism of $i_{Y,W}$ on inertia defined before. Then (9.38) gives

$$\frac{\overline{\partial}^{IY}\partial^{IY}}{2\pi i}q_{IW,IY,*}\left(\alpha q^*_{W,\mathbb{P}^1}\log(|z|^2)\right)=Ii_0^*\alpha-q_{IY_\infty,IY,*}Ii^*_\infty\alpha. \tag{9.39}$$

By (9.34) and (9.35), we know that $Ii_0^*(\alpha)=Ii^*_{Y,W}(\alpha)$ represents $\mathrm{ch}_{\mathrm{BC}}(i_{X,Y,*}\mathcal{F})$ in $H^{(=)}_{\mathrm{BC}}(IY,\mathbb{C})$, and $Ii^*_{P,W}(\alpha)$ represents $\mathrm{ch}_{\mathrm{BC}}(i_{X,P,*}\mathcal{F})$ in $H^{(=)}_{\mathrm{BC}}(IP,\mathbb{C})$. Then (9.39) gives

$$\mathrm{ch}_{\mathrm{BC}}(i_{X,Y,*}\mathcal{F})=q_{IY_\infty,IY,*}Ii^*_\infty\alpha \text{ in } H^{(=)}_{\mathrm{BC}}(IY,\mathbb{C}). \tag{9.40}$$

Recall (9.24) that $Y_\infty=P\cup\widetilde{Y}$. Let $q_{P,Y}$ and $q_{\widetilde{Y},Y}$ be the restriction of $q_{Y_\infty,Y}$ to P and $\widetilde{Y}$ respectively. Notice that

$$q_{IY_\infty,IY,*}Ii^*_\infty\alpha=q_{IP,IY,*}i^*_{IP,IW}\alpha+q_{I\widetilde{Y},IY,*}Ii^*_{\widetilde{Y},W}\alpha. \tag{9.41}$$

Since $\widetilde{Y}\cap(X\times\mathbb{P}^1)=\emptyset$, we have

$$Li^*_{\widetilde{Y},W}\circ i_{X\times\mathbb{P}^1,W,*}\circ p^*_{X\times\mathbb{P}^1,X}\mathcal{F}\simeq 0 \text{ in } D^b_{\mathrm{coh}}(\widetilde{Y}). \tag{9.42}$$

Then by Proposition 8.13 and the definition of α, (9.42) gives $Ii^*_{\widetilde{Y},W}\alpha=0$ in $H^{(=)}_{\mathrm{BC}}(I\widetilde{Y},\mathbb{C})$. Hence

$$q_{I\widetilde{Y},IY,*}Ii^*_{\widetilde{Y},W}\alpha=0 \text{ in } H^{(=)}_{\mathrm{BC}}(IY,\mathbb{C}). \tag{9.43}$$

Combining (9.35), (9.40), (9.41), and (9.43), we get

$$\mathrm{ch}_{\mathrm{BC}}(i_{X,Y,*}\mathcal{F})=q_{IP,IY,*}\mathrm{ch}_{\mathrm{BC}}(i_{X\times\infty,P,*}\mathcal{F}) \text{ in } H^{(=)}_{\mathrm{BC}}(IY,\mathbb{C}). \tag{9.44}$$

We can compute the right-hand side of (9.44) explicitly.

Recall that $(\wedge^\bullet((A\oplus\underline{\mathbb{C}})/U)^*,i_\sigma)$, with σ the image of $1\in\underline{\mathbb{C}}$ in $(A\oplus\underline{\mathbb{C}})/U$, provides a Koszul resolution of $i_{X\times\infty,P,*}\mathcal{O}_{X\times\infty}$. Let $q_{P,X\times\infty}$ be the natural projection. By the equivariant version of the projection formula [Sta22, Tag 0943], we have

$$\begin{aligned} i_{X\times\infty,P,*}\mathcal{F} &\simeq Lq^*_{P,X\times\infty}\mathcal{F}\widehat{\otimes}^L_{\mathcal{O}_P}i_{X\times\infty,P,*}\mathcal{O}_{X\times\infty}\\ &\simeq Lq^*_{P,X\times\infty}\mathcal{F}\widehat{\otimes}_{\mathcal{O}_P}(\wedge^\bullet((A\oplus\underline{\mathbb{C}})/U)^*,i_\sigma). \end{aligned} \tag{9.45}$$

Notice that since $(\wedge^\bullet((A\oplus\underline{\mathbb{C}})/U)^*, i_\sigma)$ is a complex of locally free sheaves, derived tensor product coincides with tensor product. Then by Proposition 8.13 and Proposition 8.14 we have

$$\mathrm{ch}_{\mathrm{BC}}(i_{X\times\infty,P,*}\mathcal{F}) = (Iq^*_{P,X\times\infty}\mathrm{ch}_{\mathrm{BC}}(\mathcal{F}))\mathrm{ch}_{\mathrm{BC}}(\wedge^\bullet((A\oplus\underline{\mathbb{C}})/U)^*, i_\sigma). \tag{9.46}$$

By the projection formula for Bott-Chern cohomology[4], we get

(9.47)

$$\begin{aligned} Iq_{P,Y,*}\mathrm{ch}_{\mathrm{BC}}(i_{X\times\infty,P,*}\mathcal{F}) &= Ii_{X,Y,*}\circ Iq_{P,X\times\infty,*}\Big((Iq^*_{P,X\times\infty}\mathrm{ch}_{\mathrm{BC}}(\mathcal{F}))\mathrm{ch}_{\mathrm{BC}}(\wedge^\bullet((A\oplus\underline{\mathbb{C}})/U)^*, i_\sigma)\Big)\\ &= Ii_{X,Y,*}\Big(\mathrm{ch}_{\mathrm{BC}}(\mathcal{F})Iq_{P,X\times\infty,*}\mathrm{ch}_{\mathrm{BC}}(\wedge^\bullet((A\oplus\underline{\mathbb{C}})/U)^*, i_\sigma)\Big). \end{aligned}$$

By [Bis95, Theorem 6.7], we get

$$\mathrm{ch}_{\mathrm{BC}}(\wedge^\bullet((A\oplus\underline{\mathbb{C}})/U)^*, i_\sigma) = \mathrm{Td}_{\mathrm{BC}}(N_{X\times\infty,P})^{-1}\delta_{IX\times\infty} \text{ in } H^{(=)}_{\mathrm{BC}}(IP,\mathbb{C}). \tag{9.48}$$

By (9.20) it is also clear that $N_{X\times\infty,P} = N_{X/Y}$ under the identification $X\times\infty\cong X$. Therefore (9.48) gives

$$Iq_{P,X\times\infty,*}\mathrm{ch}_{\mathrm{BC}}(\wedge^\bullet((A\oplus\underline{\mathbb{C}})/U)^*, i_\sigma) = \mathrm{Td}_{\mathrm{BC}}(N_{X/Y})^{-1} \text{ in } H^{(=)}_{\mathrm{BC}}(IX,\mathbb{C}). \tag{9.49}$$

Now (9.47) and (9.49) together give

$$\mathrm{ch}_{\mathrm{BC}}(i_{X,Y,*}\mathcal{F}) = Ii_{X,Y,*}\left(\frac{\mathrm{ch}_{\mathrm{BC}}(\mathcal{F})}{\mathrm{Td}_{\mathrm{BC}}(N_{X/Y})}\right) \text{ in } H^{(=)}_{\mathrm{BC}}(IY,\mathbb{C}). \tag{9.50}$$

□

9.5. **General case.** Combining Theorem 9.4 and Theorem 9.8 we get Theorem 1.1, the Riemann-Roch-Grothendieck theorem for embeddings of complex orbifolds.

Proof of Theorem 1.1. By Proposition 2.42, $i_{X,Y}$ is the composition of an iso-spatial embedding

$$i_1\colon X\hookrightarrow\bar{Y}$$

and a stabilizer-preserving embedding

$$i_2\colon \bar{Y}\hookrightarrow Y.$$

By Theorem 9.4 and Theorem 9.8 we get

$$\mathrm{ch}_{\mathrm{BC}}(i_{X,Y,*}\mathcal{F}) = \mathrm{ch}_{\mathrm{BC}}(i_{2,*}i_{1,*}\mathcal{F}) = Ii_{2,*}\left(\frac{\mathrm{ch}_{\mathrm{BC}}(i_{1,*}\mathcal{F})}{\mathrm{Td}_{\mathrm{BC}}(N_{\bar{Y}/Y})}\right) = Ii_{2,*}\left(\frac{i_{1,*}\mathrm{ch}_{\mathrm{BC}}(\mathcal{F})}{\mathrm{Td}_{\mathrm{BC}}(N_{\bar{Y}/Y})}\right). \tag{9.51}$$

By the projection formula for Bott-Chern cohomology, we have

$$\frac{Ii_{1,*}\mathrm{ch}_{\mathrm{BC}}(\mathcal{F})}{\mathrm{Td}_{\mathrm{BC}}(N_{\bar{Y}/Y})} = Ii_{1,*}\left(\frac{\mathrm{ch}_{\mathrm{BC}}(\mathcal{F})}{Ii_1^*\mathrm{Td}_{\mathrm{BC}}(N_{\bar{Y}/Y})}\right). \tag{9.52}$$

Therefore

$$\mathrm{ch}_{\mathrm{BC}}(i_{X,Y,*}\mathcal{F}) = Ii_{2,*}Ii_{1,*}\left(\frac{\mathrm{ch}_{\mathrm{BC}}(\mathcal{F})}{Ii_1^*\mathrm{Td}_{\mathrm{BC}}(N_{\bar{Y}/Y})}\right) = Ii_{X,Y,*}\left(\frac{\mathrm{ch}_{\mathrm{BC}}(\mathcal{F})}{Ii_1^*\mathrm{Td}_{\mathrm{BC}}(N_{\bar{Y}/Y})}\right). \tag{9.53}$$

[4] This follows from a suitable generalization of [BGV04, Eq. (1.15)] to the Bott-Chern cohomology.

We want to show that

$$I i_1^* \mathrm{Td}_{\mathrm{BC}}(N_{\bar{Y}/Y}) = \mathrm{Td}_{\mathrm{BC}}(N_{X/Y}). \tag{9.54}$$

Since $\mathrm{Td}_{\mathrm{BC}}$ is defined at the level of differential forms, the identity (9.54) can be proved by working locally on Y.

Locally on Y, the factorization $i_{X,Y} = i_2 \circ i_1$ can be described as

$$i_{X,Y} \colon G \ltimes M \xrightarrow{i_1} H \ltimes M \xrightarrow{i_2} H \ltimes \mathsf{W}, \tag{9.55}$$

where $M \subset \mathsf{W}$ is an embedding of manifolds, H is a finite group and $G \subset H$ is a subgroup. The normal bundle N_{i_2} is the normal bundle $N_{M/\mathsf{W}}$ together with the H-action. The pull-back $i_1^* N_{i_2}$ is the bundle $N_{M/\mathsf{W}}$ with the G-action induced from the H-action and $G \subset H$. Therefore the equality

$$i_1^* N_{i_2} = N_{i_{X,Y}} \tag{9.56}$$

is valid locally and globally. Therefore

$$\mathrm{Td}_{\mathrm{BC}}(N_{i_{X,Y}}) = \mathrm{Td}_{\mathrm{BC}}(i_1^* N_{i_2}). \tag{9.57}$$

It remains to show

$$\mathrm{Td}_{\mathrm{BC}}(i_1^* N_{i_2}) = I i_1^* \mathrm{Td}_{\mathrm{BC}}(N_{i_2}), \tag{9.58}$$

for which we can work locally as in (9.55). The induced map $I i_1$ between inertia orbifolds can be described locally as

$$\coprod_{(g) \in \mathrm{Conj}(G)} Z_G(g) \ltimes M^g \longrightarrow \coprod_{(h) \in \mathrm{Conj}(H)} Z_H(h) \ltimes M^h \tag{9.59}$$

where $Z_G(g) \to Z_H(h)$ is induced from $G \subset H$. Therefore, a component $Z_H(h) \ltimes M^h$ is in the image of $I i_1$ if and only if h is conjugate in H to some $g \in G \subset H$. In this situation, consider the restriction of $I i_1$:

$$i_g \colon Z_G(g) \ltimes M^g \to Z_H(h) \ltimes M^h = Z_H(g) \ltimes M^g. \tag{9.60}$$

Then we have the decomposition into eigenbundles of g:

$$N_{M/\mathsf{W}}|_{M^g} = \bigoplus_k N_k, \tag{9.61}$$

which induces isomorphic eigenbundle decomposition of $N_{i_2}|_{Z_H(g) \ltimes M^g}$ and $(i_1^* N_{i_2})|_{Z_G(g) \ltimes M^g}$. The definition of $\mathrm{Td}_{\mathrm{BC}}$ as a differential form then implies that

$$\mathrm{Td}_{\mathrm{BC}}(i_1^*(N_{i_2}))|_{Z_G(g) \ltimes M^g} = I i_g^* (\mathrm{Td}_{\mathrm{BC}}(N_{i_2})|_{Z_H(g) \ltimes M^g}). \tag{9.62}$$

This proves (9.58). Equation (9.54) follows from (9.57) and (9.58).

Finally, (9.53) and (9.54) together give (1.3). $\square$

9.6. **Uniqueness of orbifold Chern character.**

Proof of Theorem 1.2. Certainly $\mathrm{ch}_{\mathrm{BC}}$ defined in this paper satisfies the conditions listed in the Theorem. To show that any such group homomorphism must coincide with $\mathrm{ch}_{\mathrm{BC}}$, we may adopt the argument in the proof of [Gri10, Theorem 8]. This requires two general results, which we discuss below.

The first result asserts that for a coherent sheaf $\mathcal{F}$ on X, there exists a bimeromorphic map $\pi\colon \tilde{X} \to X$ which is the composition of a sequence of blow-ups along smooth centers, such that the pullbacks $\pi^*\mathcal{F}$ admits a locally free quotient of maximal rank. As indicated in [Gri10, Proposition 7], this is an immediate consequence of Hironaka's flattening result [Hir75]. To make this work for orbifolds, we simply observe that Hironaka's flattening result is valid for complex orbifolds, because the desired property (flatness) and the construction (blow-ups along smooth centers) commute with étale base changes.

The second result asserts an isomorphism between G-theory $G(D)$ of coherent sheaves on a smooth divisor $D \subset \tilde{X}$ and the G-theory $G_D(\tilde{X})$ of coherent sheaves on $\tilde{X}$ supported on D. By the argument of Proposition 7.2 in the arXiv version of [Gri10], this follows from dévissage theorem (see e.g. [Wei13, Devissage Theorem 6.3]) applied to abelian categories of coherent sheaves on orbifolds. □

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