# ON THE WARING PROBLEM FOR MATRICES OVER FINITE FIELDS

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ABSTRACT. We prove that if k is a positive integer then for every finite field  $\mathbb F$  of cardinality  $q \neq 2$  and for every positive integer n such that  $q^n > (k-1)^4$ , every  $n \times n$  matrix over  $\mathbb F$  can be expressed as a sum of three k-th powers. Moreover, if  $n \geq 7$  and k < q, every  $n \times n$  matrix over  $\mathbb F$  can be written as a sum of two k-th powers.

# 1. Introduction

Waring's problem for matrices asks to provide, for a given positive integer k, decompositions of the square matrices over a fixed ring into sums of k-th powers. Such decompositions are called Waring decompositions, and they were investigated in many papers. For instance, it was proved in [13] that if all  $n \times n$  matrices are sums of k-th powers,  $k \le n$ , then they are sums of seven k-th powers. These results were extended in [7], where the existence of Waring decompositions for a matrix is characterized by using the trace. A detailed study of Waring decompositions for finite rings, in particular for matrix rings over finite fields, is presented in [6].

For matrices over finite fields  $\mathbb{F}_q$  (of cardinality q), it was observed in [9] that the problem of finding minimal conditions for q and k such that every matrix is a sum of three or two k-th powers corresponds to a similar question, solved in [15] and [12], for non-commutative simple groups. Such minimal conditions are also described in [8], [9] and [10]. For instance, it was proved in [9, Theorem 6.1] that for every k there exists a constant  $C_k \leq k^{16}$  depending only on k such that if  $q > C_k$  then every  $n \times n$  matrix ( $n \geq 2$  in [10]) is a sum of two k-th powers. For the case  $\gcd(k,q)=1$  this was improved in [10, Corollary 1.6], where it is proved that we can take  $C_k=k^3-3k^2+3k$ . These results are connected to a conjecture of Larsen, [10, Problem 1], which states that for every k there exists a constant  $C_k$  such that if  $q^{n^2} > C_k$ , then every  $n \times n$  matrix is a sum of two k-th powers.

In this paper, we prove in Theorem 4.3 a version of Larsen's conjecture: if  $q^n > (k-1)^4$  then every  $n \times n$  matrix over a finite field  $\mathbb{F}_q \neq \mathbb{F}_2$  is a sum of three

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k-th powers. For the case of matrices over  $\mathbb{F}_2$  we also have a similar result for odd exponents. The second main result is Theorem 4.6, where it is proved that if  $n \geq 7$  and k < q is a positive integer then every  $n \times n$  matrix over  $\mathbb{F}_q$  is a sum of two k-th powers.

**Notations.** In this paper,  $\mathbb{F}$  will denote an arbitrary field, and  $\mathbb{F}_q$  will be a field with q elements. The ring af all  $n \times n$ -matrices over  $\mathbb{F}$  is denoted by  $\mathcal{M}_n(\mathbb{F})$ . If  $P = X^n - \alpha_{n-1}X^{n-1} - \cdots - \alpha_1X - \alpha_0 \in \mathbb{F}[X]$  then we call  $\alpha_{n-1}$  the trace of P, and we denote it by Tr(P). We also denote by Tr(A) the trace of every square matrix.

#### 2. Additive decompositions for non-scalar matrices

In this section, we will provide some useful decompositions for non-scalar matrices as sums of two matrices with prescribed characteristic polynomials.

Let  $\mathbb{F}$  be a field. If  $P = X^n - \alpha_{n-1}X^{n-1} - \cdots - \alpha_1X - \alpha_0 \in \mathbb{F}[X]$ , there exists a companion matrix

$$\mathbf{C}_{P} = \mathbf{C}_{\alpha_{0},...,\alpha_{n-1}} = \begin{pmatrix} 0 & 0 & \cdots & 0 & \alpha_{0} \\ 1 & 0 & \cdots & 0 & \alpha_{1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \alpha_{n-2} \\ 0 & 0 & \cdots & 1 & \alpha_{n-1} \end{pmatrix},$$

such that the characteristic polynomial of  $\mathbf{C}$  is P. Note that the minimal polynomial of  $\mathbf{C}$  is also P. A matrix  $\mathbf{A} \in \mathcal{M}_n(\mathbb{F})$  is called *non-derogatory* if the degree of its minimal polynomial is n. Every non-derogatory matrix is similar to a companion matrix. Two non-derogatory matrices are similar if and only if they have the same characteristic polynomial.

**Lemma 2.1.** Suppose that  $\mathbf{A} = (a_{ij}) \in \mathcal{M}_n(\mathbb{F})$ .

- (i) If for every  $j \in \{1, ..., n-1\}$  we have  $a_{(j+1)j} \neq 0$  and for every i > j+1 we have  $a_{ij} = 0$  then **A** is non-derogatory.
- (ii) Suppose that  $\mathbf{B} \in \mathcal{M}_n(\mathbb{F})$  is a non-derogatory matrix and that we have fixed some elements  $a_{ij}$  with  $1 \le i \le n$  and  $1 \le j \le n-1$  such that they verify the conditions from (i). Then there exist  $a_{1n}, \ldots, a_{nn} \in \mathbb{F}$  such that  $\mathbf{B}$  is similar to the matrix  $A = (a_{ij})$ .

*Proof.* (i) If  $e_1 = (1, 0, ..., 0)^t$  and  $c_j(\mathbf{A})$  represents the j-th column of  $\mathbf{A}$  then  $(e_1, c_1(\mathbf{A}), ..., c_{n-1}(\mathbf{A}))$  is a basis for  $\mathbb{F}^n$ . Using this basis, we observe that  $\mathbf{A}$  is similar to a companion matrix.

(ii) For simplicity, we can assume that  ${\bf B}$  is a companion matrix. It follows that the vectors

$$f_1 = e_1, f_2 = a_{21}^{-1}(Bf_1 - a_{11}f_1), f_3 = a_{32}^{-1}(Bf_2 - a_{12}f_2 - a_{22}f_2), \dots$$

for a basis of  $\mathbb{F}^n$ . Using this, we transform **B** into a matrix  $\mathbf{A} = (a_{ij})$  which has the desired form.

**Lemma 2.2.** Let  $\mathbb{F} \neq \mathbb{F}_2$  be a field, and  $\mathbf{A} \in \mathcal{M}_n(\mathbb{F})$  a non-scalar matrix. Suppose that  $P \in \mathbb{F}[X]$  is a polynomial of degree n-1. For every polynomial  $Q \in \mathbb{F}[X]$  of degree n there exist two (non-derogatory) matrices  $\mathbf{D} \in \mathcal{M}_{n-1}(\mathbb{F})$  and  $\mathbf{E} \in \mathcal{M}_n(\mathbb{F})$  with the minimal polynomials P and Q, respectively, such that  $\mathbf{A}$  is similar to a matrix of the form

$$\begin{pmatrix} & & & & d_0 \\ & \mathbf{D} & & \vdots \\ & & d_{n-1} \\ 0 & \dots & 0 & t \end{pmatrix} + \mathbf{E}.$$

Proof. We can assume that **A** is a Frobenius normal form diag( $\mathbf{C}_1, \ldots, \mathbf{C}_s$ ), where  $\mathbf{C}_i \in \mathcal{M}_{n_i}(\mathbb{F})$  are companion matrices,  $1 \leq n_1 \leq \cdots \leq n_s$ , and  $n_s > 1$ . Since  $\mathbb{F} \neq \mathbb{F}_2$  there exists a matrix  $\mathbf{D} = (d_{ij})$  such that it is similar to  $\mathbf{C}_P$ , satisfying condition (i) of Lemma 2.1, and  $d_{(j+1)j} - 1_{\mathbb{F}} \neq 0$  for all  $j \leq n - 1$ . Then  $\mathbf{A} - \text{Diag}(\mathbf{D}, t)$  also satisfies the conditions stated in Lemma 2.1. Therefore, there exists a non-derogatory matrix  $\mathbf{E} \in \mathcal{M}_n(\mathbb{F})$  such that its first (n-1) columns coincide with the corresponding columns of  $\mathbf{A} - \text{Diag}(\mathbf{D}, t)$ . Then  $\mathbf{A} - \mathbf{E}$  has the form

$$\mathbf{A} - \mathbf{E} = \begin{pmatrix} & & & d_0 \\ & \mathbf{D} & & \vdots \\ & & d_{n-1} \\ 0 & \dots & 0 & t \end{pmatrix}.$$

**Lemma 2.3.** Let  $\mathbf{A} \in \mathcal{M}_n(\mathbb{F})$  be a non-scalar matrix. If  $P \in \mathbb{F}[X]$  is a polynomial of degree n such that  $\operatorname{Tr}(P) = \operatorname{Tr}(\mathbf{A}) - n1_{\mathbb{F}}$  then  $\mathbf{A}$  is similar to a sum  $\mathbf{B} + \mathbf{C}_P$ , where  $\mathbf{B}$  is a matrix whose characteristic polynomial is  $(1 - X)^n$ .

*Proof.* As before, we assume that **A** is a Frobenius normal form diag( $\mathbf{C}_1, \ldots, \mathbf{C}_s$ ), where  $\mathbf{C}_i \in \mathcal{M}_{n_i}(\mathbb{F})$  are companion matrices,  $1 \leq n_1 \leq \cdots \leq n_s$ , and  $n_s > 1$ .

To each companion matrix  $\mathbf{C}=\mathbf{C}_{\alpha_0,\dots,\alpha_{n-1}}\in\mathcal{M}_n(\mathbb{F})$  , we associate the matrix

$$D(\mathbf{C}) = \begin{pmatrix} 1 & 0 & \cdots & 0 & \alpha_0 \\ 0 & 1 & \cdots & 0 & \alpha_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & \alpha_{n-2} \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

Let  $\mathbf{D}$  be the matrix

$\int D(\mathbf{C}_1)$	0		0	0
$ \begin{array}{cccccccccccccccccccccccccccccccccccc$	$D(\mathbf{C}_2)$		0	0
0	$ \begin{array}{cccccccccccccccccccccccccccccccccccc$		0	0
÷	÷	:	i:	:
0	0		$\begin{array}{cccccccccccccccccccccccccccccccccccc$	$D(\mathbf{C}_s)$

The matrix  $\mathbf{C}_P$  is similar to a matrix  $\mathbf{C}'$  whose first (n-1) columns coincide with the corresponding columns of  $\mathbf{A} - \mathbf{D}$ . It is not hard to see that the characteristic polynomial of  $\mathbf{A} - \mathbf{C}'$  is  $(1 - X)^n$ .

Remark 2.4. There are several other results regarding the decomposition of matrices into sums of two matrices with prescribed characteristic (or minimal) polynomials. For further details, we refer the reader to [1] and [4] for recent developments in this area.

# 3. k-th power companion matrices

In the beginning of this section, we recall some well-known results. A polynomial  $P \in \mathbb{F}_q[X]$  of degree n is a *primitive* polynomial if it is the minimal polynomial of a primitive element if  $\mathbb{F}_{q^n}$ .

**Theorem 3.1.** [3, Theorem 1] Let  $n \geq 2$  be an integer. If  $t \in \mathbb{F}_q$  such that  $t \neq 0$  for  $(q, n) \in \{(2, 2), (4, 3)\}$ , then there exists a primitive (and hence irreducible) polynomial  $P = X^n - tX^{n-1} - \cdots - \alpha_1 X - \alpha_0 \in \mathbb{F}_q[X]$ .

Let  $\phi: \mathbb{F}_{q^n} \to \mathbb{F}_{q^n}$ ,  $\phi(x) = x^q$ , be the Frobenius automorphism induced by the extension  $\mathbb{F}_q \leq \mathbb{F}_{q^n}$ . The following lemma is well-known.

**Lemma 3.2.** For every element  $a \in \mathbb{F}_{q^n}$  the polynomial

$$\Phi_a = (X - a)(X - \phi(a)) \dots (X - \phi^{n-1}(a))$$

belongs to  $\mathbb{F}_q[X]$ .

*Proof.* Let b be a primitive element in  $\mathbb{F}_{q^n}$ . Then its minimal polynomial over  $\mathbb{F}_q$  is  $\Phi_b$ . It follows that for every symmetric polynomial  $Q \in \mathbb{F}[X_1, \dots, X_n]$ , we have  $Q(b, \phi(b), \dots, \phi^{n-1}(b)) \in \mathbb{F}$ . Therefore,  $\Phi_a = \Phi_{b^m} \in \mathbb{F}[X]$ .

**Lemma 3.3.** Let  $a \in \mathbb{F}_{q^n}$ . For every positive integer k, if the values

$$a^k, \phi(a^k), \ldots, \phi^{n-1}(a^k)$$

are pairwise distinct (i.e., the polynomial  $\Phi_{a^k}$  is irreducible), then the companion matrix  $\mathbf{C}_{\Phi_{a^k}}$  is a k-th power in  $\mathcal{M}_n(\mathbb{F})$ .

*Proof.* According to our hypothesis,  $\mathbf{C}_{\Phi_a}$  and  $\mathbf{C}_{\Phi_{a^k}}$  are diagonalizable in  $\mathcal{M}_n(\mathbb{F}_{q^n})$ . If  $\mathbf{U} \in \mathrm{GL}_n(\mathbb{F}_{q^n})$  is a matrix such that  $\mathbf{U}^{-1}\mathrm{Diag}(a,\phi(a),\ldots,\phi^{n-1}(a))\mathbf{U} = \mathbf{C}_{\Phi_a}$  then

$$\mathbf{U}^{-1}\mathrm{Diag}(a^k,\phi(a^k),\ldots,\phi^{n-1}(a^k))\mathbf{U}=(\mathbf{C}_{\Phi_a})^k.$$

It follows that the matrix  $(\mathbf{C}_{\Phi_a})^k$  is diagonalizable in  $\mathcal{M}_n(\mathbb{F}_{q^n})$  and its eigenvalues are pairwise different. Using the Frobenius normal form, it follows that  $(\mathbf{C}_{\Phi_a})^k$  is similar to the companion matrix associated to the polynomial  $\Phi_{a^k}$ , which completes the proof.

**Lemma 3.4.** Suppose that  $a \in \mathbb{F}_{q^n}$  is an element such that

$$a, \phi(a), \ldots, \phi^{n-1}(a)$$

are not pairwise different. Then there exists  $1 \le u \le \lfloor \frac{n}{2} \rfloor$  such that  $\phi^u(a) = a$  (hence  $a^{q^u-1} = 1$ ). If  $u_0$  is minimal with this property, then  $u_0 \mid n$ .

*Proof.* Suppose that there exist  $0 \le i < j \le n-1$  such that  $\phi^i(a) = \phi^j(a)$ . It follows that  $\phi^{j-i}(a) = a$ . Since  $\phi^n$  is the identity map, we obtain  $\phi^{n-(j-i)}(a) = a$ . Therefore, there exists  $1 \le u \le \lfloor \frac{n}{2} \rfloor$  such that  $\phi^u(a) = a$ , hence  $a^{q^u-1} = 1$ .

Corollary 3.5. Suppose that  $b \in \mathbb{F}_{q^n}$  is a primitive element. Then for every positive integer  $k < q^{\frac{n}{2}} + 1$  the companion matrix  $\mathbf{C}_{\Phi_{p,k}}$  is a k-th power.

Proof. Suppose that the elements  $b^k, \phi(b^k), \ldots, \phi^{n-1}(b^k)$  are not pairwise different. Since  $b^k \notin \mathbb{F}_q$ , there exists an integer  $1 < u \le \lfloor \frac{n}{2} \rfloor$  such that  $b^{k(q^u-1)} = 1$  and  $u \mid n$ . It follows that n is not a prime and we have  $q^n - 1 \mid k(q^u - 1)$ . Using the hypothesis  $k < q^{\frac{n}{2}} + 1$  we obtain a contradiction. Then, Lemma 3.3 assures that  $\mathbf{C}_{\Phi_{b^k}}$  is a k-th power.

An irreducible polynomial P is a k-power if the companion matrix  $\mathbf{C}_P$  associated to P is a k-th power, see [11, Proposition 4.5]. In the following results, we will prove that if n is large enough,  $\gcd(k,q)=1$  and k< q, there exist k-power polynomials with arbitrary traces.

A proof for the next lemma can be consulted in [17].

**Lemma 3.6.** Let m be a positive integer and denote by d(m) the number of divisors of m. Then for every  $\epsilon > 0$  there exists a constant  $C_{\epsilon}$  such that  $d(m) \leq C_{\epsilon} m^{\epsilon}$ .

Remark 3.7. Following the proof of Lemma 3.6 and using a straightforward computation, it follows that we can take  $C_{\frac{1}{2}} \leq 3.53$ . In particular  $d(m) \leq 3.53m^{\frac{1}{3}}$ .

**Lemma 3.8.** Let N be the number of elements  $a \in \mathbb{F}_{q^n}$  such that

$$a, \phi(a), \ldots, \phi^{n-1}(a)$$

are pairwise different. Then

$$N \ge q^n - \frac{d(q^n - 1)}{2}q^{\frac{n}{2}} - 1.$$

In particular, there exists a constant  $C \leq 1.77$  that is independent on  $q^n$  such that

$$N > q^n - Cq^{\frac{5n}{6}} - 1.$$

*Proof.* We will count the elements  $a \in \mathbb{F}_{q^n}^*$  such that  $a, \phi(a), \dots, \phi^{n-1}(a)$  are not pairwise different. Let a such an element. Using Lemma 3.4, we take an integer  $1 \le k \le \lfloor \frac{n}{2} \rfloor$  such that  $\phi^k(a) = a$ , hence  $a^{q^k-1} = 1$ .

Let z be a primitive element in  $\mathbb{F}_{q^n}$  and u the smallest positive integer such that  $z^u=a$ . It follows that  $\operatorname{ord}_{\mathbb{F}_{q^n}^*}(z^u)\leq q^k-1$ , hence  $d=\gcd(u,q^n-1)\geq \frac{q^n-1}{q^k-1}\geq q^{\frac{n}{2}}+1$ . Therefore, for every divisor  $d\mid q^n-1$  with  $d\geq q^{\frac{n}{2}}+1$  we have at most  $\frac{q^n-1}{d}$  possible values for  $\frac{u}{d}$ . Hence, the number of those  $a\neq 0$  with the property that there exist  $0\leq i< j\leq n-1$  such that  $\phi^i(a)=\phi^j(a)$  is at most equal to the sum of all divisors c of  $q^n-1$  such that  $c\leq \sqrt{q^n-1}$ .

It follows that

$$N \geq q^n - \frac{d(q^n-1)}{2} \sqrt{q^n-1} - 1 \geq q^n - \frac{d(q^n-1)}{2} q^{\frac{n}{2}} - 1.$$

Taking C such that  $2C = C_{\frac{1}{3}}$ , we can use Remark 3.7 to obtain the conclusion.  $\square$ 

The following lemma is extracted from [14].

**Lemma 3.9.** Let  $\mathbb{F}_q$  be a field of cardinality  $q, n \geq 2$  an integer, and let  $\varphi : \mathbb{F}_{q^n} \to \mathbb{F}_{q^n}$ ,  $\varphi(x) = x^q$ , be the Frobenius automorphism induced by the field extension  $\mathbb{F}_q \leq \mathbb{F}_{q^n}$ . If k < q is a positive integer with  $\gcd(k, q) = 1$ ,  $t \in \mathbb{F}_q$ , and  $N_t$  denotes the number of the solutions in  $\mathbb{F}_{q^n}$  of the equation

$$x^k + \varphi(x^k) + \dots + \varphi^{n-1}(x^k) = t,$$

then

$$N_t \ge q^{n-1} + q^{\lfloor \frac{n}{2} \rfloor + 2} - q^{\lfloor \frac{n}{2} \rfloor + 3}.$$

*Proof.* This follows from the beginning of the proof of [14, Theorem 9A]. We have  $N_w \leq q^{n-1} + q^{\lfloor \frac{n}{2} \rfloor + 2}$  for all  $w \in \mathbb{F}_q$ , and

$$N_t = q^n - \sum_{w \neq t} N_w \ge q^n - (q-1)(q^{n-1} + q^{\lfloor \frac{n}{2} \rfloor + 2}) = q^{n-1} + q^{\lfloor \frac{n}{2} \rfloor + 2} - q^{\lfloor \frac{n}{2} \rfloor + 3}.$$

In the following theorem, we will prove that under a suitable hypothesis there exist k-power irreducible polynomials with arbitrary traces. For a survey on results concerning the existence of irreducible polynomials with prescribed coefficients, we refer to [2].

**Theorem 3.10.** Let  $\mathbb{F}_q$  be a field of cardinality q, and let  $n \geq 7$  be an integer. If k < q is a positive integer with gcd(k,q) = 1 then for every  $t \in \mathbb{F}_q$  there exists a k-power irreducible polynomial of degree n with trace t.

Proof. Consider the extension  $\mathbb{F}_q \leq F_{q^7}$ . Let  $t \in \mathbb{F}_q$ . The equation  $x^k + \varphi(x^k) + \cdots + \varphi^6(x^k) = t$  has at least  $q^5$  solutions in  $\mathbb{F}_{q^7}$ . If follows that the number of all k-th powers a such that t = Tr(a) is at least  $q^5/k$ , hence we can take such an element  $a \notin \mathbb{F}_q$ . Suppose that there exist  $0 \leq i < j \leq 6$  such that  $\varphi^i(a) = \varphi^j(a)$ . It follows that there exists a minimal  $d \leq 6$  such that  $a^k, \ldots, \varphi^d(a^k)$  are pairwise different. Then the degree of the extension  $\mathbb{F}_q \leq \mathbb{F}_q(a)$  is d, hence  $d \in \{1,7\}$ . Since  $a \notin \mathbb{F}_q$ , it follows that d = 7, and this implies that the polynomial  $(X - a)(X - \varphi(a)) \ldots (X - \varphi^6(a)) \in \mathbb{F}_q[X]$  is irreducible.

Now assume  $n \geq 8$  and take an element  $t \in \mathbb{F}_q$ . It follows from Lemma 3.9 that, if we denote by  $N_t(k)$  the number of all k-th powers  $a^k$  such that  $a^k + \varphi(a^k) + \cdots + \varphi^{n-1}(a^k) = t$ , then  $N_t(k) \geq \frac{q^{n-1} - q^{\lfloor \frac{n}{2} \rfloor + 2}(q-1)}{k} \geq \frac{q^{n-1} + q^{\frac{n}{2} + 2} - q^{\frac{n}{2} + 3}}{k}$ .

If N is the integer defined in Lemma 3.8, then

$$N_t(k) + N \ge q^n + \frac{q^{n-1} + q^{\frac{n}{3}+2} - q^{\frac{n}{2}+3}}{k} - \frac{d(q^n - 1)}{2}q^{\frac{n}{2}} - 1.$$

Therefore, using Lemma 3.3, it is enough to have

$$\frac{q^{n-1}+q^{\frac{n}{3}+2}-q^{\frac{n}{2}+3}}{k}>\frac{d(q^n-1)}{2}q^{\frac{n}{2}}+1,$$

since under this condition there exists a k-th power  $a^k \in \mathbb{F}_{q^n}$  such that the elements  $a^k, \varphi(a^k), \dots, \varphi^{n-1}(a^k)$  are pairwise distinct and  $a^k + \varphi(a^k) + \dots + \varphi^{n-1}(a^k) = t$ . Using Lemma 3.8, we observe that the condition  $(\sharp)$  is satisfied if q and n satisfy the equality  $q^{\frac{5n}{6}-2} + q^{\frac{n}{6}+1} - q^{\frac{n}{3}+2} > 1.77 + \frac{1}{q^{\frac{n}{6}}}$ , and this holds exactly when  $n \geq 1$ 

# 4. Waring decomposition for matrices

We begin with a very useful lemma.

**Lemma 4.1.** Let  $\mathbb{F}_q$  be a finite field of cardinality q. If k and n are positive integers and  $q^n > (k-1)^4$  then

- (1) every  $n \times n$  matrix with an irreducible characteristic polynomial is a sum of two k-th powers;
- (2) every  $n \times n$  scalar matrix is a sum of two k-th powers.

*Proof.* Suppose that **C** is an  $n \times n$  matrix over  $\mathbb{F}$  whose characteristic polynomial is irreducible. Then the  $\mathbb{F}_q$ -algebra

$$\mathbb{F}_q[\mathbf{C}] = \{\alpha_0 + \alpha_1 \mathbf{C} + \dots + \alpha_{n-1} \mathbf{C}^{n-1} \mid \alpha_0, \dots, \alpha_{n-1} \in \mathbb{F}_q\}$$

is a field of order  $q^n$ . From [16, Corollary 6.12] it follows that every non-zero element of  $\mathbb{F}_q[\mathbf{C}]$  is a sum of two k-th powers. In particular, every scalar  $n \times n$  matrix is a sum of two k-th powers and every matrix whose characteristic polynomial is irreducible is a sum of two k-th powers.

**Lemma 4.2.** Let  $\mathbf{D} \in \mathcal{M}_{n-1}(\mathbb{F})$  and k a positive integer. If  $t \neq 0$  is an element of  $\mathbb{F}$  such that  $t^k$  is not an eigenvalue for  $\mathbf{D}^k$  then every matrix of the form

$$\begin{pmatrix} & & & d_0 \\ & \mathbf{D}^k & & \vdots \\ & & d_{n-1} \\ 0 & \dots & 0 & t^k \end{pmatrix}$$

is a k-th power.

*Proof.* We consider the equation

Since  $t^k$  is not an eigenvalue for  $\mathbf{D}^k$ , the matrix  $\mathbf{D}^{k-1} + t\mathbf{D}^{k-2} + \cdots + t^{k-2}\mathbf{D} + t^{k-1}I_{n-1}$  is invertible, hence the equation has a solution.

**Theorem 4.3.** Let  $\mathbb{F}_q$  be a finite field of cardinality  $q \neq 2$ . If k and n are positive integers such that gcd(k,q) = 1 and  $q^n > (k-1)^4$  then every  $n \times n$  matrix with coefficients in  $\mathbb{F}_q$  is a sum of three k-th powers.

*Proof.* By Lemma 4.1, it is enough to prove the property for non-scalar matrices. Remark that the existence of such matrices implies  $n \geq 2$ .

If  $\mathbf{A} \in \mathcal{M}_n(\mathbb{F}_q)$  is not a scalar matrix, we observe that we can apply Corollary 3.5 to conclude that there exists a k-th power companion matrix in  $\mathcal{M}_{n-1}(\mathbb{F}_q)$ .

We use Lemma 2.2 and Lemma 4.2 to decompose  $\mathbf{A} = \mathbf{B} + \mathbf{C}$  such that B is a k-th power and the characteristic polynomial of  $\mathbf{C}$  is irreducible. The conclusion follows from Lemma 4.1.

**Corollary 4.4.** Let  $\mathbb{F}_q$  be a finite field of cardinality q and k < q. If  $n \geq 4$  is an integer then every matrix from  $\mathcal{M}_n(\mathbb{F}_q)$  is a sum of three k-th powers.

For matrices over  $\mathbb{F}_2$  we can use a similar strategy as in the proof of Theorem 4.3, but replacing Lemma 2.2 with Lemma 2.3 and Lemma 4.2 with [10, Lemma 2.6], to obtain

**Proposition 4.5.** For matrices over  $\mathbb{F}_2$ , if  $2^n > (k-1)^4$  and k is odd then every  $n \times n$  matrix is a sum of three three k-th powers.

We close the paper with a theorem that about Waring decompositions with two terms.

**Theorem 4.6.** Let  $n \geq 7$  be an integer. If  $\mathbb{F}_q$  is a finite field of cardinality q and k < q is a positive integer then every  $n \times n$  matrix over  $\mathbb{F}_q$  is a sum of two k-th powers.

*Proof.* Since the case q=2 is trivial, we assume  $\mathbb{F}_q \neq \mathbb{F}_2$ .

The conditions k < q and  $n \ge 7$  imply  $q^n > (k-1)^4$ , therefore, as in the proof of Theorem 4.3, every scalar matrix is a sum of two k-th powers, and there exists a k-th power companion matrix  $\mathbf{D} \in \mathcal{M}_{n-1}(\mathbb{F}_q)$ .

Let  $\mathbf{A} \in \mathcal{M}_n(\mathbb{F}_q)$  be a non-scalar matrix. We denote by p the characteristic of  $\mathbb{F}$ . We write  $k = p^a k'$ , with  $\gcd(k',q) = 1$ . By Theorem 3.10, there exists a k'-th power companion matrix  $\mathbf{C} \in \mathcal{M}_n(\mathbb{F}_q)$  such that the characteristic polynomial of  $\mathbf{C}$  is irreducible and  $\operatorname{Tr}(\mathbf{A}) = \operatorname{Tr}(\mathbf{C}) + \operatorname{Tr}(\mathbf{D}) + 1$ . If  $\mathbf{C} = (\mathbf{E}')^{k'}$ , it follows from [5, Corollary 1] that the order of  $\mathbf{E}'$  in  $\operatorname{GL}_n(\mathbb{F}_q)$  is coprime with p. Then there exists a matrix  $\mathbf{E}$  such that  $\mathbf{E}' = \mathbf{E}^{p^a}$ , hence  $\mathbf{C} = \mathbf{E}^k$ .

As in the proof of Theorem 4.3, we can apply Lemma 2.2 to conclude that  $\mathbf{A}$  is a sum of two k-th powers.

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