2. Probability Distribution

* Discrete Random Variable : A variable used to denote the numerical value of outcome of a experiment is called random variable and if it takes discrete values $x_1, x_2, \dots, x_n$ in the interval a, b then it is called discrete random variable.

If random variable takes all values between the interval a, b then it is called continous random variable.

If x is a random variable and $x_1, x_2, \dots, x_n$ are its all possible values having probability $Px_1, Px_2, \dots, Px_n$ then the number Px_i must satisfy the following condition.

i) $0 \leq P(x_i) \leq 1$

ii) $\sum_{i=1}^{n} P(x_i) = 1$

The function P is called probability function or probability mass function (p.m.f) or probability density function (p.d.f) and set of pairs (x_i, p_i) ~~is~~ ~~call~~ and set of function (p_i, x_i) is called probability distribution of x.

If a random variable x is continous and the function $y = f(x)$ is a continous function such that

i) $f(x)$ is integrable

ii) $f(x) > 0$

iii) $\int_{a}^{b} f(x)\,dx = 1$

iv) $\int_{\alpha}^{\beta} f(x)\,dx = P(\alpha \le x \le \beta)$

where $a < \alpha < x < \beta < b$ is called probability density function of a continous random variable x.

Ex The probability distribution of som of number appearing on the toss of two unbaiased ~~dies~~ dice.

$\Rightarrow$

x:	2	3	4	5	6	7	8	9	10	11	12
$P(X=x)$:	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

Ex The p.d.f of a random variable x is

x	0	1	2	3	4	5	6
$(X=x)$	k	$3k$	$5k$	$7k$	$9k$	$11k$	$13k$

Find

i) $P(x < 4)$

ii) $P(3 < x \le 6)$

iii) Find k

$\Rightarrow$ iii) $\sum_{i=0}^{6} P(x_i) = 1$

$P(X=0) + P(X=1) + P(X=2) + P(X=3)$
$+ P(X=4) + P(X=5) + P(X=6) = 1$

$k + 3k + 5k + 7k + 9k + 11k + 13k = 1$

$49k = 1$

$\therefore K = \frac{1}{49}$

ii) $P(x<4) = P(x=0) + P(x=1) + P(x=2) + P(x=3)$

$= k + 3k + 5k + 7k$

$= 16k$

$= \frac{16}{49}$

iii) $P(3 < x \leq 6) = P(x=4) + P(x=5) + P(x=6)$

$= 9k + 11k + 13k$

$= 33k$

$= \frac{33}{49}$ //

Ex The p.d.f of x is

x	0	1	2	3	4	5	6	7
$P(x=x)$	0	c	2c	2c	3c	c^2	$2c^2$	$7c^2+c$

find i) c

ii) $P(x \geq 6)$

iii) $P(x<6)$

iv) $P(1.5 < x < 4.5 / x > 2)$

$\Rightarrow$ i) $\sum_{i=0}^{7} P(x_i) = 1$

$P(x=0) + P(x=1) + P(x=2) + P(x=3)$

$+ P(x=4) + P(x=5) + P(x=6)$

$+ P(x=7) = 1$

$0 + c + 2c + 2c + 3c + c^2 + 2c^2 + 7c^2 + c = 1$

$9c + 10c^2 = 1$

DOMS Page No.
Date / /

By Solving

$c = 0.1$ or $c = -1$

$c \neq -1$ $(\because 0 \leq c \leq 1)$

ii) $P(X \geq 6) = P(X=6) + P(X=7)$

$= 2c^2 + 7c^2 + c$

$= 2(0.1)^2 + 7(0.1)^2 + (0.1)$

$= 0.19$

iii) $P(X<6) = P(X=0) + P(X=1) + P(X=2) + P(X=3) + P(X=4) + P(X=5)$

$= 0 + c + 2c + 2c + 3c + c^2$

$= 8c + c^2$

$= 8(0.1) + (0.1)^2$

$= 0.81$

iv) $P(1.5 < X < 4.5 / X > 2)$ ... $P(A/B) = \frac{P(A \cap B)}{P(B)}$

$= \frac{P(1.5 < x < 4.5 \cap x > 2)}{P(X > 2)}$

$= \frac{P(X=3) + P(X=4)}{1 - P(X \leq 2)}$

$= \frac{2(0.1) + 3(0.1)}{1 - [P(X=0) + P(X=1) + P(X=2)]}$

$= \frac{2(0.1) + 3(0.1)}{1 - [0 + 0.1 + 2(0.1)]}$

$= 5/7 = 0.7184$

Page No.
Date / /

Ex A random variable x has following p.d.f

x	-2	-1	0	1	2
$P(X=x)$	1/5	1/5	2/5	2/15	1/15

Find i) $V = x^2+1$ ii) $W = x^2+2x+3$

⇒ i)

x	-2	-1	0	1	2
$V=x^2+1$	5	2	1	2	5

P(V)	$P(X=2)+P(X=2)$ $\frac{1}{5}+\frac{1}{15}$ $=\frac{4}{15}$	$P(X=-1)+P(X=1)$ $\frac{1}{5}+\frac{2}{15}$ $\frac{5}{15}$	$P(X=0)$ $\frac{2}{5}$

ii)

x	-2	-1	0	1	2
$W=x^2+2x+3$	3	2	3	6	11
P(W)	$\frac{3}{5}$	$\frac{1}{5}$	$\frac{3}{5}$	$\frac{2}{15}$	$\frac{1}{15}$

$P(X=-2)+P(X=0)$

$=\frac{1}{5}+\frac{2}{5}=\frac{3}{5}$

DOMS Page No.
Date / /

Ex A random variable X takes values 1, 2, 3, 4 such that $2P(X=1) = 3P(X=2) = P(X=3) = 5P(X=4)$. Find the probability distribution & cumulative distribution function.

→ let $2P(X=1) = 3P(X=2) = P(X=3) = 5P(X=4) = k$

$\therefore P(X=1) = \frac{k}{2}, \quad P(X=2) = \frac{k}{3}$

$P(X=3) = k, \quad P(X=4) = \frac{k}{5}$

$\therefore \sum_{i=1}^{4} P(x) = 1$

$\frac{k}{2} + \frac{k}{3} + k + \frac{k}{5} = 1$

$\frac{61}{30} k = 1$

$k = \frac{30}{61}$

X	1	2	3	4
$P(X=x)$	$\frac{15}{61}$	$\frac{10}{61}$	$\frac{30}{61}$	$\frac{6}{61}$
$f(x)$	$\frac{15}{61}$	$\frac{25}{61}$	$\frac{55}{61}$	$\frac{61}{61} = 1$

Ex A shipment of 8 computers contains 3 that are defective. If a college makes a random purchase of 2 of this computers. Find probability distibution of defective computers.

X	0	1	2
$P(X=x)$	$\frac{^5C_2}{^8C_2}$	$\frac{^3C_1\,^5C_1}{^8C_2}$	$\frac{^3C_2}{^8C_2}$
	$\frac{10}{28}$	$\frac{15}{28}$	$\frac{3}{28}$

Ex Find probability distribution of no. of heads obtained when a fair coins is tossed 3 times.

X	0	1	2	3
$P(X=x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

{HHH, HHT, HTH, THH, TTH, THT, HTT, TTT}

Ex Find K if $f(x) = Kxe^{-4x^2}$, $0 \le x \le \infty$ is probability density function.

→ Since, $f(x)$ is a p.d.f

$$\therefore \int_0^\infty f(x)dx = 1$$

$$\int_0^\infty Kxe^{-4x^2} = 1$$

$$K\int_0^\infty xe^{-4x^2}dx = 1$$

put $4x^2 = t$

$8x\,dx = dt$

$x\,dx = dt/8$

x	0	∞
t	0	∞

$$K\int_0^\infty \frac{e^{-t}\,dt}{8} = 1$$

DOMS Page No.
Date / /

$$-\frac{k}{8}\left[e^{-t}\right]_0^{\infty} = 1$$

$$-\frac{k}{8}\left[e^{-\infty} - e^{0}\right] = 1$$

$$-\frac{k}{8}\left[0 - 1\right] = 1$$

$$\frac{k}{8} = 1$$

$$\boxed{k = 8}$$

Ex A function defined as $f(x) = \begin{cases} 0, & x < 2 \\ \frac{2x+3}{18}, & 2 \le x \le 4 \\ 0, & x > 4 \end{cases}$

show that $f(x)$ is p.d.f & find the probability for $2 \le x \le 3$

⇒ To show that $f(x)$ is p.d.f i.e to show that $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{2} f(x)\,dx + \int_{2}^{4} f(x)\,dx + \int_{4}^{\infty} f(x)\,dx$$

$$= \int_{-\infty}^{2} 0\,dx + \int_{2}^{4} \frac{2x+3}{18}\,dx + \int_{4}^{\infty} 0\,dx$$

$$= 0 + \left[\frac{2x^2}{36} + \frac{3x}{18}\right]_2^4 + 0$$

$$= 0 + \left[\frac{2\times16}{36} + \frac{12}{18} - \frac{8}{36} - \frac{6}{18}\right] + 0$$

$$= \left[\frac{32}{36} + \frac{12}{18} - \frac{8}{36} - \frac{6}{18}\right]$$

$$= \left[\frac{24}{36} + \frac{6}{18} \right]$$

$$= \left[\frac{28}{36} + \frac{12}{36} \right]$$

$$= \frac{36}{36}$$

$$= 1$$

$\therefore f(x)$ is p.d.f

$P(2 < x < 3)$

$$= \int_2^3 f(x)\, dx$$

$$= \int_2^3 \frac{(2x+3)}{18}\, dx$$

$$= \left[\frac{2x^2}{36} + \frac{3x}{18} \right]_2^3$$

$$= \frac{18}{36} + \frac{9}{18} - \frac{8}{36} - \frac{6}{18}$$

$$= \frac{10}{36} + \frac{3}{18}$$

$$= \frac{4}{9}$$

Ex ~~If~~ let x be a continous random variable with $f(x) = \begin{cases} x/6 + K, & 0 \le x \le 3 \\ 0, & \text{elsewhere} \end{cases}$

find K, $P(1 \le x \le 2)$

$\Rightarrow$ $\because$ $f(x)$ is p.d.f

$$\therefore \int_0^\infty f(x)\,dx = 1$$

$$\int_0^\infty \frac{x}{6} + K\,dx = 1$$

$$\left[\frac{x^2}{12} + Kx\right]_0^\infty = 1$$

$$= \left[\frac{x^2}{12} + Kx\right]_0^3 + \left[\frac{x^2}{12} + Kx\right]_3^\infty = 1$$

$$\left[\frac{x^2}{12} + Kx\right]_0^3 + 0 = 1$$

$$\left[\frac{9}{12} + 3K\right] = 1$$

$$3K = 1 - \frac{9}{12}$$

$$3K = \frac{3}{12}$$

$$K = \frac{3}{36}$$

$$K = \frac{1}{12}$$

$$P(1 \leq x \leq 2) = \int_1^2 f(x)\,dx$$

$$= \int_1^2 \frac{x}{6} + K$$

$$= \left[\frac{x^2}{12} + Kx\right]_1^2$$

$$= \left[\frac{4}{12} + 2K - \frac{1}{12} - [K]\right]$$

$$= \left[\frac{4}{12} + \frac{2}{12} - \frac{1}{12} - \frac{1}{12}\right]$$

$$= \left[\frac{6}{12} - \frac{2}{12}\right]$$

$$= \left[\frac{4}{12}\right] = \frac{1}{3}$$

Ex Find the value of K such that following $f(x)$ will be a p.d.f. Also find $P(x \leq 1.5)$

$$f(x) = \begin{cases} Kx & 0 \leq x \leq 1 \\ K & 1 \leq x \leq 2 \\ K(3-x) & 2 \leq x \leq 3 \end{cases}$$

$\Rightarrow$ $\because$ $f(x)$ is p.d.f

$$\therefore \int_0^3 f(x)\,dx = 1$$

$$\therefore \int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx + \int_2^3 f(x)\,dx = 1$$

$$\int_0^1 Kx\,dx + \int_1^2 k\,dx + \int_2^3 k(3-x)\,dx = 1$$

$$\left[\frac{Kx^2}{2}\right]_0^1 + \left[Kx\right]_1^2 + \left[3Kx - \frac{kx^2}{2}\right]_2^3 = 1$$

$$\frac{K}{2} + [2K - K] + \left[9K - \frac{9K}{2} - 6K + \frac{4K}{2}\right] = 1$$

$$\frac{K}{2} + K + 3K - \frac{5K}{2} = 1$$

$$4K - \frac{4K}{2} = 1$$

$$\frac{8K - 4K}{2} = 1$$

$$4K = 2$$

$$K = \frac{1}{2}$$

$$P(x \leq 1.5) = \int_0^1 Kx\,dx + \int_1^{1.5} k\,dx$$

$$= \left[\frac{(1/2)x^2}{2}\right]_0^1 + \left[(1/2)x\right]_1^{1.5}$$

$$= \frac{1}{4} + \left(\frac{1}{2} \times 1.5 - \frac{1}{2}\right)$$

$= \cancel{\frac{3}{4}} \cancel{+} \cancel{\frac{1}{2}}$

$= \cancel{\frac{3}{4}}\ \frac{1}{4} + \left(\frac{1}{4}\right)$

$= \frac{1}{2}$

Ex The probability that the person will die in the time interval (t_1, t_2) is given by $P(t_1 \le t \le t_2) = \int_{t_1}^{t_2} f(t)\,dt$

where $f(t) = \begin{cases} 3 \times 10^{-9}(100t - t^2)^2, & 0 < t < 100 \\ 0, & \text{elsewhere} \end{cases}$

Find i) The probability that a person will die between the age 60 - 70

ii) Find the probability that a person will die betn age 60 - 70 given that he has survived upto 60.

→ i) $P(60 \le t \le 70) = \int_{60}^{70} f(t)\,dt$

$= 3 \times 10^{-9} \int_{60}^{70} (100t - t^2)^2\,dt$

$= 3 \times 10^{-9} \int_{60}^{70} (10000t^2 - 200t^3 + t^4)\,dt$

$= 3 \times 10^{-9} \left[10^4 \frac{t^3}{3} - 200 \frac{t^4}{4} + \frac{t^5}{5} \right]_{60}^{70}$

$= 0.1543$

$$P(A/B) = \frac{P(A \cap B)}{P(B)}$$

$$P(60 \leq t \leq 70 \,/\, t \geq 66)$$

$$= \frac{P(60 \leq t \leq 70 \cap t \geq 60)}{P(t \geq 60)}$$

$$= \frac{P(60 \leq t \leq 70)}{P(t \geq 60)}$$

$$= \frac{0.1543}{\int_{60}^{100} f(t)\,dt}$$

$$= \frac{0.1543}{\left[10^4 \frac{t^3}{3} - 200 \frac{t^4}{4} + \frac{t^5}{5}\right]_{60}^{100}} \qquad = \frac{0.1543}{0.31744}$$

$$= 0.4860$$

* Mathematical Expectation

If x is a discrete random variable having values $x_1, x_2, \ldots x_n$ with the probabilities $p_1, p_2, \ldots, p_n$ then expectation of x denoted by $p(x)$ is

$$E(x) = \sum x\, p(x)$$
$$= x_1 p(x_1) + x_2 p(x_2) + \ldots\ldots + x_n p(x_n)$$

If x is continous random variable with p.d.f $f(x)$ then

$$E(x) = \int_{-\infty}^{\infty} x f(x)\, dx$$

$E(x)$ is also called as mean of x

Properties of Expectation:

i) $E(ax+b) = aE(x) + b$
ii) $E(x \pm y) = E(x) \pm E(y)$
iii) $E(x \cdot y) = E(x) \cdot E(y)$

Ex A fair coin is tossed 3 times. A person recieves ₹ x^2 if he gets x heads. Find his expectation

→

x:	0	1	2	3
$P(x)$:	1/8	3/8	3/8	1/8
x^2:	0	1	4	9
$P(x)$:	1/8	3/8	3/8	1/8

HHH, HTT, TTH, THT, TTT, HHT, THH, HTH

Date / /

$E(x) = \sum x \cdot P(x)$

$= \frac{0 \times 1}{8} + \frac{1 \times 3}{8} + \frac{4 \times 3}{8} + \frac{9 \times 1}{8}$

$= \frac{8+12+9}{8}$

$= \frac{24}{8}$

$= 3$

~~A Person will receive ₹3~~

$\therefore$ It is expected that he will get ₹3.

Ex A fair coin is tossed till a head appears. What is the expectation of the number of process required

→ X: No of tosses req to get head

X:	1	2	3	4	5 ...
P(X=x):	1/2	$\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$	$\frac{1}{8}$		

$E(x) = \sum x P(x)$

doubt

$= \frac{1 \times 1}{2} + \frac{2 \times 1}{4} + \frac{3 \times 1}{8} + \dots$

$= S$

$\frac{1}{2} S = \frac{1}{4} + \frac{2 \cdot 1}{8} + \frac{3 \cdot 1}{16} + \frac{4 \cdot 1}{32} + \dots$

$S - \frac{1}{2} S = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \dots$

[For G.P. with common ratio, r
$S = a/1-r$]

$$\therefore \frac{1}{2} S = \frac{1/2}{1 - 1/2}$$

$$\frac{1}{2} S = \frac{1/2}{1/2}$$

$$S = 2$$

$$\therefore E(X) = 2$$

Ex A continous random variable x has p.d.f $f(x) = K(1+x)$ where, $2 \leq x \leq 5$

Find i) K

ii) $P(x \leq 4)$

iii) $E(X)$.

→ ∵ It is a p.d.f

$$\int_2^5 f(x)dx = 1$$

$$\int_2^5 K(1+x) = 1$$

$$\int_2^5 K + Kx = 1$$

$$\left[Kx + \frac{Kx^2}{2}\right]_2^5 = 1$$

$$\left[5K + \frac{25K}{2} - 2K - \frac{4K}{2}\right] = 1$$

$$\left[3K + \frac{21K}{2}\right] = 1$$

$$\frac{27K}{2} = 1$$

$$K = 2/27$$

DOMS Page No.
Date / /

ii) $P(x \leq 4)$

$$= \int_2^4 \frac{2+2x}{27}\, dx$$

$$= \left[\frac{2x}{27} + \frac{2x^2}{27\times 2}\right]_2^4$$

$$= \left[\frac{2x}{27} + \frac{x^2}{27}\right]_2^4$$

$$= \left[\frac{2(4)}{27} + \frac{16}{27} - \frac{4}{27} - \frac{4}{27}\right]$$

$$P(x \leq 4) = \frac{16}{27}$$

iii) $E(x) = \int_2^5 x\,(K+Kx)\, dx$

$$= \int_2^5 Kx + Kx^2\, dx$$

$$= \left[\frac{Kx^2}{2} + \frac{Kx^3}{3}\right]_2^5$$

$$= \left[\frac{25K}{2} + \frac{125K}{3} - \frac{4K}{2} - \frac{8K}{3}\right]$$

$$= \left[\frac{25(2/27)}{2} + \frac{125(2/27)}{3} - \frac{4(2/27)}{2} - \frac{8(2/27)}{3}\right]$$

$$E(x) = \frac{11}{3}$$

Ex If mean of following distribution is 16. Find m, n and var.

x:	8	12	16	20	24
$p(x)$:	1/8	m	n	1/4	1/12

m, n, var ?

→ mean = 16

$\therefore E(x) = 16$

$\sum x.p(x) = 16$

$$\cancel{8} \times \frac{1}{\cancel{8}} + 12 \times m + 16 \times n + \frac{20}{4} + \frac{24}{12} = 16$$

$$1 + 12m + 16n + 5 + 2 = 16$$

$$12m + 16n = 8 \quad - (1)$$

$\because \sum p(x) = 1$

$$\frac{1}{8} + m + n + \frac{1}{4} + \frac{1}{12} = 1$$

$$m + n + \frac{11}{24} = 1$$

$$m + n = \frac{13}{24} \quad - (2)$$

Solving eqn (1) & (2)

$$m = \frac{1}{6} \qquad n = \frac{3}{8}$$

$$\boxed{var(x) = E(x^2) - (E(x))^2}$$

$E(x^2) = \sum x^2 \, p(x)$

$$= 64 \times \frac{1}{8} + 144 \times \frac{1}{6} + 256 \times \frac{3}{8} + 400 \times \frac{1}{4}$$

$+ 576 \times 1/12$

Date / /

$E(x^2) = 276$

$var(x) = 276 - (16)^2$

$\boxed{var(x) = 20}$

Ex A continous random variable x has p.d.f defined by $f(x) = a + bx$ where $0 \le x \le 1$ if mean of distribution is $1/3$. Find a & b.

→ mean $= 1/3$

$E(x) = 1/3$

$$E(x) = \int_0^1 x\,(a+bx)$$

$$\frac{1}{3} = \left[\frac{ax^2}{2} + \frac{bx^3}{3}\right]_0^1$$

$$\left[\frac{a}{2} + \frac{b}{3}\right] = \frac{1}{3} \quad -(1)$$

As it is a p.d.f

$$\int_0^1 f(x)\,dx = 1$$

$$\int_0^1 a + bx = 1$$

$$\left[ax + \frac{bx^2}{2}\right]_0^1 = 1$$

$$a + \frac{b}{2} = 1 \quad -(2)$$

Solving (1) & (2)

$a = 2, \; b = -2$

DOMS Page No.
Date / /

Ex Find mean & variance of the following distribution

$$f(x) = \begin{cases} (1-x), & 0<x<1 \\ x-1, & 1<x<2 \end{cases}$$

→ $E(x) = \text{mean} = \int_0^2 x\, f(x)\, dx$

$$= \int_0^1 x(1-x)\,dx + \int_1^2 x(x-1)\,dx$$

$$= \int_0^1 x - x^2\, dx + \int_1^2 x^2 - x\, dx$$

$$= \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 + \left[\frac{x^3}{3} - \frac{x^2}{2}\right]_1^2$$

$$= \frac{1}{2} - \frac{1}{3} + \frac{8}{3} - \frac{4}{2} - \frac{1}{3} + \frac{1}{2}$$

$E(x) = 1$

$$E(x^2) = \int_0^2 x^2 f(x)\,dx$$

$$= \int_0^1 x^2(1-x)\,dx + \int_1^2 x^2(x-1)\,dx$$

$$= \int_0^1 x^2 - x^3\,dx + \int_1^2 x^3 - x^2\,dx$$

$$= \left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 + \left[\frac{x^4}{4} - \frac{x^3}{3}\right]_1^2$$

$$= \frac{1}{3} - \frac{1}{4} + \frac{16}{4} - \frac{8}{3} - \frac{1}{4} + \frac{1}{3}$$

$= 3/2$

$$Var(x) = E(x^2) - E(x)^2$$

$$= \frac{3}{2} - 1$$

$$= \frac{1}{2}$$

Ex If a random variable x has following p.d.f then find $E(8x^2 - 2x + 5)$

x :	-2	-1	0	1	2
$P(x)$:	0.2	0.3	0.15	0.25	0.1

→ $E(8x^2 - 2x + 5)$

... $E(ax+b) = a E(x) + b$

$$= 3E(x^2) - 2E(x) + 5$$

$$= 3 \sum x^2 P(x) - 2 \sum x P(x) + 5$$

$$= 3[(-2)^2(0.2) + (-1)^2(0.3) + (0) + (1)^2(0.25) + (2)^2(0.1)]$$

$$- 2[(-2)(0.2) + (-1)(0.3) + 0 + (1)(0.25) + 2(0.1)]$$

$$+ 5$$

$$= \frac{21}{4} - \left(\frac{-1}{2}\right) + 5$$

$$= \frac{43}{4} = 10.75$$

DOMS Page No.
Date / /

* Variance

Properties

1. $V(c) = 0$
2. $V(ax+b) = a^2 V(x)$
3. $V(ax_1+bx_2) = a^2 V(x_1) + b^2 V(x_2)$
4. $V(ax_1-bx_2) = a^2 V(x_1) + b^2 V(x_2)$

Ex If x_1 has mean 5 & variance 5, x_2 has mean -2 & variance 3. Then, find
1. $E(x_1+x_2)$, $V(x_1+x_2)$
2. $E(x_1-x_2)$, $V(x_1-x_2)$
3. $E(2x_1+3x_2-7)$, $V(2x_1+3x_2-7)$

→ $E(x_1) = 5$ $V(x_1) = 5$
$E(x_2) = -2$ $V(x_2) = 3$

i) $E(x_1+x_2) = E(x_1) + E(x_2) = 5+(-2) = 3$
$V(x_1+x_2) = V(x_1) + V(x_2) = 5+3 = 8$

ii) $E(x_1-x_2) = E(x_1) - E(x_2) = 5-(-2) = 7$
$V(x_1-x_2) = V(x_1) + V(x_2) = 5+3 = 8$

iii) $E(2x_1+3x_2-7) = 2E(x_1) + 3E(x_2) - 7$
$= 2(5) + 3(-2) - 7$
$= 10 - 6 - 7$
$= -3$

$V(2x_1+3x_2-7) = 4V(x_1) + 9V(x_2)$
$= 4(5) + 9(3)$
$= 47$

Binomial Distribution : Consider an exp which results in either success or failure & let it be repeated n time. A random variable x is said to follow a binomial distribution ~~there~~ if

$$P(X = x) = {}^nC_x\, p^x q^{n-x}$$

$$p+q = 1 \qquad x = 0, 1, 2, \ldots n$$

When we get Binomial distribution ?

1) When a trial is repeated n times where n is finite

2) Each trial results only in 2 ways
- Success
- Failure

3) Events are independent i.e the probability of success remains constant in all trials

Mean $= np$ Variance $= npq$

Expected frequency $= Np$ where experiment of n trials repeated N times.

Ex The probability of a man hitting the target is 1/4.

i) If he fires 7 times what is the Probability of him hitting the target atleast twice.

DOMS Page No.
Date / /

ii) How many times must be fired so that the probability of him hitting the target atleast once is greater than 2/3.

→ i) $p = \frac{1}{4}$, $q = 1-p = 1-\frac{1}{4} = \frac{3}{4}$

$n = 7$

$P(x \geq 2) = {}^7C_2 p^2 q^{7-2}$

$P(x \geq 2) = 1 - P(x < 2)$

$= 1 - [P(x=0) + P(x=1)]$

$= 1 - [{}^7C_0 p^0 q^{7-0} + {}^7C_1 p^1 q^6]$

$= 1 - [{}^7C_0 (1/4)^0 (3/4)^7 + {}^7C_1 (1/4)(3/4)^6]$

$P(x \geq 2) = 0.555$

ii) $n = ?$

$P(x \geq 1) > \frac{2}{3}$

$1 - P(x < 1) > \frac{2}{3}$

$[1 - P(x=0)] > \frac{2}{3}$

$[1 - {}^nC_0 (1/4)^0 (3/4)^n] > 2/3$

$[1 - (3/4)^n] > \frac{2}{3}$

$1 - \frac{2}{3} > \left(\frac{3}{4}\right)^n$

$\frac{1}{3} > \left(\frac{3}{4}\right)^n$

$\therefore n = 4$

Ex The probability of failure in physics practical is 20%. If 25 batches of 6 students each take the examination than in how many batches 4 or more students would pass.

→ $q = 0.2$

$p = 0.8$

$n = 6$

$N = 25$

$P(x \geq 4) = P(x=4) + P(x=5) + P(x=6)$

$= [{}^6C_4 (0.8)^4 (0.2)^2] + [{}^6C_5 (0.8)^5 (0.2)^1] + [{}^6C_6 (0.8)^6 (0.2)^0]$

$= 0.901$

∴ Expected freq $= Np$

$= 25 \times 0.901$

$= 22.525$

≈ 23

∴ There are 23 batches where 4 or more students passed.

Ex An irregular six faced die is thrown. The probability that in 10 throws it will give 5 even number is twice as likley that it will give 4 even number. How many times 10000 sets 10 throws would you expect to give no even number.

⇒ $N = 10000$, $n = 10$, $p = ?$

x = getting even no.

$$P(x=5) = 2P(x=4)$$

$$^{10}C_5 (p)^5 (q)^5 = 2[^{10}C_4 (p)^4 (q)^6]$$

$$252 (p)^5 (q)^5 = 2[210 (p)^4 (q)^6]$$

$$252 (p)^5 (q)^5 = 420 \times (p)^4 \times (q)^6$$

$$252 (p)^5 \times 252 (q)^5 = 420 (p)^4 \times 840 (q)^6$$

$$252 (p)^5 \times 252 (1-p)^5 = 420 (p)^4 \times 840 (1-p)^6$$

$$\frac{252}{420} = \frac{(p)^4 (1-p)^6}{(p)^5 (1-p)^5}$$

$$\frac{252}{420} = \frac{(1-p)^6}{p(1-p)^5}$$

$$\frac{252}{420} = \frac{(1-p)}{p}$$

$$\frac{252}{420} = \frac{1}{p} - 1$$

$$\frac{252}{420} + 1 = \frac{1}{p}$$

$$\frac{1}{p} = \frac{8}{5}$$

$$p = \frac{5}{8}$$

$$q = 1 - p = 1 - \frac{5}{8} = \frac{3}{8}$$

$$P(x=0) = [^{10}C_0 (p)^0 (3/8)^{10}]$$

$$= 5.49 \times 10^{-5}$$

Expected freq $= 10000 \times 5.49 \times 10^{-5}$

$= 0.549$

≈ 1

Expected throws for not getting even number is 1

DOMS Page No.
Date / /

Ex The mean & variance of a Binomial Distribution are 3 & 1.2. Then find $P(X<4)$

→ mean = 3, var = 1.2

$\therefore np = 3$

$\therefore npq = 1.2$ $\qquad n(0.6)(0.4) = 1.2$

$3q = 1.2$ $\qquad 0.24n = 1.2$

$q = \frac{1.2}{3}$ $\qquad n = \frac{1.2}{0.24} = 5$

$q = 0.4$

$p = 0.6$

$P(X<4) = 1 - P(X \geq 4)$

$= 1 - [P(X=4) + P(X=5)]$

$= 1 - [{}^5C_4 (0.6)^4 (0.4)^1 + {}^5C_5 (0.6)^5 (0.4)^0]$

$= 0.66304$

Ex A Binomial ~~Dist~~ variant x satisfies $9P(x=4) = P(x=2)$ when $n=6$. Find the value of parameter p

→ $9P(x=4) = P(x=2)$

$9[{}^6C_4 (p)^4 (1-p)^2] = [{}^6C_2 (p)^2 (1-p)^4]$

$9[15(p)^4 (1-p)^2] = [15(p)^2 (1-p)^4]$

$9 = \frac{\cancel{15}(p)^2 (1-p)^4}{\cancel{15}(p)^4 (1-p)^2}$

$9 = \frac{(1-p)^2}{p^2}$

$9 = \frac{1 (1 - 2p + p^2)}{p^2}$

$$q = \frac{1}{p^2} - \frac{2}{p} + 1$$

$$8 = \frac{1}{p^2} - \frac{2}{p}$$

$$8 = \frac{p - 2p^2}{p^3}$$

$$8p^3 + 2p^2 - p = 0$$

$p = \frac{1}{4}$ or $p = 0$ or $p = \frac{-1}{2}$

$q = 1 - \frac{1}{4}$

$q = \frac{3}{4}$ //

Here p cannot be 0 or $\frac{-1}{2}$

Ex The ratio of probability of 3 successes in 5 independent trials to the probability of 2 successes in 5 independent trials is 1/4. What is the probability of 4 successes in 6 independent trials

→ $n = 5$

$$\frac{P(x=3)}{P(x=2)} = \frac{1}{4}$$

$$\frac{{}^5C_3\, p^3 q^2}{{}^5C_2\, p^2 q^3} = \frac{1}{4}$$

$$\frac{p}{q} = \frac{1}{4}$$

$p + q = 1$
$q = 1 - p$

$$\frac{p}{1-p} = \frac{1}{4}$$

$$4p = 1 - p$$

$$5p = 1$$

$p = 1/5 \quad \therefore q = 4/5$

Now, $n = 6$

$$P(x=4) = {}^6C_4 \, (1/5)^4 \, (4/5)^2$$

$$P(x=4) \doteq 0.01536$$

② Poisson ~~Poisson~~ Distribution : It is a limiting case of binomial distribution under the following conditions:

i) No. of trials is infinitely large i.e $n \to \infty$

ii) The probability of success in each trial is constant

iii) The average success is finite

mean = np = var

$$\boxed{P(X=x) = \frac{e^{-m} m^x}{x!}} \qquad x = 0, 1, 2, \ldots \quad (m > 0)$$

Ex A firm produces articles 0.1% of which are defective. If packed them in cases containing 500 articles & if a wholesaler purchases 100 such case, how many cases can be expected

i) To be free from defective

ii) To have 1 defective

→ $p = \frac{0.1}{100}$, $n = 500$, $N = 100$

i) $m = np = 500 \times \frac{0.1}{100} = 0.5$

$$P(X=0) = \frac{e^{-0.5}(0.5)^0}{0!}$$

$$P(X=0) = 0.6065$$

$\therefore$ Exp. no of cases to be free from defective $= Np$

$= 100 \times 0.6065$

$= 60.65$

≈ 61

ii) $P(X=1) = \frac{e^{-0.5}(0.5)}{1!}$

$= 0.3032$

$\therefore$ Exp no of cases having 1 defective article $= Np$

$= 100 \times 0.3032$

$= 30.32$

≈ 30

Ex Using posion distribution find approximate values

$$300C_2 (0.02)^2 (0.98)^{298} + 300C_3 (0.02)^3 (0.98)^{297}$$

$\rightarrow$ $300C_2 (0.02)^2 (0.98)^{298} = P(X=2)$

$300C_3 (0.02)^3 (0.98)^{297} = P(X=3)$

where $n = 300$, $p = 0.02$

$m = np = 300 \times 0.02$

$= 6$

$P(X=2) + P(X=3)$

$$= \frac{e^{-6}(6)^2}{2!} + \frac{e^{-6}(6)^3}{3!}$$

$= 0.1338$ //

Date / /

Ex Find the probability that ~~atleast~~ atmost 5 defective fuses will be found in a box of 200 fuses, if experience shows that 2% of such fuses are defective

$\Rightarrow$ $p = \frac{2}{100}$ $\quad P(x \leq 5)$ $\quad n = 200$ $\quad m = np = 200 \times \frac{2}{100}$

$m = 4$

$$P(x \leq 5) = P(X=0) + P(X=1) + P(X=2) + P(X=3) + P(X=4) + P(X=5)$$

$$= \frac{e^{-4}(4)^0}{0!} + \frac{e^{-4}(4)}{1!} + \frac{e^{-4}(4)^2}{2!} + \frac{e^{-4}(4)^3}{3!} + \frac{e^{-4}(4)^4}{4!} + \frac{e^{-4}(4)^5}{5!}$$

$$= e^{-4}\left[\frac{(4)^0}{0!} + \frac{(4)}{1!} + \frac{(4)^2}{2!} + \frac{(4)^3}{3!} + \frac{(4)^4}{4!} + \frac{(4)^5}{5!}\right]$$

$$= 0.7851$$

Ex The no. of accidents in a particular highway is a poisson variant with parameter 5. Find the probability that more than 2 accidents have occurred on the road that month.

$\Rightarrow$ $m = 5$

$$P(x > 2) = 1 - P(x \leq 2)$$

$= 1 - [P(x=0) + P(x=1) + P(x=2)]$

$$= 1 - \left[\frac{e^{-5}(5)^0}{0!} + \frac{e^{-5}(5)^1}{1!} + \frac{e^{-5}(5)^2}{2!} \right]$$

$= 1 - 0.875411$

Ex In a possion distribution $P(X=3)$ is $2/3$ of $P(x=4)$. Find mean & S.D

→ $P(X=3) = \frac{2}{3} P(x=4)$

$$\frac{P(x=3)}{P(x=4)} = \frac{2}{3}$$

$$\frac{e^{-m}(m)^3}{3!} = \frac{2}{3} \frac{e^{-m}(m)^4}{4!}$$

$$\frac{1}{3!} = \frac{2m}{3 \times 4 \times 3 \times 2}$$

$$\frac{1}{3 \times 2} = \frac{m}{3 \times 2 \times 3}$$

$$1 = \frac{m}{6}$$

$m = 6$

$Var = 6$

$S.D = \sqrt{Var} = \sqrt{6}$

Ex If x is a possion variant such that $P(x=1) = P(x=2)$. Find $E(x^2)$

→ $P(x=1) = P(x=2)$

* Normal Distribution

A continous random variable is set to follow normal distribution with the parameter m & σ.

If its p.d.f is given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2}$$

If x is a normal variant with parameters m & σ then $z = \frac{x-m}{\sigma}$ is also a normal variant with mean 0 & standard deviation 1 & it is called as standard normal variant.

~~Probability for x lying betw~~

$P(x_1 \leq x \leq x_2) = P(z_1 \leq x \leq z_2)$ = Area between $z = z_1$ & $z = z_2$

Ex If x is normally distributed with & S.D 4. Find
i) $P(5 \leq x \leq 10)$
ii) $P(x \geq 15)$
iii) $P(x \leq 15)$

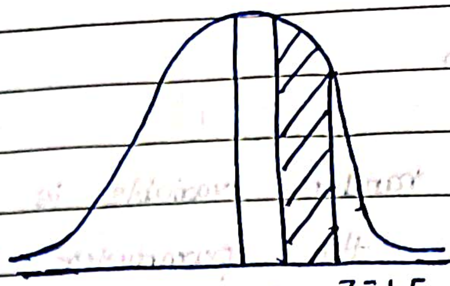
$\Rightarrow$ i) $m = 4$, $\sigma = 4$

$$z = \frac{x-m}{\sigma}$$

when $x = 5$, $z = \frac{5-4}{4} = \frac{1}{4} = 0.25$

when $x = 10$, $z = \frac{10-4}{4} = \frac{6}{4} = \frac{3}{2} = 1.5$

$\therefore P(5 \leq x \leq 10) = P(0.25 \leq z \leq 1.5)$



= (area betn z=0 to z=1.5)

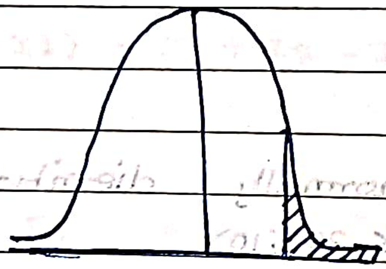
- (area betn z=0 to z=0.25)

= 0.4332 - 0.0987

= 0.3345

ii) $P(x \geq 15) =$

when $x = 15$, $z = \frac{15-4}{4} = \frac{11}{4} = 2.75$



= 0.5 - (area betn z=0 to z=2.75)

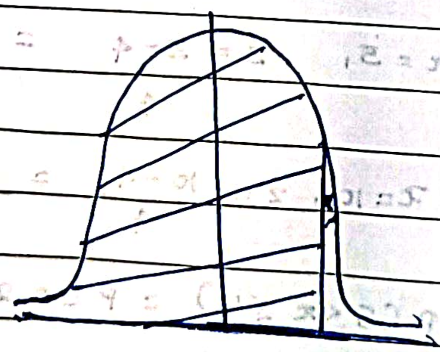
= 0.5 - (0.4970)

= 0.003

iii) $P(x \leq 15)$

when $x = 15$,

$z = 2.75$



$= 0.5 + (\text{area bet}^n\ z=0 \text{ to } z=2.75)$

$= 0.5 + (0.4970)$

$= 0.9970$

Ex The weight of 4000 students are found to be normally distributed with mean 50 Kg & S.D 5 Kg Find the probability that a student select at random will have weight

i) Less than 45 kg

ii) Between 45 & 60 kg

→ mean $= m = 50$ Kg

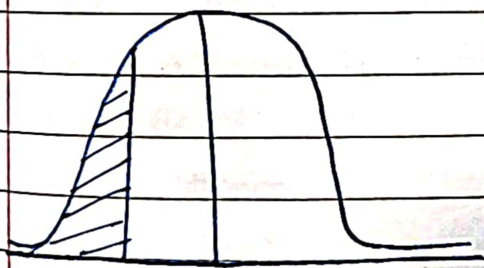
S.D $= 5$ Kg $= \sigma$

i) $P(x \leq 45)$

when $x = 45$,

$$z = \frac{x - m}{\sigma}$$

$$z = \frac{45 - 50}{5} = -1$$



$z=-1 \quad z=0$

$= 0.5 - (\text{area bet}^n\ 0 \ \& -1)$

$= 0.5 - 0.3413$

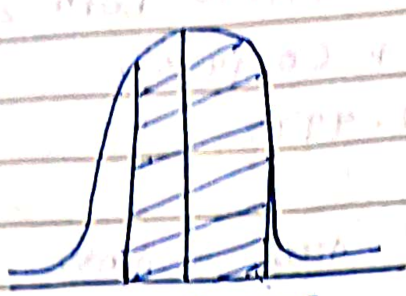
$= 0.1587$

ii) $P(45 \leq x \leq 60)$

when $x = 60$

$$z = \frac{60 - 50}{5}$$

$z = 2$



$z=-1 \quad z=0 \quad z=2$

$=$ (area betn -1 and 2)

$=$ (area betn -1 to 0) $+$ (area betn 0 to 2)

$= 0.3413 + 0.4772$

$= 0.8185$

Ex If x_1 & x_2 are 2 independent variables with mean 30 & 25 & variance 16 & 12 & if

if $y = 3x_1 - 2x_2$. Find $P(60 \le y \le 80)$?

$\Rightarrow$

$m_1 = 30 \qquad V_1 = 16 \qquad \sigma_1 = 4$

$m_2 = 25 \qquad V_2 = 12 \qquad \sigma_2 = \sqrt{12}$

$E(y) = E(3x_1 - 2x_2)$

$= 3E(x_1) - 2E(x_2)$

$= 3(30) - 2(25)$

$= 90 - 50$

$= 40$

. Mean of $Y = 40$

$var(Y) = var(3x_1 - 2x_2)$

$= 9\,var(x_1) - 4\,var(x_2)$

$= 9(16) - 4(12)$

$= 192$

$\sigma_y = \sqrt{192} = 13.85$

S.D of Y is 13.85

when $y = 60$

$$z = \frac{60-40}{13.85}$$

$$z = 1.44$$

when $y = 80$

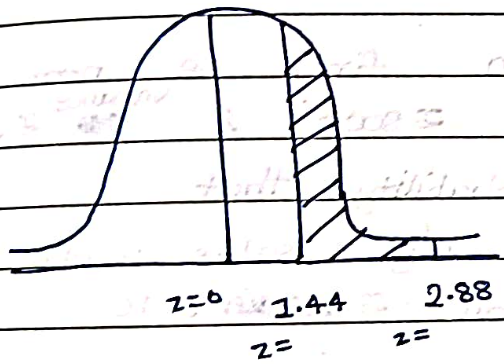
$$z = \frac{80-40}{13.85}$$

$$z = 2.888$$

$P(60 \leq y \leq 80) = P(1.44 \leq z \leq 2.88)$

= (Area betn 0 to 2.88)

− (Area betn z=0 to 1.44)



$= 0.4980 - 0.4251$

$= 0.0729$

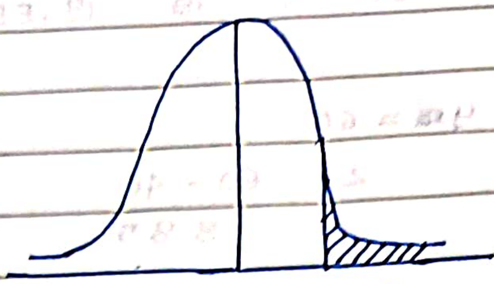
Q. The height of 1000 soldiers in a regiment are distributed normally with mean 172 cm & S.D 5cm. How many soldiers have height greater than 180 cm.

→ $m = 172$ $S.D = \sigma = 5$

~~when~~ $P(x \geq 180) = ?$

when $x = 180$

$$z = \frac{180-172}{5} = 1.6$$



$= 0.5 - (\text{area bet}^n \ 0 \text{ to } 1.6)$

$= 0.5 - 0.4452$

$P(x \geq 180) = 0.0548 \approx 0.055$

No. of soldiers $= 0.055 \times 1000$

$= 55$ soldiers

Ex - The daily sale of a firm is normally distributed with mean ₹ 8000 & variance ₹ 10000

i) what is the probability that on a certain day sales would be less ~~than~~ than ₹ 8210

ii) What is the % of days on which sales will be between ₹ 8100 & ₹ 8200

⇒ i) $m = 8000$

$var = 10{,}000$

$\sigma = \sqrt{var}$

$= \sqrt{10000}$

$= 100$

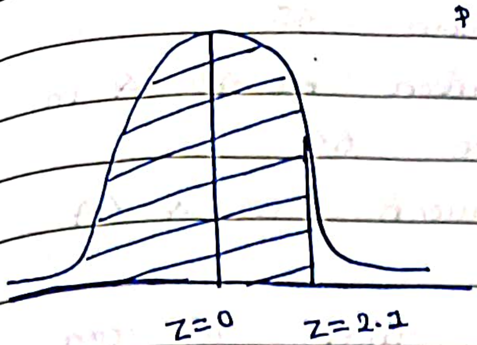
i) $P(x < 8210)$

→ when $x = 8210$

$z = \frac{x - m}{\sigma} = \frac{8210 - 8000}{100}$

$z = 2.1$

$\therefore P(x < 8210) = P(z < 2.1)$



$P(z < 2.1) = 0.5 + (\text{Area bet}^n\ z=0 \text{ to } z=2.1)$

$= 0.5 + 0.4821$

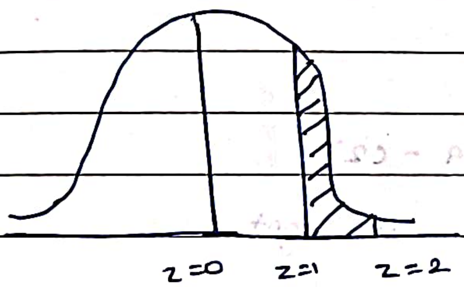
$P(z < 2.1) = 0.9821$

ii) $P(8100 < x < 8200)$

when $x = 8100$, $z = \dfrac{8100 - 8000}{100} = 1$

when $x = 8200$, $z = \dfrac{8200 - 8000}{100} = 2$

$\therefore P(8100 < x < 8200) = P(1 < z < 2)$



$= (\text{Area bet}^n\ z=0 \text{ to } z=2) - (\text{Area bet}^n\ z=0 \text{ to } z=1)$

$= 0.4772 - 0.3413$

$= 0.1359$

$\therefore$ % of Days on which sales is between 8100 & 8200

$= 0.1359 \times 100$

$= 13.59\ \%$

Ex For a normal distribution, 30% of items are below 45 & 8% items are above 64. Find mean & variance of normal distribution

→ Since 30% items are below 45

∴ 20% items are between 45 & m

Since 8% items are above 64

∴ 42% items are between m & 64

From the table we find that 0.2 area corresponds to $z = 0.52$ & 0.42 area corresponds to $z = 1.41$

But, 0.2 area is to the left of m

Hence, $z = -0.52$

$$z = \frac{x-m}{\sigma} \quad \therefore -0.52 = \frac{45-m}{\sigma}$$

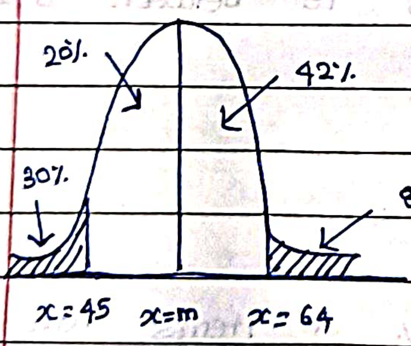
$$\therefore m - 0.52\sigma = 45 \quad -(1)$$

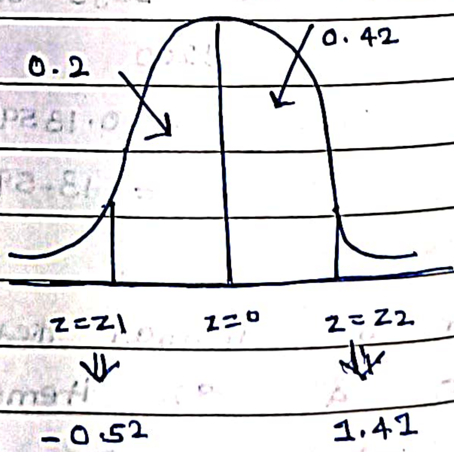
$$1.41 = \frac{64-m}{\sigma}$$

$$\therefore m + 1.41\sigma = 64 \quad -(2)$$

Solving eqn (1) & (2) we get

$$m = 50.11 \quad \sigma = 9.84 \quad \therefore var = \sigma^2 = 96.82$$





In an intelligence test of 1000 students the average score was 42 & S.D was 24 which is normally distributed. Find the number of students

i) Exceeding 50

ii) Between 30 & 54

iii) The least score of top 100 students

$m = 42$

$\sigma = 24$

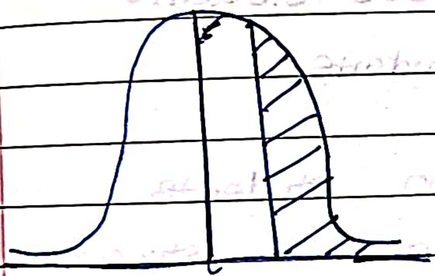
i) ~~when~~ $= P(x > 50)$

$= P(x > 50)$

when $x = 50$

$$z = \frac{50-42}{24}$$

$$z = 0.33$$



z=0 z=0.33

$P(z > 0.33) = 0.5 -$ (area betn 0 to 0.33)

$= 0.5 - 0.1293$

$= 0.3707$

No. of students having score more than 50 $= 0.3707 \times 1000$

$= 370.7$

≈ 371

ii) $P(30 \le x \le 54)$

when $x = 30$

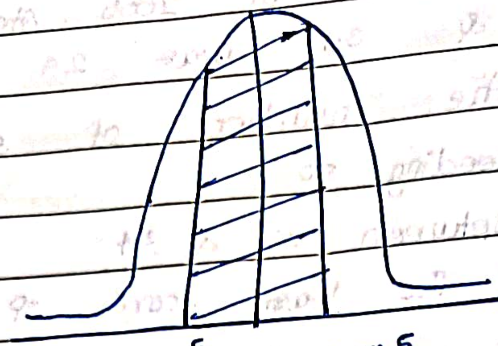
$$z = \frac{30-42}{24}$$

$= -0.5$

$x = 54$

$$z = \frac{54-42}{24}$$

$= 0.5$



$P(-0.5 \leq z \leq 0.5) = 2 * (\text{area betn } 0 \text{ to } 0.5)$

$= 2 * 0.1915$

$= 0.383$

No. of students between 30 & 54 $= 0.383 \times 1000$

$= 383$ students

iii) Least score of top 100 students

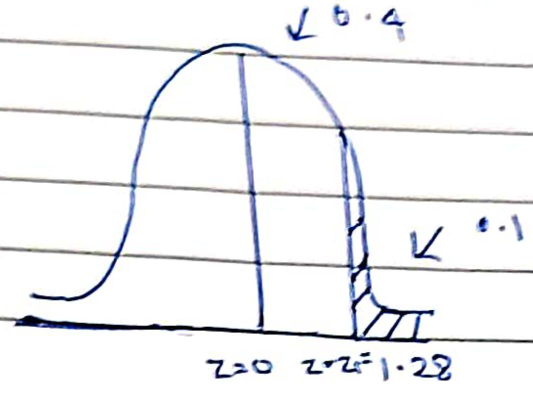
If we consider top 100 students Then the probability that a student selected at random would be one of them is $\frac{100}{1000}$ i.e 0.1. This is a reverse problem

Here we have to find the value of z to the right of which the area is 0.1
∴ Area between z=0 to this value of $z = 0.5 - 0.1 = 0.4$

From the table, for the area 0.4 The value of z is 1.28

$z = \frac{x-m}{\sigma}$ ∴ $1.28 = \frac{x-42}{24}$

$x = 72.72$


0.4
0.1
z=0
z=zα=1.28