CS 471 Homework 4

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## Abstract

In this assignment we compute derivative and integrals on a logically rectangular domain $\Omega$. We map a reference element $\Omega_{\mathrm{r}} = [-1,1]^2$ to $\Omega$ to preform computations of derivative and integrals using the reference element.

## Approximating Differentiation and Integrals

Given smooth mapping $(x, y) = (x(r, s), y(r, s))$, we use the reference element to compute

$$\frac{\partial u}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial u}{\partial r} + \frac{\partial s}{\partial x}\frac{\partial u}{\partial s}$$

$$\frac{\partial u}{\partial y} = \frac{\partial r}{\partial y}\frac{\partial u}{\partial r} + \frac{\partial s}{\partial y}\frac{\partial u}{\partial s}$$

Gives the formula:

$$\begin{bmatrix} r_x & r_y \\ s_x & s_y \end{bmatrix}\begin{bmatrix} x_{\mathrm{r}} & x_s \\ y_{\mathrm{r}} & y_s \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

We take the inverse matric $x_{\mathrm{r}}, x_s, y_{\mathrm{r}}, y_s$ which will give the metric $r_x, r_y, s_x, s_y$.

We take reference element u=1, and map it into a circle to obtain the functions:

$$x = \frac{r+1}{2}\cos\big(\pi(s+1)\big)$$

$$y = \frac{r+1}{2}\sin(\pi(s+1))$$

Taking the double integral over our domain of our reference element as a function of r and s:

$$\int_{\Omega} f(x,y)dx\,dy = \int_{-1}^{1}\int_{-1}^{1} f\big(x(r,s),y(r,s)\big)\,J(r,s)\,dr\,ds$$

Where $J(r,s) = \begin{vmatrix} x_{\mathrm{r}} & x_s \\ y_{\mathrm{r}} & y_s \end{vmatrix}$

Approximating the integral with the trapezoidal rule gives us the area of 3.440678.

The exact area being π.

To understand the error better, we try 3 functions on different mappings and approximate the error using the formula:

$$e(h_r,h_s) = (\ \int_{\Omega}\ \big(u_x(x,y) + u_y(x,y)\big) - [(u_{exact})\ _x + (u_{exact})\ _y])^2\ dxdy\ )^{\frac{1}{2}})$$

The error is plotted as a function of the effective grid size given by

$$h_{eff} = \sqrt{h_r h_s\, max\, J}$$

**Results**

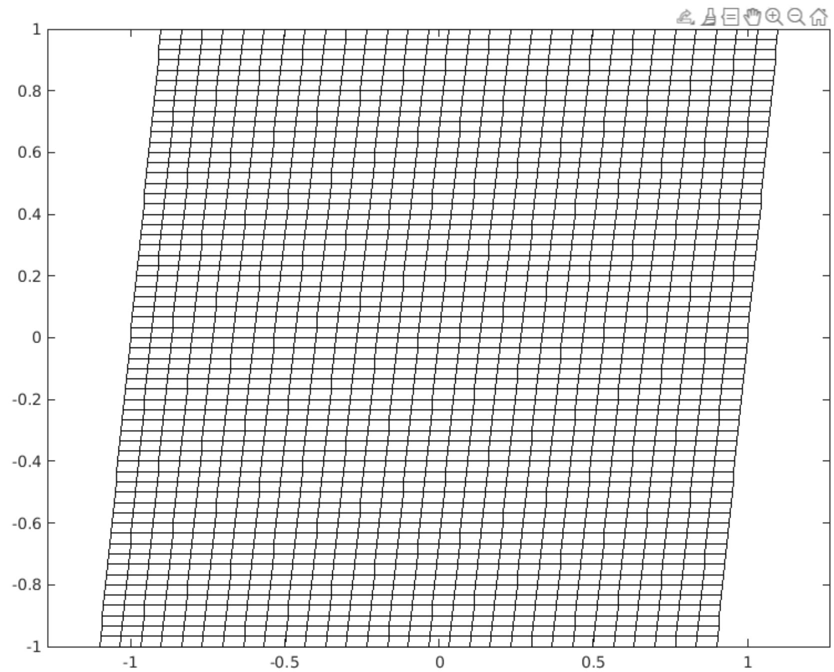
The plots of the curvilinear grids of $x, y$ and the differentiation errors of the numerical computations of $u_x$ and $u_y$ are given below. We can see that the error approaches closely to $O(h^2)$. The error lines is close to being parallel to $h^2$ on the $log\ log$ scales but it diverts when we approach to larger value. Our finding shows that our differentiation method is close to second order.

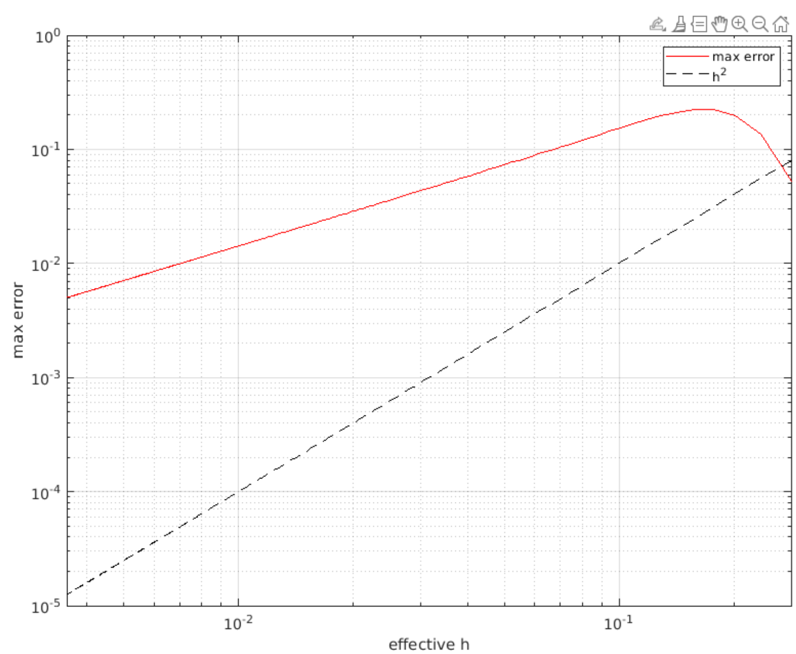
1)

$$x = r + 0.1\,s$$

$$y = s$$

$$u = sin(x)cos(y)$$



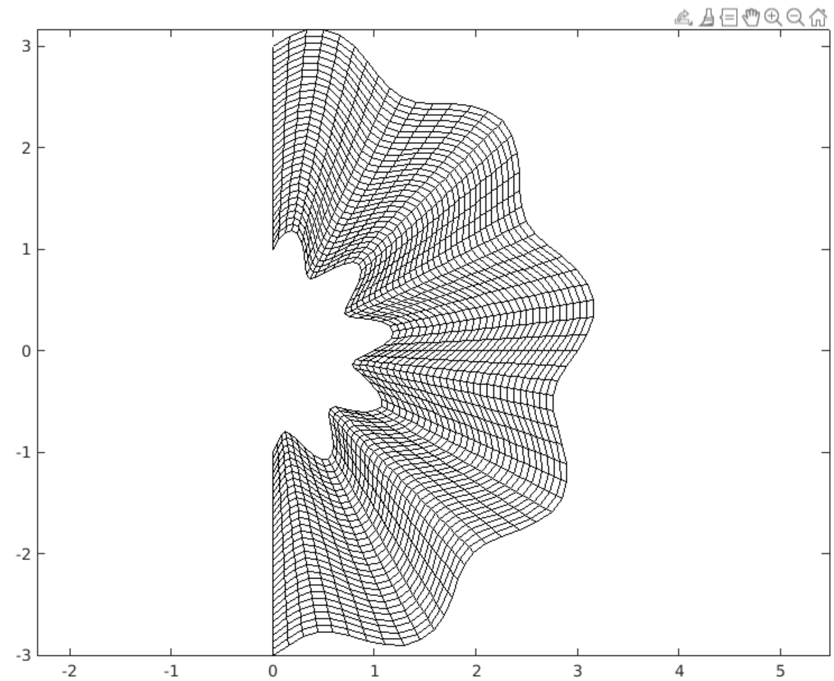



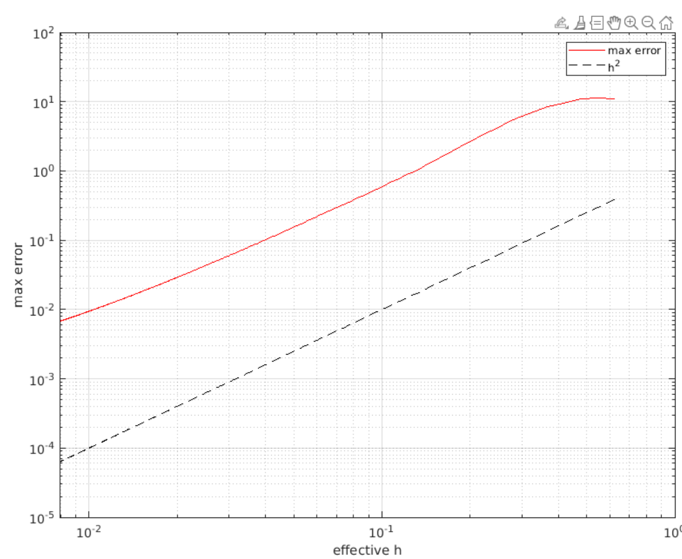
2)

$$x = 2 + r + 0.1\ sin(5\ \pi\ s)cos(0.5\ \pi\ s)$$

$$y = 2 + r + 0.1sin\ (5\ \pi\ s)sin(0.5\pi\ s)$$

$$u = e^{x+y}$$



3)

$x = r$

$y = s + s(r)^2$

$u = x^2 + y^2$