Hansen (1985) - Indivisible Labor and the Business Cycle

This replication is a copy of that in Kirkby (2017), but with more grid points.

The VFI Toolkit is now used to replicate the results of Hansen (1985). This turns out to be an interesting illustration of the influence of numerical methods on Macroeconomics. Hansen models the process on the productivity shock as an AR(1) with log-normal innovations; this works because linear-quadratic dynamic programming is used as the solution method, so the distribution of innovations is irrelevant. Many solution methods are simply unable to deal with the nonlinearities and asymmetries involved in modeling log-normal innovations. Since the VFI Toolkit is based on using robust global solution methods it is able to handle log-normal innovations, and as seen in the replication results the assumption of log-normal innovations, when modelled in full, has important implications.

The choice of productivity shock process also turns out to be an interesting illustration of the influence of numerical methods on Macroeconomics. Hansen (1985) models the process on the productivity shocks as an AR(1) with log-normal innovations, chosen as this way the productivity process is never negative. In contrast most models chose to model productivity as the having the log of productivity being AR(1) with normally distributed innovations.[1] The latter approach is more common not because it is considered in any way more realistic but because it can be more easily handled using common numerical methods, such as perturbation, which don't deal well with substantial asymmetries or nonlinearities. This shows how the widespread use of certain numerical methods, especially those only able to solve certain kinds of models, has had a strong influence on Macroeconomic modeling; including on quantitative Macroeconomic models which are commonly thought to be less susceptible to strict functional form restrictions than purely analytical models.

Before proceeding to the replication itself I give a quick explanation of why Hansen (1985)'s choice of using linear-quadratic dynamic programming to solve the model allowed modelling the process on the productivity shocks as an AR(1) with log-normal innovations: With linear-quadratic dynamic programming the assumption of log-normal innovations is irrelevant as only the conditional mean of the productivity shocks matters.[2] That only the conditional mean of variables matters is also true of first-order perturbation methods (with second-order perturbations the conditional second-moments (and hence variance) also matter, but not higher moments, etc.). While linear-quadratic dynamic programming can solve the model with log-normal innovations it implicitly involves assuming that the log-normal distribution of those shocks is entirely irrelevant. As discussed below the replication results show this is in fact untrue.

0.0.1 Model

I directly present the model as the value function problem to be solved. Readers interested in knowing the motivation behind looking at these models are referred to Hansen (1985).[3] There are two models, one with 'divisible labor', the other 'indivisible labor'. The only difference mathemat-

[1] Ie. $z_t = \rho z_{t+1} + \epsilon_t$, where $\epsilon \sim logN(0, \sigma_\epsilon)$, versus the more common $log(z_t) = \rho log(z_{t-1}) + \epsilon_t$, where $\epsilon \sim N(0, \sigma_\epsilon)$. Both satisfy that the shock process is strictly non-negative, which can be important for theoretical reasons.

[2] For more on linear-quadratic dynamic programming, see Díaz-Gímenez (2001). Since the log-normal innovations are independent and identically distributed, the conditional mean of the innovations is just equal to their unconditional mean; ie. because of the choice of numerical method only the mean value of the innovations is relevant. Hence the actual log-normality of the innovations is entirely ignored by the solution method used.

[3] Hansen (1985) uses slightly different notation, α he calls θ, z_t he calls λ_t, and ρ he calls γ.

ically is in the utility function.[4] The value function problem to be solved, with a general utility function, is given by

$$
\begin{aligned}
V(k,z) = \max_{c,i,k',y,h} & \; u(c,h) + \beta E[V(k',z')|z] \\
\text{s.t.} \quad & y = zk^{\alpha}h^{1-\alpha} \\
& c + i = y \\
& k' = i + (1-\delta)k \\
& z' = \rho z + \epsilon'
\end{aligned}
$$

where $\epsilon \sim log - normal$ with mean $1-\rho$ and standard deviation σ_ϵ.[5]. k is capital stock, z is productivity shock, c is consumption, y is output, and h is hours worked. The divisible labor economy has utility function $u(c,h) = log(c) + Alog(1-h)$, while the indivisible labor economy has utility function $u(c,h) = log(c) + B(1-h)$.

Hansen (1985) calibrates the model to quarterly US data for the period 1955:Q3-1984:Q1. This leads to the parameter values $\alpha = 0.36$, $\delta = 0.025$, $\beta = 0.99$, $A = 2$, $h_0 = 0.53$, $\rho = 0.95$, and $\sigma_\epsilon = 0.00712$; with $B = -Alog(1-h_0)/h_0$.

I also present results for an 'alternative productivity process' using the more standard approach of modelling the log of productivity as being an AR(1) process with normally distributed innovations. This method is popular as it both maintains the required assumption that productivity is always positive, and is easier to implement by using the Tauchen method to approximate the log of productivity. The parameters for this alternative productivity process are chosen to ensure that it has the same mean and variance as the original productivity process.

0.0.2 Implementation

The only difficulty arose from the productivity shocks being AR(1) with a log-normal innovation. Standard quadrature methods, such as Tauchen, cannot be used for such a shock process.[6] A large fraction of the lines of code involved in replication thus involve dealing with this. It was implemented by first simulating a lengthy time series for the productivity shocks, then using Matlab's built-in histogram routines to divide this into a grid, and then creating the transition matrix by simply counting transitions and the normalising the resulting matrix.[7]

The codes implementing the model can be found at github.com/vfitoolkit/vfitoolkit-matlab-replication. The grids used are 101 points on the hours worked choice, evenly spaced from zero to one. 501 points on the next periods capital choice, half evenly spaced from zero to steady state capital level, half evenly spaced from zero to twice the steady state capital level. 31 points on the productivity shock, either chosen by the quadrature method described above for the case of log-normal innovations, or using Tauchen method with q=3 for the alternative case of log productivity being AR(1) with normally distributed innovations. These grid sizes seem adequate for convergence,

[4]The 'indivisible labor' economy uses lotteries to ensure that the aggregate labor supply looks divisible.

[5]So $exp(\epsilon)$ is distributed as $N(\mu_{ln}, \sigma_{ln}^2)$, where $\mu_{ln} = log(1-\rho) - \sigma_{ln}^2/2$, $\sigma_{ln} = \sqrt{log(1 + \frac{\sigma_\epsilon^2}{(1-\rho)^2}}$

[6]One could use the Tauchen method to discretize the (exponential of) the lognormal innovation itself and then take the log. This would involve an increase in the state space; to z and ϵ instead of just z. In combination with the use of pure discretization it also necessitates then creating another grid for the AR(1) productivity shocks themselves, and finding the transition matrix for this shock.

[7]Counting transitions and then normalizing is a standard and well behaved estimator for Markov transition matrices. The choice of using histogram routines to choose the grid was an arbitrary assumption.

as assessed by comparing the results with a reduction of the grids to 31-351-21 points, respectively (viewed another way the limitation to 100 simulation samples, done following Hansen (1985), provides at least as much noise as the grid size).

Table 1: Replication of Table 1 from Hansen (1985)
Standard deviations in percent (a) and correlations with output (b) for US and artificial economies.

	Quarterly U.S. Time Series[a] 1955:Q3-1984:Q1		Economy with divisible labor[b]		Economy with indivisible labor[b]	
Series	(a)	(b)	(a)	(b)	(a)	(b)
Output	1.76	1.00	1.37 (0.17)	1.00 (0.00)	1.80 (0.23)	1.00 (0.00)
Consumption	1.29	0.85	0.49 (0.06)	0.70 (0.05)	0.54 (0.08)	0.78 (0.03)
Investment	8.60	0.92	4.49 (0.53)	0.97 (0.01)	5.90 (0.72)	0.98 (0.00)
Capital Stock	0.63	0.04	0.37 (0.08)	0.06 (0.07)	0.49 (0.10)	0.05 (0.07)
Hours	1.66	0.76	0.77 (0.09)	0.95 (0.01)	1.42 (0.17)	0.97 (0.01)
Productivity	1.18	0.42	0.69 (0.09)	0.94 (0.01)	0.53 (0.08)	0.79 (0.03)

Table 2: Original Table 1 from Hansen (1985)
Standard deviations in percent (a) and correlations with output (b) for US and model economies.

	Quarterly U.S. Time Series[a] 1955:Q3-1984:Q1		Economy with divisible labor[b]		Economy with indivisible labor[b]	
Series	(a)	(b)	(a)	(b)	(a)	(b)
Output	1.76	1.00	1.35 (0.16)	1.00 (0.00)	1.76 (0.21)	1.00 (0.00)
Consumption	1.29	0.85	0.42 (0.06)	0.89 (0.03)	0.51 (0.08)	0.87 (0.04)
Investment	8.60	0.92	4.24 (0.51)	0.99 (0.00)	5.71 (0.70)	0.99 (0.00)
Capital Stock	0.63	0.04	0.36 (0.07)	0.06 (0.07)	0.47 (0.10)	0.05 (0.07)
Hours	1.66	0.76	0.70 (0.08)	0.98 (0.01)	1.35 (0.16)	0.98 (0.01)
Productivity	1.18	0.42	0.68 (0.08)	0.98 (0.01)	0.50 (0.07)	0.87 (0.03)

[a] The US time series used are real GNP, total consumption expenditures, and gross private domestic investment (all in 1972 dollars). The capital stock series includes non-residential equipment and structures. The hours series includes total hours for persons at work in non-agricultural industries as derived from the *Current Population Survey*. Productivity is output divided by hours. All series are seasonally adjusted, logged and detrended.
[b] The standard deviations and correlations with output are sample means of statistics computed for each of 100 simulations. Each simulation consists of 115 periods, which is the same number of periods as the US sample. The numbers in parentheses are sample standard deviations of these statistics. Before computing any statistics each simulated time series was logged and detrended using the same procedure used for the US time series.
[c] Hansen (1985) models productivity as an AR(1) with log-normal innovations. The 'Alternative Productivity process' models the log of productivity as an AR(1) with normal innovations.

References

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