Structural Macroeconometrics

Chapter II
Basic RBC Model

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Outline

Dynamic Stochastic General Equilibrium Models

- RBC models belong to the larger class of **DSGE** models. Because today's investment will give returns tomorrow, these models are intertemporal and thus **dynamic**
- The economy is modeled as being subject to shocks, therefore the models are **stochastic**
- Not one market is looked at in isolation, but goods, capital and labor market simultaneously, i.e. only few variables are left exogenous. Thus, we have **general equilibrium** models (in contrast to partial equilibrium)
- Because of the general equilibrium structure of the model, it becomes difficult to think in the traditional framework of demand and supply, because most shocks have effects on both!

Cookbook for DSGE Models

1. **Setup: Describe the economy, i.e. technological and resource constraints, exogenous influences on the system, institutional settings.**
2. **First-Order Conditions:** Find the optimal behavior of agents (here: HHs and firms). This means setting up maximization problems of the agents and taking derivatives with respect to all variables that the agents maximize over, i.e. the control variables
3. **Solving:** Use the first-order conditions and the exogenous variables to find a solution for all variables at all times.

Problem Setup

- The HH maximizes the utility function

$$\max_{\{c_t, k_{t+1}, l_t\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t \left(\frac{c_t^{1-\sigma} - 1}{1-\sigma} + \psi \log(1 - l_t) + \theta g_t \right) \tag{1}$$

- subject to the budget constraint

$$c_t + i_t = w_t l_t + R_t k_t - T_t \; \forall \, t, \tag{2}$$

- the LOM for capital with $\gamma_x = 1 + n + g_y + n g_y$

$$\gamma_x k_{t+1} = (1-\delta) k_t + i_t \; \forall \, t, \tag{3}$$

- an initial value for capital: $k_0 > 0$,
- the government budget constraint

$$T_t = g_t \; \forall \, t, \tag{4}$$

- and the transversality constraint

$$\lim_{t \to \infty} E_0 \beta^t \lambda_t k_{t+1} = 0 \tag{5}$$

Problem Setup II

- Additionally, the HH takes into account the stochastic LOMs for z_t and g_t

$$g_t = g^* e^{\hat{g}_t} \tag{6}$$

$$\hat{g}_t = \rho_g \hat{g}_{t-1} + \varepsilon_t^G, \varepsilon_t^G \overset{iid}{\sim} \left(0, \sigma_G^2\right) \tag{7}$$

$$A_t = A^* e^{z_t} \tag{8}$$

$$z_t = \rho_z z_{t-1} + \varepsilon_t, \varepsilon_t \overset{iid}{\sim} \left(0, \sigma_\varepsilon^2\right) \tag{9}$$

- Finally, we have non-negativity constraints

$$k_t \geq 0 \tag{10}$$

$$c_t \geq 0 \tag{11}$$

$$0 \leq l_t \leq 1 \tag{12}$$

- Due to our assumptions, we will not have corner solutions and can ignore the non-negativity constraints

Variable Classification

State Variables:

- exogenous: A_t (or z_t), g_t (or $\hat{g}_t$), k_0
- endogenous: k_t

Control Variables: c_t, k_{t+1}, l_t

Other variables: w_t, R_t, z_t (or A_t), $\hat{g}_t$ (or g_t), (i_t, T_t)

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3. **Solving:** Use the first-order conditions and the exogenous variables to find a solution for all variables at all times

After having collected all the equations and assumptions describing the setup, let's turn to the first-order conditions.

Problem Setup

- This yields the Lagrangian

$$\mathcal{L} = E_0 \sum_{t=0}^{\infty} \beta^t \begin{Bmatrix} \frac{c_t^{1-\sigma} - 1}{1-\sigma} + \psi \log(1 - l_t) + \theta g_t \\ -\lambda_t \left(c_t + \gamma_x k_{t+1} - (1-\delta) k_t - w_t l_t - R_t k_t + g_t\right) \end{Bmatrix}$$

with the First Order Conditions

$$\frac{\partial L}{\partial c_t} = E_t \beta^t \left(c_t^{-\sigma} - \lambda_t\right) = 0 \Rightarrow c_t^{-\sigma} = \lambda_t \; \forall \; t \tag{13}$$

$$\frac{\partial L}{\partial l_t} = E_t \beta^t \left\{-\psi \frac{1}{1-l_t} + \lambda_t w_t\right\} = 0 \Rightarrow \psi \frac{1}{1-l_t} = \lambda_t w_t \; \forall \; t \tag{14}$$

$$\frac{\partial L}{\partial k_{t+1}} = E_t \beta^t \left\{-\lambda_t \gamma_x + \beta \lambda_{t+1} \left((1-\delta) + R_{t+1}\right)\right\} = 0 \; \forall \; t \tag{15}$$

$$\Rightarrow \lambda_t = \beta E_t \lambda_{t+1} \frac{(1-\delta) + R_{t+1}}{\gamma_x} \tag{16}$$

- plus the BC and the TV-constraint

The First Order Conditions

- The first order conditions can be written to yield

$$c_t^{-\sigma} = \beta E_t c_{t+1}^{-\sigma} \frac{(1-\delta) + R_{t+1}}{\gamma_x} \equiv \frac{\beta}{\gamma_x} E_t c_{t+1}^{-\sigma} (1 + r_{t+1}) \tag{17}$$

$$\psi \frac{1}{1-l_t} = c_t^{-\sigma} w_t \tag{18}$$

- (17) is the familiar Euler equation where the agent looks at expected returns to postponing consumption (more later)
- Due to additive separability, the labor supply decision does not impact on the consumption decision
- (18) can be interpreted as the labor supply equation, indicating the amount of labor supplied for a given real wage
- It states that the **marginal rate of substitution between consumption and leisure** equals the real wage
- Note: Due to additive separability, government spending plays no role in the **marginal** consumption and work decision

Firm Problem

- We assume a competitive representative firm
- The firm problem is static:

$$\max_{\{l_t,k_t\}} A^* e^{z_t} k_t^\alpha l_t^{1-\alpha} - w_t l_t - R_t k_t \tag{19}$$

- The resulting first order conditions again state that factors are paid their marginal products

$$w_t = (1-\alpha) A^* e^{z_t} \left(\frac{k_t}{l_t}\right)^\alpha \tag{20}$$

$$R_t = \alpha A^* e^{z_t} \left(\frac{k_t}{l_t}\right)^{\alpha-1} \tag{21}$$

Market Clearing

- Labor Market: labor supply (18) and demand (20) must be equal

$$(1-\alpha)A^* e^{z_t} \left(\frac{k_t}{l_t}\right)^\alpha = \psi \frac{1}{1-l_t} c_t^\sigma \tag{22}$$

- Capital market: from (17) and (21) follows

$$c_t^{-\sigma} = \beta E_t \left\{ c_{t+1}^{-\sigma} \frac{\alpha A^* e^{z_{t+1}} \left(\frac{k_{t+1}}{l_{t+1}}\right)^{\alpha-1} + (1-\delta)}{\gamma_x} \right\} \tag{23}$$

- Finally, the goods market equilibrium requires:

$$c_t + \gamma_x k_{t+1} - (1-\delta)k_t + g^* e^{\hat{g}_t} = A^* e^{z_t} k_t^\alpha l_t^{1-\alpha} \tag{24}$$

Characterizing the Solution

- The equilibrium can be characterized by two equations:
  1. The static labor market clearing condition
$$(1-\alpha)A_t\left(\frac{k_t}{l_t}\right)^{\alpha}=\psi\frac{1}{1-l_t}\left[A_t k_t^{\alpha} l_t^{1-\alpha}-\gamma_x k_{t+1}+(1-\delta)k_t-g_t\right]^{\sigma} \tag{22}$$
  2. The dynamic equation resulting from plugging (24) into (23)
$$\begin{aligned}&\left[A_t k_t^{\alpha} l_t^{1-\alpha}-\gamma_x k_{t+1}+(1-\delta)k_t-g_t\right]^{-\sigma}=\\ &\frac{\beta}{\gamma_x}E_t\left\{\begin{array}{c}\left[A_{t+1}k_{t+1}^{\alpha}l_{t+1}^{1-\alpha}-\gamma_x k_{t+2}+(1-\delta)k_{t+1}-g_{t+1}\right]^{-\sigma}\\ \left(\alpha A_{t+1}\left(\frac{k_{t+1}}{l_{t+1}}\right)^{\alpha-1}+(1-\delta)\right)\end{array}\right\}\end{aligned} \tag{25}$$

- Solving (22) for l_t and plugging into (25) results in a **second-order difference equation** that together with the exogenous LOMs and the TV/No-Ponzi-condition characterizes the solution

Equilibrium

- An equilibrium is an allocation $\{c_t, l_t, k_{t+1}\}_{t=0}^{\infty}$ where
  1. given the exogenous state variables $\{z_t, g_t\}_{t=0}^{\infty}$ ((6)-(7) and (8)-(9)) and k_0
  2. HH maximize their utility ((17) and (18) plus the TVC (5))
  3. Firms maximize their profits ((21) and (20))
  4. Markets clear ((22) (23), and (24))
- In our case, firms and households are price takers and have rational expectations: **Competitive Rational Expectations Equilibrium**
- Problem: Above equations only describe solution implicitly

Cookbook for DSGE Models

1. **Setup:** Describe the economy, i.e. technological and resource constraints, exogenous influences on the system, institutional settings. Here: PF, law of motion for of capital, resource constraint, government and technology, # of HHs and firms.
2. **First-Order Conditions:** Find the optimal behavior of agents (here: HHs and firms). This means setting up maximization problems of the agents and taking derivatives with respect to all variables that the agents maximize over, i.e. the control variables.
3. **Solving: Use the first-order conditions and the exogenous variables to find a solution for all variables at all times.**

Let's turn back to the path leading to the solution of the model.

Recap: Variables and Equations

- Counting leads to **5 equations**:

$$\hat{g}_t = \rho_g \hat{g}_{t-1} + \varepsilon_t^G, \varepsilon_t^G \overset{iid}{\sim} \left(0, \sigma_G^2\right) \tag{7}$$

$$z_t = \rho_z z_{t-1} + \varepsilon_t, \varepsilon_t \overset{iid}{\sim} \left(0, \sigma^2\right) \tag{9}$$

$$\psi \frac{1}{1-l_t} c_t^{\sigma} = (1-\alpha) A^* e^{z_t} \left(\frac{k_t}{l_t}\right)^{\alpha} \tag{22}$$

$$c_t^{-\sigma} = \frac{\beta}{\gamma_x} E_t \left\{ c_{t+1}^{-\sigma} \left\{ \alpha A^* e^{z_{t+1}} \left(\frac{k_{t+1}}{l_{t+1}}\right)^{\alpha-1} + (1-\delta) \right\} \right\} \tag{23}$$

$$A^* e^{z_t} k_t^{\alpha} l_t^{1-\alpha} = c_t + \gamma_x k_{t+1} - (1-\delta) k_t + g^* e^{\hat{g}_t} \tag{24}$$

- These are **5 unique variables**
  - Control variables: c_t, l_t, k_{t+1}
  - Endogenous state: k_t
  - Exogenous (state) variables: $z_t, \hat{g}_t$

Outline

Solving the Model: The Problem

- We have a system with the same number of equations as unknowns
- There are two problems:
  1. The equations are nonlinear
  2. The system is dynamic; the Euler equation is forward looking, the LOM of capital backward looking
- The second problem can be overcome by mathematical methods (remember saddle path to get c_t in the Ramsey model), the first problem is more difficult
- If the system is not convertible to a linear one, we do not know much about its solution
- There are three ways out
  1. Use **tricks** that lead to simplifications.
  2. Leave the system non-linear, and let the computer solve it (**numerical solutions**)
  3. **Linearize** the system, i.e. take linear approximations that are 'good enough' when the variables are close to the steady-state.

The Fundamental Problem

- The solution to the RBC model is characterized by a stochastic higher order non-linear difference equation
- We would like to have an explicit solution for all given states of the system, i.e. given the state of nature (state variables including shocks) we would like to directly compute the control variables

Solution Concepts

- The solution is in principle **“global”**, that is, policy function h specified over the entire state space
- However, obtaining a global solution is often difficult
- Idea: instead approximate complicated problem with an easier one that is **“locally”** accurate
- In the following we rely on a local approximation of h around a particular point of the state space

Local Approximate Solution

- We will use a first-order Taylor approximation
- Advantages
  - Robust against curse of dimensionality
  - Often a good compromise between speed and accuracy
  - Locally very accurate, provided
    - **well-defined approximation point**
    - policy function is **sufficiently smooth** in the approximation point
- Point of approximation: **Deterministic Steady State**, i.e. resting point of the system without shocks for $t \to \infty$ (and where agents take the absence of uncertainty into account)
- Denote steady state values without subscripts: c, k, etc.
- As before, variables are expressed in percentage deviations from steady state

$$\hat{x}_t = \frac{x_t - x}{x} \tag{26}$$

Linearized Equations

- The resource constraint in linearized form is:

$$\hat{k}_{t+1} = \frac{1}{\gamma_x}\left[\alpha\frac{y}{k} + (1-\delta)\right]\hat{k}_t + \frac{y}{\gamma_x k}\left(z_t + (1-\alpha)\,\hat{l}_t\right) - \frac{c}{\gamma_x k}\hat{c}_t - \frac{g}{\gamma_x k}\hat{g}_t \tag{27}$$

- Linearization of the Euler equation yields:

$$\hat{c}_t = E_t\left[\hat{c}_{t+1} - \frac{\beta}{\gamma_x\sigma}\alpha A^*\left(\frac{k}{l}\right)^{\alpha-1}\left(z_{t+1} + (\alpha-1)\,\hat{k}_{t+1} - (\alpha-1)\,\hat{l}_{t+1}\right)\right] \tag{28}$$

- The labor market condition can be approximated as

$$\hat{l}_t = \left(\frac{l}{1-l} + \alpha\right)^{-1}\left(z_t + \alpha\hat{k}_t - \sigma\hat{c}_t\right) \tag{29}$$

Linearized Equations: Substituting out l_t

- For convenience, define

$$\hat{l}_t = \left(\frac{l}{1-l} + \alpha\right)^{-1} \left(z_t + \alpha \hat{k}_t - \sigma \hat{c}_t\right) \equiv \gamma_l \left(z_t + \alpha \hat{k}_t - \sigma \hat{c}_t\right) \tag{30}$$

- Plugging into the Euler equation (28) yields:

$$\hat{c}_t = E_t \left[\begin{array}{l} \underbrace{-\frac{\beta}{\gamma_x \sigma} \alpha A^* \left(\frac{k}{l}\right)^{\alpha-1} (\alpha-1)(1-\gamma_l \alpha)}_{\alpha_5} \hat{k}_{t+1} \\ + \underbrace{\left(1 - \frac{\beta}{\gamma_x \sigma} \alpha A^* \left(\frac{k}{l}\right)^{\alpha-1} (\alpha-1) \gamma_l \sigma\right)}_{\alpha_6} \hat{c}_{t+1} \\ + \underbrace{-\frac{\beta}{\gamma_x \sigma} \alpha A^* \left(\frac{k}{l}\right)^{\alpha-1} (1-(\alpha-1)\gamma_l)}_{\alpha_7} z_{t+1} \end{array} \right] \tag{31}$$

Linearized Equations: Substituting out l_t

- Plugging into the resource constraint (27) results in

$$
\begin{aligned}
\hat{k}_{t+1} = {} & \underbrace{\frac{1}{\gamma_x}\left(\alpha\frac{y}{k}\left(1+(1-\alpha)\gamma_l\right)+(1-\delta)\right)}_{\alpha_1}\hat{k}_t \\
& + \underbrace{-\left(\frac{c}{\gamma_x k}+\frac{y}{\gamma_x k}(1-\alpha)\gamma_l\sigma\right)}_{\alpha_2}\hat{c}_t \\
& + \underbrace{\left(\frac{y}{\gamma_x k}\left(1+(1-\alpha)\gamma_l\right)\right)}_{\alpha_3} z_t \\
& + \underbrace{-\frac{g}{\gamma_x k}}_{\alpha_4}\hat{g}_t
\end{aligned}
\tag{32}
$$

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3. **Solving: Use the first-order conditions and the exogenous variables to find a solution for all variables at all times.**
   3.1 Find relevant steady-state expressions.
   3.2 Linearize the first-order conditions.
   3.3 **Find the rational-expectations equilibrium.**

Finding the Rational-Expectations Equilibrium

- Having linearized the equations (31) and (32), we have solved number one of the two problems:
  1. The equations are nonlinear
  2. The system is dynamic; the Euler equation is forward looking, the LOM of capital backward looking
- To solve the second problem, there are two possibilities:
  1. Method of undetermined coefficients
  2. Diagonalization of the transition matrix (Blanchard and Kahn 1980)
- We are looking for decisions rules of the form:

$$\hat{k}_{t+1} = \phi_{kk}\hat{k}_t + \phi_{kz} z_t + \phi_{kg}\hat{g}_t \tag{33}$$

$$\hat{c}_t = \phi_{ck}\hat{k}_t + \phi_{cz} z_t + \phi_{cg}\hat{g}_t \tag{34}$$

Generic Representation of Linear Equilibrium Conditions

- Denote the vector of control variables with u_t and the state vector with $x_t = [x_t^1; x_t^2]'$, with x_t^2 being a vector of exogenous states:

$$x_{t+1}^2 = \Gamma x_t^2 + \epsilon_{t+1} \;, \tag{35}$$

  where ϵ_t is a vector of i.i.d. innovations with bounded support, mean zero, and variance/covariance matrix Σ
- After linearizing all equilibrium conditions and applying the expectation operator, we obtained a linear system of difference equations of the type

$$AE_t a_{t+1} = B a_t, \tag{36}$$

  where $a_t = \left[\underset{(1\times n_x)}{\widehat{x}_t} \;; \underset{(1\times n_u)}{\widehat{u}_t} \right]'$
- Note: $E_t \widehat{x}_{t+1}^1 = \widehat{x}_{t+1}^1$ (endogenous state variables are predetermined)
- Solve system (36), while ruling out explosive paths

$$\lim_{t\to\infty} |a_t| < \infty \tag{37}$$

Generic Representation of Linear Equilibrium Conditions: RBC

- Our RBC example can be written as:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ \alpha_5 & \alpha_7 & 0 & \alpha_6 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} E_t \begin{pmatrix} \hat{k}_{t+1} \\ z_{t+1} \\ \hat{g}_{t+1} \\ \hat{c}_{t+1} \end{pmatrix} = \begin{bmatrix} \alpha_1 & \alpha_3 & \alpha_4 & \alpha_2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \rho^g & 0 \\ 0 & \rho_z & 0 & 0 \end{bmatrix} \begin{pmatrix} \hat{k}_t \\ z_t \\ \hat{g}_t \\ \hat{c}_t \end{pmatrix}$$

Digression: Difference equations

- Before solving the whole matrix equation (36), we first explore **univariate difference equations**
- Key distinction
  - Pre-determined variables (such as state variables)
  - Control variables (or forward-looking variables)
- Introduce **generic notation** for the univariate case:
  - y_t now denotes the endogenous variable
  - ϵ_t denotes an exogenous variable
- Both are scalars
- A **solution** relates y_t to past, present, and future values of ϵ_t (and a boundary condition)

Pre-determined Variables

- Assume y_t is a state variable determined by

$$y_t = a y_{t-1} + \epsilon_t \tag{38}$$

- Solve this equation by backward iteration

$$\begin{aligned} y_t &= a(a y_{t-2} + \epsilon_{t-1}) + \epsilon_t && (39) \\ &= a^t y_0 + a^{t-1}\epsilon_1 + a^{t-2}\epsilon_2 + ... + a\epsilon_{t-1} + \epsilon_t. && (40) \end{aligned}$$

- y_t is a function of the realizations of ϵ_s for $s = 1, \ldots, t$ and initial condition y_0

- With predetermined variables, we have a stable solution only if (we have an initial condition and) $|a| < 1$. If $|a| > 1$ there is no stable solution

Control/Forward-Looking Variables I

- Assume y_t is a control variable determined by (e.g. an Euler equation)

$$ay_t = E_t y_{t+1} + \epsilon_t \tag{41}$$

- Again solution depends on a
- Case $|a| > 1$: Rewrite and iterate forward

$$y_t = \frac{1}{a} E_t y_{t+1} + \frac{1}{a}\epsilon_t$$

$$y_t = \underbrace{\lim_{k\to\infty} \left(\frac{1}{a}\right)^{k+1} E_t(y_{t+k+1})}_{=0 \text{ by (37) and } a>1} + \sum_{i=0}^{\infty} \left(\frac{1}{a}\right)^{i+1} E_t \epsilon_{t+i} \tag{42}$$

Control/Forward-Looking Variables II

- Case $|a| < 1$: forward-iteration of (41) does not yield stable solution
- Compute backward-solution; define expectational error: $\xi_{t+1} = y_{t+1} - E_t y_{t+1}$, and rewrite (41):

$$a y_t = y_{t+1} - \xi_{t+1} + \epsilon_t \tag{43}$$

$$\text{or: } y_t = a y_{t-1} + \xi_t - \epsilon_{t-1} \tag{44}$$

- Solving backward until

$$y_t = a^t y_0 + \sum_{s=1}^{t} a^{t-s} \xi_s - \sum_{s=1}^{t} a^{t-s} \epsilon_{s-1}, \tag{45}$$

  which is a **stable solution** for $|a| < 1$.
- Note that it **holds for any realization of the sequence of expectational errors** $\{\xi_s\}_{s=1\ldots t}$, i.e. there is no unique solution (indeterminacy)

Univariate Case: Summary

- We need an explosive autoregressive coefficient ($|a| > 1$) when dealing with a forward looking variable to obtain a unique and stable solution
- We need a stable autoregressive coefficient ($|a| < 1$) when dealing with a backward looking variable to obtain a stable solution (initial condition makes it unique)

Blanchard-Kahn Conditions

- Back to the matrix equation case
- Write linearized generic DSGE model (36) as

$$E_t a_{t+1} = \underbrace{A^{-1}B}_{\equiv W} a_t \tag{46}$$

- Inverting A requires the **rank condition** to be satisfied
- Solution to (46) depends on the eigenvalues of W, as established by Blanchard and Kahn (1980)
  - If the number of eigenvalues of W outside the unit circle (explosive) is equal to the number of non-predetermined (i.e. forward-looking) variables, then there exists a **unique bounded solution**
  - If the number of eigenvalues of W outside the unit circle (explosive) exceeds the number of forward-looking variables there is **no stable solution**
  - If the number of eigenvalues of W outside the unit circle (explosive) is less than the number of forward-looking variables there is an **infinity of solutions**

Solving the System: Eigenvalue Decomposition

- Recall that a_t in (46) contains three state (backward-looking) variables: $x_t \equiv [\ \hat{k}_t \quad z_t \quad \hat{g}_t\ ]'$ and one control (forward-looking) variable $u_t = \hat{c}_t$
- Decouple the system using an **Eigenvalue (Jordan) decomposition**:

$$W = D\Lambda D^{-1}\,, \tag{47}$$

  where matrix Λ has eigenvalues on its main diagonal and D is a matrix of eigenvectors
- Then system (46), $E_t a_{t+1} = W a_t$, can be rewritten as

$$E_t \underbrace{D^{-1} a_{t+1}}_{\equiv \zeta_{t+1}} = \Lambda \underbrace{D^{-1} a_t}_{\equiv \zeta_t}$$

$$E_t \zeta_{t+1} = \Lambda \zeta_t \tag{48}$$

- This yields a set of independent univariate difference equations in $\zeta_{i,t}$

$$E_t \zeta_{1,t+1} = \lambda_1 \zeta_{1,t}$$

$$\vdots$$

$$E_t \zeta_{i,t+1} = \lambda_i \zeta_{i,t}$$

Solving the System: Reordering

- Elements in Λ can be ordered in ascending order without loss of generality (note: we need to reshuffle the eigenvectors accordingly)
- Assume that number of explosive eigenvalues equal to n_u and partition reordered D and Λ and matrix W according to stable and explosive eigenvalues

$$\Lambda = \begin{bmatrix} \underset{n_x \times n_x}{\Lambda_1} & \underset{n_x \times n_u}{0} \\ \underset{n_u \times n_x}{0} & \underset{n_u \times n_u}{\Lambda_2} \end{bmatrix}; W = \begin{bmatrix} \underset{n_x \times n_x}{w_{11}} & \underset{n_x \times n_u}{w_{12}} \\ \underset{n_u \times n_x}{w_{21}} & \underset{n_u \times n_u}{w_{22}} \end{bmatrix}; D^{-1} = \begin{bmatrix} \underset{n_x \times n_x}{q_{11}} & \underset{n_x \times n_u}{q_{12}} \\ \underset{n_u \times n_x}{q_{21}} & \underset{n_u \times n_u}{q_{22}} \end{bmatrix}$$

- Also partition $\zeta_t = \begin{bmatrix} \underset{1 \times n_x}{\zeta_{1,t}} & \underset{1 \times n_u}{\zeta_{2,t}} \end{bmatrix}'$

Solving the System: Solution for Controls

- First, consider the lower part of (48) corresponding to controls: $E_t\zeta_{2,t+1} = \Lambda_2\zeta_{2,t}$
- As in the univariate case, this part is iterated forward

$$\zeta_{2,t} = \lim_{k\to\infty} \Lambda_2^{-k} E_t\zeta_{2,t+k} = 0, \tag{49}$$

as the diagonal elements of Λ_2 are explosive and $\lim_{t\to\infty} = |\zeta_{2,t}| < \infty$ (linear combination of bounded variables)

- From the definition of ζ_t follows

$$\begin{bmatrix} \underset{n_x\times 1}{\zeta_{1,t}} \\ \underset{n_u\times 1}{\zeta_{2,t}} \end{bmatrix} = \underbrace{\begin{bmatrix} \underset{n_x\times n_x}{q_{11}} & \underset{n_x\times n_u}{q_{12}} \\ \underset{n_u\times n_x}{q_{21}} & \underset{n_u\times n_u}{q_{22}} \end{bmatrix}}_{D^{-1}} \underbrace{\begin{bmatrix} \widehat{x}_t \\ \widehat{u}_t \end{bmatrix}}_{a_t} \tag{50}$$

- Using $\zeta_{2,t} = 0$ implies we have a linear policy function for the controls:

$$0 = q_{21}\hat{x}_t + q_{22}\hat{u}_t \Rightarrow \widehat{u}_t = \underbrace{-q_{22}^{-1}q_{21}}_{\equiv\Pi}\widehat{x}_t \tag{51}$$

Solving the System: Solution for States I

- Given the partitioning of W, the upper part of

$$E_t a_{t+1} = \underbrace{A^{-1}B}_{\equiv W} a_t \tag{46}$$

is given

$$E_t \widehat{x}_{t+1} = w_{11}\widehat{x}_t + w_{12}\widehat{u}_t, \tag{52}$$

- Substituting for $\widehat{u}_t$ using

$$0 = q_{21}\hat{x}_t + q_{22}\hat{u}_t \Rightarrow \widehat{u}_t = \underbrace{-q_{22}^{-1}q_{21}}_{\equiv \Pi} \widehat{x}_t \tag{51}$$

gives

$$E_t \widehat{x}_{t+1} = \underbrace{\left(w_{11} - w_{12}q_{22}^{-1}q_{21}\right)}_{\equiv M} \widehat{x}_t \tag{53}$$

Solving the System: Solution for States II

- Denote the number of endogenous states with n_{x1} and of the exogenous states with n_{x2}
- With endogenous state variables being predetermined

$$\widehat{x}^1_{t+1} = E_t \widehat{x}^1_{t+1} \tag{54}$$

and given the exogenous LOM

$$x^2_{t+1} = \Gamma x^2_t + \epsilon_{t+1} \,, \tag{35}$$

we have

$$\widehat{x}_{t+1} = E_t \widehat{x}_{t+1} + \underbrace{\begin{bmatrix} 0_{n_{x_1} \times n_{x_2}} \\ I_{n_{x_2} \times n_{x_2}} \end{bmatrix}}_{\equiv \eta} \epsilon_{t+1} = M \widehat{x}_t + \eta \epsilon_{t+1} \tag{55}$$

The Solution

- Taken together, the solution equations (51) and (55) provide a recursive representation of the solution to (36) in **state space form**:

$$\widehat{u}_t = \Pi \widehat{x}_t \tag{56}$$

$$\widehat{x}_{t+1} = M \widehat{x}_t + \eta \epsilon_{t+1} \tag{57}$$

- The first equation is called **observation equation** or **measurement equation**, while the second one is called **state transition equation**
- Given x_0, the state-space representation of the solution can be used to compute the time series which obtains in equilibrium for a given sequence $\{\epsilon_{t+1}\}_{t=0}^{\infty}$
- Important: numerical implementation in Matlab is straightforward
- *Exercise 1:* Use the Blanchard-Kahn approach to solve the RBC-model previously solved with the MOUC

Outline

Model Calibration

- **Calibration** of the neoclassical model dates back to Kydland and Prescott (1982), who used it to evaluate the performance of the RBC model
- Idea: choose parameter values to make the model consistent with growth observations
- Important: as we want to judge the model's performance to explain business cycles, business cycle observations cannot be used to assign parameter values, i.e. not the same data should be used for calibration and evaluation
- Using growth observations to set the parameters and then looking at the fit for business cycle observations makes it more difficult for the model to fit the data. This sets a higher standard
- The model is always calibrated to a particular country, in the following the US
- Evaluation of the model on quarterly data: one period in the model corresponds to a quarter

Steady State and Calibration for the US

- In US data, we observe an investment-ratio of 25% and a capital-to-output ratio of 10.4, a capital share of 0.33, and 0.55% output per capita growth g_x and 0.27% population growth
- The LOM for capital in steady state implies:

$$i = (g_x + n + ng_x + \delta)\, k$$

- This implies:

$$\delta = \frac{i}{k} - g_x - n - ng_x = \frac{i/y}{k/y} - g_x - n - ng_x$$
$$= \frac{0.25}{10.4} - 0.55\% - 0.27\% - 0.55\% \times 0.27\% = 1.58\%$$

- The Euler equation implies for $\sigma = 1$:

$$1 = \beta \left(\frac{\alpha \frac{y}{k} + (1 - \delta)}{(1 + g_x)(1 + n)} \right)$$
$$\Rightarrow \beta = \frac{(1 + 0.55\%)(1 + 0.27\%)}{0.33/10.4 + (1 - 1.58\%)} = 0.9924$$

Steady State and Calibration for the US: Labor Supply

- Choose ψ to obtain $l = 0.33$ in steady state
- The labor FOC with $\sigma = 1$ is

$$\psi \frac{1}{1-l} c = (1-\alpha)\frac{y}{l}$$

- This implies:

$$\begin{aligned}\psi &= (1-\alpha)\frac{1-l}{l}\frac{1}{1-\frac{i}{y}-\frac{g^*}{y}} \\ &= (1-0.33)\frac{1}{1-0.25-0.2038}\frac{1-0.33}{0.33} = 2.49\end{aligned}$$

Steady State and Calibration

- The capital stock in the deterministic steady state is up to first order given by

$$k = \left(\frac{\alpha A^*}{g_x + n + \delta + \rho} \right)^{\frac{1}{1-\alpha}} l \, ,$$

where $\beta = 1/(1+\rho)$

- Consumption and output can now be computed as

$$\begin{aligned} y &= A^* k^\alpha l^{1-\alpha} \\ c &= y - (g_x + n + n g_x + \delta) k - g^* \end{aligned}$$

where g^* follows from data for G/Y

Calibration US

Parameter	Value	Target
g_x	0.0055	2.2% Output/Capita Growth
n	0.0027	1.1% Population Growth
A^*	1	Normalization
α	0.33	Capital Share
δ	1.58%	Investment/Output $=$ 0.25
β	0.9926	Capital/Output $=$ 10.4
σ	1	Model-independent evidence
ψ	2.49	$l^* = 0.33$
ρ_z	0.97	Estimate
σ_z	0.0068	Estimate
χ	0.2038	G/Y of 0.2038
ρ_g	0.98	Estimate
σ_g	0.0105	Estimate

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