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ADVANCED MACROECONOMICS

-The Algebra of the RBC-Model-

1 Household problem

The household now maximizes the utility function

$$\max_{\{c_t,k_{t+1},l_t\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t \left(\frac{c_t^{1-\sigma}}{1-\sigma} + \psi \log(1-l_t) + \theta g_t \right) \tag{1}$$

subject to the budget constraint

$$c_t + i_t = w_t l_t + R_t k_t - T_t \,\forall\, t, \tag{2}$$

the law of motion for capital with $\gamma_x = 1 + n + x + nx$ (where n and x are the growth rates of population and per-capita GDP, respectively)

$$\gamma_x k_{t+1} = (1-\delta) k_t + i_t \,\forall\, t, \tag{3}$$

an initial value for capital: $k_0 > 0$, and the transversality/No-Ponzi constraint

$$\lim_{t\to\infty} E_0 \beta^t \lambda_t k_{t+1} = 0. \tag{4}$$

Additionally the household takes into account the stochastic laws of motion for z_t

$$A_t = A^* e^{z_t} \tag{5}$$

$$z_t = \rho_z z_{t-1} + \varepsilon_t, \varepsilon_t \sim \left(0, \sigma_z^2\right) \tag{6}$$

and g_t

$$g_t = g^* e^{\hat{g}_t} \tag{7}$$

$$\hat{g}_t = \rho_g \hat{g}_{t-1} + \varepsilon_t^G, \varepsilon_t^G \sim \left(0, \sigma_G^2\right)\ , \tag{8}$$

and the government budget constraint

$$T_t = g_t \,\forall\, t \tag{9}$$

Finally, we have non-negativity constraints

$$k_t \geq 0 \tag{10}$$

$$c_t \geq 0 \tag{11}$$

$$0 \leq l_t \leq 1\ . \tag{12}$$

Due to our assumptions, we will not have corner solutions and can ignore the non-negativity constraints.
This yields the Lagrangian

$$L = E_0 \sum_{t=0}^{\infty} \beta^t \left\{ \begin{array}{l} \dfrac{c_t^{1-\sigma}}{1-\sigma} + \psi \log(1-l_t) + \theta g_t \\ -\lambda_t \left(c_t + \gamma_x k_{t+1} - (1-\delta) k_t + g_t - w_t l_t - R_t k_t \right) \end{array} \right\}$$

with the First Order Conditions ($\forall t$)

$$\frac{\partial L}{\partial c_t} = E_t \beta^t \left(c_t^{-\sigma} - \lambda_t \right) = 0 \Rightarrow c_t^{-\sigma} = \lambda_t \tag{13}$$

$$\frac{\partial L}{\partial l_t} = E_t \beta^t \left\{ -\psi \frac{1}{1-l_t} + \lambda_t w_t \right\} = 0 \Rightarrow \psi \frac{1}{1-l_t} = \lambda_t w_t \tag{14}$$

$$\frac{\partial L}{\partial k_{t+1}} = E_t \beta^t \left\{ -\lambda_t + \frac{\beta}{\gamma_x} \lambda_{t+1} \left((1-\delta) + R_{t+1} \right) \right\} = 0$$

$$\Rightarrow \lambda_t = \frac{\beta}{\gamma_x} E_t \lambda_{t+1} \left(R_{t+1} + 1 - \delta \right) . \tag{15}$$

and

$$\frac{\partial L}{\partial \lambda_t} = -\left(c_t + \gamma_x k_{t+1} - (1-\delta) k_t + g_t - w_t l_t - R_t k_t \right) = 0, \tag{16}$$

that is the budget constraint.

The first three first order conditions can be combined to yield

$$c_t^{-\sigma} = \frac{\beta}{\gamma_x} E_t c_{t+1}^{-\sigma} \left(1 - \delta + R_{t+1} \right) \tag{17}$$

$$\psi \frac{1}{1-l_t} = \frac{w_t}{c_t^{\sigma}} . \tag{18}$$

2 Firm Problem

The firm problem is again static:

$$\max_{l_t, k_t} A^* e^{z_t} k_t^{\alpha} l_t^{1-\alpha} - w_t l_t - R_t k_t$$

The resulting first order conditions again state that factors are paid their marginal products

$$w_t = (1-\alpha) A^* e^{z_t} \left(\frac{k_t}{l_t} \right)^{\alpha} \tag{19}$$

$$R_t = \alpha A^* e^{z_t} \left(\frac{k_t}{l_t} \right)^{\alpha - 1} . \tag{20}$$

3 Market Clearing

Labor Market: labor supply (18) and demand (19) must be equal:

$$(1-\alpha) A^* e^{z_t} \left(\frac{k_t}{l_t} \right)^{\alpha} = \psi \frac{1}{1-l_t} c_t^{\sigma} . \tag{21}$$

Capital market: from (17) and (20) follows

$$c_t^{-\sigma} = \frac{\beta}{\gamma_x} E_t c_{t+1}^{-\sigma} \left\{ \alpha A^* e^{z_{t+1}} \left(\frac{k_{t+1}}{l_{t+1}} \right)^{\alpha - 1} + (1-\delta) \right\} . \tag{22}$$

Finally, the goods market equilibrium requires:

$$c_t + \gamma_x k_{t+1} - (1-\delta) k_t + g_t = A^* e^{z_t} k_t^{\alpha} l_t^{1-\alpha} . \tag{23}$$

4 Approximating Technology

The Taylor approximation of technology A_t is given by

$$a e^{\hat{x}_t} \approx a + a \, e^{\hat{x}_t} \Big|_{\hat{x}_t = 0} \left(\hat{x}_t - 0 \right) = a + a \hat{x}_t ,$$

where $\hat{x}_t = z_t$ (see Linearization Handout).

5 The Labor FOC

The labor FOC is given by

$$(1-\alpha)\,A^* e^{z_t}\left(\frac{k_t}{l_t}\right)^{\alpha} = \psi\frac{1}{1-l_t}c_t{}^{\sigma} \tag{24}$$

The steady state is

$$(1-\alpha)\,A^*\left(\frac{k}{l}\right)^{\alpha} = \psi\frac{1}{1-l}c^{\sigma}\ . \tag{25}$$

Loglinearizing yields

$$(1-\alpha)\,A^*\left(\frac{k}{l}\right)^{\alpha}\left(z_t+\alpha\hat{k}_t-\alpha\hat{l}_t\right) = \psi\frac{1}{1-l}\sigma c^{\sigma}\hat{c}_t - \psi c^{\sigma}\left(\frac{1}{1-l}\right)^{2}(-1)\,l\hat{l}_t\ . \tag{26}$$

Divide by the steady state

$$z_t+\alpha\hat{k}_t-\alpha\hat{l}_t = \sigma\hat{c}_t+\frac{l}{1-l}\hat{l}_t\ . \tag{27}$$

Solving for l yields

$$\hat{l}_t = \left(\frac{l}{1-l}+\alpha\right)^{-1}\left(z_t+\alpha\hat{k}_t-\sigma\hat{c}_t\right)\ . \tag{28}$$

6 The Euler Equation

The Euler Equation is given by

$$c_t^{-\sigma} = \frac{\beta}{\gamma_x}E_t\left\{c_{t+1}^{-\sigma}\left(\alpha A_{t+1}\left(\frac{k_{t+1}}{l_{t+1}}\right)^{\alpha-1}+(1-\delta)\right)\right\}\ . \tag{29}$$

The steady state is

$$1 = \frac{\beta}{\gamma_x}\left\{\alpha A^*\left(\frac{k}{l}\right)^{\alpha-1}+(1-\delta)\right\}\ . \tag{30}$$

Loglinearizing yields

$$\begin{aligned} c^{-\sigma}-\sigma c^{-\sigma}\hat{c}_t = E_t\Bigg[&\underbrace{\frac{\beta}{\gamma_x}c^{-\sigma}\left(\alpha A^*\left(\frac{k}{l}\right)^{\alpha-1}+(1-\delta)\right)}_{c^{-\sigma}} \\ &+\underbrace{\frac{\beta}{\gamma_x}\left\{\alpha A^*\left(\frac{k}{l}\right)^{\alpha-1}+(1-\delta)\right\}}_{1}(-\sigma)\,c^{-\sigma}\hat{c}_{t+1} \\ &+\frac{\beta}{\gamma_x}c^{-\sigma}\left\{\alpha A^*\left(\frac{k}{l}\right)^{\alpha-1}\right\}\left(z_{t+1}+(\alpha-1)\,\hat{k}_{t+1}-(\alpha-1)\,\hat{l}_{t+1}\right)\Bigg]\ . \end{aligned}$$

Subtracting the steady state and dividing by $-\sigma c^{-\sigma}$ yields

$$\hat{c}_t = E_t\left[\hat{c}_{t+1}-\frac{\beta}{\gamma_x\sigma}\alpha A^*\left(\frac{k}{l}\right)^{\alpha-1}\left(z_{t+1}+(\alpha-1)\,\hat{k}_{t+1}-(\alpha-1)\,\hat{l}_{t+1}\right)\right]\ . \tag{31}$$

7 The Resource Constraint

The resource constraint is given by

$$c_t + \gamma_x k_{t+1} - (1-\delta)\,k_t + g_t = A_t k_t^{\alpha} l_t^{1-\alpha} \tag{32}$$

In steady state we have

$$c + \gamma_x k - (1-\delta)\,k + g = A^* k^{\alpha} l^{1-\alpha}\,. \tag{33}$$

Loglinearizing the single terms yields

$$\begin{align}
c_t &\approx c + c\hat{c}_t \tag{34}\\
\gamma_x k_{t+1} &\approx \gamma_x k + \gamma_x k \hat{k}_{t+1} \tag{35}\\
(1-\delta)\,k_t &\approx (1-\delta)\,k + (1-\delta)\,k\hat{k}_t \tag{36}\\
A_t k_t^{\alpha} l_t^{1-\alpha} &\approx A^* k^{\alpha} l^{1-\alpha} + A^* k^{\alpha} l^{1-\alpha} z_t + \alpha A^* k^{\alpha} l^{1-\alpha} \hat{k}_t + (1-\alpha)\,A^* k^{\alpha} l^{1-\alpha} \hat{l}_t \tag{37}\\
g e^{\hat{g}_t} &\approx g + g\hat{g}_t\,. \tag{38}
\end{align}$$

Together

$$\begin{split}
&c + c\hat{c}_t + \gamma_x k + \gamma_x k\hat{k}_{t+1} - (1-\delta)\,k - (1-\delta)\,k\hat{k}_t + g + g\hat{g}_t = \\
&A^* k^{\alpha} l^{1-\alpha} + A^* k^{\alpha} l^{1-\alpha} z_t + \alpha A^* k^{\alpha} l^{1-\alpha} \hat{k}_t + (1-\alpha)\,A^* k^{\alpha} l^{1-\alpha} \hat{l}_t\,.
\end{split} \tag{39}$$

Rewrite as

$$\begin{split}
&\underbrace{c + \gamma_x k - (1-\delta)\,k + g}_{A^* k^{\alpha} l^{1-\alpha}} + c\hat{c}_t + \gamma_x k\hat{k}_{t+1} - (1-\delta)\,k\hat{k}_t + g\hat{g}_t = \\
&A^* k^{\alpha} l^{1-\alpha} + A^* k^{\alpha} l^{1-\alpha}\left(z_t + \alpha\hat{k}_t + (1-\alpha)\,\hat{l}_t\right)\,.
\end{split} \tag{40}$$

Solving for $\hat{k}_t$ yields

$$\begin{align}
c\hat{c}_t &= A^* k^{\alpha} l^{1-\alpha} z_t + \alpha A^* k^{\alpha} l^{1-\alpha}\hat{k}_t + (1-\alpha)\,A^* k^{\alpha} l^{1-\alpha}\hat{l}_t - \gamma_x k\hat{k}_{t+1} + (1-\delta)\,k\hat{k}_t - g\hat{g}_t \notag\\
c\hat{c}_t &= \left[\alpha y + (1-\delta)\,k\right]\hat{k}_t + y\left(z_t + (1-\alpha)\,\hat{l}_t\right) - \gamma_x k\hat{k}_{t+1} - g\hat{g}_t \notag\\
\hat{k}_{t+1} &= \frac{1}{\gamma_x}\left[\alpha\frac{y}{k} + (1-\delta)\right]\hat{k}_t + \frac{y}{\gamma_x k}\left(z_t + (1-\alpha)\,\hat{l}_t\right) - \frac{c}{\gamma_x k}\hat{c}_t - \frac{g}{\gamma_x k}\hat{g}_t\,. \tag{41}
\end{align}$$

8 Eliminating l

Write (28) as

$$\hat{l}_t = \left(\frac{l}{1-l} + \alpha\right)^{-1}\left(z_t + \alpha\hat{k}_t - \sigma\hat{c}_t\right) \equiv \gamma_l\left(z_t + \alpha\hat{k}_t - \sigma\hat{c}_t\right) \tag{42}$$

and plug in into (41)

$$\begin{align}
\hat{k}_{t+1} =& \frac{1}{\gamma_x}\left[\alpha\frac{y}{k} + (1-\delta)\right]\hat{k}_t + \frac{y}{\gamma_x k}z_t + \frac{y}{\gamma_x k}(1-\alpha)\,\gamma_l\left(z_t + \alpha\hat{k}_t - \sigma\hat{c}_t\right) - \frac{c}{\gamma_x k}\hat{c}_t - \frac{g}{\gamma_x k}\hat{g}_t \notag\\
\hat{k}_{t+1} =& \frac{1}{\gamma_x}\left(\alpha\frac{y}{k} + (1-\delta) + \frac{y}{k}(1-\alpha)\,\gamma_l\alpha\right)\hat{k}_t + \left(\frac{y}{\gamma_x k} + (1-\alpha)\,\gamma_l\frac{y}{\gamma_x k}\right)z_t \notag\\
&- \left(\frac{c}{\gamma_x k} + \frac{y}{\gamma_x k}(1-\alpha)\,\gamma_l\sigma\right)\hat{c}_t - \frac{g}{\gamma_x k}\hat{g}_t \notag\\
\hat{k}_{t+1} =& \underbrace{\frac{1}{\gamma_x}\left(\alpha\frac{y}{k}\left(1 + (1-\alpha)\,\gamma_l\right) + (1-\delta)\right)}_{\alpha_1}\hat{k}_t + \underbrace{-\left(\frac{c}{\gamma_x k} + \frac{y}{\gamma_x k}(1-\alpha)\,\gamma_l\sigma\right)}_{\alpha_2}\hat{c}_t \notag\\
&+ \underbrace{\left(\frac{y}{\gamma_x k}\left(1 + (1-\alpha)\,\gamma_l\right)\right)}_{\alpha_3}z_t + \underbrace{-\frac{g}{\gamma_x k}}_{\alpha_4}\hat{g}_t \tag{43}
\end{align}$$

and into (31)

$$
\begin{aligned}
\hat{c}_t &= E_t\left[\hat{c}_{t+1} - \frac{\beta}{\gamma_x\sigma}\alpha A^*\left(\frac{k}{l}\right)^{\alpha-1}\left(z_{t+1} + (\alpha-1)\,\hat{k}_{t+1} - (\alpha-1)\,\gamma_l\left(z_{t+1} + \alpha\hat{k}_{t+1} - \sigma\hat{c}_{t+1}\right)\right)\right] \\
\hat{c}_t &= E_t\left[\begin{array}{c}\left(1 - \frac{\beta}{\gamma_x\sigma}\alpha A^*\left(\frac{k}{l}\right)^{\alpha-1}(\alpha-1)\,\gamma_l\sigma\right)c_{t+1} - \frac{\beta}{\gamma_x\sigma}\alpha A^*\left(\frac{k}{l}\right)^{\alpha-1}\left(1-(\alpha-1)\,\gamma_l\right)z_{t+1} \\
-\frac{\beta}{\gamma_x\sigma}\alpha A^*\left(\frac{k}{l}\right)^{\alpha-1}\left((\alpha-1) - \gamma_l\,(\alpha-1)\,\alpha\right)\hat{k}_{t+1}\end{array}\right] \\
\hat{c}_t &= E_t\left[\begin{array}{l}\underbrace{-\frac{\beta}{\gamma_x\sigma}\alpha A^*\left(\frac{k}{l}\right)^{\alpha-1}(\alpha-1)\left(1-\gamma_l\alpha\right)}_{\alpha_5}\hat{k}_{t+1} + \underbrace{\left(1 - \frac{\beta}{\gamma_x}\alpha A^*\left(\frac{k}{l}\right)^{\alpha-1}(\alpha-1)\,\gamma_l\right)}_{\alpha_6}c_{t+1} + \\
\underbrace{-\frac{\beta}{\gamma_x\sigma}\alpha A^*\left(\frac{k}{l}\right)^{\alpha-1}\left(1-(\alpha-1)\,\gamma_l\right)}_{\alpha_7}z_{t+1}\end{array}\right].
\end{aligned}
\tag{44}
$$

9 Method of Undetermined Coefficients

Guess a solution of the form

$$
\hat{k}_{t+1} = \phi_{kk}\hat{k}_t + \phi_{kz}z_t + \phi_{kg}\hat{g}_t \tag{45}
$$

$$
\hat{c}_t = \phi_{ck}\hat{k}_t + \phi_{cz}z_t + \phi_{cg}\hat{g}_t\,. \tag{46}
$$

Plugging in Equation (43) yields

$$
\begin{aligned}
\hat{k}_{t+1} &= \alpha_1\hat{k}_t + \alpha_2\hat{c}_t + \alpha_3 z_t + \alpha_4\hat{g}_t \\
\phi_{kk}\hat{k}_t + \phi_{kz}z_t + \phi_{kg}\hat{g}_t &= \alpha_1\hat{k}_t + \alpha_2\left(\phi_{ck}\hat{k}_t + \phi_{cz}z_t + \phi_{cg}\hat{g}_t\right) + \alpha_3 z_t + \alpha_4\hat{g}_t\,.
\end{aligned}
$$

This equation has to hold for all values of the state variables $\hat{g}_t, \hat{k}_t, z_t$. In turn setting all states to 0 except for one, which is set to 1, yields the following equations (this is equivalent to simply comparing the sides):

$$
\phi_{kk} = \alpha_1 + \alpha_2\phi_{ck} \tag{47}
$$

$$
\phi_{kz} = \alpha_3 + \alpha_2\phi_{cz} \tag{48}
$$

$$
\phi_{kg} = \alpha_4 + \alpha_2\phi_{cg} \tag{49}
$$

Equation (44):

$$
\begin{aligned}
\hat{c}_t &= E_t\left[\alpha_5\hat{k}_{t+1} + \alpha_6\hat{c}_{t+1} + \alpha_7 z_{t+1}\right] \\
\phi_{ck}\hat{k}_t + \phi_{cz}z_t + \phi_{cg}\hat{g}_t &= E_t\left[\alpha_5\hat{k}_{t+1} + \alpha_6\left(\phi_{ck}\hat{k}_{t+1} + \phi_{cz}z_{t+1} + \phi_{cg}\hat{g}_{t+1}\right) + \alpha_7 z_{t+1}\right] \\
\phi_{ck}\hat{k}_t + \phi_{cz}z_t + \phi_{cg}\hat{g}_t &= E_t\left[\left(\alpha_6\phi_{cz} + \alpha_7\right)z_{t+1} + \left(\alpha_6\phi_{ck} + \alpha_5\right)\hat{k}_{t+1} + \alpha_6\phi_{cg}\hat{g}_{t+1}\right]
\end{aligned}
$$

Plugging in the laws of motion for z_{t+1} (6) and g_{t+1} (8) and using the expectations operator yields:

$$\begin{aligned}
\phi_{ck}\hat{k}_t + \phi_{cz}z_t + \phi_{cg}\hat{g}_t =& E_t \begin{bmatrix} (\alpha_6\phi_{cz} + \alpha_7)(\rho z_t + \varepsilon_{t+1}) \\ + (\alpha_6\phi_{ck} + \alpha_5)\left(\phi_{kk}\hat{k}_t + \phi_{kz}z_t + \phi_{kg}\hat{g}_t\right) \\ + \alpha_6\phi_{cg}\left(\rho_g g_t + \varepsilon^G_{t+1}\right) \end{bmatrix} \\
\phi_{ck}\hat{k}_t + \phi_{cz}z_t + \phi_{cg}\hat{g}_t =& \left(\rho\left(\alpha_6\phi_{cz} + \alpha_7\right) + \left(\alpha_6\phi_{ck} + \alpha_5\right)\phi_{kz}\right) z_t \\
& + \left(\alpha_6\phi_{ck} + \alpha_5\right)\phi_{kk}\hat{k}_t \\
& + \left(\left(\alpha_6\phi_{ck} + \alpha_5\right)\phi_{kg} + \alpha_6\phi_{cg}\rho_g\right)\hat{g}_t \, .
\end{aligned}$$

Comparing sides or in turn setting all states to 0 except for one, which is set to 1, yields:

$$\phi_{ck} = \left(\alpha_6\phi_{ck} + \alpha_5\right)\phi_{kk} \tag{50}$$
$$\phi_{cz} = \rho\left(\alpha_6\phi_{cz} + \alpha_7\right) + \left(\alpha_6\phi_{ck} + \alpha_5\right)\phi_{kz} \tag{51}$$
$$\phi_{cg} = \left(\alpha_6\phi_{ck} + \alpha_5\right)\phi_{kg} + \alpha_6\phi_{cg}\rho_g \, . \tag{52}$$

We now have 6 equations in six unknown coefficients ϕ. Hence, we need to solve this system. Use (47) and (50)

$$\phi_{kk} = \alpha_1 + \alpha_2\phi_{ck} \Rightarrow \phi_{ck} = \frac{\phi_{kk}}{\alpha_2} - \frac{\alpha_1}{\alpha_2} \tag{53}$$
$$\phi_{ck} = \left(\alpha_6\phi_{ck} + \alpha_5\right)\phi_{kk} \, . \tag{50}$$

Set (53) and (50) equal and transform to get quadratic equation

$$\begin{aligned}
&\frac{\phi_{kk}}{\alpha_2} - \frac{\alpha_1}{\alpha_2} = \left(\alpha_6\left(\frac{\phi_{kk}}{\alpha_2} - \frac{\alpha_1}{\alpha_2}\right) + \alpha_5\right)\phi_{kk} = \frac{\alpha_6}{\alpha_2}\phi_{kk}^2 + \left(\alpha_5 - \frac{\alpha_6\alpha_1}{\alpha_2}\right)\phi_{kk} \\
&\frac{\alpha_6}{\alpha_2}\phi_{kk}^2 + \left(\alpha_5 - \frac{\alpha_6\alpha_1}{\alpha_2} - \frac{1}{\alpha_2}\right)\phi_{kk} + \frac{\alpha_1}{\alpha_2} = 0 \\
&\phi_{kk}^2 + \left(\frac{\alpha_5\alpha_2}{\alpha_6} - \frac{\alpha_6\alpha_1}{\alpha_6} - \frac{1}{\alpha_6}\right)\phi_{kk} + \frac{\alpha_1}{\alpha_6} = 0 \qquad .
\end{aligned}$$

The solution for ϕ_{kk} is given as

$$\phi_{kk} = \left(\frac{1 - \alpha_5\alpha_2 + \alpha_6\alpha_1}{2\alpha_6}\right) \pm \sqrt{\left(\frac{1 - \alpha_5\alpha_2 + \alpha_6\alpha_1}{2\alpha_6}\right)^2 - \frac{\alpha_1}{\alpha_6}} \, . \tag{54}$$

ϕ_{ck} then follows from:

$$\phi_{ck} = \frac{\phi_{kk}}{\alpha_2} - \frac{\alpha_1}{\alpha_2} \, . \tag{53}$$

Second, use

$$\phi_{kz} = \alpha_3 + \alpha_2\phi_{cz} \tag{48}$$
$$\phi_{cz} = \rho\left(\alpha_6\phi_{cz} + \alpha_7\right) + \left(\alpha_6\phi_{ck} + \alpha_5\right)\phi_{kz} \tag{51}$$

to get

$$\phi_{kz} = \alpha_3 + \alpha_2\phi_{cz} \tag{55}$$
$$\phi_{cz} = \rho\left(\alpha_6\phi_{cz} + \alpha_7\right) + \left(\alpha_6\phi_{ck} + \alpha_5\right)\phi_{kz} \tag{56}$$
$$\phi_{cz} = \alpha_6\rho\phi_{cz} + \rho\alpha_7 + \alpha_6\phi_{ck}\alpha_3 + \alpha_6\phi_{ck}\alpha_2\phi_{cz} + \alpha_3\alpha_5 + \alpha_5\alpha_2\phi_{cz} \tag{57}$$
$$\phi_{cz}\left(1 - \alpha_6\rho - \alpha_6\phi_{ck}\alpha_2 - \alpha_5\alpha_2\right) = \rho\alpha_7 + \alpha_6\phi_{ck}\alpha_3 + \alpha_3\alpha_5 \tag{58}$$
$$\phi_{cz} = \frac{\rho\alpha_7 + \alpha_6\phi_{ck}\alpha_3 + \alpha_3\alpha_5}{\left(1 - \alpha_6\rho - \alpha_6\phi_{ck}\alpha_2 - \alpha_5\alpha_2\right)} \, . \tag{59}$$

Given ϕ_{cz}, ϕ_{kz} can be computed from

$$\phi_{kz} = \alpha_3 + \alpha_2 \phi_{cz} \, . \tag{48}$$

Finally, plug

$$\phi_{kg} = \alpha_4 + \alpha_2 \phi_{cg} \tag{49}$$

into

$$\phi_{cg} = (\alpha_6 \phi_{ck} + \alpha_5)\, \phi_{kg} + \alpha_6 \phi_{cg} \rho_g \tag{52}$$

to get

$$\begin{aligned} \phi_{cg} &= (\alpha_6 \phi_{ck} + \alpha_5)\,(\alpha_4 + \alpha_2 \phi_{cg}) + \alpha_6 \phi_{cg} \rho_g \\ \phi_{cg} &= \frac{(\alpha_6 \phi_{ck} + \alpha_5)\, \alpha_4}{1 - (\alpha_6 \phi_{ck} + \alpha_5)\, \alpha_2 - \alpha_6 \rho_g} \, . \end{aligned} \tag{60}$$