Structural Macroeconometrics

Chapter III
Kalman Filter

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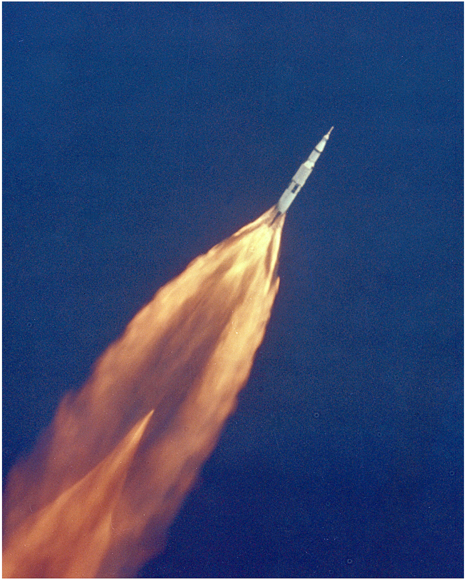
Outline

Cool Stuff



The Basic Concept

- named after seminal work of Kálmán (1960) and Kálmán and Bucy (1961)

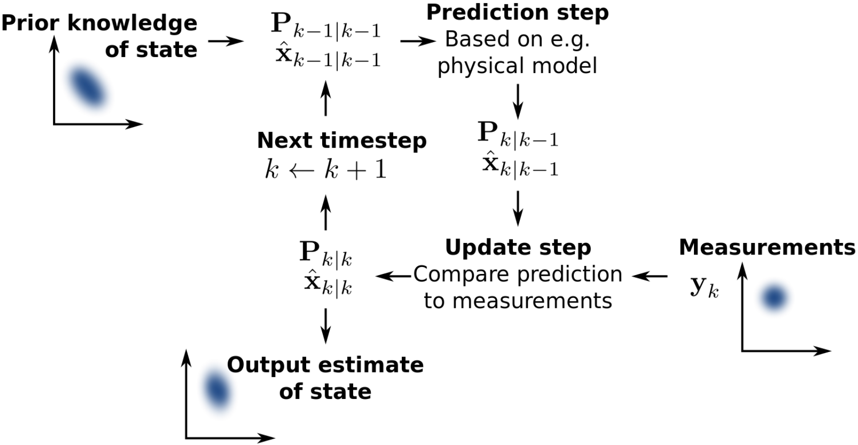


Figure 1: Source: "Basic concept of Kalman filtering" by Petteri Aimonen

References

- Handout online
- Ljungqvist and Sargent (2012, Chapter 2.5)
- Hamilton (1994, Chapter 13)
- Durbin and Koopman (2012)
- Canova (2007, Chapter 6)

Gaussian State Space

- Solution to linearized RBC model took **state space form**, which can be written as

$$x_{t+1} = Fx_t + w_{t+1},\ w_{t+1} \overset{iid}{\sim} \mathcal{N}(0, Q) \tag{1}$$

$$y_t = Hx_t + \nu_t,\ \nu_t \overset{iid}{\sim} \mathcal{N}(0, R) \tag{2}$$

- F, H, Q, R are functions of the model parameters θ
- x_t is an $n_x \times 1$ vector of states
- y_t is an $n_y \times 1$ vector of observables
- w_t is a $p \times 1$ vector of structural errors
- ν_t a vector of measurement errors
- Assumption: w_t and ν_t are orthogonal

$$E_t\left(w_{t+1}\nu_s\right) = 0 \quad \forall\ t+1 \text{ and } s \geq 0$$

Fundamental Problem: Unobserved States

- This implies that
$$y_t = H\left(Fx_{t-1} + w_t\right) + \nu_t \tag{3}$$
- Thus, y_t is normally distributed:
$$y_t \sim \mathcal{N}(HFx_{t-1}, HQH' + R) \tag{4}$$
- If all states were observed, we could directly construct the likelihood $f(y_T, \ldots, y_1|\theta)$
- We could then run optimizer over our estimated parameter set $\tilde{\theta} \subseteq \theta$ to get ML estimate of $\tilde{\theta}$
- Problem: we have **unobserved states** and cannot use equation (4)
- Solution: turn to Kalman filter to back out states from the observed data $\rightarrow$ **Filtering problem**

Initial Values

- initial value x_0 distributed as

$$x_0 \sim \mathcal{N}\left(\hat{x}_{0|-1}, \Sigma_{0|-1}\right) \tag{5}$$

- subscript -1 denotes information at the beginning of times
- particular observation: y_t
- complete history of observations up to a certain point in time: $y^t = \{y_t, \ldots, y_0\}$
- What does this imply for $y_0 = Hx_0 + \nu_0$?

Initial Values

- Let's define

$$\hat{y}_{0|-1} \equiv E(y_0|y^{-1}) = E(Hx_0 + \nu_0|y^{-1}) = H\hat{x}_{0|-1} \tag{6}$$

- **Mean Squared Error** of y_0 given information up to $t = -1$

$$\begin{aligned}
&E\left[\left(y_0 - \hat{y}_{0|-1}\right)\left(y_0 - \hat{y}_{0|-1}\right)' |y^{-1}\right] \\
&= E\left[\left(Hx_0 + \nu_0 - H\hat{x}_{0|-1}\right)\left(Hx_0 + \nu_0 - H\hat{x}_{0|-1}\right)' |y^{-1}\right] \\
&= E\left[H(x_0 - \hat{x}_{0|-1})(x_0 - \hat{x}_{0|-1})'H' + \nu_0\nu_0'|y^{-1}\right] \\
&= H\Sigma_{0|-1}H' + R
\end{aligned} \tag{7}$$

- Hence:

$$y_0 \sim N\left(H\hat{x}_{0|-1}, H\Sigma_{0|-1}H' + R\right) \tag{8}$$

What's the Goal?

- Economist only observes the observables: $y^t = \{y_t, \ldots, y_0\}$
- Wants to infer the **unobserved state variables** $x_t, \ldots, x_0$
- Assumption: Economist knows state space structure, i.e. H, F, R, Q
- Aim: find **recursive** formulas for the state forecast

$$\hat{x}_{t|t-1} \equiv E[x_t | y^{t-1}] \tag{9}$$

  and the Mean Squared Error/covariance matrices of the forecast error (FE)

$$\Sigma_{t|t-1} \equiv E\left[\left(x_t - \hat{x}_{t|t-1}\right)\left(x_t - \hat{x}_{t|t-1}\right)' | y^{t-1}\right] \tag{10}$$

- Recursiveness allows for online tracking

Idea

- regression of the (unknown) state FE on observation FE

$$x_t - \hat{x}_{t|t-1} = L_t \left(y_t - \hat{y}_{t|t-1} \right) + \eta_t \tag{11}$$

L_t: regression coefficient; η_t: regression residual

- Forecast $\hat{y}_{t|t-1}$ of y_t given by

$$\hat{y}_{t|t-1} = H\hat{x}_{t|t-1} \tag{12}$$

- regression equation can then be written as

$$x_t - \hat{x}_{t|t-1} = L_t \left(y_t - H\hat{x}_{t|t-1} \right) + \eta_t \tag{13}$$

- Problem: don't know the left-hand side $\Rightarrow$ can't compute L_t
- But: if we somehow knew L_t, could form a forecast of our state FE

Computing L_t

- Goal: compute L_t at time t
- General formula for a **population regression**

$$\beta = E\left(YX'\right)\left[E\left(XX'\right)\right]^{-1}$$

- In our case

$$\begin{aligned} L_t =& E\left[\left(x_t - \hat{x}_{t|t-1}\right)\left(y_t - H\hat{x}_{t|t-1}\right)'\right] \\ & \times \left(E\left[\left(y_t - H\hat{x}_{t|t-1}\right)\left(y_t - H\hat{x}_{t|t-1}\right)'\right]\right)^{-1} \\ =& \Sigma_{t|t-1}H'\left(H\Sigma_{t|t-1}H' + R\right)^{-1} \end{aligned} \tag{14}$$

Stochastic Singularity

- Computation requires $\left(H\Sigma_{t|t-1}H' + R\right)^{-1}$
- For the inversion, state-space model must not feature **stochastic singularity**, i.e. the forecast error matrix of the observables must have full rank
- Typical requirement: as least as many shocks as observables
- Having more shocks than observables is not a problem
- Also requires that there is no collinearity between observables

$\rightarrow$ Problem if all components of budget constraint are observed

- Way out: add measurement error (see Altug 1989; Ireland 2004; Sargent 1989)
- May also help with model misspecification (Del Negro and Schorfheide 2009)

Forecasting x_1

- Rewrite state transition equation

$$\begin{aligned} x_1 &= F x_0 + w_1 \\ &= F\hat{x}_{0|-1} + F\left(x_0 - \hat{x}_{0|-1}\right) + w_1 \\ &\overset{(13)}{=} F\hat{x}_{0|-1} + F\left(L_0\left(y_0 - H\hat{x}_{0|-1}\right) + \eta_0\right) + w_1 \end{aligned} \tag{15}$$

- Forecast tomorrow's state just based on yesterday's forecast and today's observation

$$\begin{aligned} \hat{x}_{1|0} = E\left[x_1|y^0\right] &= F\hat{x}_{0|-1} + \underbrace{FL_0}_{K_0}\left(y_0 - H\hat{x}_{0|-1}\right) \\ &= F\hat{x}_{0|-1} + K_0\left(y_0 - H\hat{x}_{0|-1}\right) \end{aligned} \tag{16}$$

- **Kalman Gain**

$$K_0 = FL_0 = F\Sigma_{0|-1}H'\left(H\Sigma_{0|-1}H' + R\right)^{-1} \tag{17}$$

$\rightarrow$ how much is state estimate updated based on previous FE

x_2, x_3,...?

- We want to use population regression

$$L_1 = \Sigma_{1|0} H' \left(H \Sigma_{1|0} H' + R \right)^{-1}$$

- For this, we need covariance/mean squared error matrix $\Sigma_{1|0}$

$$\begin{aligned} \Sigma_{1|0} =& E\left[\left(x_1 - \hat{x}_{1|0} \right) \left(x_1 - \hat{x}_{1|0} \right)' |y^0 \right] \\ =& \left(F - K_0 H \right) \Sigma_{0|-1} \left(F - K_0 H \right)' + Q + K_0 R K_0{}' \end{aligned} \tag{18}$$

- From this, we now know that

$$x_1 \sim \mathcal{N} \left(\hat{x}_{1|0}, \Sigma_{1|0} \right) \tag{19}$$

- We can now start over again, delivering the recursion we were looking for

Summary

At time t, given $\hat{x}_{t|t-1}, \Sigma_{t|t-1}$ and observing y_t

1. Compute the forecast error in the observations using

$$a_t = y_t - H\hat{x}_{t|t-1} \tag{20}$$

2. Compute the **Kalman Gain** K_t using

$$K_t = F\Sigma_{t|t-1}H'\left(H\Sigma_{t|t-1}H' + R\right)^{-1} \tag{21}$$

3. Compute the state forecast for next period given today's information

$$\hat{x}_{t+1|t} = F\hat{x}_{t|t-1} + K_t\left(y_t - H\hat{x}_{t|t-1}\right) = F\hat{x}_{t|t-1} + K_t a_t \tag{22}$$

4. Update the covariance matrix

$$\Sigma_{t+1|t} = (F - K_t H)\,\Sigma_{t|t-1}\,(F - K_t H)' + Q + K_t R K_t' \tag{23}$$

Initialization

- How to initialize filter at $t = 0$ where no observations are available?

  $\rightarrow$ start with **unconditional** mean $E(x)$ and Variance Σ

- Given covariance stationarity, the unconditional mean is

$$E(x) = Ex_{t+1} = E(Fx_t + w_{t+1}) = FE(x) \Rightarrow (I - F)E(x) = 0$$

  hence, $E(x) = 0$

- For the covariance matrix, we have

$$\begin{aligned} \Sigma &= E\left[(Fx_t + w_t)\,(Fx_t + w_t)'\right] \\ &= E\left[Fx_t{x_t}'F' + w_t{w_t}'\right] \\ &= F\Sigma F' + Q \end{aligned} \tag{24}$$

  $\rightarrow$ so-called **Lyapunov-equation**

Initialization

- If A, B, C are conformable matrices

$$vec(ABC) = (C' \otimes A)vec(B)$$

- Hence

$$vec(\Sigma) = (F \otimes F)vec(\Sigma) + vec(Q)$$

- This has solution

$$vec(\Sigma) = \left(I_{n_x^2} - (F \otimes F)\right)^{-1} vec(Q) \tag{25}$$

- Problem: Inversion of n_x^2 matrix $\rightarrow$ slooooow
- Alternative: so-called **doubling algorithm** or other algorithms Dynare offers in `lyapunov_symm`

Initialization: Diffuse Kalman Filter

- Consider model with unit root: does not violate Blanchard and Kahn (1980) conditions
- Problem: unconditional variance does not exist for some variables, only the conditional one

→ cannot use unconditional one for initialization

- Alternative: **diffuse Kalman filter** (De Jong 1991; Koopman and Durbin 2003) that considers initial state as diffuse vector (infinite variance)

  → requires specifying the `diffuse_filter`-option of Dynare

Kalman Smoothing and other Outputs

- The Kalman filter recursions provide us with **filtered variables** $E_t x_{t+1}$, i.e. the best prediction for tomorrow's state given information up to today
- We can also compute **updated variables** $E_t x_t$, i.e. our best estimate of the state today given information up to today (remember that x_t is not in the information set here!)
- Note: the terminology here follows Dynare as there seems to be no standard for naming these two objects
- More importantly, we are regularly interested in our best estimates of shocks and states given the full observed data up to time T
- The **Kalman Smoother** provides recursions for obtaining these estimates $E_T(x_t)$ and $E_T(w_t)$ by working backwards in time (for details, see the handout)

From Filtering to Estimation

- Typically we are not interested in filtering per se, but rather in estimating a model
- Start with complete history of observables up to time t: $y^t = \{y_t, \ldots, y_0\}$
- Likelihood function $f(y_T, \ldots, y_0)$ can be factored as

$$f(y_T, \ldots, y_0) = f\left(y_T | y^{T-1}\right) \times f\left(y_{T-1} | y^{T-2}\right) \times \ldots \times f\left(y_1 | y^0\right) \times f(y_0)$$

- Earlier we derived

$$y_t \sim \mathcal{N}\left(H\hat{x}_{t|t-1}, \underbrace{H\Sigma_{t|t-1}H' + R}_{\equiv \Omega_t}\right)$$

Log-likelihood

- Hence, the probability density of observing y_t given y^{t-1} is given by

$$f\left(y_t|y^{t-1}\right) = \frac{1}{\sqrt{(2\pi)^{n_y}\det\left(\Omega_t\right)}} e^{-\frac{1}{2}\left(y_t - H\hat{x}_{t|t-1}\right)'\Omega_t^{-1}\left(y_t - H\hat{x}_{t|t-1}\right)},$$

- Taking logs leads to the **log-likelihood function** for each observation:

$$\log\left(f\left(y_t|y^{t-1}\right)\right) = -\frac{n_y}{2}\log\left(2\pi\right) - \frac{1}{2}\log\left(\det\left(\Omega_t\right)\right) - \frac{1}{2}a_t{}'\Omega_t^{-1}a_t \tag{26}$$

$\rightarrow$ Kalman filter delivers everything needed to compute the likelihood

- Can be easily integrated into our VAR-ML-code

Optimality

- Kalman filter is basically a least squares estimator
- If initial conditions and innovations are normal, it is the best predictor! You cannot get better!
- Otherwise, it is the best linear predictor
- Can only be used to construct likelihood function for linear (asymptotically) Gaussian state-space systems

$\rightarrow$ works perfectly for first-order approximated DSGE models

$\rightarrow$ does not work with higher order approximations/non-linearities

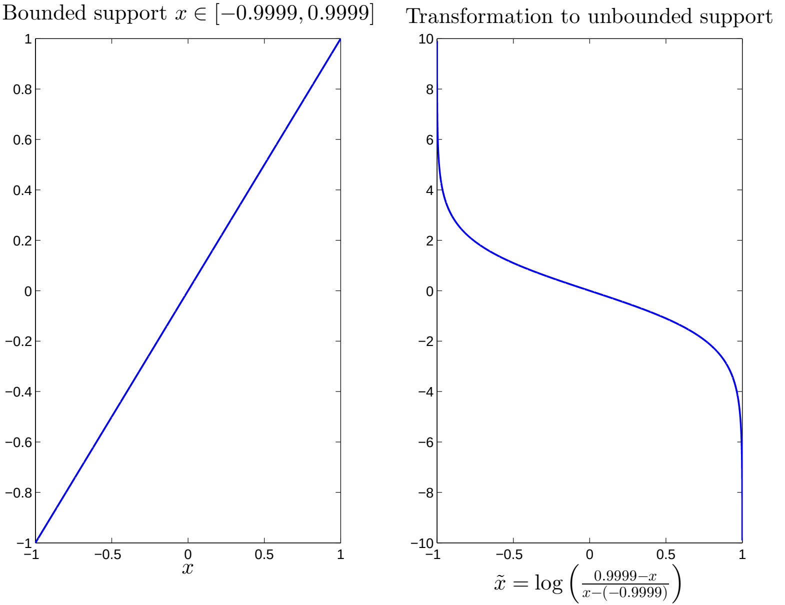
Finding the Global Maximum

- Finding a **global maximum** is hard!
- Newton-type optimizers are inherently **local**
- One way out: try numerous starting points
- Alternative: use **global optimizers** like simulated annealing (e.g. Corona, Marchesi, Martini, and Ridella 1987; Goffe, Ferrier, and Rogers 1994) or covariance matrix adaptive evolutionary strategy (CMA-ES) (Hansen, Müller, and Koumoutsakos 2003)
- Particularly CMA-ES (`mode_compute=9` in Dynare) seems to perform well in practice (Andreasen 2010)
- Dynare supports various different optimizers, calling them sequentially seems to work pretty well in hard cases

Constrained Numerical Optimization

- Most numerical optimizers are **unconstrained optimizers**
- Problem: economic parameters are typically **bounded**; e.g. $\beta \in [0, 1]$
- One way out: directly rely on constrained optimizers (e.g. `fmincon`)
- Or: use one-to-one transformation of parameter x into unbounded parameter $\tilde{x}$ before passing it to optimizer
- Then retransform $\tilde{x}$ back to x before using it in economic model inside of optimizer:
- $x \in [LB, Inf)$:
  $\tilde{x} = \log(x - LB) \Leftrightarrow x = LB + exp(\tilde{x})$
- $x \in (-Inf, UB]$:
  $\tilde{x} = \log(-x + UB) \Leftrightarrow x = UB - exp(\tilde{x})$
- $x \in [LB, UB]$:
  $\tilde{x} = \log\left(\frac{UB-x}{x-LB}\right) \Leftrightarrow x = \frac{UB+exp(\tilde{x})\times LB}{1+exp(\tilde{x})}$

Logistic Transformation for Autocorrelation Parameter



Steady State Kalman Filter

- It can be shown that the matrix recursions in the Kalman filter converge (See Hamilton 1994, Chapter 13.5)
- Kalman gain and forecast error variance matrix will settle on final values
- Continuing with recursions after they have settled within reasonable tolerance is computationally expensive

$\rightarrow$ check for convergence and then continue with fixed matrices

- Dynare by default uses this **steady state Kalman filter**

Numerical Issues and Invalid Parameter Draws

- There will generally be parameter draws for which the model cannot be solved (no valid steady state, BK conditions not satisfied)
- We could set their likelihood to 0, but many optimizers have issues with the arising cliffs
- Better: add **penalty function** that penalizes deviations from desired target so as to provide algorithms with a direction
- Also: when conducting Kalman filter recursions in the computer, various numerical problems can appear
- E.g. assuring perfect symmetry of covariance matrices in each step can improve the performance considerably
- It is often better to use existing routines than to reinvent the wheel

Observation Equations

- Up to this point, we have been silent on how to map the observed data to the model variables
- This is the topic of specifying the **observation equations**
- The biggest problem is stationarity in the model vs. trends in the data
- The guiding principle is: **the data variable needs to perfectly correspond to a variable defined in the model**
- A lengthy treatment of many possible cases is available in Pfeifer (2013)
- We will most of the time consider demeaned growth rates of variables together with a loglinear model
- This gives rise to a straightforward observation equation of the form: `y_obs=y-y(-1);`

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