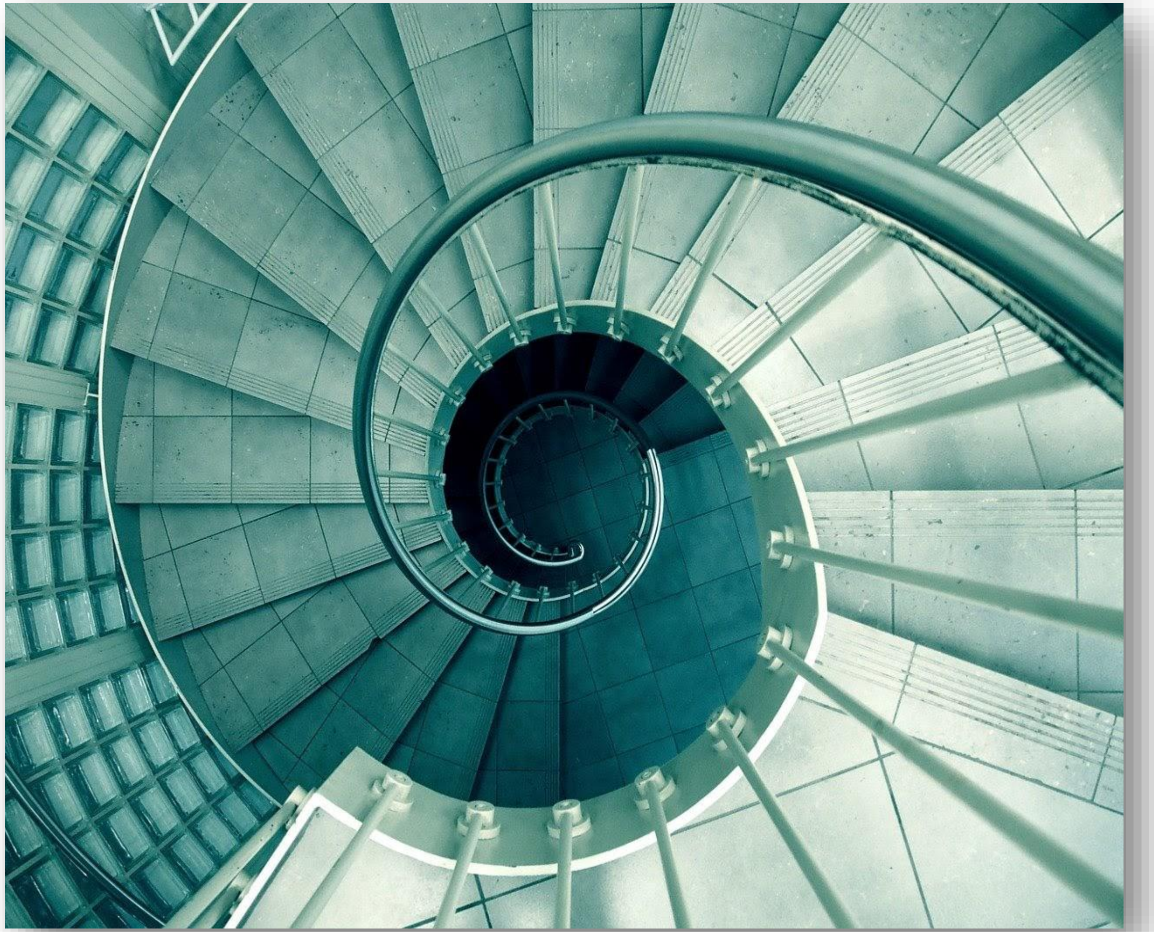
DSE 2256 DESIGN & ANALYSIS OF ALGORITHMS

Lecture 8 & 9

Mathematical Analysis of Recursive Algorithms



Recap of L6 & L7

- Mathematical analysis of non-recursive algorithms
  - General Plan for analysing time efficiency
  - Summation formulas
  - Algorithm : Max. element in an array
  - Algorithm : Unique elements in an array
  - Algorithm : Multiplication of two n x n matrices

Mathematical analysis of Recursive Algorithms I

- **Example 1: Factorial of a number**

```
ALGORITHM F(n)
    //Computes n! recursively
    //Input: A nonnegative integer n
    //Output: The value of n!
    if n = 0 return 1
    else return F(n − 1) ∗ n
```

Input size	:	**n**
Basic Operation	:	**Multiplication**
Recurrence relation	:	**M(n) = M(n-1) + 1**
		M(0) = 0

- **Definition:**

  $n! = 1 * 2 * \ldots *(n\text{-}1) * n,$ for $n \geq 1$ and,
  $0! = 1$

- **Recursive definition:**

  $F(n) = F(n\text{-}1) * n,$ for $n \geq 1$ and,

  $F(0) = 1$

$M(0) = 0.$

the calls stop when $n = 0$

no multiplications when $n = 0$

Mathematical analysis of Recursive Algorithms II

- **Example 1: Solving the recurrence for M(n)**

$M(n) = M(n-1) + 1$

$M(0) = 0$

$$
\begin{aligned}
\mathbf{M(n)} &= \mathbf{M(n-1) + 1} \\
&= \mathbf{(M(n-2) + 1) + 1 \quad = \quad M(n-2) + 2} \\
&= \mathbf{(M(n-3) + 1) + 2 \quad = \quad M(n-3) + 3} \\
&\ldots \\
&= \mathbf{M(n-i) + i} \\
&\ldots \\
&= \mathbf{M(0) + n} \\
&= \mathbf{n}
\end{aligned}
$$

$M(0) = 0$

The method is called **backward substitution.**

Mathematical analysis of Recursive Algorithms III

- **Example 1: Solving the recurrence for M(n)**

  $M(n) = M(n-1) + 1$

  $M(0) = 0$

  $\mathbf{M(n) = M(n-1) + 1}$

  $\mathbf{= (M(n-2) + 1) + 1 \quad = \quad M(n-2) + 2}$

  $\mathbf{= (M(n-3) + 1) + 2 \quad = \quad M(n-3) + 3}$

  …

  $\mathbf{= M(n-i) + i}$

  … $\longrightarrow$ M(0) = 0

  $\mathbf{= M(0) + n}$

  $\mathbf{= n}$

  The method is called **backward substitution.**

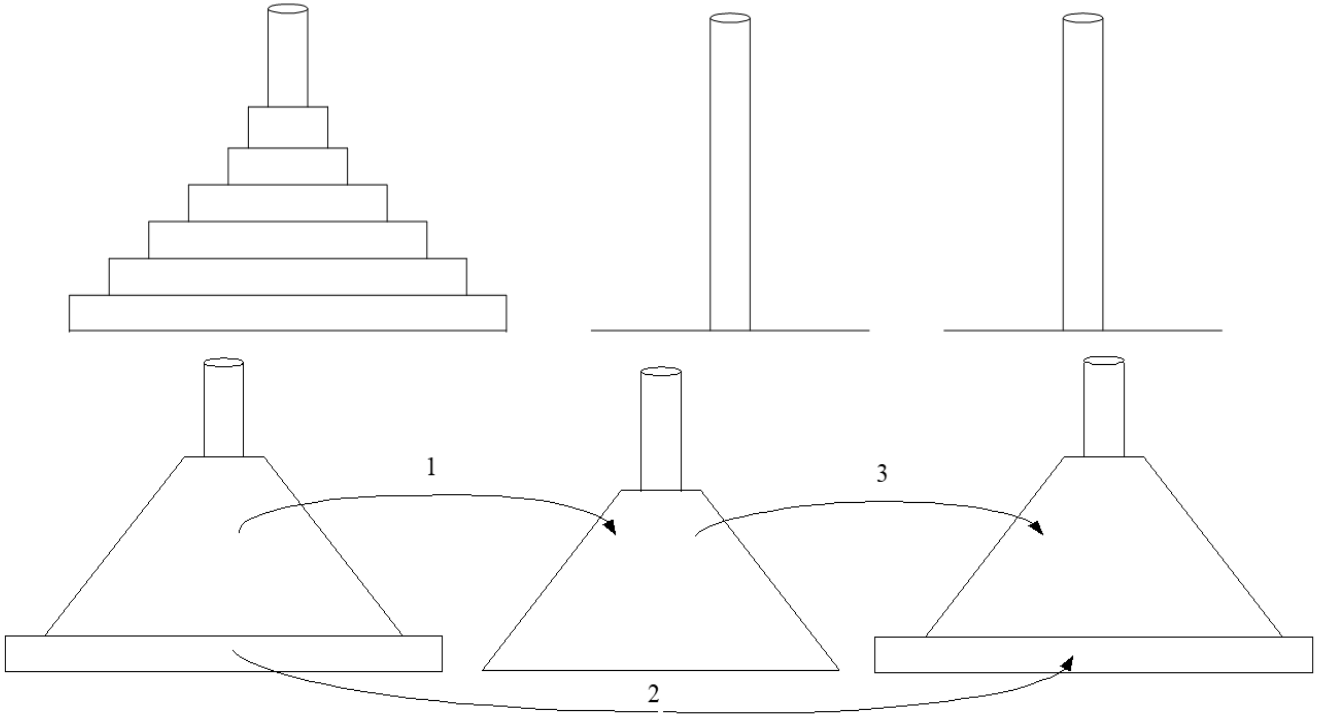
Mathematical analysis of Recursive Algorithms IV

- **General Plan for Analysis of Recursive Algorithms:**

1. Decide on a parameter indicating an input's size.
2. Identify the algorithm's basic operation.
3. Check whether the number of times the basic operation is executed may vary on different inputs of the same size.
4. Set up a recurrence relation with an appropriate initial condition expressing the number of times the basic operation is executed.
5. Solve the recurrence by backward substitutions or another method.

Mathematical analysis of Recursive Algorithms V

- **Example 2: Tower of Hanoi**



ALGORITHM Tower_of_hanoi (n,s,d,a)
// Recursively moves disks from source (s) to destination (d)

// Input : the no. of disks (n), source (s), destination (d), auxiliary (a) towers. Initially, a and d towers are empty.

// Output is the sequence of moves

```
if (n==1) then do
   print ("Move from s to d")
   return

else

Tower_of_hanoi (n-1, s, a, d)

print ("Move from s to d")

Tower_of_hanoi (n-1, a, d, s)

end if
```

Mathematical analysis of Recursive Algorithms VI

Solving recurrence for number of moves:

$M(n) = 2M(n-1) + 1,$
$M(1) = 1$

$M(n) = 2M(n-1) + 1$

$= 2(2M(n-2) + 1) + 1 = 2^2*M(n-2) + 2^1 + 2^0$

$= 2^2 *(2M(n-3) + 1) + 2^1 + 2^0$

$= 2^3*M(n-3) + 2^2 + 2^1 + 2^0$

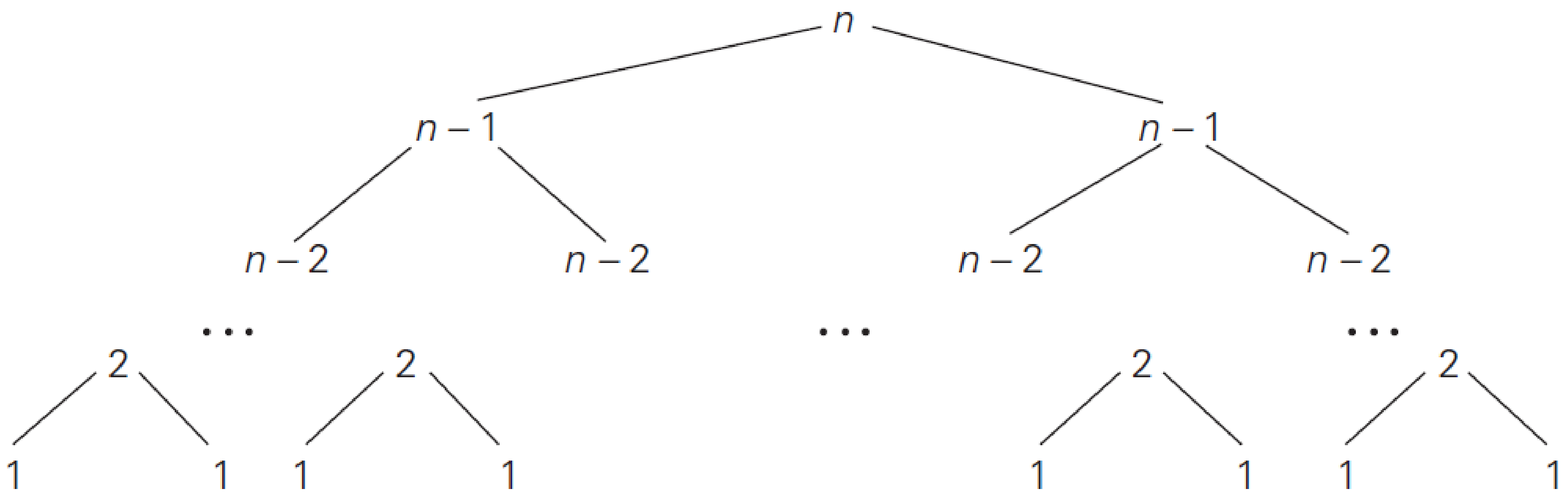
$= \ldots$

$= 2^{n-1}*M(1) + 2^{n-2} + \ldots + 2^1 + 2^0$

$= 2^{n-1} + 2^{n-2} + \ldots + 2^1 + 2^0$

$= \mathbf{2^n - 1}$

Mathematical analysis of Recursive Algorithms VII



Total no. of calls made by the Tower of Hanoi algorithm:

$$C(n) = \sum_{l=o}^{n-1} \mathbf{2}^{l} = 2^{n} - 1$$

Thank you!

Any queries?