Intro to Differential Equations: A Summary

William Boyles

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Chapter 0

Background & Review

Everything mentioned in this chapter should already be familiar to you from other math classes. These topics span three main areas: algebra/pre-calculus, single variable calculus, and matrices. These topics will be used either implicitly or with only a passing reference.

If you are unfamiliar with anything mentioned, you can use many of the great online resources, like Khan Academy, to familiarize yourself before moving forward.

0.1 Algebra and Pre-Calculus

0.1.1 Complex Numbers

Definition. *i is called the imaginary unit. It's defined by $i^2 = -1$.*

Complex numbers ($\mathbb{C}$) have the form $z = \alpha + \beta i$, where α and β are real numbers. The α part of z is called the real part, so $\Re(z) = \alpha$. The β part of z is called the imaginary part, so $\Im(z) = \beta i$.

Often, complex numbers are visualized as points or vectors in a 2D plane, called the complex plane, where α is the x-component, and β is the y-component. Thinking of complex numbers like points helps us define the magnitude of complex numbers and compare them. Since a point (x, y) has a distance $\sqrt{x^2 + y^2}$ from the origin, we can say the magnitude of z, $|z|$ is $\sqrt{\alpha^2 + \beta^2}$. Thinking of complex numbers like vectors helps us understand adding two complex numbers, since you just add the components like vectors.

A common operation on complex numbers is the complex conjugate. The complex conjugate of $z = \alpha + \beta i$ is $\overline{z} = \alpha - \beta i$. z and $\overline{z}$ are called a conjugate pair.

Conjugate pairs have the following properties. Let $z, w \in \mathbb{C}$.

$$\begin{aligned}
\overline{z \pm w} &= \overline{z} \pm \overline{w} \\
\overline{zw} &= \overline{zw} \\
\overline{z} &= z \Leftrightarrow z \in \mathbb{R} \\
z\overline{z} &= |z|^2 = |\overline{z}|^2 \\
\overline{\overline{z}} &= z \\
\overline{z}^n &= \overline{z^n} \\
z^{-1} &= \frac{\overline{z}}{|z|^2}
\end{aligned}$$

0.1.2 Factoring Polynomials

We want to break up a polynomial like $f(x) = a_0 + a_1x^1 + \ldots a_nx^n$ into linear factors so that $f(x) = c(x - b_1) \cdot \ldots \cdot (x - b_n)$. This form makes it simple to see that the roots of f, solutions to $f(x) = 0$, are $x = b_1 \ldots b_n$.

For quadratics, $f(x) = ax^2 + bx + c$, there exists a simple formula that will give us both roots, the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

We can see that when $b^2 - 4ac < 0$, like for $f(x) = x^2 + 5x + 10$, we will get complex roots $\alpha \pm \beta i$. For any polynomial, these roots come in pairs, so if $\alpha + \beta i$ is a root, then so is $\alpha - \beta i$. This means that every conjugate pair $\alpha \pm \beta i$ has a quadratic equation with those roots. Sometimes we will not factor quadratics with complex roots into linear terms.

Although there do exist explicit formulas for finding roots for cubic (degree 3) and quartic (degree 4) equations, they are too long and not useful enough to memorize. When working by hand, we instead use other tricks to find roots.

There are a few useful tricks that can help. If the polynomial doesn't have a constant term, then 0 is a root. If all the coefficients sum to 0, then 1 is a root. For certain polynomials with an even number of terms, like all cubics of the form $ax^3 + bx^2 + cax + cb$ we can factor out a term from the first two and last two terms to get $x^2(ax + b) + c(ax + b) = (ax + b)(x^2 + c)$. For other polynomials, we might just try guessing and checking values. However, we need a more efficient way that works in general.

Since we are looking to find linear factors $f(x) = (x - b_1) \cdot \ldots \cdot (x - b_n)$, we can see that the constant term in the polynomial is the product of the roots $b_1 \ldots b_n$. In fact, since the coefficients of polynomials are completely determined by the roots and the leading coefficient, all the coefficients are sums and products of roots. You might remember when factoring quadratics that the coefficient of x term is the sum of the two roots. These rules are called

Vieta's formulas.

So, if we have the constant term, we can check all of its integer factors to see if any are roots. For each root, we can divide, using a technique like synthetic division, to continue finding the rest of the roots. This method is especially useful on tests because the roots tend to be integers.

Example. *Factor the polynomial* $x^5 + x^4 - 2x^3 + 4x^2 - 24x$.

We can immediately see that there is no constant term, so $x = 0$ is a root. Now we need to work on factoring $x^4 + x^3 - 2x^2 + 4x - 24$.
The factors of -24 are: -24, -12, -8, -6, -4, -3, -2, -1, 1, 2, 3, 4, 6, 8, 12, and 24. Starting from roots close to 0 and working outwards, we find that $x = 2$ is a root. So, we synthetic divide like so

$$\begin{array}{r|ccccc} x=2 & 1 & 1 & -2 & 4 & -24 \\ & \downarrow & 2 & 6 & 8 & 24 \\ \hline & 1 & 3 & 4 & 12 & |\,0 \end{array}$$

to see that now we need to work on factoring $x^3 + 3x^2 + 4x + 12$. $x^3 + 3x^2 + 4x + 12 = x^2(x+3) + 4(x+3) = (x+3)(x^2+4)$, so $x = -3$ is a root, and we need to work on factoring $x^2 + 4$. $x^2 + 4$ has two complex roots $\pm 2i$, so we'll leave it as a quadratic.

$$x^5 + x^4 - 2x^3 + 4x^2 - 24x = x(x-2)(x-3)(x^2+4)$$

0.1.3 Trig Functions & The Unit Circle

Imagine aa circle of radius 1 centered at the origin that we'll call the unit circle. The x and y coordinates of a point on the unit circle are completely determined by the angle θ in radians between the x-axis and a line from the origin to the point.

The function $\cos\theta$ tells us x-coordinate of the point, while $\sin\theta$ tells us the y-coordinate of the point. The function $\tan\theta = \frac{\sin\theta}{\cos\theta}$ tells us the slope of the line from the origin to the point. Most of the trig functions have geometric interpretations as shown below. The most used ones are sin, cos, tan $= \frac{\sin}{\cos}$, cot $= \frac{\cos}{\sin}$, csc $= \frac{1}{\sin}$, and sec $= \frac{1}{\cos}$.

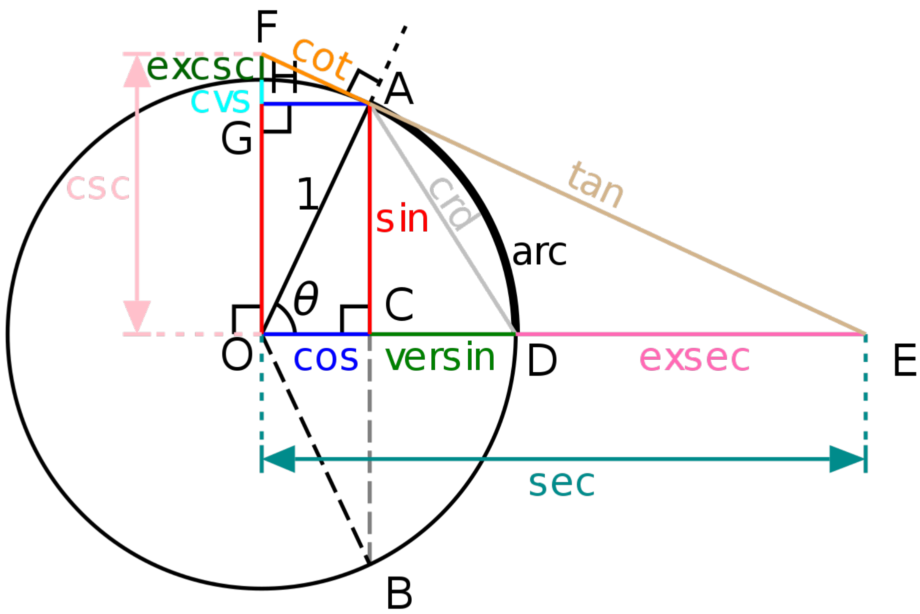


Figure 1: Wikipedia - Unit circle

We can also think about the inverses of these trig functions. These are either notated with a -1 exponent on the function, or the prefix arc in front of the function name. Many of these functions are only defined on a part of the domain $[0, 2\pi]$. Below is a table of the inverse trig functions and their domains.

Function	Domain
arcsin	$[-1, 1]$
arccos	$[-1, 1]$
arctan	$(-\infty, \infty)$
arccot	$(-\infty, \infty)$
arccsc	$(-\infty, -1] \cup [1, \infty)$
arcsec	$(-\infty, -1] \cup [1, \infty)$

0.1.4 Trig Identities

As we could see in Figure 0.1.3, sin and cos form a right triangle with hypotenuse 1. So, using the Pythagorean Theorem,

$$\sin^2\theta + \cos^2\theta = 1.$$

By dividing by $\sin^2$ or $\cos^2$, we can also get

$$1 + \cot^2\theta = \csc^2\theta \text{ and } \tan^2\theta + 1 = \sec^2\theta.$$

Together, these 3 identities are called the Pythagorean Identities.

We can also relate functions and co-functions.

$$\mathrm{xxx}(\theta) = \mathrm{coxxx}\left(\frac{\pi}{2} - \theta\right).$$

Some of the most useful and used identities are the sum and difference.

$$\begin{aligned}
\sin(\alpha \pm \beta) &= \sin\alpha\cos\beta \pm \cos\alpha\sin\beta \\
\cos(\alpha \pm \beta) &= \cos\alpha\cos\beta \mp \sin\alpha\sin\beta \\
\tan(\alpha \pm \beta) &= \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta} \\
\sin\alpha \pm \sin\beta &= 2\sin\left(\frac{\alpha \pm \beta}{2}\right)\cos\left(\frac{\alpha \mp \beta}{2}\right) \\
\cos\alpha + \cos\beta &= 2\cos\left(\frac{\alpha + \beta}{2}\right)\cos\left(\frac{\alpha - \beta}{2}\right) \\
\cos\alpha - \cos\beta &= -2\sin\left(\frac{\alpha + \beta}{2}\right)\sin\left(\frac{\alpha - \beta}{2}\right)
\end{aligned}$$

0.1.5 Exponentials & Logarithms

Definition. *e is the base of the natural logarithm. It's defined by the limit*

$$e = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n.$$

$\exp x = e^x$ and $\ln x$ are inverse functions of each other such that

$$e^{\ln x} = x \text{ and } \ln e^x = x.$$

Just like other exponentials, the normal rules for adding, subtracting, and multiplying exponents apply:

$$e^x e^y = e^{x+y},\ \frac{e^x}{e^y} = e^{x-y},\text{ and } (e^x)^k = e^{xk}.$$

Similar rules apply for logarithms:

$$\ln x + \ln y = \ln xy,\ \ln x - \ln y = \ln\left(\frac{x}{y}\right),\text{ and } \ln\left(a^b\right) = b\ln a.$$

We can also write a logarithm of any base using natural logarithms:

$$\log_b a = \frac{\ln a}{\ln b}.$$

e is also unique in that it is the only real number a satisfying the equation

$$\frac{\mathrm{d}}{\mathrm{d}x}a^x = a^x,$$

meaning e^x is its own derivative.

0.1.6 Partial Fractions

If we have a function of two polynomials $f(x) = \frac{P(x)}{Q(x)}$, it's often easier to break this quotient into a sum of parts where the denominator is a linear or quadratic factor and the numerator is always a smaller degree than the denominator.

Example.

$$\frac{2x-1}{x^3-6x^2+11x-6} = \frac{1/2}{x-1} + \frac{-3}{x-2} + \frac{5/2}{x-3}.$$

One natural way to find these small denominators comes from the linear factors of the denominator where we keep quadratics with complex roots. This way, when making a common denominator, we get back the original big denominator. However, there are a few special cases we have to take care of.

Linear Factors

This is the the most basic type where the degree of the numerator is less than the degree of the denominator and the denominator factors into all linear factors with no repeated roots. In this case we can write

$$\frac{P(x)}{Q(x)} = \frac{A_1}{(x-a_1)} + \ldots + \frac{A_n}{(x-a_n)}.$$

Multiplying each side by $Q(x)$,

$$P(x) = A_1(x-a_2)\ldots(x-a_n) + \ldots + A_n(x-a_1)\ldots(x-a_{n-1}).$$

We can then find each A_i by evaluating both sides at $x = a_i$, since every term except the ith has an $(x-a_i)$ factor that will go to 0. So,

$$A_i = \frac{P(a_i)}{(x-a_i)\ldots(x-a_{i-1})(x-a_{i+1})\ldots(x-a_n)}.$$

Example. *Find the partial fraction decomposition of the following expression:*

$$\frac{2x-1}{x^3-6x^2+11x-6}.$$

Factoring,

$$x^3-6x^2+11x-6 = (x-1)(x-2)(x-3).$$

So,

$$\frac{2x-1}{x^3-6x^2+11x-6} = \frac{A_1}{x-1} + \frac{A_2}{x-2} + \frac{A_3}{x-3}.$$

Multiplying each side by the denominator,

$$2x-1 = A_1(x-2)(x-3) + A_2(x-1)(x-3) + A_3(x-1)(x-2).$$

At $x = 1$,

$$1 = A_1(1-2)(1-3) \implies A_1 = \frac{1}{2}.$$

At $x = 2$,

$$3 = A_2(2-1)(2-3) \implies A_2 = -3.$$

At $x = 3$,

$$5 = A_3(3-1)(3-2) \implies A_3 = \frac{5}{2}.$$

So,

$$\frac{2x-1}{x^3-6x^2+11x-6} = \frac{1/2}{x-1} + \frac{-3}{x-2} + \frac{5/2}{x-3},$$

just as was shown in the previous example.

Repeated Linear Factors

If $Q(x)$ has repeated roots, it factors into

$$Q(x) = R(x)(x-a)^k,\ k \geq 2 \text{ and } R(a) \neq 0.$$

When making the common denominator for each repeated root of multiplicity k, we do

$$\frac{P(x)}{R(x)(x-a)^k} = (\text{Decomposition of } R(x)) + \frac{A_1}{x-a} + \ldots + \frac{A_k}{(x-a)^k}.$$

You would then multiply each side by the denominator like in the linear factors case and solve for the coefficients. The only additional difficulty is that you might have to use previous results or solve a system of linear equations to get some of the constants.

Example. *Find the partial fraction of the following expression:*

$$\frac{x^2+5x-6}{x^3-7x^2+16x-12}.$$

Factoring,

$$x^3-7x^2+16x-12 = (x-3)(x-2)^2.$$

So,

$$\frac{x^2+5x-6}{x^3-7x^2+16x-12} = \frac{A_1}{x-3} + \frac{A_2}{x-2} + \frac{A_3}{(x-2)^2}.$$

Multiplying each side by the denominator,

$$x^2+5x-6 = A_1(x-2)^2 + A_2(x-2)(x-3) + A_3(x-3).$$

At $x = 2$,

$$8 = A_3(2-3) \implies A_3 = -8.$$

At $x = 3$,
$$18 = A_1(3-2)^2 \implies A_1 = 18.$$
Now we'll use our results for A_1 and A_3 to find A_2 using a value for x that isn't 2 or 3 so the A_2 term doesn't become 0. A good choice is $x = 0$.
At $x = 0$,
$$-6 = 18(0-2)^2 + A_2(0-2)(0-3) + -8(0-3) \implies A_2 = -17.$$
So,
$$\frac{x^2+5x-6}{x^3-7x^2+16x-12} = \frac{18}{x-3} - \frac{17}{x-2} - \frac{8}{(x-2)^2}.$$

Quadratic Factors

If a quadratic doesn't have real roots, then we have a quadratic factor. Here, we'll assume that the quadratic factor isn't repeated. So, $Q(x) = R(x)(ax^2+bx+c)$, $b^2-4ac<0$, and $R(x)$ is not evenly divisible by ax^2+bx+c. In this case, we say
$$\frac{P(x)}{R(x)(ax^2+bx+c)} = (\text{Decomposition of } R(x)) + \frac{A_1x+B_1}{ax^2+bx+c}.$$
We then solve for the constants in the numerator, possibly having to solve a system of equations or using previous results and less convenient values for x.

Example. *Find the partial fraction decomposition of the following expression:*
$$\frac{6x^2+21x+11}{x^3+5x^2+3x+15}.$$
Factoring,
$$x^2+5x^2+3x+15 = (x+5)(x^2+3).$$
So,
$$\frac{6x^2+21x+11}{x^3+5x^2+3x+15} = \frac{A_1}{x+5} + \frac{A_2x+B_2}{x^2+3}.$$
Multiplying each side by the denominator,
$$6x^2+21x+11 = A_1(x^2+3) + (A_2x+B_2)(x+5).$$
At $x = -5$,
$$56 = 28A_1 \implies A_1 = 2.$$
Now we'll use the previous result and another value for x. We can use $x = 0$ to not have to worry about the A_2 term. At $x = 0$,
$$11 = 2(3) + (B_2)(5) \implies B_2 = 1.$$

Now we'll use the previous 2 results to find A_2. $x = 1$ is a good choice to keep the numbers small. At $x = 1$,

$$38 = 2(1 + 3) + (A_2 + 1)(6) \implies A_2 = 4.$$

So,

$$\frac{6x^2 + 21x + 11}{x^3 + 5x^2 + 3x + 15} = \frac{2}{x + 5} + \frac{4x + 1}{x^2 + 3}.$$

Repeated Quadratic Factors

If a quadratic factor that can't be broken into linear factors is repeated, then we can write $Q(x) = R(x)(ax^2 + bx + c)^k$, $k \geq 0$, and $R(x)$ is not divisible by $(ax^2 + bx + c)^k$. Now we have to do a combination of what we did for repeated linear factors and quadratic factors. We say

$$\frac{P(x)}{R(x)(ax^2 + bx + c)^k} = (\text{Decomposition of } R(x)) + \frac{A_1 x + B_1}{ax^2 + bx + c} + \ldots + \frac{A_k x + B_k}{(ax^2 + bx + c)^k}.$$

We then solve for the coefficients in the numerator.

Example. *Find the partial fraction decomposition of* $\frac{3x^4 - 2x^3 + 6x^2 - 3x + 3}{x^5 + 3x^4 + 4x^3 + 12x^2 + 4x + 12}$.

Factoring,

$$x^5 + 3x^4 + 4x^3 + 12x^2 + 4x + 12 = (x + 3)(x^2 + 2)^2.$$

So,

$$\frac{3x^4 - 2x^3 + 6x^2 - 3x + 3}{x^5 + 3x^4 + 4x^3 + 12x^2 + 4x + 12} = \frac{A_1}{x + 3} + \frac{A_2 x + B_2}{x^2 + 2} + \frac{A_3 x + B_3}{(x^2 + 2)^2}.$$

Multiplying each side by the denominator,

$$3x^4 - 2x^3 + 6x^2 - 3x + 3 = A_1(x^2 + 2)^2 + (A_2 x + B_2)(x^2 + 2)(x + 3) + (A_3 x + B_3)(x + 3).$$

At $x = -3$,

$$363 = 121A_1 \implies A_1 = 3.$$

Now, we'll use our result for A_1 and pick a value for x that minimizes the number of things we need to solve for. We'll have to solve a linear system with 4 unknowns, so we'll need up to 4 values. At $x = 0$,

$$3 = 3(2)^2 + B_2(2)(3) + B_3(3) \implies 2B_2 + B_3 = -3.$$

At $x = 1$,

$$7 = 3(3)^2 + (A_2 + B_2)(3)(4) + (A_3 + B_3)(4) \implies 3A_2 + A_3 + 3B_2 + B_3 = -5.$$

At $x = -1$,

$$17 = 3(3)^2 + (-A_2 + B_2)(3)(2) + (-A_3 + B_3)(2) \implies -3A_2 - A_3 + 3B_2 + B_3 = -5.$$

At $x = 2$,

$$53 = 3(6)^2 + (2A_2 + B_2)(6)(5) + (2A_3 + B_3)(5) \implies 12A_2 + 2A_3 + 6B_2 + B_3 = -11.$$

Now we have the following system of equations:

$$\begin{cases} 0A_2 + 0A_3 + 2B_2 + B_3 & = -3 \\ 3A_2 + A_3 + 3B_2 + B_3 & = -5 \\ -3A_2 - A_3 + 3B_2 + B_3 & = -5 \\ 12A_2 + 2A_3 + 6B_2 + B_3 & = -11 \end{cases}.$$

Solving,

$$A_2 = 0, A_3 = 0, B_2 = -2, \text{ and } B_3 = 1.$$

So,

$$\frac{3x^4 - 2x^3 + 6x^2 - 3x + 3}{x^5 + 3x^4 + 4x^3 + 12x^2 + 4x + 12} = \frac{3}{x+3} - \frac{2}{x^2+2} + \frac{1}{(x^2+2)^2}.$$

Improper Fractions

If the degree of the numerator is greater than or equal to the degree of the denominator, we have a case of improper fractions. In this case, we have to do polynomial long division to get a quotient and remainder and then decompose the remainder if necessary. So,

$$\frac{P(x)}{Q(x)} = R(x) + \frac{S(x)}{Q(x)}.$$

Example. *Find the partial fraction decomposition of the following expression:*

$$\frac{x^3 + 3}{x^2 - 2x - 3}.$$

First we do polynomial long division to find that

$$\frac{x^3 + 3}{x^2 - 2x - 3} = x + 2 + \frac{7x + 9}{x^2 - 2x - 3}.$$

Now that the numerator is of a lesser degree than the denominator, we can decompose it normally.

$$x^2 - 2x - 3 = (x - 3)(x + 1).$$

So,

$$\frac{7x + 9}{x^2 - 2x - 3} = \frac{A_1}{x - 3} + \frac{A_2}{x + 1}.$$

Multiplying each side by the denominator,

$$7x + 9 = A_1(x + 1) + A_2(x - 3).$$

At $x = -1$,

$$2 = -4A_2 \implies A_2 = \frac{-1}{2}.$$

At $x = 3$,

$$30 = 4A_1 \implies A_1 = \frac{15}{2}.$$

So,

$$\frac{x^3 + 3}{x^2 - 2x - 3} = x + 2 + \frac{15/2}{x - 3} + \frac{-1/2}{x + 1}.$$

0.2 Single Variable Calculus

0.2.1 Derivatives and Integrals

Derivatives

The derivative of a function $y = f(x)$, notated $f'(x)$, gives the slope of the tangent line to f at x.

Definition.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Below are some properties of the derivative. Let f and g be functions of x and p a scalar.

Linearity

$$(pf \pm g)' = pf' \pm g'$$

Product Rule

$$(fg)' = f'g + fg'$$

Quotient Rule

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

Chain Rule

$$(f \circ g)' = (f' \circ g) \cdot g'$$

Power Rule

$$\frac{\mathrm{d}x^x}{\mathrm{d}p} = px^{p-1},\, p \neq 0$$

Exponent Rule

$$\frac{\mathrm{d}x^x}{\mathrm{d}p} = p^x \ln p,\, p > 0$$

The Power Rule and Exponent Rule are two cases of the same rule

$$\frac{\mathrm{d}}{\mathrm{d}x}f^g = gf^{g-1}f' + f^g \ln f g'.$$

Using the definition of the derivative and these rules, we can find the derivatives to some common functions.

$$\begin{array}{r|l} \frac{\mathrm{d}}{\mathrm{d}x}p = 0 & \frac{\mathrm{d}}{\mathrm{d}x}e^x = e^x \\ \frac{\mathrm{d}}{\mathrm{d}x}\ln x = \frac{1}{x} & \frac{\mathrm{d}}{\mathrm{d}x}\sin x = \cos x \\ \frac{\mathrm{d}}{\mathrm{d}x}\cos x = -\sin x & \frac{\mathrm{d}}{\mathrm{d}x}\tan x = \sec^2 x \end{array}$$

Integrals

The definite integral of a function $f(x)$ from $x = a$ to $x = b$ where $a \leq b$ is the area between $f(x)$ and the x-axis bounded by the lines $x = a$ and $x = b$ where area above the x-axis is positive, and area below the x-axis is negative.

Definition.

$$\int_a^b f(x)\mathrm{d}x = \lim_{h\to 0}\sum_{n=1}^{\frac{b-a}{h}} f(a + (n-1)h) \cdot h.$$

We also define an indefinite integral, or antiderivative of $f(x)$, notated $F(x)$ where

$$F'(x) = f(x) \implies \int f(x)\mathrm{d}x = F(x).$$

Note that there are infinitely many such functions F, since adding a constant to F does not affect its derivative. To notate this, we add a constant C to the indefinite integral. Given an initial condition for f, we can solve for C.

Below are some properties of the integral. Let f and g be functions of x and p, a, b, and c where $a < b < c$, and f and g are continuous on the closed interval $[a, c]$.

Linearity

$$\int (pf \pm g)\mathrm{d}x = p\int f\mathrm{d}x \pm \int g\mathrm{d}x$$

Flipped Bounds

$$\int_a^b f\mathrm{d}x = -\int_b^a f\mathrm{d}x$$

Union of Intervals

$$\int_a^b f\mathrm{d}x + \int_b^c f\mathrm{d}x = \int_a^c f\mathrm{d}x$$

Power Rule

$$\int x^n \mathrm{d}x = \frac{x^{n+1}}{n+1} + C,\, n \neq -1$$

U-Substitution

$$\int (f' \circ g)\, g' \mathrm{d}x = f \circ g + C$$

Integration by Parts

$$\int f' g \mathrm{d}x = fg - \int f g' \mathrm{d}x$$

Fundamental Theorem of Calculus

$$\frac{\mathrm{d}}{\mathrm{d}x} \int_a^x f(s) \mathrm{d}s = f(x)$$

Using the definition of the integral and the above rules, we can find the indefinite integral of some common functions.

$$\int \frac{1}{x} \mathrm{d}x = \ln |x| + C$$
$$\int \sin x \mathrm{d}x = -\cos x + C$$
$$\int \cos x \mathrm{d}x = \sin x + C$$
$$\int \tan x \mathrm{d}x = -\ln |\cos x| + C$$

0.2.2 Taylor Series

A Taylor series as a way of approximating a function about a point $x = a$ using polynomials. The first approximation just keeps the same value at $x = a$, the second approximation keeps the same value and first derivative at $x = a$, etc.

Definition.

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \ldots$$

If we approximate a function about $x = 0$, we call this a Maclaurin series. Below are some common Maclaurin series, and their radii of convergence if applicable.

$$
\begin{aligned}
e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \\
\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \\
\cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots \\
\frac{1}{1+x} &= 1 - x + x^2 - \dots, \text{ where } \left|x\right| < 1 \\
\ln(1+x) &= x - \frac{x^2}{2} + \frac{x^3}{3} - \dots, \text{ where } \left|x\right| < 1
\end{aligned}
$$

Euler's Identity

Let's see what happens when we look at the Maclaurin series for e^{ix}.

$$
\begin{aligned}
e^{ix} &= 1 + (ix) + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} \dots \\
&= 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} - \dots \\
&= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right).
\end{aligned}
$$

The two expressions in parenthesis are exactly the Maclaurin series for $\cos x$ and $\sin x$. So,

$$e^{ix} = \cos x + i \sin x.$$

In the case that $x = \pi$,

$$e^{i\pi} = \cos \pi + i \sin \pi = -1 + 0.$$

So,

$$e^{i\pi} + 1 = 0.$$

0.3 Vectors and Matrices

0.3.1 Vectors

A vector is a quantity with both direction and magnitude. One can think of it as a directed line segment. In multivariable calculus, we mostly will work with vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$, but vectors can exist in other dimensions.

Numerical (scalar) quantities have vector analogues, many of which show up in physics. Speed becomes velocity, distance becomes displacement, and mass becomes weight.

Say we have a 2D vector, $\vec{v} = \langle v_x, v_y \rangle$.

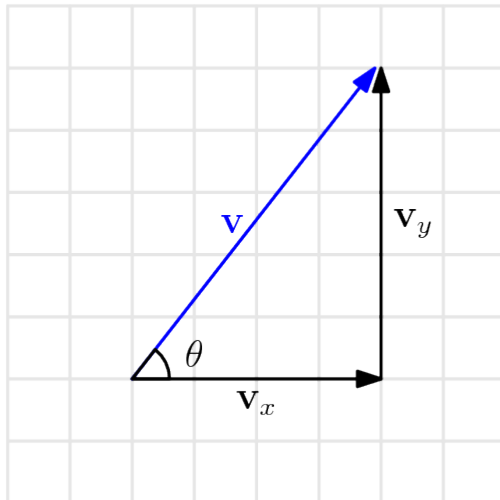


Figure 2: The x and y components of a vector v

Its length, also called magnitude or norm, is notated $\left|\left|\vec{v}\right|\right| = \sqrt{v_x^2 + v_y^2}$. This pattern of the norm being equal to the square-root of the sum of the squares of the vector's components continues into higher dimensions.

The angle a 2D vector forms with the horizontal axis is $\theta = \tan^{-1}\left(\frac{v_y}{v_x}\right)$. There is not a useful version of this formula in higher dimensions. Using θ and $\left|\left|\vec{v}\right|\right|$, we can see that $v_x = \left|\left|\vec{v}\right|\right| \cos\theta$ and $v_y = \left|\left|\vec{v}\right|\right| \sin\theta$.

Vectors can be added and subtracted from each other in a way that the result is another vector. We do this numerically by adding the corresponding components of each vector. For example, if $\vec{a} = \langle 1, 3 \rangle$ and $\vec{b} = \langle 4, 7 \rangle$, then $\vec{a} + \vec{b} = \langle 1+4, 3+7 \rangle = \langle 5, 10 \rangle$ and $\vec{b} - \vec{a} = \langle 4-1, 7-3, \rangle = \langle 3, 4 \rangle$.
Visually, you can think of $\vec{v}+\vec{w}$ as the vector connecting the tail of $\vec{v}$ with the tip of $\vec{w}$ where the tail of $\vec{v}$ is on the tip of $\vec{w}$.

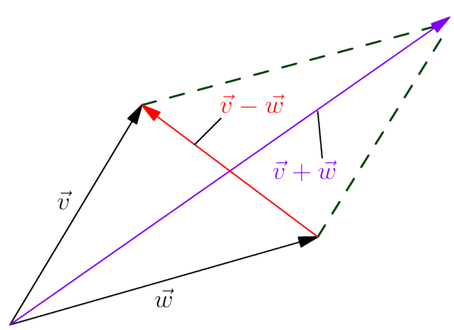


Figure 3: Visualization of $\vec{v} + \vec{w}$ and $\vec{v} - \vec{w}$

We can also multiply vectors by scalars and get another vector as a result. We do this by multiplying each component of the vector by the scalar. This has the effect of stretching or shrinking the vector and possibly changing the vector's direction if the scalar is negative.

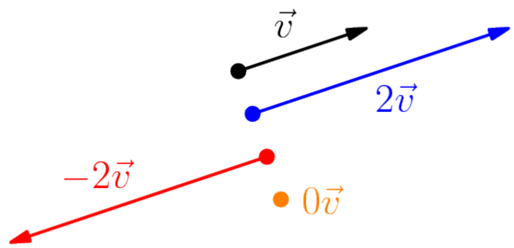


Figure 4: A vector $\vec{v}$ scaled by different constants

A unit vector is any vector with magnitude 1. Rather than using an arrow like for other vectors, unit vectors are notated with a carat ($\wedge$) over top, like $\hat{\imath}$, which is read as "i hat". We can transform any vector with non-zero magnitude into a unit vector by dividing the vector by its norm. This normalized vector will point in the same direction as the original vector.

It is common in mathematics for $\hat{\imath} = \langle 1, 0, 0 \rangle$ to be the unit vector in the x-direction, $\hat{\jmath} = \langle 0, 1, 0 \rangle$ to be the unit vector in the y-direction, and $\hat{k} = \langle 0, 0, 1 \rangle$ to be the unit vector in the z-direction. Together, $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$ are called the standard basis vectors because all other vectors in $\mathbb{R}^3$ can be written as linear combination of these.

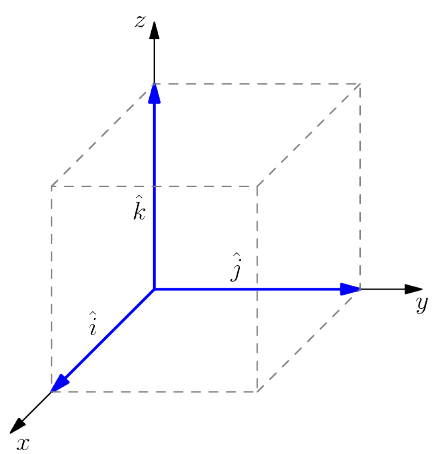


Figure 5: The standard basis vectors $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$

0.3.2 Dot Products

A dot product is a way of multiplying two vectors so that the result is a scalar. $\vec{a} \cdot \vec{b} = ||\vec{a}||||\vec{b}|| \cos\theta$ where θ is the angle between $\vec{a}$ and $\vec{b}$. One way to think of the dot product is as a measure of how much two vectors point in the same direction. We can also show using the law of cosines that $\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2 + ... + a_nb_n$. Knowing the lengths of two vectors and their dot product we can calculate the angle between them as

$$\theta = \arccos\left(\frac{\vec{a} \cdot \vec{b}}{||\vec{a}||||\vec{b}||}\right).$$

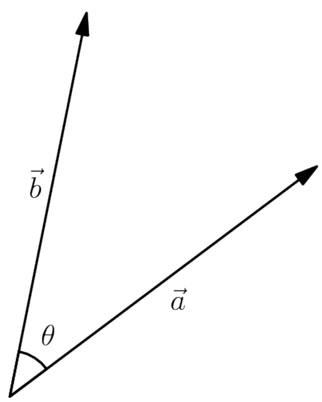


Figure 6: Two vectors and the angle between them

Although similar to scalar multiplication, dot products have some properties that set them apart.

Commutative

$$\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$$

the same as scalar multiplication.

Distributive

$$\vec{a} \cdot \left(\vec{b} + \vec{c}\right) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$$

the same as scalar multiplication.

<u>**NOT**</u> **Associative** $\left(\vec{a} \cdot \vec{b}\right) \cdot \vec{c}$ is a nonsense expression. However, like scalar multiplication, dot products are scalar associative.

$$(c \cdot \vec{a}) \cdot \vec{b} = \vec{a} \cdot \left(c \cdot \vec{b}\right)$$

0.3.3 Cross Products

A cross product is a way of multiplying two vectors so that the result is a vector. Although the cross product technically only works for 3D vectors, we will first look a a "fake" 2D version to build an intuition.

$$\vec{a} \times \vec{b} = a_1b_1 - a_2b_2.$$

This "fake" 2D cross product gives the area of the parallelogram spanned by $\vec{a}$ and $\vec{b}$.

$$\vec{a} \times \vec{b} = ||\vec{a}||||\vec{b}|| \sin\theta$$

where θ is the angle between $\vec{a}$ and $\vec{b}$. Another way to think of the magnitude of the cross product, both in 2D and 3D, is as a measure of how perpendicular two vectors are.

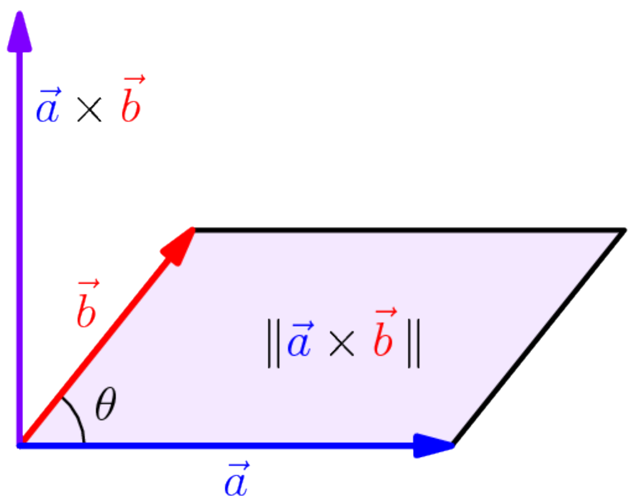


Figure 7: Visualization of the cross product

In 3D, $\vec{a} \times \vec{b}$ is a vector, and similar to the 2D case, the magnitude of $\vec{a} \times \vec{b}$ is equal to the area of the parallelogram spanned by $\vec{a}$ and $\vec{b}$.

$$\vec{a} \times \vec{b} = \langle a_2b_3 - b_2a_3, a_3b_1 - b_3a_1, a_1b_2 - b_1a_2 \rangle$$

and

$$||\vec{a} \times \vec{b}|| = ||\vec{a}||||\vec{b}|| \sin\theta$$

where θ is the angle between $\vec{a}$ and $\vec{b}$. Each component of $\vec{a} \times \vec{b}$ gives the area of the parallelogram spanned by $\vec{a}$ and $\vec{b}$ in some plane: The x-component of $\vec{a} \times \vec{b}$ gives the area in the yz-plane ($x = 0$ plane). $\vec{a} \times \vec{b}$ is perpendicular, also called "normal," to the plane containing $\vec{a}$ and $\vec{b}$. It's direction, is determined by the right hand rule.

This cross product table of the standard basis vectors is useful for providing some insight into the properties of the cross product.

$\overrightarrow{\text{row}} \times \overrightarrow{\text{col}}$	$\hat{i}$	$\hat{j}$	$\hat{k}$
$\hat{i}$	0	$\hat{k}$	$-\hat{j}$
$\hat{j}$	$-\hat{k}$	0	$\hat{i}$
$\hat{k}$	$\hat{j}$	$-\hat{i}$	0

NOT Commutative, but is antisymmetric

$$\vec{a} \times \vec{b} = -\left(\vec{b} \times \vec{a}\right)$$

Scalar Associative

$$(c \cdot \vec{a}) \times \vec{b} = \vec{a} \times \left(c \cdot \vec{b}\right)$$

Distributive

$$\vec{a} \times \left(\vec{b} \times \vec{c}\right) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$$

One can also think of the cross product as the determinant of a matrix.

$$\vec{a} \times \vec{b} = \det \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_2 \\ b_1 & b_2 & b_3 \end{bmatrix}$$

Now that we have defined the dot product and cross product, we can put the two together as the scalar triple product, which gives the volume of the parallelepiped spanned by $\vec{a}$, $\vec{b}$, and $\vec{c}$.

$$\vec{a} \cdot \left(\vec{b} \times \vec{c}\right) = \det \begin{bmatrix} a_1 & a_2 & a_2 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$$

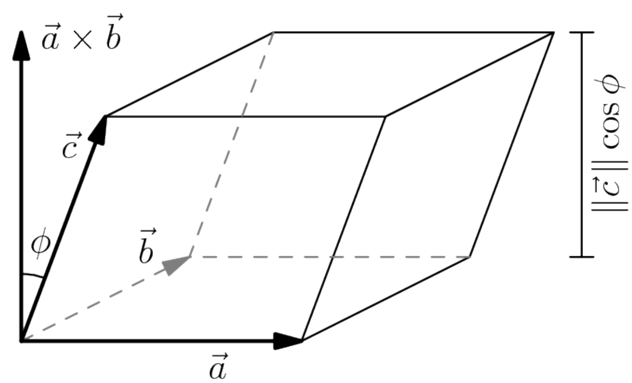


Figure 8: Scalar triple product gives volume of parallelepiped spanned by three vectors.

0.3.4 Matrices

Matrices are an array of mathematical objects, most often numbers. They are often used to represent linear transformations between two spaces and systems of linear equations. We denote the size of a matrix by saying the number of rows followed by the number of columns.

Example. *Below is a 2 x 4 matrix.*

$$\begin{bmatrix} 1 & 3 & 2 & -1 \\ -5 & 7 & 3 & 0 \end{bmatrix}$$

0.3.5 Types of Matrices

Below is a list of different types of matrices and their special properties.

- A square matrix has the same number of rows as columns.

$$\begin{bmatrix} 1 & 3 & 7 \\ 0 & 2 & -1 \\ 2 & 7 & 9 \end{bmatrix}$$

- Row vectors have one column. Column vectors have one row.

$$\begin{bmatrix} 1 & 3 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$$

- Upper triangular matrices have all 0's below the main diagonal. Lower triangular matrices have all 0's above the main diagonal.

$$\begin{bmatrix} 1 & 3 & 7 \\ 0 & 2 & -1 \\ 0 & 0 & 9 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 2 & 7 & 9 \end{bmatrix}$$

- Diagonal matrices are both upper and lower triangular. They only have non-zero entries on the main diagonal.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

- The identity matrix is one of the most common matrices. It is square, diagonal, and has all 1's on the main diagonal. It's the multiplicative identity for matrices.

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- The inverse matrix of A, A^{-1}, is such that

$$A^{-1}A = AA^{-1} = I.$$

- The transpose matrix of A, A^T, is where the rows and columns of A are swapped.

$$A = \begin{bmatrix} 1 & 3 & 2 & -1 \\ -5 & 7 & 3 & 0 \end{bmatrix} \implies A^T = \begin{bmatrix} 1 & -5 \\ 3 & 7 \\ 2 & 3 \\ -1 & 0 \end{bmatrix}$$

0.3.6 Row Reduction

Row reduction is a way of solving a system of linear equations by representing the system as a matrix and altering the rows of the matrix until we get as close as possible to an identity matrix.

Below is a list of legal row operations. Doing these does not change the solution to the system of equations.

- Multiplying or dividing each item in a row by a scalar,

$$\begin{bmatrix} 1 & 3 & 7 \\ 0 & 2 & -1 \\ 2 & 7 & 9 \end{bmatrix} \overset{R_2 = R_2/2}{\rightarrow} \begin{bmatrix} 1 & 3 & 7 \\ 0 & 1 & -1/2 \\ 2 & 7 & 9 \end{bmatrix}.$$

- Adding a multiple of one row to another row,

$$\begin{bmatrix} 1 & 3 & 7 \\ 0 & 2 & -1 \\ 2 & 7 & 9 \end{bmatrix} \overset{R_3 = R_3 - 2R_1}{\rightarrow} \begin{bmatrix} 1 & 3 & 7 \\ 0 & 2 & -1 \\ 0 & 1 & -5 \end{bmatrix}.$$

- Swapping two rows,

$$\begin{bmatrix} 1 & 3 & 7 \\ 0 & 2 & -1 \\ 0 & 1 & -5 \end{bmatrix} \overset{\text{swap } R_2,\ R_3}{\rightarrow} \begin{bmatrix} 1 & 3 & 7 \\ 0 & 1 & -5 \\ 0 & 2 & -1 \end{bmatrix}.$$

Using these rules, we solve a system of linear equations using a process called Gauss-Jordan Elimination.

A system may have a contradiction, meaning no solution exists. This will look like a row of 0's on the left and a non-zero term on the far right of the row.

$$\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 0 \end{array}\right] \implies \text{No solution}$$

A system may be underdetermined, meaning one or more variables can be any number. This will look a non-zero column on the left without a leading 1 (bolded).

$$\left[\begin{array}{ccccc|c} 1 & \mathbf{-2} & 0 & 0 & \mathbf{-3} & 2 \\ 0 & \mathbf{0} & 1 & 0 & \mathbf{1} & 5 \\ 0 & \mathbf{0} & 0 & 1 & \mathbf{2} & 4 \\ 0 & \mathbf{0} & 0 & 0 & \mathbf{0} & 0 \end{array}\right] \implies \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 5 \\ 4 \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 3 \\ 0 \\ -4 \\ -2 \\ 1 \end{bmatrix},\ \alpha,\ \beta \in \mathbb{R}.$$

Example. *Solve the following linear system of equations using row reduction.*

$$\begin{bmatrix} 1 & 3 & 0 \\ 0 & 2 & 5 \\ 2 & 7 & 7 \end{bmatrix} \vec{x} = \begin{bmatrix} 4 \\ 13 \\ 16 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 7 & 0 \\ 0 & 2 & -1 & 5 \\ 2 & 7 & 9 & 7 \end{array}\right] \overset{R_3=R_3-2R_1}{\rightarrow} \left[\begin{array}{ccc|c} 1 & 3 & 7 & 0 \\ 0 & 2 & -1 & 5 \\ 0 & 1 & -5 & 7 \end{array}\right] \overset{\text{swap } R_2,\, R_3}{\rightarrow} \left[\begin{array}{ccc|c} 1 & 3 & 7 & 0 \\ 0 & 1 & -5 & 7 \\ 0 & 2 & -1 & 5 \end{array}\right]$$

$$\overset{R_3=R_3-2R_2}{\rightarrow} \left[\begin{array}{ccc|c} 1 & 3 & 7 & 0 \\ 0 & 1 & -5 & 7 \\ 0 & 0 & 9 & -9 \end{array}\right] \overset{R_3=R_3/9}{\rightarrow} \left[\begin{array}{ccc|c} 1 & 3 & 7 & 0 \\ 0 & 1 & -5 & 7 \\ 0 & 0 & 1 & -1 \end{array}\right] \overset{R_2=R_2+5R_3}{\rightarrow} \left[\begin{array}{ccc|c} 1 & 3 & 7 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

$$\overset{R_1=R_1-7R_3}{\rightarrow} \left[\begin{array}{ccc|c} 1 & 3 & 0 & 7 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right] \overset{R_1=R_1-2R_2}{\rightarrow} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right].$$

So,

$$\vec{x} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}.$$

0.3.7 Determinants

The determinant of a matrix is a signed number that tells by how much the transformation represented by a matrix scales volumes in a space. The number is negative if the space was “flipped“ during a transformation. ”The number is zero if the dimension of the output space is less than that of the input space.

The determinant is only defined for square matrices. It’s easiest to understand the definition of a determinant recursively.

$$\det [a] = |a| = a$$
$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

We can define a_{ij} as the entry in the ith row and jth column of matrix A and A_{ij} as the adjudicate matrix, which is the matrix A if row i and column j were removed. This allows us to write a general formula for the determinant.

Definition.

$$\det A = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} A_{ij} \;\; \textit{(for fixed i)} = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} A_{ij} \;\; \textit{(for fixed j)}$$

This formula allows us to use any row or column to calculate the determinant, which is especially useful if a certain row contains lots of 0’s.

Below are some properties of the determinant for some $n \times n$ matrix A and scalar λ.

$$\begin{aligned} \det I_n &= 1 \\ \det\left(A^T\right) &= \det A \\ \text{If } A \text{ is invertible, } \det\left(A^{-1}\right) &= \frac{1}{\det A} \\ \det(\lambda A) &= \lambda^n \det A \\ \det(AB) &= \det A \det B \\ \text{If } A \text{ is triangular, } \det A &= \prod_{i=1}^{n} a_{ii} \end{aligned}$$

Example. *Find the determinant of the following 3 x 3 matrix.*

$$A = \begin{bmatrix} 1 & 3 & 7 \\ 0 & 2 & -1 \\ 2 & 7 & 9 \end{bmatrix}$$

We'll use the first column since it has only two non-zero entries.

$$\begin{bmatrix} 1 & 3 & 7 \\ 0 & 2 & -1 \\ 2 & 7 & 9 \end{bmatrix} = 1 \begin{vmatrix} 2 & -1 \\ 7 & 9 \end{vmatrix} + 2 \begin{vmatrix} 3 & 7 \\ 2 & -1 \end{vmatrix} = (18 + 7) + 2(-3 - 14) = -9.$$

0.3.8 Eigenvalues & Eigenvectors

Definition. *Let A be an $n \times n$ matrix. A scalar λ and a vector $\vec{v}$ are an eigenvalue and eigenvector of A if*

$$A\vec{v} = \lambda\vec{v}.$$

We call $p(\lambda) = \det(A - \lambda I)$ the characteristic polynomial of A. The eigenvalues for A are the solutions to the equation

$$p(\lambda) = \det(A - \lambda I) = 0.$$

Once we have an eigenvalue, we can find the basis vectors for the corresponding eigenspace by solving the equation

$$(A - \lambda I)\,\vec{v} = \vec{0}.$$

The basis vectors of the eigenspace for A are the union of the basis vectors of each eigenspace corresponding to each eigenvalue.

Example. *Find the eigenvalues and eigenvectors of*

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 2 & 3 \\ 3 & 3 & 20 \end{bmatrix}.$$

$$p(\lambda) = \begin{vmatrix} 2-\lambda & 1 & 3 \\ 1 & 2-\lambda & 3 \\ 3 & 3 & 20-\lambda \end{vmatrix} = -(\lambda-21)(\lambda-2)(\lambda-1) = 0 \implies \lambda = 1,\ 2,\ \text{and } 21.$$

When $\lambda = 1$,

$$A - \lambda I = \left[\begin{array}{ccc|c} 2-1 & 1 & 3 & 0 \\ 1 & 2-1 & 3 & 0 \\ 3 & 3 & 20-1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \implies \vec{v_1} = t \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}.$$

When $\lambda = 2$,

$$A - \lambda I = \left[\begin{array}{ccc|c} 2-2 & 1 & 3 & 0 \\ 1 & 2-2 & 3 & 0 \\ 3 & 3 & 20-2 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \implies \vec{v_2} = \begin{bmatrix} -3 \\ -3 \\ 1 \end{bmatrix}.$$

When $\lambda = 21$,

$$A - \lambda I = \left[\begin{array}{ccc|c} 2-21 & 1 & 3 & 0 \\ 1 & 2-21 & 3 & 0 \\ 3 & 3 & 20-21 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -1/6 & 0 \\ 0 & 1 & -1/6 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \implies \vec{v_{21}} = \begin{bmatrix} 1 \\ 1 \\ 6 \end{bmatrix}.$$

Bonus: A's diagonalization is

$$A = PDP^{-1} \implies \begin{bmatrix} 2 & 1 & 3 \\ 1 & 2 & 3 \\ 3 & 3 & 20 \end{bmatrix} = \begin{bmatrix} -1 & -3 & 1 \\ 1 & -3 & 1 \\ 0 & 1 & 6 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 21 \end{bmatrix} \begin{bmatrix} -1/2 & 1/2 & 0 \\ -3/19 & -3/19 & 1/19 \\ 1/38 & 1/38 & 3/19 \end{bmatrix}.$$

Chapter 1

The Basics of Differential Equations

A differential equation is an equation that relates a function to its derivatives.

1.1 Classifying Differential Equations

Below is a list of differential equations

1. $$t = 7\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + x\frac{\mathrm{d}x}{\mathrm{d}t}$$

2. $$\frac{\mathrm{d}y}{\mathrm{d}x} = x^2 + 3xy - 7y$$

3. $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

4. $$\frac{\partial^2 y}{\partial t^2} = 4\frac{\partial^2 y}{\partial x^2} + e^{t-x}$$

5. $$\frac{\mathrm{d}y}{\mathrm{d}x} + 5y = e^x$$

6. $$\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 + y^2$$

7. $$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = \frac{\mathrm{d}x}{\mathrm{d}t} + \cos x$$

8.
$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 2x\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = 0$$

9.
$$\frac{\partial^2 x}{\partial t \partial z} + xt = 5$$

10.
$$\frac{\mathrm{d}x}{\mathrm{d}y} + xy + ty = \ln y$$

Let's think about some of the ways we can classify these equations.

1.1.1 Order

One way is by this highest order derivative that appears in the equation.

Definition. *The order of a differential equation is the order of the highest derivative in the equation.*

Below is a table of orders and equation numbers.

Order	Equation Number
1	2, 5, 6, 10
2	1, 3, 4, 7, 8, 9

1.1.2 Linearity

Another useful way to classify differential equations is by linearity.

Definition. *A differential equation is linear if all terms in the equation involving the dependent varaibles and its derivatives are in linear terms.*

"Linear terms" means that dependent varaibles should all be of degree 1, not be multiplied by a derivative involving the same variable, and not be in other functions like sin or ln. However, this does not exclude differential equations from having parts that are functions of only independent variables, like in equation 4.

Below is a table of linearity and equation numbers.

Linearity	Equation Number
Linear	2, 3, 4, 5, 8, 9
Nonlinear	1, 6, 7, 10

We'll focus a lot of time on linear equations because we have some mathematical tools that are good at dealing with them.

1.1.3 Ordinary vs. Partial

Another useful classification is by the type. There are two broad types of differential equation: ordinary and partial.

Definition. *An ordinary differential equation (ODE) is a differential equation where only 1 independent variable involved in the derivatives.*

Definition. *A partial differential equation (PDE) is a differential equation where 2 or more independent variables are involved in the derivative.*

Below is a table of types and equation numbers.

Type	Equation Number
ODE	1, 2, 5, 6, 7, 8, 10
PDE	3, 4, 9

We'll work mostly with ODEs.

1.1.4 Homogeneity

Definition. *A differential equation is homogeneous if all terms involve the dependent variable or its derivatives.*

Below is a table of homogeneity and equation numbers.

Homogeneity	Equation Number
Homogeneous	3, 7, 8
Heterogeneous	1, 2, 4, 5, 6, 9, 10

Homogeneous equations have some nice properties, like always having at least a trivial solution of 0.
The three classes of order, linearity, and type are how we describe differential equations in words. Equation 1 would be described as a "1st order linear ODE". These definitions also allow us to define explicitly how an nth order linear ODE looks.

Definition. *An nth order linear ODE has the form*

$$a_n(x)\frac{\mathrm{d}^n y}{\mathrm{d}x^n} + a_{n-1}(x)\frac{\mathrm{d}^{n-1} y}{\mathrm{d}x^{n-1}} + \ldots + a_0(x)y = b(x).$$

1.2 Solutions to Differential Equations

Definition. *A solution to an ODE on an interval I is a function f that makes the differential equation true on I when f is substituted into the equation.*

Although this definition seems straightforward it gives us a formal way to check if a function is a solution to a differential equation.

Example. *Check that $y = t^2 \ln t$ $(t > 0)$ and $y = t^2$ are solutions to the differential equation*

$$t^2y'' - 3ty' + 4y = 0.$$

We'll check $y = t^2 \ln t$ $(t > 0)$ first.

$$\begin{aligned}
t^2\left(t^2 \ln t\right)'' - 3t\left(t^2 \ln t\right)' + 4\left(t^2 \ln t\right) &= 0,\ t > 0\\
t^2\left(3 + 2\ln t\right) - 3t\left(t + 2t\ln t\right) + 4\left(t^2 \ln t\right) &= 0,\ t > 0\\
3t^2 + 2t^2 \ln t - 3t^2 - 6t^2 \ln t + 4t^2 \ln t &= 0,\ t > 0\\
0 &= 0.
\end{aligned}$$

So $y = t^2 \ln t$ $(t > 0)$ is a solution. Now we'll check $y = t^2$.

$$\begin{aligned}
t^2\left(t^2\right)'' - 3t\left(t^2\right)' + 4\left(t^2\right) &= 0\\
t^2\left(2\right) - 3t\left(2t\right) + 4t^2 &= 0\\
2t^2 - 6t^2 + 4t^2 &= 0\\
0 &= 0.
\end{aligned}$$

So $y = t^2$ is also a solution.

1.3 Initial Value Problems

We can see that since solving a differential equation will mean integrating to get rid of derivatives, the $+C$ from integration will gives us multiple solutions. We call these sets of solutions that differ only in these constants "solution families". If we want to find one specific solution, we need more information about the value of the function and it's derivatives. This type of problem where a differential equation is coupled with function values is called an initial value problem (IVP).

Definition. *An initial value problem has the general form.*

$$\begin{cases}
a_ny^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_0y = f(x)\\
y(x_0) = y_0\\
y'(x_1) = y_1\\
\vdots\\
y^{(n)}(x_n) = y_n
\end{cases}.$$

Often, each x_i is 0.

Example. *The general solution to the IVP*

$$\begin{cases} y'' - 5y' + 6y = 0 \\ y(0) = 3 \\ y'(0) = 1 \end{cases}$$

is $y = c_1 e^{2x} + c_2 e^{3x}$. *Find the specific solution.*

Evaluating y at $x = 0$,

$$y(0) = c_1 + c_2 = 3.$$

Evaluating y' at $x = 0$,

$$y'(0) = 2c_1 + 3c_2 = 1.$$

To find c_1 and c_2, we need to solve the system of linear equations

$$\begin{cases} c_1 + c_2 = 3 \\ 2c_1 + 3c_2 = 1 \end{cases} \implies \begin{cases} c_1 = 8 \\ c_2 = -5 \end{cases}.$$

So, our specific solution to the IVP is

$$y = 8e^{2x} - 5e^{3x}.$$

Theorem (Existence and Uniqueness of Solutions to 1st Order IVPs)**.** *Consider the IVP*

$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}y} = f(x, y) \\ y(x_0) = y_0 \end{cases}.$$

If $f(x, y)$ and $\frac{\partial}{\partial y} f$ are both continuous on some rectangular region containing the point (x_0, y_0), then the IVP has a unique solution $y = y(x)$ on some open interval containing x_0.

Example. *Does a solution to the following IVP exist? Is it unique?*

$$\begin{cases} \frac{\mathrm{d}y}{\mathrm{d}x} = x^2 - xy^3 \\ y(1) = 6 \end{cases}$$

$f(x, y) = x^2 - xy^3$ and $\frac{\partial f}{\partial y} = -3xy^2$ are continuous on all of $\mathbb{R}^2$. So, the existence and uniqueness theorem tells us that the IVP has a unique solution on an open interval containing $x_0 = 1$.

Example. *Does a solution to the following IVP exist? Is it unique?*

$$\begin{cases} \frac{\mathrm{d}y}{\mathrm{d}x} = 3y^{2/3} \\ y(2) = 0 \end{cases}$$

$f(x, y) = 3y^{2/3}$ is continuous on $y \in \mathbb{R}$, and $\frac{\partial f}{\partial y} = 2y^{-1/3}$ is continuous on $x \in (-\infty, 0) \cup (0, \infty)$. Since $\frac{\partial f}{\partial y}$ is not continuous on a domain containing $(2, 0)$, the existence and uniqueness theorem does not guarantee a solution.

Chapter 2

1st Order Linear ODE's

1st order linear ODEs are among the simplest differential equations, but learning different techniques to solve them will allow us to develop techniques for solving more complicated types of equations later on.

2.1 Separable Differential Equations

The most basic approach for solving a 1st-order differential equation is simply integrating both sides. You're probably already familiar with this technique from taking indefinite integrals. This approach only works when the independent and dependent variables can be arranged on different sides of the equation. We'll formalize this idea with separability.

Definition. *A 1st order ODE is separable if it can be written in the form*

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)g(y).$$

Separable equations provide a special way of solving them that can be useful. If we treat the derivative like a fraction (which is not formally allowed but OK here),

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)g(y) \implies \frac{\mathrm{d}y}{g(y)} = f(x)\mathrm{d}x \implies \int \frac{\mathrm{d}y}{g(y)} = \int f(x)\mathrm{d}x.$$

We then have a function in y on the left and a function in x on the right, meaning we only have to solve for y to get the solution.

Example. *Solve the following 1st order ODE using separation of variables.*

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{5}{xy^3}$$

Separating so all terms involving y are on the left,

$$y^3\mathrm{d}y = \frac{5}{x}\mathrm{d}x.$$

Integrating,

$$\int y^3 \mathrm{d}y = \int \frac{5}{x} \mathrm{d}x \implies \frac{y^4}{4} = 5 \ln x + C \implies y = \sqrt[4]{20 \ln x + C}$$

Example. *Solve the following IVP using separation of variables.*

$$\begin{cases} \frac{\mathrm{d}y}{\mathrm{d}x} = 2y^2 + xy^2 \\ y(0) = 1 \end{cases}$$

Separating,

$$\frac{\mathrm{d}y}{y^2} = (2 + x)\, \mathrm{d}x, \text{ or } y = 0.$$

We assume that $y \neq 0$ when dividing, but we need to be careful to include $y = 0$ as a possible solution and check if it satisfies the differential equation and initial conditions.

Integrating,

$$\int \frac{\mathrm{d}y}{y^2} = \int (2 + x)\, \mathrm{d}x \implies \frac{-1}{y} = 2x + \frac{x^2}{2} + C.$$

Solving for y,

$$y = \frac{-2}{x^2 + 4x + C}.$$

We ignored $y = 0$, so we need to go back and check if it's a solution to the differential equation.

$$0 = 2(0)^2 + x(0)^2.$$

Since it's a solution to the differential equation, we'll see if its satisfies the initial conditions of the IVP.

$$y(0) = 0 \neq 1.$$

So, $y = 0$ is not a solution to the IVP because it does not satisfy the initial conditions. Solving for C using the initial conditions,

$$y(0) = \frac{-2}{(0)^2 + 4(0) + C} = 1 \implies C = -2.$$

So,

$$y = \frac{-2}{x^2 + 4x - 2}$$

is the solution to the IVP.

2.2 Integrating Factor Method

All 1st order linear differential equations have the form

$$a_1(x)\frac{\mathrm{d}y}{\mathrm{d}x} + a_0(x)y = b_1(x),$$

which can be rewritten as

$$\frac{\mathrm{d}y}{\mathrm{d}x} + a(x)y = b(x).$$

This equation isn't always separable, and we can't just integrate both sides unless $a_1(x)y = 0$.

If $a_0(x) = a_1'(x)$, then we could rewrite the equation and solve by doing the product rule in reverse.

$$(a_1(x)y)' = b_1(x) \implies y = \frac{\int b_1(x)\mathrm{d}x}{a_1(x)}$$

It's possible to rearrange into this form by multiplying the equation by some function. Specifically, what were looking for is a function $\mu(x)$ such that

$$\mu(x)\frac{\mathrm{d}y}{\mathrm{d}x} + \mu(x)a(x)y = \mu(x)b(x) \text{ and } \mu'(x) = \mu(x)a(x).$$

This equation involving $\mu(x)$ is one that we know how to solve because it's separable.[1]

$$\mu'(x) = \mu(x)a(x) \implies \mu(x) = e^{\int a(x)\mathrm{d}x}.$$

Substituting the solution for $\mu(x)$ back,

$$e^{\int a(x)\mathrm{d}x}\frac{\mathrm{d}y}{\mathrm{d}x} + e^{\int a(x)\mathrm{d}x}a(x)y = e^{\int a(x)\mathrm{d}x}b(x) \implies e^{\int a(x)\mathrm{d}x}\frac{\mathrm{d}y}{\mathrm{d}x} + \mu'(x)y = e^{\int a(x)\mathrm{d}x}b(x).$$

Applying the product rule in reverse,

$$y = \frac{\int \mu(x)b(x)\mathrm{d}x}{\mu(x)}, \ \mu(x) = e^{\int a(x)\mathrm{d}x}.$$

Example. *Solve the following 1st order linear ODE.*

$$y' - y = 2e^x$$

$a(x) = -1$, so

$$\mu(x) = e^{\int -1\mathrm{d}x} = e^{-x}.$$

Applying to our equation and solving,

$$y'e^{-x} - ye^{-x} = 2 \implies \left(ye^{-x}\right)' = 2 \implies ye^{-x} = 2x + C \implies y = \frac{2x + C}{e^{-x}}.$$

[1] Although we are taking an indefinite integral to find $\mu(x)$, we do not have a $+C$ term.

Chapter 3

Higher Order Linear ODE's

3.1 Constant Coefficients

Definition. *The general form of an nth order linear equation is*

$$a_n(x)y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_1(x)y' + a_0y = b(x).$$

If each $a_i(x)$ is a constant, then the equation has constant coefficients.

We already know how to solve linear first order differential equations using an integrating factor, but let's see if we can develop a method that can solve any order linear, homogeneous differential equation with constant coefficients.

Example. *Let's try to solve the following equation by guessing and checking likely solutions.*

$$y'' - 3y' + 2y = 0$$

Exponentials seem like good guesses. Let's try an exponential of the form $y = Ce^{rx}$ first.

$$(Ce^{rx})'' - 3\,(Ce^{rx})' + 2\,(Ce^{rx}) = 0$$

$$Cr^2e^{rx} - 3rCe^{rx} + 2Ce^{rx} = Ce^{rx}\left(r^2 - 3r + 2\right) = 0.$$

Since $Ce^{rx} \neq 0$ unless $C = 0$, we only need to solve the quadratic. Note that the coefficients of the quadratic are the same as the coefficients in the original differential equation.

$$r^2 - 3r + 2 = 0 \implies r = 1,\, 2.$$

So, our two fundamental solutions are

$$\begin{cases} y = C_1e^x \\ y = C_2e^{2x} \end{cases}.$$

Since these two fundamental solutions are linearly independent, the general solution is the sum of the fundamental solutions.

$$y = C_1e^x + C_2e^{2x}$$

3.1.1 The Auxiliary Equation

It's not a coincidence that the coefficients of the polynomial that we had to find the 0's of in the above example matched the coefficients of the differential equation. We call this polynomial the auxiliary equation, and it can help us solve linear, homogeneous differential equations with constant coefficients of any order.

Definition. *A nth order, linear, homogeneous differential equation with constant coefficients has the form*

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y' + a_0 y = 0.$$

The corresponding auxiliary equation is

$$a_n r^n + a_{n-1} r^{n-1} + \ldots + a_1 r + a_0 = 0.$$

We now have a method for solving these equations with the roots of the auxiliary equation are all unique.

Theorem. *Let $\{r_1, \ldots, r_n\}$ be the set of unique roots to an auxiliary equation corresponding to a nth order, linear, homogeneous differential equation with constant coefficients. The set of fundamental solutions are $\{C_1 e^{r_1 x}, \ldots, C_n e^{r_n x}\}$, and the general solution is*

$$y = C_1 e^{r_1 x} + \ldots + C_n e^{r_n x}.$$

We can easily extend this method to deal with roots of higher multiplicities.

Theorem. *Let α be a root with multiplicity k to an auxiliary equation corresponding to a nth order, linear, homogeneous differential equation with constant coefficients. Then $e^{\alpha x}, x e^{\alpha x}, \ldots, x^{k-1} e^{\alpha x}$ are fundamental solutions.*

Complex Roots

Although the previous two theorems already cover complex roots, we can simplify solutions that have complex roots into functions we more easily understand.

Theorem. *If an auxiliary equation has roots $\alpha \pm \beta i$, then $C_1 e^{\alpha x} \cos(\beta x)$ and $C_2 e^{\alpha x} \sin(\beta x)$ are fundamental solutions.*

Proof. The two corresponding fundamental solutions are

$$C_1 e^{(\alpha + \beta i)x},\ C_2 e^{(\alpha - \beta i)x}$$

so the part of the general solution for these two fundamental solutions are

$$C_1 e^{(\alpha + \beta i)x} + C_2 e^{(\alpha - \beta i)x}.$$

Using Euler's formula, we can also write this as

$$C_1 e^{\alpha x} \left(\cos\left(\beta t\right) + i \sin\left(\beta x\right)\right) + C_2 e^{\alpha x} \left(\cos\left(-\beta x\right) + i \sin\left(-\beta x\right)\right).$$

Using the fact that $\sin\left(-x\right) = -\sin x$, $\cos\left(-x\right) = -\cos x$ and separating into real and imaginary parts,

$$= \left[C_1 e^{\alpha x} \cos\left(\beta x\right) + C_2 e^{\alpha x} \cos\left(\beta x\right)\right] + i \left[C_1 e^{\alpha x} \sin\left(\beta x\right) - C_2 e^{\alpha x} \sin\left(\beta x\right)\right].$$

Simplifying [1],

$$= e^{\alpha x} \left(C_1 \cos\left(\beta x\right) + i C_2 \sin\left(\beta x\right)\right).$$

At this point, we need to consider if the constants are real or imaginary. We would then take the real part as a fundamental solution.
If they are both real, then a fundamental solution is[2]

$$C_1 e^{\alpha x} \cos\left(\beta x\right)$$

where C_1 is a real constant. If they are both imaginary, the a fundamental solution is[3]

$$C_2 e^{\alpha x} \sin\left(\beta x\right)$$

where C_2 is a real constant. ■

If complex roots are repeated, we just add the appropriate number of powers of x in front of both the $\sin\left(\beta x\right)$ and $\cos\left(\alpha a\right)$ parts.

[1]In this step the values of C_1 and C_2 might have changed, but they are still constants.
[2]See footnote 1
[3]See footnote 1

Example. *Find the general solution to the following differential equation.*

$$y^{(4)} + 2y'' + y = 0$$

First we extract the auxiliary equation and find its roots.

$$r^4 + 2r^2 + 1 = 0 \implies r = i(\text{double root}), -i(\text{double root}).$$

Since we have complex roots $0 \pm 1i$, we know that the following are fundamental solutions

$$C_1 e^{0x} \cos 1x \to C_1 \cos x \text{ and } C_2 e^{0x} \sin 1x \to C_2 \sin x.$$

Since both roots are double roots the following are also fundamental solutions:

$$C_3 x \cos x \text{ and } C_4 x \sin x.$$

So, the general solution is

$$y = C_1 \cos x + C_2 \sin x + C_3 x \cos x + C_4 x \sin x.$$

3.2 Free Vibrations

Free damped vibrations, like in a massed spring system, are a common application of second order linear ODEs. In a massed spring system, there are three main forces acting on the mass that make up external forces.

1) Acceleration of The Mass – Since acceleration is the 2nd derivative of position $y(t)$, and Newton's Second Law tells us that $F = ma$, the force from the acceleration of the mass is my''.

2) Dampening – We'll assume that this term is proportional to the velocity, y', and a term b. So, the force from dampening is by'.

3) Spring Stretch – Hooke's Law tells us that the force from a spring is ky, where k is some term that gives the spring's "stiffness"

Since we assume that the net force is 0 (that's what free means), our equations is

$$my'' + by' + ky = 0.$$

Extracting the coefficients and solving the auxiliary equation,

$$mr^2 + br + k = 0 \implies r = \frac{-b \pm \sqrt{b^2 - 4mk}}{2m}.$$

We will consider two cases. One in which there is no damping ($b = 0$), and one in which there is damping ($b > 0$).

3.2.1 Free Undamped Vibrations ($b = 0$)

In this case, our equation simplifies to

$$my'' + ky = 0.$$

The two roots of ou auxiliary equation are

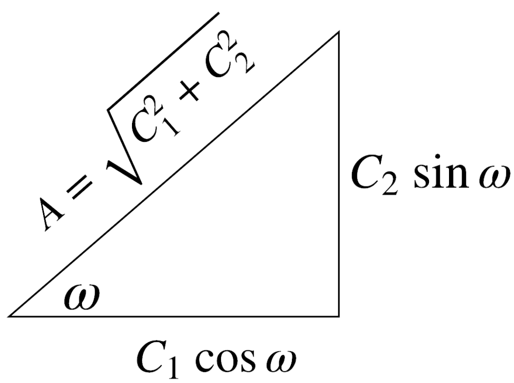
$$r = \pm i\sqrt{\frac{k}{m}} = \pm i\omega.$$

So, our solution becomes

$$y = C_1 \cos(\omega t) + C_2 \sin(\omega t).$$

This is the same ω from physics that means angular frequency, so the same physics formulas apply, like $T = \frac{2\pi}{\omega}$ for the period of the oscillation.

We can simplify this a bit further. If we think of the cos and sin components as being sides of a right triangle like so,



then we rewrite our equation as

$$y = A\left(\frac{C_1}{\sqrt{C_1^2 + C_2^2}} \cos(\omega t) + \frac{C_2}{\sqrt{C_1^2 + C_2^2}} \sin(\omega t)\right).$$

Note that since $\left(\frac{C_1}{A}\right)^2 + \left(\frac{C_2}{A}\right)^2 = 1$, we can rewrite these coefficients as $\cos\phi$ and $\sin\phi$ respectively where

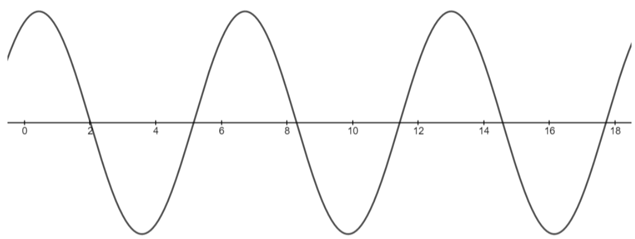
$$\phi = \begin{cases} \arctan\left(\frac{C_2}{C_1}\right) & C_1 > 0 \\ \arctan\left(\frac{C_2}{C_1}\right) + \pi & C_1 \leq 0 \end{cases}.$$

So, our equation becomes

$$y = A\left(\cos\left(\omega t\right)\cos\phi + \sin\left(\omega t\right)\sin\phi\right).$$

Using the cos angle addition formula,

$$y = A\cos\left(\omega t - \phi\right).$$



As we can see, an undamped free vibration will simply oscillate back and forth without decay.

Example. *A 2kg mass in an undamped system is attached to a spring with* $k = 50N/m$*. The initial position of the masss is* $y_0 = -0.25m$*. The initial velocity is* $v_0 = -1m/s$*. Find an expression for* $y(t)$*, the position of the mass at time* t*. Write your answer in terms of a* cos *and a phase shift. Find the period and frequency in proper units.*

The IVP describing this problem is

$$\begin{cases} 2y'' + 50y = 0 \\ y'(0) = -1 \\ y(0) = -0.25 \end{cases}.$$

Extracting the auxiliary equation and finding the roots,

$$2r^2 + 50 = 0 \implies r = \pm 5i.$$

So, our general solution is

$$y = C_1\cos\left(5t\right) + C_2\sin\left(5t\right).$$

Solving for C_1 and C_2,

$$\begin{aligned} y(0) = -0.25 = C_1 &\implies C_1 = -0.25 \\ y' = -5C_1\sin\left(5t\right) &+ 5C_2\cos\left(5t\right) \\ y'(0) = -1 = 5C_2 &\implies C_2 = -0.2. \end{aligned}$$

Solving for ϕ, keeping in mind that $C_1 < 0$,

$$\phi = \arctan\frac{C_2}{C_1} + \pi = \arctan\frac{4}{5} + \pi.$$

So, our answer is (in units of meters)

$$y = \sqrt{(-0.25)^2 + (-0.2)^2} \cos\left(5t - \arctan\left(\frac{4}{5}\right) - \pi\right) \approx 0.32 \cos(5t - 3.82).$$

Solving for the period,

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{5}\text{s.}$$

Solving for the frequency,

$$f = \frac{1}{T} = \frac{5}{2\pi}\text{Hz.}$$

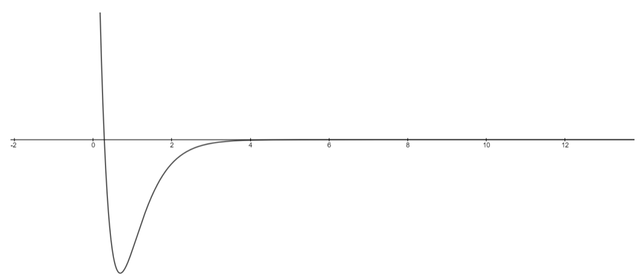
3.2.2 Free Damped Vibrations ($b > 0$)

We need to consider three cases where the discriminant $\Delta = b^2 - 4mk$ is positive, zero, and negative.

Overdamped ($\Delta > 0$)

This is the simplest and easiest case to deal with because our two roots, r_1 and r_2, are real and distinct. So, out solution is

$$y = C_1 e^{r_1 t} + C_2 e^{r_2 t}.$$



We know that $r_1, r_2 < 0$, so

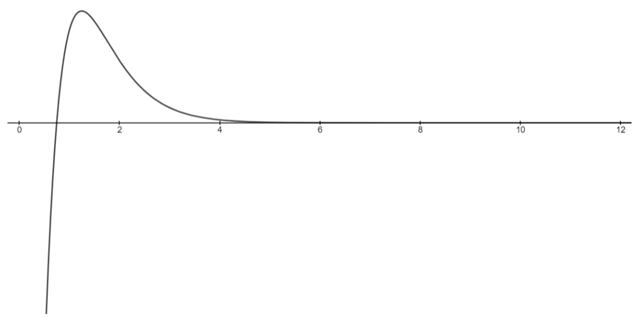
$$\lim_{t \to 0} C_1 e^{r_1 t} + C_2 e^{r_2 t} = 0$$

meaning the mass's oscillation decays over time.

Critically Damped ($\Delta = 0$)

This case isn't much more difficult. The only difference is that because both roots r_1 and r_2 are $\frac{-b}{2m}$, we need an extra t term in the solution. So, out solution is

$$y = C_1 e^{r_1 t} + C_2 t e^{r_2 t}.$$



Since both roots are once again negative,

$$\lim_{t\to 0} C_1 e^{r_1 t} + C_2 t e^{r_2 t} = 0$$

meaning the mass's oscillation decays over time.

Underdamped ($\Delta < 0$)

This is probably the most complicated case. Here, both roots are complex. Specifically,

$$r = \frac{-b}{2m} \pm i\frac{\sqrt{|\Delta|}}{2m}.$$

Letting the coefficient of the imaginary part be ω,

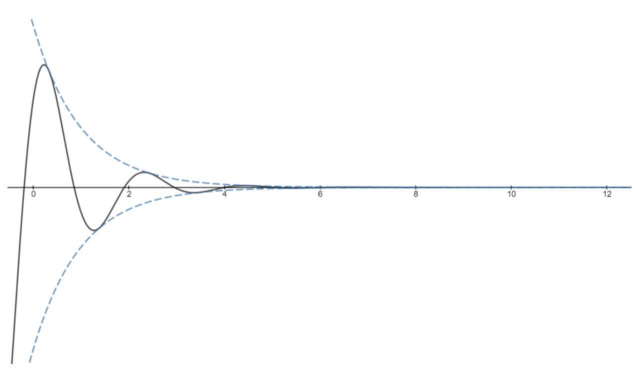
$$r = \frac{-b}{2m} \pm i\omega.$$

So, our solution becomes

$$y = e^{\frac{-b}{2m}t}\left(C_1 \cos(\omega t) + C_2 \sin(\omega t)\right).$$

Rewriting in terms of cos and a phase shift,

$$y = Ae^{\frac{-b}{2m}t}\cos(\omega t - \phi) \text{ where } A = \sqrt{A^2 + B^2},\ \phi = \begin{cases} \arctan\left(\frac{B}{A}\right) + \pi & A \leq 0 \\ \arctan\left(\frac{B}{A}\right) & A > 0 \end{cases}.$$



Here, the exponential term dominates the limit, so

$$\lim_{t\to 0} Ae^{\frac{-b}{2m}t}\cos(\omega t-\phi)=0$$

meaning the mass's oscillation decays over time, bounded by the exponential curves.

Example. *A 250g mass is attached to a spring with a constant of $k = 10N/m$. The mechanical impedance is 3kgs. Initially, the mass is at $y(0) = -1m$ and $y'(0) = 2m/s$. Find an expression for $y(t)$, the position of the mass at time t. Write any oscillations as a* cos *and a phase shift. Is the system underdamped, critically damped, or overdamped?*

The IVP describing this scenario is

$$\begin{cases} \frac{1}{4}y''+3y'+10y=0 \\ y(0)=-1 \\ y'(0)=2 \end{cases}.$$

Solving the auxiliary equation,

$$\frac{1}{4}r^2+3r+10=0 \implies r=\frac{-3\pm\sqrt{3^2-4(1/4)(10)}}{2(1/4)}=-6\pm 2i.$$

So, the general solution is

$$y=e^{-6t}\left(C_1\cos(2t)+C_2\sin(2t)\right).$$

Solving for C_1 and C_2,

$$y(0)=-1=C_1 \implies C_1=-1.$$

$$y'=e^{-6t}\left(-2C_1\sin(2t)+2C_2\cos(2t)\right)+\left(C_1\cos(2t)+C_2\sin(2t)\right)\cdot -6e^{-6t}.$$

$$y'(0)=2=-6C_1+2C_2 \implies 2C_2=-4 \implies C_2=-2.$$

Solving for ϕ, keeping in mind that $C_1<0$,

$$\phi=\arctan\frac{C_2}{C_2}+\pi=\arctan 2+\pi.$$

So, our answer is (in units of meters)

$$y=\sqrt{(-1)^2+(-2)^2}e^{-6t}\cos(2t-\arctan 2-\pi)\approx 2.24e^{-6t}\cos(2t-4.25).$$

Since our roots were complex, $\Delta<0$. So, the system is underdamped.

3.3 Higher Order Heterogeneous Equations

If we modify our equation for free vibrations to have a function as the net force, then out equations becomes

$$my'' + by' + ky = b(x).$$

Depending on the form of $b(x)$, like a sin or cos curve or an exponential, we might be able to guess the form of the solution. However, there is an important thing to keep in mind.

Theorem. *If $f(x)$ is a solution to the above equation, and $g(x)$ is a solution to the homogeneous form of the equation $b(x) = 0$, then $f(x) + g(x)$ is also a solution to the above equation.*

Proof. If $f(x) + g(x)$ is a solution, then

$$m\left(f(x) + g(x)\right)'' + b\left(f(x) + g(x)\right)' + k\left(f(x) + g(x)\right) = b(x).$$

Rearranging,

$$\left(mf''(x) + bf'(x) + kf(x)\right) + \left(mg''(x) + bg'(x) + kg(x)\right) = 0.$$

Using the definitions of $f(x)$ and $g(x)$,

$$b(x) + 0 = b(x).$$

■

$f(x)$ and $g(x)$ actually have special meanings in terms of solving these higher-order equations.

Definition. *$f(x)$ is called the particular solution to the heterogeneous equation, and $g(x)$ is called a homogeneous solution.*

3.3.1 Method of Undetermined Coefficients

The type of equation we're trying to solve is a heterogeneous linear ODE with constant coefficients. These have the form

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y' + a_0 y = b(x).$$

We will assume that the solution has the form $y = y_h + y_p$, where y_h is the general solution to the homogeneous equation ($b(x) = 0$), and y_p is the particular solution.

We know how to solve for y_h exactly without guessing. However, we will make a guess for the form of y_p based on the form of $b(x)$ and the form of y_h using the rules described in the below examples. For each term in our guess for y_p, we will solve for a constant.

When solving, we'll first solve the homogeneous equation to find the general homogeneous solution y_h. Then, we'll guess a form for the particular solution y_p based on the form of $b(x)$ and solve. Finally, we'll add these two solutions together to get the full general solution. We'll see that the constants come from y_h and not y_p.

Example. *Find the general solution to the following equation.*

$$y'' + 2y' + y = 27e^{2x}$$

First, we'll solve the homogeneous equation

$$y'' + 2y' + y = 0.$$

Extracting the auxiliary equation and finding the roots,

$$r^2 + 2r + 1 = (r+1)^2 \implies r = -1\,(\text{double root})\,.$$

So, our general solution is

$$y_h = e^{-x}\left(C_1 + C_2 x\right).$$

Since $b(x)$ is an exponential with a power of $2x$, it's safe guess to say that the particular solution is also an exponential with a power of $2x$. So, we'll guess that $y_p = Ae^{2x}$ and solve for A.

$$\begin{aligned} \left(Ae^{2x}\right)'' + 2\left(Ae^{2x}\right)' + Ae^{2x} &= 2te^{2x} \\ A\left(4e^{2x} + 4e^{2x} + e^{2x}\right) &= 27e^{2x} \implies A = 3. \end{aligned}$$

So,

$$y_p = 3e^{2x}.$$

Putting y_h and y_p together,

$$y = y_h + y_p = e^{-x}\left(C_1 + C_2 x\right) + 3e^{2x}.$$

There are a couple of catches we need to think about with it comes to guessing the form of the particular solution.

Example. *Find the general solution to the following equation.*

$$y'' + 2y' + y = 2e^{-x}.$$

We already know from the previous example what y_h is.

$$y_h = C_1e^{-x} + C_2xe^{-x}.$$

However, even though $b(x)$ is an exponential with power $-x$, guessing that y_p is of the form Ae^{-x} won't work, since that is already covered in y_h. So, we instead include factors of x until we hit a factor not already covered by y_h. In this case, we need up to x^2, so we guess that $y_p = Ax^2e^{-x}$.

$$y_p'' + 2y_p' + y_p = 2Ae^{-x} = 2e^{-x} \implies A = 1.$$

So, the general solution is

$$y = y_h + y_p = C_1e^{-x} + C_2xe^{-x} + x^2e^{-x}.$$

For certain forms of $b(x)$, like $\sin x$ or $\cos x$, our guess for y_p will have multiple terms. We also need to make sure that these terms aren't already in y_h and include factors of x.

Example. *Find the general solution to the following equation*

$$y'' + 4y = 8\cos(2t)$$

given that $y_h = C_1\cos(2t) + C_2\sin(2t)$.

When $b(x)$ has a sin or cos term in it, we need our guess for y_p to include both a sin and cos part in it, meaning there are two unknowns we'll have to solve for. However, since y_h already has these sin and cos terms, we need to include an extra factor of x. So, our guess is that $y_p = Ax\cos(2x) + Bx\sin(2x)$.

$$y_p'' + 4y_p = -4A\sin(2x) + 4B\cos(2x) = 8\cos(2t) \implies A = 0,\ B = 2.$$

Note how the sin and cos terms that have a factor of x cancel each other out. This is expected since $b(x)$ does not have any terms with a factor of x.

So, the general solution is

$$y = y_h + y_p = C_1\cos(2t) + C_2\sin(2t) + 2x\sin(2x).$$

If $b(x)$ has multiple terms, we need to include each term fully in our guess. For times when $b(x)$ has a factor that is a polynomial of degree n, our guess will also have a factor that is a polynomial of degree n and n coefficients to solve for.

Example. *Find the general solution to the following equation*

$$y'' - 3y' - 4y = 4x^2 - 1$$

given that $y_h = C_1e^{-x} + C_2e^{4x}$.

Since $b(x)$ is a degree 2 polynomial, we'll guess that $y_p = Ax^2 + Bx + C$.

$$y_p'' - 3y_p' - 4y_p = x^2(-4A) + x(-6A - 4B) + (2A - 3B - 4C) = 4x^2 - 1.$$

So, we have a system of linear equations,

$$\begin{cases} -4A = 4 \\ -6A - 4B = 0 \\ 2A - 3B - 4C = -1 \end{cases} \implies \begin{cases} A = -1 \\ B = 3/2 \\ C = -11/8 \end{cases}.$$

So, our general solution is

$$y = C_1e^{-x} + C_2e^{4x} - x^2 + \frac{3}{2}x - \frac{11}{8}.$$

3.3.2 Variation of Parameters

Although the method of undetermined coefficients is useful and relatively quick because it is algebra-based, it cannot solve many equations, even simple-looking second order equations, like

$$y'' + y = \csc x.$$

The method also requires guessing, meaning for very complicated forms of $b(x)$, things can get very messy.

Instead, we'll look at a more rigorous, calculus-based, approach developed by Lagrange called "variation of parameters." We'll first see how to apply the method to 2nd order linear ODEs with constant coefficients, like forced vibrations, and then we'll extend the method to order n.

Second Order Variation of Parameters

We'll modify our second order equation of have a 1 as the coefficient of the y'' term by dividing to get an equation of the form

$$y'' + py' + qy = g(x).$$

Just like for undetermined coefficients, we'll find homogeneous and particular solutions $y = y_h + y_p$. Since the equation is second-order, the solution to the homogeneous equation will yield two fundamental solutions y_1 and y_2 where $y_h = C_1y_1 + C_2y_2$.

So, we can write y as

$$y(x) = A(x)y_1 + B(x)y_2,$$

where

$$\begin{cases} A'y_1 + B'y_2 = 0 \\ A'y_1' + B'y_2' = g(x) \end{cases}.$$

We will then solve this system to solve for A' and B' and integrate.

Example. *Find the general solution to the following equation*

$$y'' + y = \csc x$$

given that $y_h = C_1 \cos x + C_2 \sin x$.

y_h gives us our two fundamental solutions

$$\begin{cases} y_1 = \cos x \\ y_2 = \sin x \end{cases}.$$

So, our system of equations is

$$\begin{cases} A'\cos x + B'\sin x = 0 \\ -A'\sin x + B'\cos x = \csc x \end{cases} \quad \to \begin{cases} A'\cos x + B'\sin x = 0 \\ -A'\cos x + B'\frac{\cos^2 x}{\sin x} = \frac{\cos x}{\sin^2 x} \end{cases}.$$

So,

$$B'\frac{1}{\sin x} = \frac{\cos x}{\sin^2 x} \implies B' = \frac{\cos x}{\sin x} = \cot x \implies B = \ln\big|\sin x\big| + C_2,$$

and

$$A'\cos x + \cos x = 0 \implies A' = -1 \implies A = -x + C_1.$$

So, our general solution is

$$y = (C_1 - x)\cos x + \big(\ln\big|\sin x\big| + C_2\big)\sin x = C_1\cos x + C_2\sin x - x\cos x + \sin x\ln\big|\sin x\big|.$$

Note how y_h and y_p appear together.

Higher Order Variation of Parameters

We are trying to find a solution to the nth order linear ODE

$$a_n(x)y^{(n)} + \ldots + a_0(x)y = g(x)$$

assuming that we already know the fundamental solutions for the corresponding homogeneous equation

$$y_h = C_1y_1 + \ldots + C_ny_n.$$

For this method, we'll assume that y can be written as

$$y = v_1(x)y_1 + \ldots + v_n(x)y_n$$

and we'll try to find $v_1, \ldots, v_n$.

Since there are n unknown functions, we'll need n equations to find them all. We can generate these by differentiating y_p.

$$y' = (v_1y_1' + \ldots + v_ny_n') + (v_1'y_1 + \ldots + v_n'y_n)$$

to avoid second derivatives of $v_1, \ldots, v_n$ from entering the formula for y'', we also have the condition

$$v_1'y_1 + \ldots + v_n'y_n = 0.$$

We can now continue differentiating to get $n-2$ more equations involving $v_1', \ldots, v_n'$. We also impose a final n^{th} condition that

$$v_1'y^{(n-1)} + \ldots + v_n'y_n^{(n-1)} = g.$$

This gets us a system of n equations,

$$\begin{cases} v_1' y_1 + \ldots + v_n' y_n & = 0 \\ \vdots & \vdots \\ v_1' y_1^{(n-2)} + \ldots + v_n' y_n^{(n-2)} & = 0 \\ v_1' y_1^{(n-1)} + \ldots + v_n' y_n^{(n-1)} & = g \end{cases}.$$

Hopefully this system looks familiar from second order equations.

We can rewrite this system in terms of matrices and vectors.

$$\begin{bmatrix} y_1 & \ldots & y_n \\ \vdots & & \vdots \\ y_1^{(n-2)} & \ldots & y_n^{(n-2)} \\ y_1^{(n-1)} & \ldots & y_n^{(n-1)} \end{bmatrix} \begin{bmatrix} v_1' \\ \vdots \\ v_{n-1}' \\ v_n' \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ g \end{bmatrix}.$$

It's sufficient to show that a solution to this system exists if the determinant of the square matrix on the left is non-zero. The determinant of this matrix actually has a special name.

Definition. *The Wronskian of n $n-1$ times differentiable functions $\{f_1, \ldots, f_n\}$ on an interval I is*

$$W[f_1, \ldots, f_n](x) = \begin{vmatrix} f_1(x) & \ldots & f_n(x) \\ f_1'(x) & \ldots & f_n'(x) \\ \vdots & & \vdots \\ f_1^{(n-1)}(x) & \ldots & f_n^{(n-1)}(x) \end{vmatrix}, \ x \in I.$$

Using the Wronskian, we can solve the system using Cramer's Rule.

$$v_i'(x) = \frac{g(x) W_i(x)}{W[y_1, \ldots, y_n](x)}, \ i = 1, \ldots, n,$$

where $W_i(x)$ is the determinant of the matrix obtained from the Wronskian $W(x)$ by replacing the i^{th} column with $\mathrm{col}[0, \ldots, 1]$. Using the cofactor expansion along this column, we can write $W_i(x)$ as

$$W_i(x) = (-1)^{n-i} W[y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_n](x), \ i = 1, \ldots, n.$$

Now with a solution for v_i', we can integrate to get v_i.

$$v_i = \int \frac{g(x) W_i(x)}{W[y_1, \ldots, y_n](x)} \mathrm{d}x.$$

Now with a solution for v_i, we can substitute back to find $y(x)$.

$$y(x) = \sum_{i=1}^{n} y_i(x) \int \frac{g(x) W_i(x)}{W[y_1, \ldots, y_n](x)} \mathrm{d}x$$

Example. *Find the general solution to the equation*

$$x^3y''' + x^2y'' - 2xy' = x^3 \sin x,\ x > 0$$

given that $\{x, x^{-1}, x^2\}$ *is the set of fundamental solutions.*

First we divide by x^3 to get a leading coefficient of 1.

$$y''' + x^{-1}y''' - 2x^{-2}y' = \sin x,\ x > 0.$$

Next we calculate the $W(x)$ and each $W_i(x)$ for the fundamental solution set.

$$W[x, x^{-1}, x^2](x) = \begin{vmatrix} x & x^{-1} & x^2 \\ 1 & -x^{-2} & 2x \\ 0 & 2x^{-3} & 2 \end{vmatrix} = 6x^{-1}$$

$$W_1(x) = (-1)^{3-1}W[x^{-1}, x^2](x) = (-1)^2 \begin{vmatrix} x^{-1} & x^2 \\ -x^{-2} & 2x \end{vmatrix} = 3$$

$$W_2(x) = (-1)^{3-2} \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} = -x^2$$

$$W_2(x) = (-1)^{3-2} \begin{vmatrix} x & x^{-1} \\ 1 & -x^{-2} \end{vmatrix} = -2x^{-1}.$$

Now we can calculate y.

$$\begin{aligned} y(x) &= x \int \frac{(\sin x)^3}{-6x^{-1}} \mathrm{d}x + x^{-1} \int \frac{(\sin x)(-x^2)}{-6x^{-1}} \mathrm{d}x + x^2 \int \frac{(\sin x)(-2x^{-1})}{-6x^{-1}} \mathrm{d}x \\ &= x \int \left(\frac{-1}{2} x \sin x \right) \mathrm{d}x + x^{-1} \int \left(\frac{1}{6} x^3 \sin x \right) \mathrm{d}x + x^2 \int \left(\frac{1}{3} \sin x \right) \mathrm{d}x \\ &= C_1x + c_2x^{-1} + C_3x^2 + \cos x - x^{-1} \sin x \end{aligned}$$

3.4 Forced Vibrations

Now that we have some powerful methods for solving higher order equations, we can think about forced vibrations and understand ideas like beats and resonance.

The equation we're trying to solve is

$$my'' + by' + ky = F_{\text{ext}}(t).$$

We'll assume that $F_{\text{ext}}(t) = F_0 \cos(\gamma t)$ so our equation becomes.

$$my'' + by' + ky = F_0 \cos \gamma$$

We'll look at the undamped and damped cases separately.

3.4.1 Undamped Forced Vibrations ($b = 0$)

Our equation now becomes

$$my'' + ky = F_0 \cos(\gamma t).$$

We can use the method of undetermined coefficients to solve this system.

Extracting the coefficients for the auxiliary equation and finding the roots,

$$mr^2 + k = 0 \implies r = \pm i\sqrt{\frac{k}{m}} = i\omega.$$

So, our homogeneous solution is

$$y_h = C_1 \cos(\omega t) + C_2 \sin(\omega t).$$

For guessing the form of y_p, we'll need to break into two cases depending on if $\omega = \gamma$. One case will give rise to beats and the other resonance.

Beats ($\omega \neq \gamma$)

If $\omega \neq \gamma$, then we can guess that y_p has the form

$$y_p = A\cos(\gamma t) + B\sin(\gamma t).$$

Since we don't have a term involving the 1st derivative, we can be sure that $B = 0$, since an odd number of derivatives is the only way to turn a sin term into a cos term. So,

$$y_p = A\cos(\gamma t).$$

Solving for A,

$$\begin{aligned} m\left(A\cos(\gamma t)\right)'' + k\left(A\cos(\gamma t)\right) &= F_0\cos(\gamma t) \\ -mA\gamma^2\cos(\gamma t) + kA\cos(\gamma t) &= F_0\cos(\gamma t) \\ A\left(k - m\gamma^2\right) &= F_0 \\ A &= \frac{F_0}{k - m\gamma^2} = \frac{F_0}{m(\omega^2 - \gamma^2)}. \end{aligned}$$

So, our solution is

$$y = \frac{F_0}{m(\omega^2 - \gamma^2)}\cos(\gamma t) + C_1\cos(\omega t) + C_2\sin(\omega t).$$

Let's look specifically at the IVP where $y(0) = 0$ and $y'(0) = 0$.

$$C_1 = \frac{-F_0}{m(\omega^2 - \gamma^2)} \text{ and } C_2 = 0.$$

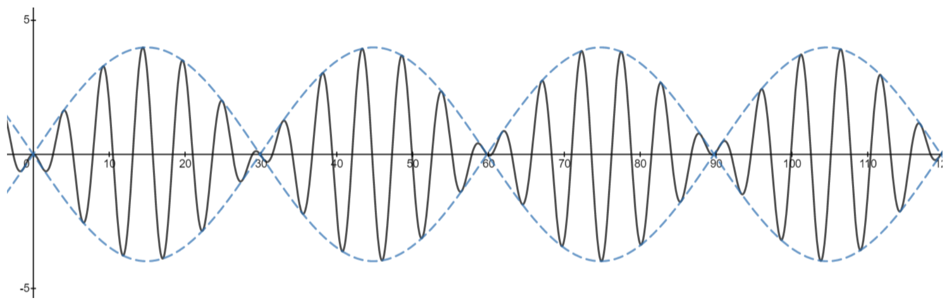
So,

$$y = \frac{F_0}{m(\omega^2 - \gamma^2)} \cos(\gamma t) - \frac{F_0}{m(\omega^2 - \gamma^2)} \cos(\omega t).$$

Using the fact that $\cos\alpha - \cos\beta = 2\sin\left(\frac{\alpha-\beta}{2}\right)\sin\left(\frac{\alpha+\beta}{2}\right)$,

$$y = \frac{2F_0}{m(\omega^2 - \gamma^2)} \sin\left(\frac{\gamma - \omega}{2} t\right) \sin\left(\frac{\gamma + \omega}{2} t\right).$$

When $\gamma \approx \omega$, the $\gamma + \omega$, with a small period, dominates the motion, and the amplitude is slowly guided by the $\gamma - \omega$ term which has a large period. This creates intervals guided by the $\gamma - \omega$ term of higher and lower amplitudes. These are beats.



Resonance $(\omega = \gamma)$

In the case where $\omega = \gamma$, we need to add an extra factor of t to our guess for y_p. So,

$$y_p = At\cos(\omega t) + Bt\sin(\omega t).$$

Solving for A and B,

$$my_p'' + ky_p = 2Bm\omega\cos(\omega t) - 2Am\omega\sin(\omega t) = F_0\cos(\omega t).$$

$$\implies A = 0 \text{ and } B = \frac{F_0}{2m\omega}.$$

So our solution is,

$$y = C_1\cos(\omega t) + C_2\sin(\omega t) + \frac{F_0}{2m\omega} t\sin(\omega t).$$

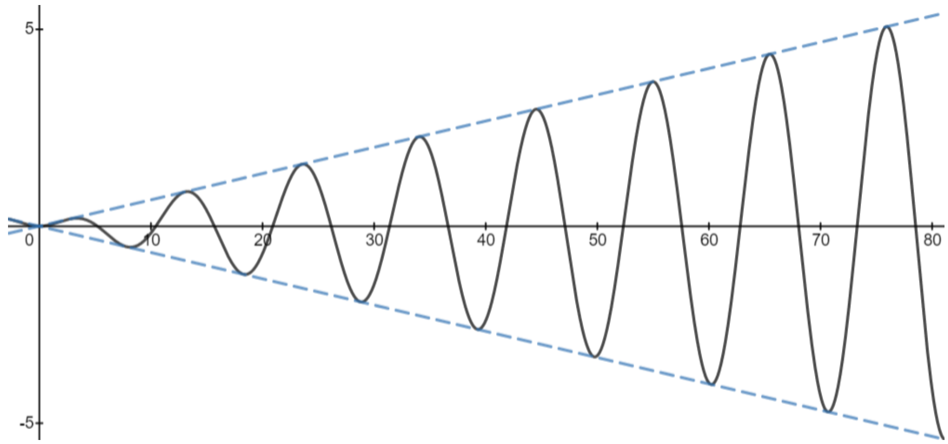
Let's look specifically at the IVP where $y(0) = 0$ and $y'(0) = 0$.

$$C_1 = C_2 = 0.$$

So,

$$y = \frac{F_0}{2m\omega} t\sin(\omega t).$$

Here, the amplitude grows with t, creating bigger and bigger waves. This is resonance, and you can see mathematically how it is responsible for one string causing another tuned the same to vibrate and the collapsing of bridges.



3.4.2 Damped Forced Vibrations ($b \neq 0$)

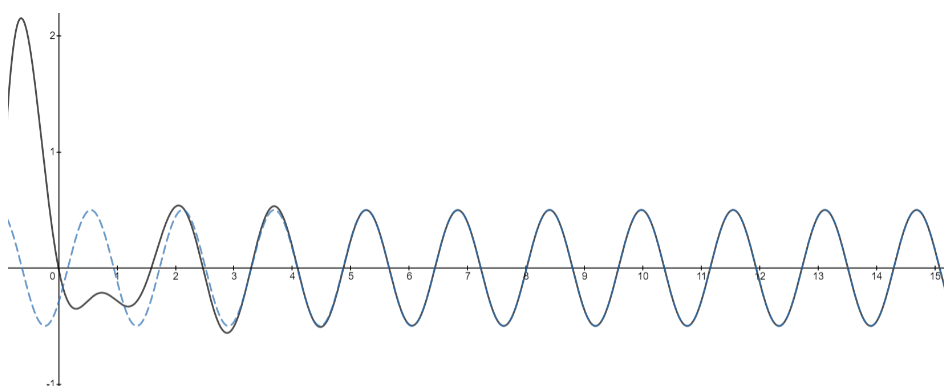
Our equation is

$$my' + by' + ky = F_0 \cos(\gamma t).$$

Depending on if $\Delta = b^2 - 4mk$ is positive, zero, or negative, we'll get different results and thus different guesses for y_p and thus different solutions. There is, however, one long, complicated and not very useful for y.

$$y = C_1 e^{\frac{-b+\sqrt{b^2-4mk}}{2m}t} + C_2 e^{\frac{-b-\sqrt{b^2-4mk}}{2m}t} + \frac{F_0\left(b\gamma \sin(\gamma t) + (k - \gamma^2 m)\cos(\gamma t)\right)}{b^2\gamma^2 + (k - \gamma^2 m)^2}.$$

The one useful thing this formula does tell us is that assuming m, b, k, F_0, and γ are all positive and non-zero, the exponential terms quickly decrease to 0, so in the limit the function looks like the particular solution part.



Example. *Solve the following IVP.*

$$\begin{cases} y'' + 2y' + 10y = 5\cos(4t) \\ y(0) = 0 \\ y'(0) = -2.6 \end{cases}$$

Some of the work has been omitted for brevity.
Solving the auxiliary equation and finding y_h,

$$r^2 + 2r + 10 = 0 \implies r = -1 \pm 3i \implies y_h = e^{-t}\left(C_1 \cos(3t) + C_2 \sin(3t)\right).$$

Guessing the form of y_p, noting that $\omega \neq \gamma$,

$$y_p = A\cos(4t) + B\sin(4t).$$

Solving for A and B,

$$y_p'' + 2y_p' + 10y_p = 5\cos(4t) \implies A = -3/10 \text{ and } B = 4/10.$$

Solving for C_1 and C_2,

$$\begin{cases} y(0) = 0 \\ y'(0) = -2.6 \end{cases} \implies C_1 = 3/10 \text{ and } C_2 = -13/10.$$

So, we have a solution for y,

$$y = e^{-t}\left(\frac{3}{10}\cos(3t) - \frac{13}{10}\sin(3t)\right) + \left(\frac{-3}{10}\cos(4t) + \frac{4}{10}\sin(4t)\right).$$

The graph of this solution was shown above before the example.

Chapter 4

Linear Systems of Differential Equations

Although we have a good idea on how to solve many single linear differential equations, it's often useful to think about linear systems of equations, like

$$\begin{cases} x_1' = 3tx_1 - x_2 + t^2 \\ x_2' = -x_1 + e^t x_2 - e^{2t} \end{cases} \quad \to \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} 3t & -1 \\ -1 & e^t \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} t^2 \\ -e^{2t} \end{bmatrix}.$$

This matrix form, $\vec{x'} = A\vec{x} + \vec{f}$, where A is a square matrix, is called normal form.

4.1 Solutions to Systems

Just like with a single differential equation, we need to define exactly what a solution to a differential equation looks like.

Definition. *A solution on an interval I is a vector $\vec{x}$ that satisfies the system of differential equations.*

We also need to define some common terms like homogeneous and heterogeneous.

Definition. *If a linear system of differential equations written in the form $\vec{x}' = A\vec{x} + \vec{f}$ where $\vec{f} = \vec{0}$, then the system is homogeneous. Else, the system is heterogeneous.*

Example. *Check that*

$$\vec{a} = \begin{bmatrix} e^{-2t} \\ -e^{-2t} \end{bmatrix} \text{ and } \vec{b} = \begin{bmatrix} 3e^{6t} \\ 5e^{6t} \end{bmatrix}$$

are the two fundamental solutions and that

$$\vec{c} = \begin{bmatrix} 3t - 4 \\ -5t + 6 \end{bmatrix}$$

is the particular solution to the system

$$\vec{x}' = \begin{bmatrix} 1 & 3 \\ 5 & 3 \end{bmatrix} \vec{x} + \begin{bmatrix} 12t - 11 \\ -3 \end{bmatrix}.$$

Use these results to write the general solution to the equation.

We'll check that $\vec{a}$ is a fundamental solution.

$$\begin{bmatrix} e^{-2t} \\ -e^{-2t} \end{bmatrix}' = \begin{bmatrix} 1 & 3 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} e^{-2t} \\ -e^{-2t} \end{bmatrix}$$
$$\begin{bmatrix} -2e^{-2t} \\ 2e^{-2t} \end{bmatrix} = \begin{bmatrix} e^{-2t} - 3e^{-2t} \\ 5e^{-2t} - 3e^{-2t} \end{bmatrix}$$
$$\begin{bmatrix} -2e^{-2t} \\ 2e^{-2t} \end{bmatrix} = \begin{bmatrix} -2e^{-2t} \\ 2e^{-2t} \end{bmatrix}.$$

Now we'll check that $\vec{b}$ is a fundamental solution.

$$\begin{bmatrix} 3e^{6t} \\ 5e^{6t} \end{bmatrix}' = \begin{bmatrix} 1 & 3 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} 3e^{6t} \\ 5e^{6t} \end{bmatrix}$$
$$\begin{bmatrix} 18e^{6t} \\ 30e^{6t} \end{bmatrix} = \begin{bmatrix} 3e^{6t} + 15e^{6t} \\ 15e^{6t} + 15e^{6t} \end{bmatrix}$$
$$\begin{bmatrix} 18e^{6t} \\ 30e^{6t} \end{bmatrix} = \begin{bmatrix} 18e^{6t} \\ 30e^{6t} \end{bmatrix}.$$

These two true results give us the homogeneous solution $\vec{x_h}$. We know that $\vec{a}$ and $\vec{b}$ make up the full homogeneous solution because the system is made up of two first order equations and thus has two fundamental solutions.

$$\vec{x_h} = C_1 \begin{bmatrix} e^{-2t} \\ -e^{-2t} \end{bmatrix} + C_2 \begin{bmatrix} 3e^{6t} \\ 5e^{6t} \end{bmatrix}.$$

Now we'll check that $\vec{c}$ is the particular solution.

$$\begin{bmatrix} 3t - 4 \\ -5t + 6 \end{bmatrix}' = \begin{bmatrix} 1 & 3 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} 3t - 4 \\ -5t + 6 \end{bmatrix} + \begin{bmatrix} 12t - 11 \\ -3 \end{bmatrix}$$
$$\begin{bmatrix} 3 \\ -5 \end{bmatrix} = \begin{bmatrix} (3t - 4) + 3(-5t + 6) + (12t - 11) \\ 5(3t - 4) + 3(-5t + 6) + (-3) \end{bmatrix}$$
$$\begin{bmatrix} 3 \\ -5 \end{bmatrix} = \begin{bmatrix} 3 \\ -5 \end{bmatrix}.$$

This true result gives us the particular solution $\vec{x_p}$.

$$\vec{x_p} = \begin{bmatrix} 3t - 4 \\ -5t + 6 \end{bmatrix}.$$

Putting $\vec{x_h}$ and $\vec{x_p}$ together gives us our general solution $\vec{x}$.

$$\vec{x} = \vec{x_h} + \vec{x_p} = C_1 \begin{bmatrix} e^{-2t} \\ -e^{-2t} \end{bmatrix} + C_2 \begin{bmatrix} 3e^{6t} \\ 5e^{6t} \end{bmatrix} + \begin{bmatrix} 3t - 4 \\ -5t + 6 \end{bmatrix}$$

4.2 Homogeneous Systems

Homogeneous systems of linear equations have the form

$$\vec{x}' = A\vec{x}.$$

We'll see how to find solutions to these types of systems using the eigenvalue method. Finding solutions to these systems will allow us to find homogeneous solutions to heterogeneous systems of equations.

4.2.1 Eigenvalue Method

This method allows us to find fundamental solutions using the eigenvalues of the matrix A. Of course, just like with roots of the auxiliary equation, we need to consider the cases of real distinct eigenvalues and eigenvectors, repeated eigenvalues, complex eigenvalues, and defective matrices.

Real Distinct Eigenvalues

Real, distinct eigenvalues are the simplest case, similar to real, distinct roots of an auxiliary equation.

Theorem. *Let $\{\lambda_1, \ldots, \lambda_n\}$ be the set of unique eigenvalues and $\{\vec{v_1}, \ldots, \vec{v_n}\}$ be the corresponding set of non-zero, unique, eigenvectors for an $n \times n$ matrix A. Then the set of fundamental solutions to the system $\vec{x}' = A\vec{x}$ is $\left\{e^{\lambda_1 t}\vec{v_1}, \ldots, e^{\lambda_n t}\vec{v_n}\right\}$*

Example. *Find the general solution to the system*

$$\vec{x}' = \begin{bmatrix} 3 & 0 & 0 \\ -5 & -2 & 0 \\ 1 & 1 & -1 \end{bmatrix} \vec{x}.$$

For a triangular matrix, the eigenvalues are simply the diagonal entries.

$$\lambda = 3, -2, -1.$$

Finding the eigenvector for $\lambda = 3$,

$$(A - 3I)\vec{v} = \vec{0} \implies \vec{v} = C_1 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}.$$

Finding the eigenvector for $\lambda = -2$,

$$(A + 2I)\vec{v} = \vec{0} \implies \vec{v} = C_2 \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}.$$

Finding the eigenvector for $\lambda = -1$,

$$(A+I)\vec{v} = \vec{0} \implies \vec{v} = C_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

So, our general solution is

$$\vec{x} = C_1 e^{3t} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + C_2 e^{-2t} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} + C_3 e^{-t} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Repeated Eigenvalues

If A has an eigenvalue with multiplicity k, that eigenvalue needs to generate k fundamental solutions. If this eigenvalue generates k linearly independent eigenvectors, then the process is much like with distinct eigenvalues. Otherwise, the matrix A is defective.

Theorem. *If λ is an eigenvalue with multiplicity k, and $\{\vec{v_1}, \ldots, \vec{v_k}\}$ are corresponding linearly independent eigenvectors, the the set of fundamental solutions generated by λ is $\left\{e^{\lambda t}\vec{v_1}, \ldots, e^{\lambda t}\vec{v_k}\right\}$.*

Example. *Find the general solution to the system*

$$\vec{x}' = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \vec{x}.$$

Finding the eigenvalues by finding the roots of the characteristic polynomial of A,

$$p(\lambda) = \det(A - \lambda I) = -\lambda^3 + 3\lambda + 2 \implies \lambda = 2, -1, -1.$$

Finding the eigenvector for $\lambda = 2$,

$$(A - 2I)\vec{v} = \vec{0} \implies \vec{v} = C_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Finding the eigenvectors for $\lambda = -1$,

$$(A+I)\vec{v} = \vec{0} \implies \vec{v} = C_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + C_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

So, our solution is

$$\vec{x} = C_1 e^{2t} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + C_2 e^{-t} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + C_3 e^{-t} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

Complex Eigenvalues

Theorem. *If an $n \times n$ matrix A is not defective, then for each pair of complex eigenvalues $\alpha \pm \beta i$ with corresponding eigenvectors $\vec{a} \pm i\vec{b}$, the corresponding fundamental solutions are $e^{\alpha t}\left(\cos(\beta t)\vec{a} - \sin(\beta t)\vec{b}\right)$ and $e^{\alpha t}\left(\sin(\beta t)\vec{a} + \cos(\beta t)\vec{b}\right)$.*

Example. *Find the general solution to the system*

$$\vec{x}' = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix} \vec{x}.$$

Finding the eigenvalues by finding the roots of the characteristic polynomial of A,

$$p(\lambda) = \det(A - \lambda I) = (2-\lambda)^2 + 9 \implies \lambda = 2 \pm 3i.$$

Finding the eigenvectors for $\lambda = 2+3i$, remembering that once we have the two eigenvectors, we don't need to find them for the conjugate,

$$(A - (2+3i)I)\vec{v} = \vec{0} \implies \vec{v} = C_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + C_2 i \begin{bmatrix} -1 \\ 0 \end{bmatrix}.$$

So, our solution is

$$\vec{x} = C_1 e^{2t}\left(\cos(3t)\begin{bmatrix} 0 \\ 1 \end{bmatrix} - \sin(3t)\begin{bmatrix} -1 \\ 0 \end{bmatrix}\right) + C_2 e^{2t}\left(\sin(3t)\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \cos(3t)\begin{bmatrix} -1 \\ 0 \end{bmatrix}\right).$$

We can rewrite this a little more elegantly as

$$\vec{x} = \begin{bmatrix} \sin(3t) & -\cos(3t) \\ \sin(3t) & \cos(3t) \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} e^{2t}.$$

Defective Matrix

If A has an eigenvalue λ with multiplicity k that does not generate k corresponding linearly independent eigenvectors, the the matrix A is defective. To generate enough vectors, we need to extend eigenvectors.

Definition. *Let A be a square matrix. A nonzero vector $\vec{v}$ satisfying*

$$(A - \lambda I)^n \vec{v} = \vec{0}$$

for some eigenvalue λ and some positive integer n is a generalized eigenvector or rank n.

These generalized eigenvectors also make up solutions.

Theorem. *Let $\vec{v}$ be an generalized eigenvector of a square matrix A corresponding to an eigenvalue λ with multiplicity k. Then*

$$e^{At}\vec{v} = e^{\lambda t}\left(\vec{v} + t(A-\lambda I)\vec{v} + \frac{t^2}{2!}(A-\lambda I)^2\vec{v} + \dots\right)$$

is the corresponding fundamental solution. Further, the above sequence will terminate after k or fewer terms (all subsequent terms are 0).

Example. *Find the general solution to the system*

$$\vec{x}' = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 3 & 0 \\ 0 & 1 & 1 \end{bmatrix} \vec{x}.$$

Since A is diagonal, the eigenvalues are simply the diagonal entries

$$\lambda = 3, 1(\text{double root}).$$

Finding the eigenvector for $\lambda = 3$,

$$(A-3I)\vec{v} = \vec{0} \implies \vec{v} = C_1 \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}.$$

Finding the eigenvector for $\lambda = 1$,

$$(A-I)\vec{v} = \vec{0} \implies \vec{v} = C_2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Since $\lambda = 1$ had multiplicity 2, we need to find a generalized eigenvector.

$$(A-I)^2\vec{v} = \vec{0} \implies \vec{v} = C_2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + C_3 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}.$$

Note how then eigenvector showed up again as a generalized eigenvector of rank 2. This is because the eigenvector makes $(A-I)\vec{v} = 0$, so $(A-I)^2\vec{v} = (A-I)\vec{0} = \vec{0}$. So, our solution is

$$\begin{aligned}
\vec{x} &= C_1 e^{3t} \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} + C_2 e^{t} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + C_3 e^{t} \left(\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + (A-I)t \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \right) \\
&= C_1 e^{3t} \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} + C_2 e^{t} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + C_3 e^{t} \begin{bmatrix} -2 \\ 1 \\ t \end{bmatrix}.
\end{aligned}$$

We can rewrite this a bit more cleanly as

$$\vec{x} = \begin{bmatrix} 0 & 0 & -2e^t \\ 2e^{2t} & 0 & e^t \\ e^{3t} & e^t & te^t \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix}.$$

This square matrix is called the fundamental matrix of the system.

This final approach for defective matrices actually gives a general way to solve $\vec{x}' = A\vec{x}$ for any real $n \times n$ matrix A.

1) Compute the characteristic polynomial $p(\lambda) = \det(A - \lambda I)$. Find the zeroes of the characteristic to find the distinct eigenvalues $\lambda_1, \ldots, \lambda_k$ with corresponding multiplicities $m_1, \ldots, m_k$.

2) For each eigenvalue λ_i find the m_i corresponding linearly independent eigenvectors of rank m_i by solving the system $(A - \lambda_i)^{m_i}\vec{v_i} = \vec{0}$.

3) For each generalized generalized eigenvector $\vec{v_i}$, compute the corresponding fundamental solution

$$\vec{x_i} = e^{At}\vec{v_i} = e^{\lambda t}\left(\vec{v_i} + t(A - \lambda I)\vec{v_i} + \frac{t^2}{2!}(A - \lambda I)^2\vec{v_i} + \ldots\right).$$

   Note that this summation terminates is at most m_i terms.

4) Combine these fundamental solutions as columns of a a square matrix to obtain the fundamental matrix of the system $X(t)$. The general solution to the system is

$$\vec{x} = X(t)\begin{bmatrix} C_1 \\ \vdots \\ C_n \end{bmatrix}.$$

4.3 Heterogeneous Systems

Now that we know how to solve the "easy case" of homogeneous systems of linear differential equations, we can tackle the general case of heterogeneous systems. Much like for single equations, we'll look for a homogeneous solution and particular solution, and we'll use undetermined coefficients or variation of parameters to find the particular solution. These methods will then lead into seeing how systems of first order equations relate to single higher order equations.

4.3.1 Method of Undetermined Coefficients

The rules for undetermined coefficients are very similar for systems and single equations. Let's say we have a heterogeneous system of the form

$$\begin{cases} x_1' = a_{11}x_1 + \ldots + a_{1n}x_n + f_1(t) \\ \vdots \\ x_n' = a_{n1}x_n + \ldots + a_{nn}x_n + f_n(t). \end{cases}.$$

Written in a matrix form,

$$\begin{bmatrix} x_1' \\ \vdots \\ x_n' \end{bmatrix} = \begin{bmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \ldots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} f_1(t) \\ \vdots \\ f_n(t) \end{bmatrix},$$

or more compactly as

$$\vec{x}' = A\vec{x} + \vec{f}.$$

All we need to do is look at each $f_i(t)$ and write in the i^{th} blank the corresponding guess. This is like doing undetermined coefficients on each equation in the system.

This is especially nice when all the $f_i(t)$'s have a similar form because we can write our guess as

$$\vec{g}(t) = f_c(t)\vec{v}$$

where $f_c(t)$ is the guess corresponding to f and $\vec{c}$ is a vector of undetermined scalars.

Example. *Find the particular solution to the system*

$$\vec{x}' = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \vec{x} + \begin{bmatrix} 2e^t \\ 4e^t \end{bmatrix}.$$

All entries in $\vec{f}$ are exponentials, so we'll guess that

$$\vec{x_p} = e^t \begin{bmatrix} A \\ B \end{bmatrix} = e^t\vec{a}.$$

Plugging our guess into the equation,

$$\begin{aligned} \left(e^t\vec{a}\right)' &= \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} e^t\vec{a} + \begin{bmatrix} 2e^t \\ 4e^t \end{bmatrix} \\ e^t \begin{bmatrix} A \\ B \end{bmatrix} &= e^t \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} + e^t \begin{bmatrix} 2 \\ 4 \end{bmatrix} \\ &. \end{aligned}$$

Dividing by e^t, which is never 0,

$$\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} A+2B \\ 3A+2B \end{bmatrix} + \begin{bmatrix} 2 \\ 4 \end{bmatrix}.$$

Rearranging,

$$\begin{bmatrix} 0 & 2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} -2 \\ -4 \end{bmatrix} \implies \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}.$$

So,

$$\vec{x_p} = e^t \begin{bmatrix} -1 \\ -1 \end{bmatrix}.$$

It's possible that we'll have to solve for multiple vectors of scalars to find $\vec{x_p}$.

Example. *Find the particular solution to the system*

$$\vec{x}' = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \vec{x} + \begin{bmatrix} -4\cos t \\ -\sin t \end{bmatrix}.$$

Just like when dealing with sin and cos in single equations, we need both sin and cos in our guess.

$$\vec{x_p} = \sin t\vec{a} + \cos t\vec{b}.$$

Plugging into the equation where A is the matrix of all 2's,

$$\cos t\vec{a} - \sin(t)\vec{b} = \sin tA\vec{a} + \cos tA\vec{b} + \sin t \begin{bmatrix} 0 \\ -1 \end{bmatrix} + \cos t \begin{bmatrix} -4 \\ 0 \end{bmatrix}.$$

This gives us a system

$$\begin{cases} -\vec{b} = A\vec{a} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} \\ \vec{a} = A\vec{b} + \begin{bmatrix} -4 \\ 0 \end{bmatrix} \end{cases}.$$

Substituting,

$$-\vec{b} = A\left(A\vec{b} + \begin{bmatrix} -4 \\ 0 \end{bmatrix}\right) + \begin{bmatrix} 0 \\ -1 \end{bmatrix} \implies (A^2+I)\vec{b} = -A\begin{bmatrix} -4 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$A^2 + I = \begin{bmatrix} 9 & 8 \\ 8 & 9 \end{bmatrix} \text{ and } -A\begin{bmatrix} -4 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 8 \\ 9 \end{bmatrix}.$$

So,

$$\begin{bmatrix} 9 & 8 \\ 8 & 9 \end{bmatrix} \vec{b} = \begin{bmatrix} 8 \\ 9 \end{bmatrix} \implies \vec{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

and

$$\vec{a} = A\vec{b} + \begin{bmatrix} -4 \\ 0 \end{bmatrix} \implies \vec{a} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}.$$

Finally we have $\vec{x_p}$,

$$\vec{x_p} = \sin t \begin{bmatrix} -2 \\ 2 \end{bmatrix} + \cos t \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

4.3.2 Variation of Parameters for Systems

Let X be a fundamental matrix for the homogeneous system.

$$\vec{x_h}' = A\vec{x_h}.$$

That is,

$$\vec{x_h} = X\vec{c}$$

where

$$\vec{c} = \begin{bmatrix} C_1 \\ \vdots \\ C_n \end{bmatrix}$$

and the entries of matrix A can be any continuous functions of t.
We are looking for the particular solution $\vec{x_p}$, to the system

$$\vec{x} = A\vec{x} + \vec{f}$$

where $\vec{x_p}$ is of the form

$$\vec{x_p} = X\vec{v}$$

where $\vec{v}$ is a vector of functions of t that we'll have to find.
Differentiating $\vec{x_p}$,

$$\vec{x_p}' = X\vec{v}' + X'\vec{v}.$$

From the system we're trying to solve we know that

$$X\vec{v}' + X'\vec{v} = A(X\vec{v}) + \vec{f}.$$

Since $X' = AX$,

$$X\vec{v}' = \vec{f}.$$

Since the columns of X are always linearly independent, we know that X^{-1} always exists.
Multiplying by X^{-1},

$$\vec{v}' = X^{-1}\vec{f}.$$

Integrating with respect to t,

$$\vec{v} = \int X^{-1}\vec{f}\mathrm{d}t.$$

So,

$$\vec{x_p} = X\int X^{-1}\vec{f}\mathrm{d}t,$$

and

$$\vec{x} = X\vec{c} + X\int X^{-1}\vec{f}\mathrm{d}t.$$

Example. *Find the general solution to the system by variation of parameters*

$$\vec{x}' = \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix} \vec{x} + \begin{bmatrix} e^{2t} \\ 1 \end{bmatrix}$$

given the fundamental matrix

$$X = \begin{bmatrix} 3e^t & e^{-t} \\ e^t & e^{-t} \end{bmatrix}.$$

Finding $\vec{x_h}$ is simply multiplying X by a vector, so we'll save that for the end and focus on $\vec{x_p}$. First we need to find X^{-1}. The details are left out, but the process is identical to that for matrices of numbers.

$$X^{-1} = \begin{bmatrix} \frac{1}{2}e^{-t} & -\frac{1}{2}e^{-t} \\ -\frac{1}{2}e^{t} & \frac{3}{2}e^{t} \end{bmatrix}.$$

Multiplying by $\vec{f}$.

$$X\vec{f} = \begin{bmatrix} \frac{1}{2}e^{t} - \frac{1}{2}e^{-t} \\ -\frac{1}{2}e^{3t} + \frac{3}{2}e^{t} \end{bmatrix}.$$

Integrating with respect to t,

$$\int X^{-1}\vec{f}\mathrm{d}t = \begin{bmatrix} \frac{1}{2}e^{t} + \frac{1}{2}e^{-t} \\ -\frac{1}{6}e^{3t} + \frac{3}{2}e^{t} \end{bmatrix}.$$

Note that although we are doing an indefinite integral, we don't have a constant (or in this case a vector) of integration. Multiplying by X to obtain $\vec{x_p}$,

$$\vec{x_p} = X\int X^{-1}\vec{f}\mathrm{d}t = \begin{bmatrix} \frac{4}{3}e^{2t} + 3 \\ \frac{1}{3}e^{2t} + 2 \end{bmatrix}.$$

Finding $\vec{x_h}$,

$$\vec{x_h} = X\vec{c} = \begin{bmatrix} 3C_1e^{t} + C_2e^{-t} \\ C_1e^{t} + C_2e^{-t} \end{bmatrix}.$$

Putting $\vec{x_h}$ and $\vec{x_p}$ together to get the general solution,

$$\vec{x} = \begin{bmatrix} 3C_1e^{3t} + C_2e^{-t} + \frac{4}{3}e^{2t} + 3 \\ C_1e^{t} + C_2e^{-t} + \frac{1}{3}e^{2t} + 2 \end{bmatrix}.$$

4.4 Systems and Higher Order Equations

There is a connection between a system of linear first order equations and a single higher order equation. We'll see how to convert between the two, which will help explain why methods like the auxiliary equation work.

4.4.1 System to Higher Order

Let's say we have the linear system

$$\vec{x}' = A\vec{x} + \vec{f}.$$

Writing the system using $x_1, \ldots x_n$ as the components of $\vec{x}$, $f_1, \ldots f_n$ as the components of $\vec{f}$, and a_{ij} as the entry in A on the i^{th} row and j^{th} column,

$$\begin{cases} x_1' = a_{11}x_1 + \ldots a_{1n}x_n + f_1 \\ \vdots \\ x_n' = a_{n1}x_1 + \ldots a_{nn}x_n + f_n \end{cases}.$$

Let's arbitrarily assign $x_1 = y$. This will allow us to find expressions for $x_2, \ldots x_n$ in terms of y and its derivatives. When we find x_n in these terms and equate x_n' with what's given in the system, we'll have an linear n^{th} order equation.

Example. *Convert the following system of equations to a single equation.*

$$\vec{x}' = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \vec{x} + \begin{bmatrix} 1 \\ t \\ t^2 \end{bmatrix}$$

Expanding the system out of matrix form,

$$\begin{cases} x_1' = x_1 + x_2 + x_3 + 1 \\ x_2' = x_1 + x_3 + t \\ x_3' = x_2 + x_3 + t^2 \end{cases}.$$

Assuming $x_1 = y$,

$$\begin{aligned} y' &= y + x_2 + x_3 + 1 \\ &\implies x_2 = y' - y - x_3 - 1. \end{aligned}$$

Taking the derivative of x_2 and equating it with what's given in the system,

$$x_2' = y'' - y' - x_3' = y + x_3 + t.$$

Solving for x_3' and equating it with what's given in the system,

$$x_3' = y'' - y' - t - x_3 = y' - y + t^2 - 1.$$

Putting this expression for x_3' back into our expression for x_2',

$$\begin{aligned} x_2' &= y'' - 2y' + y - t^2 + 1 = y + x_3 + t \\ &\implies x_3 = y'' - 2y' - t^2 - t + 1. \end{aligned}$$

Taking the derivative of x_3 and equating it to what's given in the system,

$$x_3' = y''' - 2y'' - 2t - 1 = y' - y - 1 + t^2.$$

So, we have our order 3 equation,

$$y''' - 2y'' - y' + y = t^2 + 2t.$$

The auxiliary polynomial of this higher order equation $p(r)$ and the characteristic polynomial $p(\lambda)$ of the linear system will have exactly the same roots.

4.4.2 Higher Order to System

Let's say we have the linear, n^{th} order differential equation

$$a_n y^{(n)} + \ldots + a_0 y = f$$

where $a_0, \ldots a_n$ and f are functions.

If we let $x_1 = y, x_2 = y', \ldots x_n = y^{(n-1)}$, then we have a system of equations,

$$\begin{cases} x_1' & = y' = x_2 \\ \vdots & \\ x_{n-1}' & = y^{(n-1)} = x_n \\ x_n' & = y^{(n)} = \frac{1}{a_n}\left(f - a_{n-1}x_n - \ldots - a_0 x_1\right) \end{cases}.$$

Rewriting in matrix form,

$$\vec{x}' = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ \frac{-a_0}{a_n} & \frac{-a_1}{a_n} & \frac{-a_2}{a_n} & \ldots & \frac{-a_{n-1}}{a_n} \end{bmatrix} \vec{x} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \frac{f}{a_n} \end{bmatrix}.$$

The square matrix has characteristic polynomial $p(\lambda) = \frac{(-1)^n}{a_n}\left(a_n\lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_0\right)$ which has the same zeroes as the the auxiliary equation $a_n r^n + \ldots + a_0$.

Note that if one starts with a higher order equation, converts it to a system, and the converts the system back to a higher order equation, the result is the original equation. This does not hold true when converting a system to an equation and back to a system. The two solutions will have the same eigenvalues but different eigenvectors.

Chapter 5

Laplace Transforms

Laplace transforms allow us to change differential equations into algebraic equations. We can then solve the algebraic equation and "undo" the Laplace transform to find the solution to our differential equation.

5.1 Definition

Definition. *Let $f(t)$ be a function that is defined for $t \geq 0$.*

$$\mathcal{L}\left\{f\right\}(s) = \int_0^\infty e^{-st} f(t) \mathrm{d}t.$$

The domain of F is all values of s where the integral is defined and finite.

5.1.1 Linearity

We already know from calculus that the integral is a linear operator. That is, scalar multiplication can be pulled out of the integral, and the integral of the sum of two functions is the same as the sum of the integrals of the functions. Since the Laplace transform is just an integral, it is also a linear operator. So we can say

$$\mathcal{L}\left\{cf + g\right\}(s) = c\mathcal{L}\left\{f\right\}(s) + \mathcal{L}\left\{g\right\}(s), \ c \in \mathbb{R}.$$

5.2 Derivations

We can use the definition of the Laplace transform and the fact that the Laplace transform is a linear operator to find the Laplace transform of some common functions.

5.2.1 Constant

Let a be a constant.
By definitions of a Laplace transform and an improper integral,

$$\begin{aligned}\mathcal{L}\left\{a\right\}(s) &= \lim_{n\to\infty}\int_0^n ae^{-st}\mathrm{d}t\\ &= \frac{a}{s}\lim_{n\to\infty}\left[-e^{-st}\right]_0^n,\ s\neq 0\\ &= \frac{a}{s}\lim_{n\to\infty}\left(1-e^{-sn}\right),\ s\neq 0\\ &= \frac{a}{s},\ s>0.\end{aligned}$$

So,

$$\mathcal{L}\left\{a\right\}(s) = \frac{a}{s},\ s>0.$$

5.2.2 Exponential

Let a be a constant.
By definitions of a Laplace transform and an improper integral,

$$\begin{aligned}\mathcal{L}\left\{e^{at}\right\}(s) &= \lim_{n\to\infty}\int_0^n e^{at}e^{-st}\mathrm{d}t\\ &= \lim_{n\to\infty}\int_0^\infty e^{(a-s)t}\mathrm{d}t\\ &= \frac{1}{a-s}\lim_{n\to\infty}\left[e^{(a-s)n}\right]_0^n,\ s\neq a\\ &= \frac{1}{a-s}\lim_{n\to\infty}\left(e^{(a-s)n}-1\right),\ s\neq a\\ &= \begin{cases}\frac{-1}{a-s} & s>a\\ \text{DNE} & s\leq a\end{cases}.\end{aligned}$$

So,

$$\mathcal{L}\left\{e^{at}\right\}(s) = \frac{-1}{a-s},\ s>a.$$

5.2.3 Sine and Cosine

Sine

Let a be a constant.
By definitions of a Laplace transform and an improper integral,

$$\begin{aligned}
\mathcal{L}\left\{\sin\left(at\right)\right\}(s) &= \lim_{n\to\infty}\int_0^n \sin\left(at\right)e^{-st}\mathrm{d}t \\
&= \frac{-1}{s^2+a^2}\lim_{n\to\infty}\left[e^{-st}\left(s\sin\left(at\right)+a\cos\left(at\right)\right)\right]_0^n \\
&= \frac{-1}{s^2+a^2}\left(\lim_{n\to\infty}\left(e^{-sn}\left(s\sin\left(an\right)+a\cos\left(an\right)\right)\right)-\left(e^{-s\cdot 0}\left(s\sin\left(a\cdot 0\right)+a\cos\left(a\cdot 0\right)\right)\right)\right).
\end{aligned}$$

Both sin and cos have maximum values of 1, so we can say that the left part of them expression has a maximum value at most $s+a$. For positive s, the exponential dominates and the expression goes to 0 in the limit.

$$\begin{aligned}
&= \frac{-1}{s^2+a^2}\left(0-a\right),\ s>0 \\
&= \frac{a}{s^2+a^2},\ s>0.
\end{aligned}$$

So,

$$\mathcal{L}\left\{\sin\left(at\right)\right\}(s) = \frac{a}{s^2+a^2},\ s>0.$$

Cosine

Let a be a constant.
By definitions of a Laplace transform and an improper integral,

$$\begin{aligned}
\mathcal{L}\left\{\cos\left(at\right)\right\}(s) &= \lim_{n\to\infty}\int_0^n \cos\left(at\right)e^{-st}\mathrm{d}t \\
&= \frac{1}{s^2+a^2}\lim_{n\to\infty}\left[e^{-st}\left(a\sin\left(at\right)-s\cos\left(at\right)\right)\right]_0^n \\
&= \frac{1}{s^2+a^2}\left(\lim_{n\to\infty}\left(e^{-sn}\left(a\sin sn-s\cos\left(an\right)\right)\right)-\left(e^{-s\cdot 0}\left(a\sin\left(a\cdot 0\right)-s\cos\left(a\cdot 0\right)\right)\right)\right).
\end{aligned}$$

Both sin and cos have maximum values of 1 and minimum values of -1, so we can way that the left part of the expression has a maximum value of at most $a+s$. For positive s, the exponential dominates and the expression goes to 0 in the limit.

$$\begin{aligned}
&= \frac{1}{s^2+a^2}\left(0+s\right),\ s>0 \\
&= \frac{s}{s^2+a^2},\ s>0.
\end{aligned}$$

So,

$$\mathcal{L}\left\{\cos\left(at\right)\right\}(s) = \frac{s}{s^2+a^2},\ s>0.$$

5.2.4 n^{th} Derivative

We'll use induction to show that

$$\mathcal{L}\left\{f^{(n)}(t)\right\}(s) = s^n \mathcal{L}\left\{f(t)\right\}(s) - f^{(n-1)}(0) - sf^{(n-2)}(0) - \ldots - s^{n-1}f(0).$$

We'll start with the first derivative as a base case. We could use the 0^{th} derivative, but this case will give us a little more insight into where the formula comes from.
Let f be a differentiable function.

$$\mathcal{L}\left\{f'(t)\right\}(s) = \lim_{n\to\infty} \int_0^n f'(t)e^{-st}\mathrm{d}t.$$

Integrating by parts and using the fundamental theorem of calculus,

$$= \left(\lim_{n\to\infty} \left(f(n)e^{-sn}\right) - f(0)e^{-s\cdot 0}\right) + \lim_{n\to\infty} s \int_0^n f(t)e^{-st}\mathrm{d}t.$$

Assuming that $f(n)$ grows slower than e^{-sn},

$$= (0 - f(0)) + \lim_{n\to\infty} s \int_0^n f(t)e^{-st}\mathrm{d}t.$$

Since the right part of the expression is just s times the definition of $\mathcal{L}\left\{f\right\}(s)$,

$$= s\mathcal{L}\left\{f\right\}(s) - f(0).$$

So,

$$\mathcal{L}\left\{f'(t)\right\}(s) = s\mathcal{L}\left\{f\right\}(s) - f(0).$$

Assuming the following is true,

$$\mathcal{L}\left\{f^{(n)}(t)\right\}(s) = s^n \mathcal{L}\left\{f(t)\right\}(s) - f^{(n-1)}(0) - sf^{(n-2)}(0) - \ldots - s^{n-1}f(0).$$

We'll show that the $n+1$ case follows.

$$\mathcal{L}\left\{f^{(n+1)}(t)\right\}(s) = \mathcal{L}\left\{\left(f^{(n)}\right)'\right\}(s).$$

Using our first derivative formula,

$$= s\mathcal{L}\left\{f^{(n)}(t)\right\}(s) - f^{(n)}(0).$$

Using our general formula,

$$\begin{aligned}
&= s\left(s^n \mathcal{L}\left\{f(t)\right\}(s) - f^{(n-1)}(0) - sf^{(n-2)}(0) - \ldots - s^{n-1}f(0)\right) - f^{(n)}(0)\\
&= s^{n+1}\mathcal{L}\left\{f(t)\right\}(s) - f^{(n)}(0) - sf^{(n-1)}(0)s^2 f^{(n-2)}(0) - \ldots - s^n f(0),
\end{aligned}$$

which is the $n+1$ case, meaning we have proven the general formula as correct.

5.2.5 Polynomials

We'll use our derivative formula and induction to show that

$$\mathcal{L}\left\{t^n\right\}(s) = \frac{n!}{s^{n+1}},\ n \geq 0.$$

Although the formula for $n = 0$ is clearly the same as $\mathcal{L}\left\{1\right\}(s)$, we'll use $n = 1$ as a base case to get a little more insight into where the formula comes from.

$$\mathcal{L}\left\{1\right\}(s) = \mathcal{L}\left\{t'\right\}(s) = \frac{1}{s}.$$

Using our derivative formula,

$$\begin{aligned} \frac{1}{s} &= s\mathcal{L}\left\{t\right\}(s) - 0^1 \\ &\implies \mathcal{L}\left\{t\right\}(s) = \frac{1}{s^2}. \end{aligned}$$

Assuming the following is true,

$$\mathcal{L}\left\{t^n\right\}(s) = \frac{n!}{s^{n+1}},\ n \geq 0.$$

We'll show that the $n + 1$ case follows.

$$\mathcal{L}\left\{t^n\right\}(s) = \mathcal{L}\left\{\left(\frac{t^{n+1}}{n+1}\right)'\right\}(s) = \frac{n!}{s^{n+1}}.$$

Using our derivative formula and the linearity of the Laplace transform,

$$\begin{aligned} \frac{n!}{s^{n+1}} &= \frac{s}{n+1}\mathcal{L}\left\{t^{n+1}\right\}(s) - 0^{n+1} \\ &\implies \mathcal{L}\left\{t^{n+1}\right\}(s) = \frac{(n+1)!}{s^{n+2}}, \end{aligned}$$

which is the $n + 1$ case, meaning we have proven the general formula as correct.

5.2.6 Translation

Let a be a constant.

$$\mathcal{L}\left\{e^{at}f(t)\right\}(s) = \int_0^\infty e^{at}f(t)e^{-st}\mathrm{d}t = \int_0^\infty f(t)e^{-(s-a)t}\mathrm{d}t = \mathcal{L}\left\{f(t)\right\}(s-a).$$

So,

$$\mathcal{L}\left\{e^{at}f(t)\right\}(s) = \mathcal{L}\left\{f(t)\right\}(s-a).$$

This illustrates how multiplying by e^{at} in the t domain corresponds to a translation by a in the s domain.

5.2.7 Derivative of a Laplace Transform

Consider the n^{th} derivative of the Laplace transform of f,

$$\frac{\mathrm{d}^n}{\mathrm{d}s^n}\left(\mathcal{L}\left\{f(t)\right\}(s)\right) = \frac{\mathrm{d}^n}{\mathrm{d}s^n}\left(\int_0^\infty f(t)e^{-st}\mathrm{d}t\right).$$

We're able to change the order of differentiation and integration here without affecting the result.

$$\begin{aligned} &= \int_0^\infty f(t)\frac{\mathrm{d}^n}{\mathrm{d}s^n}\left(e^{-st}\right)\mathrm{d}t \\ &= \int_0^\infty f(t)(-t)^n e^{-st}\mathrm{d}t \\ &= (-1)^n\mathcal{L}\left\{t^n f(t)\right\}(s). \end{aligned}$$

So,

$$\frac{\mathrm{d}^n}{\mathrm{d}s^n}\left(\mathcal{L}\left\{f(t)\right\}(s)\right) = (-1)^n\mathcal{L}\left\{t^n f(t)\right\}(s).$$

This formula is useful in both directions: finding the derivatives of Laplace transforms, and finding the Laplace transforms of functions multiplied by t^n.

Example. *Find the Laplace transform of* $e^{-t}\sin(3t)$.

Using the translation property,

$$\mathcal{L}\left\{e^{-t}\sin(3t)\right\}(s) = \mathcal{L}\left\{\sin(3t)\right\}(s+1).$$

We from our sin formula that

$$\mathcal{L}\left\{\sin(3t)\right\}(s) = \frac{3}{s^2+3^2}.$$

So,

$$\mathcal{L}\left\{e^{-t}\sin(3t)\right\}(s) = \mathcal{L}\left\{\sin(3t)\right\}(s+1) = \frac{3}{(s+1)^2+3^2}.$$

Example. *Find the Laplace transform of* $t^2(t^2+1)(t-1)(t+2)$.

Using the derivative of the Laplace transform property,

$$\mathcal{L}\left\{t^2(t^2+1)(t-1)(t+2)\right\}(s) = (-1)^2\frac{\mathrm{d}^2}{\mathrm{d}s^2}\mathcal{L}\left\{(t^2+1)(t-1)(t+2)\right\}(s).$$

Expanding out the polynomial,

$$= \frac{\mathrm{d}^2}{\mathrm{d}s^2}\mathcal{L}\left\{t^4+t^3-t^2+t-2\right\}(s).$$

Using the polynomial formula,

$$= \frac{\mathrm{d}^2}{\mathrm{d}s^2}\left(\frac{4!}{s^5} + \frac{3!}{s^4} - \frac{2!}{s^3} + \frac{1!}{s^2} + \frac{2\cdot 0!}{s^1}\right).$$

Taking the second derivative,

$$\mathcal{L}\left\{t^2(t^2+1)(t-1)(t+2)\right\}(s) = \frac{6!}{s^7} + \frac{5!}{s^6} - \frac{4!}{s^5} + \frac{3!}{s^4} + \frac{2\cdot 2!}{s^3}.$$

Example. *Find the Laplace transform of $(x^2)''$ using the derivative formula. Show it's equal to $\mathcal{L}\{2\}(s)$.*

Using the derivative formula,

$$\mathcal{L}\left\{\left(x^2\right)''\right\}(s) = s^2\mathcal{L}\left\{x^2\right\}(s) - s\left(x^2\right)'_{x=0} - \left(x^2\right)_{x=0}.$$

Using the polynomial formula,

$$= s^2\frac{2!}{s^3} - 0 - 0.$$

Simplifying,

$$\mathcal{L}\left\{\left(x^2\right)''\right\}(s) = \frac{2}{s} = \mathcal{L}\{2\}(s).$$

Example. *Find the Laplace transform of e^{it}. Show that it's equal to $\mathcal{L}\{\cos t + i\sin t\}(s)$.*

Using the exponential formula,

$$\mathcal{L}\left\{e^{it}\right\}(s) = \frac{1}{s-i}.$$

Now we'll work on $\cos t + i\sin t$. Using the linearity of the Laplace transform,

$$\mathcal{L}\{\cos\theta + i\sin\theta\}(s) = \mathcal{L}\{\cos t\}(s) + i\mathcal{L}\{\sin t\}(s).$$

Using the sin and cos formulas,

$$= \frac{s}{s^2+1^2} + i\frac{1}{s^2+1^2}.$$

Combining into one fraction,

$$= \frac{s+i}{s^2+1^2}.$$

Although we normally avoid it, we can factor the denominator into its two complex roots,

$$= \frac{s+i}{(s+i)(s-i)}.$$

Canceling out common factors we see that we get the same Laplace transform as we did with e^{it}.

$$\mathcal{L}\{\cos\theta + i\sin\theta\}(s) = \frac{1}{s-i} = \mathcal{L}\left\{e^{it}\right\}(s).$$

The Laplace transforms of these common functions are summarized in the appendix.

5.3 Inverse Laplace Transform

Similar to how the derivative is the inverse of the integral, we can think about an inverse Laplace transform that allows us to go back from the s domain to the t domain.

Definition. *Let $F(s) = \mathcal{L}\left\{f(t)\right\}(s)$. Then $f(t)$ is the inverse Laplace transform of $F(s)$.*

$$\mathcal{L}^{-1}\left\{F(s)\right\}(t) = \mathcal{L}^{-1}\left\{\mathcal{L}\left\{f(t)\right\}(s)\right\}(t) = f(t)$$

We don't need to work through the derivations of inverse Laplace transforms because we already did the derivations for Laplace transforms.

5.4 Solving Equations

Now that we can take the Laplace transform and its inverse of most of the common functions we've seen when solving differential equations, we can use Laplace transforms as another tool to solve differential equations.

There is a general pattern to how we'll use Laplace transforms to solve differential equations.

1) If needed, arrange the equation into a convenient form in the t domain.
2) Take the Laplace transform of both sides.
3) Use algebra, especially partial fraction decomposition and completing the square, to arrange items into a convenient form in the s domain.
4) Take the inverse Laplace transform of both sides and solve.

Example. *Solve the following IVP using a Laplace transform.*

$$\begin{cases} y'' - 2y' - 3y = 0 \\ y(0) = 1 \\ y'(0) = 1 \end{cases}$$

Taking the Laplace transform of both sides,

$$\begin{aligned} \mathcal{L}\left\{y'' - 2y' - 3y\right\}(s) = \mathcal{L}\left\{y''\right\}(s) - 2\mathcal{L}\left\{y'\right\}(s) - 3\mathcal{L}\left\{y\right\}(s) &= 0 \\ \left(s^2\mathcal{L}\left\{y\right\}(s) - sy'(0) - y(0)\right) - 2\left(s\mathcal{L}\left\{y\right\}(s) - y(0)\right) - 3\left(\mathcal{L}\left\{y\right\}(s)\right) &= 0 \\ = \mathcal{L}\left\{y\right\}(s)\left(s^2 - 2s - 3\right) &= s - 1. \end{aligned}$$

Note how the auxiliary equation appears in terms of s here.
Solving for $\mathcal{L}\left\{y\right\}(s)$,

$$\mathcal{L}\left\{y\right\}(s) = \frac{s-1}{s^2 - 2s - 3}.$$

Rearranging the right side into partial fractions,

$$\mathcal{L}\left\{y\right\}(s) = \frac{1/2}{s+1} + \frac{1/2}{s-3}.$$

Taking the inverse Laplace transform of both sides,

$$y = \frac{1}{2}e^{-t} + \frac{1}{2}e^{3t}.$$

Example. *Solve the following IVP using a Laplace transform.*

$$\begin{cases} y'' + 2y' + 5y = 0 \\ y(0) = 1 \\ y'(0) = 5 \end{cases}$$

Taking the Laplace transform of both sides and solving for $\mathcal{L}\left\{y\right\}(s)$,

$$\begin{aligned} \mathcal{L}\left\{y''\right\}(s) + 2\mathcal{L}\left\{y'\right\}(s) + 5\mathcal{L}\left\{y\right\}(s) &= 0 \\ \mathcal{L}\left\{y\right\}(s)\left(s^2 + 2s + 5\right) - y'(0) - sy(0) - 2y(0) &= 0 \\ \mathcal{L}\left\{y\right\}(s)\left(s^2 + 2s + 5\right) &= s + 7 \\ \mathcal{L}\left\{y\right\}(s) &= \frac{s+7}{s^2 + 2s + 5}. \end{aligned}$$

The denominator of the right hand side cannot be factored into linear terms. Instead, we'll complete the square.

$$= \frac{s+7}{(s+1)^2 + 2^2}.$$

If the numerator was $s+1$, we'd have a cos that's been shifted left by 1 in the s domain by -1. If the numerator were a multiple of 2, we'd have a sin that's been similarly shifted. We can rewrite the right hand side as two terms: one for cos and the other for sin with the appropriate numerators.

$$= \frac{s+1}{(s+1)^2 + 2^2} + 3\frac{2}{(s+1)^2 + 2^2}.$$

Taking the inverse Laplace transform of both sides,

$$y = e^{-t}\cos(2t) + 3e^{-t}\sin(2t).$$

We can also use the Laplace transform on non-homogeneous equations. The method will even take care of the cases where what would be our guess for the particular solution is already included in the homogeneous solution.

Example. *Solve the following IVP using a Laplace transform.*

$$\begin{cases} y'' + 2y' + 5y = e^{-t} \\ y(0) = 1 \\ y'(0) = 5 \end{cases}$$

This equation is the same as the previous example except for the e^{-t}, so we can use some of our previous work.

$$\begin{aligned}\mathcal{L}\left\{y\right\}(s)\left(s^2+2s+5\right) &= s+7+\frac{1}{s+1}\\ \mathcal{L}\left\{y\right\}(s) &= \frac{s^2+8s+8}{(s+1)(s^2+2s+5)}.\end{aligned}$$

Rearranging the right hand side into partial fractions with completed squares,

$$\mathcal{L}\left\{y\right\}(s) = \frac{1}{4}\left(\frac{1}{s+1}\right)+\frac{1}{4}\left(3\frac{s+1}{(s+1)^2+2^2}+9\frac{2}{(s+1)^2+2^2}\right).$$

Taking the inverse Laplace transform of both sides,

$$y = \frac{1}{4}e^{-t}+\frac{3}{4}e^{-t}\cos(2t)+3e^{-t}\sin(2t).$$

Example. *Solve the following IVP using a Laplace transform.*

$$\begin{cases} 8\cos(2t)-y''=4y\\ y(0)=1\\ y'(0)=0\end{cases}$$

Rearranging the equation into the standard form for a linear heterogeneous ODE,

$$y''+4y=8\cos(2t).$$

Taking the Laplace transform of both sides,

$$s^2\mathcal{L}\left\{y\right\}(s)-y'(0)-sy(0)+4\mathcal{L}\left\{y\right\}(s)=8\frac{s}{s^2+2^2}.$$

Plugging in the values given by the IVP and rearranging,

$$\mathcal{L}\left\{y\right\}(s)\left(s^2+4\right)=\frac{s^3+12s}{s^2+2^2}.$$

Dividing to solve for $\mathcal{L}\left\{y\right\}(s)$,

$$\mathcal{L}\left\{y\right\}(s)=\frac{s^3+12s}{(s^2+2^2)(s^2+2^2)}.$$

Rearranging the right side into partial fractions,

$$\mathcal{L}\left\{y\right\}(s)=\frac{s}{s^2+2^2}+2\frac{2^2s}{(s^2+2^2)^2}.$$

Taking the inverse Laplace transform of both sides,

$$y=\cos(2t)+2t\sin(2t).$$

5.5 Convolutions

5.5.1 Motivation

Taking Laplace transforms can be difficult. We already have some basic formulas derived from linearity properties of integrals that allow us to not evaluate an integral every time.

One sort of formula that would be nice to have is one relating products of functions. If we know that

$$H(s) = F(s)G(s),$$

where F and G are the Laplace transforms of f and g, it'd be nice if we could find the inverse Laplace transform of H to get h. This is one way to define the convolution.

5.5.2 Definition & Convolution Theorem

Definition. *Let $F(s)$ and $G(s)$ be the Laplace transforms of functions $f(t)$ and $g(t)$. Then*

$$F(s)G(s) = \mathcal{L}\left\{f \star g\right\}(s).$$

Note how using this definition we can see that $\star$, the convolution operator, is commutative. Defining this function that works like multiplication over Laplace transforms is only really useful if we have a formula for it, so let's derive one. Starting from the left-hand side of the equation in the definition and applying the integral definition of the Laplace transform,

$$F(s)G(s) = \int_0^\infty e^{-su} f(u)\mathrm{d}u \cdot \int_0^\infty e^{-sv} f(v)\mathrm{d}v.$$

Making this product of integrals into a double integral (we usually do this in reverse when simplifying a double integral),

$$= \int_0^\infty \int_0^\infty e^{-s(u+v)} f(u)g(v)\mathrm{d}u\mathrm{d}v.$$

Letting $t = u + v$,

$$= \int_0^\infty \int_0^t e^{-st} f(u)g(t-u)\mathrm{d}u\mathrm{d}t.$$

Bringing e^{-st} outside of the innermost integral,

$$= \int_0^\infty e^{-st} \left[\int_0^t f(u)g(t-u)\mathrm{d}u\right] \mathrm{d}t.$$

We see that we have the Laplace transform of the expression in the square brackets. So,

$$f \star g = \int_0^t f(u)g(t-u)\mathrm{d}u.$$

This is an equivalent way to define the convolution.

Example. *Compute the convolution of t^2 with t.*

Applying the formula we just derived,

$$\begin{aligned} t^2 \star t &= \int_0^t u^2(t-u)\mathrm{d}u \\ &= t\frac{u^3}{3} - \frac{u^4}{4}\bigg|_0^t \\ &= \frac{t^4}{3} - \frac{t^4}{4} \\ &= \frac{t^4}{12}. \end{aligned}$$

5.5.3 Properties

Convolution inherits all the linearity properties of integration. Let $f(t)$, $g(t)$, and $h(t)$ be piecewise continuous on $[0, \infty)$. Let a be a real constant. Then

Commutativity – $f\star = g \star f$

Associativity – $(f \star g) \star h = f \star (g \star h)$

Distributivity over Addition – $f \star (g + h) = (f \star g) + (f \star h)$

Associativity over Scalar Multiplication – $a(f \star g) = (af) \star g$

5.5.4 Applications

When we're using Laplace transforms to solve a differential equation, if we recognize $\mathcal{L}\{y\}(s)$ as the product of two functions for which we know the inverse Laplace transforms, we can compute a convolution to find y.

Example. *Solve the following differential equation using Laplace transforms and convolutions. Let a and c be real constants where $a \neq c$.*

$$y' - ay = e^{ct}.^1$$

Taking the Laplace transform of both sides,

$$\begin{aligned} s\mathcal{L}\{y\}(s) - a\mathcal{L}\{y\}(s) &= \frac{1}{s-c} \\ \mathcal{L}\{y\}(s)(s-a) &= \frac{1}{s-c}. \end{aligned}$$

[1] When we're taking the Laplace transform of both sides, we implicitly use that $y(0) = 0$.

Solving for $\mathcal{L}\left\{y\right\}(s)$,

$$\mathcal{L}\left\{y\right\}(s) = \frac{1}{s-c} \cdot \frac{1}{s-a}.$$

Recognizing the right-hand side of the equation as the product of the Laplace transforms of e^{ct} and e^{at}, we can solve for y by finding the convolution of e^{ct} and e^{at}. Applying our formula,

$$\begin{aligned}
F(s) = \frac{1}{s-c} \text{ and } G(s) &= \frac{1}{s-a} \\
f(t) = e^{ct} \text{ and } g(t) &= e^{at}. \\
y(t) &= f(t) \star g(t) \\
&= \int_0^t e^{cu} e^{a(t-u)} \mathrm{d}u \\
&= e^{at} \int_0^t e^{u(c-a)} \mathrm{d}u \\
&= e^{at} \left(\frac{e^{(c-a)t}}{c-a} - \frac{1}{c-a} \right) \\
&= \frac{e^{ct} - e^{at}}{c-a}.
\end{aligned}$$

Chapter 6

Additional Resources

6.1 Tests

The following tests are modeled off of real tests given in an undergraduate introductory differential equations course. The tests is meant to be taken during a 75 minute period without the use of notes, textbooks, the internet, or calculators. Answers may be left in unsimplified form unless the question asks for a certain form.

6.1.1 Test 1

1. Determine if $y - \ln y = x^2 + 1$ is an implicit solution to $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2xy}{y-1}$. Give reasons for your answer.

2. Find the function $y(x)$ that satisfies the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + 4y - e^{-x} = 0 \text{ and } y(0) = \frac{4}{3}.$$

3. Find the solution to the initial value problem $\frac{\mathrm{d}x}{\mathrm{d}t} = x\frac{(t-3)^2}{x+2}$, $x(3) = -1$. You may leave the solution in implicit form.

4. Find the general solution to the differential equation $y''' + y'' - 2y = 0$.

5. A tank contains 90kg of salt and 2000L of water. Pure water enters the tank at a rate of 3L/min. The solutions is mixed and drains from the tank at a rate of 6L/min.

   (a) What is the amount of salt in the tank initially?

   (b) Find the amount of salt in the tank after 2 hours.

6. A detective arrives at a murder scene at noon that has an ambient temperature of 16°C. He measures the temperature of the body as 34°C. He then returns an hour later and measures the temperature of the body as 32°C. Assuming the normal body

temperature is 37°C, write a differential equation using Newton's law of cooling that models the situation. Describe, without actually solving the equation, how one would solve the equation to find then the murder took place.

6.1.2 Test 1 Answers

1. Using implicit differentiation,
$$\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = 2x.$$
So,
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2x}{1-1/y} = \frac{2xy}{1-y}.$$
So, $y - \ln y = x^2 + 1$ is an implicit solution to $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2x}{y-1}$ as we have demonstrated by differentiating and checking.

2. We will use the integrating factor method. Rewriting the equation so all terms involving y are on the left,
$$y' + 4y = e^{-x}.$$
In this case,
$$a(x) = 4 \text{ and } b(x) = e^{-x}.$$
So,
$$\mu(x) = e^{\int a(x)\mathrm{d}x} = e^{4x}.$$
Multiplying both sides by $\mu(x)$,
$$\left(e^{4x}y\right)' = e^{3x}.$$
Integrating both sides,
$$e^{4x}y = \frac{1}{3}e^{3x} + C_1.$$
Solving for y,
$$y = \frac{1}{3}e^{-x} + C_1e^{-4x}.$$
Plugging in $x = 0$ and $y = \frac{4}{3}$ to solve for C_1,
$$\frac{4}{3} = \frac{1}{3} + C_1 \implies C_1 = 1.$$
So, our answer to the IVP is
$$y = \frac{1}{3}e^{-x} + e^{-4x}.$$

3. The equation is separable and can be rewritten as

$$\frac{x+2}{x}\mathrm{d}x = (t-3)^2\mathrm{d}t,\ x \neq 0.$$

We'll come back later to see if $x = 0$ is a solution. Integrating both sides,

$$x + 2\ln|x| = \frac{(t-3)^3}{3} + C_1,\ x \neq 0.$$

Plugging in $x = -1$ and $t = 3$ to solve for C_1,

$$1 + 2\ln 1 = 0 + C_1 \implies C_1 = -1,\ x \neq 0.$$

So, our solution to the IVP is

$$x + 2\ln|x| = \frac{(t-3)^3}{3} - 1,\ x \neq 0.$$

Checking if $x = 0$ is a solution,

$$0 = 0\frac{(t-3)^2}{2}.$$

So, $x = 0$ is a general solution. However for our solution $x(t) = 0$, $x(3) \neq -1$, to $x = 0$ is not a solution to the IVP. So, our solution to the IVP remains.

4. This equation is linear and homogeneous, so we can find the general solution using the auxiliary equation. Extracting the auxiliary equation and finding the roots,

$$r^3 + r^2 - 2 = 0 \implies r = 1, -1 \pm i.$$

So, the general solution is

$$y = C_1 e^x + C_2 e^{-x}\cos x + C_3 e^{-x}\sin x.$$

5. (a) As stated in the problem, 90kg of salt is in the tank initially.

   (b) So find the amount of salt in the tank after 2 hours we'll need to set up a differential equation that models the situation. Let $y(t)$ be the number of kgs of salt in the tank after t minutes. Let $V(t)$ be the volume of brine in the tank after t minutes. Modeling the salt,

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \text{salt rate in} - \text{salt rate out} = 0 - \text{salt rate out}.$$

   Modeling the volume,

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \text{brine rate in} - \text{brine rate out} = 3\frac{L}{min} - 6\frac{L}{min} = -3\frac{L}{min}.$$

We're also given that $V(0) = 2000L$, so we can find that

$$V(t) = 2000 - 3t.$$

Now we can write an equation to find salt rate out.

$$\text{salt rate out} = \frac{y\text{kg of salt}}{V\text{L of brine}} * \frac{6L}{\text{min}} = \frac{6y}{2000 - 3t}.$$

This allows us to write a differential equation for $y(t)$.

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -\text{salt rate out} = \frac{-6y}{2000 - 3t} = \frac{6y}{3t - 2000}.$$

This equation is separable and can be rewritten as

$$\frac{\mathrm{d}y}{y} = \frac{6\mathrm{d}t}{3t - 200}.$$

Integrating both sides,

$$\ln\left|y\right| = 2\ln\left|3t - 2000\right| + C_1.$$

We know that $y > 0$ always because we can't have a negative amount of salt. Further, we're concerned with the time between $t = 0$ and $t = 120 < \frac{2000}{3}$, so $3t - 2000 < 0$. This means we can rewrite our equation as

$$\ln y = 2\ln\left(2000 - 3t\right) + C_1.$$

Exponentiating both sides,[1]

$$y = C_1(2000 - 3t)^2.$$

Applying our initial condition of $y(0) = 90$ to solve for C_1,

$$90 = C_1(2000)^2 \implies C_1 = \frac{2000^2}{90}.$$

Plugging our solution for C_1 back into our general solution,

$$y = 90\left(\frac{2000 - 3t}{2000}\right)^2.$$

6. Newton's law of cooling is

$$\frac{\mathrm{d}T}{\mathrm{d}t} = k\left(T_e - T\right)$$

[1] C_1 is different from the C_1 previously, but it's still a constant.

where k is some constant, T_e is the temperature of the environment, and T is the current temperature of the object. In the detective's case, $T_e = 16$, so

$$\frac{\mathrm{d}T}{\mathrm{d}t} = k(16 - T)$$

with initial conditions $T(12) = 34$ and $T(13) = 32$. One could then solve this equation and apply the initial conditions to solve for k and the constant of integration C. Then one would need to solve for x in $T(x) = 37$. This value of x will tell you the number of hours after midnight one the same day that the murder took place[2].

6.1.3 Test 2

1. Find the general form of a particular solution to the differential equation

   $$y'' + 10y' + 25y = f(t)$$

   for each of the following cases:

   (a) $f(t) = \cos(-5t) + 7t\sin(-5t)$

   (b) $f(t) = te^{-5t}$.

2. Consider the matrix $A = \begin{bmatrix} 1 & 0 & 6 \\ 3 & 1 & 3 \\ -3 & 3 & -8 \end{bmatrix}$. Compute its characteristic polynomial and find all its eigenvalues. Pick one of the eigenvalues and compute a corresponding eigenvector.

3. Choose one of the two parts.

   (a) Compute the general solution for the equation $y'' - 5y' + 6y = 6e^{4t} - 10e^{t}$.

   (b) Use the method of variation of parameters to find a particular solution to $y''+9y = \csc(3t)$.

4. A 1kg mass is attached to a spring with stiffness 8N/m. The damping constant is 6Ns/m. At time $t = 0$, an external force $F(t) = 8\sin(2t)$ is applied to the system as the mass is pushed rightward from equilibrium with a velocity of 1m/s.

   (a) Find the displacement function of the mass.

   (b) Determine the steady-state solution for the system.

   (c) Write the steady-state solution in the form $A\cos(\omega t - \phi)$ indicating the amplitude, frequency, period, and phase-shift.

5. Choose one of the following two parts.

[2]For those that do solve the equation, $x \approx 10.691$, so the murder should have took place at about 10:41am.

(a) Consider the following linear system of differential equations with given initial conditions:

$$\begin{cases} x_1' = 2x_1 - 3x_2 \\ x_2' = x_1 - 2x^2 \end{cases}, \; x_1(0) = 2,\; x_2(0) = 3$$

(i) Write the system in normal form.

(ii) Show that the vectors $\begin{bmatrix} 3e^t \\ e^t \end{bmatrix}$ and $\begin{bmatrix} e^{-t} \\ e^{-t} \end{bmatrix}$ are solutions to the homogeneous system.

(iii) Write the general solution for the system and then compute the unique solution satisfying the initial conditions.

(b) The matrix $A = \begin{bmatrix} -1 & 2 \\ -1 & 3 \end{bmatrix}$ has eigenvalues $\lambda = -2 \pm i$. Find the general solution to the system $\vec{x}' = A\vec{x}$.

6. Consider the system $\vec{x}' = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \vec{x} + \begin{bmatrix} 2t \\ 3t+3 \end{bmatrix}$. It is known that the corresponding homogeneous sysstem has general solution $\vec{x_h} = C_1 e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 e^{-t} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$. Find the general solution for the given heterogeneous system.

6.1.4 Test 2 Answers

1. For both (a) and (b), we'll need to compute the general solution to make sure any terms that we'd guess as part of the particular solution aren't already part of the general solution. The auxiliary equation and its roots are

$$r^2 + 10r + 25 \implies r = -5 \text{ (double root)}.$$

So, the general solution is

$$C_1 e^{-5t} + C_2 t e^{5t}.$$

(a) We have a trig term and a linear term times a trig term, so the particular solution has the form

$$y_p = A\cos(-5t) + B\sin(-5t) + Ct\cos(-5t) + Dt\sin(-5t).$$

So, without solving for A, B, C, and D, the general form of a particular solution is

$$C_1 e^{-5t} + C_2 t e^{5t} + A\cos(-5t) + B\sin(-5t) + Ct\cos(-5t) + Dt\sin(-5t).$$

(b) We have a linear term time e^{-5t}. However, up to linear terms are already represented in the general solution, so we need to include another factor of t. So, the particular solution has the form

$$y_p = At^2 e^{-5t}.$$

So, without solving for A, the general form of a particular solution is

$$C_1e^{-5t} + C_2te^{5t} + At^2e^{-5t}.$$

2. Finding $p(\lambda)$,

$$p(\lambda) = \det\begin{bmatrix} 1 & 0 & 6 \\ 3 & 1 & 3 \\ -3 & 3 & -8 \end{bmatrix} = -\lambda^3 - 6\lambda^2 + 6\lambda + 55.$$

So, $p(\lambda) = 0$ when

$$\lambda = -5, \frac{-1 \pm 3\sqrt{5}}{2}.$$

We'll find the eigenvector for $\lambda = -5$.

$$\left[\begin{array}{ccc|c} 6 & 0 & 6 & 0 \\ 3 & 6 & 3 & 0 \\ -3 & 3 & 3 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \implies \vec{v} = t\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

3. The instructions say only do one, but we'll do both for the answers.

(a) The auxiliary equation and its roots are

$$r^2 - 5r + 6 = 0 \implies r = 2, 3.$$

So, the homogeneous solution is

$$y_h = C_1e^{2t} + C_2e^{3t}.$$

We use the method of undetermined coefficients to find y_p. We'll guess that y_p has the form

$$y_p = Ae^{4t} + Be^t.$$

Solving for A and B[3],

$$y_p'' - 5y_p' + 6y_p = 6e^{4t} - 10e^t \implies A = 3, B = -5.$$

So, we can write the particular solution

$$y_p = 3e^{4t} - 5e^t.$$

So, the general solution is

$$y = C_1e^{2t} + C_2e^{3t} + 3e^{4t} - 5e^t.$$

[3] The algebra of solving for A and B have been omitted for brevity.

(b) Although this is a second-order equation, we'll do the version of variation of parameters that also works for higher orders too. First, we need to find our fundamental solutions that are part of the homogeneous solution by finding the roots of the auxiliary equation.

$$r^2 + 9 = 0 \implies r = \pm 3i.$$

So, the homogeneous solution is

$$y_h = C_1 \cos(3t) + C_2 \sin(3t),$$

and the fundamental solutions are

$$\begin{aligned} y_1 &= \cos(3t) \\ y_2 &= \sin(3t). \end{aligned}$$

So, the Wronskian matrix and its determinant are

$$[W] = \begin{bmatrix} \cos(3t) & \sin(3t) \\ -3\sin(3t) & 3\cos(3t) \end{bmatrix}, \; W = 3\cos^2(3t) + 3\sin^2(3t) = 3.$$

The sub-matrices are

$$\begin{aligned} W_1 &= \det \begin{bmatrix} 0 & \sin(3t) \\ 1 & 3\cos(3t) \end{bmatrix} = -\sin(3t) \\ W_2 &= \det \begin{bmatrix} \cos(3t) & 0 \\ -3\sin(3t) & 1 \end{bmatrix} = \cos(3t). \end{aligned}$$

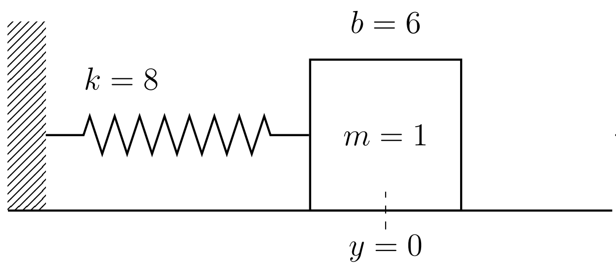
Solving for v_1 and v_2,

$$\begin{aligned} v_1 &= \int \frac{-\csc(3t)\sin(3t)}{3} \mathrm{d}t = \frac{-1}{3} t + C_1 \\ v_2 &= \int \frac{\csc(3t)\cos(3t)}{3} \mathrm{d}t = \frac{1}{9} \ln\big|\sin(3t)\big| + C_2. \end{aligned}$$

Solving for y,

$$\begin{aligned} y &= \cos(3t)\left(\frac{-1}{3} t + C_1\right) + \sin(3t)\left(\frac{1}{9}\ln\big|\sin(3t)\big| + C_2\right) \\ &= C_1\cos(3t) + C_2\sin(3t) - \frac{1}{3} t\cos(3t) + \frac{1}{9}\sin(3t)\ln\big|\sin(3t)\big|. \end{aligned}$$

4. Below is a diagram that depicts the situation.



$$\begin{cases} y'(0) = 1 \\ F_{\text{ext}}(t) = 8\sin(2t) \end{cases}$$

(a) Our IVP to model this is

$$\begin{cases} y'' + 6y' + 8y = 8\sin(2t) \\ y'(0) = 1 \\ y(0) = 0 \end{cases}.$$

Solving the auxiliary equation,

$$r^2 + 6r + 8 = 0 \implies r = -4, -2.$$

So, our homogeneous solution is

$$y_h = C_1 e^{-4t} + C_2 e^{-2t}.$$

We'll use the method of undetermined coefficients to find y_p. We'll guess that y_p has the form

$$y_p = A\cos(2t) + B\sin(2t).$$

Solving for A and B[4],

$$y_p'' + 6y_p' + 8y_p = 8\sin(2t) \implies A = \frac{-3}{5}, B = \frac{1}{5}.$$

So, our particular solution is

$$y_p = \frac{-3}{5}\cos(2t) + \frac{1}{5}\sin(2t),$$

and our general solution is

$$y = C_1 e^{-4t} + C_2 e^{-2t} - \frac{3}{5}\cos(2t) + \frac{1}{5}\sin(2t).$$

Plugging in our initial condition to solve for C_1 and C_2[5],

$$y'(0) = 1, y(0) = 0 \implies C_1 = \frac{-9}{10}, C_2 = \frac{3}{2}.$$

So, our solution to the IVP is

$$y(t) = \frac{-9}{10}e^{-4t} + \frac{3}{2}e^{-2t} - \frac{3}{5}\cos(2t) + \frac{1}{5}\sin(2t) \text{ m}.$$

[4]The algebra of solving for A and B has been omitted for brevity.
[5]The algebra of solving for C_1 and C_2 has been omitted for brevity.

(b) As t grows large, the exponential terms will decrease to 0 and have minimal effect. So, the steady-state solution y_{ss} is just the terms with cos and sin.

$$y_{ss}(t) = \frac{-3}{5}\cos(2t) + \frac{1}{5}\sin(2t) \text{ m}$$

(c)

$$y_{ss}(t) = \sqrt{\frac{2}{5}}\cos\left(2x + \arctan\left(\frac{1}{3}\right) + \pi\right) \text{ m.}$$

The amplitude, frequency, period, and phase shift are

$$\begin{aligned} A &= \sqrt{\frac{2}{5}} \text{ m} \\ f &= \frac{1}{\pi} \text{ Hz} \\ T &= \pi \text{ secs} \\ \phi &= \arctan\left(\frac{1}{3}\right) + \pi. \end{aligned}$$

5. The instructions say only do one, but we'll do both for the answers.

(a) (i) The system in matrix form is

$$\vec{x}' = \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix}\vec{x},\ \vec{x}(0) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

(ii) We'll check the first vector.

$$\begin{bmatrix} 3e^t \\ e^t \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 3e^t \\ e^t \end{bmatrix} = \begin{bmatrix} 3e^t \\ e^t \end{bmatrix}.$$

Next we'll check the second vector.

$$\begin{bmatrix} -e^{-t} \\ -e^{-t} \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} e^{-t} \\ e^{-t} \end{bmatrix} = \begin{bmatrix} -e^{-t} \\ -e^{-t} \end{bmatrix}.$$

So, both vectors are solutions to the homogeneous equation.

(iii) Since we know two linearly independent solutions from (ii), we can write the general solution as

$$\vec{x} = C_1\begin{bmatrix} 3e^t \\ e^t \end{bmatrix} + C_2\begin{bmatrix} -e^{-t} \\ -e^{-t} \end{bmatrix}.$$

Applying the initial conditions[6],

$$\vec{x}(0) = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \implies C_1 = \frac{-1}{2}, C_2 = \frac{-7}{2}.$$

[6]The algebra of solving for C_1 and C_2 has been omitted for brevity

So, the solution to the IVP is

$$\vec{x} = \frac{-1}{2}\begin{bmatrix} 3e^t \\ e^t \end{bmatrix} - \frac{7}{2}\begin{bmatrix} -e^{-t} \\ -e^{-t} \end{bmatrix}.$$

(b) Since the eigenvalues are a complex conjugate pair, we only need to consider one eigenvalue to find both corresponding eigenvectors. We'll use $\lambda = -2 + i$.

$$\left[\begin{array}{cc|c} 1-i & 2 & 0 \\ -1 & -1-i & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1-i & 0 \\ 0 & 0 & 0 \end{array}\right] \to t\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right).$$

So, the two eigenvectors are

$$\begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Remembering our solution form for complex eigenvalues, we get the solution

$$\vec{x} = C_1 e^{-2t}\left(\cos(t)\begin{bmatrix} -1 \\ 1 \end{bmatrix} - \sin(t)\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + C_2 e^{-2t}\left(\cos(t)\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \sin(t)\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right).$$

6. We'll use the method of undetermined coefficients for systems to find $\vec{x_p}$. We'll guess that $\vec{x_p}$ has the form

$$\vec{x_p} = \vec{a}t + \vec{b}.$$

Solving for $\vec{a}$ and $\vec{b}$[7],

$$\vec{x_p}' = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}\vec{x_p} \implies \vec{a} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \vec{b} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

So, we have our solution for $\vec{x_p}$,

$$\vec{x_p} = t\begin{bmatrix} -1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

The general solution is thus

$$\vec{x} = C_1 e^t\begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 e^{-t}\begin{bmatrix} 1 \\ 3 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

6.1.5 Test 3

1. Write the integral definition of the Laplace transform.

2. Find the Laplace transform of the following functions. Be sure to give the domains. Use the integral definition for (a) and (e). Be sure to state the domain of the transformed function and any rules of Laplace transforms you use.

[7] The algebra of solving for $\vec{a}$ and $vecb$ has been omitted for brevity.

(a)
$$f(x) = e^x$$

(b)
$$h(t) = \sin t + 2\cos 3t$$

(c)
$$j(t) = e^{3t}\left(t^2 + 3t + 2\right)$$

(d)
$$b(t) = \frac{\mathrm{d}}{\mathrm{d}t}(e^{3t+1} + e^{1-t})$$

3. Find the inverse Laplace transform of the following functions.

(a)
$$F(s) = \frac{1}{1+s}$$

(b)
$$H(s) = \frac{s^2 + 2s + 1}{s^3 - 4s^2 + 5s - 2}$$

(c)
$$J(s) = \frac{s-4}{s^2 - 8s + 32}$$

(d)
$$A(s) = \frac{768}{(2s+3)^5}$$

4. Find the general solution to the following differential equation using one of the methods you learned previously and by Laplace transform. Show that the two methods give the same answer.
$$2y'' - 3y' + y = 10\sin x$$

5. Solve the following IVP by Laplace transform.
$$\begin{cases} 2y'' + 4y' - 6y = te^{-3t} \\ y'(0) = 0 \\ y(0) = 1 \end{cases}$$

6.1.6 Test 3 Answers

1. Let $f(t)$ be a function defined for $t \geq 0$.

$$\mathcal{L}\left\{f\right\}(s) = \int_0^\infty e^{-st} f(t) \mathrm{d}t$$

2. (a)

$$\begin{aligned}
\mathcal{L}\left\{e^x\right\}(s) &= \int_0^\infty e^{-st} e^t \mathrm{d}t \\
&= \int_0^\infty e^{t(1-s)} \mathrm{d}t \\
&= \frac{1}{1-s} e^{t(1-s)} \Big|_0^\infty \\
&= \frac{1}{1-s} \left(\lim_{a \to \infty} e^{a(1-s)} - e^{0(1-s)} \right) \\
&= \frac{1}{1-s} (0-1), \ s > 1 \\
&= \frac{1}{s-1}, \ s > 1
\end{aligned}$$

(b)

$$\mathcal{L}\left\{\sin t + 2\cos 3t\right\}(s) = \mathcal{L}\left\{\sin t\right\}(s) + 2\mathcal{L}\left\{\cos 3t\right\}(s)$$

by the linearity of the Laplace transform.

$$= \mathcal{L}\left\{\sin t\right\}(s) + \frac{2}{3}\mathcal{L}\left\{\cos t\right\}(\frac{s}{3})$$

as specified in the table of Laplace transforms for sin and cos.

$$\begin{aligned}
&= \frac{1}{s^2+1^2} + \frac{2}{3}\left(\frac{s/3}{(s/3)^2+1^2}\right) \\
&= \frac{1}{s^2+1^2} + \frac{2}{3}\frac{3s}{s^2+3^2} \\
&= \frac{1}{s^2+1^2} + \frac{2s}{s^2+3^2} \\
&= \frac{2s^3+s^2+2s+9}{(s^2+1)(s^2+9)}
\end{aligned}$$

(c)

$$\mathcal{L}\left\{e^{3t}\left(t^2+3t+2\right)\right\}(s) = \mathcal{L}\left\{e^{3t}t^2\right\}(s) + 3\mathcal{L}\left\{e^{3t}x\right\}(s) + 2\mathcal{L}\left\{e^{3t}\right\}(s)$$

by the linearity of the Laplace transform.

$$= \frac{2!}{(s-3)^{2+1}} + 3\frac{1!}{(s-3)^{1+1}} + 2\frac{0!}{(s-3)^{0+1}}$$

as specified in the table of Laplace transforms for an exponential and power of t.

$$\begin{aligned}
&= \frac{2}{(s-3)^3} + \frac{3}{(s-3)^2} + \frac{2}{(s-3)} \\
&= \frac{2(s-3)^2 + 3(s-3) + 2}{(s-3)^3} \\
&= \frac{2s^2 - 9s + 11}{(s-3)^3}
\end{aligned}$$

(d)

$$\mathcal{L}\left\{\frac{\mathrm{d}}{\mathrm{d}t}(e^{3t+1} + e^{1-t})\right\}(s) = s\mathcal{L}\left\{e^{3t+1} + e^{1-t}\right\}(s) - \left(e^{3\cdot 0+1} + e^{1-0}\right)$$

by the Laplace transform of a derivative.

$$= s\left(e\mathcal{L}\left\{e^{3t}\right\}(s) + e\mathcal{L}\left\{e^{-t}\right\}(s)\right) - 2e$$

by the linearity of the Laplace transform.

$$= s\left(e\frac{1}{s-3} + e\frac{1}{s+1}\right) - 2e$$

as specified in the table of Laplace trasforms for the Laplace transform of an exponential.

$$\begin{aligned}
&= \frac{se}{s-3} + \frac{se}{s+1} - 2e \\
&= \frac{se(s+1) + se(s-3) - 2e(s+1)(s-3)}{(s+1)(s-3)} \\
&= \frac{2e(s+3)}{(s+1)(s-3)}
\end{aligned}$$

3. (a)

$$\mathcal{L}^{-1}\left\{\frac{1}{1+s}\right\}(t) = e{-}t$$

(b)

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{s^2+2s+1}{s^3-4s^2+5s-2}\right\}(t) &= \mathcal{L}^{-1}\left\{\frac{-8}{s-1}\right\}(t) - \mathcal{L}^{-1}\left\{\frac{4}{(s-1)^2}\right\}(t) + \mathcal{L}^{-1}\left\{\frac{9}{s-2}\right\}(t) \\
&= -8e^t - 4e^t t + 9e^{2t}
\end{aligned}$$

(c)

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{s-4}{s^2-8s+32}\right\}(t) &= \mathcal{L}^{-1}\left\{\frac{s-4}{(s-4)^2+4^2}\right\}(t) \\
&= e^{4t}\cos(4t)
\end{aligned}$$

(d)

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{768}{(2s+3)^5}\right\}(t) &= \mathcal{L}^{-1}\left\{\frac{4!}{(s+\frac{3}{2})^{4+1}}\right\}(t) \\
&= e^{-\frac{3}{2}t}t^4
\end{aligned}$$

4. We'll use method of undetermined coefficients. Extracting and solving the axillary equation,

$$2r^2 - 3r + 1 = 0 \implies r = \frac{1}{2},\, 1.$$

So, our homogeneous solution is

$$y_h = C_1 e^{\frac{x}{2}} + C_2 e^x.$$

We'll guess that the particular solution has the form

$$\begin{gathered}
y_p = A\sin x + B\cos x \\
2y_p'' - 3y_p' + y_p = 10\sin x \implies A = -1,\ B = 3.
\end{gathered}$$

So,

$$y_p = -\sin x + 3\cos x,$$

and our general solution is

$$y = C_1 e^{\frac{x}{2}} + C_2 e^x - \sin x + 3\cos x.$$

Now we'll use Laplace transform. Taking the Laplace transform of both sides,

$$\begin{aligned}
\mathcal{L}\left\{2y'' - 3y' + y\right\}(s) &= \mathcal{L}\left\{10\sin x\right\}(s) \\
\mathcal{L}\left\{y\right\}(s)\cdot(2s^2 - 3s + 1) &= \frac{10}{s^2+1} + 2y'(0) + 3y(0) - 2sy(0) \\
\mathcal{L}\left\{y\right\}(s)\cdot(2s^2 - 3s + 1) &= \frac{10 + 3y(0)s^2 + 3y(0) - 2s^3y(0) - 2sy(0) + 2s^2y'(0) + 2y'(0)}{s^2+1} \\
\mathcal{L}\left\{y\right\}(s) &= \frac{-2y(0)s^3 + (3y(0) + 2y'(0))s^2 - 2sy(0) + 3y(0) + 2y'(0) + 10}{(s^2+1)\left(2s^2 - 3s + 1\right)} \\
&= \frac{-2y(0)s^3 + (3y(0) + 2y'(0))s^2 - 2sy(0) + 3y(0) + 2y'(0) + 10}{(s^2+1)\left(2s - 1\right)\left(s - 1\right)} \\
&= \frac{-4(y(0) + y'(0) + 4)}{2s-1} + \frac{y(0) + 2y'(0) + 5}{s-1} + \frac{3s-1}{s^2+1}.
\end{aligned}$$

Let $C_1 = -2(y(0) + y'(0) + 4)$ and $C_2 = y(0) + 2y'(0) + 5$.

$$\mathcal{L}\left\{y\right\}(s) = \frac{2C_1}{2s-1} + \frac{C_2}{s-1} - \frac{1}{s^2+1} + 3\frac{s}{s^2+1}.$$

Taking the inverse Laplace transform of both sides,

$$y = C_1 e^{\frac{x}{2}} + C_2 e^x - \sin x + 3\cos x.$$

We can see that we have the same solution for y in both methods.

5. Taking the Laplace transform of both sides and solving for the Laplace transform of y,

$$\begin{aligned}
\mathcal{L}\left\{2y'' + 4y' - 6y\right\}(s) &= \mathcal{L}\left\{te^{-3t}\right\}(s) \\
\mathcal{L}\left\{y\right\}(s) \cdot (2s^2 + 4s - 6) - 2sy(0) - 2y'(0) - 4y(0) &= \frac{1}{(s+3)^2} \\
\mathcal{L}\left\{y\right\}(s) \cdot (2s^2 + 4s - 6) - 2s - 4 &= \frac{1}{(s+3)^2} \\
\mathcal{L}\left\{y\right\}(s) \cdot (2s^2 + 4s - 6) &= \frac{1}{(s+3)^2} + 2s + 4 \\
&= \frac{2s^3 + 16s^2 + 42s + 37}{(s+3)^2} \\
\mathcal{L}\left\{y\right\}(s) &= \frac{2s^3 + 16s^2 + 42s + 37}{2(s+3)^3(s-1)}.
\end{aligned}$$

Converting the right side to partial fractions,

$$\mathcal{L}\left\{y\right\}(s) = \frac{31}{128(s+3)} - \frac{1}{32(s+3)^2} - \frac{1}{8(s+3)^3} + \frac{97}{128(s-1)}.$$

Taking the inverse Laplace transform of both sides to solve for y,

$$y = \frac{31}{128}e^{-3t} - \frac{1}{32}te^{-3t} - \frac{1}{16}t^2e^{-3t} + \frac{97}{128}e^{t}.$$

6.2 Online Resources

Below is a list of other useful resources for learning differential equations. Most are freely accessible online.

- Paul's Online Notes – Goes deeper than this text and has additional practices problems.
- Khan Academy – Video lectures and practice problems.
- PatrickJMT – YouTube series focused mostly on solving example problems.
- MIT OCW 18.03SC – Complete series of lectures, recitations, assignments, practice problems, lecture notes, and exams needed for independent study.
- Jeffery R. Chasnov: Differential Equations – Online textbook from the Hong Kong University of Science and Technology. Adapted from Coursera's Differential Equations for Engineers.

6.3 Contributors

Special thanks to everyone who made contributions to this project on Github. They are listed in order of number of commits as `name (GitHub username)`.

William Boyles (wmboyles)	Nate (aneziac)	Ashwin M (Suzukazole)
Aditya Dev (dev-aditya)	Calvin McPhail-Snyder (esselltwo)	Robert Washbourne (rawsh)

Appendix A

Reference Tables

A.1 Table of Laplace Transforms

Below is a brief table of Laplace transforms. Although most differential equations textbooks have a longer table, this table covers all Laplace transforms done in this text.

$f(t)$	$F(s) = \mathcal{L}\{f\}(s)$
$f(at)$	$\frac{1}{a}F\left(\frac{s}{a}\right)$
$f'(t)$	$sF(s) - f(0)$
$f^{(n)}(t)$	$s^n F(s) - \sum_{i=0}^{n-1} s^{n-i-1} f^{(i)}(0)$
$t^n f(t)$	$(-1)^n F^{(n)}(s)$
$\frac{1}{t} f(t)$	$\int_s^\infty F(u)\mathrm{d}u$
a	$\frac{a}{s}$, $s > 0$
e^{at}	$\frac{1}{s-a}$, $s > a$
$t^n, n = 1, 2, \ldots$	$\frac{n!}{s^{n+1}}$, $s < 0$
$e^{at} t^n, n = 1, 2, \ldots$	$\frac{n!}{(s-a)^{n+1}}$, $s > a$
$\sin(bt)$	$\frac{b}{s^2+b^2}$, $s > 0$
$\cos(bt)$	$\frac{s}{s^2+b^2}$, $s > 0$
$e^{at}\sin(bt)$	$\frac{s}{(s-a)^2+b^2}$, $s > a$
$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$, $s > a$