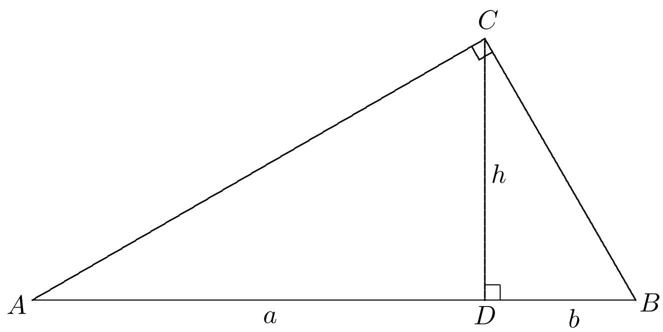
WEEK 1 EXERCISE

1. Another proof of pythagoras theorem and Pythagorean means

Problem 1. In the following diagram, $AD = a, DB = b$ and $CD = h$.

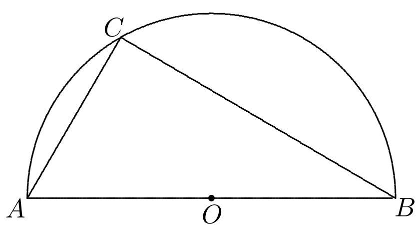


Show that $\triangle ACD \sim \triangle CBD$ and $h = \sqrt{ab}$

Remark. $\sqrt{ab}$ is called the *geometric mean (GM)* of a and b.

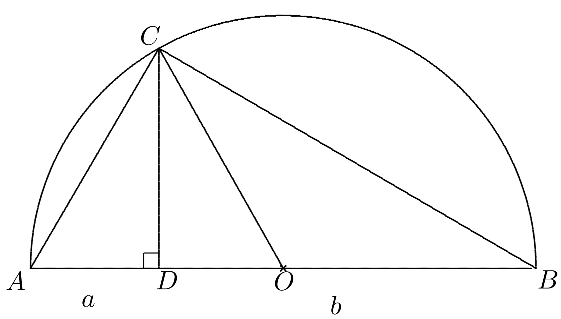
$\frac{a+b}{2}$ is called the *arithmetic mean (AM)* of a and b.

Problem 2. In the following diagram, AB is the diameter, C is a point on the semi-circle and O is the center.



Show that $\angle ACB = 90°$.

Problem 3. Point D is the foot of the perpendicular from C to AB and $AD = a$ and $DB = b$.



From the previous questions, we have $\angle ACB = 90°$ and hence $DC = \sqrt{ab}$.

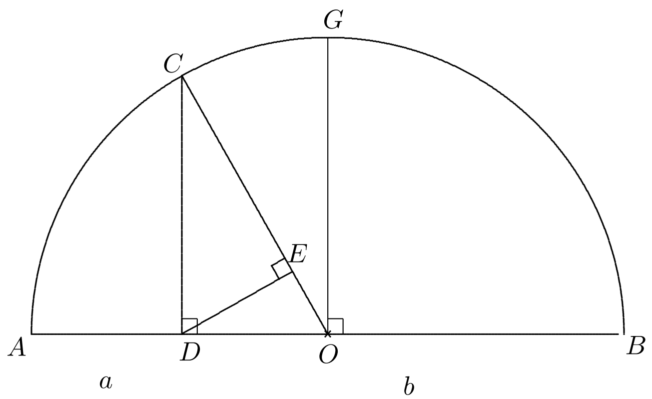
(1) Find OC and OD in terms of a and b.

(2) Prove the pythagoras theorem by considering $\triangle ODC$.

Remark. This shows that pythagoras theorem is true for every right triangle with sides $\left(\sqrt{ab}, \frac{b-a}{2}, \frac{a+b}{2}\right)$.

Problem 4 (Construction of Pythagorean mean)**.** In the diagram below, $AD = a$ and $DB = b$.

G is the intersection of the the perpendicular bisector of AB and the semi-circle.



In previous exercise, we show the constructon of arithmetic mean and geometric mean of two line segments.

Here, we are going to construct the *harmonic mean (HM)*, $\dfrac{2}{\frac{1}{a}+\frac{1}{b}}$, of a and b.

Finally, we construct the *quadratic mean (QM)*, $\sqrt{\frac{a^2+b^2}{2}}$, of a and b.

(1) Show that $\triangle CDO \sim \triangle CED$ and $CE = \dfrac{2ab}{a+b} = \dfrac{2}{\frac{1}{a}+\frac{1}{b}}$.

(2) Draw the line segment DG and show that $DG = \sqrt{\frac{a^2+b^2}{2}}$.

Theorem (Mean inequality)**.** *Let a, b be any positive real numbers, then*

$$\frac{2}{\frac{1}{a}+\frac{1}{b}} \leq \sqrt{ab} \leq \frac{a+b}{2} \leq \sqrt{\frac{a^2+b^2}{2}}$$

The equality holds if and only if $a = b$.

Proof. With refer to the diagram, the harmonic mean of a and b is the segment CE.

It is less than the hypotenuse of the triangle $\triangle CDE$, i.e. $CE \leq DC$.

DC is the geometric mean, hence we showed $HM \leq GM$.

Since DC is perpendicular to the diameter, it is less then the radius OG.

$OG = OA = OB$ hence it is the arithmetic mean.

This shows $GM \leq AM$.

Finally, a similar argument on the triangle $\triangle ODG$ shows that $AM \leq QM$

This complete the proof of the mean inequality, $HM \leq GM \leq AM \leq QM$.

If $a = b$, the above four are equal to a.

Conversely, if all of them are equal, one can show that $a = b$ algebraically. □

2. C's Syntax

Rewrite the following mathematical expressions in C's expressions.

Do not simplify any one of them.

You can assume that $\sqrt{x}$ is written as $sqrt\,(x)$ in C.

(1) $x^3 + y^3$

(2) $(x+y)^2\,(x-y)$

(3) $\sqrt{(x-y)^2}$

(4) $(\sqrt{x-y})^2$

(5) $a^2 + 2ab + b^2$

(6) $\dfrac{x^3+y^3}{x+y}$

(7) $x^2 - xy + y^2$

(8) $\left(\dfrac{x^3+y^3}{x+y}\right)^2$

Remark. Be careful, computer is so stupid.

It cannot understand simplified notation.

For example, we may write $2xy$ usually to stand for the product of $2, x$ and y.

In this case, you have to write $2 * x * y$.