习题3.1

1.

(1) $f'(x)=\lim\limits_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$

$f'_+(0)=\lim\limits_{\Delta x\to 0^+}\frac{f(\Delta x)}{\Delta x}=1$

$f'_-(0)=\lim\limits_{\Delta x\to 0^-}\frac{f(\Delta x)}{\Delta x}=\frac{-\Delta x}{\Delta x}=-1$

故不可导

(2) $f'_-(0)=0$

$f'_+(0)=\lim\limits_{\Delta x\to 0^+}\frac{f(\Delta x)-f(0)}{\Delta x}=1$

故不可导

(3) $f'_+(0)=\lim\limits_{\Delta x\to 0^+}\frac{f(\Delta x)-f(0)}{\Delta x}$

$f'_-(0)=\lim\limits_{\Delta x\to 0^-}\frac{f(\Delta x)-f(0)}{\Delta x}$

一定∃$\{x_n\}$，x_n皆为正数 $n\to\infty$ $x_n\to 0^+$

取 $y_n=-x_n$，故$\{y_n\}$ $n\to\infty$ $y_n\to 0^-$

$f'_+(0)=\lim\limits_{n\to\infty}\frac{f(x_n)-1}{x_n}=\frac{x_n^2\sin\frac{1}{x_n}-1}{x_n}$

$f'_-(0)=\lim\limits_{n\to\infty}\frac{f(y_n)-1}{y_n}=\frac{y_n^2\sin\frac{1}{y_n}-1}{y_n}=\frac{x_n^2\sin\frac{1}{x_n}+1}{x_n}$

$f'_+(0)\neq f'_-(0)$ 故不可导.

(4) $f'_+(x)=\frac{1}{x+1}$ $\quad f'_+(0)=1$

$f'_-(0)=\lim\limits_{\Delta x\to 0^-}\frac{f(\Delta x)-f(0)}{\Delta x}=\frac{\Delta x+1-0}{\Delta x}\to-\infty$

故不可导

(5) $f'_+(0)=\lim\limits_{\Delta x\to 0^+}\frac{f(\Delta x)-f(0)}{\Delta x}=\frac{\Delta x e^{\Delta x}-0}{\Delta x}=e^{\Delta x}=1$

$f'_-(0)=\lim\limits_{\Delta x\to 0^-}\frac{f(\Delta x)-f(0)}{\Delta x}=\frac{-\Delta x e^{\Delta x}-0}{\Delta x}=-e^{\Delta x}=-1$

故不可导

(6) $f'_+(0)=\lim\limits_{\Delta x\to 0^+}\frac{f(\Delta x)-f(0)}{\Delta x}=\Delta x^2=0$

$f'_-(0)=\lim\limits_{\Delta x\to 0^-}\frac{f(\Delta x)-f(0)}{\Delta x}=-\Delta x^2=0$

故 $f'_+(0)=f'_-(0)$ 可导

2. (1) $f'(1)=f'_+(1)=2=f'_-(1)=a$

$1\times a+b=1^2$ 故 $a=2$ $b=-1$

(2) $f'_+(0)=a$

$f'_-(0)=\frac{1}{1+0}=1$

$a\times 0+b=\ln(1+0)-1$

故 $a=1$ $b=-1$

3. $f'(x)=g(x)+(x-a)g'(x)$

$f'(a)=g(a)$

4.

$$\lim_{h\to 0}\frac{f(x_0+\alpha h)-f(x_0-\beta h)}{h}=\alpha\lim_{h\to 0}\frac{f(x_0+\alpha h)-f(x_0)}{\alpha h}+\beta\lim_{h\to 0}\frac{f(x_0)-f(x_0-\beta h)}{\beta h}$$

$$=\alpha\lim_{h\to 0}\frac{f(x_0+\alpha h)-f(x_0)}{\alpha h}+\beta\lim_{h\to 0}\frac{f(x_0+(-\beta)h)-f(x_0)}{(-\beta)\cdot h}$$

$$=(\alpha+\beta)f'(x_0)$$

5. $f'_+(a)=\lim_{\Delta x\to 0^+}\frac{f(a+\Delta x)-f(a)}{\Delta x}$ $\lim_{\Delta x\to 0^-}\frac{f(a+\Delta x)-f(a)}{\Delta x}$

$g(x)=|f(x)|$ $g'_+(a)=\lim_{\Delta x\to 0^+}\frac{|f(a+\Delta x)|-|f(a)|}{\Delta x}$ $g'_-(a)=\lim_{\Delta x\to 0^-}\frac{|f(a+\Delta x)|-|f(a)|}{\Delta x}$

又∵函数$f(x)$在$x=a$处可导，故$f(a)$存在，又$f(a)\neq 0$，不妨$f(a)>0$

且$f(x)$在$x=a$处连续，即$\forall\varepsilon>0$，$\exists\delta$，$|x-a|<\delta$，有$|f(x)-f(a)|<\varepsilon$

令$x=a+\Delta x$，并取$\varepsilon=\frac{f(a)}{2}$，故$|\Delta x|<\delta$，有 $f(x)>f(a)-\varepsilon=\frac{f(a)}{2}$

即 $f(a+\Delta x)$和$f(a)$始终同号

故 $g'_+(a)=g'_-(a)$ （当$f(a)>0$，$g'_+(a)=f'_+(a)=f'_-(a)=g'_-(a)$
当$f(a)<0$ $g'_+(a)=-f'_+(a)=-f'_-(a)=g'_-(a)$） □

而若$f(a)=0$，保号性不一定成立，故不一定成立，如$f(x)=|x|$在$x=0$处.

6. (1) $y'=\frac{(6x+9)(5x+8)-5(3x^2+9x-2)}{(5x+8)^2}=\frac{15x^2+48x+82}{25x^2+80x+64}$

(2) $y'=\cos x\tan x+\sin x\frac{1}{\cos^2 x}-\frac{1}{\sin^2 x}=\sin x+\frac{\tan x}{\cos x}-\frac{1}{\sin^2 x}$

(3) $y'=x\log_3 x+\frac{x^2}{x\ln 3}=x\log_3 x+\frac{x}{\ln 3}$

(4) $y'=\frac{1-\cos x-(\sin x)x}{(1-\cos x)^2}=\frac{1-\cos x-x\sin x}{1+\cos^2 x-2\cos x}$

(5) $y=1+\frac{2\ln x}{1-\ln x}$ $y'=\frac{\frac{2}{x}(1-\ln x)-(-\frac{1}{x})\cdot 2\ln x}{(1-\ln x)^2}=\frac{2}{x(\ln^2 x-2\ln x+1)}$

(6) $y'=\frac{(2x\ln x+\frac{1+x^2}{x})(\sin x+\cos x)-(\cos x-\sin x)(1+x^2)\ln x}{(\sin x+\cos x)^2}$

$=\frac{2x\sin x\ln x+2x\cos x\ln x+\frac{1+x^2}{x}\sin x+\frac{1+x^2}{x}\cos x-\cos x\ln x-x^2\cos x\ln x+\sin x\ln x+x^2\sin x\ln x}{1+\sin 2x}$

(7) $y'=2x(3x-1)(1-x^3)+3(x^2+1)(1-x^3)+(-3x^2)(x^2+1)(3x-1)$

$=(6x^2-2x)(1-x^3)+(3x^2+3)(1-x^3)+(-3x^4-3x^2)(3x-1)$

$=6x^2-2x-6x^5+2x^4+3x^2+3-3x^5-3x^3-9x^5-3x^3+3x^4+3x^2$

$=-18x^5+5x^4-6x^3+12x^2-2x+3$

(8) $y'=3x^2\tan x\ln x+x^3\cdot\frac{1}{\cos^2 x}\ln x+x^3\tan x\cdot\frac{1}{x}$

$=3x^2\tan x\ln x+\frac{1}{\cos^2 x}x^3\ln x+x^2\tan x$

7. (1) $y'=\sqrt{1-x^2}+x\cdot\frac{1}{2}\cdot(1-x^2)^{-\frac{1}{2}}=\sqrt{1-x^2}+\frac{1}{2\sqrt{1-x^2}}$

(2) $y'=\frac{1}{3}(1+\ln^2 x)^{-\frac{2}{3}}\cdot 2\cdot\frac{1}{x}\ln x$

(3) $y'=\frac{1}{-\sin\frac{2x-1}{\sqrt{3}}}\cdot\frac{2}{\sqrt{3}}$

(4) $y'=3(\sin x+\cos x)^2\cdot(\cos x-\sin x)=3(\sin x+\cos x)\cdot\cos 2x$

(5) $y'=3(\sin x^3)^2\cdot 3x^2\cdot\cos x^3=9x^2(\sin x^3)^2\cdot\cos x^3$

(6) $y'=\frac{1}{2}(x+\sqrt{x+\sqrt{x}})^{-\frac{1}{2}}\cdot\left(1+\frac{1}{2}(x+\sqrt{x})^{-\frac{1}{2}}\cdot(1+\frac{1}{2}x^{-\frac{1}{2}})\right)$

(7) $y'=\cos(\sin(\sin x))\cdot\cos(\sin x)\cdot\cos x$

(8) $y'=\cos(\cos^5(\arctan x^3))\left[\cos^5(\arctan x^3)\right]'$

$=\cos(\cos^5(\arctan x^3))\cdot 5\cos^4(\arctan x^3)\cdot\left[\cos(\arctan x^3)\right]'$

$=\cos(\cos^5(\arctan x^3))\cdot 5\cos^4(\arctan x^3)\cdot(-\sin\arctan x^3)\cdot(\cos^2 x^3)(3x^2)$

(9) $y'=3\left(\frac{x^3-1}{x^4+1}\right)^2\cdot\frac{3x^2(x^4+1)-4x^3(x^3-1)}{(x^4+1)^2}=3\frac{(x^3-1)^2\left[-x^6+4x^3+3x^2\right]}{(x^4+1)^4}$

(10) $y'=\sqrt{1+x^2}\sin x+x\cdot(\frac{1}{2})\cdot(1+x^2)^{-\frac{1}{2}}\sin x+x\cdot\sqrt{1+x^2}\cos x$

(11) $y'=e^{\sqrt{x^2+1}}\cdot(\frac{1}{2})\cdot(x^2+1)^{-\frac{1}{2}}$ $\qquad (2^x)'=(e^{x\ln 2})'=\ln 2\cdot e^{x\ln 2}$

(12) $y'=\frac{1}{\ln^2(\ln^3 x)}\cdot 2\ln(\ln^3 x)\cdot\frac{1}{\ln^3 x}\cdot 3\cdot\ln^2 x=6\frac{\ln(\ln^3 x)\cdot\ln^2 x}{\ln^2(\ln^3 x)\cdot\ln^3 x}$

$=\frac{6}{\ln(\ln^3 x)\cdot\ln x}$

(13) $y=e^{x^x\cdot\ln x}+e^{x\ln x}+e^{2^x\ln x}=e^{e^{x\ln x}\cdot\ln x}+e^{x\ln x}+e^{2^x\ln x}$

$y'=e^{e^{x\ln x}\cdot\ln x}\cdot\left(\frac{1}{x}e^{x\ln x}+e^{x\ln x}\cdot(1+\ln x)\cdot\ln x\right)+e^{x\ln x}\cdot(\ln x+1)+e^{2^x\ln x}\cdot\left(\ln 2\cdot 2^x\cdot\ln x+\frac{1}{x}\cdot 2^x\right)$

$=x^{x^x}\left(x^{x-1}+x^x(1+\ln x)\cdot(\ln x)\right)+x^x(1+\ln x)+x^{2^x}\cdot\left(\ln 2\cdot 2^x\cdot\ln x+\frac{1}{x}2^x\right)$

(14) 令 $\ln x = t$

$$y = t^{e^{e^t}} = e^{(e^{e^t})\cdot \ln t}$$

$$y' = e^{(e^{e^t})\cdot\ln t}\cdot\left(\frac{1}{t}e^{e^t} + e^{e^t}\cdot e^t\cdot\ln t\right)\cdot(t)'$$

$$= e^{e^x\cdot\ln\ln x}\cdot\left(\frac{1}{\ln x}e^x + e^x\cdot x\cdot\ln\ln x\right)\cdot\frac{1}{x}$$

法二：$e^{e^x \ln\ln x} = e^{e^x \ln\ln x}\left(e^x \ln\ln x + \frac{1}{\ln x}\cdot\frac{1}{x}\cdot e^x\right)$

(15) $y = e^{\ln(\tan x)\cdot\cot x}$

$$y' = e^{\ln(\tan x)\cdot\cot x}\cdot\left(\frac{\cot x}{\tan x}\cdot\frac{1}{\cos^2 x} - \frac{1}{\sin^2 x}\ln(\tan x)\right)$$

$$= e^{\ln(\tan x)\cdot\cot x}\cdot\frac{1}{\sin^2 x}[1-\ln(\tan x)]$$

(16) $y' = (\ln 10)\cdot 10^x\cdot(\sin x)^{\cos x} + 10^x\left(\sin x^{\cos x}\right)'$

$$= 10^x(\ln 10)(\sin x)^{\cos x} + 10^x\cdot e^{\ln\sin x\cos x}\left(\frac{1}{\sin x}\cdot\cos^2 x - \sin x\ln\sin x\right)$$

(17) $y' = 2(x+5)(x-4)^{\frac{1}{3}}\cdot(x+2)^{-\frac{1}{5}}\cdot(x+4)^{-\frac{1}{2}}$

$$+ (x+5)^2\cdot\left(\frac{1}{3}\right)\cdot(x-4)^{-\frac{2}{3}}\cdot(x+2)^{-\frac{1}{5}}\cdot(x+4)^{-\frac{1}{2}}$$

$$+ (x+5)^2\cdot(x-4)^{\frac{1}{3}}\cdot\left(-\frac{1}{5}\right)(x+2)^{-\frac{6}{5}}(x+4)^{-\frac{1}{2}}$$

$$+ (x+5)^2\cdot(x-4)^{\frac{1}{3}}\cdot\left(-\frac{1}{5}\right)(x+2)^{-\frac{1}{5}}\cdot\left(-\frac{1}{2}\right)(x+4)^{-\frac{3}{2}}$$

(18) $y' = \frac{2x(1-x)-(-x)\cdot x^2}{(1-x)^2}\sqrt{\frac{x+1}{1+x+x^2}} + \frac{1}{2}\left(\frac{1+x}{1+x+x^2}\right)^{-\frac{1}{2}}\cdot\frac{x^2}{1-x}$

$$= \frac{x^3-2x^2+2x}{(1-x)^2}\sqrt{\frac{1+x}{1+x+x^2}} + \frac{1}{2}\left(\frac{1+x}{1+x+x^2}\right)^{-\frac{1}{2}}\frac{x^2}{1-x}$$

11.

(1) $y=\begin{cases}\dfrac{x}{e^x+1} & x\neq 0\\ 0 & x=0\end{cases}$ 也即 $y=\dfrac{x}{e^x+1}$

$$y'=\frac{e^x+1-xe^x}{(e^x+1)^2}=\frac{(1-x)e^x+1}{e^{2x}+2e^x+1}$$

(2) $y=|1-2x|\sin x=\begin{cases}(1-2x)\sin x & x<\frac{1}{2}\\ 0 & x=\frac{1}{2}\\ (2x-1)\sin x & x>\frac{1}{2}\end{cases}$

故当 $x<\frac{1}{2}$ 时 $y_1'=-2\sin x+(1-2x)\cos x$

当 $x>\frac{1}{2}$ 时 $y_2'=2\sin x+(2x-1)\cos x$

而 $y_1'(\frac{1}{2})\neq y_2'(\frac{1}{2})$ 故 $f(x)$ 在 $x=\frac{1}{2}$ 处无导数

12.

(1) $x_{n_1}=\dfrac{1}{2n\pi}$ $x_{n_2}=\dfrac{1}{2n\pi+\pi/2}$，显然 $n\to\infty$ 时 $\{x_{n_1}\},\{x_{n_2}\}\to 0$

$$\lim_{\Delta x\to 0^+}\frac{f(\Delta x)-f(0)}{\Delta x}=\lim_{n_1\to\infty}\frac{f(x_{n_1})-f(0)}{x_{n_1}}=0$$

$$\lim_{\Delta x\to 0^+}\frac{f(\Delta x)-f(0)}{\Delta x}=\lim_{n_2\to\infty}\frac{f(x_{n_2})-f(0)}{x_{n_2}}=1$$ 矛盾!

故 $n=1$ 时，$f(x)$ 在 $x=0$ 处不可导

(2) 考虑 $\lim\limits_{\Delta x\to 0}\dfrac{f(\Delta x)}{\Delta x}=\lim\limits_{\Delta x\to 0}\dfrac{(\Delta x)^2\sin(\Delta x)}{\Delta x}=\lim\limits_{\Delta x\to 0}(\Delta x)\sin(\Delta x)=0$

故 $f'(0)=\lim\limits_{\Delta x\to 0}\dfrac{f(\Delta x)}{\Delta x}=0$ 可导! (0×有界)

$f(x)=\begin{cases}x^2\sin\frac{1}{x} & x\neq 0\\ 0 & x=0\end{cases}$ $f'(x)=2x\sin\frac{1}{x}+x^2\cos\frac{1}{x}\cdot(-\frac{1}{x^2})$

$=2x\sin\frac{1}{x}-\cos\frac{1}{x}$ $(x\neq 0)$

故 $f'(x)=\begin{cases}2x\sin\frac{1}{x}-\cos\frac{1}{x} & x\neq 0\\ 0 & x=0\end{cases}$

考虑 $x_n=\frac{1}{2n\pi}$ $n\to\infty$ $x_n\to 0$ $\lim\limits_{n\to\infty}f(x_n)=-1$

$y_n=\frac{1}{2n\pi+\frac{\pi}{2}}$ $n\to\infty$ $x_n\to 0$ $\lim\limits_{n\to\infty}f'(y_n)=\lim\limits_{n\to\infty}\frac{2}{2n\pi+\frac{\pi}{2}}=0$ 故为第二类间断点

(3) $n \geq 3$

考虑 $\lim\limits_{x\to 0}\frac{f(\Delta x)}{\Delta x} = (\Delta x)^{n-1}\sin\frac{1}{\Delta x} = 0$. (0 × 有界)

故 $f'(0) = \lim\limits_{x\to 0}\frac{f(\Delta x)}{\Delta x} = 0$ 在0处 $f(x)$ 可导

当 $x \neq 0$ 时. $f'(x) = nx^{n-1}\sin\frac{1}{x} + x^n\cos\frac{1}{\Delta x}(-\frac{1}{x^2})$

$= nx^{n-1}\sin\frac{1}{x} - x^{n-2}\cos\frac{1}{\Delta x}$ $\qquad g(x) = f'(x)$

考虑 $\lim\limits_{x\to 0} g(x) = \lim\limits_{x\to 0}\left(nx^{n-1}\sin\frac{1}{x} - x^{n-2}\cos\frac{1}{\Delta x}\right) = 0$ (0 × 有界)

故当 $n \geq 3$ 时, $f(x)$ 在点 $x=0$ 处可导, 且导函数在 $x=0$ 处连续

13.

$f'_+(0) = \lim\limits_{\Delta x\to 0^+}\frac{f(\Delta x)}{\Delta x} = \lim\limits_{\Delta x\to 0}\frac{(\Delta x)^2\sin\frac{1}{(\Delta x)^2}}{\Delta x} = \lim\limits_{\Delta x\to 0}\Delta x\sin\frac{1}{(\Delta x)^2} = 0$

$f'_-(0) = \lim\limits_{\Delta x\to 0^-}\frac{f(\Delta x)}{\Delta x} = \lim\limits_{\Delta x\to 0}\frac{(\Delta x)^2\sin\frac{1}{(\Delta x)^2}}{\Delta x} = \lim\limits_{\Delta x\to 0}\Delta x\sin\frac{1}{(\Delta x)^2} = 0$

故 $f'(0) = 0$.

而在 $[-1,0)$ 和 $(0,1]$, 由可导函数四则运算知其可导

$f'(x) = 2x\sin\frac{1}{x^2} + x^2\cos\frac{1}{x^2}\cdot(-2)\cdot\frac{1}{x^3}$

$= 2x\sin\frac{1}{x^2} - \frac{2}{x}\cos\frac{1}{x^2}$ $\quad (x \neq 0)$

而取 $x_n = \frac{1}{\sqrt{2n\pi}}$ $\quad n\to\infty$ 时 显然 $x_n \in [-1,1]$

$f'(x_n) = 0 - 2\sqrt{2n\pi} = -2\sqrt{2n\pi}$ 无下界

$y_n = \frac{-1}{\sqrt{2n\pi}}$

$f'(x_n) = 0 + 2\sqrt{2n\pi} = 2\sqrt{2n\pi}$ 无上界

14.

(1) $y' = (x+1)e^x$ $\qquad$ (2) $y' = \frac{1}{1+(\frac{1}{x})^2}\cdot(-\frac{1}{x^2}) = \frac{-1}{1+x^2}$

(3) $y = 2e^{-x\ln x} - e^{-2x}$

$y' = 2e^{-x\ln x}\cdot(-\ln x - 1) + 2e^{2x}$

(4) $y' = \frac{1}{e^x + \sqrt{1+e^{2x}}}\cdot\left(e^x + \frac{1}{2}(1+e^{2x})^{-\frac{1}{2}}\cdot 2e^{2x}\right)$

$= \frac{1}{e^x + \sqrt{1+e^{2x}}}\left(e^x + e^{2x}(1+e^{2x})^{-\frac{1}{2}}\right)$