14.

$F(x)=f(x)-g(x)$

结合 13题 知 $F(x)$ 严格单增

$F(a)=f(a)-g(a)=0$

由其严格单增知 $x>a$ $F(x)>0 \Rightarrow f(x)>g(x)$

$x<a$ $F(x)<0 \Rightarrow f(x)<g(x)$

15.

$\because$ $f'(x)$ 严格单增

$\forall x_1<x<x_2$ $\exists$ $\xi_1\in(x_1,x)$ $\xi_2\in(x,x_2)$

$$\frac{f(x)-f(x_1)}{x-x_1}=f'(\xi_1)<f'(\xi_2)=\frac{f(x_2)-x}{x_2-x}$$

引理：若 $\frac{a}{b}<\frac{c}{d}$ 则 $b>0$，$d>0$.

则 $\frac{a}{b}<\frac{a+c}{b+d}<\frac{c}{d}$ 此引理通分后是显然的.

故 $$\frac{f(x)-f(x_1)}{x-x_1}<\frac{f(x_2)-f(x_1)}{x_2-x_1}<\frac{f(x_2)-x}{x_2-x}$$

令 $x_1=0$. 则上述第一个不等号即 $\frac{f(x_1)}{x_1}<\frac{f(x_2)}{x_2}$ □.

18.

(1) $y' = 6x^2 - 6x = 6x(x-1) = 0$

$x<0$ 时 $y'>0$. $\uparrow$

$0<x<1$ 时 $y'<0$ $\downarrow$

$x>1$ 时 $y'>0$ $\uparrow$

极大值, $y(0) = 0$, 极小值 $y(1) = -1$

(2) $y' = \frac{2}{3}x^{-\frac{1}{3}}$

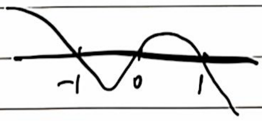
$x<0$. $y'<0$ $\downarrow$.

$x>0$ $y'>0$ $\uparrow$

极小值 $y(0)=0$, 无极大值.

(3) $y' = 2xe^{-x^2} - 2x\cdot x^2e^{-x^2}$

$= 2xe^{-x^2}(1-x^2)$




$x<-1$ $y'>0$ $\uparrow$

$-1<x<0$ $y'<0$ $\downarrow$

$0<x<1$ $y'>0$ $\uparrow$

(4) $y = x^{\frac{1}{x}} = e^{\frac{1}{x}\ln x}$

$y' = e^{\frac{1}{x}\ln x}\cdot\left(\frac{-1}{x^2}\ln x + \frac{1}{x^2}\right)$

$= \frac{1}{x^2}e^{\frac{1}{x}\ln x}(1-\ln x)$

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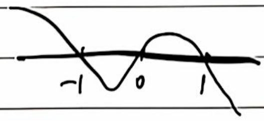

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# 习题3.4 (P115)

1\. (1) $\lim\limits_{x\to0} \alpha\cdot\frac{1}{m}(1+\alpha x)^{\frac{1}{m}-1} - \beta\cdot\frac{1}{n}(1+\beta x)^{\frac{1}{n}-1} = \frac{\alpha}{m}-\frac{\beta}{n}$

(2) $\lim\limits_{x\to0} \frac{mn(1+mx)^{n-1} - mn(1+nx)^{m-1}}{2x} = \lim\limits_{x\to0} \frac{mn(n-1)(1+mx)^{n-2} - mn(m-1)(1+nx)^{m-2}}{2} = mn^2 - m^2n$

(3) $\lim\limits_{x\to1} \frac{3x^2+1}{2x-3} = \lim\limits_{x\to1} \frac{6x}{2} = 3$

(4) $\lim\limits_{x\to0} \frac{x-\arcsin x}{x^3} = \lim\limits_{x\to0} \frac{1-\frac{1}{\sqrt{1-x^2}}}{3x^2} = \lim\limits_{x\to0} \frac{\frac{1}{2}(1-x^2)^{-\frac{3}{2}}\cdot 2x}{6x} = \lim\limits_{x\to0} \frac{(1-x^2)^{-\frac{3}{2}}}{6} = \frac{1}{6}$

(5) $\lim\limits_{x\to0} \frac{e^x-1}{x} = \lim\limits_{x\to0} \frac{e^x}{1} = 1$

(6) $\lim\limits_{x\to0} \frac{\alpha x}{x} = \alpha$

(7) $\lim\limits_{x\to0} \frac{2x^{-3}\cdot e^{-\frac{1}{x^2}}}{1} = \lim\limits_{x\to0} \frac{e^{-\frac{1}{x^2}}}{x^3} = \lim\limits_{x\to0} \frac{2e^{-\frac{1}{x^2}}}{x^6}$

$= \lim\limits_{t\to+\infty} 2t^3 e^{-t} = \lim\limits_{t\to+\infty} \frac{2t^3}{e^t} = \lim\limits_{t\to\infty} \frac{12}{e^t} = 0$

(8) $\lim\limits_{x\to0} \frac{(a+x)^x - a^x}{x^2} = \lim\limits_{x\to0} \frac{e^{x\ln(a+x)} - e^{x\ln a}}{x^2} = \lim\limits_{x\to0} \frac{e^{x\ln(a+x)-x\ln a} - 1}{e^{-x\ln a}x^2}$

$= \lim\limits_{x\to0} \frac{x\ln(a+x) - x\ln a}{e^{-x\ln a}x^2} = \lim\limits_{x\to0} \frac{\ln(1+\frac{x}{a})}{x\cdot a^{-x}} = \lim\limits_{x\to0} \frac{1}{a\cdot a^{-x}} = \frac{1}{a}$

(9) $\lim\limits_{x\to0} x\sin\frac{1}{x} = 0$ (10) $\lim\limits_{x\to0} \frac{x^2\cos\frac{1}{x}}{x} = 0$

(11) $\lim\limits_{x\to0} \frac{x^2-\arctan^2 x}{x^4} = \lim\limits_{x\to0} \frac{2x - \frac{2}{1+x^2}\arctan x}{4x^3}$

$= \lim\limits_{x\to0} \frac{x-\frac{1}{1+x^2}\arctan x}{2x^3} = \lim\limits_{x\to0} \frac{1-\frac{1}{(1+x^2)^2} + \frac{2x}{(1+x^2)^2}\arctan x}{6x^2} = \lim\limits_{x\to0} \frac{1-\frac{1}{(1+x^2)^2}}{6x^2} + \lim\limits_{x\to0} \frac{\frac{1}{(1+x^2)^2}}{3}$

$= \lim\limits_{x\to0} \frac{2(1+x^2)^{-3}\cdot 2x}{12x} + \frac{1}{3} = \frac{2}{3}$

(12) $\lim\limits_{x\to 1^-} \ln x \ln(1-x) = \lim\limits_{x\to 1^-} \dfrac{\ln(1-x)}{\log_x e} = \lim\limits_{x\to 1^-} \dfrac{\frac{-1}{1-x}}{-\frac{1}{x}\ln x}$

$= \lim\limits_{x\to 1^-} \dfrac{\frac{x}{1-x}}{\ln x} = \lim\limits_{x\to 1^-} \dfrac{\frac{1-x+x}{(1-x)^2}}{\frac{1}{x}} = \lim\limits_{x\to 1^-} \dfrac{x}{(1-x)^2} = \lim\limits_{x\to 1^-} \dfrac{1}{2(x-1)} = 0$

(13) $\lim\limits_{x\to \frac{\pi}{2}^-} (\tan x)^{2x-\pi} = e^{\lim\limits_{x\to\frac{\pi}{2}^-} (2x-\pi)\cdot \ln(\tan x)} = e^{\lim\limits_{x\to\frac{\pi}{2}^-} \frac{\ln(\tan x)}{\frac{1}{2x-\pi}}}$

$= e^{\lim\limits_{x\to\frac{\pi}{2}^-} \frac{\frac{1}{\tan x}\cdot\frac{1}{\cos^2 x}}{\frac{-2}{(2x-\pi)^2}}} = e^{\lim\limits_{x\to\frac{\pi}{2}^-} \frac{(2x-\pi)^2}{-2\tan x\cos^2 x}} = e^{\lim\limits_{x\to\frac{\pi}{2}^-} \frac{2(2x-\pi)\cdot 2}{-2+4\tan x\cos x\sin x}}$

$= e^{\lim\limits_{x\to\frac{\pi}{2}^-} \frac{4(2x-\pi)}{-2+4\sin^2 x}} = e^0 = 1$

(14) $\lim\limits_{x\to 0} \left(\dfrac{(1+x)^{\frac{1}{x}}}{e}\right)^{\frac{1}{x}} = e^{\lim\limits_{x\to 0} \frac{1}{x}\ln\left(\frac{(1+x)^{\frac{1}{x}}}{e}\right)}$

$= e^{\lim\limits_{x\to 0} \frac{\frac{1}{x}\ln(1+x)-1}{x}} = e^{\lim\limits_{x\to 0} \frac{\ln(1+x)-x}{x^2}} = e^{\lim\limits_{x\to 0} \frac{\frac{1}{1+x}-1}{2x}}$

$= e^{\lim\limits_{x\to 0} \frac{\frac{-1}{(1+x)^2}}{2}} = e^{-\frac{1}{2}}$

(15) $\lim\limits_{x\to 1^-} \dfrac{\ln(1-x)+\tan\frac{\pi}{2}x}{\cot\pi x} = \lim\limits_{x\to 1^-} \dfrac{\frac{-1}{1-x}+\frac{\pi}{2\cos^2\frac{\pi}{2}x}}{\frac{1}{-\sin^2\pi x}}$

$= \lim\limits_{x\to 1^-} \dfrac{\sin^2\pi x}{1-x} + \dfrac{\sin^2\pi x}{2\cos^2\frac{\pi}{2}x}$

$= \lim\limits_{x\to 1^-} \dfrac{\sin^2\pi x}{1-x} + \lim\limits_{x\to 1^-} \dfrac{\sin^2\pi x}{1+\cos\pi x}$

$= \lim\limits_{x\to 1^-} \dfrac{\pi\sin 2\pi x}{-1} + \lim\limits_{x\to 1^-} \dfrac{2\cancel{\pi}\cancel{\sin\pi x}\cos\pi x}{-\cancel{\pi}\cancel{\sin\pi x}}$

$= 0-2 = -2$

(16) $\lim\limits_{x\to 0} \dfrac{x^4}{\frac{1}{2}x^2 \cdot x^2} = 2$

(17) $e^{\lim\limits_{x\to+\infty} \frac{1}{x} \ln \frac{\ln(1+x)}{x}} = e^{\lim\limits_{x\to+\infty} \frac{1}{x} \ln \ln(1+x)^{\frac{1}{x}}} = e^0 = 1$

(18) $\lim\limits_{x\to 0} \dfrac{x^2}{\sqrt{1+x\sin x}-\sqrt{\cos x}} \cdot \lim\limits_{x\to 0}\left(2-\dfrac{x}{e^x-1}\right)$

$= \lim\limits_{x\to 0} 2 - \lim\limits_{x\to 0} \dfrac{x}{e^x-1} = 2-1 = 1$

$= \lim\limits_{x\to 0} \dfrac{2x}{\frac{1}{2}(1+x\sin x)^{-\frac{1}{2}} \cdot (\sin x + x\cos x) + \frac{1}{2}(\cos x)^{-\frac{1}{2}}\sin x}$

$= \lim\limits_{x\to 0} \dfrac{2}{-\frac{1}{4}(1+x\sin x)^{-\frac{3}{2}}(\sin x+x\cos x)^2 + \frac{1}{2}(1+x\sin x)^{-\frac{1}{2}}(2\cos x - x\sin x) + -\frac{1}{4}(\cos x)^{-\frac{3}{2}}(-\sin^2 x) + \frac{1}{2}(\cos x)^{\frac{1}{2}}}$

$= \lim\limits_{x\to 0} \dfrac{2}{1+\frac{1}{2}} = \dfrac{4}{3}$

(19) $\lim\limits_{x\to\infty} \dfrac{n^k}{a^n} = \lim\limits_{x\to\infty} \dfrac{k\cdot n^{k-1}}{\ln a\, a^n} = \cdots = \lim\limits_{x\to\infty} \dfrac{k!}{(\ln a)^n a^n} \overset{\forall}{=} \dfrac{}{\infty}\ 0$

(20) $\lim\limits_{x\to\infty} \dfrac{\ln n}{n^k} = \lim\limits_{x\to\infty} \dfrac{1}{k\cdot n\cdot n^{k-1}} = \lim\limits_{x\to\infty} \dfrac{1}{k\cdot n^k} \overset{\forall}{=} \dfrac{}{\infty}\ 0$

2.

(1) $x_{n+1}-x_n=f(x_n)-x_n<0$.

$x_n$ 单减　又 $0<x_n=f(x_{n-1})<x$

故 $x_n$ 有下界，根据单调有界定理，$x_n$ 收敛，设 $\lim\limits_{n\to\infty}x_n=A$

$\lim\limits_{n\to\infty}x_{n+1}=\lim\limits_{n\to\infty}f(x_n)=f(\lim\limits_{n\to\infty}x_n)$　即 $A=f(A)$

下证 $A=0$

若 $f(0)>0$　$F(x)=f(x)-x$　即 $F(0)>0$

由 $F(x)$ 连续知，$\exists\, x_1\in(0,\delta_1)$，$F(x_1)>0$.

与 $f(x)<x$　$x\in(0,a)$ 矛盾

若 $f(0)<0$，则 $\exists\, x_2\in(0,\delta_2)$　$f(x_2)<0$（连续性）.

故与 $0<f(x)$ 矛盾

故 $f(0)=0$，故 $\lim\limits_{n\to\infty}x_n=0$

我们还可以验证 $f(0)=0$ 的合理性.

$F(0)=f(0)-0=0$　$F'(0)=0$　$F''(0)\neq0$.

则当 $F''(0)<0$ 时　满足题设条件，即 $x\in(0,\delta)$　$F(x)<$

(2)

$nx_n=\dfrac{x_n}{\frac{1}{n}}$

考虑 Stolz 定理.　$\dfrac{x_n-x_{n-1}}{\frac{1}{n}-\frac{1}{n-1}}=\dfrac{-(x_n-x_{n-1})}{n(n-1)}=\dfrac{x_{n-1}-x_n}{n(n-1)}$

故 $\lim\limits_{n\to\infty}\dfrac{x_{n-1}-x_n}{n(n-1)}=\lim\limits_{n\to\infty}(x_{n-1}-x_n)\cdot\lim\limits_{n\to\infty}\dfrac{1}{n(n-1)}=0\cdot0=0$

故由 Stolz 定理，$nx_n$ 也收敛，且极限为 0.

□