11.19 作业：P154 2(2,3,6,7,9) 3(3,6,7,10) 5(3,7,9,10) 6(1) 7(2,6,7,11,12,18,23)

习题4.1

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(2) $\int \frac{1}{x^2}\sin\frac{1}{x}dx = \int \frac{1}{x^2}(-\cos\frac{1}{x})'dx = \int -\frac{1}{x^2}(\cos\frac{1}{x})'dx = \int (\cos\frac{1}{x})'d\frac{1}{x} = \cos\frac{1}{x}+C$

(3) $\int \frac{\cos x-\sin x}{1+\sin x+\cos x}dx = \int \frac{d(1+\sin x+\cos x)}{1+\sin x+\cos x} = \int \frac{1}{u}du = \ln|1+\sin x+\cos x|+C$

(6) $\int \frac{1}{\sqrt{x}(1+x)}dx = 2\int \frac{1}{1+x}d\sqrt{x} = 2\int \frac{1}{1+u^2}du = 2\arctan u = 2\arctan\sqrt{x}+C$

(7) $\int \frac{\arctan\frac{1}{x}}{1+x^2}dx = -\int \arctan\frac{1}{x}\, d\arctan\frac{1}{x} = -\frac{1}{2}\arctan^2\frac{1}{x}+C$

(9) $\int \sin^2 x\,dx = \int \frac{1-\cos 2x}{2}dx = \frac{1}{2}\int(1-\cos 2x)dx = \frac{1}{2}(x-\frac{1}{2}\sin 2x)+C$

3.

(3) $\int \frac{1}{(x^2-a^2)^{3/2}}dx \overset{x=a\sec u}{=} \int \frac{1}{(a^2\sec^2 u-a^2)^{\frac{3}{2}}}dx = \frac{1}{a^3}\int \frac{1}{\tan^3 u}dx = \frac{1}{a^3}\int\frac{1}{\tan^3 u}\,a\cdot 2\cdot\sec u\cdot\frac{\sin u}{\cos^2 u}du$

$= \frac{2}{a^2}\int\frac{1}{\sin^2 u}du = \frac{-2}{a^2}\cot u+C$

(4) $\int \frac{x^2}{\sqrt{a^2-x^2}}dx \xlongequal[t\in(-\frac{\pi}{2},\frac{\pi}{2})]{x=a\sin t} \int \frac{a^2\sin^2 t}{\cos t}\cos t\,dt = \frac{a^2}{2}\int 1-\cos 2t\,dt$

$= \frac{a^2}{2}(x-\frac{1}{2}\sin 2t)+C$

(6) $\int \frac{x\ln x}{(1+x^2)^{\frac{3}{2}}}dx \xlongequal{x=\tan y} \int \cos^3 y\tan y\ln\tan y\frac{1}{\cos^2 y}dy = \int \sin y\ln\tan y\,dy$

$= -\cos y\ln\tan y + \int\cos y\, d\ln\tan y$

$= -\cos y\ln\tan y + \int\cos y\frac{1}{\tan y}\cdot\frac{1}{\cos^2 y}dy$

$= -\cos y\ln\tan y + \int\frac{1}{\sin y}dy$

$= -\cos y\ln\tan y - \int\frac{d\cos y}{1-\cos^2 y}$

$= -\cos y\ln\tan y + \int\frac{d\cos y}{\cos^2 y-1}$

$= -\cos y\ln\tan y + \frac{1}{2}\ln\left|\frac{\cos y-1}{\cos y+1}\right| - C$

$= -\cos y\ln\tan y + \frac{1}{2}\ln\frac{(1-\cos y)^2}{\sin^2 y} - C$

$= -\cos y\ln\tan y + \ln\left|\frac{1-\cos y}{\sin y}\right| - C$

$= -\cos y\ln\tan y + \ln\left|\frac{1}{\sin y}-\frac{1}{\tan y}\right| - C$

法二：$\left(\frac{x}{\sqrt{1+x^2}}\right)' = \frac{\sqrt{1+x^2}-\frac{2x}{2\sqrt{1+x^2}}\cdot x}{(1+x^2)}$

$= \frac{1+x^2-x^2}{(1+x^2)^{\frac{3}{2}}} = \frac{1}{(1+x^2)^{\frac{3}{2}}}$

7. $$\int \frac{1-\ln x}{(x-\ln x)^2}dx = \frac{x}{x-\ln x}+C$$

$$\left(\frac{x}{x-\ln x}\right)' = \frac{x-\ln x-(1-\frac{1}{x})x}{(x-\ln x)^2} = \frac{1-\ln x}{(x-\ln x)^2}$$

10.

$$u = x^{\frac{1}{14}}$$

$$原式 = 14\int \frac{u^{14}+u^{19}}{1+u^{15}}du = \frac{14}{15}\int \frac{1+u^5}{1+u^{15}}d(1+u^{15})$$

$$= \frac{14}{15}\int (1+u^5)\, d\ln(1+u^{15})$$

$$= \frac{14}{15}(1+u^5)\ln(1+u^{15}) - \frac{14}{15}\int \ln(1+u^{15})\, d(1+u^5)$$

$$= \frac{14}{15}(1+u^5)\ln(1+u^{15}) - \frac{14}{15}\int \ln(1+u^{15})\cdot du^5$$

$$= \frac{14}{15}(1+u^5)\ln(1+u^{15}) - \frac{14}{15}\int \ln(1+t^3)\,dt \qquad t=u^5$$

$$= \frac{14}{15}(1+u^5)\ln(1+u^{15}) - \frac{14}{15}\int$$

$$\int \ln(1+x^3)dx = \int \ln(1+x^3)(x)'dx = x\ln(1+x^3) - \int x\, d(\ln(1+x^3))$$

$$= x\ln(1+x^3) - \int \frac{x\cdot 3x^2}{1+x^3}dx$$

$$= x\ln(1+x^3) - 3\int \frac{x^3}{1+x^3}dx$$

$$= x\ln(1+x^3) - 3\int \frac{1}{1+x^3}dx$$

8.

$u=\ln x \quad x=e^u$

(3) $\int \cos(\ln x)dx = \int \cos u \cdot e^u du = \cos u e^u - \int e^u d\cos u$

$= \cos u e^u + \int \sin u e^u du$

$= \cos u e^u + \sin u e^u - \int e^u d\sin u$

$= \cos u e^u + \sin u e^u - \int \cos u e^u du$

故原式 $=\frac{1}{2}(\cos u e^u + \sin u e^u) + C$

(7) $\int x \arcsin x dx = \frac{1}{2}\int \arcsin x dx^2 = \frac{1}{2}x^2\arcsin x - \frac{1}{2}\int x^2 \cdot \frac{1}{\sqrt{1-x^2}} dx$

$= \frac{1}{2}x^2\arcsin x - \frac{1}{2}\int \sin^2 t \cdot \frac{1}{\cos t} \cdot \cos t \, dt$

$x=\sin t$ $\quad = \frac{1}{2}x^2\arcsin x - \frac{1}{2}\int \frac{1-\cos 2t}{2} dt$

$= \frac{1}{2}x^2\arcsin x - \frac{1}{4}(t - \frac{1}{2}\sin 2t) + C$

(9) $\int (\arcsin x)^2 dx = \int u^2 \cos u du = \int u^2 d\sin u$

$u=\arcsin x$ $\quad = u^2\sin u - \int \sin u du^2$

$x=\sin u$ $\quad = u^2 \sin u - 2\int u \sin u du$

$= u^2\sin u + 2\int u d\cos u$

$= u^2\sin u + 2u\cos u - 2\int \cos u du$

$= u^2\sin u + 2u\cos u - 2\sin u + C$

(10) $\int \sin^n x dx = \int \sin x \cdot \sin^{n-1} x dx = -\cos x \sin^{n-1} x + \int \cos x d\sin^{n-1} x$

$= -\cos x \sin^{n-1} x + (n-1)\int \cos^2 x \sin^{n-2} x dx$

$= -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x dx - (n-1)\int \sin^n x dx$

$n\int \sin^n x dx = -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x dx$ $\quad nI_n = -\cos x \sin^{n-1} x + (n-1)I_{n-2}$.

7. (2). $\int \frac{x^2-1}{x^4+x^2+1}dx = \int \frac{1-\frac{1}{x^2}}{x^2+1+\frac{1}{x^2}}dx = \int \frac{1}{(x+\frac{1}{x})^2-1}d(x+\frac{1}{x}) = \int \frac{1}{u^2-1}du = \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right|+C$

(6) $\int \frac{xe^x}{\sqrt{e^x-2}}dx = \int \frac{\ln u \cdot u}{\sqrt{u-2}} \cdot \frac{1}{u}du = \int \frac{\ln u}{\sqrt{u-2}}du$

$= 2\int \ln u \, d(u-2)^{\frac{1}{2}} = 2\ln u(u-2)^{\frac{1}{2}} - 2\int (u-2)^{\frac{1}{2}} \cdot d\ln u$

$= 2\ln u(u-2)^{\frac{1}{2}} - 2\int \frac{\sqrt{u-2}}{u}du$ $\quad u=m^2+2$

$= 2\ln u(u-2)^{\frac{1}{2}} - 2\int \frac{m}{m^2+2} \cdot 2m\,dm$ $\quad du=2m$

$= 2\ln u(u-2)^{\frac{1}{2}} - 4\int 1-\frac{2}{m^2+2}dm$

$= 2\ln u(u-2)^{\frac{1}{2}} - 4m + \frac{8}{\sqrt{2}}\arctan\frac{m}{\sqrt{2}} + C$

(7). $\int xe^x\sin x\,dx = x\sin xe^x - \int e^x dx\sin x$

$= x\sin xe^x - \int (\sin x + x\cos x)e^x dx$

$= x\sin xe^x - (\sin x + x\cos x)e^x + \int (\cos x + \cos x - x\sin x)e^x dx$

$= x\sin xe^x - (\sin x + x\cos x)e^x + \int 2\cos xe^x dx - \int x\sin xe^x dx \cdots ①$

又$\because \int 2\cos xe^x dx = 2\int \cos xe^x dx = 2[\cos xe^x + \int \sin xe^x dx]$

$= 2\cos xe^x + 2\sin xe^x - 2\int \cos xe^x dx$

故 $2\int \cos xe^x dx = \cos xe^x + \sin xe^x$

代回①得 $\int xe^x\sin x\,dx = \frac{1}{2}\left[(x\sin x - \sin x - x\cos x)e^x + \cos xe^x + \sin xe^x\right]$

(11) $\int \frac{x\arctan x}{(1+x^2)^3}dx$

$x=\tan u \qquad 1+x^2=1+\frac{\sin^2 u}{\cos^2 u}=\frac{1}{\cos^2 u}$

$$\int \frac{u\tan u}{\cos^{-6} u}\cdot\frac{1}{\cos^2 u}du$$

$$=\int u\cdot\tan u\cdot\cos^4 u\,du=\int u\cdot\sin u\cos^3 u\,du$$

$$=\frac{1}{2}\int u\sin 2u\cos^2 u\,du$$

$$=\frac{1}{2}\int u\sin 2u\,\frac{\cos 2u-1}{2}du$$

$$=\frac{1}{4}\int u\sin 2u\cos 2u-u\sin 2u\,du$$

$$=\frac{1}{8}\int u\sin 4u\,du-\frac{1}{4}\int u\sin 2u\,du$$

$$=\frac{1}{8}\left[\int u\,d\left(\frac{-\cos 4u}{4}\right)\right]-\frac{1}{4}\int u\,d\left(\frac{-\cos 2u}{2}\right)$$

$$=-\frac{1}{32}\left[u\cos 4u-\int\cos 4u\,du\right]$$

$$+\frac{1}{8}\left[u\cos 2u-\int\cos 2u\,du\right]$$

$$=\frac{1}{8}u\cos 2u-\frac{1}{32}u\cos 4u+\frac{1}{128}\sin 4u-\frac{1}{16}\sin 2u+C$$

(12)

$$\int \frac{x}{1+\sin x}dx \qquad t=\tan\frac{x}{2} \qquad x=2\arctan t$$

$$\sin x=\frac{2t}{1+t^2} \qquad dx=2\frac{1}{1+t^2}$$

$$4\int \frac{\arctan t}{1+\frac{2t}{1+t^2}}\cdot\frac{1}{1+t^2}dt=4\int\frac{\arctan t}{(t+1)^2}dt=4\int\frac{\arctan t}{(t+1)^2}d(t+1)$$

$$=-4\int \arctan t\, d(t+1)^{-1}=-4\left[\arctan t\,(1+t)^{-1}-\int (t+1)^{-1}\cdot\frac{1}{1+t^2}dt\right]$$

$$=-4\arctan t\cdot\frac{1}{1+t}+4\int\frac{1}{(1+t^2)(1+t)}dt$$

$$=-4\arctan t\cdot\frac{1}{1+t}+4\int\left(\frac{\frac{1}{2}}{1+t}-\frac{\frac{1}{2}t-\frac{1}{2}}{1+t^2}\right)dt$$

$$=-4\arctan t\cdot\frac{1}{1+t}+2\int\frac{1}{1+t}-\frac{t-1}{1+t^2}dt$$

$$=-4\arctan t\cdot\frac{1}{1+t}+2\ln|1+t|-\int\frac{2t}{1+t^2}-\frac{2}{1+t^2}dt$$

$$=-4\arctan t\cdot\frac{1}{1+t}+2\ln|1+t|-\ln(1+t^2)+2\arctan t$$

(18) $$\int\frac{\arctan e^x}{e^x}dx \qquad x=\ln u$$

$$=\int\frac{\arctan u}{u}\cdot\frac{1}{u}du=\int\frac{\arctan u}{u^2}du=-\int\arctan u\, d\frac{1}{u}$$

$$=-\left[\arctan u\cdot\frac{1}{u}-\int\frac{1}{u}\cdot d\arctan u\right]$$

$$=\int\frac{1}{u(1+u^2)}du-\frac{\arctan u}{u}$$

$$=\int\frac{1}{u}+\frac{-u}{1+u^2}du-\frac{\arctan u}{u}$$

$$=\ln|u|-\frac{1}{2}\int\frac{2u}{1+u^2}du-\frac{\arctan u}{u}$$

$$=\ln|u|-\frac{1}{2}\ln(1+u^2)-\frac{\arctan u}{u}+C$$

23. $$\int\frac{1}{\sqrt{\sqrt{x}+1}}dx=\int\frac{2u}{\sqrt{u+1}}du=4\int u\, d\sqrt{u+1}$$

$x=u^2$ $$=4u\sqrt{u+1}+4\int\sqrt{u+1}\,du=4u\sqrt{u+1}+\frac{8}{3}(u+1)^{\frac{3}{2}}+C$$