8. $\lim_{x\to\infty} f(x) = l$ $\forall \varepsilon>0$ $\exists X$ $\forall |x|>X$ $|f(x)-l|<\varepsilon$

令 $t=\frac{1}{x}$，$x\to 0$ 即 $t\to\infty$，对上述 X，一定 $\exists X_1$，当 $|x|<X_1$ 时，$\frac{1}{|x|}>X$

$\lim_{x\to 0} f(\frac{1}{x}) = \lim_{t\to 0} f(t)$，$|t|>X$，故 $|f(t)-l|<\varepsilon$

故 $\lim_{x\to 0} f(\frac{1}{x}) = l$，反之亦成立

当 $x\to+\infty$ 时，将 $\forall |x|>X$ 改为 $\forall x>X$

$x\to-\infty$ 时，将 $\forall |x|>X$ 改为 $\forall x<-X$

9.

(1) $\lim_{x\to 0}\frac{\tan 2x}{\sin 5x} = \lim_{x\to 0}\frac{2x}{5x} = \frac{2}{5}$

(2) $\cos 3x = \cos 2x\cos x - \sin 2x \sin x$

$= (2\cos^2 x-1)\cos x - 2\sin^2 x\cos x$

$= 2\cos^3 x - \cos x - 2\cos x + 2\cos^3 x$

$= 4\cos^3 x - 3\cos x$

$\frac{\cos x-\cos 3x}{x^2} = \frac{\cos x(1-4\cos^2 x+3)}{x^2}$

$\lim_{x\to 0}\frac{\cos x-\cos 3x}{x^2} = \lim_{x\to 0}\frac{4-4\cos^2 x}{x^2} = 4\lim_{x\to 0}\frac{\sin^2 x}{x^2} = 4$

10.

(1) 无穷小·有界=无穷小 $\lim_{x\to+\infty}\frac{\arctan x}{x}=0$

(2) 无穷小·有界=无穷小 $\lim_{x\to 0} x^2\sin\frac{1}{x}=0$

(3) $\lim_{x\to 2}\frac{x^3-2x^2}{x-2} = \lim_{x\to 2} x^2 = 4$

(4) $\lim_{x\to\infty} 2x^2-x+1 = +\infty$

11. (1) $\forall M>0$, $\exists X=a^M$ $\forall x>X$ $\log_a x = |\log_a x| > M$

(2) $\forall M>0$ $\exists \delta = a^{-M}$ $\forall 0<x<\delta$. $\log_a x < -M$

(3) $\forall M>0$ $\exists \delta = \frac{\pi}{2}-\arctan M$, $\forall \frac{\pi}{2}-\delta < x < \frac{\pi}{2}$ $\tan x > M$

(4) $\forall M>1$ $\exists \delta = \frac{1}{\ln M}$ $\forall 0<x<\delta$ $e^{\frac{1}{x}} > M$

12. $a_n = 2n\pi+\frac{\pi}{2}$ $y = 2n\pi+\frac{\pi}{2}$ 故 M 无界

$b_n = 2n\pi$ $y=0$ $n\to+\infty$ 时，y 并不是无穷大量

13. 令 $t=\frac{1}{x}$ $t\in(1,+\infty)$

$y=t\cdot\cos t$ 又 $x\to0^+$, $t\to+\infty$

$a_t=2t\pi$ $b_t=2t\pi+\frac{\pi}{2}$ $y=0$, 不是无穷大量

$y=2t\pi$, 无界.

14. 不妨 $x\to+\infty$, $C: f(x)$ $y=ax+b$

$|f(x)-(ax+b)|=k\cdot|MN|$ $\Longleftrightarrow$ $\lim\limits_{x\to+\infty}(f(x)-(ax+b))=0$

$\Rightarrow$

1° 当 $b=0$, $a=\lim\limits_{x\to+\infty}\frac{f(x)}{x}$

2° 当 $b\neq0$ $b=\lim\limits_{x\to+\infty}(f(x)-ax)$

综合1°.2°知 $a=\lim\limits_{x\to+\infty}\frac{f(x)}{x}$ $b=\lim\limits_{x\to+\infty}(f(x)-ax)$ □

$\Leftarrow$

$a=\lim\limits_{x\to+\infty}\frac{f(x)}{x}$ $b=\lim\limits_{x\to+\infty}(f(x)-ax)$

故 $\lim\limits_{x\to+\infty}(f(x)-(ax+b))=\lim\limits_{x\to+\infty}0=0$. □

(1) $y=x\ln(e+\frac{1}{x})$

设 $y=ax+b$ $a=\lim\limits_{x\to+\infty}\ln(e+\frac{1}{x})=1$ 或 $a=\lim\limits_{x\to-\infty}\ln(e+\frac{1}{x})=1$

$b=\lim\limits_{x\to+\infty}(x\ln(e+\frac{1}{x})-x)=0$ 或 $\lim\limits_{x\to-\infty}(x\ln(e+\frac{1}{x})-x)=0$

故 $y=x$ 为渐进线

(2) $y=\frac{3x^2-2x+3}{x-1}$ $a=\lim\limits_{x\to+\infty}\frac{3x-2+\frac{3}{x}}{x-1}=3$ 或 $a=\lim\limits_{x\to-\infty}\frac{3x-2+\frac{3}{x}}{x-1}=3$

$b=\lim\limits_{x\to+\infty}\frac{3x^2-2x+3}{x-1}-3x=\lim\limits_{x\to+\infty}\frac{x+3}{x-1}=1$

或 $b=\lim\limits_{x\to-\infty}\frac{x+3}{x-1}=1$

故 $y=3x+1$ 为其渐进线

15.

(1) $\lim \alpha(x) = \lim \alpha(x) \cdot \frac{\alpha(x)}{\alpha(x)} = \lim \alpha(x)$

(2) $\lim \alpha(x) = \lim \alpha(x) \frac{\beta(x)}{\alpha(x)} = \lim \beta(x) = \lim \beta(x) \cdot \frac{\alpha(x)}{\beta(x)} = \lim \alpha(x)$

(3) $\lim \alpha(x) = \lim \alpha(x) \frac{\beta(x)}{\alpha(x)} = \lim \beta(x) = \lim \beta(x) \frac{\gamma(x)}{\beta(x)} = \lim \gamma(x)$

16.

(1) $\lim\limits_{x\to 0} \frac{\tan x - \sin x}{x^3} = \lim\limits_{x\to 0} \frac{\tan x(1-\cos x)}{x^3} = \lim\limits_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}$ 同阶

(2) $\lim\limits_{x\to 0} \frac{x^3+x^2}{\sin^2 x} = \lim\limits_{x\to 0} \frac{x^3+x^2}{x^2} = \lim\limits_{x\to 0} 1+x = 1$ 等阶

(3) $\lim\limits_{x\to 0} \frac{1-\cos x}{x^2} = \lim\limits_{x\to 0} \frac{2\sin^2\frac{x}{2}}{4(\frac{x}{2})^2} = \frac{1}{2}$ 同阶

17.

(1) $\lim\limits_{x\to +\infty} \frac{P_n(x)}{P_m(x)} = \frac{x^n}{x^m} = x^{n-m}$

(2) $\lim\limits_{x\to +\infty} \frac{x^\alpha}{x^\beta} = x^{\alpha-\beta}$

(3) $\lim\limits_{x\to +\infty} \frac{a^x}{b^x} = (\frac{a}{b})^x$

补充：证明 $\lim\limits_{x\to 0} \frac{x}{\arctan x} = 1$

$\lim\limits_{x\to 0} \frac{\tan x}{x} = 1$

令 $x = \arctan t$，$t \to 0$.

$\lim\limits_{t\to 0} \frac{\tan(\arctan t)}{\arctan t} = \lim\limits_{t\to 0} \frac{t}{\arctan t} = 1$

18.

(1) $\lim\limits_{x\to 0} \frac{\sin mx}{\sin nx} = \frac{mx}{nx} = \frac{m}{n}$

(2) $\lim\limits_{x\to 0} \frac{\tan ax}{x} = a$

(3) $\lim\limits_{x\to 0} \frac{\sqrt[n]{1+\sin x} - 1}{\arctan x} = \lim\limits_{x\to 0} \frac{1+\frac{\sin x}{n} - 1}{\arctan x} = \lim\limits_{x\to 0} \frac{\sin x}{n \arctan x} = \frac{1}{n}$

(4) $\lim\limits_{x\to 0} \frac{\sqrt{2} - \sqrt{1+\cos x}}{\sin^2 x} = \lim\limits_{x\to 0} \frac{\frac{3}{2} - (1+\frac{1}{2}\cos x)}{x^2} = \lim\limits_{x\to 0} \frac{\frac{1}{2}(1-\cos x)}{x^2} = \frac{1}{4}$

(5) $\lim\limits_{x\to 0} \frac{\sqrt{1+x+x^2} - 1}{\sin 2x} = \lim\limits_{x\to 0} \frac{\frac{1}{2}x + \frac{1}{2}x}{2x} = \frac{1}{4}$

(6) $\lim\limits_{x\to 0} \frac{\sqrt{1+x^2} - 1}{1-\cos x} = \frac{\frac{1}{2}x^2}{1-\cos x} = 1$