17 (3). $\forall p \in \mathbb{N}_+$.

$$|a_{n+p}-a_n| = |\alpha_{n+1}q^{n+1}+\alpha_{n+2}q^{n+2}+\cdots+\alpha_{n+p}q^{n+p}|$$

$$\le |\alpha_{n+1}|\cdot|q|^{n+1}+|\alpha_{n+2}|\cdot|q|^{n+2}+\cdots+|\alpha_{n+p}|\,|q^{n+p}|$$

$$\le M\cdot\sum_{i=n+1}^{n+p-1}|q|^i < M\cdot\sum_{i=n+1}^{\infty}|q|^i = M\cdot\frac{|q|^{n+1}}{1-|q|} < \varepsilon$$

由 Cauchy 收敛准则知 $\{a_n\}$ 收敛

17.

(4). $\forall p \in \mathbb{N}_+$. $a_{n+p}-a_n = \frac{\cos n}{(n+1)(n+2)}+\frac{\cos(n+1)}{(n+2)(n+3)}+\cdots+\frac{\cos(n+p)}{(n+p)(n+p+1)}$

$$|a_{n+p}-a_n| \le \frac{1}{(n+1)(n+2)}+\frac{1}{(n+2)(n+3)}+\cdots+\frac{1}{(n+p)(n+p+1)} = \frac{1}{n+1}-\frac{1}{n+p+1}$$

$$< \frac{1}{n+1} < \frac{1}{n}$$

$\forall \varepsilon>0$. 取 $N=[\frac{1}{\varepsilon}]$ 当 $n>N$ 时. $\forall p$. 有 $|a_{n+p}-a_n|<\varepsilon$

故 a_n 收敛 □

18.

(1) $a_n=\frac{n}{c^n}\ (c>1)$

$\because \lim\limits_{n\to\infty}\frac{(n+1)-n}{c^{n+1}-c^n} = \lim\limits_{n\to\infty}\frac{1}{c^n(c-1)}=0$

由 Stolz定理 知 $\lim\limits_{n\to\infty}\frac{n}{c^n}=0$

(2). 先证 $a_n\le c$. $0\le c\le 1$

数学归纳法. $a_1=\frac{c}{2}<c$ 成立.

假设 $n=k$ 时, $a_k\le c$

$n=k+1$ 时 $a_{k+1}=\frac{c}{2}+\frac{a_k^2}{2}=\frac{c}{2}+\frac{c^2}{2}\le\frac{c}{2}+\frac{c}{2}=c$ □

再证单增. 数学归纳法. 显然 $a_2>a_1$ 当 $n=k$ 时, 有 $a_{k+1}>a_k$

当 $n=k+1$ 时, 有 $a_{k+2}=\frac{c}{2}+\frac{a_{k+1}^2}{2}>\frac{c}{2}+\frac{a_k^2}{2}=a_{k+1}$ 成立 □.

故数列 $\{a_n\}$ 收敛. 设 $\lim\limits_{n\to\infty}a_n=a$ 则 $a=\frac{c}{2}+\frac{a^2}{2}\Rightarrow a=1-\sqrt{1-c}=\lim\limits_{n\to\infty}a_n$

23.

反证. 设 $\lim_{n\to\infty} a_n b_n = a$

则 $\forall \varepsilon>0$. $\exists N\in\mathbb{N}$ $\forall n>N$. $|a_n b_n - a|<\varepsilon$

即 $|a_n b_n|<\varepsilon+|a|$

又∵ $\lim_{n\to\infty} a_n=\infty$ 则令 $M=\dfrac{|a|+2\varepsilon}{b}$. $\exists N_2\in\mathbb{N}$ $\forall n>N_2$ 有 $|a_n|>M$

此时 $|a_n b_n| \geq \dfrac{|a|+2\varepsilon}{b}\cdot b = |a|+2\varepsilon > |a|+\varepsilon$ 矛盾!

故 $\lim_{n\to\infty} a_n b_n = \infty$ □

法一:

24. 引理: Cauchy II. $\lim_{n\to\infty}\sqrt[n]{a_n}=\frac{a_n}{a_{n-1}}$ 假设 a_n 收敛

则 $\lim_{n\to\infty}\sqrt[n]{n!}=\lim_{n\to\infty}\dfrac{n!}{(n-1)!}=\lim_{n\to\infty} n=+\infty$ 故其无上界, 趋于无穷大

法二: $\sqrt[n]{n!}=e^{n\ln n!} > e^{n}$ $(n\geq 3)$ 发散, 趋于无穷大.

再考虑 $n\sin\frac{n\pi}{2}$

当 $n=2k$ 时 $n\sin\frac{n\pi}{2}=2k\sin k\pi=0$

当 $n=4k+1$ 时 $n\sin\frac{n\pi}{2}=(4k+1)\sin(\frac{4k+1}{2})\pi=4k+1 \to +\infty$

当 $n=4k+3$ 时 $n\sin\frac{n\pi}{2}=(4k+3)\sin(\frac{4k+3}{2})\pi=-(4k+3)\to -\infty$

故当 $n\to\infty$ 时 $n\sin\frac{n\pi}{2}$ 无界, 但并不发散到无界大

25. ∵ $a_1=1$ 故 $a_n>0$. $\frac{1}{a_n}>0$

故 $a_{n+1}-a_n=\dfrac{1}{a_n}>0$ $\{a_n\}$ 单调递增.

反证. 设 $\{a_n\}$ 收敛. $\lim_{n\to\infty} a_n = a$ $(a>0)$

故 $\lim_{n\to\infty} a_{n+1}=\lim_{n\to\infty} a_n+\dfrac{1}{a_n} = a+\dfrac{1}{a}=a \Rightarrow \dfrac{1}{a}=0$

又∵ $a>0$ 故 $\frac{1}{a}>0$ 矛盾!

4. $a_n=\sqrt{n}$,

当 $n>N$ 时 $\left|\sqrt{n+1}-\sqrt{n}\right|=\left|\frac{1}{\sqrt{n+1}+\sqrt{n}}\right|<\varepsilon$

5.

$A_{n+1}-A_n=|a_{n+2}-a_{n+1}|\geq 0$ 单增

$0<A_n\leq M$ 有界 故 $\{A_n\}$ 收敛

Cauchy 收敛 $|a_{n+p}-a_n|\leq|a_{n+1}-a_n|+\cdots+|a_{n+p}-a_{n+p-1}|$

$=|A_{n+p-1}-A_{n-1}|<\varepsilon$

故 $\{a_n\}$ 也收敛.

6.

$a_{n+1}^{\alpha}-a_n^{\alpha}=(a_{n+1}-a_n)(a_{n+1}^{\alpha-1}+a_{n+1}^{\alpha-2}a_n+\cdots+a_{n+1}a_n^{\alpha-2}+a_n^{\alpha-1})$

$<(a_{n+1}-a_n)(a_{n+1}^{\alpha-1}\cdot\alpha)$ ---①

情况一. 若 a_n 收敛 则 $n>N$ 时 $a_{n+1}-a_n<\varepsilon$ $|a_n|<M$

① $<M\cdot\varepsilon$ $\Rightarrow$ $\lim\limits_{n\to\infty}(a_{n+1}^{\alpha}-a_n^{\alpha})=0$

情况二: 若 a_n 发散. 则 $n\to+\infty$ 时 $a_n\to+\infty$ $a_{n+1}^{\alpha-1}\to 0$,

故 $n>N_1$ 时 $a_{n+1}^{\alpha-1}<\varepsilon$ $(a_{n+1}-a_n)\leq T$

① $<(a_{n+1}-a_n)\cdot\alpha\cdot\varepsilon$ $<K\cdot\varepsilon$ $\Rightarrow\lim\limits_{n\to\infty}(a_{n+1}^{\alpha}-a_n^{\alpha})=0.$ □

易证 逆命题不对.

7. 令 $b_{n+1}=a_{n+1}-a_n$　$b_1=a_1$，$\lim\limits_{n\to\infty} b_{n+1}=a$

则 $a_n=b_n+b_{n-1}+\cdots+b_1$

故 $\lim\limits_{n\to\infty}\frac{a_n}{n}=\lim\limits_{n\to\infty}\frac{b_1+b_2+\cdots+b_n}{n}=a$

8. $$\frac{n}{\frac{1}{a_1}+\cdots+\frac{1}{a_n}}<\sqrt[n]{a_1a_2\cdots a_n}<\frac{a_1+a_2+\cdots+a_n}{n}$$

又∵ $\lim\limits_{n\to\infty}a_n=a \Rightarrow \lim\limits_{n\to\infty}\frac{a_1+a_2+\cdots+a_n}{n}=a$

$\Rightarrow \lim\limits_{n\to\infty}\frac{\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}}{n}=\frac{1}{a} \Rightarrow \lim\limits_{n\to\infty}\frac{n}{\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}}=a$

再由夹逼定理得. $\lim\limits_{n\to\infty}\sqrt[n]{a_1a_2\cdots a_n}=a$

9.

$\lim\limits_{n\to\infty}b_n=b$　$\lim\limits_{n\to\infty}\sqrt[n]{b_1b_2\cdots b_n}=b$

此时令 $b_1=a_1$　$b_n=\frac{a_n}{a_{n-1}}$

则 $\lim\limits_{n\to\infty}\sqrt[n]{b_1b_2\cdots b_n}=\lim\limits_{n\to\infty}\sqrt[n]{a_n}=b=\lim\limits_{n\to\infty}b_n=\lim\limits_{n\to\infty}\frac{a_n}{a_{n-1}}=\lim\limits_{n\to\infty}\frac{a_{n+1}}{a_n}$

故 $\lim\limits_{n\to\infty}\sqrt[n]{a_n}=\lim\limits_{n\to\infty}\frac{a_{n+1}}{a_n}$

补充:

10(1). $\lim_{n\to\infty} \frac{1+\sqrt{2}+\sqrt[3]{3}+\cdots+\sqrt[n]{n}}{n}$

Stolz定理 $\Leftrightarrow$ $\lim_{n\to\infty} \frac{\sqrt[n]{n}}{n-(n-1)} = \lim_{n\to\infty}\sqrt[n]{n} = 1$

10.(2) 证明

令 $a_n = \frac{n^n}{n!}$

则 $\lim_{n\to\infty}\sqrt[n]{a_n} = \lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{\frac{(n+1)^{n+1}}{(n+1)!}}{\frac{n^n}{n!}} = \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$

说明: $= e$

引理:

$\lim_{n\to\infty} b_n = b$ $\quad \lim_{n\to\infty}\sqrt[n]{b_1 b_2\cdots b_n} = b$

此时令 $b_1 = a_1$ $\quad b_n = \frac{a_n}{a_{n-1}}$

则 $\lim_{n\to\infty}\sqrt[n]{b_1 b_2\cdots b_n} = \lim_{n\to\infty}\sqrt[n]{a_n} = b = \lim_{n\to\infty} b_n = \lim_{n\to\infty}\frac{a_n}{a_{n-1}} = \lim_{n\to\infty}\frac{a_{n+1}}{a_n}$

补充:

11. Stolz定理

$\lim_{n\to\infty}\frac{n a_n}{2n-1} = \lim_{n\to\infty}\frac{n}{2n-1}\cdot\lim_{n\to\infty} a_n = \frac{a}{2}$

12. (1)

情况一: 令 $B_n = \sum_{i=1}^{n} b_i$ 若 B_n 有界 设 $|B_n| \le M$

令 $a_n = a + \alpha_n$ 则 $C_n = \frac{\sum_{k=1}^{n} a_k b_k}{B_n} = \frac{a\sum_{k=1}^{n} b_k + \sum_{k=1}^{n}\alpha_k b_k}{B_n} = a + \frac{\sum_{k=1}^{n}\alpha_k b_k}{B_n}$

Abel变换: 令 $B_0 = 0$. 则 $\sum_{k=1}^{n}\alpha_k b_k = \sum_{k=1}^{n}\alpha_k(B_k - B_{k-1}) = \sum_{k=1}^{n-1}\alpha_k(B_k - B_{k+1}) + \alpha_n B_n$

Cauchy收敛准则 令 $E_n = \sum_{k=1}^{n}\alpha_k b_k$ $\quad |E_{n+p} - E_n| = \left|\sum_{k=n+1}^{n+p}\alpha_k b_k\right|$

$= \left|\sum_{k=n+1}^{n+p}\alpha_k(B_k - B_{k-1})\right|$

$< \left|\sum_{k=n+1}^{n+p}|\alpha_k|(B_k - B_{k-1})\right|$

$< \varepsilon\left|\sum_{k=n+1}^{\infty}(B_k - B_{k-1})\right|$

$= \varepsilon\cdot(B_\infty - B_n) < \varepsilon\cdot B_\infty < T\cdot\varepsilon$ 常数 □

Ex I: 求 $\lim\limits_{n\to\infty} \dfrac{1^2+2^2\sqrt[2]{2}+\cdots+n^2\sqrt[n]{n}}{n^3}$

考虑 stolz 定理. $\Longleftrightarrow$ $\lim\limits_{n\to\infty} \dfrac{n^2\sqrt[n]{n}}{n^3-(n-1)^3} = \lim\limits_{n\to\infty} \dfrac{n^2\sqrt[n]{n}}{3n^2-3n+1}$

$= \lim\limits_{n\to\infty} \dfrac{n^2}{3n^2-3n+1} \cdot \lim\limits_{n\to\infty}\sqrt[n]{n}$

$= \dfrac{1}{3}\cdot 1 = \dfrac{1}{3}$

Ex II.

$\lim\limits_{n\to\infty} \dfrac{1+\sqrt{2}+\cdots+\sqrt{n}}{n\sqrt{n}}$ --- ①

stolz 定理. ① $\Longleftrightarrow$ $\lim\limits_{n\to\infty} \dfrac{\sqrt{n}}{n\sqrt{n}-(n-1)\sqrt{n-1}} = \dfrac{\sqrt{n}\,(n\sqrt{n}+(n-1)\sqrt{n-1})}{(n\sqrt{n}-(n-1)\sqrt{n-1})\,(n\sqrt{n}+(n-1)\sqrt{n-1})}$

$= \dfrac{n^2+(n-1)\sqrt{n-1}\sqrt{n}}{n^3-(n-1)^3} = \dfrac{n^2+(n-1)\sqrt{n-1}\sqrt{n}}{3n^2-3n+1}$

又 $2(n-1)^2 < n^2+(n-1)\sqrt{n-1}\sqrt{n} < 2n^2$

故 $\underbrace{\dfrac{2(n-1)^2}{3n^2-3n+1}}_{\to\frac{2}{3}} < \dfrac{n^2+(n-1)\sqrt{n-1}\sqrt{n}}{3n^2-3n+1} < \dfrac{2n^2}{3n^2-3n+1} \to \dfrac{2}{3}$

夹逼定理 $\Rightarrow$ $\lim\limits_{n\to\infty}$ ① $= \dfrac{2}{3}$