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(1) $y'=-2xe^{-x^2}$

$y''=-2e^{-x^2}+(-2x)\cdot(-2x)e^{-x^2}$

$=(4x^2-2)e^{-x^2}$

(2) $y'=2x\cdot 2^x+\ln 2\cdot 2^x\cdot x^2$

$y''=2^{x+1}+\ln 2\cdot 2^x\cdot 2x+\ln 2\left(2x\cdot 2^x+\ln 2\cdot 2^x\cdot x^2\right)$

(3)

$y'=2x\arctan x+\frac{1+x^2}{1+x^2}$

$y''=2\arctan x+\frac{2x}{1+x^2}$

(4) $y'=\begin{cases}2x & x\geq 0\\ -2x & x<0\end{cases}$

$y''=\begin{cases}2 & x\geq 0\\ -2 & x<0\end{cases}$

19.

(1) $y'=2xf'(x^2)$

$y''=4xf''(x^2)$

$y'''=8xf'''(x^2)$

(2) $y=(e^x+1)f'(e^x+x)$

$y''=(e^x+1)^2f''(e^x+x)$

$y'''=(e^x+1)^3f'''(e^x+x)$

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$f'(x)=\begin{cases}nx^n & x>0\\ 0 & x=0\\ -nx^n & x<0\end{cases}$

法二：$f(x)=x^n|x|=\frac{x^{n+1}}{\sqrt{x^2}}$

$f^{(n-1)}(x)=\begin{cases}n!\,x^2/2 & x>0\\ 0 & x=0\\ -n!\,x^2/2 & x<0\end{cases}$

$f_+^{n}(0)=\lim\limits_{\Delta x\to 0^+}\frac{f^{(n-1)}(\Delta x)-0}{\Delta x}=\frac{n!\,(\Delta x)^2}{\Delta x}=n!(\Delta x)=0=f_-'(0)$

故 $f^{(n)}(x)=\begin{cases}n!\,x & x>0\\ 0 & x=0\\ -n!\,x & x<0\end{cases}$

$f_+^{(n+1)}(0)=n!\neq -n!=f_-^{(n+1)}(0)$

21.

(1) $y=x^2e^x$

$y^{(n)}=C_n^0(x^2)^{(n)}e^x+C_n^1(x^2)^{(n-1)}e^x+\cdots+C_n^n(x^2)e^x$

$=(x^2+2nx+2C_n^2)\cdot e^x$

(2) $y=(x^2+1)\sin x$

$y^{(n)}=C_n^n(x^2+1)(\sin x)^n+C_n^{n-1}(2x)(\sin x)^{n-1}+C_n^{n-2}\,2\,(\sin x)^{n-2}$

$=(x^2+1)\sin(x+\frac{n\pi}{2})+2nx\cdot\sin(x+\frac{n-1}{2}\pi)+n(n-1)\cdot\sin(x+\frac{n-2}{2}\pi)$

(3) $y=\left(\frac{1}{x-1}\right)\left(\frac{1}{x-2}\right)$

$$y^{(n)}=C_n^0\left(\frac{1}{x-1}\right)^{(n)}\left(\frac{1}{x-2}\right)+\cdots+C_n^n\left(\frac{1}{x-1}\right)\cdot\left(\frac{1}{x-2}\right)$$

$$=C_n^0\left[(x-1)^{-1}\right]^{(n)}\left(\frac{1}{x-2}\right)+\cdots+C_n^n\left[(x-1)^{-1}\right]\left[(x-2)^{-1}\right]^{(n)}$$

$$=(-1)^n n!\,(x-1)^{-(n+1)}(x-2)^{-1}+\cdots+C_n^k(-1)^k(x-2)^{-(k+1)}\cdot(k!)\cdot(n-k)!\,(-1)^{n-k}\cdot(x-1)^{k-n}$$

$$+\cdots+(x-1)^{-1}\cdot(-1)^n n!\,(x-2)^{-(n+1)}$$

$$=(-1)^n\sum_{i=0}^{n}C_n^i(x-1)^{-(n+1-i)}(x-2)^{-(i+1)}\cdot(i)!\cdot(n-i)!$$

(4) $y=\sin x\cos x$

$$y^{(n)}=C_n^0(\sin x)^{(n)}\cos x+\cdots+C_n^k(\sin x)^{(n-k)}(\cos x)^{(k)}+\cdots+C_n^n\sin x(\cos x)^{(n)}$$

$$=\sin\left(x+\frac{n\pi}{2}\right)\cos x+\cdots+C_n^k\sin\left(x+\frac{n-k}{2}\pi\right)\cos\left(x+\frac{k}{2}\pi\right)$$

$$+\cdots+\sin x\cos\left(x+\frac{n\pi}{2}\right)$$

$$=\sin\left(x+\frac{n\pi}{2}\right)\cos x+\cdots+C_n^k\cdot\frac{1}{2}\left[\sin\left(2x+\frac{n}{2}\pi\right)+\sin\left(\frac{n-2k}{2}\right)\pi\right]$$

$$+\cdots+\sin x\cos\left(x+\frac{n\pi}{2}\right)$$

$$=\frac{1}{2}\sin\left(2x+\frac{n}{2}\pi\right)\left[C_n^0+C_n^1+\cdots+C_n^k+\cdots+C_n^n\right]+\sum_{k=0}^{n}C_n^k\sin\left(\frac{n-2k}{2}\right)\pi$$

$$=2^{n-1}\sin\left(2x+\frac{n}{2}\pi\right)+\sum_{k=0}^{n}C_n^k\sin\left(\frac{n}{2}-k\right)\pi$$

22. $y'=-\sin x$

$y'\left(\frac{\pi}{4}\right)=-\frac{\sqrt{2}}{2}$ $\quad y\left(\frac{\pi}{4}\right)=\frac{\sqrt{2}}{2}$

切线方程为 $y-\frac{\sqrt{2}}{2}=-\frac{\sqrt{2}}{2}\left(x-\frac{\pi}{4}\right)$

23. $y=\frac{1}{x}$ $\quad y'=\frac{-1}{x^2}$ $\quad y-\frac{1}{x_0}=-\frac{1}{x_0^2}(x-x_0)$

即 $y=-\frac{1}{x_0^2}x+\frac{2}{x_0}$ 过 $(2x_0,0)$ 和 $\left(0,\frac{2}{x_0}\right)$ 面积为 2.

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(1) $$\frac{dy}{dx}=\frac{dy}{dt}\cdot\frac{dt}{dx}=\frac{1-\frac{1}{1+t^2}}{\frac{2t}{1+t^2}}=\frac{1+t^2-1}{2t}=\frac{t}{2}$$

$$\frac{d^2y}{dx^2}=\frac{1}{2}\cdot\frac{1+t^2}{2t}=\frac{1+t^2}{4t}$$

(2). $$\frac{dy}{dx}=\frac{\sin t}{1-\cos t}=\tan\frac{t}{2}$$

$$\frac{d^2y}{dx^2}=\frac{1}{\cos^2\frac{t}{2}}\cdot\frac{1}{1-\cos t}=\frac{1}{\cos^2\frac{t}{2}\cdot 2\sin^2\frac{t}{2}}=\frac{2}{\sin^2 t}$$

(3) $$\frac{dy}{dx}=\frac{\sin\varphi+\varphi\cos\varphi}{\cos\varphi-\varphi\sin\varphi}=\frac{\tan\varphi+\varphi}{1-\varphi\tan\varphi}$$

$$\frac{d^2y}{dx}=\frac{\left[(\frac{1}{\cos\varphi})^2+1\right](1-\varphi\tan\varphi)-\left[-\tan\varphi-\frac{\varphi}{\cos^2\varphi}\right](\tan\varphi+\varphi)}{(1-\varphi\tan\varphi)^2}\cdot\frac{1}{\cos\varphi-\varphi\sin\varphi}$$

(4) $$\frac{dy}{dx}=\frac{3\sin^2\varphi\cdot\cos\varphi}{3\cos^2\varphi\cdot(-\sin\varphi)}=-\tan\varphi$$

$$\frac{d^2y}{dx^2}=-\frac{1}{\cos^2\varphi}\cdot\frac{1}{3\cos^2\varphi(-\sin\varphi)}=\frac{1}{3\cos^4\varphi\sin\varphi}$$

4.

(1) $$\frac{dy}{dx}=\frac{\cos t}{-\sin t}=-\cot t$$

l: $y-\sin t=-\cot t\,(x-\cos t)$

即 $y-\frac{\sqrt{2}}{2}=-(x-\frac{\sqrt{2}}{2})$ 即 $y+x-\sqrt{2}=0$

(2) $$\frac{dy}{dx}=\frac{\frac{6t(1+t^2)-2t\times3t^2}{(1+t^2)^2}}{\frac{3(1+t^2)-2t\cdot3t}{(1+t^2)^2}}=\frac{6t}{3(1-t^2)}=\frac{2t}{1-t^2}$$

l: $y-\frac{12}{5}=-\frac{4}{3}(x-\frac{6}{5})$ 即 $3y+4x-12=0$

习题 3.3

1. $f'(x) = (x-2)(x-3)(x-4) + (x-1)(x-3)(x-4)$
$+ (x-1)(x-2)(x-4) + (x-1)(x-2)(x-3)$

$f'(1)<0 \quad f'(2)>0 \quad f'(3)<0 \quad f'(4)>0$

故 $(1,2)$, $(2,3)$, $(3,4)$ 中各有一零点

又由 $f(x)$ 是三次多项式 至多有三个实根, 故

共有3个实根, 分别分布在 $(1,2)$, $(2,3)$, $(3,4)$

6. 令 $F(x) = f(x) - x$

$\because F(0) = f(0) - 0 > 0$ $\quad f(x) \in (0,1)$

$F(1) = f(1) - 1 < 0$

由零值定理 $\Rightarrow \exists \xi \quad F(\xi) = 0$ 即 $f(\xi) = \xi$

Largrange 中值定理:

$f(x_1) - f(x_2) = f'(\xi)(x_1 - x_2) \neq (x_1 - x_2)$

故若 $\exists\ a, b \in [0,1]$

$F(a) = 0 = F(b)$ 则 $f(a) - f(b) = a - b$ 矛盾

综上, 在 $(0,1)$ 内有且仅有一个 $x = \xi$, 使 $f(x) = x$

$\square$