习题1.3

1. (1) $\forall \varepsilon>0$，取 $X=\log_a \varepsilon$，$\forall x<X$，有 $|a^x|<\varepsilon$

故 $\lim\limits_{x\to-\infty} a^x=0$. □

(2) $\forall \varepsilon>0$ 取 $X=\frac{2}{\varepsilon}-1$ $\forall |x|>X$，有 $\left|\frac{x-1}{x+1}-1\right|=\left|\frac{2}{x+1}\right|<\varepsilon$ □

(3)

$\forall \varepsilon>0$ 不妨 $|x+1|<\frac{1}{2}$ 则 $-\frac{3}{2}<x<-\frac{1}{2}$，$|x|>\frac{1}{2}$

取 $\delta=\min\left\{\frac{1}{2},\frac{\varepsilon}{2}\right\}$，对于 $\forall\ 0<|x+1|<\delta$

有 $\left|\frac{x^2-1}{x^2+x}-2\right|=\left|\frac{-x^2-2x-1}{x^2+x}\right|=\left|\frac{x+1}{x}\right|<2|x+1|<2\cdot\frac{\varepsilon}{2}=\varepsilon$ □

(4)

$\forall \varepsilon>0$. 取 $\delta=\varepsilon^q$ $\forall\ 0<x<\delta$

有 $\left|x^{\frac{1}{q}}-0\right|<\varepsilon$ □

2.

(1) $\lim\limits_{x\to1}\left(x^5-5x+2+\frac{1}{x}\right)=\lim\limits_{x\to1}\frac{x^6-5x^2+2x+1}{x}=\frac{\lim\limits_{x\to1}x^6-5x^2+2x+1}{\lim\limits_{x\to1}x}=\frac{-1}{1}=-1$

(2) $\lim\limits_{x\to1}\frac{x^n-1}{x-1}=\lim\limits_{x\to1}\left(1+x+\cdots+x^{n-1}\right)=n$

(3) $\lim\limits_{x\to1}\frac{(x-1)(x+1)}{(x-1)(2x+1)}=\lim\limits_{x\to1}\frac{x+1}{2x+1}=\frac{2}{3}$

(4) $\lim\limits_{x\to1}\frac{(3x+6)^{70}}{(5x-1)^{70}}\cdot\frac{(8x-5)^{20}}{(5x-1)^{20}}=\left(\lim\limits_{x\to1}\frac{3x+6}{5x-1}\right)^{70}\cdot\left(\lim\limits_{x\to1}\frac{8x-5}{5x-1}\right)^{20}=\left(\frac{9}{4}\right)^{70}\cdot\left(\frac{3}{4}\right)^{20}=\frac{3^{160}}{4^{90}}$

3.(补充)

(1). $a_n=2n\pi$ $b_n=2n\pi+\frac{\pi}{2}$

$\lim\limits_{n\to+\infty}a_n=0\neq\lim\limits_{n\to\infty}b_n=1$

故 $\lim\limits_{x\to+\infty}\sin x$ 无极限 □

(2) $\lim\limits_{x\to0^+}\frac{|x|}{x}=1$

$\neq\lim\limits_{x\to0^-}\frac{|x|}{x}=-1$

故 $\lim\limits_{x\to0}\frac{|x|}{x}$ 无极限 □

4. $\lim\limits_{x\to+\infty} f(x)=l$ $\quad\forall\varepsilon>0\ \ \exists X,\ \forall x>X$ 时 $|f(x)-l|<\varepsilon$

对上述 ε，取 $N_1\in N_+$，当 $n>N_1$ 时，有 $a_n>X$

$|f(a_n)-l|<\varepsilon$ 故 $\lim\limits_{n\to\infty} f(a_n)=l$

取 $a_n=n$，满足上述条件，故 $\lim\limits_{n\to\infty} f(n)=l$

5. (1) $\lim\limits_{x\to0^+}[x]=0$ $\quad\lim\limits_{x\to0^-}[x]=-1$ 故 $f(x)=[x]$ 在 $x=0$ 处无极限

(2) $\lim\limits_{x\to0^+}\operatorname{sgn}x=1$ $\quad\lim\limits_{x\to0^-}\operatorname{sgn}x=-1$ 故 $f(x)=\operatorname{sgn}x$ 在 $x=0$ 处无极限

(3) $\lim\limits_{x\to0^+}f(x)=1$ $\quad\lim\limits_{x\to0^-}f(x)=1$ 故 $\lim\limits_{x\to0}f(x)=1$

(4) 考虑 $\lim\limits_{x\to0^+}\cos\frac1x$ $\quad$ 取 $a_n=\frac{1}{2n\pi}$ $\quad b_n=\frac{1}{2n\pi+\frac{\pi}{2}}$

$\lim\limits_{n\to\infty}a_n=-1\ne\lim\limits_{n\to\infty}b_n=0$ $\quad$ 故 $\lim\limits_{x\to0^+}\cos\frac1x$ 无极限

故 $\lim\limits_{x\to0^+}f(x)$ 无极限

6. $\cos\frac x2\cos\frac{x}{2^2}\cdots\cos\frac{x}{2^n}=\dfrac{\cos\frac x2\cdots\cos\frac{x}{2^n}\sin\frac{x}{2^n}}{\sin\frac{x}{2^n}}=\dfrac{\sin x}{2^n\sin\frac{x}{2^n}}$

考虑 $\lim\limits_{n\to\infty}2^n\sin\frac{x}{2^n}=\lim\limits_{n\to\infty}2^n\cdot\frac{x}{2^n}\cdot\frac{2^n}{x}\sin\frac{x}{2^n}=x\cdot\lim\limits_{n\to\infty}\frac{2^n}{x}\sin\frac{x}{2^n}$

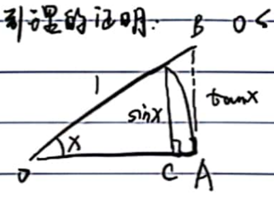
令 $\frac1t=\frac{2^n}{x}$ $\quad\lim\limits_{n\to\infty}\frac{2^n}{x}\sin\frac{x}{2^n}=\lim\limits_{t\to0}\frac1t\sin t=1$ 故 $\lim\limits_{n\to\infty}2^n\sin\frac{x}{2^n}=x$

(引理I)

故 $\lim\limits_{n\to\infty}\cos\frac x2\cos\frac{x}{2^2}\cdots\cos\frac{x}{2^n}=\dfrac{\sin x}{x}$

引理I：$\lim\limits_{x\to0}\dfrac{\sin x}{x}=1$

引理的证明：$0<x<\frac{\pi}{2}$ 时，由 $S_{\triangle AOD}<S_{扇AOD}<S_{\triangle ABO}$ 知



$\frac{\sin x}{2}<\frac{x}{2}<\frac{\tan x}{2}\Rightarrow\sin x<x<\tan x=\frac{\sin x}{\cos x}$

则 $\cos x<\frac{\sin x}{x}<1$

且 $\lim\limits_{x\to0}\cos x=1$ $\quad$ 故 $\lim\limits_{x\to0}\frac{\sin x}{x}=1$

现取 $t=\frac1x$ 故 $\lim\limits_{t\to\infty}t\sin\frac1t=1$

扫描全能王 创建

7.

当$\alpha=0$时，显然成立，下证$\alpha\neq0$时成立.

$$2\left(\sin\frac{\alpha}{n^2}+\cdots+\sin\frac{n\alpha}{n^2}\right)=\left(\sin\frac{\alpha}{n^2}+\sin\frac{n\alpha}{n^2}\right)+\cdots+\left(\sin\frac{n\alpha}{n^2}+\sin\frac{\alpha}{n^2}\right)$$ # 逆序配对

$$=2\left[\sin\frac{(n+1)\alpha}{2n^2}\cos\frac{(n-1)\alpha}{2n^2}+\sin\frac{(n+1)\alpha}{2n^2}\cos\frac{(n-3)\alpha}{2n^2}+\cdots+\sin\frac{(n+1)\alpha}{2n^2}\cos\frac{(1-n)\alpha}{2n^2}\right]$$

故 $$\sin\frac{\alpha}{n^2}+\cdots+\sin\frac{n\alpha}{n^2}=\left[\sin\frac{(n+1)\alpha}{2n^2}\cos\frac{(n-1)\alpha}{2n^2}+\sin\frac{(n+1)\alpha}{2n^2}\cos\frac{(n-3)\alpha}{2n^2}+\cdots+\sin\frac{(n+1)\alpha}{2n^2}\cos\frac{(1-n)\alpha}{2n^2}\right]$$

$$=\sin\frac{(n+1)\alpha}{2n^2}\left[\cos\frac{n-1}{2n^2}\alpha+\cdots+\cos\frac{1-n}{2n^2}\alpha\right]$$

考虑 $$\lim_{n\to\infty}\sin\frac{(n+1)\alpha}{2n^2}\left[\cos\frac{n-1}{2n^2}\alpha+\cdots+\cos\frac{1-n}{2n^2}\alpha\right]=\lim_{n\to\infty}\sin\frac{(n+1)\alpha}{2n^2}\cdot[1+1+\cdots+1]=\lim_{n\to\infty}n\sin\frac{(n+1)\alpha}{2n^2}$$

$$=\lim_{n\to\infty}n\cdot\sin\left(\frac{n\alpha}{2n^2}+\frac{\alpha}{2n^2}\right)=\lim_{n\to\infty}n\cdot\sin\left(\frac{\alpha}{2n}+\frac{\alpha}{2n^2}\right)=\lim_{n\to\infty}n\cdot\left[\sin\frac{\alpha}{2n}\cos\frac{\alpha}{2n^2}+\sin\frac{\alpha}{2n^2}\cos\frac{\alpha}{2n}\right]$$

$$=\lim_{n\to\infty}\left(n\sin\frac{\alpha}{2n}\cos\frac{\alpha}{2n^2}+n\sin\frac{\alpha}{2n^2}\cos\frac{\alpha}{2n}\right)=\lim_{n\to\infty}n\sin\frac{\alpha}{2n}\cos\frac{\alpha}{2n^2}+\lim_{n\to\infty}n\sin\frac{\alpha}{2n^2}\cos\frac{\alpha}{2n}$$

$$=\lim_{n\to\infty}n\sin\frac{\alpha}{2n}+\lim_{n\to\infty}n\cdot\sin\frac{\alpha}{2n^2}=\frac{1}{2}\lim_{n\to\infty}2n\sin\frac{\alpha}{2n}+\lim_{n\to\infty}\frac{1}{2n}\cdot2n^2\sin\frac{\alpha}{2n^2}$$

$$=\frac{1}{2}\lim_{n\to\infty}2n\sin\frac{\alpha}{2n}+\lim_{n\to\infty}\frac{1}{2n}\cdot\lim_{n\to\infty}2n^2\sin\frac{\alpha}{2n^2}$$

$$=\frac{1}{2}\lim_{n\to\infty}2n\cdot\frac{\alpha}{2n}\cdot\frac{2n}{\alpha}\sin\frac{\alpha}{2n}+\lim_{n\to\infty}\frac{1}{2n}\cdot\lim_{n\to\infty}2n^2\cdot\frac{\alpha}{2n^2}\cdot\frac{2n^2}{\alpha}\cdot\sin\frac{\alpha}{2n^2}$$

$$=\frac{\alpha}{2}\lim_{n\to\infty}\frac{2n}{\alpha}\sin\frac{\alpha}{2n}+\alpha\cdot\lim_{n\to\infty}\frac{1}{2n}\lim_{n\to\infty}\frac{2n^2}{\alpha}\cdot\sin\frac{\alpha}{2n^2}$$

又由引理I知: $\lim_{t\to\infty}t\cdot\sin\frac{1}{t}=1$

故 $$\lim_{n\to\infty}\frac{2n}{\alpha}\sin\frac{\alpha}{2n}=\lim_{n\to\infty}\frac{2n^2}{\alpha}\cdot\sin\frac{\alpha}{2n^2}=1$$

故原极限 $=\frac{\alpha}{2}\cdot1+\alpha\cdot0\cdot1=\frac{\alpha}{2}$ □

8. $\lim_{x\to\infty} f(x) = l \quad \forall \varepsilon>0 \quad \exists X \quad \forall |x|>X \quad |f(x)-l|<\varepsilon$

令 $t=\frac{1}{x}$，$x\to 0$ 即 $t\to\infty$，对上述 X，一定 $\exists X_1$，当 $|x|<X_1$ 时，$\frac{1}{|x|}>X$

$\lim_{x\to 0} f(\frac{1}{x}) = \lim_{t\to 0} f(t)$，$|t|>X$，故 $|f(t)-l|<\varepsilon$

故 $\lim_{x\to 0} f(\frac{1}{x}) = l$，反之亦成立

当 $x\to+\infty$ 时，将 $\forall |x|>X$ 改为 $\forall x>X$

$x\to-\infty$ 时，将 $\forall |x|>X$ 改为 $\forall x<-X$

9.

(1) $\lim_{x\to 0}\frac{\tan 2x}{\sin 5x} = \lim_{x\to 0}\frac{2x}{5x} = \frac{2}{5}$

(2) $\cos 3x = \cos 2x\cos x - \sin 2x \sin x$ $\qquad \frac{\cos x-\cos 3x}{x^2} = \frac{\cos x(1-4\cos^2 x+3)}{x^2}$

$=(2\cos^2 x-1)\cos x - 2\sin^2 x\cos x$

$=2\cos^3 x-\cos x-2\cos x+2\cos^3 x$ $\qquad \lim_{x\to 0}\frac{\cos x-\cos 3x}{x^2} = \lim_{x\to 0}\frac{4-4\cos^2 x}{x^2} = 4\lim_{x\to 0}\frac{\sin^2 x}{x^2} = 4$

$=4\cos^3 x-3\cos x$

10.

(1) 无穷小·有界=无穷小 $\lim_{x\to+\infty}\frac{\arctan x}{x}=0$

(2) 无穷小·有界=无穷小 $\lim_{x\to 0} x^2\sin\frac{1}{x}=0$

(3) $\lim_{x\to 2}\frac{x^3-2x^2}{x-2} = \lim_{x\to 2} x^2 = 4$

(4) $\lim_{x\to\infty} 2x^2-x+1 = +\infty$

11. (1) $\forall M>0, \exists X=a^M \quad \forall x>X \quad \log_a x = |\log_a x| > M$

(2) $\forall M>0 \quad \exists \delta = a^{-M} \quad \forall 0<x<\delta. \quad \log_a x < -M$

(3) $\forall M>0 \quad \exists \delta = \frac{\pi}{2}-\arctan M, \ \forall \frac{\pi}{2}-\delta<x<\frac{\pi}{2} \quad \tan x > M$

(4) $\forall M>1 \quad \exists \delta = \frac{1}{\ln M} \quad \forall\ 0<x<\delta \quad e^{\frac{1}{x}} > M$

12. $a_n = 2n\pi+\frac{\pi}{2} \qquad y=2n\pi+\frac{\pi}{2}$ 故 M 无界

$b_n = 2n\pi \qquad y=0 \qquad n\to+\infty$ 时，y 并不是无穷大量