习题3.6.

1. $f(x)=f(0)+xf'(0)+\frac{x^2}{2}f''(0)+\cdots+\frac{x^n}{n!}f^{(n)}(0)+o(x^n)$

(1) $y=x^2+x+3+\frac{4}{x-1}$

(2) $\sin^2 x=\sum_{n=1}^{n}\frac{(-1)^{k-1}2^{2k-1}}{(2k)!}x^{2k}+o(x^{2k})$

6.

(3). $\lim_{x\to\infty}[x-x^2\ln(1+x)]=\lim_{x\to\infty}[x-x^2\ln(1+\frac{1}{x})]$

$\ln(1+x)=0+x-\frac{x^2}{2}+o(x^2)$

(4) $\lim_{x\to 0}\frac{\cos(\sin x)-\cos x}{x^4}=\lim_{x\to 0}\frac{1-\frac{1}{2}\sin^2 x+\frac{1}{24}\sin^4 x-1+\frac{1}{2}x^2-\frac{1}{24}x^4+o(x^4)}{x^4}$

$=\lim_{x\to 0}\frac{\frac{1}{2}(x^2-\sin^2 x)+\frac{1}{24}(\sin^4 x-x^4)+o(x^4)}{x^4}=\lim_{x\to 0}\frac{x(x-\sin x)}{x^4}=\frac{1}{6}$

8. $f(y)=f(x)+f'(x)(y-x)+\frac{f''(\xi)}{2!}(y-x)^2$

$f(0)=f(x)+f'(x)(-x)+\frac{f''(\xi_1)}{2!}x^2$

$f(2)=f(x)+f'(x)(2-x)+\frac{f''(\xi_2)}{2!}(2-x)^2$

$f(2)-f(0)=2f'(x)+\frac{f''(\xi_2)}{2}(2-x)^2-\frac{f''(\xi_1)}{2}x^2$

$\left|\frac{f(2)-f(0)+\frac{f''(\xi_1)}{2}x^2-\frac{f''(\xi_2)}{2}(2-x)^2}{2}\right|=f'(x)\leq\frac{1+1+2}{2}=2$

10、 $f'(x)=\frac{2}{x^3}e^{-\frac{1}{x^2}}$ $f''(x)=(\frac{4}{x^6}-\frac{6}{x^4})e^{-\frac{1}{x^2}}$

设 $\forall m\in N_+$ $P_m(\frac{1}{x})$ 表示 $\frac{1}{x}$ 的 m 次多项式，从而

$f'(x)=P_3(\frac{1}{x})e^{-\frac{1}{x^2}}$ $f''(x)=P_6(\frac{1}{x})e^{-\frac{1}{x^2}}$

故 $f^{(k)}(x)=P_{3k}(\frac{1}{x})e^{-\frac{1}{x^2}}$ 设 $f^{(k-1)}(x)=Q_{3(k-1)}(\frac{1}{x})e^{-\frac{1}{x^2}}$

$f^{(k)}=Q_{3(k-1)}(\frac{1}{x})\frac{2}{x^3}e^{-\frac{1}{x^2}}+Q'_{3(k-1)}(\frac{1}{x})(-\frac{1}{x^2})e^{-\frac{1}{x^2}}$

$=[Q_{3k}(\frac{1}{x})-Q'_{3(k-1)}(\frac{1}{x})\frac{1}{x^2}]e^{-\frac{1}{x^2}}$

$=P_{3k}(\frac{1}{x})e^{-\frac{1}{x^2}}$

所以 $f(x)$ 在 $x=0$ 处存在任意阶导数

且 $f^{(n)}(0)=\lim_{x\to 0}f^{(n)}(x)=0$

11.

$$f(x)=f(x_0)+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n+o(x-x_0)^n$$

(1) n为奇数时，$(x-x_0)^n$在x_0两侧异号，$f(x)-f(x_0)$在x_0处不取极值

(2) n为偶数时，在x_0两侧$(x-x_0)^n>0$

若$f^{(n)}(x_0)<0$ 从而 $f(x)-f(x_0)<0$

即 $f(x_0)$为极大值

若$f^{(n)}(x_0)>0$ 从而 $f(x)-f(x_0)>0$

故 $f(x_0)$为极小值