(5) $\cos^2 1+\cos^2 2+\cdots+\cos^2 100>1$

故 $1<\cos^2 1+\cos^2 2+\cdots+\cos^2 n<1+1+\cdots+1=n$

又 $\because \lim\limits_{n\to\infty}\sqrt[n]{n}=1$

$\therefore \lim\limits_{n\to\infty}\sqrt[n]{\cos^2 1+\cos^2 2+\cdots+\cos^2 n}=1$

16. 令 $a_k=\max\{a_1,a_2,\cdots,a_m\}$ $1\le k\le m$

$\sqrt[n]{a_k^n}\le\sqrt[n]{a_1^n+a_2^n+\cdots+a_m^n}<\sqrt[n]{m a_k^n}$

且 $\lim\limits_{n\to\infty}\sqrt[n]{a_k^n}=a_k=\lim\limits_{n\to\infty}\sqrt[n]{m a_k^n}$

故 $\lim\limits_{n\to\infty}\sqrt[n]{a_1^n+a_2^n+\cdots+a_m^n}=a_k$ □

17.

(1) 证: $a_n>0$.

$\frac{a_{n+1}}{a_n}=1-\frac{1}{2^{n+1}}<1$ 故 $a_{n+1}<a_n$ a_n 单调递减

$\therefore$ a_n 单减有下界. a_n 收敛 □

(2) $a_{n+1}-a_n=\frac{1}{3^{n+1}+1}>0$, $\therefore a_n$ 单增.

$a_n<\frac{1}{3}+\frac{1}{3^2}+\cdots+\frac{1}{3^n}=\frac{1}{3}\cdot\frac{1-(\frac{1}{3})^n}{1-\frac{1}{3}}<\frac{1}{3}\cdot\frac{1}{1-\frac{1}{3}}=\frac{1}{2}$

$\therefore$ a_n 单增有上界, 收敛 □

(3). $\forall p \in \mathbb{N}_+$.

$$|a_{n+p}-a_n| = |\alpha_{n+1}q^{n+1}+\alpha_{n+2}q^{n+2}+\cdots+\alpha_{n+p}q^{n+p}|$$

$$\le |\alpha_{n+1}|\cdot|q|^{n+1}+|\alpha_{n+2}|\cdot|q|^{n+2}+\cdots+|\alpha_{n+p}||q^{n+p}|$$

$$\le M\cdot\sum_{i=n+1}^{n+p-1}|q|^i < M\cdot\sum_{i=n+1}^{\infty}|q|^i = M\cdot\frac{|q|^{n+1}}{1-|q|} < \varepsilon$$

由 Cauchy 收敛准则知 $\{a_n\}$ 收敛

(4). $\forall p \in \mathbb{N}_+$. $a_{n+p}-a_n = \frac{\cos n}{(n+1)(n+2)}+\frac{\cos(n+1)}{(n+2)(n+3)}+\cdots+\frac{\cos(n+p)}{(n+p)(n+p+1)}$

$$|a_{n+p}-a_n| \le \frac{1}{(n+1)(n+2)}+\frac{1}{(n+2)(n+3)}+\cdots+\frac{1}{(n+p)(n+p+1)} = \frac{1}{n+1}-\frac{1}{n+p+1}$$

$$< \frac{1}{n+1} < \frac{1}{n}$$

$\forall \varepsilon>0$. 取 $N=[\frac{1}{\varepsilon}]$ 当 $n>N$ 时. $\forall p$. 有 $|a_{n+p}-a_n|<\varepsilon$

故 a_n 收敛 □

18.

(1) $a_n = \frac{n}{c^n}$ $(c>1)$

$\because \lim_{n\to\infty}\frac{(n+1)-n}{c^{n+1}-c^n} = \lim_{n\to\infty}\frac{1}{c^n(c-1)} = 0$

由 Stolz定理知 $\lim_{n\to\infty}\frac{n}{c^n}=0$

(2). 先证 $a_n \le c$. $0\le c\le 1$

数学归纳法. $a_1=\frac{c}{2}<c$ 成立.

假设 $n=k$ 时, $a_k\le c$

$n=k+1$ 时 $a_{k+1}=\frac{c}{2}+\frac{a_n^2}{2}=\frac{c}{2}+\frac{c^2}{2}\le\frac{c}{2}+\frac{c}{2}=c$ □

再证单增. 数学归纳法. 显然 $a_2>a_1$ 当 $n=k$ 时, 有 $a_{k+1}>a_k$

当 $n=k+1$ 时, 有 $a_{k+2}=\frac{c}{2}+\frac{a_{k+1}^2}{2}>\frac{c}{2}+\frac{a_k^2}{2}=a_{k+1}$ 成立 □.

故数列 $\{a_n\}$ 收敛. 设 $\lim_{n\to\infty}a_n=a$ 则 $a=\frac{c}{2}+\frac{a^2}{2} \Rightarrow a=1-\sqrt{1-c}=\lim_{n\to\infty}a_n$

(3) $a_{n+1}=\frac{1}{2}(a_n+\frac{a}{a_n})\geqslant\frac{1}{2}\cdot 2\cdot\sqrt{a}=\sqrt{a}$

$\therefore a_n^2\geqslant a$

$a_{n+1}-a_n=\frac{1}{2}(\frac{a}{a_n}-a_n)=\frac{a-a_n^2}{2a_n}<0$

$\therefore a_{n+1}<a_n$ $\quad\therefore a_n$单减，且有下界$\sqrt{a}$

下证 a_n极限为$\sqrt{a}$

$a_n-\sqrt{a}=\frac{1}{2}(a_n+\frac{a}{a_n})-\sqrt{a}=\frac{1}{2}(a_n-\sqrt{a}+\frac{a}{a_n}-\sqrt{a})=\frac{1}{2}(a_n-\sqrt{a})+\frac{1}{2}(\frac{a}{a_n}-\sqrt{a})$

$\leqslant\frac{1}{2}(a_n-\sqrt{a})$

$\therefore |a_n-\sqrt{a}|\leqslant\frac{1}{2}|a_n-\sqrt{a}|\leqslant\cdots\leqslant(\frac{1}{2})^{n+1}|a_1-\sqrt{a}|$.

$\therefore |a_n-\sqrt{a}|<\varepsilon$ $\qquad\lim\limits_{n\to\infty}a_n=\sqrt{a}$

用了结论① 当 $|a_n-a|\leqslant q|a_{n-1}-a|$ $\quad 0<q<1$ 时 $\Rightarrow$ a_n的极限为a

此结论课上讲过，陈老师说可以直接使用，

(4) $a_n\geqslant 1$ $\qquad a_n=2-\frac{1}{a_{n-1}+1}$

考虑 $\left|a_n-\frac{1+\sqrt{5}}{2}\right|=\left|2-\frac{1}{a_{n-1}+1}-\frac{1+\sqrt{5}}{2}\right|$

$=\left|\frac{3-\sqrt{5}}{2}-\frac{1}{a_{n-1}+1}\right|=\left|\frac{(3-\sqrt{5})(a_{n-1}+1)-2}{2(a_{n-1}+1)}\right|=\left|\frac{(3-\sqrt{5})a_{n-1}-(\sqrt{5}-1)}{2(a_{n-1}+1)}\right|$.

$=\frac{3-\sqrt{5}}{2(a_{n-1}+1)}\left|a_{n-1}-\frac{\sqrt{5}-1}{3-\sqrt{5}}\right|=\frac{3-\sqrt{5}}{2(a_{n-1}+1)}\left|a_{n-1}-\frac{1+\sqrt{5}}{2}\right|\leqslant\frac{3-\sqrt{5}}{4}\left|a_{n-1}-\frac{1+\sqrt{5}}{2}\right|$

$\because 0<\frac{3-\sqrt{5}}{4}<1$

由结论1 知 $\lim\limits_{n\to\infty}a_n=\frac{1+\sqrt{5}}{2}$

(5) 考虑函数 $f(x)=\sin x$ 当 $x\in(0,\frac{\pi}{2})$ 时，$0<f(x)<1$ 且 $f(x)\uparrow$ 且 $f(x)<x$

又 $0<1<\frac{\pi}{2} \Rightarrow 0<\sin 1<1$. $\Rightarrow 0<\sin\sin 1<\sin 1<1$ $\Rightarrow 0<\sin\sin\cdots\sin 1<1$

$a_{n+1}=\underbrace{\sin\sin\cdots\sin}_{n+1个\sin} 1<\underbrace{\sin\sin\cdots\sin}_{n个\sin} 1=a_n$ 且 $a_{n+1}=\sin a_n$ 故 a_n 单减有下界

故 a_n 收敛，设 $\lim\limits_{n\to\infty}a_n=a$

故 $a=\lim\limits_{n\to\infty}a_{n+1}=\lim\limits_{n\to\infty}\sin a_n=\sin a$ $\Rightarrow a=0$ $\therefore \lim\limits_{n\to\infty}a_n=0$

19 $\forall\varepsilon>0,\exists N_1\in\mathbb{N}\ \forall n>N_1$ 有 $|b_n-a|\leq|b_n-a_n|<\varepsilon$,

故 $\lim\limits_{n\to\infty}b_n=a$

同理 $\forall\varepsilon>0.\ \exists N_2\in\mathbb{N}\ \forall n>N_2$ 有 $|a-a_n|\leq|b_n-a_n|<\varepsilon$

故 $\lim\limits_{n\to\infty}a_n=a$ □

补充

20. 先证 $\{a_n\}$ 一定收敛 ……②

反证：假设 $\{a_n\}$ 发散到无穷大 由数列无界知，$\forall N\in\mathbb{N}_+$ $\exists n_1>N.\ a_{n_1}\leq a_{n_1+1}$

又 $\because \lim\limits_{n\to\infty}\frac{a_n}{a_{n+1}}=l>1$ 故 $\forall\varepsilon>0,\exists N_1\in\mathbb{N}_+\ \forall n>N_1$ 有 $\left|\frac{a_n}{a_{n+1}}-l\right|<\varepsilon$

即 $l-\varepsilon<\frac{a_n}{a_{n+1}}<l-\varepsilon$ 故 $\frac{a_n}{a_{n+1}}\geq 1$，否则 取 $\varepsilon=1-\frac{a_n}{a_{n+1}}$

有 $l-\varepsilon=l-1+\frac{a_n}{a_{n+1}}>\frac{a_n}{a_{n+1}}$ 矛盾

又由②知 $\exists n_1$.. $\frac{a_{n_1}}{a_{n_1+1}}<1$ 矛盾！ 故 $\{a_n\}$ 收敛.

再证 $\lim\limits_{n\to\infty}a_n=0$.

反证，设 $\lim\limits_{n\to\infty}a_n=a\neq 0$ 则 $\lim\limits_{n\to\infty}a_{n+1}=a$

$\forall\varepsilon>0.$ $\frac{a_n}{a_{n+1}}-1=\frac{a_n-a_{n+1}}{a_{n+1}}=\frac{a_n-a+a-a_{n+1}}{a_{n+1}}\leq\frac{|a_n-a|+|a-a_{n+1}|}{a_{n+1}}\leq\frac{2\varepsilon}{\frac{|a|}{2}}=\frac{4\varepsilon}{|a|}$

故 $\lim\limits_{n\to\infty}\frac{a_n}{a_{n+1}}=1\neq l$. 矛盾！ 故 $\lim\limits_{n\to\infty}a_n=0$ □

21. 设 $\lim\limits_{n\to\infty} b_n = L$ ∵ $\{b_n\}$ 为正数列，故 $L\geq 0$.

情况一，$L>0$. 则 $\lim\limits_{n\to\infty}\frac{b_{n+1}}{b_n} = \frac{\lim\limits_{n\to\infty} b_{n+1}}{\lim\limits_{n\to\infty} b_n} = 1$

下证 $\frac{a_{n+1}}{a_n}\leq 1$ ($n>N_0$ 时) → $n>N_0$ 时 $\left|\frac{b_{n+1}}{b_n}-1\right|<\varepsilon$.

反证. 设 $\exists N_1>N_0$，$\frac{a_{N_1+1}}{a_{N_1}}>1$ 则 $\frac{b_{N_1+1}}{b_n}\geq\frac{a_{N_1+1}}{a_{N_1}}>1$

则取 $\varepsilon = \frac{\frac{a_{N_1+1}}{a_{N_1}}-1}{2}$，$\exists N_1$，$\left|\frac{b_{N_1+1}}{b_{N_1}}-1\right| = \frac{b_{N_1+1}}{b_N}-1 > \frac{a_{N_1+1}}{a_{N_1}}-1>\varepsilon$ 矛盾!

故当 $n>N_0$ 时 $\frac{a_{n+1}}{a_n}\leq 1$ 此时 $a_{n+1}\leq a_n$ 单减且有界 → 子列收敛.

故 $|a_{n+p}-a_n|<\varepsilon$ Cauchy 收敛准则 ⇒ a_n 收敛 □

情况二. $L=0$. 即 $\lim\limits_{n\to\infty} b_n = 0$

$$\frac{a_n}{a_1} = \frac{a_n}{a_{n-1}}\cdot\frac{a_{n-1}}{a_{n-2}}\cdots\frac{a_2}{a_1} \leq \frac{b_n}{b_{n-1}}\cdot\frac{b_{n-1}}{b_{n-2}}\cdots\frac{b_2}{b_1} = \frac{b_n}{b_1} \Rightarrow a_n \leq \frac{a_1}{b_1}\cdot b_n < M\cdot\varepsilon$$ □

22.

(1) $\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$

令 $n=2k+1$

$\lim\limits_{2k+1\to\infty}\left(1+\frac{1}{2k+1}\right)^{2k+1} = e$

有 $\lim\limits_{k\to\infty}\left(1+\frac{1}{2k+1}\right)^{2k+1} = e$

故 $\lim\limits_{n\to\infty}\left(1+\frac{1}{2n+1}\right)^{2n+1} = e$

(2) $a_n = \left(\frac{n-3}{n-2}\right)^{n+1}$

$\frac{1}{a_n} = \left(\frac{n-2}{n-3}\right)^{n+1} = \left(1+\frac{1}{n-3}\right)^{n+1}$

$= \left(1+\frac{1}{n-3}\right)^{n-3}\cdot\left(1+\frac{1}{n-3}\right)^4$

$\lim\limits_{n\to\infty}\frac{1}{a_n} = e\cdot 1^4 = e = \frac{1}{\lim\limits_{n\to\infty} a_n}$

故 $\lim\limits_{n\to\infty} a_n = \frac{1}{e}$

(3) $\frac{1}{a_n} = \left(\frac{n+2}{n+1}\right)^n = \left(1+\frac{1}{n+1}\right)^n$

$= \left(1+\frac{1}{n+1}\right)^{n+1}\cdot\frac{1}{1+\frac{1}{n+1}}$

$= \left(\frac{n+1}{n+2}\right)\cdot\left(1+\frac{1}{n+1}\right)^{n+1}$

$= \left(1-\frac{1}{n+2}\right)\left(1+\frac{1}{n+1}\right)^{n+1}$

$\lim\limits_{n\to\infty}\frac{1}{a_n} = 1\cdot e = \frac{1}{\lim\limits_{n\to\infty} a_n}$ 故 $\lim\limits_{n\to\infty} a_n = \frac{1}{e}$

(4) $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty}\left[\left(1+\frac{1}{n^3}\right)^{n^3}\right]^2$

$= \left[\lim\limits_{n\to\infty}\left(1+\frac{1}{n^3}\right)^{n^3}\right]^2 = e^2$

23.

反证. 设 $\lim_{n\to\infty} a_n b_n = a$

则 $\forall \varepsilon>0$. $\exists N\in\mathbb{N}$ $\forall n>N$. $|a_nb_n-a|<\varepsilon$

即 $|a_nb_n|<\varepsilon+|a|$

又∵ $\lim_{n\to\infty} a_n=\infty$ 　则令 $M=\frac{|a|+2\varepsilon}{b}$. $\exists N_2\in\mathbb{N}$ $\forall n>N_2$ 有 $|a_n|>M$

此时 $|a_nb_n|\geq \frac{|a|+2\varepsilon}{b}\cdot b=|a|+2\varepsilon>|a|+\varepsilon$ 矛盾!

故 $\lim_{n\to\infty} a_nb_n=\infty$ □

法一:

24. 引理: Cauchy II. $\lim_{n\to\infty}\sqrt[n]{a_n}=\frac{a_n}{a_{n-1}}$ 　假设 a_n 收敛.

则 $\lim_{n\to\infty}\sqrt[n]{n!}=\lim_{n\to\infty}\frac{n!}{(n-1)!}=\lim_{n\to\infty} n=+\infty$ 　故其无上界, 趋于无穷大

法二: $\sqrt[n]{n!}=e^{\frac{\ln n!}{n}}>e^{n}$ $(n\geq 3)$ 　发散. 趋于无穷大.

再考虑 $n\sin\frac{n\pi}{2}$

当 $n=2k$ 时 　$n\sin\frac{n\pi}{2}=2k\sin k\pi=0$

当 $n=4k+1$ 时 　$n\sin\frac{n\pi}{2}=(4k+1)\sin(\frac{4k+1}{2})\pi=4k+1\to+\infty$

当 $n=4k+3$ 时 　$n\sin\frac{n\pi}{2}=(4k+3)\sin(\frac{4k+3}{2})\pi=-(4k+3)\to-\infty$

故当 $n\to\infty$ 时 $n\sin\frac{n\pi}{2}$ 无界, 但并不发散到无界大

25. ∵ $a_1=1$ 故 $a_n>0$. $\frac{1}{a_n}>0$

故 $a_{n+1}-a_n=\frac{1}{a_n}>0$ 　$\{a_n\}$ 单调递增.

反证. 设 $\{a_n\}$ 收敛. $\lim_{n\to\infty}a_n=a$ $(a>0)$

故 $\lim_{n\to\infty}a_{n+1}=\lim_{n\to\infty}a_n+\frac{1}{a_n}=a+\frac{1}{a}=a \Rightarrow \frac{1}{a}=0$

又∵ $a>0$ 故 $\frac{1}{a}>0$ 矛盾!