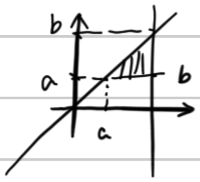
第8周作业

(1) $\int_a^b dy \int_y^b f(x,y)\,dx$



$y=x$

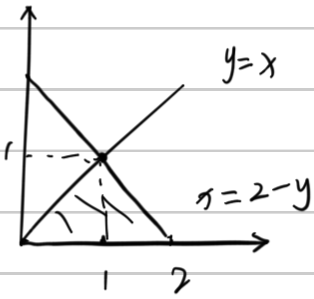
$x=b$

$a \le y \le x$

$= \iint_D f(x,y)\,d\sigma = \int_a^b dx \int_a^x f(x,y)\,dy$

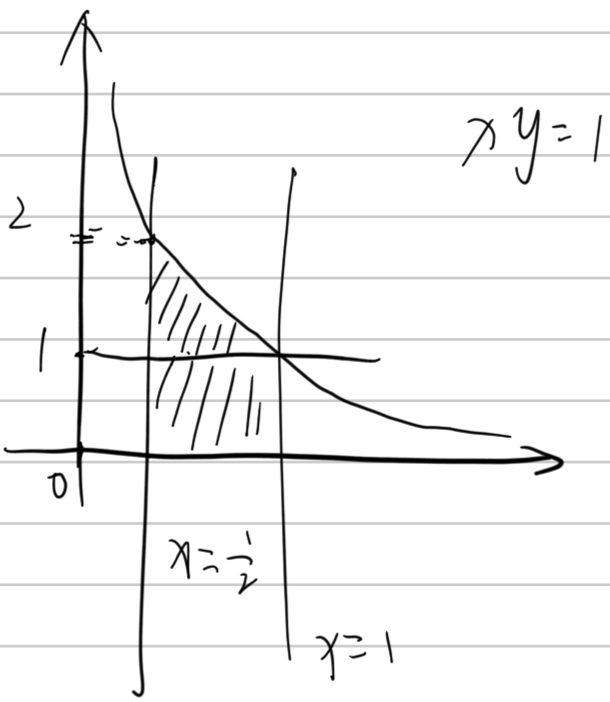
(5) $\int_0^1 dx \int_0^x f(x,y)\,dy + \int_1^2 dx \int_0^{2-x} f(x,y)\,dy$

$= \int_0^1 dy \int_y^{2-y} f(x,y)\,dx$



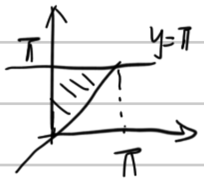
(6) $\int_0^1 dy \int_{\frac{1}{2}}^1 f(x,y)\,dx + \int_1^2 dy \int_{\frac{1}{2}}^{\frac{1}{y}} f(x,y)\,dx$

$= \int_{\frac{1}{2}}^1 dx \int_0^{\frac{1}{x}} f(x,y)\,dy$

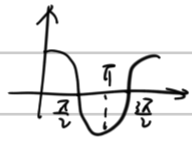


2.

(3) $\iint_D \cos(x+y)\,dx\,dy$

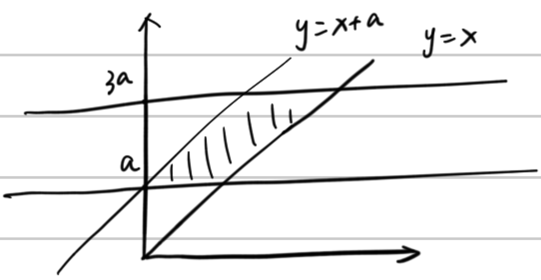


$$\int_0^{\pi} dy \int_0^{y} \cos(x+y)\,dx$$



$$= \int_0^{\pi} (\sin 2y - \sin y)\,dy = \left(-\frac{1}{2}\cos 2y + \cos y\right)\Big|_0^{\pi} = \left(-\frac{1}{2}-1\right) - \left(-\frac{1}{2}+1\right) = -2$$

(5) $\iint_D (x+y-1)\,dx\,dy$ 由 $y=x$，$y=x+a$，$y=a$，$y=3a$ 围成.



$$\int_a^{3a} dy \int_{y-a}^{y} (x+y-1)\,dx$$

$$= \int_a^{3a} \frac{1}{2}x^2 + (y-1)x \Big|_{x=y-a}^{x=y} dy$$

$$= \int_a^{3a} \frac{1}{2}y^2 + (y-1)y - \frac{1}{2}(y-a)^2 - (y-1)(y-a)\,dy$$

$$= \int_a^{3a} \frac{1}{2}y^2 + y^2 - y - \frac{1}{2}(y^2 - 2ay + a^2) - (y^2 - (a+1)y + a)\,dy$$

$$= \int_a^{3a} -y + ay - \frac{1}{2}a^2 + (a+1)y + a\,dy$$

$$= \int_a^{3a} 2ay - \frac{1}{2}a^2 + a\,dy$$

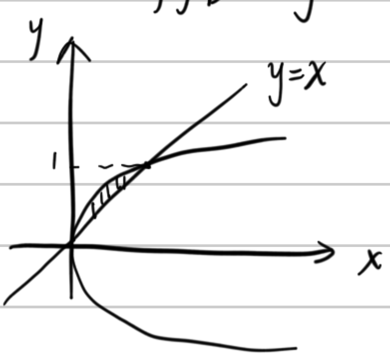
$$= ay^2 + \left(a - \frac{1}{2}a^2\right)y \Big|_{y=a}^{y=3a}$$

$$= 9a^3 + \left(a - \frac{1}{2}a^2\right)3a - a^3 - \left(a - \frac{1}{2}a^2\right)a$$

$$= 9a^3 + 3a^2 - \frac{3}{2}a^3 - a^3 - a^2 + \frac{1}{2}a^3$$

$$= \frac{13}{2}a^3 + \frac{5}{2}a^2$$

(6) $\iint_D \frac{\sin y}{y}\,dx\,dy$ 由 $y=x$ 和 $x=y^2$ 围成



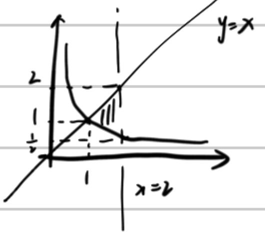
$$\int_0^1 dy \int_{y^2}^{y} \frac{\sin y}{y}\,dx = \int_0^1 \frac{\sin y}{y}x \Big|_{x=y^2}^{y=y} dy$$

$$= \int_0^1 y\sin y - \sin y\,dy$$

$$= \frac{1}{2}y^2 \sin y - \sin y \cdot y \Big|_0^1$$

$$= \frac{1}{2}\sin 1 - \sin 1 = -\frac{1}{2}\sin 1.$$

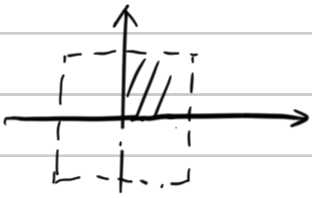
(7) $\iint_D \frac{x^2}{y^2}dxdy$ 由 $x=2$ $y=x$ 及 $xy=1$ 围成



$$\int_1^2 dx\int_{\frac{1}{x}}^{x}\frac{x^2}{y^2}dy$$
$$=\int_1^2 x^2\left(-\frac{1}{y}\right)\Big|_{y=\frac{1}{x}}^{y=x} dx$$
$$=\int_1^2 -x-x^3\ dx$$
$$=-\frac{1}{2}x^2-\frac{1}{4}x^4\Big|_{x=1}^{x=2}$$
$$=-6+\frac{3}{4}=\frac{-21}{4}$$

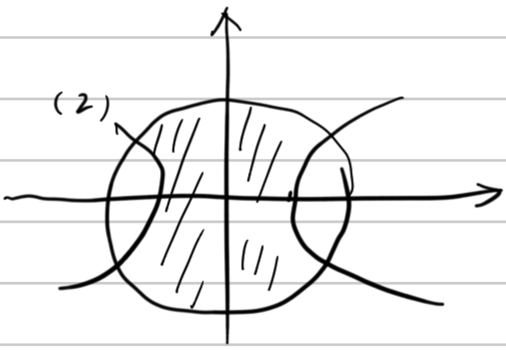
(1)

3. 偶函数 $\iint_D (x^2+y^2)dxdy = 4\iint_{D'}(x^2+y^2)dxdy$



D': $\{0\le x\le 1,\ 0\le y\le 1\}$
$$=4\int_0^1 dx\int_0^1 (x^2+y^2)dy$$
$$=4\int_0^1 \left(x^2y+\frac{1}{3}y^3\right)\Big|_0^1 dx$$
$$=4\int_0^1 x^2+\frac{1}{3}\ dx = 4\left(\frac{1}{3}x^3+\frac{1}{3}x\Big|_0^1\right)$$
$$=4\cdot\frac{2}{3}=\frac{8}{3}$$

(2)



关于 y 对称

关于 x 奇函数

$\iint_D \sin x\sin y\,dxdy = 0.$

6. $$\int_a^b dx\int_c^d\frac{\partial^2 f(x,y)}{\partial x\partial y}dy=\int_a^b\frac{\partial f(x,y)}{\partial x}\Big|_{y=c}^{y=d}dx$$
$$=\int_a^b \frac{\partial f(x,d)}{\partial x}-\frac{\partial f(x,c)}{\partial x}\ dx$$
$$=f(b,d)-f(b,c)-f(a,d)+f(a,c)$$

7. $\lim\limits_{r\to 0}\frac{1}{\pi r^2}\iint_{x^2+y^2\le r^2} f(x,y)\,dx\,dy$ $\quad D=\{x^2+y^2\le r^2\}$

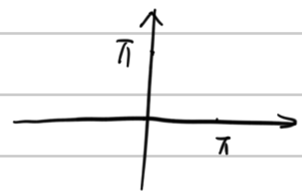
中值定理 $\exists\, x_0, y_0 \in D$ $\quad \iint_{x^2+y^2\le r^2} f(x,y)\,dx\,dy = f(x_0,y_0)\cdot r^2\cdot \pi$

原式 $=\lim\limits_{r\to 0}\frac{1}{\pi r^2} f(x_0,y_0)\, r^2 \pi = \lim\limits_{r\to 0} f(x_0,y_0)$

连续 $\Rightarrow$ 原式 $= f(0,0)$

习题 10.2

1. (3) $\int_0^{\pi}\int_0^{\pi}\cos(x+y)\,dx\,dy$



$= \int_0^{\pi} dx \int_0^{\pi}\cos(x+y)\,dy$

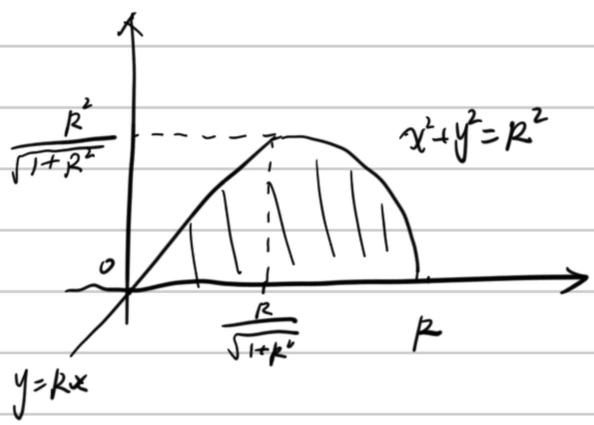
$= \int_0^{\pi} \sin(x+y)\Big|_{y=0}^{y=\pi} dx$

$= \int_0^{\pi} \sin(x+\pi)-\sin x \, dx$

$= -2\int_0^{\pi}\sin x\, dx$

$= 2\cos x\Big|_{x=0}^{x=\pi} = -4$

(5) $\int_0^{\frac{R}{\sqrt{1+R^2}}} dx \int_0^{Rx}\left(1+\frac{y^2}{x^2}\right)dy + \int_{\frac{R}{\sqrt{1+R^2}}}^{R} dx \int_0^{\sqrt{R^2-x^2}}\left(1+\frac{y^2}{x^2}\right)dy$



$\int_0^{\frac{R^2}{\sqrt{1+R^2}}} dy \cdot \int_{\frac{y}{R}}^{\sqrt{R^2-y^2}} \left(1+\frac{y^2}{x^2}\right)dx$

$= \int_0^{\frac{R^2}{\sqrt{1+R^2}}} x-\frac{y^2}{x}\Big|_{x=\frac{y}{R}}^{x=\sqrt{R^2-y^2}} dy$

$= \int_0^{\frac{R^2}{\sqrt{1+R^2}}} \sqrt{R^2-y^2}-\frac{y^2}{\sqrt{R^2-y^2}}-\left(\frac{y}{R}-Ry\right) dy$

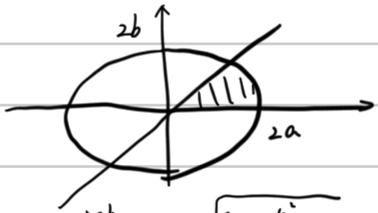
$y\sqrt{R^2-y^2}-\frac{\frac{1}{2}y^2}{R}+R\cdot\frac{1}{2}y^2$

$= \frac{R^2}{\sqrt{1+R^2}}\cdot\frac{R}{\sqrt{1+R^2}} - \frac{1}{2R}\cdot\frac{R^4}{1+R^2} + R\cdot\frac{\frac{R^4}{1+R^2}}{2}$

$= \frac{2R^3-R^2+R^5}{2(1+R^2)}$

2.

(2)



$x^2(\frac{1}{a^2}+\frac{1}{b^2})=4$

$x^2=\frac{4a^2b^2}{a^2+b^2}$

$x=\frac{2ab}{\sqrt{a^2+b^2}}$

$$\int_0^{\frac{2ab}{\sqrt{a^2+b^2}}} dy \int_y^{\sqrt{(4-\frac{y^2}{b^2})a^2}} \sqrt{\frac{x^2}{a^2}+\frac{y^2}{b^2}}\,dx$$

$$x\sqrt{\frac{x^2}{a^2}+\frac{y^2}{b^2}} - \int x\cdot\frac{x}{\sqrt{\frac{x^2}{a^2}+\frac{y^2}{b^2}}}$$

$$x\sqrt{\frac{x^2}{a^2}+\frac{y^2}{b^2}} - \int \frac{x^2}{\sqrt{\frac{x^2}{a^2}+\frac{y^2}{b^2}}}\,dx$$

令 $x=ar\cos\theta$ $\quad y=br\sin\theta$

$$\frac{\partial(x,y)}{\partial(r,\theta)}=\begin{vmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial\theta}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial\theta}\end{vmatrix}=\begin{vmatrix}a\cos\theta & -ar\sin\theta\\ b\sin\theta & br\cos\theta\end{vmatrix}=abr.$$

故 $\iint_{D'}\sqrt{r^2}\,|abr|\,drd\theta$

边界 $r^2=4 \Rightarrow r=2$

$br\sin\theta=0 \Rightarrow \theta=0$

$br\sin\theta=ar\cos\theta \Rightarrow \tan\theta=\frac{a}{b}$

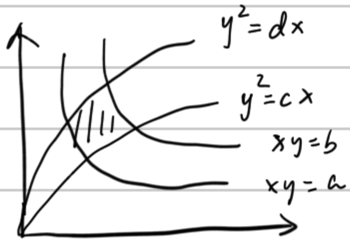
$\theta=\arctan\frac{a}{b}$

$$=\int_0^{\arctan\frac{a}{b}} d\theta\int_0^2 r^2\,dr$$

$$=\frac{8}{3}\arctan\frac{a}{b}$$

故 D': $\{0\le r\le 2,\ 0\le\theta\le\arctan\frac{a}{b}\}$

(5) $\iint_D xy\,dxdy$



令 $u=xy$

$v=\frac{y^2}{x}$

$$\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}$$

$$=\begin{vmatrix}\frac{2}{3}u^{-\frac{1}{3}}\cdot v^{-\frac{1}{3}} & -\frac{1}{3}u^{\frac{2}{3}}v^{-\frac{4}{3}}\\ \frac{1}{3}u^{-\frac{2}{3}}v^{\frac{1}{3}} & \frac{1}{3}u^{\frac{1}{3}}v^{-\frac{2}{3}}\end{vmatrix}$$

$$=\frac{2}{9}\cdot\frac{1}{v}+\frac{1}{9}\cdot\frac{1}{v}=\frac{1}{3v}$$

D': $\{a\le u\le b \quad c\le v\le d\}$

$\frac{u^2}{v}=x^3 \qquad uv=y^3$

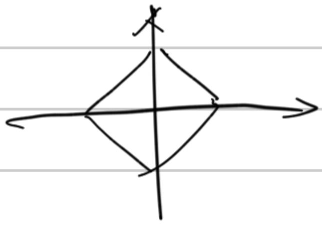
$x=\sqrt[3]{\frac{u^2}{v}}=\frac{u^{\frac{2}{3}}}{v^{\frac{1}{3}}} \qquad v^{-\frac{1}{3}}$

$y=u^{\frac{1}{3}}v^{\frac{1}{3}}$

$$\iint_D xy\,dxdy=\iint_{D'} u\cdot\frac{1}{|3v|}\,dudv=\int_a^b du\int_c^d\frac{1}{3v}\,dv$$

$$=\frac{(b-a)}{3}\ln\frac{d}{c}$$

(7) $\iint_D \frac{x^2-y^2}{\sqrt{x+y+3}}\,dxdy$ $D: |x|+|y|\le 1$



$u=x+y$ $v=x-y$ $\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{\frac{\partial(u,v)}{\partial(x,y)}} = \frac{1}{\begin{vmatrix}1 & 1\\ 1 & -1\end{vmatrix}} = -\frac{1}{2}$

$\left|\frac{u+v}{2}\right|+\left|\frac{u-v}{2}\right|\le 1$ 原式 $= \iint_{D'} \frac{\frac{1}{2}uv}{\sqrt{u+3}}\,dudv$

$\Leftrightarrow$ $|u|+|v|\le 1$ (旋转视角) v. 奇函数 + 对称. 上式 $=0$

圆: $0\le\theta\le\frac{\pi}{2}$ $0\le r\le 1$

$\int_0^1 r^3\,dr\int_0^{\frac{\pi}{2}} d\theta$

$\frac{1}{4}\cdot\frac{\pi}{2}=\frac{\pi}{8}$

正方形: $\int_0^{\frac{\sqrt{2}}{2}}dx\int_0^{\frac{\sqrt{2}}{2}}dy$

$=\frac{\pi}{4}$

(9) $\iint_D |xy|\,dxdy$ $D: x^2+y^2\le a^2$

$= 4\iint_{D'} xy\,dxdy$ $D'\{x^2+y^2\le a^2, x>0, y>0\}$

$x=r\cos\theta$ $y=r\sin\theta$ $D'': \{0\le r\le a, 0\le\theta\le\frac{\pi}{2}\}$

上式 $= 4\iint_{D'} r^2\sin\theta\cos\theta\, r\,d\theta dr$

$= 4\int_0^a r^3\,dr\int_0^{\frac{\pi}{2}}\frac{1}{2}\sin 2\theta\,d\theta$

$= 2\int_0^a r^3\,dr\int_0^{\frac{\pi}{2}}\sin 2\theta\,d\theta$

$= \int_0^a r^3\,dr\,(-\cos 2\theta)\Big|_0^{\frac{\pi}{2}} = 2\int_0^a r^3\,dr = 2\cdot\frac{1}{4}r^4\Big|_0^a = \frac{1}{2}a^4$

3. (2)

$u=x-y$ $v=x$

$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix} = \begin{vmatrix}1 & 1\\ -1 & 0\end{vmatrix} = 1$

$-\iint_{D'} 1\,d\sigma$ $D'=\{u^2+v^2=a^2\}$

$0\le r=a$ $0\le\theta\le\frac{\pi}{2}$

$4\iint_{D''} 1\,d\sigma$

$=4\int_0^a r\,dr\int_0^{\frac{\pi}{2}}d\theta = 4\cdot\frac{1}{2}a^2\cdot\frac{\pi}{2}$

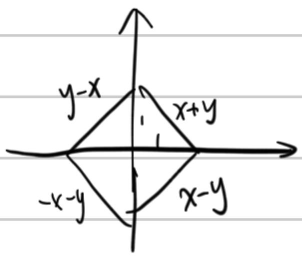
$=\pi a^2$

$\Rightarrow S_1 = S_2$ $S_2: u^2+v^2=a^2$ $\Rightarrow$ 面积为 πa^2

$e^x+e^{-x}\ge 2$

6. $\iint_{|x|+|y|\le 1} e^{f(x+y)}\,dxdy = \iint_{D_1} e^{f(x+y)}+e^{-f(x+y)}\,dxdy$

$+\iint_{D_2} e^{f(x-y)}+e^{-f(x-y)}\,dxdy$



$\ge \iint_{D_1} 2\,dxdy + \iint_{D_2} 2\,dxdy$

$= 2\iint_D dxdy = 2\cdot 1 = 2$

7. $u = x-y \quad v = x+y$

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{2}$$

$$\iint_D f(x-y)\,dx\,dy = \frac{1}{2}\iint_{D'} f(u)\,du\,dv$$

$-A-u \le v \le A-u$

$$\left|\frac{u+v}{2}\right| \le \frac{A}{2} \Rightarrow |u+v| \le A$$

$$\Rightarrow |u|-A \le v \le A-|u|$$

$$\left|\frac{v-u}{2}\right| \le \frac{A}{2} \Rightarrow |v-u| \le A$$

$-A+u \le v \le A+u$

$$原式 = \frac{1}{2}\int_{-A}^{A} f(u)\,du \int_{|u|-A}^{A-|u|} dv$$

$$= \frac{1}{2}\int_{-A}^{A} f(u)\cdot 2(A-|u|)\,du$$

$$= \int_{-A}^{A} f(u)(A-|u|)\,du$$

$$\int_{-1}^{1} e^{f(u)}du = \int_{-1}^{1} e^{-f(-u)}du = \int_{1}^{-1} e^{-f(-u)}d(-u) = \int_{-1}^{1} e^{-f(u)}du$$