1. $x - \lfloor x \rfloor = m \quad 0 \le m < 1$

$\lfloor x \rfloor = nq + r \quad 0 \le r < n \quad \Rightarrow \quad r \le n-1 \quad$ so $\quad r + m < r+1 \le n$

$$\frac{x}{n} = \frac{\lfloor x \rfloor}{n} + \frac{m}{n} = \frac{nq+r}{n} + \frac{m}{n} = q + \frac{m+r}{n} < q+1$$

so $\quad \lfloor \frac{x}{n} \rfloor < q+1$

for $\quad \frac{x}{n} > \frac{\lfloor x \rfloor}{n} \quad$ so $\quad \lfloor \frac{x}{n} \rfloor \geqslant \lfloor \frac{\lfloor x \rfloor}{n} \rfloor = q$

and $\quad \lfloor \frac{x}{n} \rfloor < q+1$

then $\quad \lfloor \frac{x}{n} \rfloor = q = \lfloor \frac{\lfloor x \rfloor}{n} \rfloor$

2. $\gcd(a, b) = 1$

Bezout theory, $\exists\, q.p \in \mathbb{N} \quad pa + qb = 1$

so. $\quad n = anp + qnb$

for $\quad b \mid n \quad$ so. $\quad ab \mid an$

$a \mid n \quad$ so $\quad ab \mid bn$

so $\quad ab \mid anp + qnb \quad$ so $\quad ab \mid n$

3. if $p^k < n$ but $p^{k+1} > n$

$n!$ contains $\lfloor \frac{n}{p} \rfloor$ numbers that are mutiples of p

$\lfloor \frac{n}{p^2} \rfloor$ of p^2

$\lfloor \frac{n}{p^k} \rfloor$ of p^k

So the ~~largestest~~ largest integer $\alpha = \sum_{i=1}^{k} \lfloor \frac{n}{p^i} \rfloor$

when $i > k \quad p^i > n \quad$ so $\quad \lfloor \frac{n}{p^i} \rfloor = 0.$

So $\quad \alpha = \sum_{i=1}^{\infty} \lfloor \frac{n}{p^i} \rfloor$

4\. because $a \equiv b \pmod{n}$

$n \mid b-a$.

for $(c_0 + c_1 b + c_2 b^2 + c_3 b^3) - (c_0 + c_1 a + c_2 a^2 + c_3 b^3)$

$= c_1(b-a) + c_2(b-a)(b+a) + c_3(b-a)(b^2+ab+a^2)$

$= (b-a)\left[c_1 + c_2(a+b) + c_3(b^2+ab+a^2)\right]$

$\equiv 0 \pmod{n}$

so $c_0 + c_1 a + c_2 a^2 + c_3 a^3 \equiv c_0 + c_1 b + c_2 b^2 + c_3 b^3 \equiv 0 \pmod{n}$

5\. if $u \equiv 0 \pmod{3}$

let $u = 3k$

then $9k^2 + 3kv + v^3 \equiv 0 \pmod{9}$

then $9k^2 + 3kv + v^3 \equiv 0 \pmod{3}$

then $v^3 \equiv 0 \pmod{3}$

then $3 \mid v$.

if $v \equiv 0 \pmod{3}$ similarily $u \equiv 0 \pmod{3}$

if $u \not\equiv 0 \pmod{3}$ and $v \not\equiv 0 \pmod{3}$.

situation 1 (S_1) $v \equiv u \equiv 1 \pmod 3$

Let $v = 3k_2+1$ $u = 3k_1+1$

then $u^2+uv+v^2 = 9k_1^2+9k_2^2+6k_1+6k_2+9k_1k_2+3k_1+3k_2+3$

$= 9k_1^2+9k_2^2+9k_1k_2+9k_1+9k_2+3 \equiv 3 \pmod 9$

lead to contradiction

S2. $v \equiv u \equiv 2 \pmod 3$

let $u = 3t_1+2$ $v = 3t_2+2$

then $u^2+uv+v^2 = 9t_1^2+9t_2^2+9t_1t_2+18t_1+18t_2+12 \equiv 3 \pmod 9$

lead to contradiction

S_3 $v \equiv 1 \pmod 3$ $u \equiv 2 \pmod 3$

Let $u = 3q_1+1$ $v = 3q_2+2$

then $u^2+uv+v^2 = 9q_2^2+9q_1^2+9q_1q_2+12q_1+15q_2+7$

$\equiv 1 \pmod 3$

but $u^2+uv+v^2 \equiv 0 \pmod 9 \Rightarrow u^2+uv+v^2 \equiv 0 \pmod 3$

that lead to contradiction

Similarly when $u \equiv 1 \pmod 3$ $v \equiv 2 \pmod 3$

Summarizing: $u \equiv 0 \pmod 3$ $v \equiv 0 \pmod 3$ i.e. $u, v \in [0]_3$