Hw 2.

补充习题：

求方程 $y'' + y = \sec x$ 的通解

$y'' + y = 0 \qquad y_1 = \cos x \quad y_2 = \sin x$

$$c_1' = \frac{-\sin x \cdot \sec x}{w(x)} = -\tan x \qquad c_2' = \frac{\cos x \cdot \sec x}{w(x)} = 1$$

$$c_1 = \int -\tan x\,dx = \int \frac{d\cos x}{\cos x} = \ln|\cos x| \qquad c_2 = x$$

$$y^* = \cos x \ln|\cos x| + x\sin x$$

$$y(x) = \cos x \ln|\cos x| + x\sin x + C_1\cos x + C_2\sin x$$

习题 6.2.

1. (1) $$y_2 = y_1 \int \frac{e^{-\int p(x)dx}}{y_1^2} dx = \frac{\sin x}{x} \int \frac{x^2}{\sin^2 x} e^{-\int \frac{2}{x}dx} dx$$

$$= \frac{\sin x}{x} \int \frac{x^2}{\sin^2 x} \cdot \frac{1}{x^2} dx = \frac{\sin x}{x} \cdot \int \frac{1}{\sin^2 x} dx$$

$$= \frac{\sin x}{x} \cdot \left(-\frac{\cos x}{\sin x}\right) = -\frac{\cos x}{x}$$

通解 $y = C_1\frac{\sin x}{x} + C_2\frac{\cos x}{x}$

(3) $$y'' - \frac{2x}{1-x^2}y' + \frac{2}{1-x^2}y = 0$$

$$y_2 = x \cdot \int \frac{e^{\int \frac{2x}{1-x^2}dx}}{x^2} dx = x \cdot \int \frac{1}{x^2(1-x^2)} dx = x \cdot \int \frac{1}{x^2} + \frac{1}{1-x^2} dx$$

$$= x \cdot \left(-\frac{1}{x} - \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right|\right) = -1 - \frac{1}{2x}\ln\left|\frac{x-1}{x+1}\right|$$

$$y = C_1 x + C_2\left(1 + \frac{1}{2x}\ln\frac{|x-1|}{|x+1|}\right)$$

2. (2) $xy'' - (1+x)y' + y = 0 \qquad x \neq 0$

特解：$y_1 = e^x$

$$y'' - \frac{1+x}{x}y' + \frac{1}{x}y = 0$$

$$y_2 = e^x \int \frac{e^{\int \frac{1+x}{x}dx}}{e^{2x}} dx = e^x \int \frac{x \cdot e^x}{e^{2x}} dx = e^x \int \frac{x}{e^x} dx$$

$$= e^x \int x e^{-x} dx = -e^x \int x \, de^{-x} = -e^x \left[x e^{-x} - \int e^{-x} dx \right]$$

$$= -x - 1$$

通解 $y = C_1 e^x + C_2(x+1)$

4. (1) $\lambda^2 - 2\lambda - 1 = 0$

$$\lambda_1 = \frac{2+\sqrt{8}}{2} = 1+\sqrt{2} \quad \lambda_2 = 1-\sqrt{2}$$

基础解系 $e^{(1+\sqrt{2})x} \quad e^{(1-\sqrt{2})x}$

通解 $y = C_1 e^{(1+\sqrt{2})x} + C_2 e^{(1-\sqrt{2})x}$

(2) $\lambda^2 + 2\lambda + 2 = 0$

$$\lambda_1 = \frac{-2+\sqrt{-4}}{2} = -1+i$$

$$\lambda_2 = -1-i$$

基础解系 $e^{-x}\cos x \quad e^{-x}\sin x$

通解 $y = C_1 e^{-x}\cos x + C_2 e^{-x}\sin x$

(3) $\lambda^2 + \lambda - 6 = 0 \quad (\lambda-2)(\lambda+3) = 0$

$\lambda_1 = 2 \quad \lambda_2 = -3$

通解 $y = C_1 e^{2x} + C_2 e^{-3x}$

5. (1) $y'' + y = 0$ 有解 $y_1 = \cos x \quad y_2 = \sin x \qquad w(x) = 1$

$$C_1' = \frac{-y_2 \cdot 2\sin\frac{x}{2}}{w(x)} = -\sin x \cdot 2\sin\frac{x}{2} \qquad C_2' = \frac{y_1 \cdot 2\sin\frac{x}{2}}{w(x)} = 2\cos x \cdot \sin\frac{x}{2}$$

$$C_1 = \int -\sin x \, 2\sin\frac{x}{2} dx = -4\int \cos\frac{x}{2}\sin^2\frac{x}{2} dx = -8\int \sin^2\frac{x}{2} \, d\sin\frac{x}{2} = -\frac{8}{3}\sin^3\frac{x}{2}$$

$$C_2 = \int 2\cos x \sin\frac{x}{2} dx = -4\int (2\cos^2\frac{x}{2} - 1) \, d\cos\frac{x}{2} = -4\left[\frac{2}{3}\cos^3\frac{x}{2} - \cos\frac{x}{2}\right] = -\frac{8}{3}\cos^3\frac{x}{2} + 4\cos\frac{x}{2}$$

$y^* = -\frac{8}{3}\sin^3\frac{x}{2}\cos x - \frac{8}{3}\cos^3\frac{x}{2}\cdot\sin x + 4\cos\frac{x}{2}\sin x$

(2) $y'' - 6y' + 9y = 0$ $\quad \lambda^2 - 6\lambda + 9 = 0 \quad (\lambda-3)^2 = 0 \quad \lambda = 3$

$y_1 = e^{3x} \quad y_2 = xe^{3x}$ $\quad w(x) = \begin{vmatrix} e^{3x} & xe^{3x} \\ 3e^{3x} & e^{3x}+3xe^{3x} \end{vmatrix} = e^{6x}$

$C_1' = \frac{-xe^{3x}\cdot(x+1)e^{2x}}{e^{6x}} = \frac{-x(x+1)}{e^x}$ $\quad C_1 = \int x(x+1)\,de^{-x}$

$= e^{-x}x(x+1) - \int (2x+1)e^{-x}dx$

$C_2' = \frac{e^{3x}(x+1)e^{2x}}{e^{6x}} = \frac{x+1}{e^x}$ $\quad = e^{-x}(x^2+x) + 2\cancel{\int x\,de^{-x}}$

$= e^{-x}(x^2+x) + 2xe^{-x} - 3\int e^{-x}dx$

$C_2 = \int (x+1)e^{-x}dx = -\int (x+1)\,de^{-x}$ $\quad = e^{-x}(x^2+x) + 2xe^{-x} + 3e^{-x}$

$= -[(x+1)e^{-x} - \int e^{-x}d(x+1)]$ $= e^{-x}(x^2+3x+3)$

$= -(x+1)e^{-x} - e^{-x}$

$= e^{-x}(-x-2)$

$y^* = e^{-x}(x^2+3x+3)\cdot e^{3x} - (x+2)x\cdot e^{3x}$

$= (x+3)e^{2x}$

9. (2) $\lambda^3 - 2\lambda^2 + \lambda - 2 = 0$

$(\lambda^2+1)(\lambda-2) = 0 \quad \lambda_1 = 2 \quad \lambda_2 = i \quad \lambda_3 = -i$

基础解系: $e^{2x} \quad e^0(\cos x) \quad e^0\sin x$

通解 $y = C_1e^{2x} + C_2\cos x + C_3\sin x$

(4) $\lambda^4 + 2\lambda^2 + 1 = 0 \quad (\lambda^2+1)^2 = 0 \quad \lambda^2 = -1$

$\lambda_1 = -i \quad \lambda_2 = i$ 基础解系 $\cos x, \sin x, x\cos x, x\sin x$

通解 $y = C_1\cos x + C_2\sin x + C_3x\cos x + C_4x\sin x$

《第二册》

习题 8.1

3. (2) × $\vec{a}=\vec{0}$

(3) ×  $\vec{a}\cdot\vec{b}=0$ 向量积可能为0

(4) × $\vec{a}$和$\vec{c}$方向不一定相同

(6) × $(\vec{a}+\vec{b})\times(\vec{a}+\vec{b})=0$.

6. $(\vec{a}+\vec{b}+\vec{c})^2=2(\vec{a}\vec{b}+\vec{a}\vec{c}+\vec{b}\vec{c})+\vec{a}^2+\vec{b}^2+\vec{c}^2=0$

即 $\vec{a}\vec{b}+\vec{a}\vec{c}+\vec{b}\vec{c}=-\frac{3}{2}$

7. $(\vec{a}+3\vec{b})\cdot(7\vec{a}-5\vec{b})=7\vec{a}^2-15\vec{b}^2+16\vec{a}\vec{b}=0$ ---①

$(\vec{a}-4\vec{b})\cdot(7\vec{a}-2\vec{b})=7\vec{a}^2+8\vec{b}^2-30\vec{a}\vec{b}=0$

故 $23\vec{b}^2=46\vec{a}\vec{b}$ $\vec{b}^2=2\vec{a}\vec{b}$ $\vec{b}(\vec{b}-2\vec{a})=0$

→ 代入① $7\vec{a}^2-30\vec{a}\vec{b}+16\vec{a}\vec{b}=0$

$\vec{a}^2=2ab=\vec{b}^2$

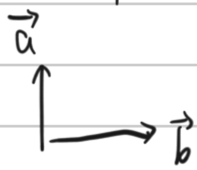
故 $|\vec{a}|=|\vec{b}|$

则 $8|\vec{a}|^2=16|\vec{a}||\vec{b}|\cos\theta$

$=16|\vec{a}|^2\cos\theta$

$\cos\theta=\pm\frac{1}{2}$ $\theta=60^\circ$ / 120°

8. $|(a+b)\times(a-b)|=|a+b|\cdot|a-b|\cdot\sin\theta(a+b,a-b)$



不妨设 $\vec{a}=(0,3)$ $\vec{b}=(4,0)$

$\vec{a}+\vec{b}=(4,3)$ $\vec{a}-\vec{b}=(-4,3)$

法一: 夹角 $\cos\theta=\frac{(\vec{a}+\vec{b})(\vec{a}-\vec{b})}{|\vec{a}+\vec{b}||\vec{a}-\vec{b}|}=\frac{-7}{5^2}=-\frac{7}{25}$

$\sin\theta=\frac{24}{25}$

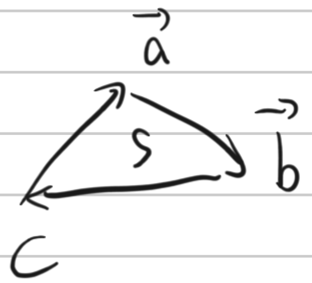
上式 $=5\times5\times\frac{24}{25}=24$

法二: 构成平行四边形的面积. $S=3\times(4-(-4))=24$

(2) $\vec{m}=3\vec{a}-\vec{b}=(-4,9)$ $\vec{n}=\vec{a}-2\vec{b}=(-8,3)$

$|\vec{m}\times\vec{n}| = S = \begin{vmatrix} -4 & 9 \\ -8 & 3 \end{vmatrix} = -12+72=60$

10.



$\vec{a}\times\vec{b}$ 相当于平行四边形面积
$=2S$

故 $|\vec{a}\times\vec{b}|=2S=|\vec{b}\times\vec{c}|=|\vec{c}\times\vec{a}|$

又∵ 都是左手系(上图)，方向相同，故为同一个向量

即 $\vec{a}\times\vec{b}=\vec{b}\times\vec{c}=\vec{c}\times\vec{a}$

13. $\vec{a}=(1,-3,1)$ $\vec{b}=(2,1,-3)$

$\vec{c}=(1,2,1)$

平行六面体体积 $|\vec{a}\ \vec{b}\ \vec{c}| = \begin{vmatrix} 1 & 2 & 1 \\ -3 & 1 & 2 \\ 1 & -3 & 1 \end{vmatrix}$

$= 7+3(2+3)+(4-1)$

$=7+15+5=27$

21. $\vec{a}=(5,2,5)$ $\vec{b}=(2,-1,2)$

$|\vec{a}|=\sqrt{54}$ $|\vec{b}|=3$ $e_b=\frac{\vec{b}}{|\vec{b}|}=(\frac{2}{3},-\frac{1}{3},\frac{2}{3})$

$\vec{a}\cdot\vec{e_b}=\frac{10}{3}-\frac{2}{3}+\frac{10}{3}=6.$

12.(4)

$$y''+(y')^2=2e^{-y} \quad ---①$$

法一：令 $p=y'$

$$\frac{dp}{dy}\cdot p+p^2=2e^{-y}$$

$$\frac{1}{2}\frac{dp^2}{dy}+p^2=2e^{-y}$$

令 $t=p^2$

$$\frac{1}{2}\frac{dt}{dy}+t=2e^{-y}$$

$$t'+2t=4e^{-y}$$

$$t=e^{-\int 2dy}\left(\int 4e^{-y}e^{\int 2dy}+c\right)$$

$$=e^{-2y}\left(4e^{y}+C\right)$$

$$(y')^2=4e^{-y}+e^{-2y}c$$

法二：令 $m=e^{y}$

$$m'=\frac{dm}{dx}=y'\cdot e^{y}$$

$$m''=\frac{dm}{dx^2}=(y')^2e^{y}+y''\cdot e^{y}$$

①式左右乘 e^{y}

$$\Rightarrow m''=2$$

则 $m'=2x+C_1$

$$e^{y}=m=x^2+C_1x+C_2$$