第七周作业

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3}$$

习题9.5

4. (1) $f(x,y) = \left(1+x+\frac{x^2}{2!}+o(x^2)\right)\left(y-\frac{y^2}{2}+\frac{y^3}{3}+o(y^3)\right)$

$$= y - \frac{y^2}{2} + \frac{y^3}{3} + xy - \frac{xy^2}{2} + \frac{x^2y}{2} + o(\rho^3)$$

(5) $\sin x^2 \cos y^2 + \sin y^2 \cos x^2$

$$\sin x^2 = x^2 - \frac{(x^2)^3}{3!} + \frac{(x^2)^5}{5!} + \cdots + \frac{(-1)^{n-1}(x^2)^{2n-1}}{(2n-1)!} + o(x^{4n})$$

$$\cos x^2 = 1 - \frac{(x^2)^2}{2!} + \frac{(x^2)^4}{4!} + \cdots + \frac{(-1)^n (x^2)^{2n}}{(2n)!} + o(x^{4n})$$

故 $f(x,y) = x^2 - \frac{x^6}{3!} + \cdots +$ ——

$= 0$

(7) $f'_x = 4x - y - 6$ $\quad f''_{xy} = -1$ $\quad f''_{xx} = 4$ $\quad f'''_{xxx} = 0$

$f'_y = -x - 2y - 3$ $\quad f''_{yx} = -1$ $\quad f''_{yy} = -2$ $\quad f'''_{yxy} = 0$

$= 0$

$$f(x,y) = f(1,-2) + \left((x-1)f'_x + (y+2)f'_y\right) + \frac{1}{2}\left((x-1)^2 f''_{xx} + (y+2)^2 f''_{y} + 2(x-1)(y+2) f''_{xy}\right)$$

$$= 5 + 2(x-1)^2 - (y+2)^2 - (x-1)(y+2)$$

5. $(3z^2 - 2x)\,dz - 2z\,dx + dy = 0$.

$$dz = \frac{2z}{3z^2-2x}dx - \frac{1}{3z^2-2x}dy \qquad x = y = z = 1$$

$$\frac{\partial z}{\partial x} = \frac{2z}{3z^2-2x} \qquad \frac{\partial^2 z}{\partial x^2} = \frac{2\frac{\partial z}{\partial x}(3z^2-2x) - \left(6z\frac{\partial z}{\partial x} - 2\right)(2z)}{(3z^2-2x)^2} = \frac{-6\frac{\partial z}{\partial x}z^2 + 4z - 4x\frac{\partial z}{\partial x}}{(3z^2-2x)^2}$$

$$= -16$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{6z\frac{\partial z}{\partial y}}{(3z^2-2x)^2} = -6 \qquad \frac{\partial^2 z}{\partial x \partial y} = \frac{6\frac{\partial z}{\partial x} - 2}{(3z^2-2x)^2} = 10$$

$$z = 1+2(x-1)-(y-1)+\frac{1}{2!}\frac{\partial^2 z}{\partial x^2}(x-1)^2+\frac{1}{2!}\frac{2\partial^2 z}{\partial x\partial y}(x-1)(y-1)$$
$$+\frac{1}{2!}\frac{\partial^2 z}{\partial x\partial y}(y-1)^2+R_2$$

$$\Rightarrow z = 1+2(x-1)-1(y-1)-8(x-1)^2+10(x-1)(y-1)$$
$$-3(y-1)^2+R_2$$

$\frac{50}{x}=\frac{20}{y}$

$\frac{y}{x}=\frac{2}{5}$

7.

(1) $f_x'=y-\frac{50}{x^2}=0$
$f_y'=x-\frac{20}{y^2}=0$ $\Rightarrow \begin{cases} x^2y=50 \\ y^2x=20 \end{cases}$ $\Rightarrow y=2 \quad x=5.$

$$H=\begin{vmatrix} \frac{100}{x^3} & 1 \\ 1 & \frac{40}{y^3} \end{vmatrix} = 3 > 0$$

$\frac{100}{x^3}=\frac{4}{5}>0$ 正定 极小值 $f(5,2)=30$

(4) $2(x^2+y^2)(2x+2y\cdot y') = a^2(2x-2yy')$

$y'=0$ $x^2+y^2=a^2$ 或 $x=0$ (舍)

原方程

代入得 $x^2-y^2=a^2$

$(x,y)=(\pm a,0)$

(5) $(x-1)^2+(y+1)^2+(z-2)^2=16$

$z'_x=\frac{1-x}{z}=0 \quad z'_y=\frac{-1-y}{z}=0$

$\Rightarrow x=1 \quad y=-1 \quad z=6/-2$

$H=\begin{vmatrix} z''_{xx} & z''_{xy} \\ z''_{xy} & z''_{yy} \end{vmatrix}>0.$

故有极值 $z''_{xx}=\frac{-(z+(1-x)z'_x)}{z^2}=-\frac{1}{z}$

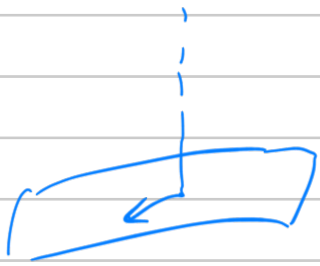
故 $z=6$ 极大值 $z=-2$ 极小值

$F(x,y,z)=0$

两边微分 $\Rightarrow (z^2-2x)dz+(x^2+y^2)dy+(z^2-y^2)dx=0$

能进一步通过微分操作得到 $\frac{d^2z}{dx^2}$ 和 $\frac{d^2z}{dy^2}$ 吗

$z=4\sqrt{1-\frac{x^2}{16}-\frac{y^2}{36}}$ 水滴下落.



(1) $\vec{v}=\left(\frac{x}{z},\frac{y}{\frac{9}{4}z}\right)$

沿最快方向.
(梯度)

$\frac{dy}{dx}=\frac{4y}{9x}$ $\quad \vec{u}=(dx,dy)$

$\vec{u}\parallel\vec{v}$

$\frac{1}{y}dy=\frac{1}{x}dx\cdot\frac{4}{9}$

$\ln y=\frac{4}{9}\ln x+C$

$y(1)=3$ $\quad \underline{C=\ln 3.}$

10. 条件极值 $F(x,y,\lambda)=f(x,y)+\lambda\varphi(x,y)$

(1) $F=x^2+y^2+\lambda(\frac{x}{a}+\frac{y}{b}-1)$

$F_x'=2x+\frac{\lambda}{a}=0$ $\quad F_y'=2y+\frac{\lambda}{b}=0$ $\quad F_\lambda'=\frac{x}{a}+\frac{y}{b}-1=0$

$\Rightarrow x=\frac{-\lambda}{2a}$ $\quad y=\frac{-\lambda}{2b}$ $\quad$ 且 $-\frac{\lambda}{2a^2}-\frac{\lambda}{2b^2}=1$ $\quad \lambda=-\frac{1}{\frac{1}{2a^2}+\frac{1}{2b^2}}=\frac{-2a^2b^2}{a^2+b^2}$

极值为 $\quad u=x^2+y^2=\frac{a^2b^4+a^4b^2}{(a^2+b^2)^2}=\frac{a^2b^2}{a^2+b^2}$ $\quad \Rightarrow x=\frac{ab^2}{a^2+b^2}$

极小值 $\quad y=\frac{a^2b}{a^2+b^2}$

(2) $F(x,y,z,\lambda)=\sin x\sin y\sin z+\lambda(x+y+z-\frac{\pi}{2})$

$F_x'=\cos x\sin y\sin z+\lambda=0$

$F_y'=\sin x\cos y\sin z+\lambda=0$

$F_z'=\sin x\sin y\cos z+\lambda=0$

$F_\lambda'=x+y+z-\frac{\pi}{2}=0$.

$\cos x\sin y=\sin x\cos y \Rightarrow \tan x=\tan y$ 轮换

$\Rightarrow \tan x=\tan y=\tan z$.

$\Rightarrow \tan x=\tan y=\tan(\frac{\pi}{2}-x-y)=\frac{1}{\tan(x+y)}$

$\tan(x+y)=\frac{2\tan x}{1-\tan^2x}$

$\Rightarrow \tan x=\frac{1-\tan^2x}{2\tan x} \Rightarrow 3\tan^2x=1$

$\Rightarrow \tan^2x=\frac{1}{3}$ 轮换 $\Rightarrow \tan^2y=\tan^2z=\frac{1}{3}$

又 $\because x>0\ y>0,\ z>0$ 且 $x+y+z=\frac{\pi}{2} \Rightarrow x=y=z$.

$\Rightarrow \tan x=\frac{\sqrt{3}}{3} \Rightarrow \sin x=\frac{1}{2} \Rightarrow u=\frac{1}{8}$ 极大值

(4) $F(x,y,z,\lambda,u) = xyz - \lambda(x+y+z) - u(x^2+y^2+z^2-1)$

$F'_x = yz - \lambda - 2ux = 0 \quad F'_y = xz - \lambda - 2uy = 0 \quad F'_z = xy - \lambda - 2uz = 0$

$F'_\lambda = x+y+z = 0 \quad F'_u = x^2+y^2+z^2-1 = 0$

$$yz - 2ux = xz - 2uz = xy - 2uz = \lambda$$

$$3\lambda = xy + yz + zx - 2u(x+y+z) = xy + yz + zx$$

$$(F'_\lambda)^2 = x^2+y^2+z^2 + 2(xy+yz+zx) = 1 + 2(xy+yz+zx) = 0$$

$$\Rightarrow 3\lambda = -\frac{1}{2} \quad \lambda = -\frac{1}{6}$$

$$\Rightarrow xyz = 2ux^2 - \frac{1}{6}x = 2uy^2 - \frac{1}{6}y = 2uz^2 - \frac{1}{6}z$$

$$3xyz = 2u(x^2+y^2+z^2) - \frac{1}{6}(x+y+z) = 2u.$$

$$xyz = \frac{2}{3}u.$$

$$u = \frac{yz+\frac{1}{6}}{2x} = \frac{xz+\frac{1}{6}}{2y} = \frac{xy+\frac{1}{6}}{2z} = \frac{yz+xz+\frac{1}{3}}{2x+2y} = \frac{yz+xz+\frac{1}{3}}{-2z}$$

$$\Rightarrow xy + \frac{1}{6} = -z(x+y) - \frac{1}{3}$$

$$\frac{1}{2} = (x+y)^2 - xy = x^2+y^2+xy = 1 - z^2 + xy$$

$$\Rightarrow z^2 = xy + \frac{1}{2} \Rightarrow xyz = z^3 - \frac{1}{2}z = \frac{2}{3}u = \frac{2}{3}\cdot\frac{z^2-\frac{1}{2}+\frac{1}{6}}{2z}$$

$$= \frac{z^2-\frac{1}{3}}{3z}$$

$$z^3 - \frac{1}{2}z = \frac{z^2-\frac{1}{3}}{3z}$$

$$\Rightarrow 3z^4 - \frac{3}{2}z^2 = z^2 - \frac{1}{3} \Rightarrow 3z^4 - \frac{5}{2}z^2 + \frac{1}{3} = 0$$

$$\Rightarrow 18z^4 - 15z^2 + 2 = 0 \quad \Rightarrow (3z^2-2)(6z^2-1) = 0$$

$$\begin{matrix} 3 & -2 \\ 6 & -1 \end{matrix} \quad \Rightarrow z^2 = \frac{2}{3} \quad z^2 = \frac{1}{6}$$

$z^2 = \frac{2}{3}$ 时 $xy = \frac{1}{6} > 0$ 同号 $\Rightarrow xyz = \pm\frac{\sqrt{6}}{18}$

$z^2 = \frac{1}{6}$ 时 $xy = -\frac{1}{3} < 0$ 异号 $\Rightarrow xyz = \pm\frac{\sqrt{6}}{18}$

有极值 $\pm\frac{\sqrt{6}}{18}$

11.

(2). $z'_x = 2x - y = 0$ $z'_y = 2y - x = 0$ $\Rightarrow x = y = 0 \Rightarrow z = 0$

边界: $|x| + |y| = 1$

由对称性, 考虑 $x + y = 1$ 和 $x - y = 1$

$z(x, y, \lambda) = x^2 - xy + y^2 - \lambda(x + y - 1)$

$z'_x = 2x - \overset{-y}{\lambda} = 0$ $z'_y = 2y - x - \lambda = 0$ $z'_\lambda = x + y - 1 = 0$

$2x - y = 2y - x \Rightarrow \lambda = y \Rightarrow x = y = \frac{1}{2}$ $z = x^2 - xy + y^2 = \frac{1}{4}$

若 $x - y = 1$ 无解. 故 $\min = 0$ $\max = \frac{1}{4}$

(4) $z = x^2 y(4 - x - y)$ $z'_x = 2xy(4 - x - y) - x^2 y = 0$

$z'_y = x^2(4 - x - y) - x^2 y = 0$

$x = 2(4 - x - y) = y$ $x = 2(4 - 2x)$ $5x = 8 \Rightarrow x = y = \frac{8}{5}$

$\Rightarrow z = \frac{64 \times 8}{125}\left(4 - \frac{16}{5}\right) = \frac{32 \times 64}{125 \times 5} = \frac{2048}{625}$

边界条件. ① $x = 0$ 或 $y = 0$ 时 $z = 0$.

② $x > 0, y > 0$. $x + y = 6$

$z = -2x^2 y$. $z(x, y, \lambda) = -2x^2 y - \lambda(x + y - 6)$

$z'_x = -4xy - \lambda = 0$ $z'_y = -2x^2 - \lambda = 0$ $z'_\lambda = x + y - 6 = 0$

$2x^2 = 4xy \Rightarrow x = 2y \Rightarrow y = 2$ $x = 4$

$\Rightarrow z = -32$.

故 $\min = -32$ $\max = \frac{2048}{625}$

13 $x^2+2y^2=6-2x^2-y^2$

$\Rightarrow x^2+y^2=2$

$z(x,y,\lambda)=x^2+2y^2-\lambda(x^2+y^2-2)$

$z'_x=2x-2x\lambda=0$ $z'_y=4y-2y\lambda=0$ $z'_\lambda=x^2+y^2-2=0$

$\Rightarrow$ ① $\begin{cases}x=0\\y^2=2\end{cases}\Rightarrow z=4$ ② $\begin{cases}x^2=2\\y=0\end{cases}\Rightarrow z=2$

min=2 max=4

14. $f'_x=6xy-4x^3$ $f'_y=3x^2-4y$

$f''_{xx}=6x-12x^2$ $f''_{xy}=6x$ $f''_{yy}=-4$ 在(0,0)点 det=0 非极值点

$f(0,y)=-2y^2$ $f(x,0)=-x^4$ 沿每条线为极值点,

三阶导 $f'''_{xxx}=6-24x$ $f'''_{xyx}=6$ $f'''_{xyy}=0$ $f'''_{yyy}=0$

Talar $\Rightarrow \frac{(6-24x)x^3+6x^2y}{3!}$ $f(x,y)$

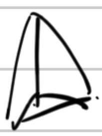
当y取0时 Talor 值会因x变化而变化,非极值点。

21 $-\frac{1}{2\sqrt{x}}dx-\frac{1}{2\sqrt{y}}dy-\frac{1}{2\sqrt{z}}dz=0$ 法向量$(\frac{1}{\sqrt{x_0}},\frac{1}{\sqrt{y_0}},\frac{1}{\sqrt{z_0}})$

切平面 $\frac{1}{\sqrt{x_0}}(x-x_0)+\frac{1}{\sqrt{y_0}}(y-y_0)+\frac{1}{\sqrt{z_0}}(z-z_0)=0$

截距 $\frac{z}{\sqrt{z_0}}=\frac{x_0}{\sqrt{x_0}}+\frac{y_0}{\sqrt{y_0}}+\frac{z_0}{\sqrt{z_0}}=\sqrt{a}$ $\Rightarrow z=\sqrt{a}\cdot\sqrt{z_0}$

$3\sqrt[3]{\sqrt{x_0y_0z_0}}\le\sqrt{a}$



$x_0=y_0=z_0=\frac{a}{9}$

$x+y+z=\sqrt{a}(\sqrt{x_0}+\sqrt{y_0}+\sqrt{z_0})=\sqrt{a}\cdot\sqrt{a}=a$

$V=\frac{1}{3}\times\frac{1}{2}xyz=\frac{1}{6}(\sqrt{a})^3\cdot\sqrt{x_0y_0z_0}\le\frac{1}{6}(\sqrt{a})^3\cdot\frac{1}{27}(\sqrt{a})^3=\frac{1}{162}a^3$

习题9.6.

5.(1) 散度 $\operatorname{div} V = \nabla\cdot V = 6x+3y^2+z^2+xy-6xz \Big|_{(1,-2,2)} = 8$

5.(2) 旋度 $\operatorname{rot} V = \nabla\cdot V = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = (1+2ze^y-1)\vec{i} - (0-0)\vec{j} + (0-xe^y-1)\vec{k}$

$= 2ze^y\vec{i} - (xe^y+1)\vec{k}$

8. $\nabla\cdot(\varphi\vec{a}) = \frac{\partial}{\partial x}(\varphi a_1) + \frac{\partial}{\partial x}(\varphi a_2) + \frac{\partial}{\partial x}(\varphi a_3)$

$\vec{a} = (a_1, a_2, a_3)$

$= \sum \frac{\partial\varphi}{\partial x}\cdot a_1 + \frac{\partial a_1}{\partial x}\cdot\varphi$

$= \varphi\cdot\left(\frac{\partial a_1}{\partial x} + \frac{\partial a_2}{\partial y} + \frac{\partial a_3}{\partial z}\right) + (a_1, a_2, a_3)\nabla\varphi$

$= \varphi\nabla\cdot a + a\cdot\nabla\varphi$

$\nabla\times(\varphi a) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \varphi a_1 & \varphi a_2 & \varphi a_3 \end{vmatrix} = \sum \frac{\partial}{\partial y}(\varphi a_3) - \frac{\partial}{\partial z}(\varphi a_2)$

$= \sum \frac{\partial\varphi}{\partial y}\cdot a_3 + \frac{\partial a_3}{\partial y}\cdot\varphi - \frac{\partial\varphi}{\partial z}a_2 - \frac{\partial a_2}{\partial z}\cdot\varphi$

$= \nabla\varphi\times\vec{a} + \varphi\nabla\times\vec{a}$

9. $\operatorname{div}\operatorname{rot}\vec{a} = \nabla\cdot(\nabla\times a) = \nabla\cdot \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_1 & a_2 & a_3 \end{vmatrix}$

$= \nabla\left[\left(\frac{\partial}{\partial y}a_3 - \frac{\partial}{\partial z}a_2\right)\vec{i} + \cdots\cdots\right]$ $\frac{\partial}{\partial z}a_1 - \frac{\partial}{\partial x}a_3 + \frac{\partial}{\partial x}a_2 - \frac{\partial}{\partial y}a_1$

$= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\cdot\left(\qquad\right)$

$= \sum \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}a_3 - \frac{\partial}{\partial z}a_2\right)$

$= \frac{\partial^2}{\partial x\partial y}a_3 - \frac{\partial^2}{\partial x\partial z}a_2 + \frac{\partial^2}{\partial y\partial z}a_1 - \frac{\partial^2}{\partial y\partial x}a_3$

$+ \frac{\partial^2}{\partial z\partial x}a_2 - \frac{\partial^2}{\partial z\partial y}a_1 = 0$