习题 6.1

2.18 作业：P234 1(3,4) 2(2,4) 3(1) 5(2)

1. (3) $xy'+y=y^2$

$x\cdot\frac{dy}{dx}=y^2-y$ $\qquad \frac{dy}{y^2-y}=\frac{1}{x}dx$

$\int\frac{1}{y(y-1)}dy=\int\frac{1}{x}dx$

$\int\frac{1}{y-1}-\frac{1}{y}dy=\int\frac{1}{x}dx$

$\ln(y-1)-\ln y=\ln x+C$

(4) $yy'=\frac{1-2x}{y}$

$\int y^2dy=\int(1-2x)dx$

$\frac{1}{3}y^3=x-x^2+C$

2. (2) $y'=\frac{y}{x}+\frac{x}{y}$

令 $u=\frac{y}{x}$ $\qquad y'=u+u'x$

$u+u'x=u+\frac{1}{u}$

$\frac{du}{dx}x=\frac{1}{u}$

$\int u\,du=\int\frac{1}{x}dx$

$\frac{1}{2}u^2=\ln x+C$

(4) $(x^2+3y^2)dx-2xydy=0$

$\frac{dy}{dx}=\frac{x}{2y}+\frac{3y}{2x}$

令 $u=\frac{y}{x}$ $\qquad y'=u+u'x$

$u+u'x=\frac{1}{2u}+\frac{3}{2}u$

$u'x=\frac{1}{2u}+\frac{1}{2}u$

$2u'x=\frac{1}{u}+u$

$\int\frac{2}{\frac{1}{u}+u}du=\int\frac{1}{x}dx$

$\int\frac{2u}{1+u^2}du=\ln x+C$

$\ln(1+u^2)=\ln x+C$

3. 证明：设 $\begin{cases}a_1x+b_1y+c_1=0\\a_2x+b_2y+c_2=0\end{cases}$ 有解 (x_0,y_0)

故令 $u=x-x_0$ $\quad v=y-y_0$ $\quad$ 有 $\frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}=\frac{a_1u+b_1v}{a_2u+b_2v}$

即化为齐次方程

3.(1) $\frac{dy}{dx}=\frac{x+y+3}{x-y+1}$ $x_0=-2$ $y_0=-1$

令 $u=x+2$ 令 $v=y+1$

$$\frac{dy}{dx}=\frac{u+v}{u-v}=\frac{1+\frac{v}{u}}{1-\frac{v}{u}}$$ 令 $t=\frac{v}{u}$

$$\frac{dy}{dx}=\frac{dv}{du}=v'=t+t'u=\frac{1+t}{1-t}$$

故 $\frac{dt}{du}\cdot u=\frac{1+t^2}{1-t}$

$$\int\frac{1-t}{1+t^2}dt=\int\frac{1}{u}du$$

$$\arctan t-\frac{1}{2}\ln(1+t^2)=\ln u+C$$

$$\arctan\frac{v}{u}-\frac{1}{2}\ln\left(1+\frac{v^2}{u^2}\right)=\ln u+C$$

$$\arctan\frac{y+1}{x+2}-\frac{1}{2}\ln\left(1+\frac{(y+1)^2}{(x+2)^2}\right)=\ln(x+2)+C$$

5.(2)

$$y'+\frac{y}{x}=\frac{\sin x}{x}$$

$$y=e^{-\int\frac{1}{x}dx}\left(\int\frac{\sin x}{x}\cdot e^{\int\frac{1}{x}dx}dx+C\right)$$

$$=\frac{1}{x}(-\cos x+C)$$

又 ∵ $y(\pi)=1$ 故 $1=\frac{1}{\pi}(1+C)$

$C=\pi-1$ 即 $y=\frac{1}{x}(-\cos x+\pi-1)$

2.20 作业 P234 6(2,3) 8 9 12(2,4) 13(2)
P247 6 8

6. (2) $y' = \cos(x-y)$

令 $t = x-y$ $\quad t' = 1-y'$

故 $1-t' = \cos t$

$t' = 1-\cos t = \frac{dt}{dx}$

$\int \frac{dt}{1-\cos t} = \int 1\, dx$

$\frac{1}{2}\int \frac{dt}{\cos^2\frac{t}{2}} = x+C$

$\int \frac{1}{\cos^2\frac{t}{2}}\, d\frac{t}{2} = x+C$

$2\tan\frac{t}{2} = x+C$

即 $2\tan\frac{x-y}{2} = x+C$

(3) $y' - e^{x-y} + e^x = 0$

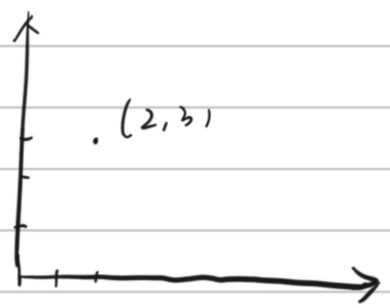
$\frac{y'}{e^x} - \frac{1}{e^y} + 1 = 0$

$\frac{dy}{dx}\cdot\frac{1}{e^x} = \frac{1}{e^y} - 1 = \frac{1-e^y}{e^y}$

$\int \frac{e^y}{1-e^y}\, dy = \int e^x\, dx$

$-\ln(1-e^y) = e^x + C$

8.



$y - y_0 = y'(x-x_0)$

$(0,\ y_0 - y'x_0) \quad (-\frac{y_0}{y'}+x_0,\ 0)$

$2x_0 = -\frac{y_0}{y'} + x_0$

$2y_0 = y_0 - y'x_0$

$x_0 = -\frac{y_0}{y'}$

$y_0 = -y'x_0$

即 $y'(x_0) = \frac{-y(x_0)}{x_0}$

故 $y' = \frac{-y}{x}$ $\quad \frac{-1}{y}dy = \frac{1}{x}dx$

$\ln\frac{1}{y} = \ln x + C \Rightarrow \frac{1}{y} = C_0 x$ 故 $xy = 6$

又∵过 $(2,3)$ 故 $y = \frac{3}{2}x$

9. $f'(x) = f(x)$ $\quad \frac{df(x)}{dx} = f(x)$ 即 $\int \frac{1}{f(x)}\, df(x) = \int dx$

$\ln f(x) = x + C \quad f(x) = C_0 e^x$

又 $f(0) = 0$ 故 $C_0 = 0$ $\quad f(x) \equiv 0$

12. (2) $y'' = \frac{y'}{x} + x$

$y'' - \frac{1}{x} y' = x$

$y' = e^{-\int -\frac{1}{x}dx} \left(\int x \cdot e^{\int -\frac{1}{x}dx} dx + c_0 \right)$

$= x \cdot (x + c_0)$

即 $\int dy = \int (x^2 + c_0 x)\, dx$

故 $y = \frac{1}{3}x^3 + \frac{c_0}{2}x^2 + C$

(4) $y'' + (y')^2 = 2e^{-y}$

令 $e^{y} = t$

$t' = e^{y} \cdot y' = t y'$

$t'' = (y')^2 \cdot e^{y} + e^{y} \cdot y''$

原式为 $e^{y} \cdot y'' + (y')^2 e^{y} = 2$

也即 $t'' = 2$

$t' = 2x + C_0$

$e^{y} = t = x^2 + C_0 x + C_1$

13. (2)

$y^3 y'' = -1$

令 $y' = p$ $\quad y'' = \frac{dp}{dx} = \frac{dp}{dy} \cdot p$

故 $y^3 \cdot \frac{dp}{dy} p = -1$ $\quad \int p\, dp = -\int \frac{1}{y^3} dy$

即 $\frac{1}{2}p^2 = \frac{1}{2} \frac{1}{y^2} + C$

也即 $p^2 = \frac{1}{y^2} + C_0$ $\quad$ 当 $x=1$ 代入 $\quad C_0 = -1$

故 $(y')^2 = \frac{1}{y^2} - 1$ $\quad y' = \pm\sqrt{\frac{1}{y^2} - 1} = \frac{dy}{dx}$

$\int \frac{y}{\sqrt{1-y^2}} dy = \pm \int dx$

$-\sqrt{1-y^2} = \pm x + C$

$\pm\sqrt{1-y^2} = x + C_1$ $\quad$ 取 $x=1$ 得 $C_1 = -1$

故方程解为 $\pm\sqrt{1-y^2} = x - 1$

习题 6.2

6. $C_0+C_1x+C_2x^2+\cdots+C_nx^n=0$.

令 $x=0$. $\Rightarrow C_0=0$

故 $C_1x+C_2x^2+\cdots+C_nx^n=0$.

两边同除 x $\Rightarrow C_1+C_2x+\cdots+C_nx^{n-1}=0$

再取 $x=0$. $\Rightarrow C_1=0$

重复上述操作 $\Rightarrow C_0=C_1=\cdots=C_n=0$

故线性无关

$1-\cos^2x-\sin^2x=0$ $\exists$ $C_0=1$ $C_1=-1$ $C_2=-1$

使 $C_0+C_1\cos^2x+C_2\sin^2x=0$ 故线性相关

8. 设 $C_1y_1(x)+C_2y_2(x)\equiv0$. $x\in[0,2]$

当 $x\in[0,1]$ 时, $C_1(x-1)^2=0$ $\Rightarrow C_1=0$.

当 $x\in[1,2]$ 时 $C_2(x-1)^2=0$ $\Rightarrow C_2=0$

故 $y_1(x)$ 和 $y_2(x)$ 线性无关

当 $x\in[0,1]$ 时 $y_1'(x)=2x-2$ $y_2'(x)=0$.

$$W(x)=\begin{vmatrix}(x-1)^2 & 0\\ 2x-2 & 0\end{vmatrix}=0$$

当 $x\in(1,2]$ 时 $y_2'(x)=2x-2$ $y_1'(x)=0$

$$W(x)=\begin{vmatrix}0 & (x-1)^2\\ 0 & 2x-2\end{vmatrix}=0$$

即 $W(x)\equiv0$.