Assignment 3
Optimisation for Computer Science
Group 33

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1 Position estimation

1. State vector x is an approximation of the position of the boat regarding the lighthouses l^j and angles to those lighthouses z_j^t in 2D grid. Start position of the x is [2,2].

$$x^i = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

We update position of the x by using Kalman filter with measurements of the angle from position x to the lighthouse position l. Position changes for value of the step t with every iteration. We start from position [2,2] as previously stated and finish at position [-4,-20]. We did our estimations for $\lambda = 0.6$ and $\lambda = 0.9$ that yielded different results because λ value affects noise n_j. The formula for extended Kalman filter is:

$$\xi_i = \xi_{i-1} + \mathbf{H}_i^{-1}\mathbf{C}_i^\top(z_i - \mathbf{C}_i\xi_{i-1}),$$

Where ξ_i is our new best estimate of position. ξ_{i-1} is a previous best estimate from the last step. H_i^{-1} is a correction for known external influences. $C_i^\top$ is a derivative of our g function ($\arctan \frac{l_2-x_2}{l_1-x_1}$)

2. For the $\lambda = 0.6$ we get slower descent to solution. The reason is that for the small λ we only take in account small percentage of previous solution that diverges. We have three important variables that we take (λ coefficient, H_i and n_i) and if only small portion of that value is gotten from previous solution we can't diverge fast. The other case is for the $\lambda = 0.9$, where we take the value from the previous step more in account.

$$\begin{aligned} f(x) &= \arctan \frac{l_2 - x_2}{l_1 - x_1}, \\ \frac{\partial f(x)}{\partial x_1} &= \frac{1}{1 + (\frac{l_2-x_2}{l_1-x_1})^2} \frac{l_2 - x_2}{(l_1 - x_1)^2}, \\ \frac{\partial f(x)}{\partial x_2} &= \frac{1}{1 + (\frac{l_2-x_2}{l_1-x_1})^2} \frac{-1}{l_1 - x_1} \end{aligned}$$

Detailed derivation calculation is appended in the end of this document.

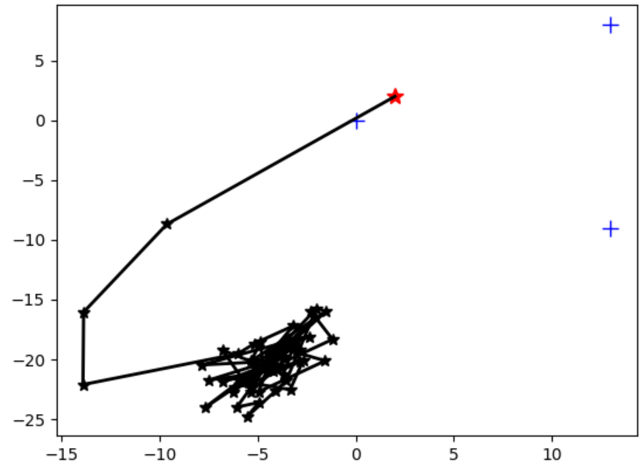

Image 1: $\lambda = 0.6$

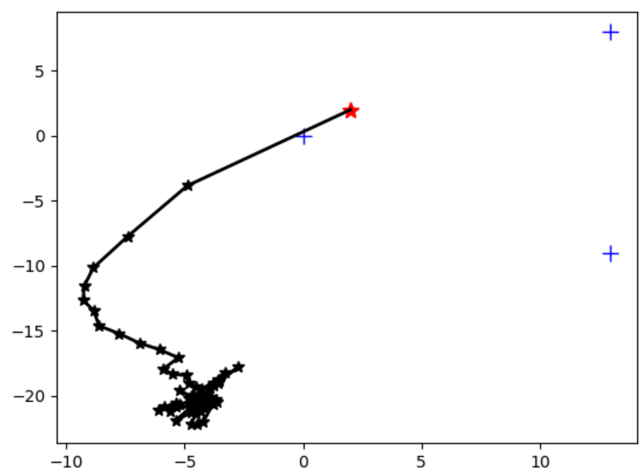

Image 2: $\lambda = 0.9$

On these graphs we can notice that our position for the $\lambda = 0.6$ oscillates more. Our result in both scenarios is the same, but we got there more smoothly with bigger lambda.

2 Motion estimation

1. Same as the first assignment, state vector x describes position of the ship in 2D grid. This time the start position is [-4,-20] which we got from the first task. In second task we use extended Kalman filter with the measured distances of the position x and the lighthouse positions l.

$$x_t = x_0 + t * v$$

Where x_0 is initial position, t is time and v is the vector valued course. Every step t we measure the distance to the towers.

$$z_j^t = \|l^j - x_t\|_2 + n_j^t$$

This z function is our g function in this task. We also need a derivative for it. For the easier written derivations, $f(x) = z$.

$$\frac{\partial f(x)}{\partial x_1} = \frac{x_1 - l_1}{\|l^j - x_t\|_2} = \frac{x_1 - l_1}{f(x)},$$
$$\frac{\partial f(x)}{\partial x_2} = \frac{x_1 - l_1}{\|l^j - x_t\|_2} = \frac{x_2 - l_2}{f(x)},$$

Detailed derivation calculation is appended in the end of this document.

After 200 iterations, our x position changes to the position [22, -7]. Again, the same as in the first task, we did those measurements with two values for λ. When $\lambda = 0.6$

View of a state vector x:

$$x^i = \begin{pmatrix} x_0^1 \\ x_0^2 \end{pmatrix} + t \begin{pmatrix} v^1 \\ v^2 \end{pmatrix}$$
$$\xi^i = \begin{pmatrix} x_0^1 \\ x_0^2 \\ v^1 \\ v^2 \end{pmatrix}$$

2. From the graphs we can conclude that the ship has to make three course changes. Firstly, it starts with going north, then it moves to the east (right), then it sales to the north again and ends with going in the east direction.

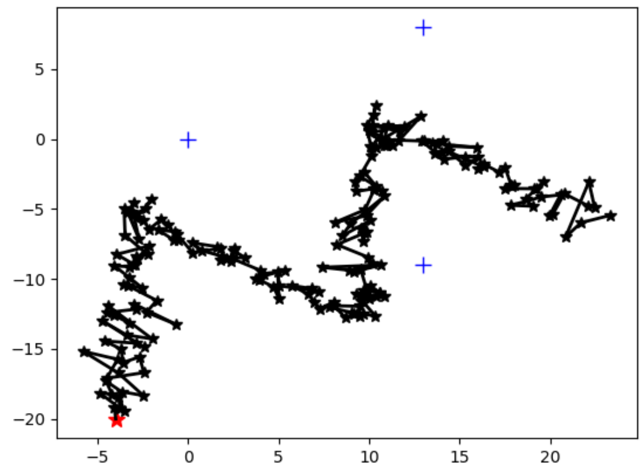

Image 4: $\lambda = 0.6$

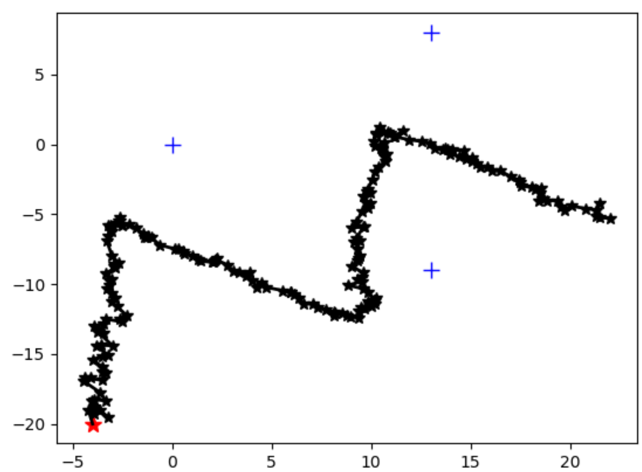

Image 5: $\lambda = 0.9$

3. We created 2D scatter plot with values for v vector course. From the graph we can see that the density of the dots is around v values $0 < v_1 < 0.14$ and $0.3 > v_2 > 0.14$. These are the courses which are required to sail through the narrow shoal.

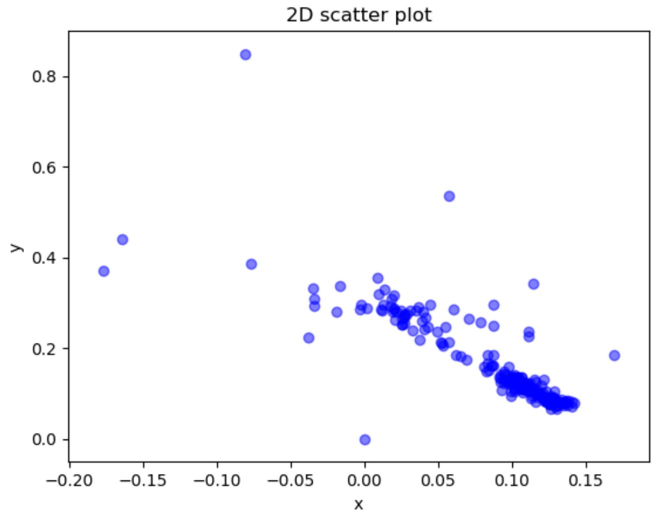

Image 6: $\lambda = 0.6$

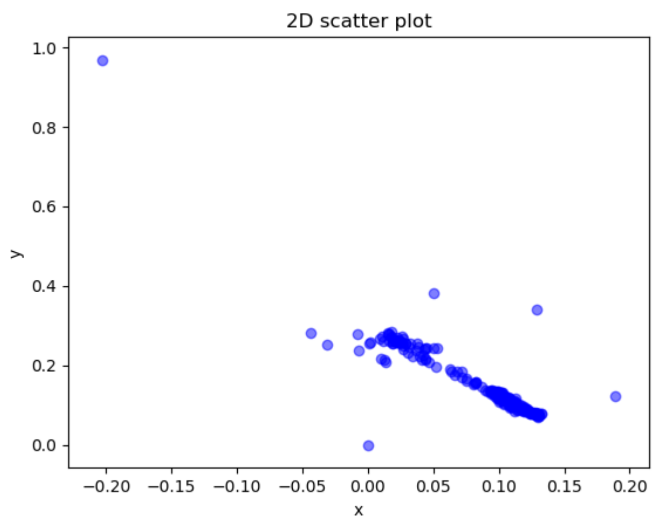

Image 7: $\lambda = 0.9$

$$f(x) = \| l^j - x_t \|_2$$

$$f(x) = \sqrt{(l_1 - x_1)^2 + (l_2 - x_2)^2} = \left[(l_1 - x_1)^2 + (l_2 - x_2)^2 \right]^{\frac{1}{2}}$$

$$\frac{\partial f(x)}{\partial x_1} = \frac{1}{2} \cdot \left[(l_1 - x_1)^2 + (l_2 - x_2)^2 \right]^{-\frac{1}{2}} \cdot \left[(l_1 - x_1)^2 + (l_2 - x_2)^2 \right]'$$

$$= \frac{1}{2} \cdot \left[(l_1 - x_1)^2 + (l_2 - x_2)^2 \right]^{-\frac{1}{2}} \cdot \left(l_1^2 - 2 l_1 x_1 + x_1^2 + l_2^2 - 2 l_2 x_2 + x_2^2 \right)'$$

$$= \frac{1}{2} \cdot \frac{1}{\sqrt{(l_1 - x_1)^2 + (l_2 - x_2)^2}} \cdot (-2 l_1 + 2 x_1)$$

$$= \frac{1}{\cancel{2}} \cdot \frac{1}{\| l - x \|_2} \cdot \cancel{2} (x_1 - l_1) = \frac{x_1 - l_1}{\| l - x \|_2} = \frac{x_1 - l_1}{f(x)}$$

$$\frac{\partial f(x)}{\partial x_2} = \frac{1}{2} \cdot \left[(l_1 - x_1)^2 + (l_2 - x_2)^2 \right]^{-\frac{1}{2}} \cdot \left[(l_1 - x_1)^2 + (l_2 - x_2)^2 \right]'$$

$$= \frac{1}{2} \cdot \frac{1}{\sqrt{(l_1 - x_1)^2 + (l_2 - x_2)^2}} \cdot \left(l_1^2 - 2 l_1 x_1 + x_1^2 + l_2^2 - 2 l_2 x_2 + x_2^2 \right)'$$

$$= \frac{1}{2} \cdot \frac{1}{\| l - x \|_2} \cdot (-2 l_2 + 2 x_2)$$

$$= \frac{1}{\cancel{2}} \cdot \frac{1}{\| l - x \|_2} \cdot \cancel{2} (x_2 - l_2)$$

$$= \frac{x_2 - l_2}{\| l - x \|_2} = \frac{x_2 - l_2}{f(x)}$$

$$f(x) = \arctan\left(\frac{l_2 - x_2}{l_1 - x_1}\right) \qquad \arctan(x) = \frac{1}{1+x^2} \cdot x'$$

$$\frac{\partial f(x)}{\partial x_1} = \frac{1}{1+\left(\frac{l_2-x_2}{l_1-x_1}\right)^2} \cdot \left(\frac{l_2-x_2}{l_1-x_1}\right)'$$

$$= \frac{1}{1+\left(\frac{l_2-x_2}{l_1-x_1}\right)^2} \cdot \frac{(l_1-x_1)(l_2-x_2)' - (l_2-x_2)(l_1-x_1)'}{(l_1-x_1)^2}$$

$$= \frac{1}{1+\left(\frac{l_2-x_2}{l_1-x_1}\right)^2} \cdot \frac{-(l_2-x_2)\cdot(-1)}{(l_1-x_1)^2}$$

$$= \frac{1}{1+\left(\frac{l_2-x_2}{l_1-x_1}\right)^2} \cdot \frac{l_2-x_2}{(l_1-x_1)^2}$$

$$\frac{\partial f(x)}{\partial x_2} = \frac{1}{1+\left(\frac{l_2-x_2}{l_1-x_1}\right)^2} \cdot \left(\frac{l_2-x_2}{l_1-x_1}\right)'$$

$$= \frac{1}{1+\left(\frac{l_2-x_2}{l_1-x_1}\right)^2} \cdot \frac{(l_1-x_1)(l_2-x_2)' - (l_2-x_2)(l_1-x_1)'}{(l_1-x_1)^2}$$

$$= \frac{1}{1+\left(\frac{l_2-x_2}{l_1-x_1}\right)^2} \cdot \frac{\cancel{(l_1-x_1)}\cdot(-1)}{(l_1-x_1)^{\cancel{2}}}$$

$$= \frac{1}{1+\left(\frac{l_2-x_2}{l_1-x_1}\right)^2} \cdot \left(-\frac{1}{l_1-x_1}\right)$$