Assignment 1

Numerical Optimization / Optimization for CS WS2019

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Submission: Upload your report as a single pdf–file, including all your derivations, plots, and discussions. Also upload your python implementations to the TU-Graz TeachCenter. Please do not zip the files and include your names and group number in your report.

Deadline: Nov 6, 2019 at 18:00

1 Characterization of Functions (10P.)

Plot the level sets of the following functions using a contour plot[1] and visualize the functions in 3D using a surface plot[2] Compute the gradient and the Hessian for all functions and determine the set of stationary points and characterize every stationary point whether it is a saddle point, (strict) local/global minimum or maximum.

1. $f(x, y) = 2x^3 - 6y^2 + 3x^2y$
2. $f(x, y) = (x - 2y)^4 + 64xy$
3. $f(x, y) = 2x^2 + 3y^2 - 2xy + 2x - 3y$
4. $f(x, y) = \ln\left(1 + \frac{1}{2}(x^2 + 3y^2)\right)$

2 Numerical Gradient Approximation (2P.)

Validate your derived gradients of Section 1 by computing a numerical approximation using central differences

$$\nabla_x f(x, y) \approx \frac{f(x + \epsilon, y) - f(x - \epsilon, y)}{2\epsilon}$$
$$\nabla_y f(x, y) \approx \frac{f(x, y + \epsilon) - f(x, y - \epsilon)}{2\epsilon}\,.$$

Write a short python script to check the gradients for all the functions in Section 1 numerically. To do so, set $\epsilon = 0.001$ and choose random points $(x, y)^T$. For each point compare the result of your analytically computed gradient with the numerically approximation.

3 Vectors, Norms and Matrices (5P.)

1. Show that $\|\cdot\|_{\frac{1}{2}}$ is not a norm.

[1] https://matplotlib.org/3.1.1/api/_as_gen/matplotlib.pyplot.contour.html
[2] https://matplotlib.org/mpl_examples/mplot3d/surface3d_demo.py

2. Show that for any $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n$

$$\|x - z\| \leq \|x - y\| + \|y - z\|$$

holds.

3. Let $\mathbf{A} \in \mathbb{R}^{n \times n}$ be symmetric. Show that $\mathbf{A}$ is positive semidefinite if and only if there exists a matrix $\mathbf{B}$ such that $\mathbf{A} = \mathbf{B}\mathbf{B}^T$.

4. Let $\mathbf{x} \in \mathbb{R}^n$ and let $\mathbf{A}$ be given by the outer vector product, i.e.,

$$\mathbf{A} = \mathbf{x}\mathbf{x}^T \iff a_{ij} = x_i x_j, \quad i, j = 1, 2, \ldots, n\,.$$

Show that $\mathbf{A}$ is positive semidefinite. Moreover, show that $\mathbf{A}$ is not positive definite for $n > 1$.

5. Let $\mathbf{Q} \in \mathbb{R}^{n \times n}$ be a positive definite matrix. Show that

$$\|\mathbf{x}\|_Q = \sqrt{\mathbf{x}^T \mathbf{Q} \mathbf{x}}$$

is a norm.

4 Matrix Calculus (4P.)

Calculate the first and second derivatives of the following functions analytically. First, compute the derivative for each element using the summation forumlation as shown in the exercise lecture. Second, convert your result back to linear algebra notation. Include all your steps in your report.

1. $f(\mathbf{x}) = \frac{1}{2}\|\mathbf{A}\mathbf{x} - \mathbf{b}\|^2$ for $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{b} \in \mathbb{R}^m$ and $\mathbf{A} \in \mathbb{R}^{m \times n}$
2. $f(\alpha) = \frac{1}{2}\|\mathbf{A}(\mathbf{x} + \alpha\mathbf{y}) - \mathbf{b}\|^2$ for $\alpha \in \mathbb{R}$, $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $\mathbf{b} \in \mathbb{R}^m$ and $\mathbf{A} \in \mathbb{R}^{m \times n}$

5 Student Task Selection Problem (4P.)

Assume there are $I = 2$ students that need to complete an assignment sheet, which consists of $J = 15$ tasks. Each student has a total time budget which can be assigned to the individual tasks. To account for the different abilities of the students, they estimated for each student the required time for each task in hours as well as their time budget and inserted it into the following table:

	Task															
Student	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	Time budget
1	0.5	0.25	0.25	0.25	1.0	1.0	0.5	0.5	1.0	0.5	1.5	2.5	1.0	2.5	3.5	9
2	0.75	1.0	0.75	0.5	0.5	0.25	0.25	0.25	2.0	1.25	1.0	4.0	2.5	3.0	2.0	6

Given the time budgets and the estimated task time of each student, they need to figure out the distribution of students to tasks that minimizes the time to complete the whole assignment sheet.

To do so, you have to complete the following steps:

1. Cast the student task selection problem into a linear program.
2. Convert the linear program into the standard linear program form.
3. Use `scipy`'s linear program solver[3] to solve the linear program.
4. Report the student to task assignment in your report and document each step. In addition, include your `python` script in your submission.

[3] `https://docs.scipy.org/doc/scipy/reference/generated/scipy.optimize.linprog.html`