Part 2: Reconstruction

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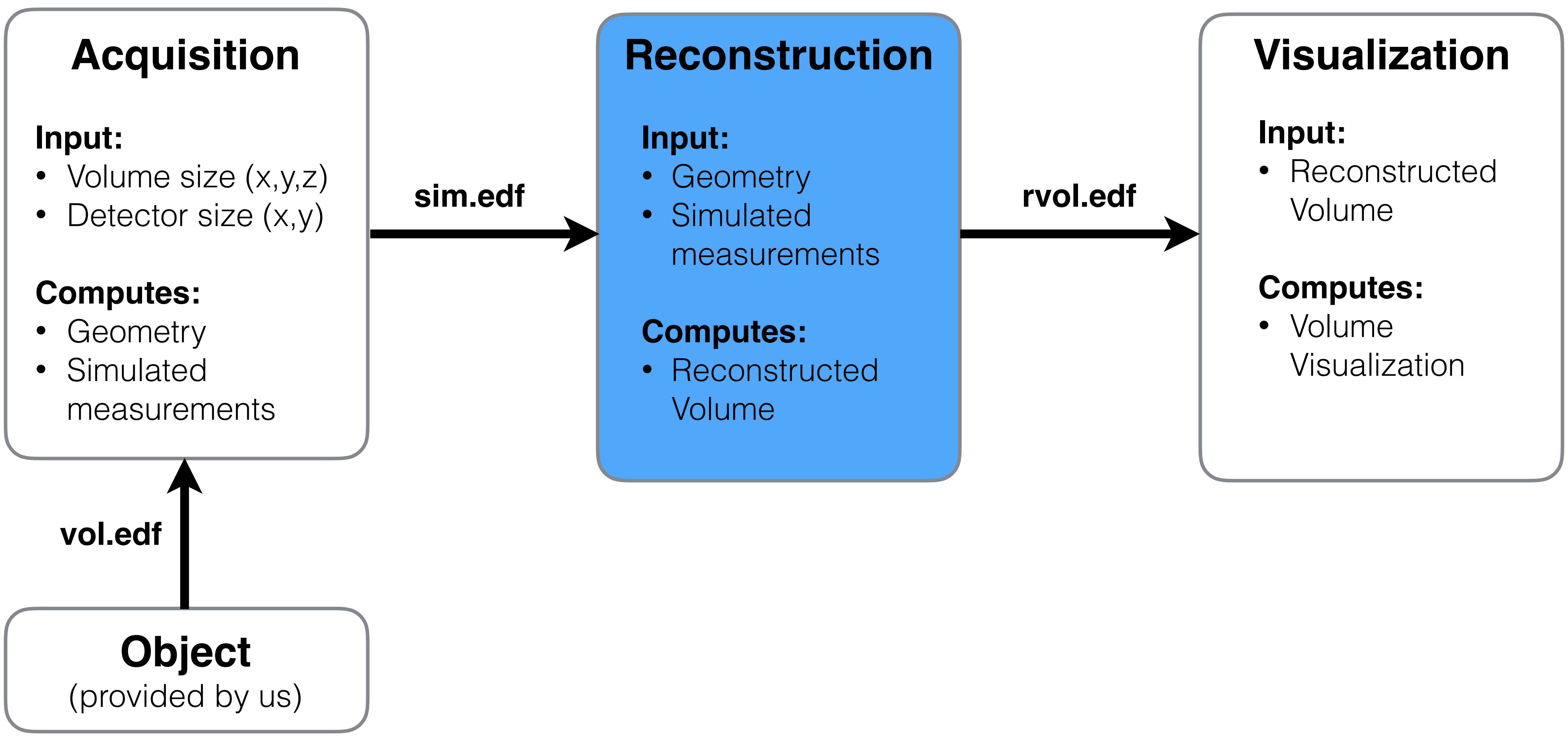
Initial Comments

- Commit your progress often!
- Use your(!) account to commit

Part 2: Flow-diagram

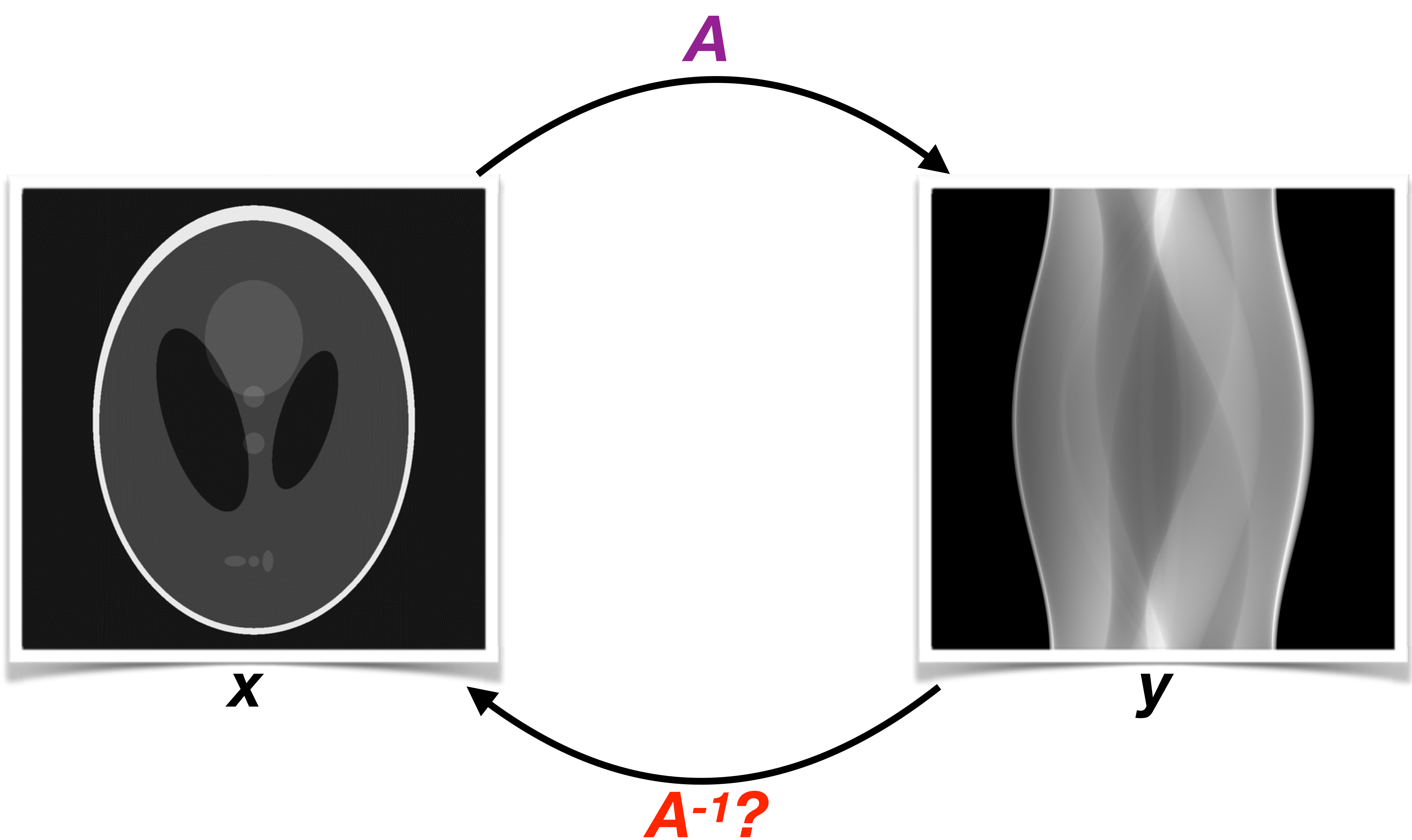


Part 2: Reconstruction

- Computed tomography: motivation (Recap)
- Forward model (Recap)
- Problem formulation and solving

How to compute tomographic reconstructions?

Model and inversion



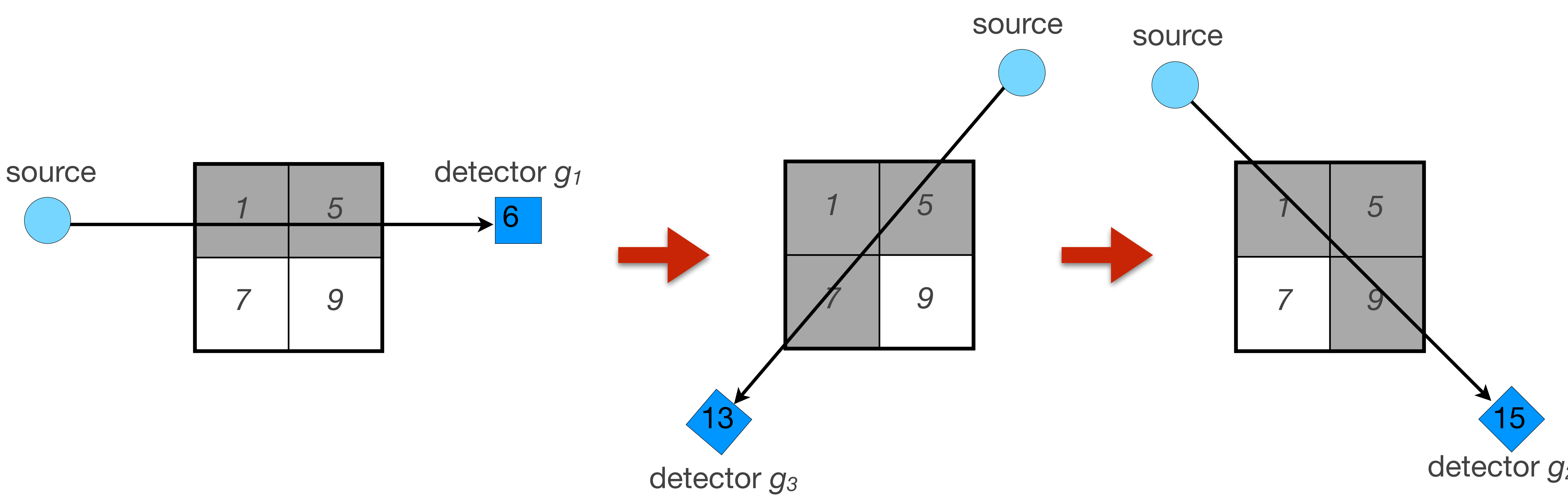
On-the-fly system-matrix

- Raytracing
- Forward/back projection
- (See last session)

What to do with those voxels?

Forward projection: Computation of *Ax*

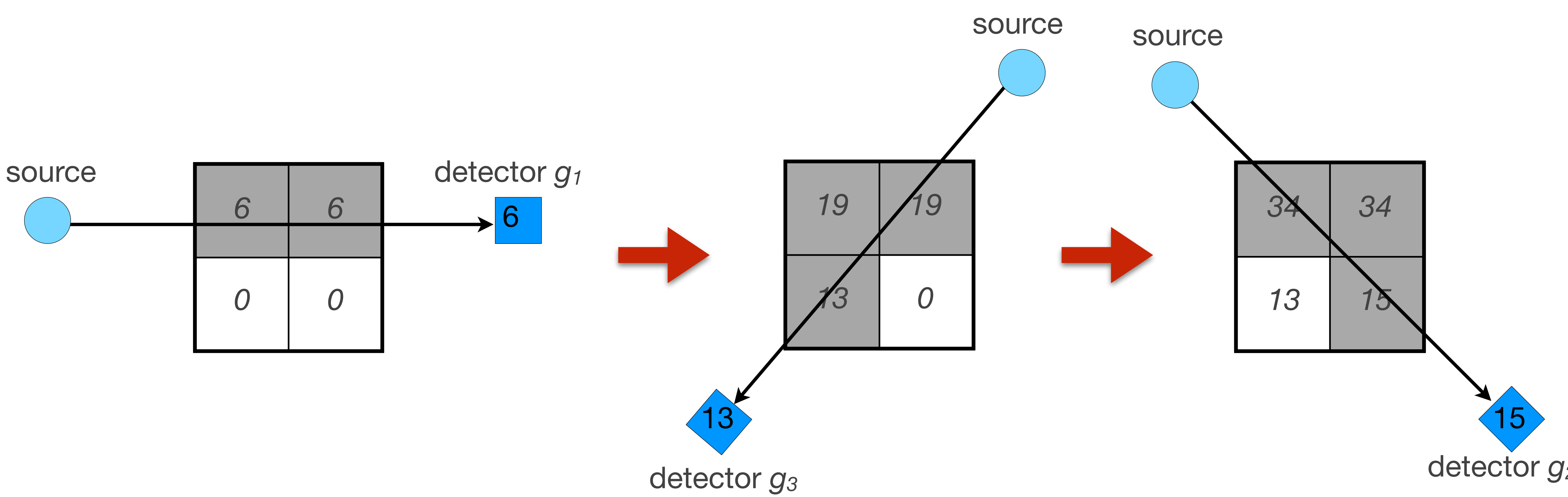
- Accumulate all voxel values along the ray



What to do with those voxels?

Back projection: Computation of $A^T z$

- Initial: Zero volume
- Successively add the element value of z to each value of a voxel hit by ray



Part 2: Reconstruction

- Computed tomography: motivation (Recap)
- Forward model (Recap)
- Problem formulation and solving

The tomographic problem
(in a perfect world)

- We have to solve a linear equation system

$$\mathrm{Ax} = \mathrm{y}$$

with A being the system-matrix, x the volume and y the measurements.

The tomographic problem (in reality)

- We do not have an exact linear equation
- Causes:
  - Noise
  - Patient movement
  - Beam hardening
  - and many more

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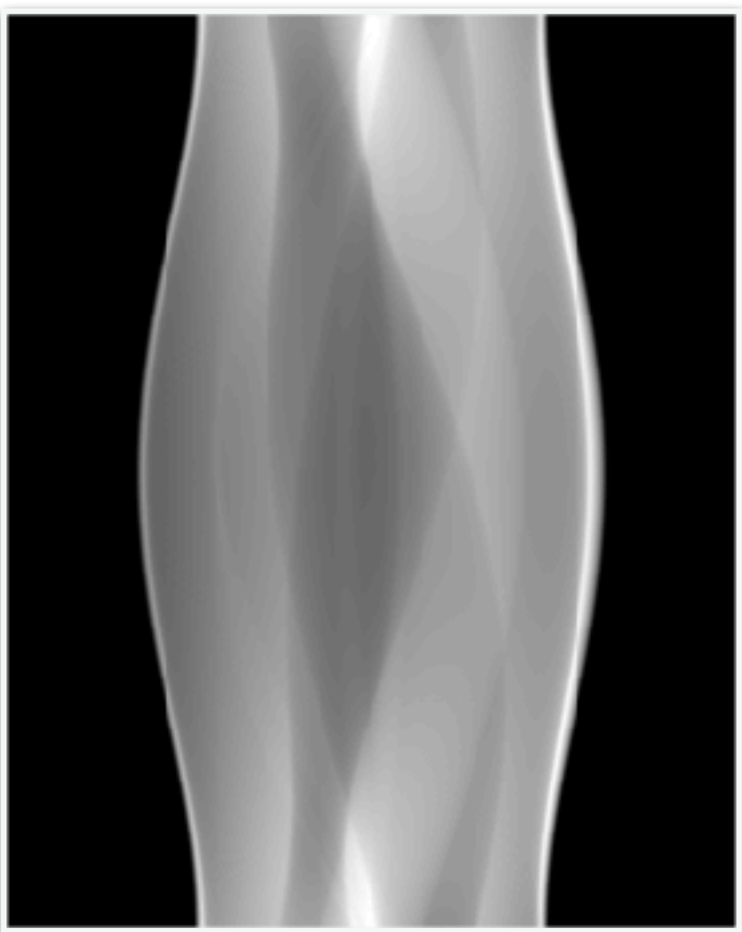
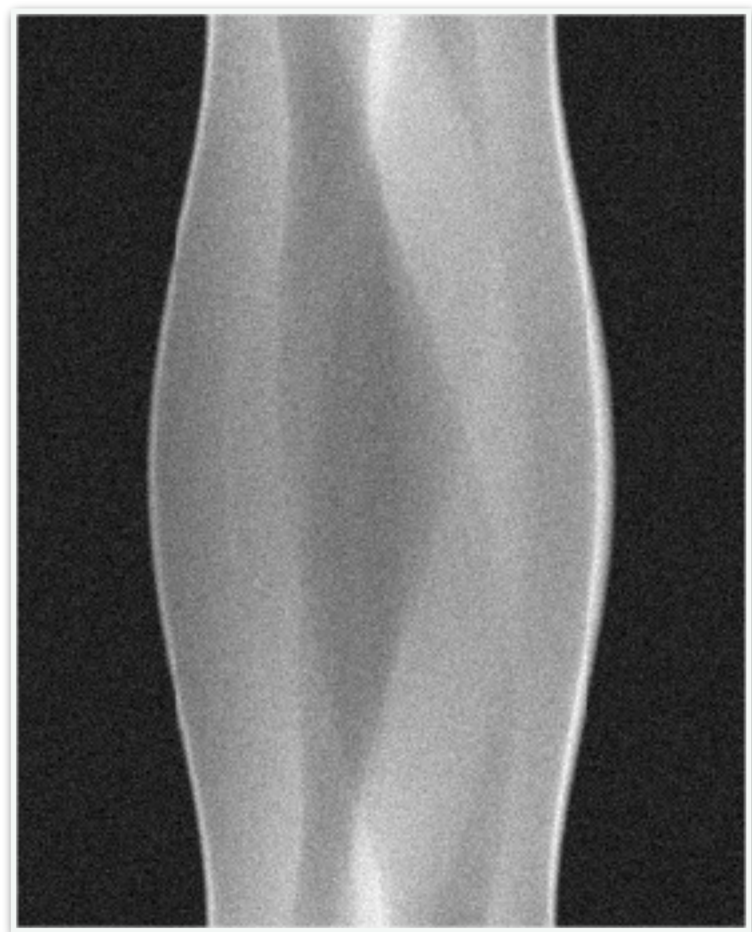
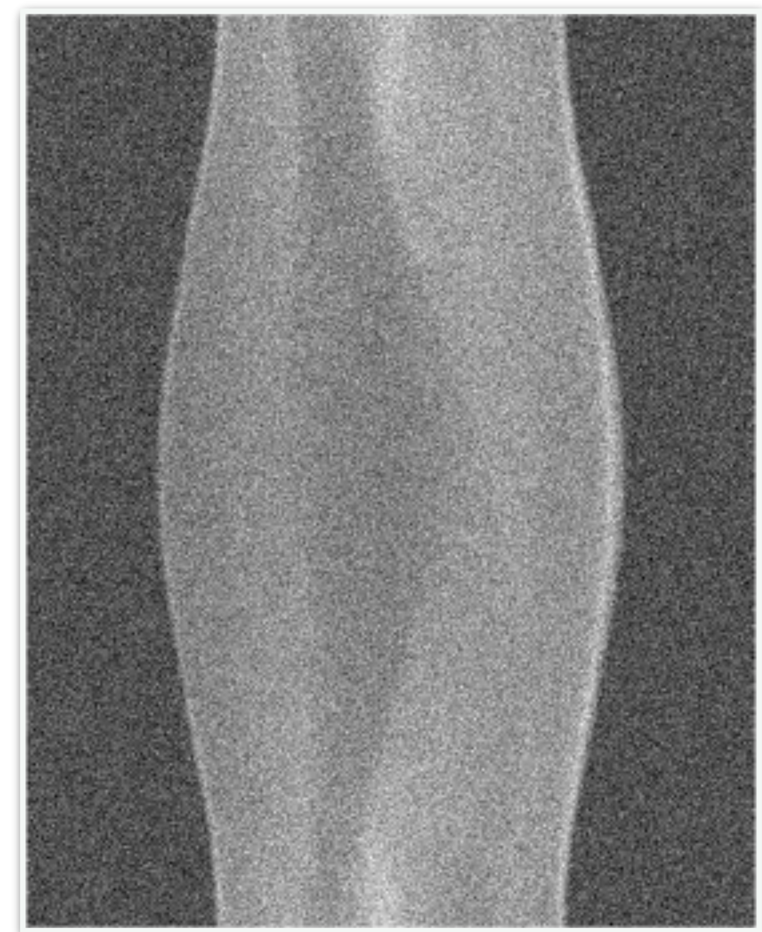


perfect



4% noise



16% noise

The tomographic problem (with noise)

- We have to solve a linear equation system

$$\mathsf{A}\mathsf{x} + \eta = \mathsf{y}$$

  with *A* being the system-matrix, *x* the volume, *y* the measurements and η an **unknown** noise influence.

- Solution
  - Solve for a minimal least squares error instead

$$\underset{\mathsf{x}}{\operatorname{argmin}} \frac{1}{2} \|\mathsf{A}\mathsf{x} - \mathsf{y}\|_2^2$$

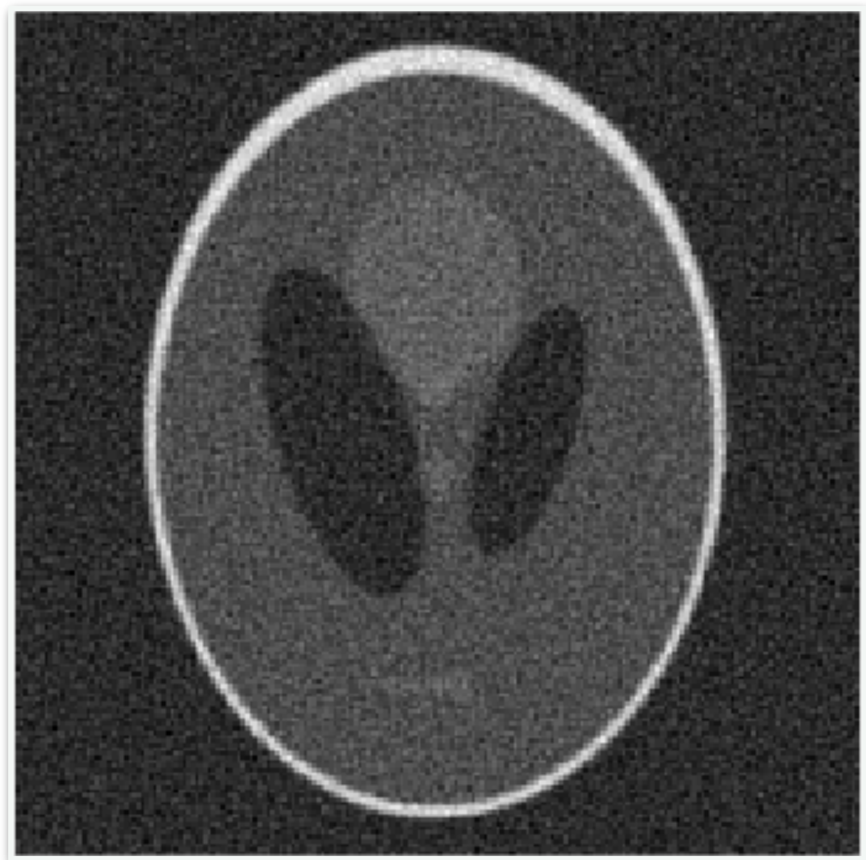
$$\Big\Updownarrow \ \text{equivalent}$$

$$\mathsf{A}^\top \mathsf{A}\mathsf{x} = \mathsf{A}^\top \mathsf{y}$$

The tomographic problem
(with noise)

- Still — Noise affects reconstruction quality

perfect



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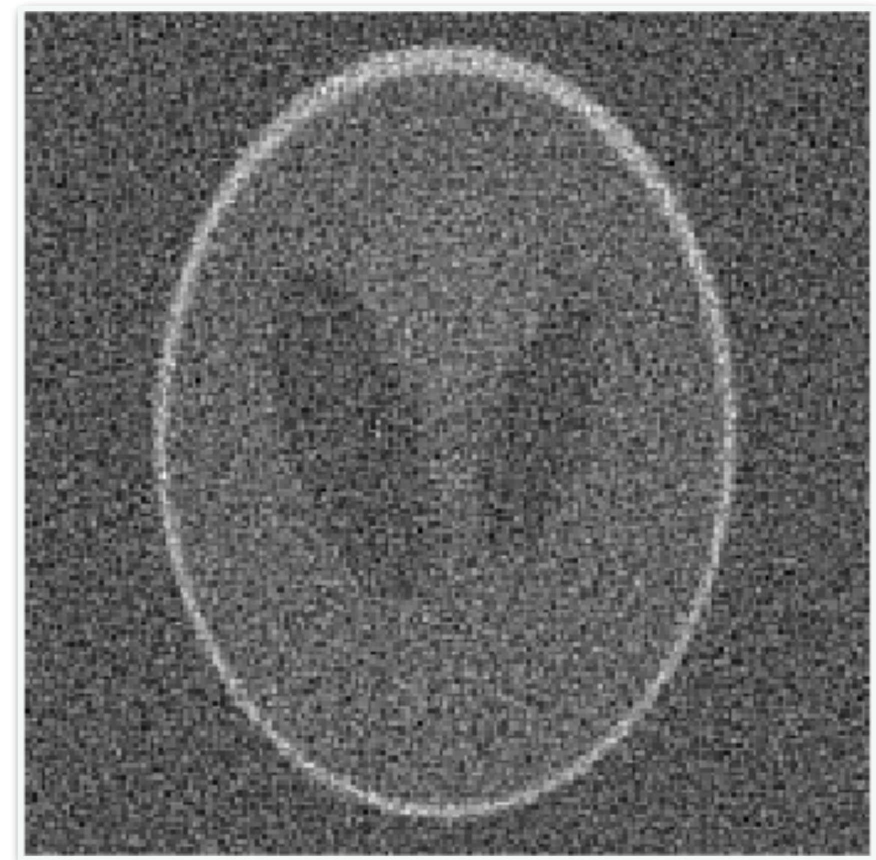

16% noise

Regularization

- Incorporate prior knowledge:
  - smoothness (Sobolev / **Tikhonov** / Total Variation)
  - sparsity (wavelets / framelet / dictionary decomposition)
  - …

Regularization — Problem formulation

- Regularized least-squares problem:

$$\underset{x}{\operatorname{argmin}} \frac{1}{2}\|Ax - y\|_2^2 + \lambda \mathcal{R}(x)$$

- We will consider generalized Tikhonov regularization:

$$\underset{x}{\operatorname{argmin}} \frac{1}{2}\|Ax - y\|_2^2 + \frac{\lambda}{2}\|x\|_2^2$$

$$\Big\Updownarrow \text{ equivalent}$$

$$(A^\top A + \lambda I)x = A^\top b$$

Advanced iterative solver

Conjugate gradient

- Solves system of linear equations Ba=b, provided B is spd
- Pseudocode [1,p.50]:

```
result = vector(0); // zero vector as starting value

r = b - B * result;
d = r;
rtr = r.dot(r);

for #iterations
   q = B * d;

   alpha = rtr / d.dot(q);
   result += alpha * d;
   r -= alpha * q;

   oldRtr = rtr;
   rtr = r.dot(r)
   d = r + rtr / oldRtr * d;
end
```

[1] J. R. Shewchuk, “An Introduction to the Conjugate Gradient Method Without the Agonizing Pain,” Mar. 1994.