

Differential equations, 1 lab

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169 - 171 Having found a particular solution by selection, bring this Riccati equation to the Bernoulli equations and solve them

$$\mathbf{169} \quad xy' - (2x + 1)y + y^2 = -x^2$$

$$xy' - (2x + 1)y + y^2 = -x^2$$

$$y_0 = x - \text{particular solution}$$

Examination:

$$x - (2x + 1)x + x^2 = -x^2$$

$$-x^2 = -x^2$$

Replacement:

$$y = U + y_0 = U + x$$

$$y' = U' + 1 = \frac{2x + 1}{x}y - \frac{y^2}{x} - x$$

$$U' = -1 + \frac{2x + 1}{x}(U + x) - \frac{(y + x^2)}{x} - x$$

$$U' = -1 + 2U + 2x + \frac{U}{x} + 1 - \frac{U^2}{x} - x - 2U - x$$

$U' = -\frac{U^2}{x} + \frac{U}{x} - \text{Bernoulli equation}$

Replacement for Bernoulli's equation:

$$m = 2; v = U^{1-m} = \frac{1}{U}$$

$$U = \frac{1}{v}$$

$$U' = -\frac{v'}{v^2}$$

$$-\frac{v'}{v^2} = -\frac{1}{v^2 x} + \frac{1}{vx}$$

$$-v' = -\frac{1}{x} + \frac{v}{x}$$

$$-v' - \frac{v}{x} = 0$$

$$-v' = \frac{v}{x}$$

$$-\frac{dv}{v} = \frac{dx}{x}$$

$$\ln(v) = -\ln(Cx)$$

$$v = C\frac{1}{x}$$

$$v' = C'\frac{1}{x} - \frac{1}{x^2}C$$

$$C'\frac{1}{x} - \frac{1}{x^2}C + C\frac{1}{x^2} = \frac{1}{x}$$

$$C' = 1 \Rightarrow C = \tilde{C} + x$$

$$v = \frac{\tilde{C}}{x} + 1$$

$$U = \frac{x}{\tilde{C} + x}$$

$$\boxed{y = x + \frac{x}{\tilde{C} + x}}$$

$$\mathbf{170} \quad y' - 2xy + y^2 = 5 - x^2$$

privat desision:

$$y = ax + b$$

$$a - 2ax^2 - 2xb + a^2x^2 + b^2 + 2axb = 5 - x^2$$

$$\begin{cases} a^2 - 2a = -1 \Rightarrow (a - 1)^2 = 0 \Rightarrow a = 1 \\ -2b + 2ab = 0 \\ a + b^2 = 5 \Rightarrow b^2 = 5 - a = 5 - 1 = 4 \Rightarrow b = \pm 2 \end{cases}$$

$$y_0 = x + 2 - \text{privat desision}$$

Replacement:

$$y = U + x + 2$$

$$y' = U' + 1 = 5 - x^2 + 2xy - y^2$$

$$y' = 5 - x^2 + 2xU + 2x^2 + 4x - U^2 - x^2 - 4 - 2Ux - 4x - 4U$$

$U' + 1 = -U^2 - 4U - \text{Bernoulli equation}$
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Replacement for Bernoulli's equation:

$$m = 2; v = U^{1-m} = \frac{1}{U}$$

$$U = \frac{1}{v}$$

$$U' = -\frac{v'}{v^2}$$

$$-\frac{v'}{v^2} = -\frac{1}{v^2} - \frac{4}{v}$$

$$-v' = -1 - 4v$$

$$-v' + 4v = 0$$

$$\frac{dv}{v} = 4dx$$

$$\ln(v) = 4\ln(Ce^x)$$

$$v = Ce^{4x}$$

$$v' = C'e^{4x} + 4Ce^{4x}$$

$$-C'e^{4x} - 4Ce^{4x} = -1 - 4Ce^{4x}$$

$$C' = e^{-4x} \Rightarrow C = -\frac{1}{4}e^{-4x} + \tilde{C}$$

$$v = -\frac{1}{4} + \tilde{C}e^{4x}$$

$$U = \frac{1}{-\frac{1}{4} + \tilde{C}e^{4x}}$$

$$\boxed{y = x + 2 + \frac{4}{\tilde{C}e^{4x} - 1}}$$

171 $y' + 2ye^x - y^2 = e^{2x} + e^x$

privat desision:

$$y = e^x + a$$

$$e^x + 2e^{2x} + 2e^xa - e^{2x} - a^2 - 2e^xa = e^{2x} + e^x$$

$$-a^2 = 0 \Rightarrow a = 0$$

$$y_0 = e^x - \text{privat desision}$$

Replacement:

$$y = y_0 + U = e^x + U$$

$$y' = U' + e^x$$

$$U' + e^x = e^{2x} + e^x - 2e^{2x} - 2Ue^x + e^{2x} + U^2 + 2Ue^x$$

$$\boxed{U' = U^2 - \text{Bernoulli equation}}$$

Replacement for Bernoulli's equation:

$$m = 2; v = U^{1-m} = \frac{1}{U}$$

$$\begin{aligned}
U &= \frac{1}{v} \\
U' &= -\frac{v'}{v^2} \\
-\frac{v'}{v^2} &= \frac{1}{v^2} \\
-v' &= 1 \Rightarrow v' = -1 \Rightarrow v = -x + C \\
U &= \frac{1}{-x + C} \\
\boxed{y = e^x - \frac{1}{x - C} = e^x - \frac{1}{x + \tilde{C}}}
\end{aligned}$$

179 $ay' + ay = f(x)$; $a = \text{const} > 0$; $\lim_{x \rightarrow 0} f(x) = b$

Show that only one solution remains bounded at $x \rightarrow 0$ and find the limit of this solution at $x \rightarrow 0$

$$ay' + ay = f(x)$$

$$p(x)y + q(x) + r(x)y' = 0$$

$$r(x) = x$$

$$p(x) = a$$

$$q(x) = f(x)$$

Common decision:

$$y = e^{-\int_{x_0}^x \frac{p(\tau)}{r(\tau)} d\tau} \left(C + \int_{x_0}^x e^{\int_{x_0}^t \frac{p(\tau)}{r(\tau)} d\tau} \frac{q(t)}{r(t)} dt \right)$$

$$y = e^{-\int_{x_0}^x \frac{a}{\tau} d\tau} \left(C + \int_{x_0}^x e^{\int_{x_0}^t \frac{a}{\tau} d\tau} \frac{f(t)}{t} dt \right)$$

$$x_0 = 0$$

$$y = e^{-a \ln(x)} \left(C + \int_0^x \frac{1}{t^{-a}} \frac{f(t)}{t} dt \right)$$

$$y = \frac{1}{x^a} \left(C + \int_0^x f(t) t^{a-1} dt \right)$$

$$\lim_{x \rightarrow 0} \frac{C}{x^a} = \infty \Rightarrow C = 0$$

$$y = \frac{1}{x^a} \int_0^x f(t) t^{a-1} dt$$

$$\lim_{x \rightarrow 0} y = \lim_{x \rightarrow 0} \frac{1}{x^a} \int_0^x f(t) t^{a-1} dt$$

$$\lim_{x \rightarrow 0} y = \lim_{x \rightarrow 0} \frac{1}{x^a} \int_0^x b t^{a-1} dt$$

$$\lim_{x \rightarrow 0} y = \lim_{x \rightarrow 0} \frac{1}{x^a} \frac{b}{a} x^a$$

$$\boxed{\lim_{x \rightarrow 0} y = \frac{b}{a}}$$

180 $ay' + ay = f(x)$; $a = \text{const} < 0$; $\lim_{x \rightarrow 0} f(x) = b$

Show that all solutions of this equation have the same finite limit at $x \rightarrow 0$. Find him.

From 179 follows the general solution of the equation:

$$y = \frac{1}{x^a} \left(C + \int_0^x f(t) t^{a-1} dt \right)$$

$$y = \frac{b}{a} + \frac{1}{x^a} \left(C + \int_0^x \epsilon(t) t^{a-1} dt \right)$$

$$\int_0^x \epsilon(t) t^{a-1} dt \text{ limited } \forall C \Rightarrow \boxed{\lim_{x \rightarrow 0} y = \frac{b}{a}}$$

$$\int_0^x \epsilon(t) t^{a-1} dt \text{ is not limited } \Rightarrow$$

$$\lim_{x \rightarrow 0} \frac{C + \int_0^x \epsilon(t) t^{a-1} dt}{x^a} = 0 + \frac{\int_0^x \epsilon(t) t^{a-1} dt}{x^a}$$

$$\lim_{x \rightarrow 0} \frac{\epsilon(x) x^a}{a x^a} = 0$$

$$\boxed{\lim_{x \rightarrow 0} y = \frac{b}{a} \forall C}$$

181 $\frac{dx}{dt} + x = f(t)$; $f(t)$ - continuous function $|f(f)| \leq M$ at $-\infty < t < +\infty$

Show that the equation has one bounded solution. Find this solution. Show that the found solution is periodic if the function $f(x)$ is periodic.

$x' + x = f(t)$ — first-order linear differential equation with constant coefficients of the form: $x' - \lambda x = f(t)$

$$\lambda = -1$$

Common decision:

$$x(t) = e^{\lambda t} \left(C + \int_s^t f(\tau) e^{-\lambda \tau} d\tau \right)$$

$$x(t) = e^{-t}C + e^{-t} \int_{-\infty}^t f(\tau)e^{\tau} d\tau$$

$$| \int_{-\infty}^t f(\tau)e^{\tau} d\tau | \leq Me^t$$

$$e^{-t} \int_{-\infty}^t f(\tau)e^{\tau} d\tau \leq M \Rightarrow C = 0 \quad \forall t - \infty < t < +\infty$$

$$\boxed{x(t) = e^{-t} \int_{-\infty}^t f(\tau)e^{\tau} d\tau}$$

Periodicity:

T-period

$$x(t) = e^{-t} \int_{-\infty}^t f(\tau + T)e^{\tau} d\tau$$

$$\begin{bmatrix} \tau + T = z \\ d\tau = dz \end{bmatrix}$$

$$x(t) = e^{-t} \int_{-\infty}^{t+T} f(z)e^{z-T} dz$$

$$x(t - T) = e^{-t} \int_{-\infty}^{t+T} f(z)e^z dz = x(t + T)$$