Differential equations

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169 - 171 Having found a particular solution by selection, bring this Riccati equation to the Bernoulli equations and solve them

169 $xy' - (2x+1)y + y^2 = -x^2$

$$xy' - (2x+1)y + y^2 = -x^2$$
$$y_0 = x - \text{privat desision}$$

Examination:

$$x - (2x+1)x + x^2 = -x^2$$
$$-x^2 = -x^2$$

Replacement:

$$y = U + y_0 = U + x$$
$$y' = U' + 1 = \frac{2x+1}{x}y - \frac{y^2}{x} - x$$
$$U' = -1 + \frac{2x+1}{x}(U+x) - \frac{(y+x^2)}{x} - x$$
$$U' = -1 + +2U + 2x + \frac{U}{x} + 1 - \frac{U^2}{x} - x - 2U - x$$
$$\boxed{U' = -\frac{U^2}{x} + \frac{U}{x} - \text{ Bernoulli equation}}$$

Replacement for Bernoulli's equation:

$$m = 2;\ \upsilon = U^{1-m} = \frac{1}{U}$$

$$U = \frac{1}{\upsilon}$$

$$U' = -\frac{\upsilon'}{\upsilon^2}$$

$$-\frac{\upsilon'}{\upsilon^2} = -\frac{1}{\upsilon^2 x} + \frac{1}{\upsilon x}$$

$$-\upsilon' = -\frac{1}{x} + \frac{\upsilon}{x}$$

$$-\upsilon' - \frac{\upsilon}{x} = 0$$

$$-\upsilon' = \frac{\upsilon}{x}$$

$$-\frac{d\upsilon}{\upsilon} = \frac{dx}{x}$$

$$\ln(\upsilon) = -\ln(Cx)$$

$$\upsilon = C\frac{1}{x}$$

$$\upsilon' = C'\frac{1}{x} - \frac{1}{x^2}C$$

$$C'\frac{1}{x} - \frac{1}{x^2}C + C\frac{1}{x^2} = \frac{1}{x}$$

$$C' = 1 \Rightarrow C = \widetilde{C} + x$$

$$\upsilon = \frac{\widetilde{C}}{x} + 1$$

$$U = \frac{x}{\widetilde{C} + x}$$

$$\boxed{y = x + \frac{x}{\widetilde{C} + x}}$$

170 $y' - 2xy + y^2 = 5 - x^2$

privat desision:

$$y = ax + b$$

$$a - 2ax^2 - 2xb + a^2x^2 + b^2 + 2axb = 5 - x^2$$

$$\begin{cases} a^2 - 2a = -1 \Rightarrow (a-1)^2 = 0 \Rightarrow a = 1 \\ -2b + 2ab = 0 \\ a + b^2 = 5 \Rightarrow b^2 = 5 - a = 5 - 1 = 4 \Rightarrow b = \pm 2 \end{cases}$$

$$y_0 = x + 2 - \text{privat desision}$$

Replacement:

$$y = U + x + 2$$

$$y' = U' + 1 = 5 - x^2 + 2xy - y^2$$

$$y' = 5 - x^2 + 2xU + 2x^2 + 4x - U^2 - x^2 - 4 - 2Ux - 4x - 4U$$

$$\boxed{U' + 1 = -U^2 - 4U - \text{ Bernoulli equation}}$$

Replacement for Bernoulli's equation:

$$m = 2;\ \upsilon = U^{1-m} = \frac{1}{U}$$

$$U = \frac{1}{\upsilon}$$

$$U' = -\frac{\upsilon'}{\upsilon^2}$$

$$-\frac{\upsilon'}{\upsilon^2} = -\frac{1}{\upsilon^2} - \frac{4}{\upsilon}$$

$$-\upsilon' = -1 - 4\upsilon$$

$$-\upsilon' + 4\upsilon = 0$$

$$\frac{d\upsilon}{\upsilon} = 4dx$$

$$ln(\upsilon) = 4ln(Ce^x)$$

$$\upsilon = Ce^{4x}$$

$$\upsilon' = C'e^{4x} + 4Ce^{4x}$$

$$-C'e^{4x} - 4Ce^{4x} = -1 - 4Ce^{4x}$$

$$C' = e^{-4x} \Rightarrow C = -\frac{1}{4}e^{-4x} + \widetilde{C}$$

$$\upsilon = -\frac{1}{4} + \widetilde{C}e^{4x}$$

$$U = \frac{1}{-\frac{1}{4} + \widetilde{C}e^{4x}}$$

$$\boxed{y = x + 2 + \frac{4}{\widetilde{C}e^{4x} - 1}}$$

171 $y' + 2ye^x - y^2 = e^{2x} + e^x$

privat desision:

$$y = e^x + a$$

$$e^x + 2e^{2x} + 2e^x a - e^{2x} - a^2 - 2e^x a = e^{2x} + e^x$$

$$-a^2 = 0 \Rightarrow a = 0$$

$$y_0 = e^x - \text{privat desision}$$

Replacement:

$$y = y_0 + U = e^x + U$$
$$y' = U' + e^x$$
$$U' + e^x = e^{2x} + e^x - 2e^{2x} - 2Ue^x + e^{2x} + U^2 + 2Ue^x$$
$$\boxed{U' = U^2 - \text{ Bernoulli equation}}$$

Replacement for Bernoulli's equation:

$$m = 2;\ \upsilon = U^{1-m} = \frac{1}{U}$$
$$U = \frac{1}{\upsilon}$$
$$U' = -\frac{\upsilon'}{\upsilon^2}$$
$$-\frac{\upsilon'}{\upsilon^2} = \frac{1}{\upsilon^2}$$
$$-\upsilon' = 1 \Rightarrow \upsilon' = -1 \Rightarrow \upsilon = -x + C$$
$$U = \frac{1}{-x + C}$$
$$\boxed{y = e^x - \frac{1}{x - C} = e^x - \frac{1}{x + \widetilde{C}}}$$

179 $ay' + ay = f(x);\ a = const > 0;\ \lim\limits_{x \to 0} f(x) = b$

Show that only one solution remains bounded at $x \to 0$ and find the limit of this solution at $x \to 0$

$$ay' + ay = f(x)$$

$$p(x)y + q(x) + r(x)y' = 0$$
$$r(x) = x$$
$$p(x) = a$$
$$q(x) = f(x)$$

Common decision:

$$y = e^{-\int\limits_{x_0}^{x} \frac{p(\tau)}{r(\tau)}\,d\tau}(C + \int\limits_{x_0}^{x} e^{\int\limits_{t_0}^{t} \frac{p(\tau)}{r(\tau)}\,d\tau} \frac{q(t)}{r(t)}\,dt)$$
$$y = e^{-\int\limits_{x_0}^{x} \frac{a}{\tau}\,d\tau}(C + \int\limits_{x_0}^{x} e^{\int\limits_{t_0}^{t} \frac{a}{\tau}\,d\tau} \frac{f(t)}{t}\,dt)$$

$x_0 = 0$

$$y = e^{-aln(x)}(C + \int\limits_{0}^{x} \frac{1}{t^{-a}} \frac{f(t)}{t}\,dt)$$
$$y = \frac{1}{x^a}(C + \int\limits_{0}^{x} f(t)t^{a-1}\,dt)$$
$$\lim_{x\to 0} \frac{C}{x^a} = \infty \Rightarrow C = 0$$
$$y = \frac{1}{x^a}\int\limits_{0}^{x} f(t)t^{a-1}\,dt$$
$$\lim_{x\to 0} y = \lim_{x\to 0} \frac{1}{x^a}\int\limits_{0}^{x} f(t)t^{a-1}\,dt$$
$$\lim_{x\to 0} y = \lim_{x\to 0} \frac{1}{x^a}\int\limits_{0}^{x} bt^{a-1}\,dt$$

$$\lim_{x\to 0} y = \lim_{x\to 0} \frac{1}{x^a}\frac{b}{a}x^a$$

$$\boxed{\lim_{x\to 0} y = \frac{b}{a}}$$

180 $ay' + ay = f(x);\ a = const < 0;\ \lim\limits_{x\to 0} f(x) = b$

Show that all solutions of this equation have the same finite limit at $x \to 0$. Find him.

From 179 follows the general solution of the equation:

$$y = \frac{1}{x^a}(C + \int\limits_0^x f(t)t^{a-1}\,dt)$$

$$y = \frac{b}{a} + \frac{1}{x^a}(C + \int\limits_0^x \epsilon(t)t^{a-1}\,dt)$$

$$\int\limits_0^x \epsilon(t)t^{a-1}\,dt \text{ limited } \forall C \Rightarrow \boxed{\lim_{x\to 0} y = \tfrac{b}{a}}$$

$$\int\limits_0^x \epsilon(t)t^{a-1}\,dt \text{ is not limited} \Rightarrow$$

$$\lim_{x\to 0} \frac{C + \int\limits_0^x \epsilon(t)t^{a-1}\,dt}{x^a} = 0 + \frac{\int\limits_0^x \epsilon(t)t^{a-1}\,dt}{x^a}$$

$$\lim_{x\to 0} \frac{\epsilon(x)x^a}{ax^a} = 0$$

$$\boxed{\lim_{x\to 0} y = \tfrac{b}{a}\ \forall C}$$

181 $\dfrac{dx}{dt} + x = f(t)$; $f(t)$- continuous function $|f(f)| \leq M$ at$-\infty < t < +\infty$

Show that the equation has one bounded solution. Find this solution. Show that the found solution is periodic if the function $f(x)$ is periodic.

$$x' + x = f(t) - \text{ first-order linear differential equation with constant}$$

$$\text{coefficients of the form: } x' - \lambda x = f(t)$$

$$\lambda = -1$$

Common decision:

$$x(t) = e^{\lambda t}(C + \int\limits_{s}^{t} f(\tau)e^{-\lambda\tau}\, d\tau)$$

$$x(t) = e^{-t}C + e^{-t}\int\limits_{-\infty}^{t} f(\tau)e^{\tau}\, d\tau$$

$$|\int\limits_{-\infty}^{t} f(\tau)e^{\tau}\, d\tau| \leq Me^{t}$$

$$e^{-t}\int\limits_{-\infty}^{t} f(\tau)e^{\tau}\, d\tau \leq M \Rightarrow C = 0\ \forall t - \infty < t < +\infty$$

$$\boxed{x(t) = e^{-t}\int\limits_{-\infty}^{t} f(\tau)e^{\tau}\, d\tau}$$

Periodicity:
T-period

$$x(t) = e^{-t} \int\limits_{-\infty}^{t} f(\tau + T) e^{\tau} \, d\tau$$

$$\begin{bmatrix} \tau + T = z \\ d\tau = dz \end{bmatrix}$$

$$x(t) = e^{-t} \int\limits_{-\infty}^{t+T} f(z) e^{z-T} \, dz$$

$$x(t-T) = e^{-t} \int\limits_{-\infty}^{t+T} f(z) e^{z} \, dz = x(t+T)$$