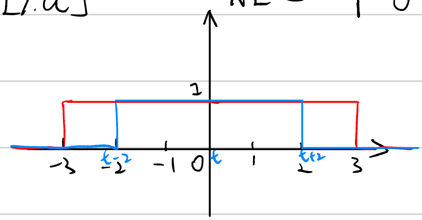
Q1

[1.a]

$$x[n] = \begin{cases} 1 & -3 \le n \le 3 \\ 0 & \text{otherwise} \end{cases} \qquad h[n] = \begin{cases} 1 & -2 \le n \le 2 \\ 0 & \text{otherwise} \end{cases}$$



Case 1: $t < -5$

$$y(t) = 0$$

Case 2: $-5 \le t < -1$

$$y(t) = \int_{-5}^{t} 1\, d\tau = t + 5$$

Case 3: $-1 \le t < 1$

$$y(t) = \int_{t-2}^{t+2} 1\, d\tau = t+2-t+2 = 4$$

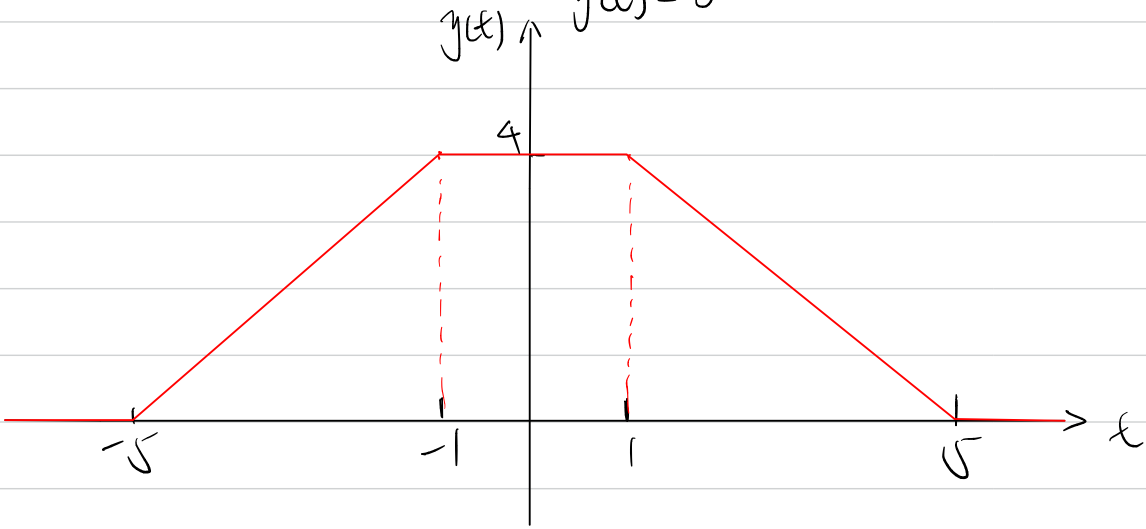
Case 4: $3 \le t < 5$

$$y(t) = \int_{t-2}^{3} 1\, d\tau = 3-t+2 = 5-t$$

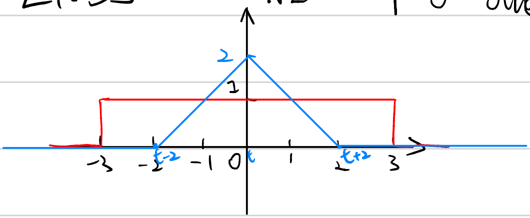
Case 5: $t \ge 5$

$$y(t) = 0$$

$$y(t) = \begin{cases} 0, & t < -5 \\ t+5, & -5 \le t < -1 \\ 4, & -1 \le t < 1 \\ 5-t, & 3 \le t < 5 \\ 0, & t \ge 5 \end{cases}$$



[1.b] $x[n] = \begin{cases} 1 & -3 \le n \le 3 \\ 0 & \text{otherwise} \end{cases}$ $\quad h[n] = \begin{cases} 2-|n| & -2 \le n \le 2 \\ 0 & \text{otherwise} \end{cases}$



Case 1: $t < -5$ $\quad y(t) = 0$

Case 2: $-5 \le t < -3$

$$y(t) = \int_{-3}^{t+2} 2-|t-\tau|\,d\tau = 2\tau + t\tau - \tfrac{1}{2}\tau^2 \Big|_{-3}^{t+2} = \tfrac{1}{2}(t+5)^2$$

Case 3: $-3 \le t < -1$

$$y(t) = \int_{-3}^{t+2} 2-|t-\tau|\,d\tau = \tfrac{1}{2}(-t^2-2t+7)$$

Case 4: $-1 \le t < 1$

$$y(t) = \int_{t-2}^{t+2} 2-|t-\tau|\,d\tau = t+2-t+2 = 4$$

Case 5: $1 \le t < 3$

$$y(t) = \int_{t-2}^{3} 2-|t-\tau|\,d\tau = \tfrac{1}{2}(-t^2+2t+7)$$

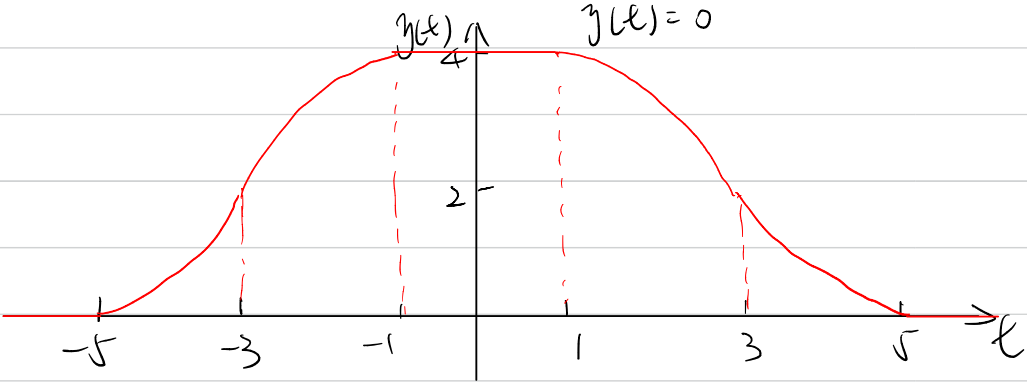
Case 6: $3 \le t < 5$

$$y(t) = \int_{t-2}^{3} 2-|t-\tau|\,d\tau = \tfrac{1}{2}(t-5)^2$$

Case 7: $t > 5$

$y(t) = 0$

$$y(t) = \begin{cases} 0, & t < -5 \\ \tfrac{1}{2}(t+5)^2, & -5 \le t < -3 \\ \tfrac{1}{2}(-t^2-2t+7), & -3 \le t < -1 \\ 4, & -1 \le t < 1 \\ \tfrac{1}{2}(-t^2+2t+7), & 1 \le t < 3 \\ \tfrac{1}{2}(t-5)^2, & 3 \le t < 5 \\ 0, & t > 5 \end{cases}$$



Q2

[2.a]

Proof: $x(n) = \sum_{i=-\infty}^{\infty} x(i)\delta(n-i)$

$$
\begin{aligned}
T[x(n)] &= T\left[\sum_{i=-\infty}^{\infty} x(i)\delta(n-i)\right] \\
&= \sum_{i=-\infty}^{\infty} x(i)\, T[\delta(n-i)] \\
&= \sum_{i=-\infty}^{\infty} x(i)\, h(n-i) \\
&= h(n) * x(n)
\end{aligned}
$$

(Since time-invariant $h(n-i) = T[\delta(n-i)]$)

$\boxed{M}$

[2.b] Consider one dimension Gaussian function:

$$G(x) = \frac{1}{2\pi\sigma^2} e^{-\frac{x^2}{2\sigma^2}}$$

Gaussian blurring is neither linear nor time-invariant because the input x is going through a non-linear transform, So $G[a_1x_1(n) + a_2x_2(n)] \neq a_1G(x_1(n)) + a_2G(x_2(n))$ and $G(x(n-n_0)) \neq y(n-n_0)$.

[2.c] $T[x(n)] = x(-n)$

$\Rightarrow \quad T[a_1 x_1(n) + a_2 x_2(n)] = a_1 x_1(-n) + a_2 x_2(-n)$

$= a_1 T[x_1(n)] + a_2 T[x_2(n)]$

$\Leftarrow \quad a_1 T[x_1(n)] + a_2 T[x_2(n)] = a_1 x_1(-n) + a_2 x_2(-n)$

$= T[a_1 x_1(n) + a_2 x_2(n)]$

$\therefore$ Time reversal is linear.

$T[x(n-n_0)] = x(-n+n_0) \neq x(-n-n_0)$

$\therefore$ Time reversal is not time-invariant.

Q3 Define $U(x) = \sum_{i=0}^{m} u_i x^i$, $V(x) = \sum_{j=0}^{n} v_j x^j$

$$U(x)V(x) = \sum_{i=0}^{m}\sum_{j=0}^{n} u_i v_j x^i x^j$$

Let $i+j=k$

$$\Rightarrow \sum_{i=0}^{m}\sum_{j=0}^{n} u_{k-j} v_j x^k$$

$$= \sum_{k=0}^{m+n} \Big(\underbrace{\sum_{j=0}^{k} u_{k-j} v_j}_{\equiv \vec{u} * \vec{v}} \Big) x^k$$

$\therefore$ Convolving them is equivalent to multiplying the two polynomials.

Q4 proof:

define $\begin{cases} u = x\cos\theta - y\sin\theta \\ v = x\sin\theta + y\cos\theta \end{cases}$

Apply chain rule:

$$\frac{\partial}{\partial x} = \frac{\partial u}{\partial x}\frac{\partial}{\partial u} + \frac{\partial u}{\partial x}\frac{\partial}{\partial v}$$
$$= \cos\theta\frac{\partial}{\partial u} + \sin\theta\frac{\partial}{\partial v}$$
$$\frac{\partial}{\partial y} = -\sin\theta\frac{\partial}{\partial u} + \cos\theta\frac{\partial}{\partial v}$$

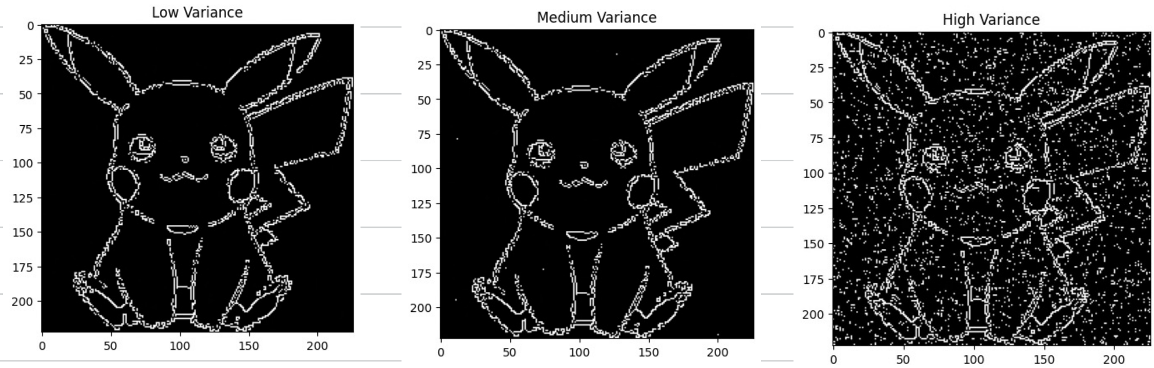
$$\frac{\partial^2 I}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial I}{\partial x}\right) \qquad \text{Let } g = \frac{\partial I}{\partial x} = \frac{\partial u}{\partial x}\frac{\partial I}{\partial u} = \cos\theta\frac{\partial I}{\partial u} + \sin\theta\frac{\partial I}{\partial v}$$
$$= \frac{\partial g}{\partial x}$$
$$= \frac{\partial u}{\partial x}\cdot\frac{\partial g}{\partial u} + \frac{\partial v}{\partial x}\cdot\frac{\partial g}{\partial v}$$
$$= \cos\theta\frac{\partial g}{\partial u} + \sin\theta\frac{\partial g}{\partial v}$$
$$= \cos^2\theta\frac{\partial^2 I}{\partial u^2} + 2\cos\theta\sin\theta\frac{\partial^2 I}{\partial u\partial v} + \sin^2\theta\frac{\partial^2 I}{\partial v^2}$$

$$\frac{\partial^2 I}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial I}{\partial y}\right) \qquad \text{Let } h = \frac{\partial I}{\partial y} = -\sin\theta\frac{\partial I}{\partial u} + \cos\theta\frac{\partial I}{\partial v}$$
$$= \frac{\partial h}{\partial y}$$
$$= \sin^2\theta\frac{\partial^2 I}{\partial u^2} - 2\sin\theta\cos\theta\frac{\partial^2 I}{\partial u\partial v} + \cos^2\theta\frac{\partial^2 I}{\partial v^2}$$

$$\frac{\partial^2 I}{\partial x^2} + \frac{\partial^2 I}{\partial y^2} = \cos^2\theta\frac{\partial^2 I}{\partial u^2} + \cancel{2\cos\theta\sin\theta\frac{\partial^2 I}{\partial u\partial v}} + \sin^2\theta\frac{\partial^2 I}{\partial v^2} + \sin^2\theta\frac{\partial^2 I}{\partial u^2} - \cancel{2\sin\theta\cos\theta\frac{\partial^2 I}{\partial u\partial v}} + \cos^2\theta\frac{\partial^2 I}{\partial v^2}$$
$$= \frac{\partial^2 I}{\partial u^2}\cancel{(\cos^2\theta + \sin^2\theta)} + \frac{\partial^2 I}{\partial v^2}\cancel{(\sin^2\theta + \cos^2\theta)}$$
$$= \frac{\partial^2 I}{\partial u^2} + \frac{\partial^2 I}{\partial v^2} = I_{rr} + I_{r'r'}$$

$\blacksquare$

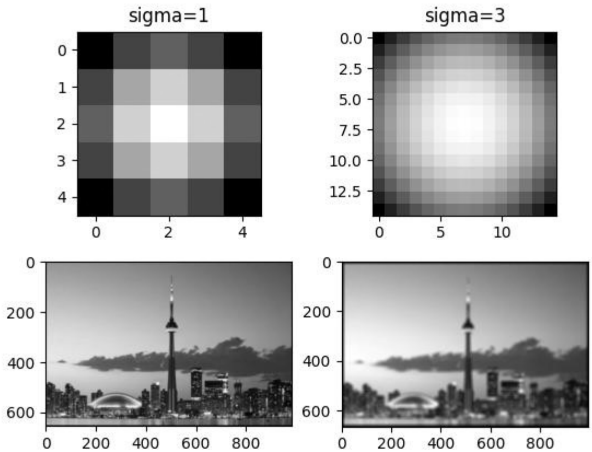
Q5



The edge detector are sensitive to the noise, the higher variance lead to the lower performance of the detector.

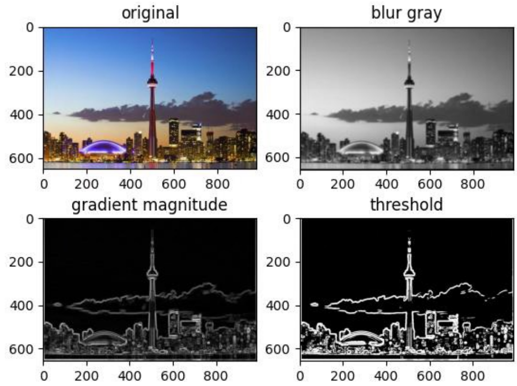
Q6

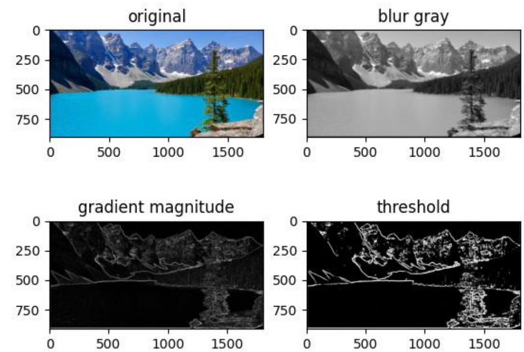
[Step I]


sigma=1
sigma=3
0
1
2
3
4
0
2
4
0.0
2.5
5.0
7.5
10.0
12.5
0
5
10
0
200
400
600
0
200
400
600
800
0
200
400
600
0
200
400
600
800

[Step IV]

The algorithm first blurs the image with gaussian kernel, and get the gradient magnitude using Sobel operator, then we calculate the threshold and apply it to get the white and black edge image.





From these inputs, we can tell the detector can capture all the main edges.

But, from the two given complex images test, we can see some small details are also marked as edges. The pikachu lose the edge of its lip. So we need to tune the sigma of gaussian filter based on the complexity of the image.