Learning Module 2: The Time Value of Money in Finance

LOS 2a: calculate and interpret the present value (PV) of fixed-income and equity instruments based on expected future cash flows

The time value of money (TVM) is a fundamental financial concept. It emphasizes that a sum of money is worth more in the present than in the future. There are three key reasons supporting this principle:

- **The concept of opportunity** cost suggests that money available today can be invested and generate interest, increasing its value over time. By delaying the use of money, one forgoes potential investment opportunities and the growth they offer.

- **Inflation** poses a threat to the purchasing power of money in the future. Due to inflation, the same amount of money may buy fewer goods or services in the future compared to its present value. Consequently, having money now is advantageous since its purchasing power diminishes as time progresses.

- There is an element of **uncertainty regarding future cash flows**. Unexpected events or circumstances may prevent the receipt of money as planned, rendering it less reliable. Until the money is obtained, there is a level of uncertainty attached to its availability and utility.

Time value of money calculations allow us to establish the future value of a given amount of money.

Key Components of Time Value of Money

- **Discount rate or interest rate**: The rate of discounting or compounding that you apply to an amount of money to calculate its present or future value.

- **Time periods**: The whole number of time periods over which the present or future value of a sum is being calculated. These periods can be annually, semi-annually, quarterly, monthly, weekly, etc.

- **Present value (PV)**: The amount of money you have today (or at time T = 0) is referred to as the present value.
- **Future value (FV):** The accumulated amount of money you get after investing the original sum at a specific interest rate and for a given time period, say, two years.

Fundamental Formulas in Time Value of Money Calculations:

Let,

FV = Future value.

PV = Present value.

r = Stated discount rate per period.

N = Number of periods (Years).

Then the future value (FV) of an investment is given by:

$$FV = PV(1+r)^N$$

If N is large such that N → ∞ the initial cashflow is compounded continuously:

$$FV = PVe^{rN}$$

To find the present value of the investment, we rewrite the above formula so that:

$$PV = FV(1+r)^{-N}$$

And for the continuous compounding, we have,

$$PV = FV_t e^{-rN}$$

Example: Calculating the Present Value of continuously Compounded Cashflows

A fund continuously accumulates to $4,000 over ten years at a 10% annual interest rate.

Calculate the closest present value of this fund.

Solution

From the question, $FV = 4{,}000$, $r_s = 10\%$, $N = 10$

So,

$$PV = FVe^{-Nr_s} = \$4,000 \times e^{-10 \times 0.1} = \$1,471.5178$$

Frequency of Compounding

When the frequency of compounding is more than once per year (quarterly, monthly, etc.), the formulas are analogously defined as:

$$FV_N = PV(1 + \frac{r_s}{m})^{mN}$$

Where:

m = Number of compounding periods per year.

N = Number of years.

r_s = Annual stated rate of interest.

Intuitively, the formula for the PV is given by:

$$PV = FV(1 + \frac{r_s}{m})^{-mN}$$

In the following discussion, we shall let $t = mN$ denote the number of compounding periods and $\frac{r_s}{m} = r$ denote the stated discount rate per period.

Calculation using a Financial Calculator

For calculating FV and PV using the BA II Plus[TM] Financial Calculator, use the following keys:

N = Number of compounding periods.

I/Y = Rate per period.

PV = Present value.

FV = Future value.

PMT = Payment.

CPT = Compute.

It is important to note that the sign of PV and FV will be opposite. For example, if PV is negative, then FV will be positive. Generally, an inflow is entered with a positive sign, while an outflow is entered as a negative sign in the calculator.

Time Value of Money in Fixed-income Instruments

Fixed-income instruments are debt securities where an issuer borrows money from an investor (lender) in exchange for a promised future payment. Examples of fixed-income instruments are bonds, loans, and notes.

The market discount rate for fixed-income instruments is also known as yield-to-maturity (YTM). It's the interest rate investors require to invest in a specific fixed-income instrument.

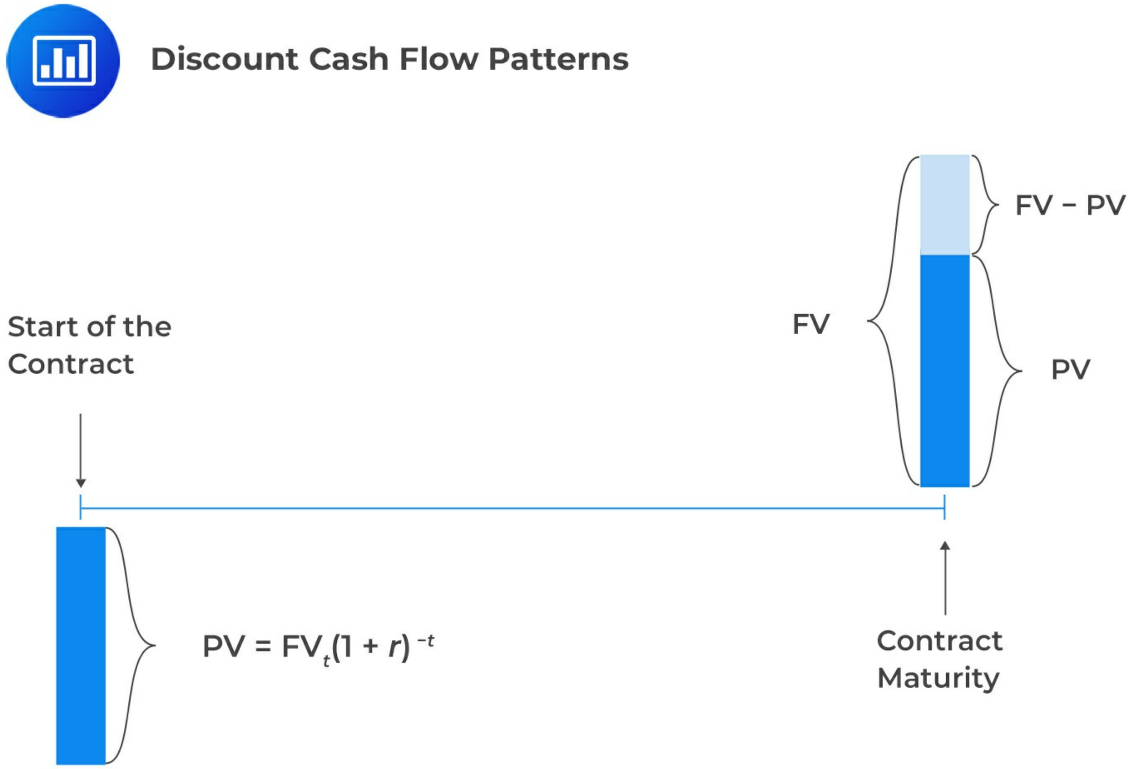
Cash Flow Patterns Associated with Fixed-Income Instruments

The cash flows in fixed-income instruments occur in three general patterns: Discount, periodic interest, and level payments.

Discount Cash Flow Patterns

For discount cashflow patterns, an investor pays an initial discounted price (PV) for the instrument (such as a bond or a loan) and gets one payment (FV) at the end maturity. The investor's return is the interest earned, that is, the difference between the initial price and principle ($FV - PV$).

The discount bonds are also called zero-coupon bonds – they do not have periodic interest payments.



The price of a discount bond can be calculated using the formula for the present value (PV) of a single cash flow, which is as follows:

$$PV = FV_t(1 + r)^{-t}$$

Where:

FV = Future value.

PV = Present value.

r = Stated discount rate per period.

t = Number of compounding periods.

Example: Calculating the Future Value of a Zero-Coupon Bond

Assume Chad invests $8,000 in a zero-coupon bond that yields 8% annually and matures in four years. The maturity value of this bond is *closest to*:

Solution

Recall that:

$$FV = PV(1 + r)^t$$

In this case, we have PV=8,000, r=8%, t=4 so that:

$$FV = 8,000(1 + 8\%)^4 = 10,883.91$$

Using the BA II Plus[TM] Financial Calculator

Steps	Explanation	Display
[2nd][QUIT]	Return to standard calc Mode	0
[2nd][CLR TVM]	Clears TVM Worksheet	0
2[N]	Years/periods	$N = 4$
10[1/Y]	Set interest rate	$PV = -8,000$
0[PMT]	Set payment	$PMT = 0$
[CPT][FV]	Compute future value	$FV = 10,883.91$

Note that zero-coupon bonds can be issued at negative interest rates. In this case, the price (PV) of the bond is higher than the face value (FV).

Example: Calculating the Price of a Discount Bond Issued at Negative Interest Rates

In January 2018, the Swiss government issued 15-year sovereign bonds at a negative yield of -0.08%. The present value (PV) of the bond per CHF100 of principal (FV) at the time of issuance is *closet to*:

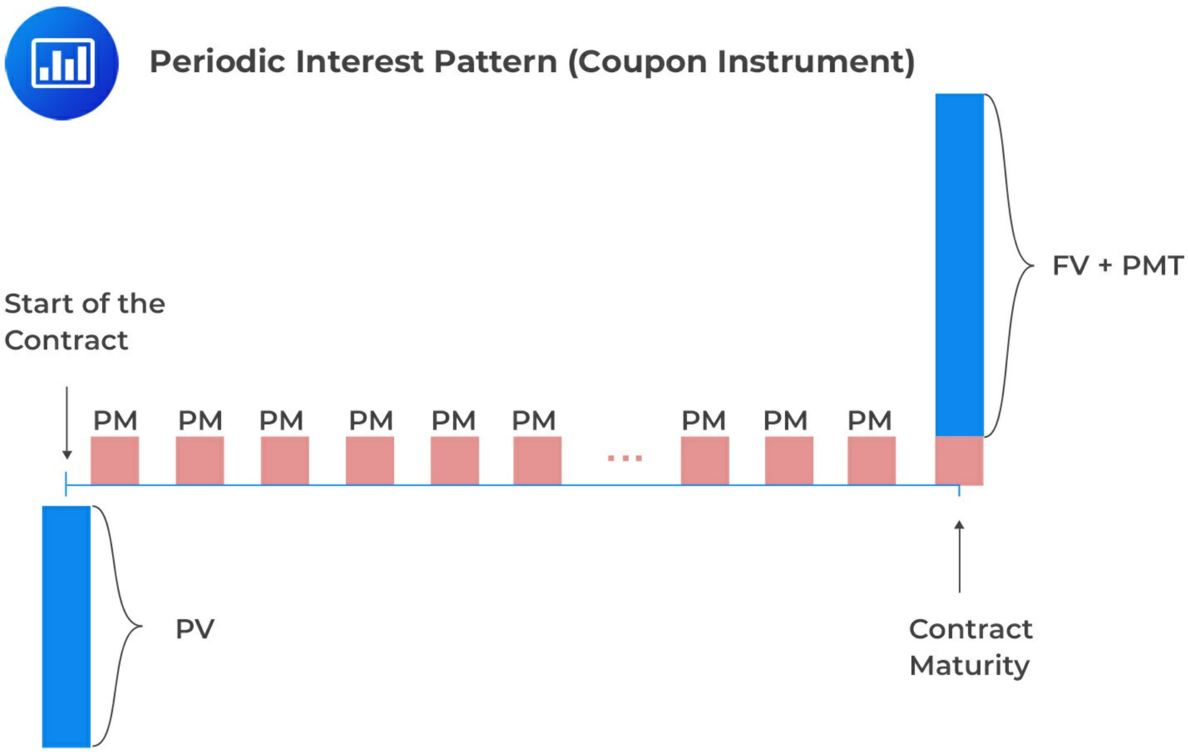
Solution

Recall for a zero-coupon bond,

$$\begin{aligned} PV &= FV_t(1 + r)^{-t} \\ &= 100(1 - 0.0008)^{-15} = 101.21 \end{aligned}$$

Periodic Interest Pattern (Coupon Instrument)

A coupon instrument is a fixed-income investment. It includes periodic cash flows called coupons and repays the principal at maturity. People often use these in coupon bond investments. These instruments have a set schedule with regular, equal payments.



The pricing of a coupon bond involves calculating its present value (PV) based on the market discount rate. The general formula for calculating the bond's price is derived from the discounted cash flow model. It considers the coupon payments (PMTs) and the final principal payment (FV) at maturity. The bond's price is determined by discounting each cash flow using the market discount rate (r).

The formula used to calculate the present value (PV) of a coupon bond is as follows:

$$\text{PV(Coupon Bond)} = \frac{PMT}{(1+r)^1} + \frac{PMT}{(1+r)^2} + \ldots + \frac{(PMT_N + FV_N)}{(1+r)^N}$$

Where:

PMT = Coupon payment.

FV = Future value.

r = Market discount rate (YTM).

N = Number of periods.

Example 1: Pricing a Coupon Bond on an Annual Basis

Suppose we have a 5-year bond with a face value of $1,000 and an annual coupon rate of 5%. The market discount rate is 6%. The bond's price is *closest to*:

Solution

$$\text{PV(Coupon Bond)} = \frac{PMT}{(1+r)^1} + \frac{PMT}{(1+r)^2} + \cdots + \frac{(PMT_N + FV_N)}{(1+r)^N}$$

In this case, we have PMT=5% of $1,000=$50, r=6%, N=5 years, FV=$1,000 so that:

$$\begin{aligned} PV &= \frac{\$50}{(1+0.06)^1} + \frac{\$50}{(1+0.06)^2} + \frac{\$50}{(1+0.06)^3} + \frac{\$50}{(1+0.06)^4} \\ &+ \frac{(\$50 + \$1,000)}{(1+0.06)^5} \\ PV &= \$47.17 + \$44.50 + \$41.98 + \$39.60 + \$784.62 = \$957.88 \end{aligned}$$

Therefore, the price of the bond would be $957.88

You could use a BA II Plus Calculator to solve the above question:

Steps	Explanation	Display
[2nd][QUIT]	Return to standard calc Mode	0
[2nd][CLR TVM]	Clears TVM Worksheet	0
5[N]	Years/periods	N = 5
6[1/Y]	Set interest rate	I/Y = 6.00
50[PMT]	Set payment	PMT = 50.00
1000[FV]	Set the face value	FV = 1000.00
[CPT][PV]	Compute the present value	PV = −957.88

Example 2: Pricing a Coupon Bond With a Single Cash Flow on a semi-annual Basis

Suppose an investor has a 2-year bond with a face value of $1000 and an annual coupon rate of 6%, paid semi-annually. The market discount rate is 5%. The price of the bond is *closest to*:

Solution

Recall that:

$$\text{PV(Coupon Bond)} = \frac{PMT}{(1+r)^1} + \frac{PMT}{(1+r)^2} + \ldots + \frac{(PMT_N + FV_N)}{(1+r)^N}$$

Where:

PMT = Coupon payments ($\$1,000 \times \frac{6\%}{2}$) = $30 *in this case.*

FV = Future value ($1,000 *in this case*).

r = Market discount rate (YTM), ($\frac{(5\%)}{2}$ = 2.5%) *per period in this case.*

N = Number of periods (4 *periods in this case*).

Plugging these values into the formula, we get:

$$PV = \frac{\$30}{(1.025)^1} + \frac{\$30}{(1.025)^2} + \frac{\$30}{(1.025)^3} + \frac{(\$30 + \$1,000)}{(1.025)^4}$$
$$PV = \$29.27 + \$28.55 + 27.85 + \$933.13 = \$1,018.81$$

Therefore, the bond's price is $\$1,018.81$

You can easily use the BA II Plus calculator (or any other allowed financial calculator) to solve the above question.

Steps	Explanation	Display
[2nd][QUIT]	Return to standard calc Mode	0
[2nd][CLR TVM]	Clears TVM Worksheet	0
4[N]	Years/periods	N = 4
2.5[1/Y]	Set interest rate	I/Y = 2.50
30[PMT]	Set payment	PMT = 30.00
1000[FV]	Set the face value	FV = 1000.00
[CPT][PV]	Compute the present value	PV = −1,018.81

Perpetual Bonds

Perpetual bonds are rare types of coupon bonds that do not have a stated date of maturity. They are generally issued by firms seeking equity-like financing and usually include redemption provisions.

The formula present value of perpetual bonds is obtained as follows: As $N \rightarrow \infty$, the formula for calculating PV of coupon changes as follows:

$$\begin{aligned} &\text{PV (perpetual bond)} \\ &= \lim_{(N \rightarrow \infty)} \left[\frac{PMT}{(1+r)^1} + \frac{PMT}{(1+r)^2} + \cdots + \frac{(PMT_N + FV_N)}{(1+r)^N}\right] \\ &= \frac{PMT}{r} \end{aligned}$$

So, the present value of a perpetuity is given by:

$$PV = \frac{PMT}{r}$$

Example: Perpetual Bond

In 2021, XYZ Financial (the holding company for XYZ Bank) issued \$500 million in perpetual bonds with a 4.00 percent semi-annual coupon. Calculate the bond's yield to maturity (YTM) if the market price was \$98.50 (per \$100).

Solution

Recall,

$$PV = \frac{PMT}{r}$$

Hence,

$$r = \frac{PMT}{PV}$$

To solve this problem, we first need to calculate the semi-annual coupon payment, which is,

$$\text{PMT(semi-annual coupon payment)} = \frac{\$100 \times 4\%}{2} = \$2, \text{PV} = \$98.50$$

Therefore,

$$r = \frac{\$2}{\$98.50} = 0.0203 = 2.03\%$$

The annualized yield-to-maturity is:

$$r = 0.0203 \times 2 \approx 4.06\%$$

Level Payments (Annuity Instruments) Patterns

An annuity is a finite series of cash flows, all with the same value. **A fixed-income instrument** with annuity payments provides a stream of periodic equal cash inflows over a finite period.

The level payments consist of interest and principal payments. Fixed income instruments with level payments include fully amortizing loans such as mortgages.

There are two types of annuities: ordinary annuities and annuities due. Annuity due is a type of annuity where payments start immediately at the beginning of time, at time $t = 0$. In other words, payments are made at the beginning of each period.

On the other hand, an ordinary annuity is an annuity where the cashflows occur at the end of each period. Such payments are said to be made in arrears (beginning at time t = 1). We shall consider ordinary annuity in this section.

Ordinary Annuity

Remember that in an ordinary annuity, the series of payments does not begin immediately. Instead, payments are made at the end of each period. It is further worth noting that the present value of an annuity is equal to the sum of the current value of each annuity payment:

$$
\begin{aligned}
PV &= A(1+r)^{-1} + A(1+r)^{-2} + \ldots + A(1+r)^{-N-1} + A(1+r)^{-N} \\
&= A(1+r)^{-1} + (1+r)^{-2} + \ldots + (1+r)^{-(N-1)} + (1+r)^{-N} \\
PV &= A\frac{1-(1+r)^{N}}{r}
\end{aligned}
$$

Where:

A = Periodic cash flow.

r = Market interest rate per.

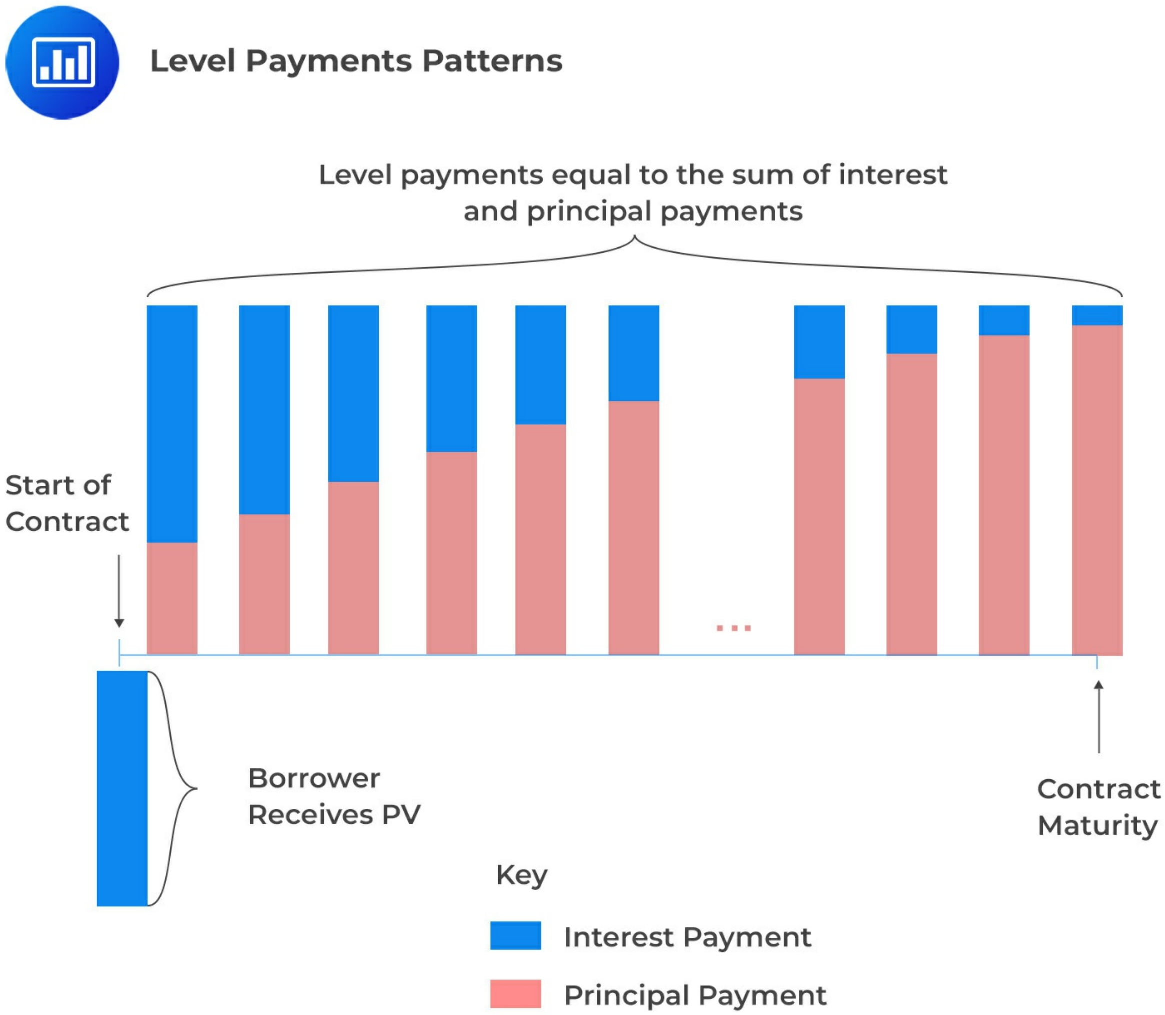
PV = Present value/ Principal Amount of the loan or bond.

N = Number of payment periods.

The periodic annuity is calculated as follows:

$$A = \frac{r(PV)}{(1-(1+r)^{-N}}$$

Consider a fully amortizing mortgage loan. In this case, the borrower receives the mortgage loan now and promises to make periodic payments equal to the sum of interest and principal payments.



Note that the periodic mortgage payment is constant, but the proportion of the interest payment decreases while the principal payment increases.

The cash flow pattern of a fully amortizing mortgage follows the pattern of an ordinary annuity with a series of equal cash flows. As such, the periodic annuity (periodic payment) of a fully amortizing mortgage is given by:

$$A = \frac{r(PV)}{1-(1+r)^{-t}}$$

Where:

A = Periodic cash flow.

r = Market interest rate per period.

PV = Present value or principal amount of loan or bond.

t = Number of payment periods.

Example: Calculating the Periodic Payment of a Mortgage

Jake is looking to secure a fixed-rate 25-year mortgage to finance 75% of the value of an $800,000 residential property. If the annual interest rate on the mortgage is 4.5%, Jake's monthly mortgage payment is *close to*:

Solution

Remember,

$$A = \frac{r(PV)}{1-(1+r)^{-t}}$$

Where:

A = Periodic cash flow.

r = Market interest rate per period.

PV = Present value/ Principal Amount of the loan or bond.

t = Number of payment periods.

In this case, we have:

- $r = 0.375\% \,(= \frac{4.5\%}{12})$
- $N = 300 \text{ months}(= 25 \text{ years} \times 12 \text{ months/year})$
- $PV = \$600,000(= 75\% \times \$800,000)$

Plugging these values into the formula, we get:

$$A = \frac{0.00375 \times \$600,000}{1-(1+0.00375)^{-300}} = \$3,334.995 \approx \$3,335$$

Using a BA II Plus financial calculator:

Steps	Explanation	Display
[2nd][QUIT]	Return to standard calc Mode	0
[2nd][CLR TVM]	Clears TVM Worksheet	0
300[N]	Years/periods	N = 300
0.375[1/Y]	Set interest rate	I/Y = 0.375
−600,000[PV]	Set the present value of the mortgage	PV = −600,000.00
0[FV]	Set the face value	FV = 0.00
[CPT][PMT]	Compute the periodic payment	PMT = 3,334.99

Time Value of Money in Equity Instruments

Equity investments, such as stocks, enable an investor to acquire a fractional share/ownership by the issuing company. This gives investors the right to receive a share of the company's available cash flows as dividends.

In the context of equity instruments, the time value of money (TVM) is used to discount expected future cash flows to determine their present value. This allows investors to value the company shares.

The present value of expected future cash flows is calculated using a discount rate, r, which represents the expected rate of return on the investment.

Common Approaches for Valuing Equity Instruments

Valuing equity investments depends on the dividends cashflows which can take one of three forms: constant dividends, constant dividend growth rate, and changing dividend growth rate.

1. Valuing Equity Instruments based on Constant Dividend: The Constant Dividends model values stocks based on the assumption that dividends will remain constant over time. The preferred or common share dividend cash flows are in the form of an infinite series that is valued like perpetuity. The formula for the constant dividends model is as follows:

$$PV_t = \sum_{i=1}^{\infty} \frac{D_t}{(1+r)^i} = \frac{D_t}{r}$$

Where: PV_t = Present value at time t

D_t = Dividend payment at time t

r = Discount rate.

Example: Valuing Equity Instruments based on Constant Dividend

Assuming we have a preferred stock with a dividend payment of $5 per year. The discount rate is 8%. The present value of the stock is *closest to*:

Solution

Recall,

$$PV_t = \frac{D_t}{r}$$

In this case, D=$5, r=8%, $PV=?$

So,

$$PV = \frac{5}{0.08} = \$62.5$$

This means that the present value of the stock is $62.5.

2. Valuing Equity Instruments Based on Constant Dividend Growth Rate The constant dividend growth model is a method used to estimate the value of a stock based on its future dividends. This model assumes that dividends will grow at a constant rate (g) forever. To derive the formula for this model, we start by considering that the **present value of a stock is equal to the sum of its future dividends**, discounted by the required rate of return r. If dividends are assumed to grow at a constant rate, then each future dividend can be calculated by multiplying the previous dividend by $(1 + g)$.

Let D_t represent the expected dividend in the next period. The present value of the stock can then be expressed as:

$$PV_t = \frac{D_t}{(1+r)} + \frac{D_t(1+g)}{(1+r)^2} + \frac{D_t(1+g)^2}{(1+r)^3} + \ldots$$
$$= \sum_{i=1}^{\infty} \frac{D_t(1+g)}{(1+r)^i}$$

This is an infinite geometric series with a common ratio of $\frac{(1+r)}{(1+g)}$. Using the formula for the sum of an infinite geometric series, we can simplify this equation to:

$$PV_t = \frac{D_t(1+g)}{r-g} = \frac{D_{t+1}}{r-g}$$

Where:

PV_t = Present value at time t.

D_{t+1} = Expected Dividend in the next period.

r = Required rate of return.

g = Constant growth rate.

$r - g > 0$

Therefore, this is the formula for calculating the present value of a stock using the constant dividend growth rate. This model can help estimate the value of a stock when its future dividends are expected to grow steadily.

Example: Valuing Equity Instruments based on Constant Dividend Growth Rate

Suppose a stock currently pays an annual dividend of $2.00 per share. The required rate of return for this stock is 10%, and the dividends are expected to grow at a constant rate of 5% per year indefinitely. Using the constant dividend growth model, the present value of this stock is *closest to*:

Solution

Recall that,

$$PV_t = \frac{D_t(1+g)}{r-g} = \frac{D_{t+1}}{r-g}$$

In this case, we know that D_t=$2.00, r=10%, g=5%

So,

$$PV = \frac{2 \times 1.05}{r-g} = \frac{2.10}{0.10-0.05} = \$42$$

Therefore, the present value of the stock is $42

3. Valuing Equity Instruments with Changing Dividend Growth Rates is a dynamic process. It begins with the investor buying a stock at an initial price and getting an initial dividend. The unique aspect is that the dividend is expected to grow at a rate that evolves as the company matures and shifts from high growth to slower growth. This valuation doesn't have a single formula because it relies on assumptions about future dividend growth. However, a common method is to use a multi-stage dividend discount model. This model assumes that dividends will grow at different rates during various stages of the company's growth. To find the stock's present value, you sum up the present values of dividends at each stage.
The Multi-Stage Dividend Discount Model builds on the Constant Dividend Growth Model. It accommodates a company's transition from high initial growth to lower, more stable growth.

Let's say a company has a high short-term growth rate g_s followed by a perpetual lower growth rate g_l. To find the present value (PV) of the stock at time t using this model, we compute it in two stages:

I. **First Part**: The first part calculates the present value of dividends during the initial n periods of higher growth (g_s). This is done by discounting the dividends for each period by the required rate of return r using the following formula:

$$PV_t = \sum_{i=1}^{n} \frac{D_t(1+g_s)^i}{(1+r)^i}$$

Where:PV = Present value.n = Number of periods.D_t = Dividend at time (t).g_s = Initial higher dividend growth rate.r = Required rate of return.

II. **Second Part**: The second part calculates the present value of dividends after the initial n periods, assuming constant growth at a lower long-term rate (g_l). This can be simplified using the geometric series simplification, where $E(S_t + n)$ represents the terminal value or stock value in n periods:

$$PV_t = \frac{E(S_t + n)}{(1+r)^n}$$

Where:$E(S_t + n) = \frac{D_{t+n+1}}{(r-g_l)}$ and g_l is the lower, more stable dividend growth rate.

Example: Valuing Equity Instruments based on Changing Dividend Growth Rate

Assuming we have a stock with an expected dividend payment of $2 in one period. The discount rate is 10%. The stock is expected to have a high dividend growth rate of 20% for the first three years, followed by a slower growth rate of 5% thereafter. Calculate the present value of the stock.

Solution

First, we calculate the present value of the dividends during the high growth period:

Recall that,

$$PV_t = \sum_{i=1}^{n} \frac{D_t(1+g_s)^i}{(1+r)^i}$$

In this case, D_t=\$2, g_s=0.20, r=0.10, n=3

So,

$$PV_1 = \frac{2}{(1+0.10)^1} + \frac{2 \times (1+0.20)^1}{1+0.10)^2} + \frac{2 \times (1+0.20)^2}{1+0.10)^3}$$
$$PV_1 = 1.818 + 1.983 + 2.163 = 5.965$$
$$PV_1 = \$5.97$$

Next, we calculate the present value of the dividends during the slower growth period, assuming that dividends will grow at a constant rate of 5% thereafter:

Recall that,

$$E(S_t + n) = \frac{D_{t+n+1}}{(r - g_l)}$$

So,

$$E(S_t + n) = \frac{2(1+0.20)^3 \times (1+0.05)}{0.10 - 0.05} = \frac{3.629}{0.05} = \$72.578$$

Finally, we calculate the present value of P_4

$$PV_2 = \frac{\$72.578}{(1+0.10)^3} = \$54.527$$

The total present value of the stock is the sum of PV_1 and PV_2

$$\begin{aligned} PV_{total} &= PV_1 + PV_2 \\ &= \$5.965 + 54.527 \\ &= \$60.493 \approx \$60.49 \end{aligned}$$

Question

Five years ago, Milton Inc. issued corporate bonds with a 15-year maturity. The bonds have a semi-annual coupon rate of 7.8% per annum, and the current yield to maturity is 8.5% per annum. The current price of Milton Inc's bonds (per CAD100 of par value) is *closest to*:

A. CAD91.23.
B. CAD95.35.
C. CAD96.15.

The correct answer is B.

Solution

Recall the formula for calculating the price of a bond:

$$\text{PV(Coupon Bond)} = \frac{\text{PMT}}{(1+r)^1} + \frac{\text{PMT}}{(1+r)^2} + \ldots + \frac{(\text{PMT}_N + \text{FV}_N)}{(1+r)^N}$$

First, let's get the semi-annual equivalent rates:

- The semi-annual coupon rate is $\frac{7.8\%}{2} = 3.9\%$.
- The semi-annual yield to maturity is $\frac{8.5\%}{2} = 4.25\%$.

Next, we find the number of periods remaining until the bond matures:

Since the bonds were issued 5 years ago and have a 15-year maturity, $10(= 15 - 5)$ years remain to maturity. Since interest is paid semi-annually, this equates to $10 \times 2 = 20$ periods.

You can plug the above values into the general formula, consuming valuable time. Therefore, using BA II plus a financial calculator,

Steps	Explanation	Display
[2nd][QUIT]	Return to standard calc Mode	0
[2nd][CLR TVM]	Clears TVM Worksheet	0
20[N]	Years/periods	$N = 20$
4.25[1/Y]	Set interest rate	$I/Y = 4.25$
3.9[PMT]	Set the periodic coupon payment	$PMT = 3.90$
100[PV]	Set the face value of the bond	$FV = 100.00$
[CPT][PV]	Compute the present value	$PV = -95.35$

LOS 2b: Calculate and interpret the implied return of fixed-income instruments and the required return and implied growth of equity instruments given the present value (PV) and cash flows

Implied Return for Fixed-Income Instruments

The growth rate is the rate at which the market expects an asset to grow. On the other hand, implied return reflects a return based on the current price and future security cash flows.

Consider a fixed-income instrument. If we have its present value and assume all future cash flows happen as expected, the discount rate rrr, or yield-to-maturity (YTM), shows the implied return under these assumptions for the cash flow pattern.

Now, take an equity investment. If we have the present value, future value, and discount rate, we can find the implied growth rate that aligns with these values.

The implied return or growth rate provides a view of the market expectations incorporated into an asset's market price. It is useful for investors to understand these expectations when making investment decisions.

Calculating the Implied Return for Fixed-Income Instruments

Discount Bond

In the case of a discount bond or instrument, recall that an investor receives a single principal cash flow (FV) at maturity, with ($FV - PV$) representing the implied return.

To solve for the implied return earned over the life of an instrument (N periods), we can rearrange the single cash flow present value formula.

Recall that the single cash flow present value formula is:

$$PV = FV_t(1 + r)^{-t}$$

Where:

FV = Future value.

PV = Present value.

r = Stated discount rate per period.

t = Number of compounding periods.

To solve for r, we can rearrange this formula as follows:

$$r = \sqrt[t]{\frac{FV_t}{PV}} - 1 = \left(\frac{FV_t}{PV}\right)^{\frac{1}{t}} - 1$$

We use this formula to calculate the periodic return earned during the life of the instrument (t periods) based on the present value (or price) and future value of the instrument.

Example: Calculating the Implied Return for a Discount Bond

Consider a zero-coupon bond with price of $900, a future value of $1,000, and a maturity of 5 years. Calculate the implied annualized return r.

Solution

Recall that,

$$r = \left(\frac{FV_t}{PV}\right)^{\frac{1}{t}} - 1$$

In this case, t=5, FV_t=$1,000, PV=$900

So,

$$r = \left(\frac{1000}{900}\right)^{\frac{1}{5}} - 1 = 2.13\%$$

This means that an investor who purchases this zero-coupon bond at a price of $900 and holds it for five years would earn an annualized return of 2.13%.

Coupon Bonds

Recall that fixed-income instruments that pay periodic interest have cash flows throughout their life until maturity. The **yield-to-maturity (YTM)** is a single implied market discount rate for all cash flows, regardless of timing. It assumes an investor expects to receive all promised cash flows through maturity and reinvest any cash received at the same YTM.

The present value of a fixed-income instrument with periodic interest can be calculated using the following formula:

$$PV = \frac{PMT_1}{(1+r)^1} + \frac{PMT_2}{(1+r)^2} + \cdots + \frac{(PMT_N + FV_N)}{(1+r)^N}$$

Where:

PV = Present value (or price) of the instrument.

PMT = Periodic payment.

FV = Bond's principal.

N = Number of periods to maturity.

r = Discount rate (or internal rate of return) (YTM).

Example: Implied Return for Fixed-income Instruments With Periodic Interest

Consider a five-year corporate bond issued in 2023 with a 4.00 percent annual coupon and a price of USD110.00 per USD100 principal three years later. If Milka can reinvest periodic interest at the original YTM of 4.00 percent, the implied three-year return is *closest to*:

Solution

We can calculate the future value (FV) after three years, including the future price of 110.00 and all cash flows reinvested to that date:

$$\begin{aligned} FV_3 &= PMT_1(1+r)^2 + PMT_2(1+r) + PMT_3 + PV_3 \\ &= 4 \times (1.04)^2 + 4 \times (1.04) + 4 + 110.00 \\ &= \$122.49 \end{aligned}$$

We can then solve for Milka's annualized return r using the formula for implied return since we have PV=100, FV=122.49, N= 3 as follows:

$$r = \sqrt[t]{\frac{FV_t}{PV}} - 1 = \sqrt[3]{\frac{122.49}{110}} - 1 = 3.65\%$$

This means that Milka, who purchased the corporate bond at a price of 100 and held it for three years, would earn an annualized return of 3.65%.

Example: Calculating the Yield-to-Maturity of a Coupon Bond

CityGroup Corp. issued a corporate bond 7 years ago with a face value of $1,000 and a 20-year maturity. The bond pays annual interest at a coupon rate of 6%. Currently, the bond is trading at $1,120. The yield to maturity (YTM) of CityGroup Corp.'s bond is *closest to*:

Solution

Using the BA II Plus calculator, we solve the question as follows:

We have

Steps	Explanation	Display
[2nd][QUIT]	Return to standard calc Mode	0
[2nd][CLR TVM]	Clears TVM Worksheet	0
13[N]	Years/periods	$N = 13$
$-1,120$[PV]	Set the present value of the bond	$PV = -1,120$
60[PMT]	Set the periodic coupon payment	$PMT = 3.90$
1,000[PV]	Set the face value of the bond	$FV = 1000.00$
[CPT][I/Y]	Compute the YTM	$I/Y = 4.74\%$

Therefore, the YTM is 4.74%.

Implied Return and Growth for Equity Instruments

The value of a stock is determined by both the expected return and the growth of its cash flows.

By assuming a constant growth rate for dividends, we can use the formula for the present value of an equity investment to calculate the stock's implied return or growth rate.

Implied Return

Recall that the present value of a stock for constant growth of dividends is given by:

$$PV_t = \frac{D_t(1+g)}{r-g} = \frac{D_{t+1}}{r-g}$$

Where:

PV_t = Present value at time t.

D_t = Expected Dividend in the next period.

r = Required rate of return.

g = Constant growth rate.

$r - g > 0$

Therefore, we can calculate the implied return on a stock given its expected dividend yield and implied growth by rearranging the above formula as follows:

$$r = \frac{D_t(1+g)}{PV_t} + g = \frac{D_{t+1}}{PV_t} + g$$

In simple terms, if we assume a stock's dividends will grow at a steady rate forever, the implied return is the combination of its expected dividend yield and the constant growth rate.

Example: Implied Return and Growth

Suppose Apple Inc. stock is trading at a share price of USD150.00, and its annualized expected dividend per share during the next year is USD2.00.

Moh, an analyst, projects that Apple's dividend per share will increase at a constant rate of 5% per year indefinitely. The required return expected by investors on the stock is *closest to*:

Solution

Recall that the implied return formula is,

$$r = \frac{D_t(1+g)}{PV_t} + g = \frac{D_{t+1}}{PV_t} + g$$

In this case, D=\$2.00, PV=\$150, g=5%

Therefore,

$$r = \frac{2.00(1.05)}{150} + 0.05 = 6.4\%$$

Implied Growth

We can also solve for a stock's implied growth rate, which is given by the following formula:

$$g = \frac{r \times PV_t - D_t}{PV_t} + D_t = \frac{r - D_{t+1}}{PV_t}$$

Example: Calculating the Implied

Consider the previous example. Suppose Moh believes that Apple stock investors should expect a return of 8%. Calculate the implied dividend growth rate for Apple Inc.

Solution

Recall that the formula for calculating implied growth is as follows.

$$g = \frac{r \times PV_t - D_t}{PV_t} + D_t = \frac{r - D_{t+1}}{PV_t}$$

So,

$$g = 0.08 - \frac{2.00 \times 1.05}{150} = 0.066 = 6.60\%$$

Price-to-Earnings Ratio

In equity instruments, it is common practice to compare the price-to-earnings ratio.

The **price-to-earnings (P/E) ratio** is a valuation metric that compares the current share price of a stock to its earnings per share. Investors and analysts use it to determine the relative value of a company’s shares compared to other companies or the market.

A stock with a higher price-to-earnings ratio is more expensive than a lower one, as investors are willing to pay more for each unit of earnings. This ratio is also a valuation metric for stock indexes, such as S&P 500.

Relating P/E Ratio to Expected Future Cash Flows

The P/E ratio can relate to our earlier discussion on a stock's price (PV) to the expected future cash flow relationship. Recall the following equation:

$$PV_t = \frac{D_t \times (1+g)}{r-g}$$

By dividing both sides of the equation by E_t, which represents earnings per share for period t, we get the following equation:

$$\frac{PV_t}{E_t} = \frac{\frac{D_t}{E_t} \times (1-g)}{r-g}$$

Where:

$\frac{PV_t}{E_t}$ = Price-to-earnings (P/E) ratio.

$\frac{D_t}{E_t}$ = Dividend payout ratio.

g = Growth rate.

r = Required rate of return.

The dividend payout ratio represents the percentage of a company's earnings paid out to shareholders in the form of dividends.

Typically, the **forward P/E ratio**, which is based on a projection of a company's earnings per share for the next period $(t+1)$, is used. This ratio is positively correlated with higher expected dividend payouts and growth rates but negatively correlated with the required return.

Therefore, the equation:

$$\frac{PV_t}{E_t} = \frac{\frac{D_t}{E_t} \times (1-g)}{r-g}$$

Can be simplified as below to find the forward P/E ratio:

$$\frac{PV_t}{E_{t+1}} = \frac{\frac{D_{t+1}}{E_{t+1}}}{r-g} = \frac{D_{t+1}}{E_{t+1}} \times \frac{1}{r-g}$$

Example: Solving for Implied Dividend Growth Rate

Suppose a company has a forward P/E ratio of 15, a dividend payout ratio of 40%, and a required return of 10%. The implied dividend growth rate for this company is *closest to*:

Solution:

First, we can use the formula for the forward P/E ratio to solve for the implied dividend growth rate:

$$\frac{PV_t}{E_{t+1}} = \frac{D_{t+1}}{E_{t+1}} \times \frac{1}{r-g}$$

Where:

PV_t = Present value at time t.

E_{t+1}= Earnings per share for the next period.

D_{t+1}= Dividend payout for the next period.

r = Required return.

g = Implied dividend growth rate.

Substituting the given values into the formula, we get:

$$15 = \frac{0.4}{0.1 - g}$$

Solving for g, we get:

$$g = 0.1 - \frac{0.4}{15} = 0.0733$$

Therefore, the implied dividend growth rate for this company is 7.33%

Example: Solving for Required Return

Let's assume you are not given the required rate of return in the question above so that the company has a forward P/E ratio of 15, a dividend payout ratio of 40%, and an implied dividend growth rate of 7.33%. What is the required return for this company?

Solution

Recall the formula for the forward P/E ratio to solve for the required return:

$$\frac{PV_t}{E_{t+1}} = \frac{D_{t+1}}{E_{t+1}} \times \frac{1}{r - g}$$

Substituting the given values into the formula, we get:

$$15 = \frac{0.4}{r - 0.0733}$$

Solving for r, we get:

$$r = \frac{0.4}{15} + 0.0733 = 0.1000$$

Therefore, the required return for this company is 10%.

Question

Edmund company's stock trades at USD50.00. The company pays an annual dividend to its shareholders, and its most recent payment of USD 2.00 occurred yesterday. Analysts following the company expect its dividend to grow at a constant rate of 4 percent per year. What is the company's required return?

A. 8.16%.
B. 8.48%.
C. 9.16%.

The correct answer is A.

Solution

Recall that:

$$PV = \frac{D_{t+1} \times (1+g)}{r-g}$$

Where:

PV = Current stock price.

D_{t+1} = Recent dividend payout.

g = Expected dividend growth rate.

r = Required return.

Substituting the given value into the formula, we get:

$$50 = \frac{2 \times (1 + 0.04)}{r - 0.04}$$

Solving for r, we get:

$$r = \frac{2 \times (1 + 0.04)}{50} + 0.04 = 0.0816$$
$$r = 0.0816$$

Edmund's required return is 8.16%.

LOS 2c: explain the cash flow additivity principle, its importance for the no-arbitrage condition, and its use in calculating implied forward interest rates, forward exchange rates, and option values

A timeline is a physical illustration of the amounts and timing of cashflows associated with an investment project. For cashflows that are regular and of equal amounts, the standard annuity formula or the financial calculator can be used. However, a timeline is preferred for irregular, unequal, or both cashflows.

Remember that the general formula that relates the present value and the future value of an investment is given by:

$$FV_N = PV(1 + r)^N$$

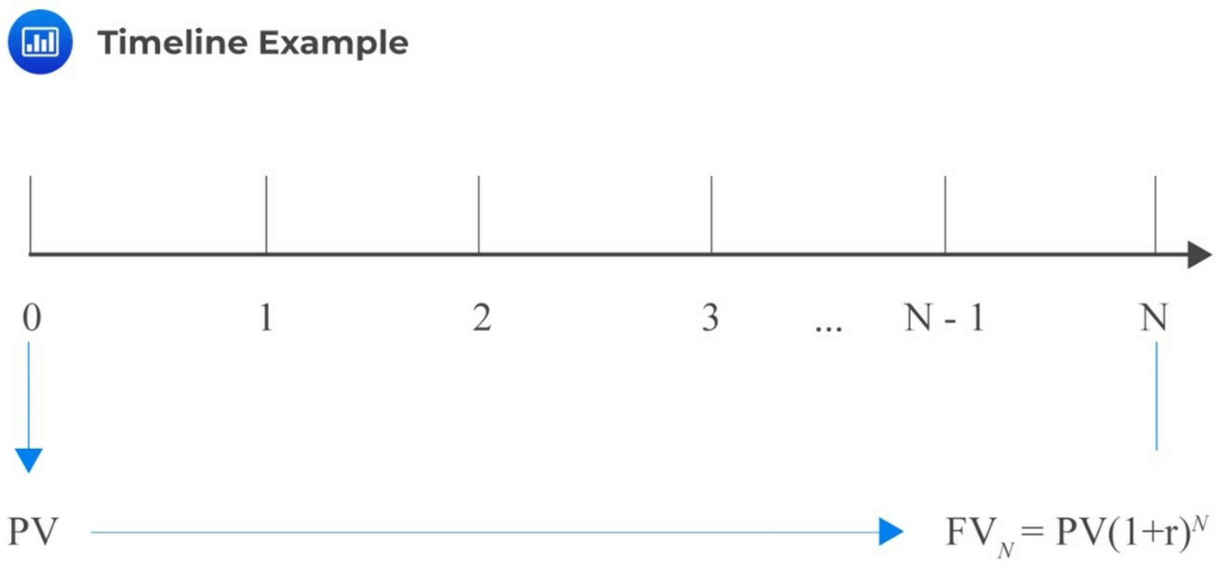
Where:

PV = Present value of the investment.

FV_N = Future value of the investment N periods from today.

r = Rate of interest per period.

We can represent this in a timeline:



In a particular timeline, a time index, t, represents a particular point in time, a specified number of periods from today. Therefore, the present value is the investment amount today (t = 0), and by using this amount, we can calculate the future value (t = N). Alternatively, we can use the future value to calculate the present value.

The above argument can be written in terms of the present value. That is:

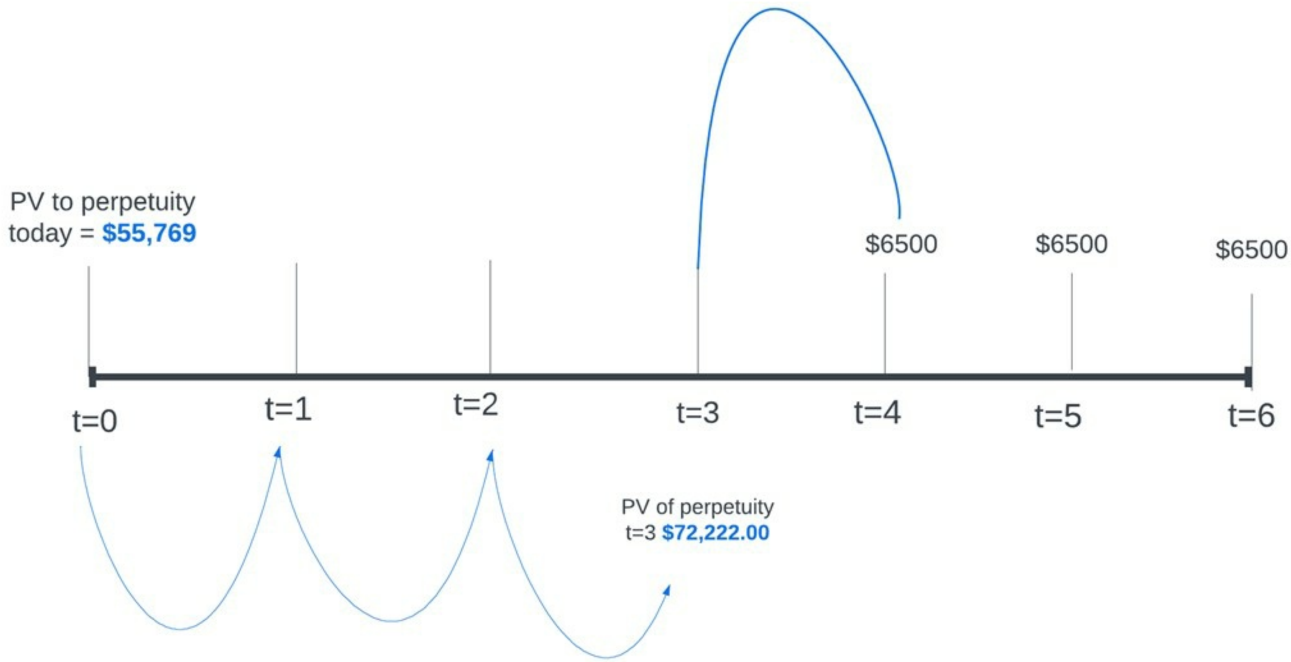
$$PV = FV_N(1 + r)^{-N}$$

Example: Applying a Timeline to Model Cashflows

A fixed-income investor receives a series of payments, each amounting to $6,500, set to be received in perpetuity. Payments are to be made at the end of each year, starting at the end of year 4. If the discount rate is 9%, then what is the present value of the perpetuity at t = 0?

Solution

We would then draw a timeline to understand the problem better:



Here, we can see that the investor is receiving $6,500 in perpetuity. Recall that the PV of aperpetuity is given by:

$$\text{PV of a perpetuity} = \frac{C}{r}$$

So, in this case:

$$PV_3 = \frac{\$6,500}{9\%} = \$72,222$$

This is the value of the perpetuity at $t = 3$, so we need to discount it for three more periods to get the value at $t = 0$. Using the formula:

$$PV_0 = FV_N(1 + r)^{-N}$$

The PV at time zero is $\frac{\$72,222}{(1+0.09)^3} = \$55,769$

Why is the Use of a Timeline Recommended?

There are many instances in real life when cashflows are uneven. A good example is a pension contribution that varies with age. Applying one of the basic time value formulae is impossible in such cases. You are advised to draw a timeline even if the question appears relatively straightforward. It will help you understand the question structure better. A timeline also helps candidates add cashflows indexed to the same period and apply the value additivity principle.

Cashflow Additivity Principle

According to the cashflow additivity principle, the present value of any stream of cashflows indexed at the same point equals the sum of the present values of the cashflows. This principle has different applications in time value of money problems. Besides, this principle can be applied to in different economic scenarios.

Application of Cashflow Additivity

Investing in Different Currencies

The principle of cash flow additivity can be applied to scenarios involving different currencies by converting all cash flows to a common currency using the appropriate exchange rates. Doing so allows us to compare and combine cash flows from different currencies and make investment decisions based on their combined value.

For example, suppose we have two investment opportunities, one in US dollars and one in Japanese yen. In that case, we can convert the expected cash flows from the Japanese yen investment into US dollars using the appropriate exchange rates. Then, we can compare the combined value of the two investments and decide based on their relative values.

Dealing with different currencies assumes continuous compounding. Recall that the present value of a continuous compounding is given by:

$$PV = FV_N e^{-N r_s}$$

Example: Applying the Cash Flow Additivity Principle

Consider an investor with USD 2,000 who wants to invest it for three months. The investor can choose between two options: investing in the US government debt or German government debt.

Option 1: Investing in US Government Debt

The investor can invest his USD 2,000 in a three-month US Treasury bill. This means that he lends the government USD 2,000, and it promises to pay him back with interest in three months. The interest is 3%, so after three months, he will receive:

Recall that,

$$\text{FV} = 2,000 \times e^{0.03 \times \frac{3}{12}} = \text{USD } 2,015$$

Option 2: Investing in German Government Debt

The investor chooses to invest in German government debt. To do this, the investor must convert his USD 2,000 into Euros at the current exchange of EUR/USD = 0.92 (1 USD =0.92EUR). This means that the investor will receive $(2,000 \times 0.92) = \text{EUR } 1840$. He can then lend this money to the German government by investing in a three-month German Treasury bill. Assuming the interest rate is 0.06 percent, after three months, the investor will receive:

$$\begin{aligned} \text{FV} &= \text{EUR } 1840 \times e^{0.06 \times \frac{3}{12}} \\ &= \text{EUR } 1,867.81 \end{aligned}$$

Assuming the investor wants his money in US dollars, we need to convert the EUR1867.81 back into USD at the forward exchange rate of USD/EUR = 1.0788. This means that the investor will receive:

$$\frac{\text{EUR } 1{,}867.81 \times \text{USD } 1.0788}{1 \text{ EUR}} = \$2,014.99 \approx 2,015$$

Both options give you the same amount of money after three months: USD 2,015. The difference is that one option involves investing in US dollars, and the other involves converting your money into Euros and back into US dollars.

The forward exchange rate of 1.0788 USD/EUR is important because it determines how much money you, the investor, will receive when converting your Euros back into US dollars. If this rate differs from 1.0788, there would be an arbitrage opportunity in converting Euros to dollars.

More on foreign exchange rates will be discussed later in the curriculum.

Implied Forward Rates

Consider two zero-coupon bonds. Bond A has a maturity of two years and a yield of 2% per annum, while bond B has a maturity of four years and a yield of 3%. An investor, who doesn't seek to take advantage of price differences and is risk-neutral, has $1,000 to invest. The investor has two investment options that earn the same return.

Option 1: The investor can put their money into bond B now, which has an annual yield of 3%, and will pay out at the end of the four years. The Future Value (FV) of this investment in four years, using the formula for compound interest, is:

$$FV_4 = PV_0(1 + r_4)^4 = 1000(1.03)^4 = 1,125.51$$

Option 2: Alternatively, the investor can initially invest in bond A and, after two years, reinvest the proceeds at a forward rate $F_{2,2}$ which represents a two-year forward rate starting in year two.

By the principle of cash flow additivity, a risk-neutral investor will not prefer one option over the other - they are indifferent between Options 1 and 2. This is because the Future Values of both investments at the end of four years should be the same:

$$FV_4 = PV_0(1 + r_4)^4 = PV_0(1 + r_2)(1 + F_{2,2})$$

In this scenario, this simplifies to:

$$1,125.51 = 1,000(1.02)^2(1 + F_{2,2})$$

Solving this equation for the forward rate gives:

$$\Rightarrow F_{2,2} = \frac{1,125.51}{1,000(1.02)^2} - 1 = 8.18\%$$

Therefore, to prevent arbitrage opportunities, the forward rate $F_{2,2}$ should be set to 8.18%. This ensures that there is no potential for risk-free profits, maintaining market efficiency.

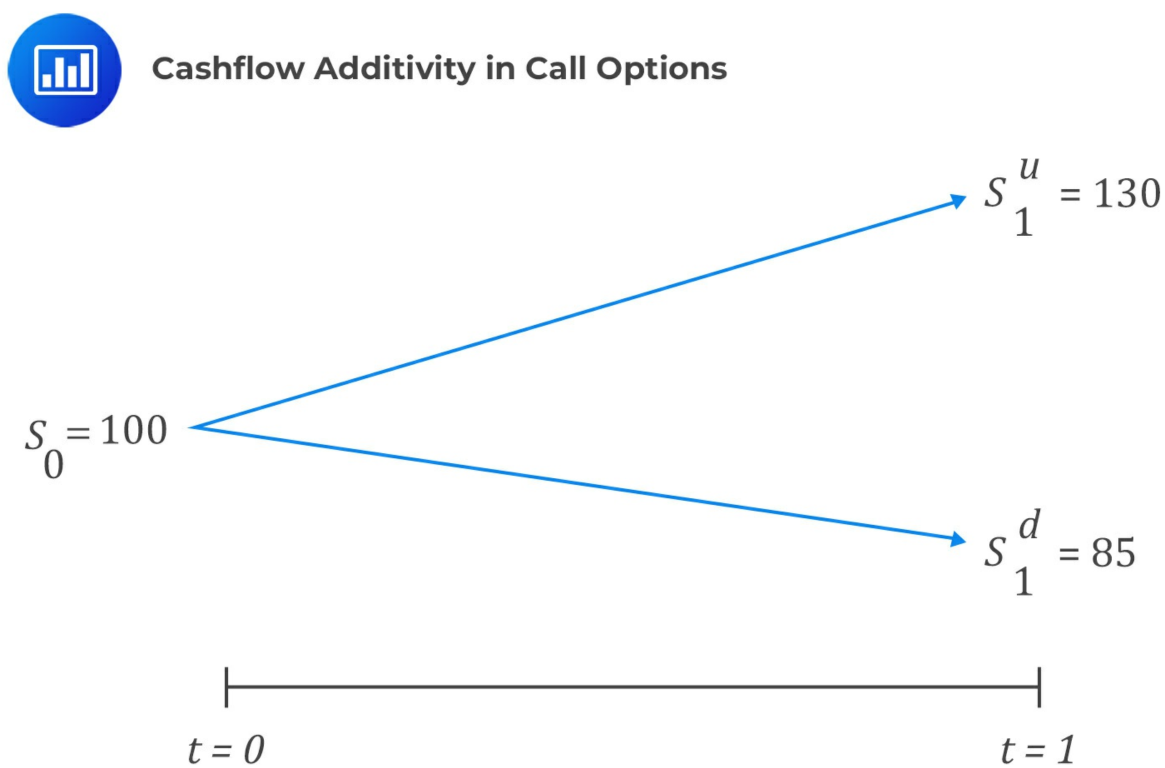
Cashflow Additivity and Option Pricing

Cash flow additivity can be used to determine the fair price of an option contract. An option contract gives the buyer the right, but not the obligation, to buy (call) or sell (put) an underlying asset at a specified price within a certain period.

Cash flow additivity allows investors to compare different strategies and determine a no-arbitrage price for a financial instrument.

Example: Illustrating Cashflow Additivity in Call Option

Consider a stock that costs $100 now. Its price might increase by 30% to $130 or decrease by 15% to $85 in one year.



Let's say an investor wants to sell a call on the stock that gives the buyer the right, but not the obligation, to purchase the asset for $120. The principle of cash flow additivity can be used to determine the contract's no-arbitrage price.

If the stock price goes up, the contract is worth $c_1^u = 10$. That's because the buyer can use the contract to buy the asset for $120 and then sell it for $130, making a profit of $10. But if the price goes down, the contract is worth nothing. The buyer wouldn't want to use the contract to buy the asset for $120 when they could just buy it for $85 without the contract.

The underlying argument here is that the value of the option is each movement of the stock option may be used to construct a risk-free portfolio (the value of the portfolio is the same in both scenarios).

Denote the initial value of the call option by c_0, which we wish to determine using cash flow additivity and no-arbitrage pricing. Also, denote the value of the portfolio at $t = 0$ by V_0, when the stock price increases by V_1^u and when the stock price decreases by V_1^d

Assume that at $t = 0$ creates a risk-free portfolio by selling a call option at c_0 and buying 0.22 units of underlying assets. Then, the value of the portfolio at inception is:

$$V_0 = 0.22 \times 100 - c_0$$

This portfolio is called a **replicating portfolio** because it is designed to create a matching future cash flow stream to that of a risk-free asset.

Similarly, in each scenario of stock price decrease and increase:

$$V_1^u = 0.22 \times 130 - 10 = 18.89$$
$$V_1^d = 0.22 \times 85 - 0 = 18.89$$

Intuitively, the value replicating portfolio equals 18.89, whether the stock prices rise or decline. As such, the replicating portfolio is risk-free and can be discounted as a risk-free asset. Assuming that risk-free rate is r = 2.5%, then:

$$\begin{aligned} V_0 &= V_1^u(1 + r)^{-1} = V_1^d(1 + r)^{-1} \\ &= 18.89(1.025)^{-1} = 18.43 \end{aligned}$$

At this point, we can calculate the value of c_0 rearranging the initial portfolio value equation:

$$\begin{aligned} V_0 &= 0.22 \times 100 - c_0 \\ \Rightarrow 18.43 &= 0.22 \times 100 - c_0 \\ \therefore c_0 &= 3.57 \end{aligned}$$

As such, the fair price of the call option is \$3.57, which the seller expects to receive from the buyer.

Question

The current USD/CHF exchange rate is 0.9. The risk-free interest rates for one year are 2% for the US dollar and 1% for the Swiss franc. Which of the following one-year USD/CHF forward rates would best prevent arbitrage opportunities?

A. USD/CHF 0.909.
B. USD/CHF 0.099.
C. USD/CHF 0.891.

Solution

The correct Answer is A.

Dealing with different currencies assumes continuous compounding. Recall that the future value of a continuous compounding is given by:

$$FV = PV_N e^{Nr_s}$$

So,

In one year, a single unit of Swiss franc invested is:

$$e^{0.01} = \text{CHF } 1.0101$$

In one year, a single unit of Swiss Franc converted to US dollars and then invested risk-free is worth;

$$0.9e^{0.02} = \text{USD } 0.9182$$

Therefore, to convert USD 0.9182 into CHF 1.0101 requires a forward exchange rate of:

$$\frac{0.9182}{1.0101} = \text{USD/CHF } 0.909$$