LOS 4a: calculate expected values, variances, and standard deviations and demonstrate their application to investment problems

Expected Value

Expected value is an essential quantitative concept investors use to estimate investment returns and analyze any factor that may impact their financial position.

Mathematically, the expected value is the probability-weighted average of the possible outcomes of the random variable. For a random variable X, the expected value of X is denoted E(X). More specifically,

$$E(X) = P(X_1)X_1 + P(X_2)X_2 + \cdots + P(X_n)X_n$$
$$= \sum_{i=1}^{n} P(X_i)X_i$$

Where,

X_i = One of n possible outcomes of the discrete random variable X.

$P(X_i) = P(X_i = x_i)$ = Probability of X taking the value x.

Note that the expected can be a forecast (looking into the future) or the true value of the population mean.

The sample mean differs from the expected value. The sample mean is a central value for a specific set of observations, calculated as an equally weighted average of those observations.

Example: Calculating Expected Value

An analyst anticipates the following returns from an asset:

Return	Probability
5%	65%
7%	25%
8%	10%

The expected value of the investment is *closest to*:

Solution

Recall that,

$$
\begin{aligned}
E(X) &= \sum_{i=1}^{n} P(X_i)X_i \\
&= 0.05 \times 0.65 + 0.07 \times 0.25 + 0.10 \times 0.08 \\
&= 0.0325 + 0.0175 + 0.008 \\
&= 0.058 = 5.8\%
\end{aligned}
$$

Variance and Standard Deviation

Consider expected value as a forecast of the outcome of an investment. Then, variance and standard deviation measure the risk of an investment. That is the dispersion of outcomes around the mean.

The variance of a random variable is the expected value (the probability-weighted average) of squared deviations from the random variable's expected value. Denoted by $\sigma^2(X)$ or $Var(X)$, its formula is given by:

$$
\begin{aligned}
Var(X) &= E[X - E(X)]^2 \\
&= P(X_1)[X_1 - E(X)]^2 + P(X_2)[X_2 - E(X)]^2 + \cdots \\
&\quad + P(X_n)[X_n - E(X)]^2 \\
\Rightarrow Var(X) &= \sum_{i=1}^{n} P(X_i)[X_i - E(X)]^2
\end{aligned}
$$

Since variance is in squared terms, it can take any number greater than or equal to $0(Var(X) \geq 0)$. Intuitively, if $Var(X) = 0$, there is no risk (dispersion). On the other hand, if $Var(X) > 0$, it signifies the dispersion of outcomes.

Moreover, $Var(X)$ is a quantity given in square units of X. That is, if the X is given in percentage, then $Var(X)$ is given in squared percentage.

The standard deviation is the square root of variance:

$$\sigma(X) = \sqrt{\sigma^2(X)} = \sqrt{Var(X)}$$

The standard deviation is given in the same units as the random variance; hence it is easy to interpret.

Question

An analyst anticipates the following returns from an asset:

Return	Probability
5%	65%
7%	25%
8%	10%

The variance and standard deviation of the investment are *closest to*:

Solution

We know that,

$$
\begin{aligned}
Var(X) = \sum_{i=1}^{n} P(X_i)[X_i - E(X)]^2 \\
= 0.65(0.05 - 0.058)^2 + 0.25(0.07 - 0.058)^2 \\
+ 0.10(0.08 - 0.058)^2 \\
= 0.000126
\end{aligned}
$$

For standard deviation,

$$
\sigma(X) = \sqrt{Var(X)} = \sqrt{0.000126} = 0.0112
$$

LOS 4b: Formulate an investment problem as a probability tree and explain the use of conditional expectations in investment application

A tree diagram is a visual representation of all possible future outcomes and the associated probabilities of a random variable. Tree diagrams are handy when we have several possible outcomes.

They facilitate the recording of all the possibilities in a clear, uncomplicated manner. Each branch in a tree diagram represents an outcome.
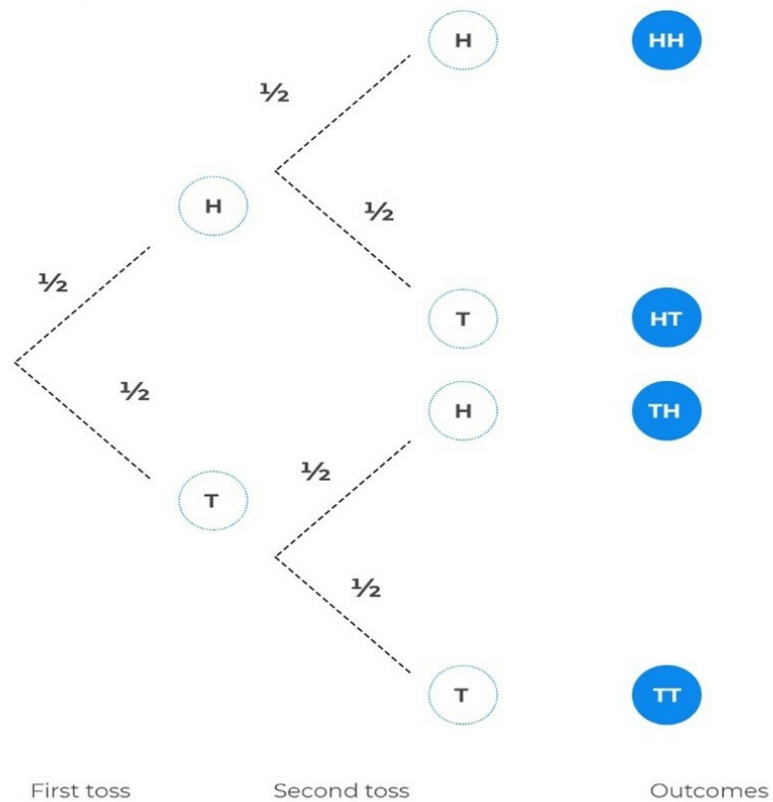
Example: Probability Tree

Let's consider a scenario where we toss a fair coin twice. The outcomes of these tosses are independent, meaning the first toss doesn't influence the second one.

For the first toss, we have two possibilities: It can result in either a head or a tail. Similarly, we still have the same two possibilities for the second toss: head or tail. Importantly, the outcome of the second toss is not affected by what happened in the first toss because coins don't have memory.

We can visualize these probabilities using a tree diagram like this:

Tree Diagram

First toss   Second toss   Outcomes

Please, note the following:

- The tree diagram must include all the possible outcomes.

- The sum of the probabilities must add up to 1.

- The number of branches is the number of different possibilities.

- The numbers on the branches present probabilities.

To calculate probabilities, we follow the tree branches from left to right and multiply any probabilities we encounter.

So, to find the probability of getting two heads (HH), we multiply the probabilities along the path.

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$$P(HH) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$

If we sum up the probabilities of all possibilities, we get 1.

Conditional Expectations

In investments, conditional expectation means predicting the expected value of an investment based on specific real-world events. Analysts consider the probability and impact of future events when calculating this value.

Competitors, governments, and other financial institutions keep releasing new information. Such pieces of information may have a positive or a negative impact on investment. This means that a project's expected value must be based on real-world dynamics.

In statistics, the conditional expected value is the expected value of a random variable X given an event or scenario S, denoted by $E(X \mid S)$.

Now, assume that X can take on any of n different outcomes $X_1, X_2, \ldots, X_n$, which are outcomes from a set of mutually exclusive and exhaustive events. Then,

$$E(X \mid S) = X_1 \cdot P(X_1 \mid S) + X_2 \cdot P(X_2 \mid S) + \cdots + X_n \cdot P(X_n \mid S)$$
$$= \sum_{i=1}^{n} X_i \cdot P(X_i \mid S)$$

Stating Unconditional Expected Value in Terms of Conditional Expected Value

To state the unconditional expected values in terms of conditional expected value, we use the total probability rule for expected value, which is built from the following formula:

$$E(X) = P(S) \cdot E(X \mid S) + P(S^C) \cdot E(X \mid S^C)$$

Where S^C is the complement (event or scenario "S" does not occur) of S.

Now assume that S can take on any of $S_1, S_2, \ldots, S_n$ mutually exclusive and exhaustive scenarios or events, then.

$$
\begin{aligned}
E(X) &= P(S_1) \cdot E(X \mid S_1) + P(S_2) \cdot E(X \mid S_2) + \cdots + P(S_n) \cdot E(X \mid S_n) \\
&= \sum_{i=1}^{n} P(S_i) \cdot E(X \mid S_i)
\end{aligned}
$$

Example: Conditional Expectation

The probability of relaxed trade restrictions in a given country is 40%. Therefore, shareholders of XYZ Company Limited expect a 5% share return if trade restrictions are maintained and a loss of 8% if they are relaxed. The expected change in return is *closest to*:

Solution

We must take every possibility into account. We have a 40% chance of relaxed trade restrictions in this case. Intuitively, this means there is a 60% chance that the current restrictions will be maintained. Therefore:

$$
\begin{aligned}
E(X) &= \sum_{i=1}^{n} P(S_i) \cdot E(X \mid S_i) \\
&= 0.6(0.05) \times 0.4(-0.08) \\
&= -0.002
\end{aligned}
$$

Example: Total Probability Rule for Expected Value

BlueChip Inc.'s profits are sensitive to economic growth, benefitting significantly during periods of high economic growth. Suppose there is a 0.70 probability that BlueChip Inc. will operate in a high-growth economic environment in the next fiscal year and a 0.30 probability that it will operate in a moderate-growth environment. Assume that the chance of a recession is negligible.
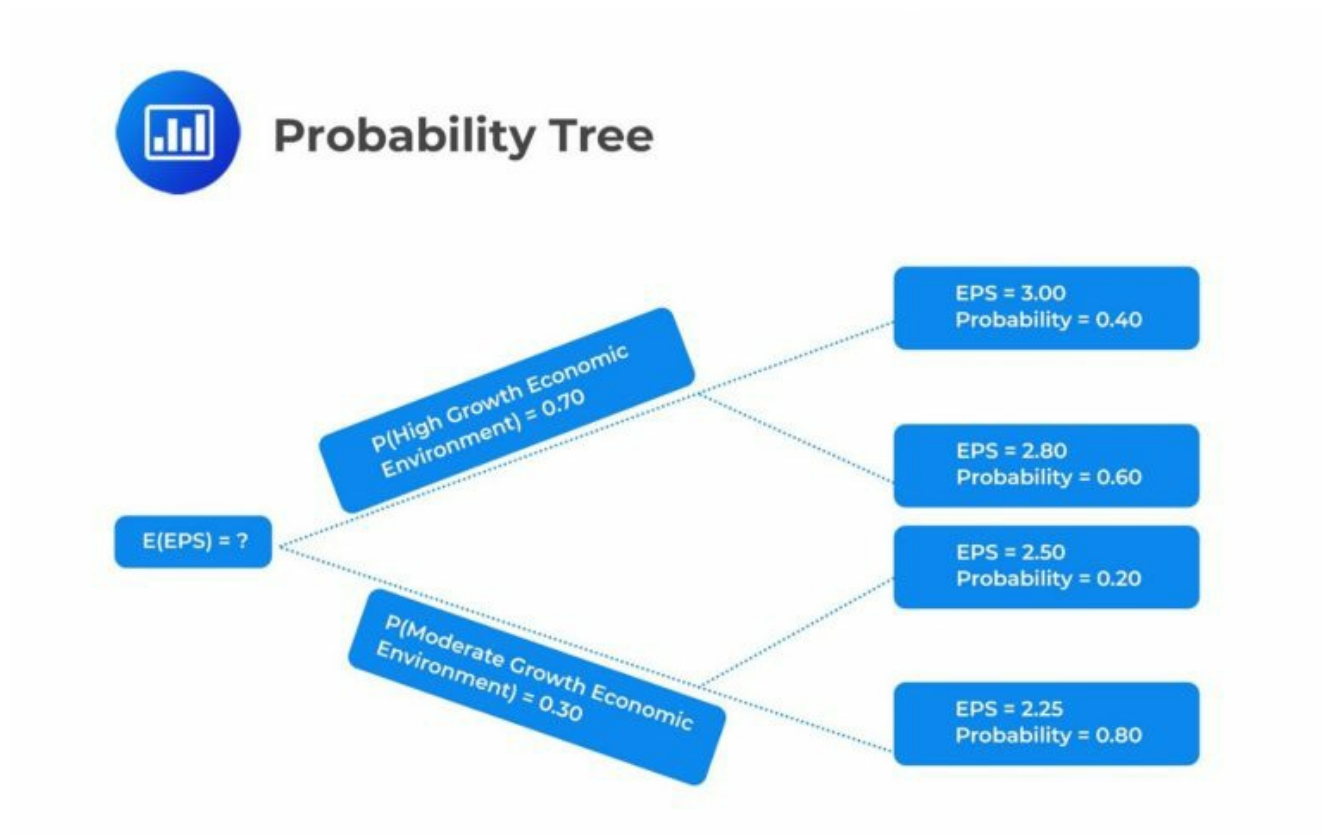
If a high-growth economic environment occurs, the probability that EPS will be USD 3.00 is estimated at 0.40, and the probability that EPS will be USD 2.80 is estimated at 0.60.

On the other hand, if the company operates in a moderate-growth environment, the probabilities that the EPS will be USD 2.50 and USD 2.25 are 20% and 80%, respectively.

Calculate the expected value of EPS for BlueChip Inc. in the next fiscal year.

Solution

We start by drawing a probability tree:

We first need to calculate the conditional expectations of EPS for each scenario: High-growth environment and moderate-growth environment. That is,

$$
\begin{aligned}
\text{E(EPS | High-growth environment)} &= 0.40 \times 3.00 + 0.60 \times 2.80 \\
&= \text{USD } 2.88 \\
\text{E(EPS | Moderate-growth environment)} &= 0.20 \times 2.50 + 0.80 \times 2.25 \\
&= \text{USD } 2.30
\end{aligned}
$$

Using the total probability for the expected value, we have:

$$
\begin{aligned}
E(EPS) =\ & P(\text{High-growth environment}) \\
& \cdot\ E(EPS \mid \text{High-growth environment}) \\
& +\ P(\text{High-growth environment}) \\
& \cdot\ E(EPS?\text{Moderate-growth environment}) \\
& =\ 0.70 \times 2.88 + 0.30 \times 2.30 \\
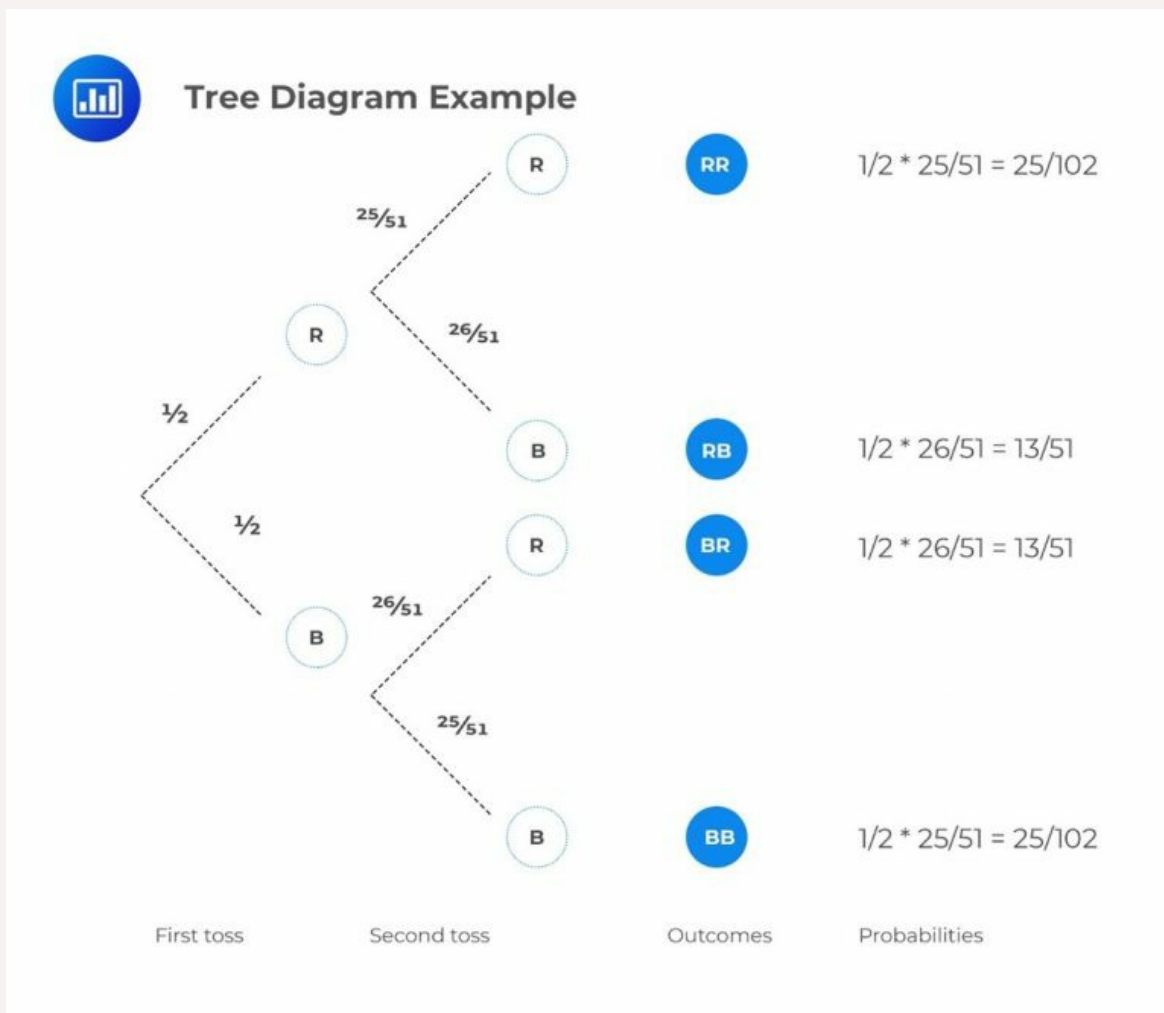& =\ \text{USD } 2.71
\end{aligned}
$$

Question 1

Someone picks a card from an ordinary pack of 52 playing cards **without replacement**. He then picks another one. Draw a probability tree and use it to calculate the probability of picking two red cards.

 A. $\frac{25}{102}$.

 B. $\frac{13}{51}$.

 C. $\frac{26}{51}$.

Solution

The correct answer is A.

Tree Diagram Example

		Outcomes	Probabilities
R → 25/51 → R		RR	1/2 * 25/51 = 25/102
R → 26/51 → B		RB	1/2 * 26/51 = 13/51
B → 26/51 → R		BR	1/2 * 26/51 = 13/51
B → 25/51 → B		BB	1/2 * 25/51 = 25/102

First toss — Second toss — Outcomes — Probabilities

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Question 2

There is a 20% chance that the government will impose a tariff on imported cars. A company that assembles cars locally expects returns of 14% if the tariff is imposed and returns of 11% otherwise. The (unconditional) expected return is *closest to*:

    A. 11.6%.

    B. 12.8%.

    C. 12.5%.

Solution

The correct answer is A.

The unconditional expected return will be the sum of:

1. The expected return **given** no tariff times the probability that a tariff will not be imposed.
2. The expected return **given** tariff times the probability that the tariff will be imposed. Therefore,

$$
\begin{aligned}
E(X) &= \sum_{i=1}^{n} P(S_i) + E(X \mid S_i) \\
&= 0.11(0.8) + 0.14(0.2) \\
&= 0.116 = 11.6\%
\end{aligned}
$$

LOS 4c: calculate and interpret an updated probability in an investment setting using Bayes' formula

Investors make investment decisions based on their experience and expertise. Their decisions may change in the wake of new knowledge and observations.

Bayes' formula allows us to update our decisions as we receive new information. In other words, Bayes' formula is used to calculate an updated or posterior probability given a set of prior probabilities for a given event.

Given a set of prior probabilities for an event, if we receive new information, the updated probability is as follows:

$$\text{Updated probability of an event given the new information} = \frac{\text{Probability of the new information gi}}{\text{Unconditional probability of the new}} \times \text{Prior probability of event.}$$

The above equation can be written as:

$$P(\text{Event} \mid \text{Information}) = \frac{P(\text{Information} \mid \text{Event})}{P(\text{Information})} \cdot P(\text{Event})$$

Deriving Bayes' Formula

Let $B_1, B_2, B_3, \ldots, B_n$ be a set of mutually exclusive and exhaustive events.

Using the conditional probability:

$$P(B_i \mid A) = \frac{P(B_i \cap A)}{P(A)} \ldots \ldots (1)$$

And also, the relationship:

$$P(B_i \cap A) = P(A \cap B_i) = P(B_i) \cdot P(A \mid B_i) \ldots \ldots (2)$$

Also, using the total probability rule:

$$P(A) = \sum_{i=1}^{n} P(A \cap B_i) = \sum_{i=1}^{n} P(B_i) \cdot P(A \mid B_i) \ldots \ldots (3)$$

Substituting equations (2) and (3) in (1), we have:

$$P(B_i \mid A) = \frac{(P(A \mid B_i)}{\sum_{i=1}^{n} P(B_i) \cdot P(A \mid B_i)} \cdot P(B_i)$$

This is the Bayes' formula, and it allows us to 'turnaround' conditional probabilities, i.e., we can calculate $P(B_i \mid A)$ if given information only about $P(A \mid B_i)$.

Note that:

1. $P(B_i)$ are known as **prior probabilities**.
2. Event A is some event known **to have occurred**.
3. $P(B_i \mid A)$ is the **posterior probability**.

Example: Bayes' Formula

An Investment Analyst wishes to investigate the performance of stocks by considering a number of stocks listed on different exchanges. In the sample, 50% of stocks were listed on the New York Stock Exchange (NYSE), 30% on the London Stock Exchange (LSE), and 20% on the Tokyo Stock Exchange (TSE).

The probability of a stock posting a negative return on the NYSE, LSE, and TSE is 40%, 35%, and 25%, respectively.

If the Analyst picks a stock at random from this group, what is the probability that it has a negative return on the NYSE?

Solution

We are looking for P(NYSE | Negative Return).

Let's define the following events:

NYSE is the event "A stock chosen at random is listed on the NYSE."

LSE is the event "A stock chosen at random is listed on the LSE."

TSE is the event "A stock chosen at random is listed on the TSE."

Finally, let NR be the event "A randomly chosen stock posts a negative return."

Therefore,

$$
\begin{aligned}
P(NYSE|NR) &= \frac{P(NYSE)\,P(NR|NYSE)}{P(NYSE)\,P(NR|NYSE) + P(LSE)\,P(NR|LSE) + P(TSE)\,P(NR|TSE)} \\
&= \frac{0.5 \times 0.4}{0.5 \times 0.4 + 0.3 \times 0.35 + 0.2 \times 0.25} \\
&= \frac{0.2}{0.355} \\
&= 0.5634 \approx 56.3\%
\end{aligned}
$$

Question

You have developed a set of criteria for assessing potential investments in growth-stage companies. Companies not meeting these criteria are predicted to be insolvent within 24 months. You gathered the following information when validating your criteria:

- Fifty percent of the companies that have been assessed will become insolvent within 24 months: $P(\text{insolvency}) = 0.50$.

- Sixty-five percent of the companies assessed meet the criteria: $P(\text{meet criteria}) = 0.65$.

- The probability that a company will meet the criteria given that it remains solvent for 24 months is 0.80: $P(\text{meet criteria} \mid \text{solvency}) = 0.80$.

The probability that a company will remain solvent, given that it meets the criteria, that is, $P(\text{solvency} \mid \text{meet criteria})$, is *closest to*:

A. 20%.
B. 50%.
C. 62%.

Solution

Using Bayes' formula, we have:

$$
\begin{aligned}
&P(\text{solvency} \mid \text{meet criteria}) \\
&= \frac{P(\text{meet criteria} \mid \text{solvency})P(\text{solvency})}{[P(\text{meet criteria} \mid \text{solvency})P(\text{solvency}) \\ +P(\text{meet criteria} \mid \text{insolvency})P(\text{insolvency})]} \\
&= \frac{0.80 \times 0.50}{0.80 \times 0.50 + P(\text{meet criteria} \mid \text{insolvency}) \times 0.50}
\end{aligned}
$$

Clearly, we need to calculate the $P(\text{meet criteria} \mid \text{insolvency})$. Using the total probability:

$$P(\text{meet criteria}) = P(\text{meet criteria} \mid \text{solvency})P(\text{solvency})$$
$$+ P(\text{meet criteria} \mid \text{insolvency})P(\text{insolvency})$$
$$\Rightarrow 0.65 = 0.80 \times 0.50 + P(\text{meet criteria} \mid \text{insolvency}) \times 0.50$$

$$\therefore P(\text{meet criteria} \mid \text{insolvency}) = \frac{0.65 - 0.80 \times 0.50}{0.50} = 0.50$$

As such,

$$P(\text{solvency} \mid \text{meet criteria}) = \frac{0.80 \times 0.50}{0.80 \times 0.50 + 0.50 \times 0.50} = 0.6153 \approx 62\%$$