Learning Module 5: Pricing and Valuation of Forward Contracts and For An Underlying with Varying Maturities

LOS 5a: Explain how the value and price of a forward contract are determined at initiation, during the life of the contract, and at expiration

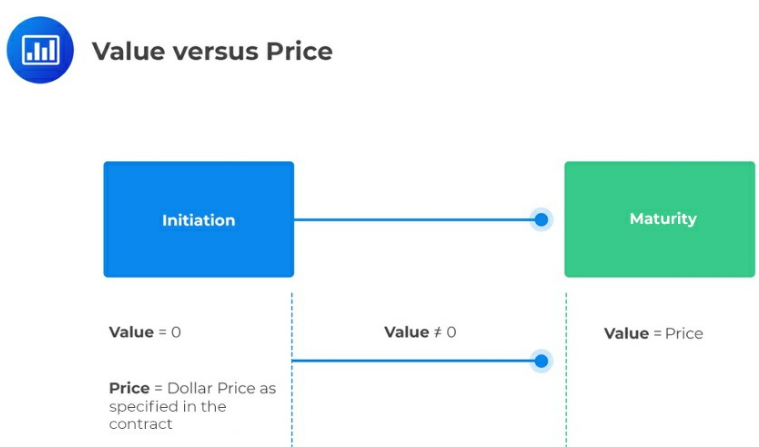
The price of a forward commitment is agreed upon at the contract's initiation and remains fixed until the contract's maturity. Moreover, the price is also used in determining the basis on which the underlying will be traded in the future against the spot price at maturity.

The value of a forward commitment changes over the life of the forward contract as the spot price and other factors change. For instance, consider a forward contract on an underlying that does not generate cash flows.

- **At the inception of the forward contract:** The value is zero since there is no down payment; that is, investors are willing to sign off on the contract for $0.00.

- **During a forward contract's life:** The value of the forward contract is calculated as spot price minus the present value of the forward price discounted at the risk-free rate.

- **At maturity:** The value of the forward contract is calculated as the spot price at maturity minus the forward price.



Forward Contract Pricing and Valuation

Define the following:

S_0 = Spot price of the underlying at the initiation.

S_t = Spot price of the underlying at time *t*, during the contract's life.

r = Risk-free rate of return.

$F_0(T)$ = Forward price (satisfies the no-arbitrage conditions).

$V_0(T)$ = Value of the forward contract at the initiation.

$V_t(T)$ = Value of the forward contract during the life of the contract.

$V_T(T)$ = Value of the forward contract at expiration.

Price and Value of Forward Contract at Initiation

Assume that there are no costs or benefits associated with the underlying. At the initiation of the forward contract, no money is exchanged, and the forward contract's value is zero:

$$V_0(T) = 0$$

The forward price that parties have agreed upon at initiation satisfies no arbitrage opportunities. As such, the forward price at initiation is the spot price of the underlying compounded at the risk-free rate over the contract's life:

$$F_0(T) = S_0(1 + R)^T$$

Price and Value of Forward Contract During the Life of the Contract

After its initiation, the value of the forward contract changes due to the change in the underlying's spot price, among other factors.

As such, the mark-to-market value of the forward contract at any time, $V_t(T)$, captures the spot price at time t and the present value of the forward price. More specifically, the value of a forward contract during the contract's life is the spot price at time t of the underlying asset minus the present value of the forward price (long position).

And the short forward position value is calculated as follows:

$$V_T(T) = S_t - F_0(T)(1+r)^{(-T-t)} - S_t$$

Remember that this is a zero-sum game. The value of the contract to the short position is the negative value of the long position, and thus the sum of both is always zero.

Price and Value of Forward Contract at Expiration

Forward contract settlement depends on the difference between the forward price, $F_0(T)$, and the spot price of the underlying at expiration, S_T. Therefore, the value of the forward contract for the long position will be:

$$V_T(T) = S_T - F_0(T)$$

And the value of the forward contract for the short position will be:

$$-V_T(T) = F_0(T) - S_T$$

Assuming there are no costs or benefits associated with the underlying, study the following table.

Outcome	Value of the Long Position	Value of the Short Position
$S_t > F_0(T)(1+r)^{-(T-t)}$	Mark-to-market gain	Mark-to-market loss
$S_t < F_0(T)(1+r)^{-(T-t)}$	Mark-to-market loss	Mark-to-market gain
$S_t = F_0(T)(1+r)^{-(T-t)}$	No mark-to-market gain/loss	No mark-to-market gain/loss

Example: Pricing vs. Valuation of Forward Commitments

Ali Muhamud currently owns 6,000 shares at Unilever Limited, whose spot price is $134 per share. Muhamud agrees to enter a forward contract to sell 2,000 shares to a financial intermediary at $178 per share in nine months. The spot price at maturity is $197.

The contract’s value at maturity from the financial intermediary’s perspective is *closest* to:

Solution

Muhamud is the seller (shot position), so the financial intermediary has a long forward position. Since we're at maturity, there is no need to take any present value. The value of the contract for the financial intermediary will simply be:

$$\begin{aligned} V_T(T) &= S_T - F_0(T) \\ &= \$197 - \$178 \\ &= \$19 \end{aligned}$$

Price and Value of a Forward Contract with Cost of Carry

At Initiation

For a forward contract whose underlying is associated with the cost of carry, the forward price is adjusted as follows:

$$F_0(T) = [S_0 - PV_0(I) + PV_0(C)](1 + r)^T$$

Where:

$PV_0(I)$ = Present value of income or benefits at time $t = 0$.

$PV_0(C)$ = Present value of costs at time $t = 0$.

The forward price above satisfies the no-arbitrage condition at time $t = 0$. Moreover, the forward contract, whose underlying has associated benefits and costs, is neither a liability nor an asset to the buyer or seller. As such, the value of the contract at initiation is zero:

$$V_0(T) = 0$$

During the Contract's Life

During the forward contract’s life with the cost of carry, the mark-to-market (MTM) value is determined as the difference between the current spot price adjusted for the cost of carry and the present value of the forward price.

This is expressed as:

$$V_t(t) = (S_t - PV_t(I) + PV_t(C)) - F_0(T)(1 + r)^{-(T-T)}$$

Where:

$PV_t(I)$ = Present value of income or benefits at any time t.

$PV_t(C)$ = Present value of costs at any time t.

At Maturity

Since the forward price includes the cost of carry, the value of the underlying asset at maturity is calculated as the difference between the spot price at maturity, S_T, and the forward price agreed upon at inception, $F_0(T)$.

$$V_0(T) = S_T - F_0(T)$$

Example: Price and Value of a Forward Contract with Cost of Carry

Nico Haas has entered a forward contract with a seller of an asset whose spot price is \$62, and the net cost of carry is \$5. The risk-free rate is 5% and the life of the contract is nine months. Suppose the spot price maturity remained at \$62, the contract’s value at maturity is closest to:

Solution

First, let’s find the forward price:

$$\begin{aligned} F_0(T) &= S_0(1-r)^T - (PV_t(I) + PV_t(C))(1-r)^T \\ &= (S_0 - \text{Cost of carry})(1+r)^T \\ &= (57)(1+0.05)^{\frac{9}{12}} \\ &= \$59.12 \end{aligned}$$

Therefore, the value of the contract is:

$$\begin{aligned} V_0(T) &= S_T - F_0(T) \\ &= \$62 - \$59.12 \\ &= \$2.87 \end{aligned}$$

Foreign Exchange Forward: Continuous Compounding

The spot price, ($S_{0,f/d}$), and the forward price, ($F_{0,f/d}(T)$), are expressed in terms of price currency (foreign, *f*) per single unit of base currency (domestic currency, *d*).

The difference between the spot price and forward price depicts the difference between the risk-free foreign rate (r_f) and the risk-free domestic rate (r_d) as expressed below:

$$F_{0,f/d}(T) = S_{0,f/d}e^{(r_f - r_d)T}$$

At initiation, the currency with higher risk-free rates is said to trade at a **forward discount**, while a currency with a low risk-free rate for the forward period trades a **forward premium**.

At any time, t before maturity, the mark-to-market (MTM) value of the forex exchange forward is the spot price, $S_{t,f/d}$, minus the PV of the forward price, $F_{0,f/d}(T)$, discounted by the difference in the risk-free rates $r_f - r_d$ over the remainder of the contract using continuous compounding, $e^{-(r_f - r_d)(T-t)}$:

$$V_t(T) = S_{t,f/d} - F_{0,f/d}(T)e^{-(r_f - r_d)(T-t)}$$

The fluctuation between the foreign and domestic risk-free rates indicates changes in the available opportunity costs between the two currencies.

Question

Suppose Ceriotti Cosmas has 20,000 shares of VIVO, and he agrees to enter a 3-month forward contract with Lumis to sell his shares at CAD 239 per share. The risk-free rate is 3.5%, and a spot rate at $t = 0$ is CAD 225 per share. If VIVO's spot price falls to CAD 215 per share in one month, the forward contract's mark-to-market (MTM) value in VIVO's perspective in one-month is *closest to:*

A. 21.27

B. 22.63

C. 32.36

Solution

The correct answer is **B**

From VIVO's perspective:

$$V_T(T) = F_0(T)(1 + r)^{-(T-t)} - S_t$$

Where, $F_0(T) = \text{CAD}239$, $r = 0.035$, $T = 0.25, 0.0833$, $S_t = \text{CAD}215$

$$\begin{aligned} V_t(T) &= \text{CAD } 239(1.035)^{-(0.1670)} - 215 \\ &= \text{CAD } 22.63 \text{ MTM gain} \end{aligned}$$

LOS 5b: Explain how forward rates are determined for an underlying with a term structure and describe their uses

The link between the spot and forward prices is determined by a risk-free interest rate which is regarded as the opportunity cost of holding an asset. **Term structure** implies that various interest rates are available depending on the time to maturity. Given an upward-sloping yield curve, which is the most usual case, investors would charge a higher yearly interest rate for five-year loans than for a one-year loan.

Spot Rates

A spot rate, also known as zero rates, is defined as the yield to maturity of a zero-coupon security maturing at the date of each cash flow.

Example: Spot Rate

A 4-year $2,000 par value bond pays 10% annual coupons. The spot rate for year 1 is 8%, the 2-year spot rate is 13%, the 3-year spot rate is 14%, and the 4-year spot rate is 16%. The price of the bond is *closest* to:

Solution

Here, we have an upward-sloping yield curve since the 2-, 3-, and 4-year interest rates are higher than the 1-year interest rate.

$$\begin{aligned} \text{Annual coupon} &= \$2000 \times 10\% = \$200 \\ \text{Price} &= \left(\frac{\$200}{(1.08)^1}\right) + \left(\frac{\$200}{(1.13)^2}\right) + \left(\frac{\$200}{(1.14)^3}\right) + \left(\frac{\$2200}{(1.16)^4}\right) \\ &= \$1691.85 \end{aligned}$$

Discount Factor

The discount factor is the price equivalent of a zero/spot rate, which, when multiplied by the total amount of money to be received (principal + interest), gives the bond's price (present value).

$$\text{Discount factor }(\text{DF}_\text{i}) = \frac{1}{(1 + \text{Discount rate}(\text{Z}_\text{i}))^{\text{Period number(i)}}}$$

Example: Discount Factor

Suppose that an investor sells a four-year zero-coupon bond at par with a price of 94.78; the four-year zero rates is *closest* to:

Solution

$$\text{DF}_\text{i} = \frac{1}{(1 + \text{Z}_\text{i})^\text{i}}$$

Where:

DF_i = The discount factor for a given period.

Z_i = The zero rate for a given period.

i = The period.

To solve for the four-year zero rate z_4 solve the equation:

$$94.78 = \frac{100}{(1 + z_4)^4}$$

We can see that:

$$0.9478 = \frac{1}{(1 + z_4)^4}$$
$$Z_4 = 0.013493 \approx 1.3493\%$$

Forward Rates vs. Spot Rates

A forward rate indicates the interest rate on a loan beginning at some time in the future. A spot rate, on the other hand, is the interest rate on a loan beginning immediately.

In an interest rate forward contract, the most common market practice is to name forward rates. For instance, “3y7y” is also denoted as $F_{3,7}$, which means “3-year into 7-year rate”. The first number refers to the length of the forward period from today, while the second one refers to the tenor or time-to-maturity of the underlying bond.

For short-term market reference rates (MRRs), forward rates are normally named in months. For example, (F_{2m,5m}\) means the 5-month forward period starting at the end of 2 months, and the time-to-maturity is seven months from today.

Implied Forward Rate (IFR)

The implied forward rate is the breakeven rate that links a short-date and long-dated zero-coupon bond. It is an interest rate at a future period where an investor breaks even and can earn the same return as today.

The general formula for the relationship between the two spot rates and the implied forward rate (IFR) is:

$$(1 + Z_B)^B = (1 + Z_A)^A \times (1 + IFR_{A,B-A})^{B-A}$$

Where:

Z_A = Yield-to-maturity per period of short-term bond.

Z_B = Yield-to-maturity per period of long-term bond.

$IFR_{A,B-A}$ = Implied forward rate between period A and period B, with a tenor of B-A.

The implied forward rate (IFR) is the interest rate for a period in the future where an investor can earn a return:

- By investing now until the forward rate reaches its final maturity.
- By investing now until the forward rate’s start date. Besides, the investor increases returns at the implied forward rate by rolling over the proceeds.

Example: Implied Forward Rate

An investor notes that the two-year and the three-year zero-coupon bonds yield 6% and 8%, respectively. The investor enters a one-year fixed-rate FRA agreement to hedge against rate fluctuations.

The implied forward rate applicable to the investor's position is *closest* to:

Solution

$$
\begin{aligned}
(1+Z_A)^A \times (1+IFR_{A,B-A})^{B-A} &= (1+Z_B)^B \\
(1.06)^2 \times (1+IFR_{2,1})^1 &= (1.08)^3 \\
1.1236 \times (1+IFR_{2,1}) &= 1.2597 \\
1.1236 + 1.1236IFR_{2,1} &= 1.2597 \\
1.1236IFR_{2,1} &= 0.136112 \\
IFR_{2,1} &= \frac{0.136112}{1.1236} \\
&= 0.1211 \approx 12.11\%
\end{aligned}
$$

Forward Rate Agreements (FRAs)

A forward rate agreement (FRA) is a cash-settled over-the-counter (OTC) contract between two counterparties. In this contract, the buyer (long position) is borrowing and the seller (short position) is lending a notional sum (underlying) at a fixed interest rate (the FRA rate) and for a specified period starting at an agreed date, e.g., period A in the future and ends in period B.

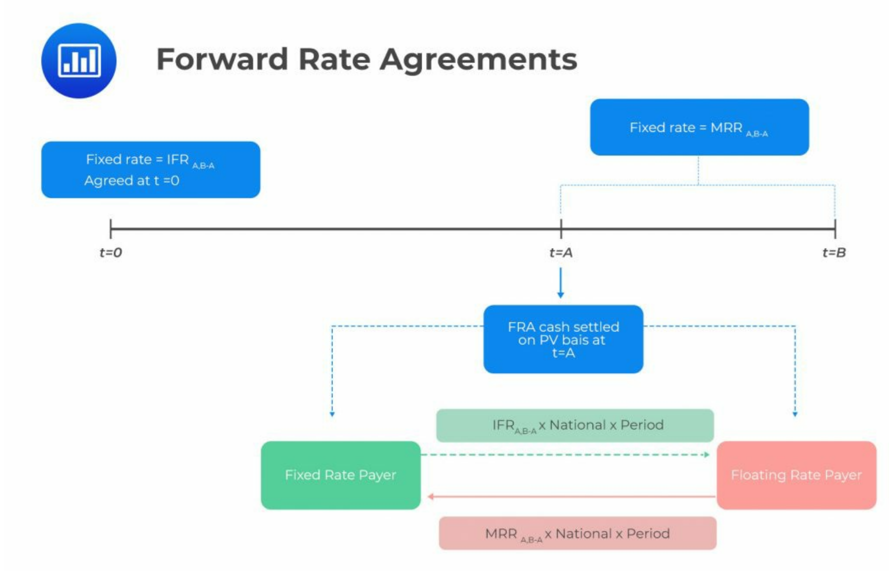
The seller deposits interest based on the market reference rate (MRR), where the MMR is established before the settlement dates, at time $t = A$.

The FRA settlement amount is a function of the difference between the forward interest rate ($IFR_{A,B-A}$) and the market reference rate (MRR_{B-A}) or the MMR for $B - A$ periods, which ends at period B.

FRAs are usually cash-settled at the beginning of the period during which the reference rate applies.

The net payment from the long position's perspective would be:

$$\text{Net payment} = (\text{MRR}_{\text{B}-\text{A}} - \text{IFR}_{\text{A,B}-\text{A}}) \times \text{Notional principal} \times \text{Period}$$



FRA can be seen as a one-period interest swap. They are mainly utilized by investors and issuers to manage interest rate risk. The notional amount is not exchanged but used for interest calculations. Also, fixed and floating payments occur on a net basis. Financial intermediaries use FRAs to protect rate-sensitive balance sheets, assets, or liabilities against interest rate risk.

Example: Forward Rate Agreements

James Malcolm enters an FRA agreement with Fred Green on a notional amount of USD 50,000. Malcolm will receive a fixed rate in 4 months, and a two-month MRR is set at 1.3%. If the $\text{IFR}_{3\text{m},1\text{m}}$ at the initiation of the contract is 2.5%, the settlement amount is *closest* to:

Solution

$$\begin{aligned}\text{Net payment at the end of the period} &= (\text{MRR}_{\text{B}-\text{A}} - \text{IFR}_{\text{A,B}-\text{A}}) \times \text{Notional principal} \times \text{Period} \\ &= (2.5\% - 1.3\%) \times \text{USD } 50{,}000 \times \frac{2}{12} \\ &= \text{USD } 100\end{aligned}$$

$$\begin{aligned}\text{Cash settlement (PV)} &= \text{USD} \frac{100}{1 + 0.013/12} \\ &= \text{USD } 99.89\end{aligned}$$

Question

Vanessa Raquela is an analyst at Money Wise Capital. She analyses two-year and three-year government zero-coupon bonds whose prices are 95 and 92 per 100 face value, respectively. The implied two-year forward rate in five years' time is *closest* to:

A. 2.60%.

B. 2.81%.

C. 3.26%.

Solution

$$DF_i = \frac{1}{(1+Z_i)^i}$$

Thus:

$$0.95 = \frac{1}{(1+z_2)^2} = 2.5978\%$$
$$0.92 = \frac{1}{(1+z_3)^3} = 2.81845\%$$

Now we need to solve for $IFR_{2,5}$ as follows:

$$(1+Z_A)^A \times (1+IFR_{A,B-A})^{B-A} = (1+Z_B)^B$$
$$(1+Z_2)^2 \times (1+IFR_{2,5}) = (1+z_3)^3$$
$$1.025978^2 \times (1+IFR_{2,5}) = 1.0281845^3$$

Where:

$$1.025978^2(1+IFR_{2,5}) = 1.0281845^3$$
$$1.025978^2(1)+(1.025978^2)(IFR_{2,5}) = 1.0281845^3$$
$$1.052631+1.052631IFR_{2,5} = 1.086959$$
$$1.052631IFR_{2,5} = 0.034328$$
$$IFR_{2,5} = 0.032612 \approx 3.2612\%$$