Learning Module 9: The Term Structure of Interest Rates: Spot, Par and Forward Curves

LOS 9a: define spot rates and the spot curve, and calculate the price of a bond using spot rates.

Spot Rates

Spot rates are the market discount rates for default-risk-free zero-coupon bonds. Unlike typical bonds that offer periodic interest payments, these bonds are sold at a discount and repaid at face value upon maturity. Sometimes referred to as "zero rates," using a sequence of spot rates ensures a bond price that prevents arbitrage opportunities. In finance, this no-arbitrage condition ensures consistent asset pricing across markets, eliminating the chance for investors to gain risk-free profit from price differentials.

Spot Curve

The spot curve visually charts the yield-to-maturity of default-risk-free zero-coupon bonds against their maturities. Often termed the "zero" or “strip” curve, the "strip" terminology originates from the stripping of periodic coupon payments, converting bonds to zero-coupon status. An example of this is the spot curve of Canadian Government bonds shown below:

Types of Spot Curves

- Upward sloping spot curve: this is observed when longer-term government bonds yield higher than shorter-term bonds. It is a typical pattern under normal market conditions.
- Downward sloping (inverted) yield curve: this rarer configuration, where shorter-term yields are higher than longer-term yields, can signal impending economic downturns. It suggests that investors anticipate lower future rates, often due to expected economic slowdowns, and are thus more inclined to accept lower yields for longer-term bonds.

The spot curve is pivotal for maturity structure analysis, especially with government bonds that standardize elements like currency, credit risk, liquidity, and tax status. Notably, the absence of coupon reinvestment risk in zero-coupon bonds simplifies their evaluation.

Calculating the Price of a Bond Using Spot Rates

To determine bond prices using the spot curve, each cash flow date corresponds to a specific discount rate. The goal is to achieve "no-arbitrage" prices. The bond's price is determined by discounting its cash flows with the corresponding spot rates. For bonds with periodic payments and a final principal repayment, the price is:

$$PV = \frac{PMT}{(1 + Z_1)^1} + \frac{PMT}{(1 + Z_2)^2} + \ldots + \frac{PMT + FV}{(1 + Z_N)^N}$$

Where:

- PV is the present value or price of the bond.
- PMT is the periodic payment or coupon.
- FV is the bond's face value.
- $Z_1, Z_2, \ldots Z_N$ are the spot rates for periods $1, 2, \ldots N$ respectively.

This approach ensures that the bond price remains consistent, whether discounted using spot rates or yield-to-maturity.

Example: Calculating the Price of a Bond Using Spot Rates

Given the term structure of government bonds:

Maturity	Yield-to-maturity
$1 - \text{Year}$	1.5000%
$2 - \text{Year}$	1.2500%
$3 - \text{Year}$	1.0000%
$4 - \text{Year}$	0.7500%
$5 - \text{Year}$	0.5000%

Calculate the price of a 1.00% coupon, four-year government bond.

Formula:

$$PV = \frac{PMT}{(1+Z_1)^1} + \frac{PMT}{(1+Z_2)^2} + \ldots + \frac{PMT+FV}{(1+Z_N)^N}$$

$$PMT = 1\% \times 100 = 1$$

$$PV = \frac{1}{(1+0.015)^1} + \frac{1}{(1+0.0125)^2} + \frac{1}{(1+0.01)^3} + \frac{1+100}{(1+0.0075)^4} = 100.957$$

Question

Which of the following best describes a spot rate?

A. The yield-to-maturity of a coupon-bearing bond.
B. The market discount rate is applied to default-risk-free zero-coupon bonds.
C. The annual interest rate of a bond with periodic payments.

Solution

The correct answer is **B.**

Spot rates are market discount rates applied to default-risk-free zero-coupon bonds.

A is incorrect: The yield-to-maturity usually applies to coupon-bearing bonds, not specifically zero-coupon bonds.

C is incorrect: Spot rates are particularly associated with zero-coupon bonds and not bonds with periodic payments.

LOS 9b: define par and forward rates, and calculate par rates, forward rates from spot rates, spot rates from forward rates, and the price of a bond using forward rates.

Par Rates

A par rate is the yield-to-maturity that equates the present value of a bond's cash flows to its par value (typically 100% of face value). Spot rates play a pivotal role in determining par rates. For a bond to be priced at par, its coupon rate and yield-to-maturity must be identical. This is depicted in the equation:

$$100 = \frac{PMT}{(1+z_1)^1} + \frac{PMT}{(1+z_2)^2} + \ldots + \frac{PMT+100}{(1+z_N)^N}$$

Here, PMT represents the periodic payment, and $z_1, z_2, \ldots, z_N$ are the sequence of spot rates for respective periods. By solving for PMT, we obtain the yield-to-maturity that would make the bond trade at par, which is the par rate.

Example: Calculating Par Rate Given Spot Rates

Given the following spot rates for government bonds. Note that these are effective annual rates:

Term	Spot Rate
$1-\text{Year}$	4.50%
$2-\text{Year}$	4.90%
$3-\text{Year}$	5.25%

One-Year Par Rate

Given the spot rate for one year is 4.50%, the one-year par rate will also be 4.50%.

$$100 = \frac{PMT+100}{(1.0450)^1}$$

Two-Year Par Rate:

Using the spot rates for one year and two years, we can derive the two-year par rate:

$$100 = \frac{PMT}{(1.0450)^1} + \frac{PMT + 100}{(1.0490)^2}$$

PMT has been calculated as 4.8904, which translates to a two-year par rate of 4.8904%.

Three-Year Par Rate

Using the spot rates for one, two, and three years:

$$100 = \frac{PMT}{(1.0450)^1} + \frac{PMT}{(1.0490)^2} + \frac{PMT + 100}{(1.0525)^3}$$

PMT has been calculated as 5.2252, which translates to a two-year par rate of 5.2252%.

Forward Rates

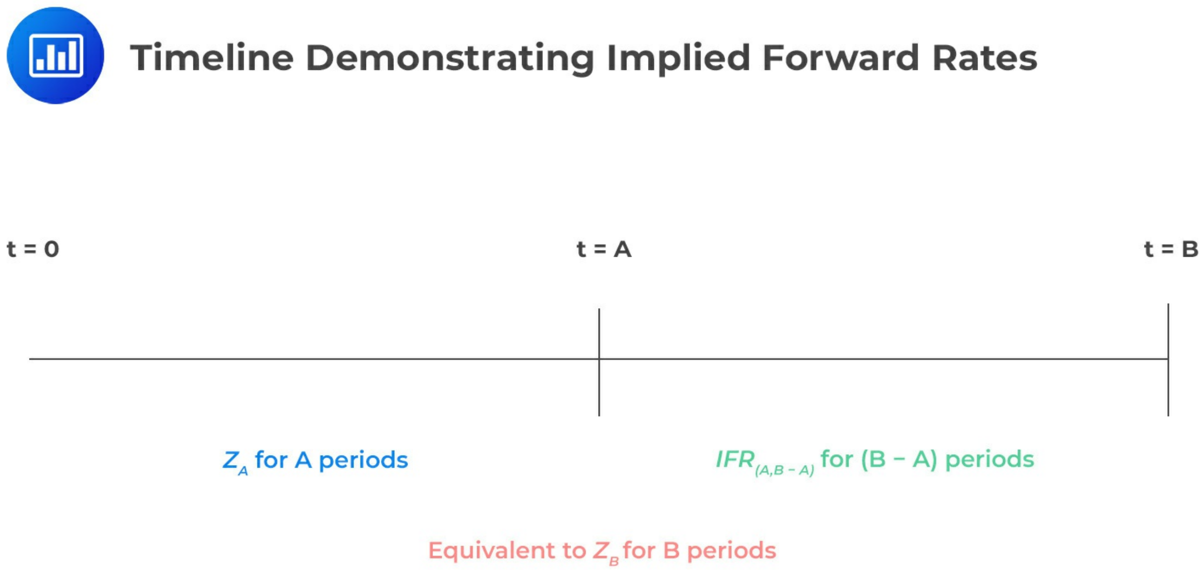
Forward rates, often termed implied forward rates or forward yields, act as breakeven reinvestment rates. They establish a connection between the return on an investment in a shorter-term zero-coupon bond to the return on a longer-term zero-coupon bond. The most common market practice is to name forward rates by, for instance, "2y5y", which means "2-year into 5-year rate". The first number refers to the length of the forward period from today, while the second number refers to the tenor or time-to-maturity of the underlying bond.

Forward rates can be derived from spot rates and vice versa. The general formula to compute an implied forward rate between two periods is:

$$(1 + Z_A)^A \times (1 + IFR_{(A,B-A)})^{(B-A)} = (1 + Z_B)^B$$

Here, $IFR_{(A,B-A)}$ denotes the implied forward rate for a bond beginning at time $t = A$ and maturing at t=B. Z_A and Z_B represent spot rates for periods A and B, respectively.

The following figure demonstrates the implied forward rates:



- The blue segment (from $t = 0$ to $t = A$) represents investing at the spot rate Z_A for A periods.
- The green segment (from $t = A$ to $t = B$) represents investing at the implied forward rate $IFR_{(A,B-A)}$ for $B - A$ periods.
- The red segment at the bottom (spanning the entire timeline) signifies that the compounded return of the above two rates is equivalent to investing at the spot rate Z_B for the entire B periods.

Forward rates are pivotal for investors and analysts as they provide insights into market expectations of future interest rate movements. They serve as breakeven rates, implying that if an investor's expectations align with the forward rate, they would be indifferent between investing in a longer-term bond now or investing in shorter-term bonds successively.

Example: Calculating Implied Forward Rates from Spot Rates

Suppose that the yields-to-maturity on five-year and seven-year zero-coupon bonds are 4.85% and 5.45%, respectively, stated on a quarterly bond basis. An analyst wants to know the "5y2y" implied forward rate, which is the implied two-year forward yield five years into the future.

Given:

- $A = 20$ (periods, since 5 years $\times 4$ quarters per year)
- $B = 28$ (periods, since 7 years $\times 4$ quarters per year)
- $B - A = 8$ (periods)
- $z_{20} = \frac{0.0485}{4}$ (per period)
- $z_{28} = \frac{0.0545}{4}$ (per period)

Let's solve for $\text{IFR}_{20,8}$ (the implied forward rate from period 20 to period 28).

Formula:

$$(1 + z_{20})^{20} \times (1 + \text{IFR}_{20,8})^{8} = (1 + z_{28})^{28}$$

$$\left(1 + \frac{0.0485}{4}\right)^{20} \times (1 + \text{IFR}_{20,8})^{8} = \left(1 + \frac{0.0545}{4}\right)^{28}$$

$$\left(1 + \frac{0.0485}{4}\right)^{20} \times (1 + \text{IFR}_{20,8})^{8} = \left(1 + \frac{0.0545}{4}\right)^{28}$$

$$\text{IFR}_{20,8} = 1.738\%$$

The "5y2y" implied forward rate is approximately 1.738% on a quarterly basis. Annualized, the "5y2y" implied forward yield is 6.952%.

Calculating Spot Rates from Forward Rates

Suppose the current forward curve for one-year rates is as follows:

Time Period	Forward Rate
0y1y	1.50%
1y1y	2.20%
2y1y	2.80%

The provided rates are expressed on an annual basis with a period of one year, making them effective yearly rates. The initial rate, termed "0y1y," represents the spot rate for one year. The subsequent rates are forward rates for one-year durations. Using these rates, the spot curve can be determined by taking the geometric average of the forward rates. The two-year implied spot rate can be calculated as:

$$(1.0150 \times 1.0220) = (1 + z_2)^2$$

where z_2 is the two-year implied spot rate.

$$z_2 = 1.84940\%$$

Using this, and the 2y1y forward rate, we can then determine the three-year implied spot rate:

$$(1.0150 \times 1.0220 \times 1.0280) = (1 + z_3)^3$$

where z_3 is the three-year implied spot rate.

$$z_3 = 2.16529\%$$

Suppose an analyst needs to value a three-year, 2.50% annual coupon payment bond that has the same risks as the bonds used to obtain the forward curve. Using the implied spot rates, we can determine the value of the bond.

$$PV = \frac{PMT}{(1+z_1)^1} + \frac{PMT}{(1+z_2)^2} + \ldots + \frac{PMT + 100}{(1+z_N)^N}$$

$$PV = \frac{2.5}{(1.0150)^1} + \frac{2.5}{(1.084940)^2} + \frac{2.5 + 100}{(1.0216529)^3}$$

$$PV = 100.993$$

This bond can also be valued using the forward rates and generate the same result.

Bond Pricing Using Forward Rates

Bonds can also be valued using forward rates. The bond's future cash flows are discounted at the product of the sequence of one-year forward rates leading up to each cash flow. The summation of these discounted cash flows gives the bond's price. The bond price remains consistent regardless of whether spot or forward rates are used.

Example: Bond Pricing Using Forward Rates

Suppose the current forward curve for one-year rates is as follows:

Time Period	Forward Rate
0y1y	1.50%
1y1y	2.20%
2y1y	2.80%

Suppose an analyst needs to value a three-year, 2.50% annual coupon payment bond that has the same risks as the bonds used to obtain the forward curve. Determine the value of the bond using the forward rates above.

$$PV = \frac{2.5}{(1.0150)^1} + \frac{2.5}{(1.0150 \times 1.0220)} + \frac{2.5 + 100}{(1.0150 \times 1.0220 \times 1.0280)}$$

$$PV = 100.993$$

Question

Given a three-year spot rate of 3.5% and a four-year spot rate of 4%, what is the one-year forward rate three years from now (3yly)?

A. 2.720%
B. 3.75%
C. 5.515%

Solution

The correct answer is **C.**

Using the formula:

$$(1+Z_A)^A \times (1+IFR_{A,B-A})^{B-A} = (1+Z_B)^B$$

Where $A = 3, B = 4, Z_3 = 3.5\%$, and $Z_4 = 4\%$

$$(1+0.035)^3 \times (1+IFR_{3,1})^1 = (1+0.04)^4$$

Solving for $IFR_{3,1}$ will give 5.515%

LOS 9c: compare the spot curve, par curve, and forward curve.

Yields-to-maturity for zero-coupon government bonds could be analyzed for a full range of maturities called the government bond spot curve (or zero curves). Government spot rates are assumed to be risk-free.

Spot Curve

The spot curve is upward-sloping and flattens for longer times-to-maturity. As a result, longer-term government bonds usually have higher yields than shorter-term bonds. The hypothetical spot curve is ideal for analyzing the maturity structure because it meets the "all other things being equal" assumption. The spot curve can also be inverted. This implies that one-year rates are expected to be lower in the future.

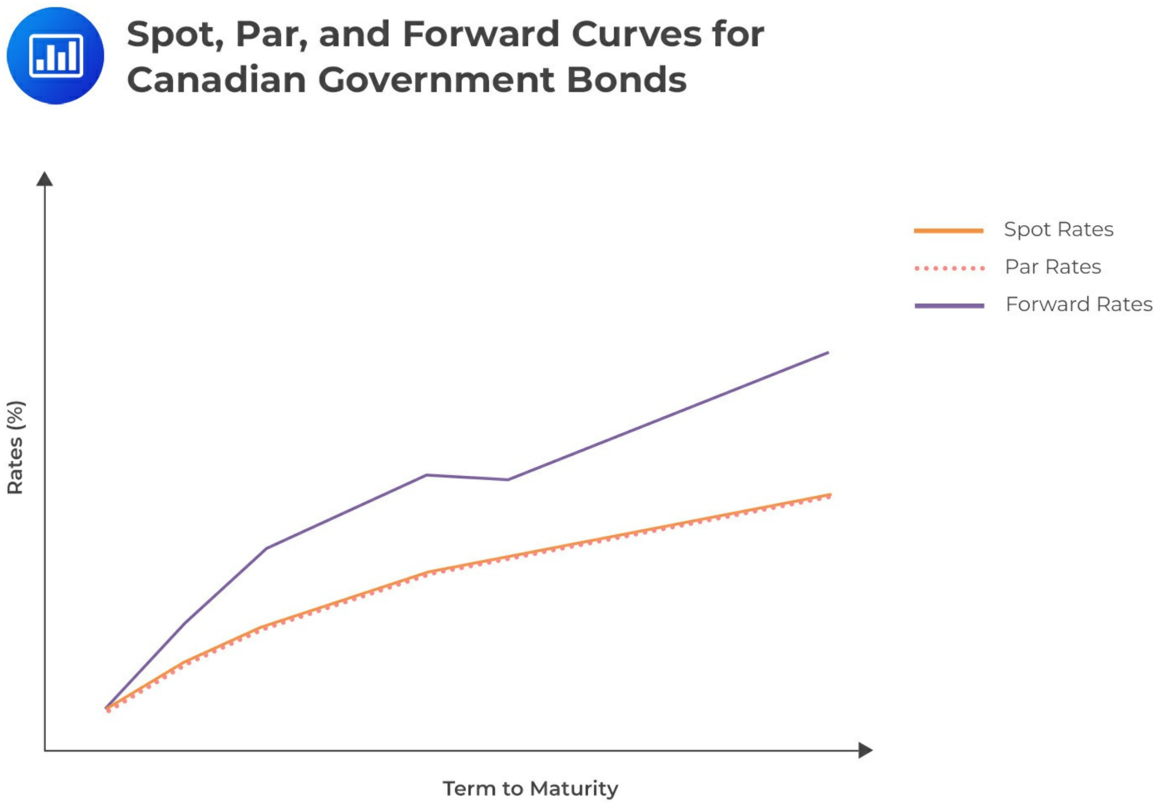
Par Curve

The par curve differs from the spot curve because it is a sequence of yields-to-maturity, and each bond is priced at a par value. The par curve is obtained from the spot curve. All bonds on the par curve are supposed to have the same credit risk, periodicity, currency, liquidity, tax status, and annual yields. Between coupon payment dates, the flat price (not full price) is equal to the par value.

Forward Curve

The forward curve is a series of forward rates, each of which has the same time frame.

The following diagram demonstrates a comparison of the Spot curve, Par Curve, and Forward Curve for the Canadian Government Bonds.



We can deduce the following from the figure above:

1. The spot rates exhibit a positive trend, leading to an upward-sloping spot curve.
2. The spot and par curves closely align; however, par rates are a bit lower than spot rates. This difference becomes more pronounced for longer maturities.
3. Forward rates are greater than both spot and par rates.

These insights are rooted in the inherent relationships between curve patterns. When the spot curve trends upwards, par rates tend to be close to, but slightly beneath, spot rates, especially towards the long end of the curve. This is attributed to the influence of lower short-term spot rates, which increase the bond prices, especially for bonds with longer-term maturities. This increase, in turn, results in reduced par rates when calculations are based on a price equivalent to 100% of the par value.

The following table summarizes the relationship between spot, par, and forward curves.

Spot Curve Shape	Par Curve	Forward Curve
Upward Sloping	Below spot curve	Above spot curve
Flat	Equal to spot curve	Equal to spot curve
Downward Sloping (Inverted)	Above spot curve	Below spot curve

Question

When the spot curve is upward-sloping, how do the par rates typically compare to the spot rates?

A. Par rates are above spot rates.
B. Par rates are equal to spot rates.
C. Par rates are below spot rates.

Solution

The correct answer is **C**.

When the spot curve is upward-sloping, par rates tend to be close to, but slightly beneath, spot rates.

A is incorrect: Par rates are not typically above spot rates when the spot curve is upward-sloping.

B is incorrect: Par rates are not typically equal to spot rates; they are slightly below.