Learning Module 4: Arbitrage, Replication, and the Cost of Carry in Pricing Derivatives

LOS 4a: explain how the concepts of arbitrage and replication are used in pricing derivatives

Arbitrage refers to buying an asset in a cheaper market and simultaneously selling it in a more expensive market to make a risk-free profit.

Traders endeavor to exploit arbitrage opportunities when there are short-lived market differences between assets in the same or different markets. An arbitrageur will buy assets in a market with low prices and sell in another market at a higher price to make a profit. Arbitrage opportunities disappear quickly.

When multiple traders buy low-priced assets and sell high-priced assets simultaneously, it increases the demand and price for the former while decreasing the demand and price for the latter. The markets will continue to work in that fashion until prices converge, thereby eliminating arbitrage.

Example: Arbitrage Opportunity

Company ABC's stock trades on the New York Stock Exchange for $10.15 and the equivalent of $10.25 on the London Stock Exchange. How does this set up a perfect, risk-free arbitrage opportunity?

Solution

The 'arbitrageur' can buy ABC's stock on the New York Stock Exchange for $10.15 and simultaneously sell the stock on the London Stock Exchange for $10.25, making a 'riskless' profit of $0.10 per share. This action by other market participants would force the two prices to converge to one price.

Arbitrage and the Law of One Price

The law of one price postulates that assets that produce identical results have only one true market price. In layman's language, it means *identical things should have the same prices*.

Intuitively, arbitrage opportunities exist if the law of one price does not hold.

Arbitrage Opportunities in Derivative Contracts

Remember that the value of derivative contracts is derived from future cash flows linked to the underlying assets. As such, arbitrage opportunity results in the following ways:

- **Case 1**: Two assets with identical future cashflow trade at different prices, or

- **Case 2**: An asset with a definite future price does not trade at the present value of its future price, calculated at an appropriate discount rate.

Example: Arbitrage Opportunity (Case 1)

Bonds X and Y have the same maturity dates, payment at par, and default risk. Bond X has a price of $80 at the initiation. Bond Y has a price of $80.30 at the initiation.

If both bonds have an expected price of $100, show how arbitrage opportunity is created.

Solution

At initiation, sell bond Y at $80.30 and buy bond X at $80 to receive a cash inflow of $0.30. At maturity, receive $100 from bond X and buy bond Y at $100 to cover the short position. Cashflows offset each other, earning an investor a riskless profit of $0.30.

Example: Arbitrage Opportunity (Case 2)

Mkate Bakeries wishes to enter a one-year forward contract to buy 100 bags of wheat at an agreed price of $40 per bag. Today's spot price for wheat is $35 per bag, and the applicable risk-free interest rate is 5%.

Assume that Mkate Bakeries can borrow at the risk-free rate of interest, and the wheat is stored

at no cost.

Show how Mkate Bakeries can make a riskless profit.

Solution

Note that the future price of a bag of wheat does not equate its present value. Using discrete compounding:

$$
\begin{aligned}
PV &= FV(1 + r)^{-N} \\
&= 40(1.05)^{-1} \\
&= \$38.10 \neq \$35
\end{aligned}
$$

At time $t = 0$, Mkate Bakeries borrows $\$3,500 (= 35 \times 100)$ and buys 100 bags of wheat at today's spot price.

Then, Mkate Bakeries enters a forward contract to sell the wheat at $40 per bag in one year.

At maturity ($t = T$), Mkate Bakeries delivers 100 bags of wheat and receives $\$4000 (= 40 \times 100)$.

Moreover, Mkate Bakeries repays the loan of $\$3,675 (= 3,500(1.05)^1)$.

Riskless profit is equal to the forward sale proceeds minus the repayment of the loan:

$$
\text{Riskless profit} = \$4,000 - \$3,675 = \$325
$$

The riskless profit is equivalent to $3.25 per bag of wheat. Therefore, for Mkate Bakeries to earn a riskless profit, it must enter a forward contract to sell the wheat due to the discrepancy between spot and future wheat prices.

In conclusion, the no-arbitrage conditions for pricing derivatives with the underlying with no additional cash flows include:

- Identical assets (assets with identical cashflows) traded at the same time must have the same price.

- Assets with known future prices must have a spot price equal to the present value of

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the future price discounted at risk-free of interest.

Replication

Replication refers to a **strategy** in which a derivative's cash flow stream may be recreated using a combination of long or short positions in an underlying asset and borrowing or lending cash.

Replication mirrors or offsets a derivative position, given that the law of one price holds and arbitrage does not exist. It implies that a trader can take opposing positions in a derivative and the underlying, creating a default risk-free hedge portfolio and replicating the payoff of a risk-free asset.

For example, the following combinations produce the equivalent single asset:

Long asset + Short derivatives = Long risk-free asset

We can rearrange the same formula as:

Long asset + Short risk-free asset = Long derivatives

Short derivative + Short risk-free asset = Short asset

If assets are priced correctly to prohibit arbitrage, replication would seem to be a pointless exercise. However, if we relax the no-arbitrage assumption, we may identify opportunities where replication may be more profitable or have lower transaction costs.

Example: Replicating Long Forward Commitment

Consider a long forward contract with a forward price of $1,600. The underlying spot price is $1,560, and the risk-free interest rate is r% (r > 0).

Show how the cash flow stream of the forward contract can be replicated using borrowing funds at a risk-free interest rate.

Solution

At time $t = 0$:

- Borrow $1,560 at a risk-free rate of interest and buy the underlying at $S_0 = \$1,560$.

$$\text{Cashflow} = \$1,560 - \$1,560 = \$0$$

- Enter a forward contract to buy the underlying at $F_0(T) = \$1,600$

$$\text{Cashflow} = \$0$$

At Maturity t=T

- Sell the underlying at the S_T and repay the loan of $S_0(1 + r)^T) = \$1,600$.

$$\text{Cashflow} = S_T - \$1,600$$

- Settle the forward contract at $S_T - F_0(T) = S_T - \$1,600$, and thus,

$$\text{Cashflow} = S_T - \$1,600$$

As such, we have replicated a long derivative with a long asset plus a short risk-free asset:

Cash Market		Long Forward Contract

Time t=0:
    Borrow $1,560 and buy at the
    underlying at $S_0 = \$1,560$
    $\text{Cashflow} = \$1,560 - \$1,560 = \$0$
Time t = T:
    Repay $S_0(1 + r)^T = \$1,560(1.026)^1$
    $= \$1,600$ and sell at spot, S_T
    $\text{Cashflow} = S_T - \$1,600$
	=	Time t = 0:
		    Agree to buy the underlying at
		    $F_0(T) = \$1,600$
		    $\text{Cashflow} = \$0$
		Time t =T:
		    Settle the contract and sell at spot, S_T
		    $S_T - F_0(T) = S_T - \$1,600$
		    $\text{Cashflow} = S_T - \$1,600$

↑ Long Asset + Short Risk-Free Asset		↑ Long Derivative

Question

Which statement *best* describes arbitrage?

A. Arbitrage is the opportunity to make consistent abnormal returns due to market inefficiency.

B. Arbitrage refers to the ability to profit from price mismatches that last a very short time.

C. Arbitrage allows market participants to recreate using a combination of long or short positions in an underlying asset and borrowing or lending cash.

Solution

The correct answer is **B**.

Arbitrage refers to buying an asset in the cheaper market and simultaneously selling that asset in the more expensive market to make a risk-free profit.

A is incorrect. Arbitrage opportunities allow investors to make risk-free returns without capital commitment. However, such opportunities do not persist for any length of time and cannot be consistently captured.

C is incorrect. It's a description of replication.

LOS 4b: explain the difference between the spot and expected future price of an underlying and the cost of carry associated with holding the underlying asset

The spot price is the price an investor must pay immediately to acquire the asset. In other words, it is the asset's current value or the amount that sellers and buyers agree it is worth. On the other hand, the future price refers to the projected price of an asset at a later date, say, in 6 months.

The Link between Spot and Expected Future Prices

Assuming there are no costs and benefits associated with the underlying asset, spot and forward prices are related as follows under **discrete compounding**:

$$F_0(T) = S_0(1 + r)^T$$

Where:

$F_0(T)$ = Forward price.

S_0 = Spot price.

r = Risk-free rate of return.

T = Time to maturity.

Under **continuous compounding**:

$$F_0(T) = S_0e^{(r)T}$$

Where e is Euler's constant = 2.71828...

Example: Discrete Compounding

Assume that ThinkCare Capital enters a forward contract with Sky Capital to sell 12,500 shares

in its possession in nine months. Sky Capital's spot price per share is USD 68, and the risk-free rate of 6%. If there is no cash flow associated with the underlying, the forward price per share is *closest* to:

Solution

We know that:

$$F_0(T) = S_0(1 + r)^T$$

Thus,

$$= 68(1 + 0.06)^0.75$$
$$= 71.04$$

Foreign Exchange Forward: Continuous Compounding

An FX (foreign exchange) forward contract involves an agreement to buy a particular amount of foreign currency on a future date at a forward price $F_{0,f/d}$. The transaction is made at a pre-agreed exchange rate and is meant to protect the investor from changes in the exchange rates of that foreign currency.

The foreign exchange spot rate is denoted as $S_{0,f/d}$ where the foreign currency f is taken as the ***price currency,*** while the domestic currency d is considered the ***base currency***.

For example, given a EUR/JPY spot rate of 1.60, the Euro is the price currency (f), and the Japanese Yen is the base currency (d), where $EUR1.60 = JPY1$. A **long foreign exchange forward position** implies that an investor purchases a base currency and sells the price currency.

There exists an opportunity cost for the foreign currency referred to as a foreign risk-free rate (r_f) and domestic currency referred to as the domestic risk-free rate (r_d).

A forward price $F_{0,f/d}$ reflects the difference between risk-free foreign rates (r_f) and the domestic

risk-free rate (r_d) as expressed below:

$$F_{0,f/d}(T) = S_{0,f/d}e^{(r_f - r_d)T}$$

Example: Continuous Compounding

Assume that the current EUR/JPY is 0.8762. In this case, Euros is the price currency, and the Japanese Yen is the base currency. If the 4-month Euros risk-free rate is 0.08% and the 4-month Japanese yen risk-free rate is 0.04%, the EUR/JPY forward price is *closest to*:

Solution

$$
\begin{aligned}
F_{0,f/d}(T) &= S_{0,f/d}e^{(r_f - r_d)T} \\
&= 0.8762e^{(0.0008 - 0.0004) \times \frac{1}{3}} \\
&= 0.8763
\end{aligned}
$$

Cost of Carry

Underlying assets may be associated with the costs or benefits of ownership, which must be included in the pricing of the forward commitments to avoid arbitrage opportunities.

Costs include storage, transportation, insurance, and spoilage costs associated with holding the underlying asset, such as warehouse costs (rent) and insurance costs. If the asset owner incurs costs (in addition to opportunity cost), compensation is done through a higher forward price to cover the added costs.

Benefits (or income) refer to monetary returns (such as interest and dividends) and non-monetary returns (such as convenience yield) associated with holding the underlying asset. Benefits decrease the forward price since it accrues to the underlying.

Convenience yield is a non-monetary benefit of holding a physical asset rather than a contract (derivative).

Cost of carry is the net of the costs and benefits associated with owning an underlying asset for a period.

Cost of Carry in Pricing Forward Contracts

Denotes the costs (C) and benefits/income (I). Considering the cost of carry, the relationship between the spot price and futures price changes as follows:

$$F_0(T) = [S_0 - PV_0(I) + PV_0(C)](I + R)^T$$

Under continuous compounding, the costs (c) and income (i) are expressed as rates of return so that the futures price is given by:

$$F_0(T) = S_0 e^{(r+c-i)T}$$

Note that the risk-free rate (r) is the **opportunity cost** of holding an asset. Intuitively, the greater the risk-free rate, the higher the forward price.

Example: Discrete Compounding

Asset ABC has a spot price of USD 89 with a present value of the cost of carry of USD 5. Suppose the risk-free rate is 4.5% (with discrete compounding). The no-arbitrage forward price for half a year contract is *closest* to:

Solution

$$
\begin{aligned}
F_0(T) &= [S_0 - PV_0(I) + PV_0(C)](1 + R)^T \\
&= (USD\ 89 - USD\ 5)(1 + 0.045)^{0.5} \\
&= 85.87
\end{aligned}
$$

Note: The net cost of carry is positive. This means that the benefit is higher than the costs of storing and insurance of the underlying asset.

In summary, the relationship between costs and benefits versus the relationship between forward and spot prices can be outlined as follows:

Relationship Between Costs and Benefits	Relationships Between Forward and Spot Prices
Costs > Benefits	$F_0(T) > S_0$
Costs < Benefits	$F_0(T) < S_0$
Costs = Benefits	$F_0(T) = S_0$

Question

A financial institution enters into a 2-year interest rate swap agreement with a corporation, where the institution will pay a fixed rate of 3% annually and receive a floating rate based on the 6-month LIBOR, which is currently at 2.5%. The notional amount of the swap is USD 10 million. If the 6-month LIBOR rate increases to 3.5% at the end of the first year, what is the net cash flow for the financial institution at that time?

A. The institution receives USD 350,000.

B. The institution pays USD 50,000.

C. The institution receives USD 50,000.

Solution

The correct answer is **C**.

The net cash flow for the financial institution in the swap can be calculated as the difference between the floating rate payment received and the fixed rate payment made, based on the notional amount.

At the end of the first year:

- Fixed rate payment made by the institution = 3% of USD 10 million = USD 300,000.

- Floating rate payment received by the institution = 3.5% of USD 10 million = USD 350,000.

Therefore,

$$\text{Net cash flow} = \text{Floating rate received - Fixed rate paid}$$
$$= \text{USD } 350,000 - \text{USD} 300,000$$
$$= \text{USD } 50,000$$