Learning Module 9: Option Replication Using Put-Call Parity

LOS 9a: explain put–call parity for European options

Put-call parity is a no-arbitrage concept. It involves a combination of cash and derivative instruments in a portfolio. Put-call parity allows pricing and valuation of these positions without directly modeling them using non-arbitrage conditions.

Deriving Put-Call Parity

Consider an investor whose main objective is to benefit from an increase in the underlying value and hedge an investment against a decrease in underlying value. Consider the following portfolios:

Portfolio A

At time $t = 0$, an investor buys a call option at a price of c_0 on an underlying with an exercise price of X and a risk-free bond that is redeemable at X at time $t = T$. Intuitively, assuming the call option expires at time t=T, the cost of this strategy is

$$c_0 + X(1 + r)^{-T}$$

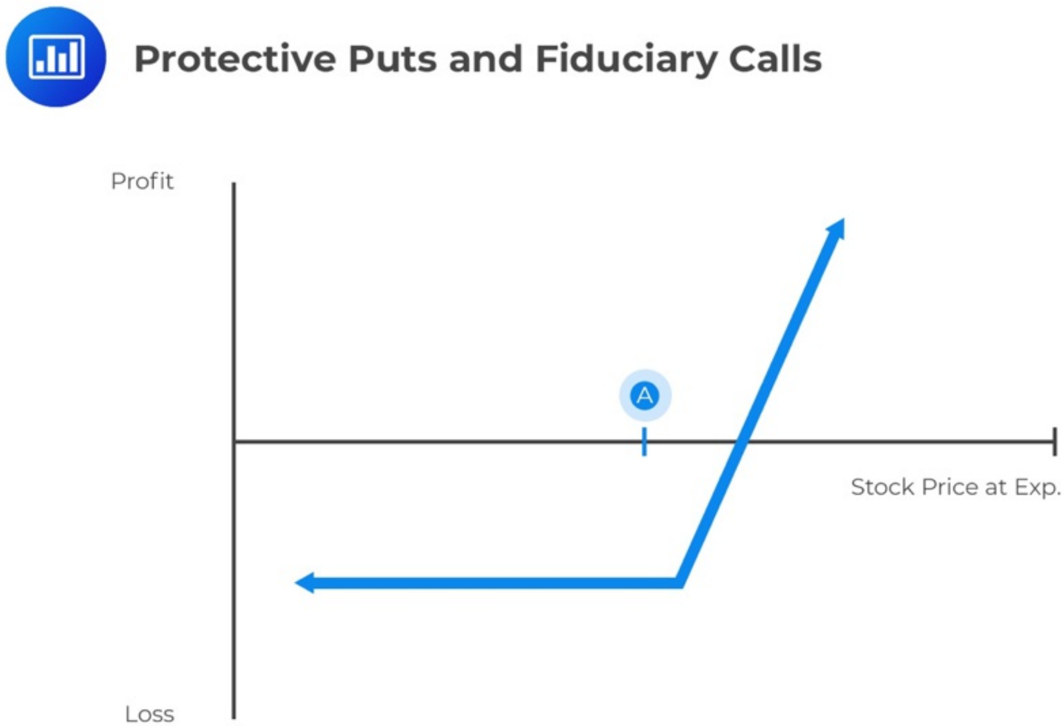
In this portfolio, the investor buys a call option with a positive payoff if the underlying price exceeds the exercise price ($S_T > X$) and invests cash in a risk-free bond. This strategy is called the **fiduciary call**.

Portfolio B

At time $t = 0$, an investor buys an underlying at a price of S_0 and a put option on the underlying price of p_0 whose exercise price is X at time $t = T$. Intuitively, the cost of this strategy is

$$p_0 + S_0$$

The strategy applied in portfolio B is called **protective put**. Protective put involves holding an asset and buying a put on the same asset.



Both portfolios allow the investor to benefit from the rise in underlying price without exposure to a decrease below the exercise price. Moreover, portfolios A and B have identical profiles. Based on the no-arbitrage condition, assets with similar future payoff profiles must trade at the same price, ignoring associated transaction costs. Consider the following table:

Portfolio Position	Put exercised $S_T < X$	No Exercise $S_T = X$	Call Exercised $S_T > X$
Fiduciary Call:			
Call Option	0	0	$S_T - X$
Risk-free Asset	X	X	X
Total:	X	$X(= S_T)$	S_T
Protective Put:			
Underlying Asset	S_T	S_T	S_T
Put option	$X - S_T$	0	0
Total:	X	$S_T(= X)$	S_T

Therefore, since portfolios A and B have identical payoffs at time $t = T$, the costs of these portfolios must be similar at time $t = 0$. For this reason, the put-call parity equation:

$$S_0 + p_0 = c_0 + X(1 + r)^{-T}$$

Where:

S_0 = Price of the underlying asset. p_0 = Put premium. c_0 = Call option premium. X = Exercise price. r = Risk-free rate. T = Time to expiration.

Put-call parity holds for European options that have similar exercise prices and expiration times. These similarities ensure a no-arbitrage relationship between the put option, call option, the underlying asset, and risk-free asset prices. Put-call parity implies that at time $t = 0$, the price of the long underlying asset plus the long put must be equal to the price of the long call option plus the risk-free asset.

Example: Put-Call Parity

Consider European put and call options, where both have an exercise price of $50 and expire in 3 months. The underlying asset is priced at $52 and makes no cash payments during the life of the options.

If the put is selling for $3.80 and the risk-free rate is 4.5%, the price of the call option is *closest* to:

Solution

The put-call parity is given by:

$$S_0 + p_0 = c_0 + X(1 + r)^{-T}$$

We need to rearrange the formula to make the subject of the formula so that:

$$\begin{aligned} c_0 &= S_0 + p_0 - X(1 + r)^{-T} \\ &= 52 + 3.80 - 50(1.045)^{-0.25} \\ &= \$6.35 \end{aligned}$$

Option Replication Using Put-Call Parity

We can rearrange the put-call parity equation to solve for the put option premium, p_0:

$$p_0 = c_0 + X(1 + r)^{-T} - S_0$$

The right side of this equation is referred to as a **synthetic put**. It consists of a long call, a short position in the underlying, and a long position in the risk-free bond.

We can make another re-arrangement to solve for a long call, c_0:

$$c_0 = p_0 + S_0 - X(1 + r)^{-T}$$

The right side of this equation is equivalent to a call option and is referred to as a **synthetic call**. It consists of a long put, a long position in the underlying asset, and a short position in the risk-free bond.

Also, we can further rearrange the put-call parity as follows:

$$S_0 - c_0 = X(1 + r)^{-T} - p_0$$

The right-hand side of the above equation is called the **covered call position**. Intuitively, a covered call is equivalent to a long risk-free bond and short put option.

In summary, synthetic relationships with options occur by replicating a one-part position under put-call parity. Study the following table.

Position	Underlying (S_0)	Risk-free Bond ($(1 + r)^{-T}$)	Call Option (c_0)	Put Option (p_0)
Underlying (S_0)	−	Long	Long	Short
Risk-free bond ($\frac{x}{(1+r)^T}$)	Long	−	Short	Long
Call option (c_0)	Long	Short	−	Long
Put Option (p_0)	Short	Long	Long	−

If the put-call parity does not hold, an arbitrage opportunity exists. The arbitrage opportunity can be exploited by selling the most expensive portfolio and purchasing the cheaper one.

Example: Arbitrage Opportunity

A European call option with a strike price of $25 sells at $7. The price of a European put option with the same strike price is also $7. If the underlying stock sells for $28, and the one-year risk-free rate is 4%, determine if there is an arbitrage opportunity.

Solution

The put-call parity equation:

$$\begin{gathered} p_0 + s_0 ? c_0 + X(1+r)^{-T} \\ 7 + 28 ? 7 + 25(1.04)^{-1} \\ 35 \neq 31.0385 \end{gathered}$$

To exploit the opportunity, we need to:

- Sell the right side (**Protective put**) for $35.
- Buy the left side (**fiduciary call**) for $31.0385.

We get a cash inflow of $\$35 - \$31.0385 = \$3.9615$. Thus, the strategy provides cash inflow ($3.9615) today and no cash outflow at expiration.

Question

European put and call options have an exercise price of \$50 and expire in three months. The underlying asset is priced at \$52 and makes no cash payments during the option's life. The risk-free rate is 4.5%, and the put is selling for \$3.80. According to the put-call parity, the price of the call option should be *closest* to:

A. \$5.25.

B. \$6.35.

C. \$7.12.

Solution

The correct answer is **B**.

The put-call parity is given by:

$$S_0 + p_0 = c_0 + X(1 + r)^{-T}$$

Where:

S_0 =Price of the underlying asset.

p_0 = Put premium.

c_0= Call option premium.

X = Exercise price.

r= Risk-free rate.

T= Time to expiration.

Making c_0 the subject, we have:

$$
\begin{aligned}
c_0 &= S_0 + p_0 - X(1+r)^{-T} \\
&= 52 + 3.80 - 50(1.045)^{-0.25} \\
&= 6.35
\end{aligned}
$$

LOS 9b: explain put–call forward parity for European options

The put-call forward parity extends the put-call parity to include the forward contracts. To get the put-call forward parity, we substitute the present value of the forward price, $F_0(T)$, for the underlying price:

$$F_0(T)(1+r)^{-T} + p_0 = c_0 + X(1+r)^{-T}$$

Deriving Put-Call Forward Parity

Consider an investor whose main objective is to benefit from an increase in underlying value and hedge against a decrease in underlying value. Consider the following portfolios:

Portfolio A

At time $t = 0$, an investor buys a forward contract and a risk-free bond whose face value is equal to the forward price $F_0(T)$. The investor then puts an option on the underlying at a price of p_0 whose exercise price is X at $t = T$. The cost of this strategy is:

$$F_0(T)(1+r)^{-T} + p_0$$

Portfolio B

At time $t = 0$, an investor buys a call option at a price of c_0 on the same underlying exercise price of X and a risk-free bond redeemable at a price of X at time $t = T$. The cost of this transaction is:

$$c_0 + X(1+r)^{-T}$$

Portfolio A is called **synthetic protective put**. Compared to a synthetic put, a synthetic protective put replaces the underlying cash position with a synthetic position using forward purchase and a risk-free bond.

Portfolio B is the same fiduciary call as in the put-call parity seen previously.

Cash flows at time $t = T$ for the synthetic protective put and the fiduciary call are shown in the following table:

Portfolio Position	Put exercised $S_T < X$	No Exercise $S_T = X$	Call Exercised $S_T > X$
Fiduciary Call:			
Purchased Call Option	0	0	$S_T - X$
Risk-free Asset	X	X	X
Total:	X	$X (= S_T)$	S_T
Synthetic Protective Put:			
Purchased Put at p_0	$X - S_T$	0	0
Purchased Forward Contract	$S_T - F_0(T)$	$S_T - F_0(T)$	$S_T - F_0(T)$
Risk-free bond are currently priced as $F_0(T)(1+r)^{-T}$	$F_0(T)$	$F_0(T)$	$F_0(T)$
Total:	X	$S_T (= X)$	S_T

Since portfolios A and B have identical payoffs at time $t = T$, the costs of these portfolios must be identical at time $t = 0$. Therefore, based on no-arbitrage conditions, the put-call forward parity is given by:

$$F_0(T)(1+r)^{-T} + p_0 = c_0 + X(1+r)^{-T}$$

Example: Put-Call Forward Parity

Capital Investments would like to buy a 6-month put option on a company's shares, whose current price is $195 per share. The exercise price of the put options is $190.00 per share.

The 6-month call option on the same shares trades at $64 per share with the same exercise price of $190.00. Using the put-call forward parity and assuming a 1.5% risk-free rate, the price of the put option is *closest* to:

Solution

Using the put-call forward parity

$$F_0(T)(1+r)^{-T} + p_0 = c_0 + X(1+r)^{-T}$$

Making p_0 the subject of the formula, we get:

$$p_0 = c_0 + (X - F_0(T))(1 + r)^{-T}$$

$

We need to solve for $F_0(T)$, which, if you recall, is given by:

$$\begin{aligned} F_0(T) &= S_0(1+r)^T \\ &= 195(1.015)^{0.5} \\ &= \$196.4571 \end{aligned}$$

As such,

$$\begin{aligned} p_0 &= c_0 + (X - F_0(T))(1.015)^{-T} \\ &= 64 + (190 - 196.4571)(1.015)^{-0.5} \\ &= \$57.59 \end{aligned}$$

Option Put-Call Parity Applications: Firm Value

The put-call parity relationship can be used to define a firm's value based on equity holders' and debt holders' interests.

As a rule of thumb, at time $t = 0$, a company's market value, V_0, is equivalent to the present value of its outstanding debt obligations, $PV(D)$, and equity, E_0, where the borrowed funds are in zero-coupon debt with a face value of D.

In an equation, we can express this relationship as:

$$V_0 = E_0 + PV(D)$$

In the event of debt maturity at $t = T$, the assets and debts of the company will be split between debtholders and shareholders, with two possible outcomes based on the company's value at that given time:

Solvency ($V_T > D$)

Recall that solvency refers to a company's ability to meet its financial obligations and long-term debt. If at time T, a firm's value (V_T) is greater than the face value of the debt, ($V_T > D$) the firm is solvent, and thus able to return capital to both the shareholders and the debtholders.

Debt holders come first when distributing capital returns. As such, they receive the debt repayments (D) in full. On the other hand, the shareholders receive what remains. That is $E_T = V_T - D$.

In summary, we've established that shareholders benefit if a company can meet its debt obligation and maintain solvency. On the other hand, debt holders benefit when a company is solvent and hence meets its debt obligations.

Insolvency ($V_T < D$)

Insolvency refers to a company's inability to meet its financial obligations and long-term debts. This occurs if, at debt maturity (T), a company's value is less than the debt's face value, $V_T < D$.

When a firm is insolvent, the shareholders receive the residual, which is equal to zero ($E_T = 0$), and the debtholders are owed more than the firm's total assets. As such, the debtholders receive $V_T < D$ to cover the debt of D at time T.

Payoff in Terms of Options

Note that shareholders retain the unlimited upside potential in solvency and limited downside potential in insolvency. On the other hand, the debtholders are limited to receiving debt repayment in the case of solvency and principal and interest in the case of insolvency.

Intuitively, the payoff profiles can be mathematically represented as follows:

- At time $t = T$, the payoff of the shareholders can be expressed as $\max(0, D - V_T)$.
- On the other hand, the debt holder's payoff is expressed as $\min(V_T, D)$.

In terms of options, the payoff profiles can be expressed as follows:

- **Shareholders' payoff:** They hold a long position in the underlying firm's assets value.

For instance, assume that we have bought a put option on the firm value, V_T, with the exercise price of D. The payoff is $\max(0, D - V_T)$.

- **Debtholders' payoff:** The hold a long position in the risk-free bond, valued at D, and have sold a put option to the shareholders on the firm value, V_T, with an exercise price of D.

Remember the put-call parity relationship:

$$S_0 + p_0 = c_0 + PV(X)$$

If we replace the underlying asset, (S_0), for the company's value at time 0, (V_0), and further replace the risk-free bond, (X), with debt, \(\left(D\right)), the equation becomes:

$$V_0 + p_0 = c_0 + PV(D)$$

We can also rearrange the formula to solve for the value of the company, (V_0):

$$V_0 = c_0 + PV(D) - p_0$$

From the above results, the shareholders have a payoff equivalent to that of a call option (c_0) on the firm's value. On the other hand, the debtholders hold a position of $(D) - p_0$, which is the risk-free debt plus a short position in a put option.

This put option may be seen as a **credit spread** on a company's debt or the premium above the risk-free rate a company must pay debtholders to bear insolvency risk. The value of the put option to shareholders rises as the probability of insolvency grows.

Question

Which of the following *best* describes the replication of a risk-free bond under the put-call parity?

A. Long underlying, short call option, and long put option.

B. Long underlying, short risk-free bond, and long put option.

C. Short underlying, long risk-free bond, and long call option

Solution

The correct answer is **A.**

Recall that the put-call parity relationship may be expressed as:

$$\begin{aligned} c_0 + X(1+r)^{-T} &= p_0 + s_0 \\ \Rightarrow X(1+r)^{-T} &= p_0 + s_0 - c_0 \end{aligned}$$

The risk-free bond replicating individual positions under put call parity is a long underlying, short call option and long put option.

B is incorrect: Call option individual replication position equals long underlying, short risk-free bond, and long put option.

$$c_0 = p_0 + s_0 - X(1+r)^{-T}$$

C is incorrect: Put option position equals short underlying, long risk-free bond, and long call option.

$$p_0 = c_0 + X(1+r)^{-T} - s_0$$