Level I of the CFA® 2025 Exam

Study Notes - Derivatives

Offered by AnalystPrep

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<u>**Learning Module 1: Derivative Instrument and Derivative Market Features**</u>

LOS 1a: define a derivative and describe basic features of a derivative instrument

What is a Derivative?

A derivative is a financial instrument that derives (obtains) its value from the performance of an underlying. The underlying may be a single asset, a group of assets, or variables such as interest rates.

Creation of Derivatives

A derivative is created in the form of a derivative contract involving two counterparties: a buyer and a seller. A derivative contract is a legal agreement between counterparties that defines the rights and obligations of each party. It contains specific maturity or settlement.

The buyer is the **long** or the **holder**, owns the derivative, and is said to hold a long position. On the other hand, the seller is referred to as **short** and holds a short position.

Derivatives can be stand-alone or embedded. Stand-alone derivatives are distinct. An example, a is a derivative on a bond. On the other hand, embedded derivatives are derivatives within an underlying, for example, callable, puttable, and convertible bonds.

Derivatives can be classified into either one of two categories:

1. Firm Commitment (Forward Commitment)

In firm commitment, an amount is pre-determined, and the parties involved make an agreement to exchange it at a future date. Firm commitments include **forward contracts**, **futures contracts**, and **swaps**.

2. Contingent Claim

In a contingent claim, the settlement of the trade depends on one of the counterparties. **Options** are the primary contingent claims.

Benefits of Using Derivatives

Investors can get access to broad opportunities by creating or modifying exposures in the following ways:

1. Earning profits by short selling an underlying whose value is expected to decline.
2. Diversifying a portfolio.
3. Offsetting the financial market exposure that comes with a commercial transaction.
4. Creating large exposures to an underlying using relatively low amounts of cash.
5. Either increasing or decreasing financial market exposure. For instance, hedging uses derivatives to offset (neutralize) existing (anticipated) exposure to an underlying.

Uses of Derivatives

1. **Hedging**: Reduce or eliminate certain forms of risk.
2. **Speculation**: Derivatives have, as an inherent feature, a high degree of leverage. This means that investors only commit small amounts of money to a derivative position relative to the equivalent position in the underlying asset. Small movements in the underlying can lead to large movements in the derivative – both positive and negative.
3. **Arbitrage**: Simultaneous buying and selling to take advantage of varying prices for the same asset to earn a riskless profit.

Note: Unlike the spot markets, derivative markets have lower transaction costs and are more liquid.

Derivative Underlyings

One way of classifying a derivative is by using the underlying from which the derivative derives its value.

Commonly used underlyings are equities, fixed income, interest rates, currencies, commodities, and credit.

Equities

Derivatives that use equities as the underlying may reference a single stock, a group of stocks, or a stock index. Options are predominantly associated with individual stocks, while index derivatives are mostly traded as options, futures, forwards, and swaps.

In index or equity swaps, an investor can receive a return on one index or interest rate and pay the return on one stock index. Investment managers can also use index swaps to increase or decrease exposure to an equity market without trading in individual shares.

Investors trade options on individual stocks. Also, issuers may use stock options to compensate their executives and employees as a motivation for greater corporate performance, which leads to higher stock prices.

Besides, companies may issue warrants. Warrants are stock options that give their holders the right to purchase shares at a fixed price directly from the issuer in the future.

Fixed-Income Instruments and Interest Rates

Fixed income instruments mostly use bonds as the underlying. Associated derivatives include futures, swaps, and options.

Interest rate is a fixed income underlying used by interest rate derivatives, such as forwards, futures, and options. Note that interest rate is not considered an asset. Interest rate swaps are usually used to convert from a fixed interest rate to a floating interest rate exposure – or vice versa – over a certain period. Interest rate swaps mostly use a **market reference rate** (MRR) as the underlying. The most common market reference rate is the secured overnight financing rate (SOFR).

Currencies

Derivatives can be used to hedge foreign exchange risk in commercial and financial transactions. For example, exporters may use forward contracts to sell domestic currency and buy foreign currency in a way that coincides with the delivery of goods or services in a foreign country.

Commodities

Commodities are classified into hard or soft commodities. Hard commodities are natural resources such as crude oil. Soft commodities, on the other hand, are agricultural products such as crops and cattle.

Derivatives on commodities are usually used to manage the price risk of an individual commodity or a commodity index separate from physical delivery.

Credit

Derivative contracts that use credit as an underlying are based on the default risk of either a single or a group of issuers. For instance, Credit Default Swaps (CDS) help manage the risk of loss if a borrower defaults.

Others

Others include weather, cryptocurrencies, and longevity. Derivatives that use such underlyings are less common, and their pricing is challenging.

Question

Which of the following derivatives *most likely* represents a contingent claim?

A. Futures.

B. Options.

C. Forwards.

The correct answer is **B**.

Options are the primary contingent claims. For a contingent claim, trade occurrence depends on one of the counterparties.

A and C are incorrect. For firm commitments, an amount is pre-determined and an agreement is made to exchange it at a future date. Forward or firm commitments include forwards swaps, and futures.

LOS 1b: describe the basic features of derivative markets, and contrast over-the-counter and exchange-traded derivative markets

Over-the-Counter (OTC) Derivative Markets

OTC derivative markets can be formal institutions such as NASDAQ or an information connection of parties who buy from and sell to one another.

In OTC derivative markets, **derivatives end-users** enter contracts with **dealers** or a financial intermediary such as a bank. The dealers (also regarded as the market markers) engage in bilateral transactions to transfer risk to other parties.

Terms of OTC can be modified to match a desired risk exposure profile. This is a beneficial feature to derivative end users who want to hedge existing or expected exposure.

Exchange-Traded Derivative (ETD) Markets

In ETD markets, derivatives are traded in more formal and standardized contracts, promoting higher liquidity and transparency. Such derivatives include futures, options, and other financial contracts at the exchange.

The exchange determines the terms and conditions, including the size of each contract, type, quality, and location of the underlying.

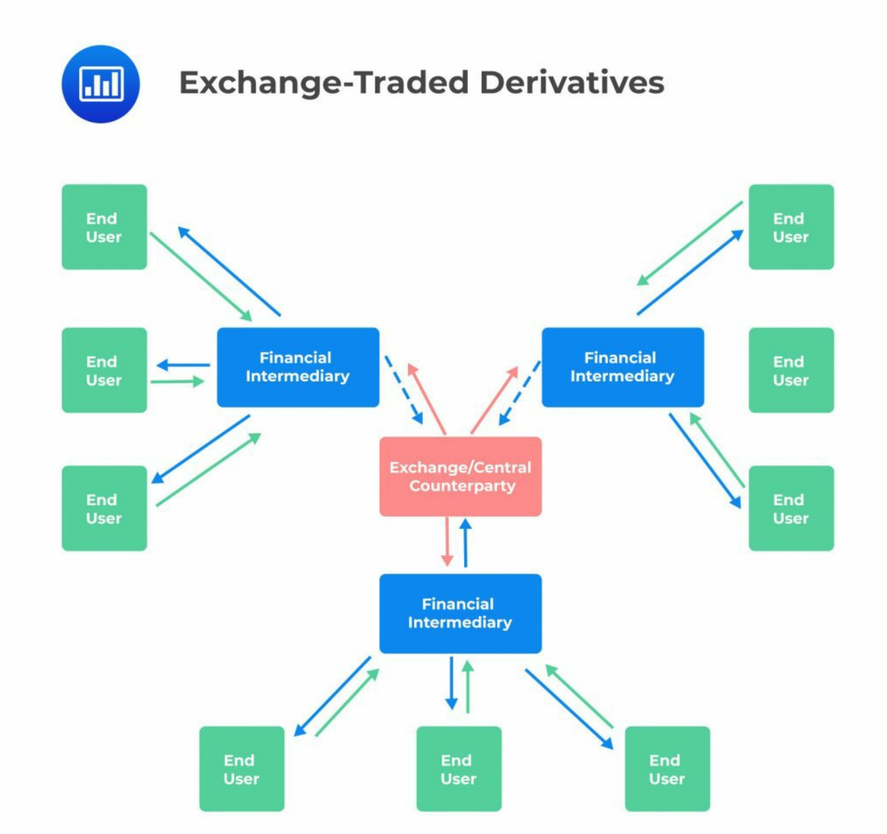
The exchange members consist of dealers (market markers) who are prepared to buy at one price and sell at a higher price. If they cannot find counterparties to trade, risk takers such as speculators may be willing to assume an exposure in the underlying.

Exchange-traded derivatives have standardized terms and conditions. As such, clearing and settlement are done efficiently.

Clearing is a process where the exchange/central counterparty verifies the execution of a transaction, exchange of payments, and records of the participants. On the other hand,

settlement refers to the payment of final amounts and/or delivery of securities or physical commodities between the counterparties based upon exchange rules.

Exchange-traded derivatives demand collateral on deposit upon initiation and during the life of a contract to reduce counterparty risk. The deposit is paid through a financial intermediary, which assures counterparty default.



Differences between Over-the-Counter (OTC) Derivative and Exchange-Traded Derivative (ETD) Markets

- OTC offers more flexibility and customizability.
- OTC is less transparent than ETD.
- OTC involves more counterparty risk, and it is less liquid than ETD.

Question 1

Which statement *best* describes the OTC derivatives market?

A. Contracts are flexible, and there is a high degree of reporting to the regulatory authorities.

B. Contracts are standardized, cleared, and settled through a centralized clearing house.

C. Contracts are flexible, often cleared and settled between transacting parties with a low level of regulatory oversight.

Solution

The correct answer is C.

Exchange-traded derivative contracts are standardized, cleared, and settled through a centralized clearinghouse and accompanied by a high level of regulatory reporting. OTC contracts are far more flexible and less regulated.

Question 2

Consider the following draft commercial contract extracted from Clap company's records.

Contract date	Today
Goods seller	ABZ Limited, Japan
Goods buyer	Clap Company, USA
Goods description	Oil drilling machine
Quantity	Two
Delivery date	150 days from the contract date
Delivery terms	Delivered by ferry. Costs to be paid by the buyer
Payment terms	The amount is payable by the buyer upon delivery
Contract price	$17,525

Which derivative market should ABZ Limited *most likely* use to hedge its financial risk under this commercial contract?

A. An exchange-traded market since it is standardized and transparent.

B. An OTC market, since the contract can be customized to match ABC's desired risk profile.

C. The market with the best price regardless of whether it is an OTC or an exchange-traded market.

Solution

The correct answer is **B**.

An over-the-counter market allows the customization of risk to suit a client's risk exposure profile.

It would be difficult for ABZ Limited to find a contract that matches the desired 150 days from contract date delivery and the exact contract price in an exchange-traded market.

A and C are incorrect. As seen above, the over-the-counter market is the best-suited market for ABZ Limited.

Learning Module 2: Forward Commitment and Contingent Claim Features and Instruments

LOS 2a: define forward contracts, futures contracts, swaps, options (calls and puts), and credit derivatives and compare their basic characteristics

A forward contract is an **over-the-counter (OTC) derivative** contract. In this contract, two parties agree that one party, the buyer (long), will purchase an underlying asset from the other party, the seller (short), at a later date at a fixed price (the **forward price**) agreed upon when the contract is initiated.

A forward contract is suitable for hedging an existing or expected underlying exposure based on specified terms. For example, an importer may use a forward contract to hedge against foreign exchange by entering a forward contract to buy foreign currency to fulfill a future goods delivery contract.

Payoff Profile of a Forward Contract

Assume that we are currently at time $t = 0$, where the price of the underlying is S_0. The forward contract expires at a future date $t = T$, where the underlying price is now S_T. The price S_T is unknown at the initiation of the contract.

At time $t = 0$, the **long (buyer)** and **the short (seller)** agree that the seller will deliver the underlying asset for the price of $F_0(T)$, the **forward price**, at time $t = T$, the expiration date.

However, an important element of a forward contract is that no money is exchanged when the contract is initiated. Thus, forward contracts can be considered to have zero value at the start and are neither assets nor liabilities. The value deviates from zero as the price of the underlying moves. The ability to “lock in” a future price for an asset has important practical benefits and is used as an instrument for financial speculation.

Outcomes of a Forward Contract at Maturity

If, at the expiration date, the current spot price is greater than the forward price [$S_T > F_0(T)$], the buyer (long) receives a payoff of

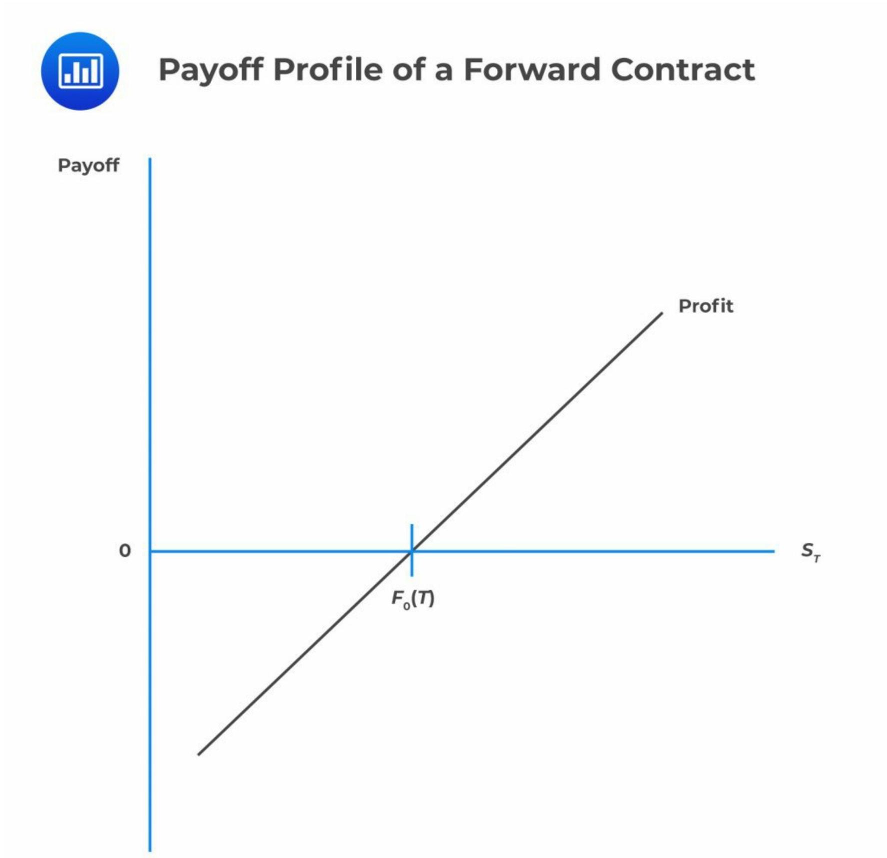
$$\text{Payoff} = S_T - F_0(T)$$

Intuitively, the short incurs a loss of $-(S_T - F_0(T))$ because the seller must deliver an asset at S_T and receive less amount $F_0(T)$.

The table below gives a summary of the outcomes:

Outcome at Expiry	Buyer (long) Payoff	Seller (Short) Payoff
$S_T > F_0(T)$	$[S_T - F_0(T)] > 0$	$[F_0(T) - S_T] < 0$
$S_T < F_0(T)$	$[S_T - F_0(T)] < 0$	$[F_0(T) - S_T] > 0$

We can represent the above results in a graph:



From the graph above, it is easy to see that the price of the forward contract is a linear function of the underlying. As such, forward commitments are also called **linear derivatives**.

Example: Calculating the Forward Contract

Minners Inc. enters a forward contract with a financial intermediary to buy 80 kilos of gold at USD 53,000 per kilo. The spot price of gold is USD 52,780 per kilo.

How much will Minners Inc. pay (receive) to (from) the financial intermediary?

Solution

$$\begin{aligned}\text{Payoff at maturity} &= S_T - F_0(T) \\ &= 52,780 - 53,000 \\ &= -\text{USD } 220\end{aligned}$$

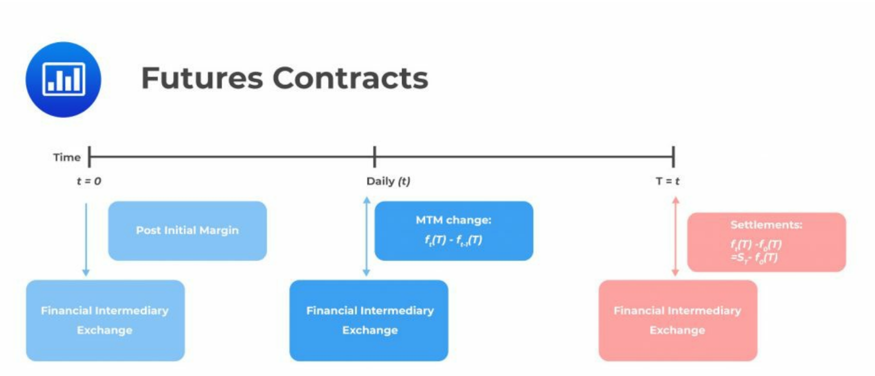
Therefore, Minners Inc. (buyer) must pay USD 17,600 (= 80 × 220) to the financial intermediary (seller).

Futures Contracts

Futures contracts are a standardized variation of forward contracts. The buyer of the futures contracts agrees to buy the underlying in the future at a pre-agreed price (futures price). On the other hand, the seller agrees today to sell the underlying asset in the future at a price agreed upon at the initiation of the contract.

The exchange determines expiration dates, underlying assets, the size of the contracts, and other details.

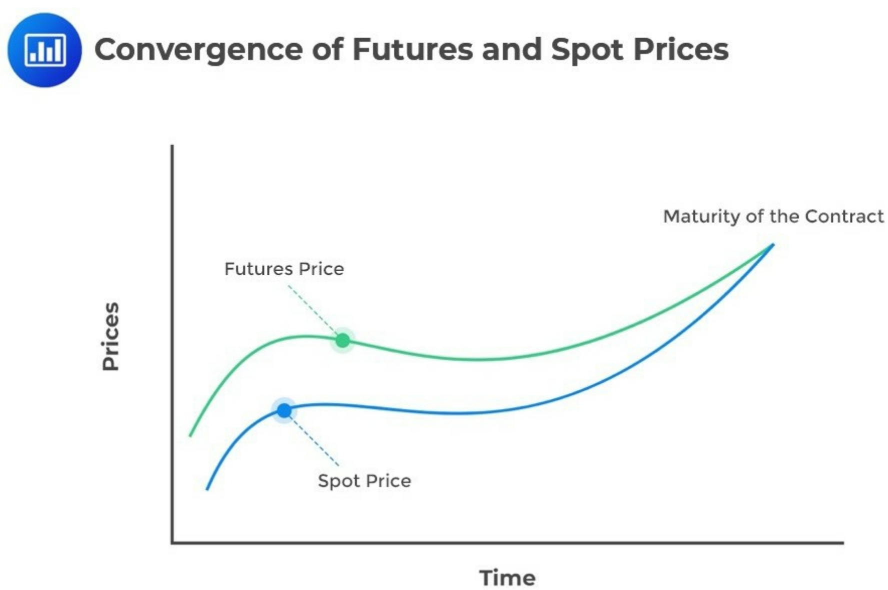
A distinguishing element of futures contracts compared to forward contracts is the **mark-to-market** feature (also called **daily settlement**), where the exchange determines an average of the final futures trades of the day (**settlement price**). Each party's account in the transaction will be debited or credited with the losses or gains for the day.



Like forward contracts, the payoff is based on the difference between the futures price and the underlying price at the expiration date. For example, from the buyer's perspective, the payoff is given by:

$$\text{Payoff} - S_T - F_0(T)$$

Additionally, the futures price converges towards the spot price at expiration. In cash-settled transactions, there is a final mark-to-market at expiration, with the futures price set to the spot price to ensure convergence:



Futures Margin and Settlement Process

Denote the futures price by $f_0(T)$ and the current spot price at expiry by S_T. Note that, like forward contracts, no cash changes hands at the initiation of the futures contract. However, both counterparties must deposit the **initial margin** into a **futures margin account** at the exchange.

The exchange uses the futures margin account to settle daily market changes. Special financial intermediaries usually execute futures contracts on behalf of the counterparties.

When the futures margin account funds fall below the initial margin, the seller receives a **margin call** to top up the account back to the initial amount. The added sum is called the **variation margin**.

If a counterparty cannot replenish the margin account, it must close out the contract as soon as possible and incur additional costs in the process. In extreme cases where a counterparty cannot meet the obligations, the exchange covers the losses through an **insurance fund**.

At maturity, the outstanding contracts (collectively called open interest) are settled through cash or physical delivery (whichever is stated in the contract). However, a counterparty may elect to enter an offsetting future contract before expiration to close out a position.

How Exchanges Limit Losses Due to Defaults in Futures Contracts

An exchange might impose additional requirements to limit potential default-occasioned losses. These include:

- **Increasing required margins**: Due to an increase in price volatility.
- **Price limits**: Establishing a band relative to the previous day's settlement price in which all trades take place.
- **Circuit breaker**: Stopping intraday trading for a short period if the price limit has been reached.

Futures Contracts Accounts (Summary)

Futures Margin Accounts

Futures margin accounts are transactional accounts of buyer and seller held by the exchange. If the end-of-day settlement price decreases, the buyer (long) loses money, which is charged from the margin account and transferred to the seller's account (short). The opposite is true.

Let's assume that at maturity, the futures price is greater than the underlying price. In that case, the profit is transferred to the buyer's account (short) from the seller's (short) account, which equals total settlements.

Initial Margin

This is the minimum amount of money (typically less than 10% of the futures price) that is deposited by both parties to cover possible future losses.

Maintenance Margin

This refers to the sum of money (lower than the initial margin) that each party must maintain in the margin account from the initiation to the maturity of the trade.

Margin Call

This is the request to deposit additional funds (variation margin) into the account if it falls below the maintenance margin.

Example: Margin Call

Minners Inc. enters a 3-month futures contract on an exchange through a financial intermediary to buy 80 kilos of gold with an opening price of USD 53,000 per kilo.

The exchange requires an initial margin of USD 127,200 and a maintenance margin of USD 115,750.

Below is an excerpt of mark-to-market (MTM) details:

Day	Futures Price	Day Gain/Loss	Net Gain/Loss	Margin Balance	Margin Call Variation Margin
T_{90}	$53,000			$127,200	
T_{89}	$53,124	$9,920	$9.920	$137,120	
T_{88}	$53,080	($3,520)	$6,400	$133,600	
T_{87}	$52,600	($38,400)	($32,000)	$95,200	$32,000
T_{88}	$53,024	$33,920	$1,920	$161,120	
T	$53,129	$10,020	($2,020)	$134,125	

Day 89:

- The gold futures price increases by USD 124, so the gain is USD 9,920($= 80 \times 124$).
- Margin balance $= \$127,200 + \$9,920 = \$137,120$
- No margin call since the margin balance is higher than the maintenance margin.

Day 87

- The gold futures price decreases by USD 480, so the loss is USD 38,400($= 80 \times 480$).
- Margin balance $= \$133,600?\$38,400 = \$95,200$ (**below the maintenance margin**).
- Margin call amount (variation margin) $= \$127,200?\$95,200 = \$32,000$

Swaps

A swap is an over-the-counter derivative contract in which two parties agree to **exchange a series of cash flows** whereby one party pays a variable (floating) series that an underlying asset or rate will determine. The other party either pays (1) a variable series determined by a different underlying asset or rate or (2) a fixed rate.

For instance, in interest rate swaps, the floating rate payer pays a market reference rate (MRR) which resets every period, and the fixed rate payer pays a fixed-rate (swap rate), which is constant, as shown below.

As is the case in forwards and futures, no money changes hands at the initiation of the contract. However, as time passes and market conditions vary, the MTM value of the swap contract changes. Note that counterparties exchange a net payment on fixed-and floating payments. The counterparties privately negotiate the credit terms of a swap rate. The credit terms may range from uncollateralized exposure to margining like the futures contract. Swaps may be centrally settled between financial intermediaries by a central counterparty (CCP), where they involve margins like those of futures contracts. The **notional amount** is a sum of money that is used to calculate fixed and floating interest payments.

Example: Calculating Swap Contract Net Cashflow

FinnLay LTD has entered a 10-year interest rate swap with a financial institution with a notional amount of USD 100 million.

The contract states that FinnLay signed to receive a semiannual USD fixed rate of 5% and, in turn, pay a semiannual market reference rate (MRR). For the first six months, MRR is 2%.

Calculate the first swap cash flow.

Solution

The amount owed to FinnLay by the financial intermediary: $\frac{5\%}{2} \times 100\text{m} = 2.5\text{m}$

Amount paid by FinnLay to the Financial intermediary: $\frac{2\%}{2} \times 100\text{m} = 1.0\text{m}$

Therefore, the netting is: $2.5\text{m} - 1.0\text{m} = 1.5\text{m}$

Net Result: Financial intermediary pays FinnLays USD 1.5 million after six months.

Common Features among Forwards, Futures, and Swaps

- Defined contract size.
- Defined underlying.
- There is one or more exchanges of cash flows or underlying on a given date or dates.
- The exchanges are based on pre-agreed prices.

Option Contracts

Options are derivative instruments that give the option buyer the right, but not the obligation, to buy (call) or sell (put) an asset from (or to) the option seller at a fixed price on or before expiration.

In other words, options are contingent claims that give the option buyer the right but not the obligation to transact the underlying, and the option seller is obligated to meet the obligation chosen by the buyer. As such, the **payoff** of an option is positive or zero.

However, the **profit** can be negative since it takes into account the payoff plus (minus) the **premium** received (paid) by (to) the option buyer (seller). If the option expires out-of-the-money, the seller of the option receives the full amount of the premium.

More on Options

When the option buyer decides to transact the underlying, it is referred to as **exercising the option**. The pre-agreed price at which the option buyer exercises the underlying is called the **exercise price**. It is the fixed price at which the underlying asset can be bought or sold at expiry.

The option buyer pays the seller an **option premium** for the right to exercise the option in the future. It is the **fair price** of an option in a **well-functioning** market. The option buyer (long) is not obligated to exercise the option beyond the initial payment of the premium.

Options can be traded on over-the-counter (OTC) markets or exchanges based on standardized terms.

Options can be American options or European options. European-style options can only be **exercised at expiry**, while American-style options are exercisable **before expiry**. This reading primarily dwells on European options.

There are two types of options: call option and put option. A put option is a financial contract that gives the buyer the right, but not the obligation, to sell an underlying asset at a predetermined price within a specified period of time to the seller of the option while a call option is a financial contract that gives the buyer the right, but not the obligation, to buy a specified amount of an underlying asset at a predetermined price within a specified period of time from the seller of the option.

Consequently, the option will only be exercised if the payoff is positive; otherwise, the option expires worthless, and the buyer incurs a loss equal to the option premium.

More information on payoff profile options is given in the next reading.

Credit Derivatives

Credit derivative contracts, like credit default swaps (CDS), manage default risk from single or multiple debt issuers. CDS contracts trade based on credit spreads, influenced by default probability and loss severity. Unlike standard options, exercise timing and payment vary in CDS contracts, which resemble firm commitments.

The buyer pays the seller to assume default risk, with the seller paying in the event of an issuer credit event. CDS can hedge existing credit exposure or speculate on credit spreads, with buyers seeking protection and sellers receiving fixed payments. An issuer credit event triggers termination, with the seller compensating the buyer based on the loss severity.

Question

Which statement is *most* accurate when the stock price is above the exercise price ($S_T > X$) on a put option at expiration?

A. The option seller will suffer a loss equivalent to the difference between the stock price and the exercise price.

B. The option buyer will suffer a loss equivalent to the difference between the stock price and the exercise price.

C. The option seller will show a profit equivalent to the option premium amount, the option buyer will show a loss equivalent to the option premium amount.

Solution

The correct answer is **C**.

If the stock price is above the exercise price at expiration, the put option expires out-the-money and is worthless. The option buyer has lost the premium paid while the option seller has made a gain equivalent to the premium received.

LOS 2b: determine the value at expiration and profit from a long or a short position in a call or put option

Define the following:

c_T = Value of the call at expiration.

p_T = Value of a put option at expiration.

S_T = Price of the underlying at time T.

X = Exercise price.

c_0 = Call option premium.

p_0 = Put option premium.

Π = Profit from an option strategy.

Payoff Profile of a Call Option

Recall that in call options, the buyer has the right but not the obligation to buy the underlying. Moreover, the call option will only be exercised if the payoff is positive; otherwise, the option expires worthless, and the option buyer incurs a loss equal to the option premium.

Intuitively for a call option, the buyer would only exercise the option if $S_T > X$. As such, the payoff to the buyer at expiration is given by:

$$C_T = \max(0, S_T - X)$$

Conversely, the payoff to the seller at expiration is:

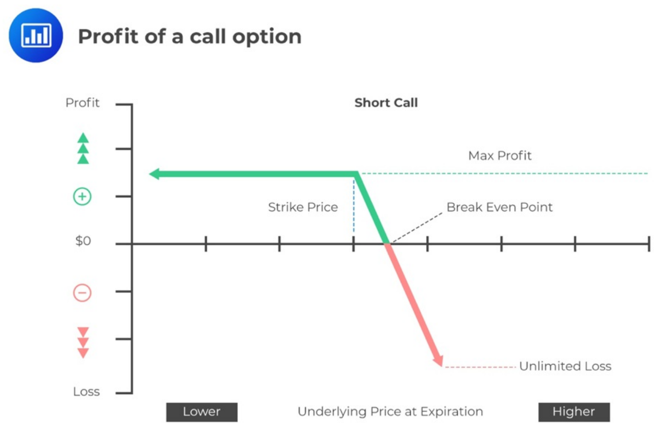
$$-C_T = -(\max(0, S_T - X))$$

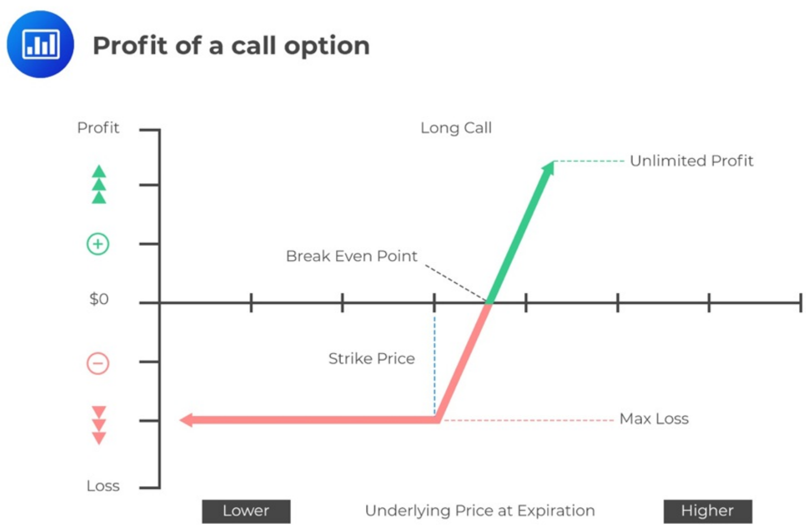
Note also that the option buyer pays the seller the call option premium (c_0) at time $t = 0$ for the right to buy the underlying S_T at an exercise price of X at time $t = T$. Therefore, the profit the buyer will earn is calculated as follows:

$$\Pi = \max(0, S_T - X) - C_0$$

For the call option seller's profit, it is given by:

$$\Pi = -\max(0, S_T - X) + C_0$$





From the above graphs, the following points can be deduced:

- **The maximum loss** for the long (buyer) of the call option is the **premium**, and the **profit** for the **buyer** is **unlimited**.
- **The maximum loss** for the short (seller) of the call option is **unlimited**, and the **greatest profit** the **seller** can make is the premium.

- The **breakeven point** for both the long (buyer) and short (seller) is the strike **price plus the premium**.

- The sum of the profits between the long and the short equals zero since options trading is a **zero-sum game**.

Example: Calculating the Value (Payoff) of a Call Option at Expiration

Consider a one-year call option with a premium of $2 and a strike price of $30. If the price of the underlying at expiration is $40, the value at expiration is *closest* to:

Solution

At $40, the stock price is above the exercise price. Therefore, the option has a value of:

$$\begin{aligned} c_T &= \max(0, S_T - X) \\ &= \$40 - \$30 \\ &= \$10 \end{aligned}$$

Example: Calculating the Profit/Loss of a Call Option at Expiration

Consider a one-year call option with a premium of $2 and a strike price of $30. If the price of the underlying at expiration is $40, the value and profit/loss at expiration is *closest* to:

Solution

The buyer of the call option will exercise the option and make a profit of:

$$\begin{aligned} \Pi &= \max(0, S_T - X) - C_0 \\ &= \$10 - \$2 \\ &= \$8 \end{aligned}$$

Intuitively the seller is at a loss of:

$$\begin{aligned} \Pi = & -\max(0, S_T - X) + C_0 \\ & -10 + 2 \\ & -\$8 \end{aligned}$$

Payoff Profile of a Put Option

For a put option, the buyer has the right but not an obligation to exercise the option at expiry. Exercising the option means that at expiration, the buyer sells the underlying S_T at the exercise price X. As such, the put option is only exercisable if $S_T < X$.

Therefore, the payoff to the buyer is given by:

$$p_T = \max(0, X - S_T)$$

Conversely, the payoff to the put option seller is:

$$-p_T = -(\max(0, X - S_T))$$

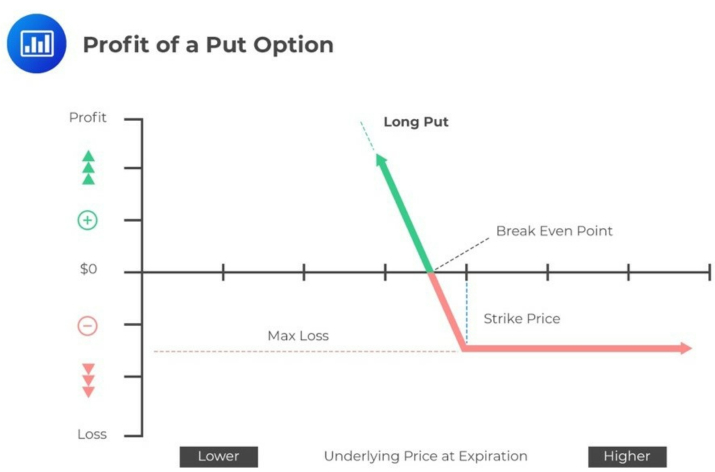
Recall that the put option buyer pays the seller a put option premium (p_0). Therefore, the profit to the option buyer is given by:

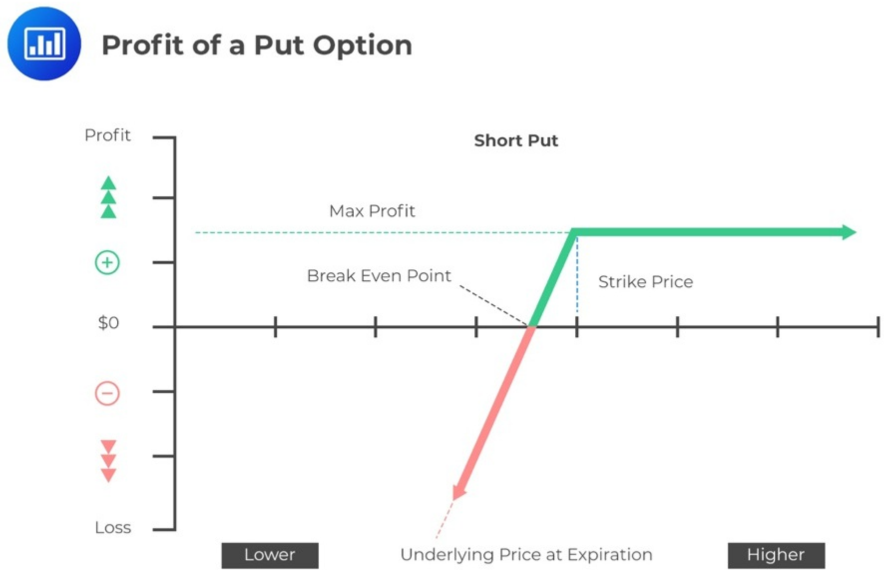
$$\Pi = \max(0, S_T - X) - p_0$$

Conversely, the profit to the options' seller is given by:

$$\Pi = -\max(0, S_T - X) + p_0$$

The following graphs can represent the profit of a put option to both buyer and the seller:





From the above graphs and formulas, the following can be deduced:

- The **maximum loss** for the long (buyer) of the put option is the **premium** paid, and the **profit** for the **buyer** is **limited** to the difference between the strike price and the underlying asset's price (or strike price if the underlying price falls to 0) at expiration. The price of the underlying asset cannot drop below zero, so the maximum profit for the buyer cannot be unlimited.
- The **maximum loss** for the short (seller) of the put option is **limited** to the strike price, and the **greatest profit** the **seller** can make is the premium.
- The **breakeven point** for both the long (buyer) and short (seller) is the strike **price minus the premium**.
- The sum of the profits between the long and the short party equals zero. Options trading is a **zero-sum game**.

Example: Calculating the Payoff (Value) and Profit/Loss of Put Options

Consider a one-year put option with a premium of $3 and a strike of $30. If the underlying price at expiration is $20, the value and profit/loss at expiration is *closest* to:

Solution

At $20, the stock price is below the exercise price. Therefore, the options have a value of:

$$\begin{aligned} p_T &= \max(0, X - S_T) \\ &= \$30 - \$20 \\ &= \$10 \end{aligned}$$

The buyer of the put option will exercise the option; therefore, he makes a profit of:

$$\begin{aligned} \Pi &= \max(0, X - S_T) - p_0 \\ &= \$10 - \$3 \\ &= \$7 \end{aligned}$$

The seller/writer makes a loss of:

$$\begin{aligned} \Pi &= -(\max(0, X - S_T)) + p_0 \\ &= -\$10 + \$3 \\ &= -\$7 \end{aligned}$$

Question

If a put option has a premium of $3 and the exercise price is $100, and the price of the underlying is $105, the value at expiration and the profit to the option seller are *closest to*:

A. Value = -$3; Profit = $0

B. Value = $0; Profit = $8

C. Value = $0; Profit = $3

Solution

The correct answer is **C**.

Note that the exercise price ($100) is less than the underlying price ($105), so we have a situation where $S_T \geq X$. Therefore, the option expires worthless, so the value (payoff) at maturity is zero ($p_T = 0$).

Intuitively, the profit to the seller is equal to the option premium paid by the option buyer ($\Pi = p_0$), which is $3. From the perspective of the put buyer (long put), $p_T = 0$ and $\Pi = -p_0$ or a loss of $3.

LOS 2c: contrast forward commitments with contingent claims

Derivatives typically fall into two classes: forward commitments or contingent claims. The primary difference between the two is based on rights and obligations. Forward commitments carry an obligation to transact, whereas contingent claims confer the right to transact **but not the obligation**.

Forward Commitments

Forward commitments are derivative contracts between two parties that require both parties to transact in the future at a pre-specified price. The parties are obligated to transact, and a legal remedy may be enforced in the event of non-performance.

The payoff profiles of forward commitments are linear. That is, the payoff moves upwards or downwards in direct relation to the underlying asset's price. In other words, the payoff of forward commitments is a linear function of the underlying price.

Forward commitments include forward contracts, futures contracts, and swaps.

Contingent Claims

A contingent claim is a type of derivative where the payoff profile is dependent on the outcome of the underlying asset or **conditional** on the occurrence of some events. With a contingent claim, there is the right to transact but not the obligation. As such, contingent claims have become synonymous with the term "option."

The payoff profile of a contingent claim is non-linear. That is, the payoff of an option is asymmetric (limits losses in one direction).

Contingent claims include options, credit derivatives, and asset-backed securities.

Question

Which statement *best* describes the key difference between a forward commitment and a contingent claim?

A. A forward commitment creates an obligation to transact, whereas a contingent claim allows a transaction to be optional.

B. A forward commitment allows the holder to choose whether to transact, whereas a contingent claim is always enforceable.

C. A forward commitment is enforceable, and a party must transact, whereas a contingent claim allows the seller to choose whether to enforce the transaction.

Solution

The correct answer is **A.**

A forward commitment creates an obligation between the transacting parties, whereas a contingent claim creates the right but not the obligation to transact at a future date.

Learning Module 3: Derivative Benefits, Risks, and Issuer and Investor Uses

LOS 3a: describe the benefits and risks of derivative instruments

Benefits

1. Risk Allocation, Transfer, and Management

Derivative instruments allow allocation, transfer, and management of risks without trading an underlying. The information on cash or spot market prices for financial instruments, goods, and services may assist an investor or issuer in buying and selling. However, issuers and investors are affected by the timing difference between the economic decision and the ability to transact in a cash market.

The ability to buy or sell a derivative instrument today at a pre-agreed price reduces the time between the economic decision and transacting in price risk under different scenarios. For instance, forward commitments or contingent claims can allocate or transfer risk across time and among investors willing to assume those exposures.

2. Information Discovery

Derivative instrument prices provide a price discovery function outside cash or spot markets. More specifically, futures prices may give information about future cash market movement. For instance:

- Analyzing equity index futures prices before the stock market's opening may indicate the direction of cash market prices in early trading.
- Analysts often use the interest rate futures market to predict expectations of the central bank's benchmark interest rate movement.
- Prices for commodity futures serve as a proxy for supply and demand patterns among producers, consumers, and investors across maturities.

Options prices mirror underlying features such as implied volatility, which measures the expected price risk of the underlying.

3. Operational Advantages Compared to Cash or Spot Markets

The operational benefits of derivatives that differentiate them from the cash or spot market include the following:

- **Lower transaction costs**: Derivatives remove the need for insurance, transportation, and storage costs before taking a short position in an underlying.
- **High liquidity**: Derivative markets are associated with reduced capital needed to trade derivatives as compared to a position in cash position in the underlying.
- **Low upfront cash requirements**: Derivatives are associated with low initial margins and premiums compared to cash market transaction costs.
- **Ability to take short positions** with low associated costs.

4. Market Efficiency

Operational efficiency of derivative markets natures greater market efficiency. Derivative markets offer an effective way to exploit mispricing (deviation of prices from the fundamental value). Consequently, fundamental values are frequently reflected earlier in the derivative market than in the cash market. As such, derivative markets lead to more efficient financial markets.

Risks of Derivative Instruments

Derivative instruments and positions are complex. As a result, there are potential risks associated with their usage. These include:

i. High Potential for Speculative Use

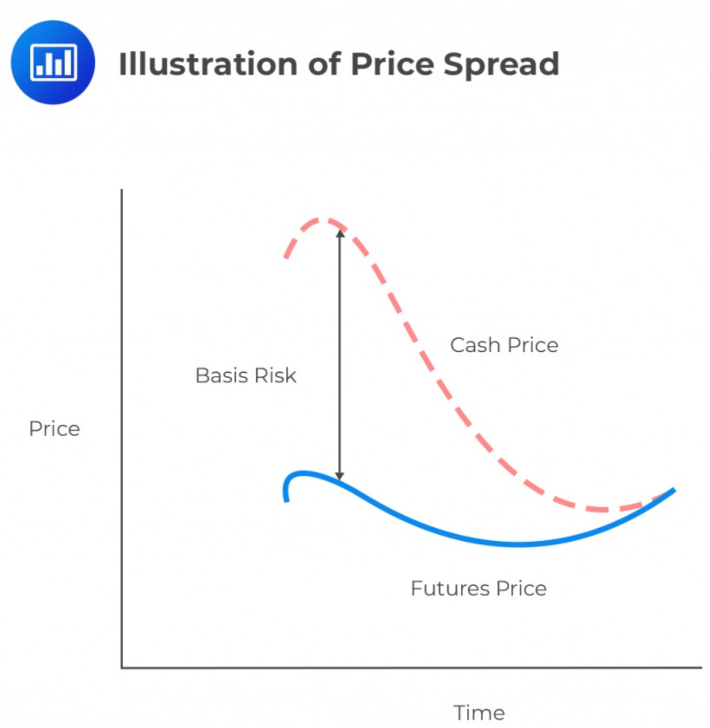
High operational efficiency in derivative instruments limits an investor's initial cash outlay. Consequently, this feature attracts a high degree of implicit leverage compared to cash or spot markets. A high degree of leverage may increase the chances of financial distress.

ii. Lack of Transparency

Derivatives are used to create exposures not found in cash or spot markets. This results in greater portfolio complexity and risks unknown to the investors.

iii. Basis Risk

Basis risk occurs when the expected value of a derivative instrument suddenly deviates from that of the underlying or hedged transaction. Basis risk may manifest when a derivative instrument references a price of an index that is similar but does not precisely match the underlying exposure.



iv. Liquidity Risk

Liquidity risk arises when the cash flow timing of a derivative differs from that of the underlying transaction. For instance, if an investor or issuer fails to honor the margin call requirements, its position is closed out.

v. Counterparty Credit Risk

Counterparty credit risk is the risk of one or more parties in a financial transaction failing to fulfill their side of the contractual agreement. Derivative instruments are associated with counterparty credit exposure leading to differences in the current price compared with the expected future settlement price. However, counterparty credit risk varies with derivatives instrument types and markets in which they are traded.

For example, counterparty credit risk is prevalent in over-the-counter (OTC) markets since credit terms are privately negotiated between counterparties. On the other hand, exchange-traded derivatives are associated with low counterparty risk due to the mark-to-market (MTM) process and margining procedures.

vi. Destabilization and Systemic Risk

Systemic risk occurs due to extensive risk-taking and the use of leverage in derivative markets (which may lead to market stress). As such, financial market supervisory has increased, with a heightened focus on the effect of financial innovation and financial conditions necessary for market stability.

Question 1

Which of the following *best* describes basis risk? The risk that:

A. cash flow timing of a derivative instrument differs from that of an underlying transaction.

B. the expected value of the derivative deviates unexpectedly from that of the underlying.

C. arise due to imprudent risk-taking and utilization of leverage that play a part in market stress.

Solution

The correct answer is B.

Basis risk occurs when the **expected value** of a derivative instrument suddenly deviates from that of the **underlying or hedged transaction**.

A is incorrect. It describes liquidity risk.

C is incorrect. It describes the systemic risk.

LOS 3b: compare the use of derivatives among issuers and investors

Financial intermediaries, investors, and issuers use derivative products to increase, reduce, or alter their exposure to an underlying to achieve their financial goals. With the development of derivatives accounting, these instruments are now reported on the balance sheet at their fair market value instead of using off-balance-sheet reporting.

Use of Derivatives among Issuers

Issuers use derivatives to primarily hedge market-based underlying exposures. Issuers often use hedge accounting, which allows them to offset a hedging instrument – such as derivatives – against a hedged transaction or balance sheet item to decrease financial statement volatility.

In other words, hedge accounting allows issuers to recognize derivative gains, losses, and associated underlying hedged transactions. According to derivative accounting standards, any derivative bought or sold must be marked to market via the income statement through earnings unless it is embedded in an asset or liability or qualifies for hedge accounting.

Types of Hedge Accounting

1. Cash Flow Hedges

Cash flow hedges absorb **variable cash flow** of floating-rate assets or liabilities such as interest rates and foreign exchanges. Cash flow hedges use either forward commitments or contingent claims. For example, a currency forward contract to hedge estimated future sales.

2. Fair Value Hedge

A fair value hedge occurs when a derivative is used to **offset** fluctuation in the **fair value** of an asset or liability. For example, commodities futures may be used to hedge an inventory.

3. Net Investment Hedges

Net investment hedges arise when a **foreign currency bond** or a **derivative** such as forward is used to **offset** the **exchange rate risk** of equity of a foreign operation. Using a currency forward, in this case, could be an effective way to offset foreign exchange risk associated with equity in a foreign company.

Use of Derivatives among Investors

Investors use derivatives to:

i. Replicate a Cash Market Strategy

A derivative market has greater liquidity and reduced capital requirements to trade. This feature allows investors to replicate a chosen position using derivatives easily.

ii. Perform Derivative Hedges

Derivative hedges allow an investor to isolate specific underlying exposures while retaining other positions.

iii. Add or Modify Exposures

Derivative markets are associated with the flexibility to take positions. As such, an investor can use a derivative to add or modify an exposure beyond cash market alternatives.

Note that investors are less concerned about hedge accounting treatment than issuers. This is because an investment fund’s position is usually marked to market daily and included in the daily net asset value (NAV) of the portfolio or fund. This explains why investors transact more frequently in exchange-traded derivatives markets than issuers.

Question

Derivatives intended to withstand a company’s fluctuating cash flows are *most likely* referred to as:

A. cash flow hedges.

B. fair value hedges.

C. new investment hedges.

Solution

The correct answer is **A**.

Cash flow hedges are derivatives intended to withstand a company’s fluctuating cash flows.

Fair value hedges are derivatives considered to balance off the variation in the fair value of an asset or liability.

Net investment hedges happen when a derivative is used to offset the exchange rate risk of the equity of a foreign operation.

Learning Module 4: Arbitrage, Replication, and the Cost of Carry in Pricing Derivatives

LOS 4a: explain how the concepts of arbitrage and replication are used in pricing derivatives

Arbitrage refers to buying an asset in a cheaper market and simultaneously selling it in a more expensive market to make a risk-free profit.

Traders endeavor to exploit arbitrage opportunities when there are short-lived market differences between assets in the same or different markets. An arbitrageur will buy assets in a market with low prices and sell in another market at a higher price to make a profit. Arbitrage opportunities disappear quickly.

When multiple traders buy low-priced assets and sell high-priced assets simultaneously, it increases the demand and price for the former while decreasing the demand and price for the latter. The markets will continue to work in that fashion until prices converge, thereby eliminating arbitrage.

Example: Arbitrage Opportunity

Company ABC's stock trades on the New York Stock Exchange for $10.15 and the equivalent of $10.25 on the London Stock Exchange. How does this set up a perfect, risk-free arbitrage opportunity?

Solution

The 'arbitrageur' can buy ABC's stock on the New York Stock Exchange for $10.15 and simultaneously sell the stock on the London Stock Exchange for $10.25, making a 'riskless' profit of $0.10 per share. This action by other market participants would force the two prices to converge to one price.

Arbitrage and the Law of One Price

The law of one price postulates that assets that produce identical results have only one true market price. In layman's language, it means *identical things should have the same prices*.

Intuitively, arbitrage opportunities exist if the law of one price does not hold.

Arbitrage Opportunities in Derivative Contracts

Remember that the value of derivative contracts is derived from future cash flows linked to the underlying assets. As such, arbitrage opportunity results in the following ways:

- **Case 1**: Two assets with identical future cashflow trade at different prices, or
- **Case 2**: An asset with a definite future price does not trade at the present value of its future price, calculated at an appropriate discount rate.

Example: Arbitrage Opportunity (Case 1)

Bonds X and Y have the same maturity dates, payment at par, and default risk. Bond X has a price of $80 at the initiation. Bond Y has a price of $80.30 at the initiation.

If both bonds have an expected price of $100, show how arbitrage opportunity is created.

Solution

At initiation, sell bond Y at $80.30 and buy bond X at $80 to receive a cash inflow of $0.30. At maturity, receive $100 from bond X and buy bond Y at $100 to cover the short position. Cashflows offset each other, earning an investor a riskless profit of $0.30.

Example: Arbitrage Opportunity (Case 2)

Mkate Bakeries wishes to enter a one-year forward contract to buy 100 bags of wheat at an agreed price of $40 per bag. Today's spot price for wheat is $35 per bag, and the applicable risk-free interest rate is 5%.

Assume that Mkate Bakeries can borrow at the risk-free rate of interest, and the wheat is stored at no cost.

Show how Mkate Bakeries can make a riskless profit.

Solution

Note that the future price of a bag of wheat does not equate its present value. Using discrete compounding:

$$\begin{aligned} PV &= FV(1+r)^{-N} \\ &= 40(1.05)^{-1} \\ &= \$38.10 \neq \$35 \end{aligned}$$

At time $t = 0$, Mkate Bakeries borrows $\$3,500(= 35 \times 100)$ and buys 100 bags of wheat at today's spot price.

Then, Mkate Bakeries enters a forward contract to sell the wheat at $40 per bag in one year.

At maturity ($t = T$), Mkate Bakeries delivers 100 bags of wheat and receives $\$4000(= 40 \times 100)$.

Moreover, Mkate Bakeries repays the loan of $\$3,675(= 3,500(1.05)^1)$.

Riskless profit is equal to the forward sale proceeds minus the repayment of the loan:

$$\text{Riskless profit} = \$4,000 - \$3,675 = \$325$$

The riskless profit is equivalent to $3.25 per bag of wheat. Therefore, for Mkate Bakeries to earn a riskless profit, it must enter a forward contract to sell the wheat due to the discrepancy between spot and future wheat prices.

In conclusion, the no-arbitrage conditions for pricing derivatives with the underlying with no additional cash flows include:

- Identical assets (assets with identical cashflows) traded at the same time must have the same price.
- Assets with known future prices must have a spot price equal to the present value of

the future price discounted at risk-free of interest.

Replication

Replication refers to a **strategy** in which a derivative's cash flow stream may be recreated using a combination of long or short positions in an underlying asset and borrowing or lending cash.

Replication mirrors or offsets a derivative position, given that the law of one price holds and arbitrage does not exist. It implies that a trader can take opposing positions in a derivative and the underlying, creating a default risk-free hedge portfolio and replicating the payoff of a risk-free asset.

For example, the following combinations produce the equivalent single asset:

$$\text{Long asset} + \text{Short derivatives} = \text{Long risk-free asset}$$

We can rearrange the same formula as:

$$\text{Long asset} + \text{Short risk-free asset} = \text{Long derivatives}$$

$$\text{Short derivative} + \text{Short risk-free asset} = \text{Short asset}$$

If assets are priced correctly to prohibit arbitrage, replication would seem to be a pointless exercise. However, if we relax the no-arbitrage assumption, we may identify opportunities where replication may be more profitable or have lower transaction costs.

Example: Replicating Long Forward Commitment

Consider a long forward contract with a forward price of $1,600. The underlying spot price is $1,560, and the risk-free interest rate is r% (r > 0).

Show how the cash flow stream of the forward contract can be replicated using borrowing funds at a risk-free interest rate.

Solution

At time $t = 0$:

- Borrow $\$1,560$ at a risk-free rate of interest and buy the underlying at $S_0 = \$1,560$.

$$\text{Cashflow} = \$1,560 - \$1,560 = \$0$$

- Enter a forward contract to buy the underlying at $F_0(T) = \$1,600$

$$\text{Cashflow} = \$0$$

At Maturity t=T

- Sell the underlying at the S_T and repay the loan of $S_0(1+r)^T) = \$1,600$.

$$\text{Cashflow} = S_T - \$1,600$$

- Settle the forward contract at $S_T - F_0(T) = S_T - \$1,600$, and thus,

$$\text{Cashflow} = S_T - \$1,600$$

As such, we have replicated a long derivative with a long asset plus a short risk-free asset:

Cash Market		Long Forward Contract
Time t=0:		Time t = 0:
Borrow $\$1,560$ and buy at the underlying at $S_0 = \$1,560$	=	Agree to buy the underlying at $F_0(T) = \$1,600$
Cashflow = $\$1,560 - \$1,560 = \$0$		Cashflow = $\$0$
Time t = T:		Time t =T:
Repay $S_0(1+r)^T = \$1,560(1.026)^1 = \$1,600$ and sell at spot, S_T		Settle the contract and sell at spot, S_T $S_T - F_0(T) = S_T - \$1,600$
Cashflow = $S_T - \$1,600$		Cashflow = $S_T - \$1,600$
↑ Long Asset + Short Risk-Free Asset		↑ Long Derivative

Question

Which statement *best* describes arbitrage?

A. Arbitrage is the opportunity to make consistent abnormal returns due to market inefficiency.

B. Arbitrage refers to the ability to profit from price mismatches that last a very short time.

C. Arbitrage allows market participants to recreate using a combination of long or short positions in an underlying asset and borrowing or lending cash.

Solution

The correct answer is **B**.

Arbitrage refers to buying an asset in the cheaper market and simultaneously selling that asset in the more expensive market to make a risk-free profit.

A is incorrect. Arbitrage opportunities allow investors to make risk-free returns without capital commitment. However, such opportunities do not persist for any length of time and cannot be consistently captured.

C is incorrect. It’s a description of replication.

LOS 4b: explain the difference between the spot and expected future price of an underlying and the cost of carry associated with holding the underlying asset

The spot price is the price an investor must pay immediately to acquire the asset. In other words, it is the asset's current value or the amount that sellers and buyers agree it is worth. On the other hand, the future price refers to the projected price of an asset at a later date, say, in 6 months.

The Link between Spot and Expected Future Prices

Assuming there are no costs and benefits associated with the underlying asset, spot and forward prices are related as follows under **discrete compounding**:

$$F_0(T) = S_0(1+r)^T$$

Where:

$F_0(T)$ = Forward price.

S_0 = Spot price.

r = Risk-free rate of return.

T = Time to maturity.

Under **continuous compounding**:

$$F_0(T) = S_0 e^{(r)T}$$

Where e is Euler's constant = 2.71828...

Example: Discrete Compounding

Assume that ThinkCare Capital enters a forward contract with Sky Capital to sell 12,500 shares in its possession in nine months. Sky Capital's spot price per share is USD 68, and the risk-free rate of 6%. If there is no cash flow associated with the underlying, the forward price per share is *closest* to:

Solution

We know that:

$$F_0(T) = S_0(1 + r)^T$$

Thus,

$$= 68(1 + 0.06)^0.75$$
$$= 71.04$$

Foreign Exchange Forward: Continuous Compounding

An FX (foreign exchange) forward contract involves an agreement to buy a particular amount of foreign currency on a future date at a forward price $F_{0,f/d}$. The transaction is made at a pre-agreed exchange rate and is meant to protect the investor from changes in the exchange rates of that foreign currency.

The foreign exchange spot rate is denoted as $S_{0,f/d}$ where the foreign currency f is taken as the ***price currency,*** while the domestic currency d is considered the ***base currency***.

For example, given a EUR/JPY spot rate of 1.60, the Euro is the price currency (f), and the Japanese Yen is the base currency (d), where $EUR1.60 = JPY\,1$. A **long foreign exchange forward position** implies that an investor purchases a base currency and sells the price currency.

There exists an opportunity cost for the foreign currency referred to as a foreign risk-free rate (r_f) and domestic currency referred to as the domestic risk-free rate (r_d).

A forward price $F_{0,f/d}$ reflects the difference between risk-free foreign rates (r_f) and the domestic risk-free rate (r_d) as expressed below:

$$F_{0,f/d}(T) = S_{0,f/d}e^{(r_f - r_d)T}$$

Example: Continuous Compounding

Assume that the current EUR/JPY is 0.8762. In this case, Euros is the price currency, and the Japanese Yen is the base currency. If the 4-month Euros risk-free rate is 0.08% and the 4-month Japanese yen risk-free rate is 0.04%, the EUR/JPY forward price is *closest to*:

Solution

$$\begin{aligned} F_{0,f/d}(T) &= S_{0,f/d}e^{(r_f - r_d)T} \\ &= 0.8762e^{(0.0008-0.0004)\times\frac{1}{3}} \\ &= 0.8763 \end{aligned}$$

Cost of Carry

Underlying assets may be associated with the costs or benefits of ownership, which must be included in the pricing of the forward commitments to avoid arbitrage opportunities.

Costs include storage, transportation, insurance, and spoilage costs associated with holding the underlying asset, such as warehouse costs (rent) and insurance costs. If the asset owner incurs costs (in addition to opportunity cost), compensation is done through a higher forward price to cover the added costs.

Benefits (or income) refer to monetary returns (such as interest and dividends) and non-monetary returns (such as convenience yield) associated with holding the underlying asset. Benefits decrease the forward price since it accrues to the underlying.

Convenience yield is a non-monetary benefit of holding a physical asset rather than a contract (derivative).

Cost of carry is the net of the costs and benefits associated with owning an underlying asset for a period.

Cost of Carry in Pricing Forward Contracts

Denotes the costs (*C*) and benefits/income (*I*). Considering the cost of carry, the relationship between the spot price and futures price changes as follows:

$$F_0(T) = [S_0 - PV_0(I) + PV_0(C)](I + R)^T$$

Under continuous compounding, the costs (*c*) and income (*i*) are expressed as rates of return so that the futures price is given by:

$$F_0(T) = S_0 e^{(r+c-i)T}$$

Note that the risk-free rate (r) is the **opportunity cost** of holding an asset. Intuitively, the greater the risk-free rate, the higher the forward price.

Example: Discrete Compounding

Asset ABC has a spot price of USD 89 with a present value of the cost of carry of USD 5. Suppose the risk-free rate is 4.5% (with discrete compounding). The no-arbitrage forward price for half a year contract is *closest* to:

Solution

$$\begin{aligned} F_0(T) &= [S_0 - PV_0(I) + PV_0(C)](1 + R)^T \\ &= (\text{USD } 89 - \text{USD } 5)(1 + 0.045)^{0.5} \\ &= 85.87 \end{aligned}$$

Note: The net cost of carry is positive. This means that the benefit is higher than the costs of storing and insurance of the underlying asset.

In summary, the relationship between costs and benefits versus the relationship between forward and spot prices can be outlined as follows:

Relationship Between Costs and Benefits	Relationships Between Forward and Spot Prices
Costs > Benefits	$F_0(T) > S_0$
Costs < Benefits	$F_0(T) < S_0$
Costs = Benefits	$F_0(T) = S_0$

Question

A financial institution enters into a 2-year interest rate swap agreement with a corporation, where the institution will pay a fixed rate of 3% annually and receive a floating rate based on the 6-month LIBOR, which is currently at 2.5%. The notional amount of the swap is USD 10 million. If the 6-month LIBOR rate increases to 3.5% at the end of the first year, what is the net cash flow for the financial institution at that time?

A. The institution receives USD 350,000.

B. The institution pays USD 50,000.

C. The institution receives USD 50,000.

Solution

The correct answer is **C**.

The net cash flow for the financial institution in the swap can be calculated as the difference between the floating rate payment received and the fixed rate payment made, based on the notional amount.

At the end of the first year:

- Fixed rate payment made by the institution = 3% of USD 10 million = USD 300,000.
- Floating rate payment received by the institution = 3.5% of USD 10 million = USD 350,000.

Therefore,

$$\begin{aligned}\text{Net cash flow} &= \text{Floating rate received - Fixed rate paid}\\ &= \text{USD } 350,000 - \text{USD}300,000\\ &= \text{USD } 50,000\end{aligned}$$

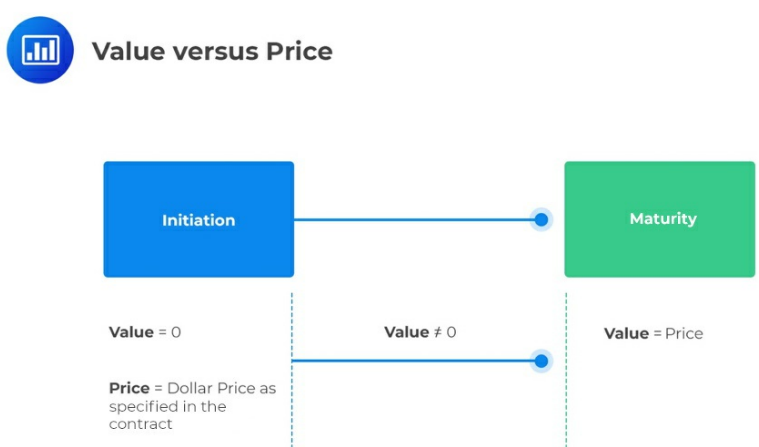
Learning Module 5: Pricing and Valuation of Forward Contracts and For An Underlying with Varying Maturities

LOS 5a: Explain how the value and price of a forward contract are determined at initiation, during the life of the contract, and at expiration

The price of a forward commitment is agreed upon at the contract's initiation and remains fixed until the contract's maturity. Moreover, the price is also used in determining the basis on which the underlying will be traded in the future against the spot price at maturity.

The value of a forward commitment changes over the life of the forward contract as the spot price and other factors change. For instance, consider a forward contract on an underlying that does not generate cash flows.

- **At the inception of the forward contract:** The value is zero since there is no down payment; that is, investors are willing to sign off on the contract for $0.00.
- **During a forward contract's life:** The value of the forward contract is calculated as spot price minus the present value of the forward price discounted at the risk-free rate.
- **At maturity:** The value of the forward contract is calculated as the spot price at maturity minus the forward price.



Forward Contract Pricing and Valuation

Define the following:

S_0 = Spot price of the underlying at the initiation.

S_t = Spot price of the underlying at time *t*, during the contract's life.

r = Risk-free rate of return.

$F_0(T)$ = Forward price (satisfies the no-arbitrage conditions).

$V_0(T)$ = Value of the forward contract at the initiation.

$V_t(T)$ = Value of the forward contract during the life of the contract.

$V_T(T)$ = Value of the forward contract at expiration.

Price and Value of Forward Contract at Initiation

Assume that there are no costs or benefits associated with the underlying. At the initiation of the forward contract, no money is exchanged, and the forward contract's value is zero:

$$V_0(T) = 0$$

The forward price that parties have agreed upon at initiation satisfies no arbitrage opportunities. As such, the forward price at initiation is the spot price of the underlying compounded at the risk-free rate over the contract's life:

$$F_0(T) = S_0(1 + R)^T$$

Price and Value of Forward Contract During the Life of the Contract

After its initiation, the value of the forward contract changes due to the change in the underlying's spot price, among other factors.

As such, the mark-to-market value of the forward contract at any time, $V_t(T)$, captures the spot price at time t and the present value of the forward price. More specifically, the value of a forward contract during the contract's life is the spot price at time t of the underlying asset minus the present value of the forward price (long position).

And the short forward position value is calculated as follows:

$$V_T(T) = S_t - F_0(T)(1+r)^{(-T-t)} - S_t$$

Remember that this is a zero-sum game. The value of the contract to the short position is the negative value of the long position, and thus the sum of both is always zero.

Price and Value of Forward Contract at Expiration

Forward contract settlement depends on the difference between the forward price, $F_0(T)$, and the spot price of the underlying at expiration, S_T. Therefore, the value of the forward contract for the long position will be:

$$V_T(T) = S_T - F_0(T)$$

And the value of the forward contract for the short position will be:

$$-V_T(T) = F_0(T) - S_T$$

Assuming there are no costs or benefits associated with the underlying, study the following table.

Outcome	Value of the Long Position	Value of the Short Position
$S_t > F_0(T)(1+r)^{-(T-t)}$	Mark-to-market gain	Mark-to-market loss
$S_t < F_0(T)(1+r)^{-(T-t)}$	Mark-to-market loss	Mark-to-market gain
$S_t = F_0(T)(1+r)^{-(T-t)}$	No mark-to-market gain/loss	No mark-to-market gain/loss

Example: Pricing vs. Valuation of Forward Commitments

Ali Muhamud currently owns 6,000 shares at Unilever Limited, whose spot price is $134 per share. Muhamud agrees to enter a forward contract to sell 2,000 shares to a financial intermediary at $178 per share in nine months. The spot price at maturity is $197.

The contract’s value at maturity from the financial intermediary’s perspective is *closest* to:

Solution

Muhamud is the seller (shot position), so the financial intermediary has a long forward position. Since we're at maturity, there is no need to take any present value. The value of the contract for the financial intermediary will simply be:

$$\begin{aligned} V_T(T) &= S_T - F_0(T) \\ &= \$197 - \$178 \\ &= \$19 \end{aligned}$$

Price and Value of a Forward Contract with Cost of Carry

At Initiation

For a forward contract whose underlying is associated with the cost of carry, the forward price is adjusted as follows:

$$F_0(T) = [S_0 - PV_0(I) + PV_0(C)](1 + r)^T$$

Where:

$PV_0(I)$ = Present value of income or benefits at time $t = 0$.

$PV_0(C)$ = Present value of costs at time $t = 0$.

The forward price above satisfies the no-arbitrage condition at time $t = 0$. Moreover, the forward contract, whose underlying has associated benefits and costs, is neither a liability nor an asset to the buyer or seller. As such, the value of the contract at initiation is zero:

$$V_0(T) = 0$$

During the Contract's Life

During the forward contract’s life with the cost of carry, the mark-to-market (MTM) value is determined as the difference between the current spot price adjusted for the cost of carry and the present value of the forward price.

This is expressed as:

$$V_t(t) = (S_t - PV_t(I) + PV_t(C)) - F_0(T)(1 + r)^{-(T-T)}$$

Where:

$PV_t(I)$ = Present value of income or benefits at any time t.

$PV_t(C)$ = Present value of costs at any time t.

At Maturity

Since the forward price includes the cost of carry, the value of the underlying asset at maturity is calculated as the difference between the spot price at maturity, S_T, and the forward price agreed upon at inception, $F_0(T)$.

$$V_0(T) = S_T - F_0(T)$$

Example: Price and Value of a Forward Contract with Cost of Carry

Nico Haas has entered a forward contract with a seller of an asset whose spot price is $62, and the net cost of carry is $5. The risk-free rate is 5% and the life of the contract is nine months. Suppose the spot price maturity remained at $62, the contract's value at maturity is closest to:

Solution

First, let's find the forward price:

$$\begin{aligned} F_0(T) &= S_0(1 - r)^T - (PV_t(I) + PV_t(C))(1 - r)^T \\ &= (S_0 - \text{Cost of carry})(1 + r)^T \\ &= (57)(1 + 0.05)^{\frac{9}{12}} \\ &= \$59.12 \end{aligned}$$

Therefore, the value of the contract is:

$$\begin{aligned} V_0(T) &= S_T - F_0(T) \\ &= \$62 - \$59.12 \\ &= \$2.87 \end{aligned}$$

Foreign Exchange Forward: Continuous Compounding

The spot price, ($S_{0,f/d}$), and the forward price, ($F_{0,f/d}(T)$), are expressed in terms of price currency (foreign, *f*) per single unit of base currency (domestic currency, *d*).

The difference between the spot price and forward price depicts the difference between the risk-free foreign rate (r_f) and the risk-free domestic rate (r_d) as expressed below:

$$F_{0,f/d}(T) = S_{0,f/d}e^{(r_f - r_d)T}$$

At initiation, the currency with higher risk-free rates is said to trade at a **forward discount**, while a currency with a low risk-free rate for the forward period trades a **forward premium**.

At any time, t before maturity, the mark-to-market (MTM) value of the forex exchange forward is the spot price, $S_{t,f/d}$, minus the PV of the forward price, $F_{0,f/d}(T)$, discounted by the difference in the risk-free rates $r_f - r_d$ over the remainder of the contract using continuous compounding, $e^{-(r_f - r_d)(T-t)}$:

$$V_t(T) = S_{t,f/d} - F_{0,f/d}(T)e^{-(r_f - r_d)(T-t)}$$

The fluctuation between the foreign and domestic risk-free rates indicates changes in the available opportunity costs between the two currencies.

Question

Suppose Ceriotti Cosmas has 20,000 shares of VIVO, and he agrees to enter a 3-month forward contract with Lumis to sell his shares at CAD 239 per share. The risk-free rate is 3.5%, and a spot rate at $t = 0$ is CAD 225 per share. If VIVO's spot price falls to CAD 215 per share in one month, the forward contract's mark-to-market (MTM) value in VIVO's perspective in one-month is *closest to:*

A. 21.27

B. 22.63

C. 32.36

Solution

The correct answer is **B**

From VIVO's perspective:

$$V_T(T) = F_0(T)(1+r)^{-(T-t)} - S_t$$

Where, $F_0(T) = \text{CAD}239$, $r = 0.035$, $T = 0.25, 0.0833$, $S_t = \text{CAD}215$

$$\begin{aligned} V_t(T) &= \text{CAD } 239(1.035)^{-(0.1670)} - 215 \\ &= \text{CAD } 22.63 \text{ MTM gain} \end{aligned}$$

LOS 5b: Explain how forward rates are determined for an underlying with a term structure and describe their uses

The link between the spot and forward prices is determined by a risk-free interest rate which is regarded as the opportunity cost of holding an asset. **Term structure** implies that various interest rates are available depending on the time to maturity. Given an upward-sloping yield curve, which is the most usual case, investors would charge a higher yearly interest rate for five-year loans than for a one-year loan.

Spot Rates

A spot rate, also known as zero rates, is defined as the yield to maturity of a zero-coupon security maturing at the date of each cash flow.

Example: Spot Rate

A 4-year $2,000 par value bond pays 10% annual coupons. The spot rate for year 1 is 8%, the 2-year spot rate is 13%, the 3-year spot rate is 14%, and the 4-year spot rate is 16%. The price of the bond is *closest* to:

Solution

Here, we have an upward-sloping yield curve since the 2-, 3-, and 4-year interest rates are higher than the 1-year interest rate.

$$\begin{aligned}\text{Annual coupon} &= \$2000 \times 10\% = \$200\\ \text{Price} &= \left(\frac{\$200}{(1.08)^1}\right) + \left(\frac{\$200}{(1.13)^2}\right) + \left(\frac{\$200}{(1.14)^3}\right) + \left(\frac{\$2200}{(1.16)^4}\right)\\ &= \$1691.85\end{aligned}$$

Discount Factor

The discount factor is the price equivalent of a zero/spot rate, which, when multiplied by the total amount of money to be received (principal + interest), gives the bond's price (present value).

$$\text{Discount factor } (DF_i) = \frac{1}{(1 + \text{Discount rate}(Z_i))^{\text{Period number}(i)}}$$

Example: Discount Factor

Suppose that an investor sells a four-year zero-coupon bond at par with a price of 94.78; the four-year zero rates is *closest* to:

Solution

$$DF_i = \frac{1}{(1 + Z_i)^i}$$

Where:

DF_i = The discount factor for a given period.

Z_i = The zero rate for a given period.

i = The period.

To solve for the four-year zero rate z_4 solve the equation:

$$94.78 = \frac{100}{(1 + z_4)^4}$$

We can see that:

$$0.9478 = \frac{1}{(1 + z_4)^4}$$
$$Z_4 = 0.013493 \approx 1.3493\%$$

Forward Rates vs. Spot Rates

A forward rate indicates the interest rate on a loan beginning at some time in the future. A spot rate, on the other hand, is the interest rate on a loan beginning immediately.

In an interest rate forward contract, the most common market practice is to name forward rates. For instance, “3y7y” is also denoted as $F_{3,7}$, which means “3-year into 7-year rate”. The first number refers to the length of the forward period from today, while the second one refers to the tenor or time-to-maturity of the underlying bond.

For short-term market reference rates (MRRs), forward rates are normally named in months. For example, (F_{2m,5m}\) means the 5-month forward period starting at the end of 2 months, and the time-to-maturity is seven months from today.

Implied Forward Rate (IFR)

The implied forward rate is the breakeven rate that links a short-date and long-dated zero-coupon bond. It is an interest rate at a future period where an investor breaks even and can earn the same return as today.

The general formula for the relationship between the two spot rates and the implied forward rate (IFR) is:

$$(1 + Z_B)^B = (1 + Z_A)^A \times (1 + IFR_{A,B-A})^{B-A}$$

Where:

Z_A = Yield-to-maturity per period of short-term bond.

Z_B = Yield-to-maturity per period of long-term bond.

$IFR_{A,B-A}$ = Implied forward rate between period A and period B, with a tenor of B-A.

The implied forward rate (IFR) is the interest rate for a period in the future where an investor can earn a return:

- By investing now until the forward rate reaches its final maturity.
- By investing now until the forward rate’s start date. Besides, the investor increases returns at the implied forward rate by rolling over the proceeds.

Example: Implied Forward Rate

An investor notes that the two-year and the three-year zero-coupon bonds yield 6% and 8%, respectively. The investor enters a one-year fixed-rate FRA agreement to hedge against rate fluctuations.

The implied forward rate applicable to the investor's position is *closest* to:

Solution

$$\begin{aligned}
(1+Z_A)^A \times (1+IFR_{A,B-A})^{B-A} &= (1+Z_B)^B \\
(1.06)^2 \times (1+IFR_{2,1})^1 &= (1.08)^3 \\
1.1236 \times (1+IFR_{2,1}) &= 1.2597 \\
1.1236 + 1.1236 IFR_{2,1} &= 1.2597 \\
1.1236 IFR_{2,1} &= 0.136112 \\
IFR_{2,1} &= \frac{0.136112}{1.1236} \\
&= 0.1211 \approx 12.11\%
\end{aligned}$$

Forward Rate Agreements (FRAs)

A forward rate agreement (FRA) is a cash-settled over-the-counter (OTC) contract between two counterparties. In this contract, the buyer (long position) is borrowing and the seller (short position) is lending a notional sum (underlying) at a fixed interest rate (the FRA rate) and for a specified period starting at an agreed date, e.g., period A in the future and ends in period B.

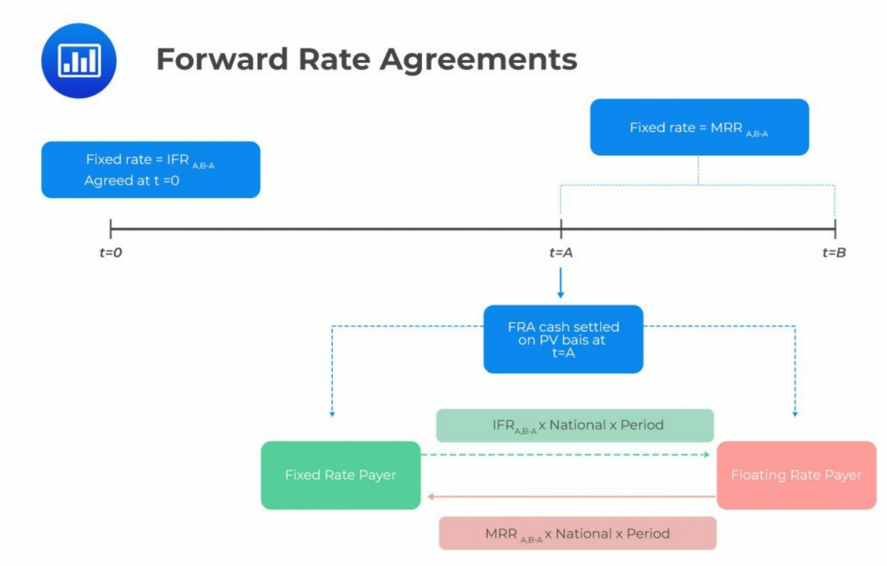
The seller deposits interest based on the market reference rate (MRR), where the MMR is established before the settlement dates, at time $t = A$.

The FRA settlement amount is a function of the difference between the forward interest rate ($IFR_{A,B-A}$) and the market reference rate (MRR_{B-A}) or the MMR for $B - A$ periods, which ends at period B.

FRAs are usually cash-settled at the beginning of the period during which the reference rate applies.

The net payment from the long position's perspective would be:

$$\text{Net payment} = (\text{MRR}_{\text{B}-\text{A}} - \text{IFR}_{\text{A,B}-\text{A}}) \times \text{Notional principal} \times \text{Period}$$



FRA can be seen as a one-period interest swap. They are mainly utilized by investors and issuers to manage interest rate risk. The notional amount is not exchanged but used for interest calculations. Also, fixed and floating payments occur on a net basis. Financial intermediaries use FRAs to protect rate-sensitive balance sheets, assets, or liabilities against interest rate risk.

Example: Forward Rate Agreements

James Malcolm enters an FRA agreement with Fred Green on a notional amount of USD 50,000. Malcolm will receive a fixed rate in 4 months, and a two-month MRR is set at 1.3%. If the $\text{IFR}_{3\text{m},1\text{m}}$ at the initiation of the contract is 2.5%, the settlement amount is *closest* to:

Solution

$$\begin{aligned}\text{Net payment at the end of the period} &= (\text{MRR}_{\text{B}-\text{A}} - \text{IFR}_{\text{A,B}-\text{A}}) \times \text{Notional principal} \times \text{Period} \\ &= (2.5\% - 1.3\%) \times \text{USD } 50{,}000 \times \frac{2}{12} \\ &= \text{USD } 100\end{aligned}$$

$$\begin{aligned}\text{Cash settlement (PV)} &= \text{USD} \frac{100}{1 + 0.013/12} \\ &= \text{USD } 99.89\end{aligned}$$

Question

Vanessa Raquela is an analyst at Money Wise Capital. She analyses two-year and three-year government zero-coupon bonds whose prices are 95 and 92 per 100 face value, respectively. The implied two-year forward rate in five years' time is *closest* to:

A. 2.60%.

B. 2.81%.

C. 3.26%.

Solution

$$DF_i = \frac{1}{(1+Z_i)^i}$$

Thus:

$$0.95 = \frac{1}{(1+z_2)^2} = 2.5978\%$$
$$0.92 = \frac{1}{(1+z_3)^3} = 2.81845\%$$

Now we need to solve for $IFR_{2,5}$ as follows:

$$(1+Z_A)^A \times (1+IFR_{A,B-A})^{B-A} = (1+Z_B)^B$$
$$(1+Z_2)^2 \times (1+IFR_{2,5}) = (1+z_3)^3$$
$$1.025978^2 \times (1+IFR_{2,5}) = 1.0281845^3$$

Where:

$$1.025978^2(1+IFR_{2,5}) = 1.0281845^3$$
$$1.025978^2(1) + (1.025978^2)(IFR_{2,5}) = 1.0281845^3$$
$$1.052631 + 1.052631IFR_{2,5} = 1.086959$$
$$1.052631IFR_{2,5} = 0.034328$$
$$IFR_{2,5} = 0.032612 \approx 3.2612\%$$

Learning Module 6: Pricing and Valuation of Futures Contracts

LOS 6a: compare the value and price of forward and futures contracts

Recall that during the initiation of a forward commitment, no cash changes hands. Further, the forward commitment is neither a liability nor an asset to a buyer or the seller. As such, the value of both the forward contract and futures contract is zero:

$$V_0(T) = 0$$

Consider an underlying with no associated costs or benefits. Like forward contracts, the futures price is calculated by compounding the spot price of the underlying using the risk-free rate:

$$f_T(0) = S_0(1 + r)^T$$

Where:

$f_T(0)$ = Futures forward price.

S_0 = Spot price of the underlying at time $t = 0$.

r = Risk-free rate.

T = Time to maturity.

Note that, like forward contracts, we have used discrete compounding. However, continuous compounding is also preferred in futures contracts if the underlying assets comprise a portfolio, such as commodities, fixed income, and equity. Also, continuous compounding is preferred when the underlying is foreign exchange denominated in two currencies.

Using continuous compounding, the future price is given by:

$$f_T(0) = S_0e^{rT}$$

Cost of Carry and Futures Price

Like forward contracts, the price of futures whose underlying has income (I) and costs (C) is adjusted as follows:

$$f_T(0) = [S_0 - PV_0(I) + PV_0(C)](1 + r)^T$$

Where:

$PV_0(I)$ = Present value of income or benefit associated with the underlying at time $t = 0$.

$PV_0(C)$ = Present value of costs associated with the underlying at time $t = 0$

Example: Futures Price Valuation

Minners Inc. enters a futures contract on an exchange via a financial intermediary to buy 80 kilos of gold. The current spot price is $52,950 per kilo.

If the risk-free rate of return is 3%, what is the no-arbitrage futures price for settlement in 95 days?

Solution

The futures price is equal to the compounded value of the spot price of the underlying at the risk-free rate for a period T:

$$\begin{aligned} f_T(T) &= S_0(1 + r)^T \\ &= \$52,950(1.03)^{\frac{95}{365}} \\ &= \$53,358.94 \end{aligned}$$

Mark-to-Market Valuation of a Future Contract Compared to a Forward Contract

As time passes, the value of futures and forward contracts changes. However, the forward contract price remains constant until maturity.

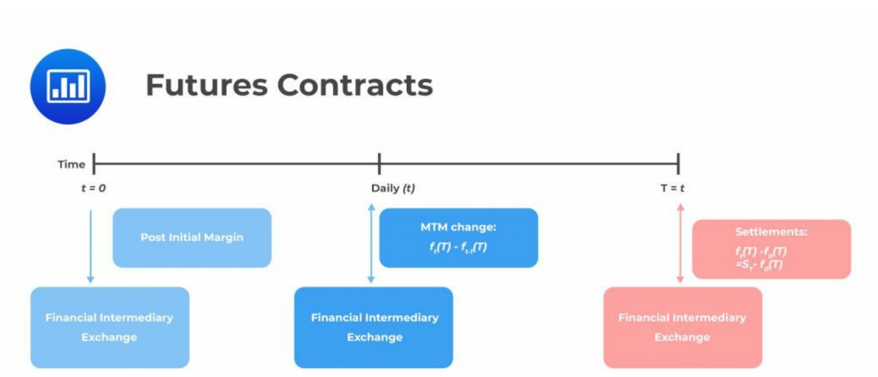
As seen previously, for the long position, the value of a forward contract during its life is calculated as the difference between the current spot price and the present value of the original forward price:

$$V_t(T) = S_t - F_0(T)(1+r)^{-(T-t)}$$

The MTM value of the forward contract is not settled until its expiration date, which causes counterparty risk.

On the other hand, the futures price changes depending on market conditions. Moreover, the daily settlement resets the MTM value to zero. Besides, the variation margin is exchanged to cover the difference, decreasing counterparty risk.

Note that the cumulative MTM gain or loss is approximately equal to that of a comparable forward contract.



Interest Rate Futures and Forward Contracts

Remember that a forward rate agreement (FRA) uses implied forward rates as a no-arbitrage fixed rate. In this instance, the counterparties exchange fixed for floating payments at a specified time in the future.

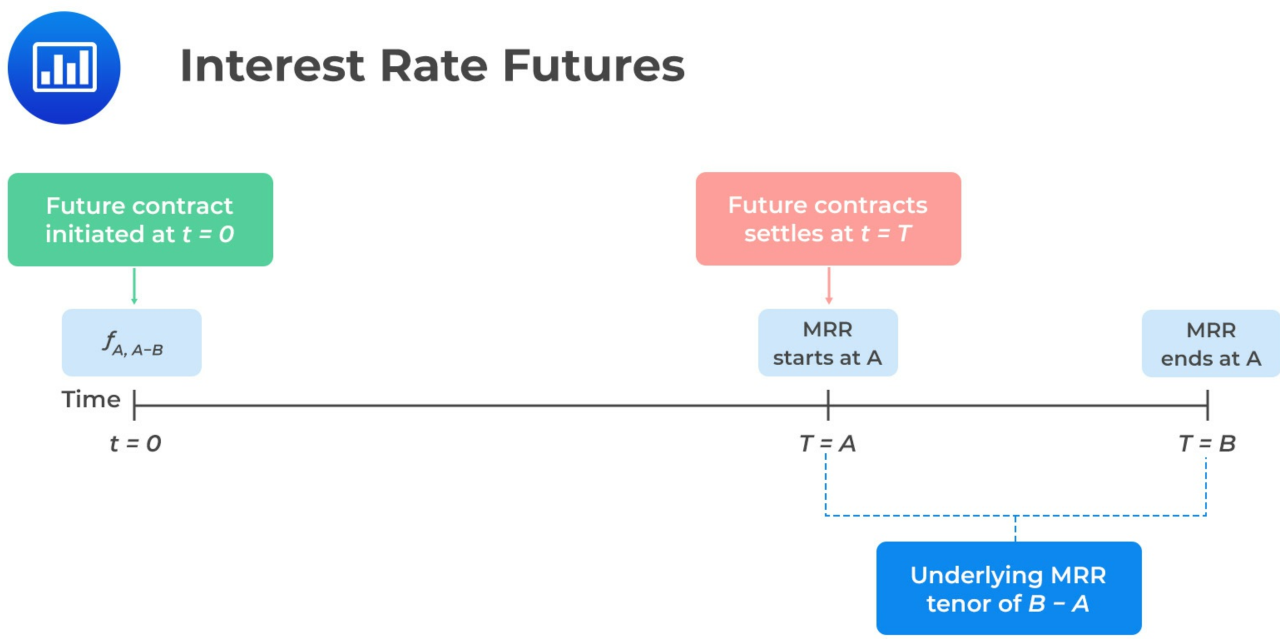
The futures contracts on short-term interest rates are more liquid and standardized than FRAs. These contracts are often available for monthly and quarterly market reference rates (MRRs).

Description of Interest Rate Futures

As is the case in FRA, the underlying of the interest rate futures is the market reference rate on a hypothetical amount of money at a future date. However, interest rate futures trade on a price basis, given by the following formula:

$$f_{A,B-A} = 100 - (100 \times MRR_{A,B-A})$$

Where: $f_{A,B-A} =$ futures price for the market reference rate for B–A periods that begin in A period ($MRR_{A,B-A}$).



Note that the formula $f_{A,B-A} = 100 - (100 \times MRR_{A,B-A})$ can be written as $f_{A,B-A} = 100 - \text{yield}$.

Intuitively, the (100 – Yield) price convention leads to an inverse price versus yield relationship that differs from the price of a zero-coupon bond at a contract rate. As such, a long futures position receives MRR in A period while the short position pays MRR in A period.

In summary, the long position (lender) gains as prices rise and future MRR falls. In contrast, the short position (borrower) gains as prices fall and future MRR rises.

Interest Rate Futures Settlement

The **daily settlement** of the interest futures occurs depending on the price changes, regarded as **futures contract basis point value** (BPV) and calculated as follows:

$$\text{Futures contract BPV} = \text{Notional principlal} \times 0.01\% \times \text{Period}$$

For instance, consider USD 50 million for a 3-month MRR of 3% for 180/360 days. The futures contract BPV is:

$$\text{Futures contract BPV} = 50,000,000 \times 0.01\% \times \left(\frac{180}{360}\right) = \$2,500$$

Example: Calculating Futures Contracts Gains or Losses

A&M Bank has issued its clients a USD 10 million three-month loan at a fixed rate. To finance the loan, the bank has borrowed a one-month variable MRR. To hedge against interest rate risk, the bank sells futures contracts on two-month MRR. Assume that the bank agrees to sell the futures at $97.75, but the actual settlement price is $96.75.

The cumulative gain/loss to the contract from the bank's perspective is *closest* to:

Solution

We need to start by calculating the contract's BPV:

$$\begin{aligned}\text{Futures contract BPV} &= \text{Notional principal} \times 0.01\% \times \text{Period}\\ &= 10,000,000 \times 0.01\% \times \frac{2}{12}\\ &= \$166.67\end{aligned}$$

We need to calculate corresponding market reference rates (MRRs) for both prices. Note that

$$f_{A,B-A} = 100 - (100 \times \text{MRR}_{A,B-A})$$

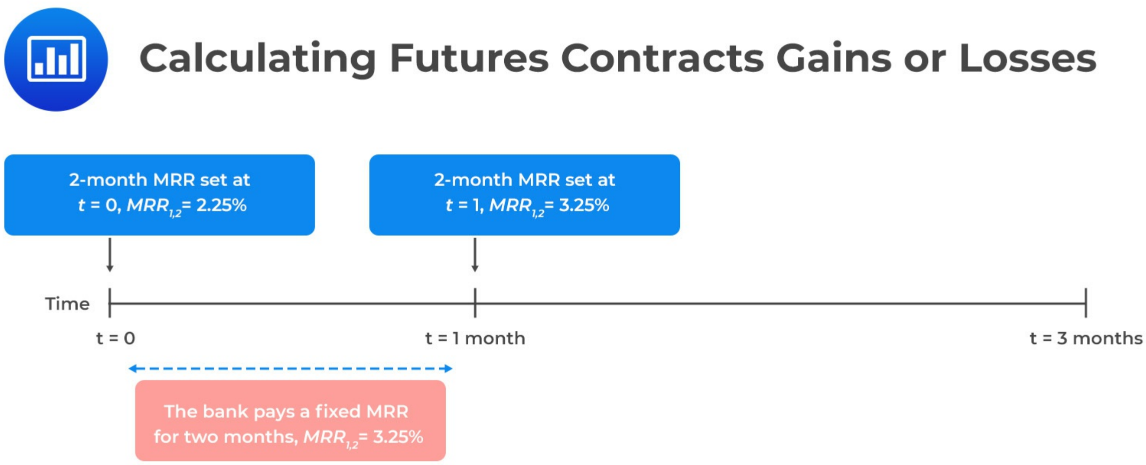
Therefore;

$$\begin{aligned}96.75 = 100 - (100 \times \text{MRR}_{A,B-A}) &\rightarrow \text{MRR}_{1,2} = 3.25\%\\ 97.75 = 100 - (100 \times \text{MRR}_{A,B-A}) &\rightarrow \text{MRR}_{1,2} = 2.25\%\end{aligned}$$

Now, we calculate the cumulative gain/loss:

- Cumulative gain/loss = Change in interest rates × Contract size
- Cumulative gain/loss = 1% × USD 10,000,000 = USD 100,000

So, the cumulative gain/loss to the contract from the bank's perspective is a loss of USD 100,000.



Question

Which of the following *best describes* the difference between the price of a futures contract and its value?

A. The price determines the profit to the buyer, and the value determines the profit to the seller.

B. The futures price is fixed at the start, and the value starts at zero and changes throughout the contract’s life.

C. The futures contract value is a benchmark against which the price is compared to determine whether a trade is advisable.

Solution

The correct answer is **B**.

The futures price is fixed at the start, whereas the value starts at zero and then changes, either positively or negatively, throughout the contract’s life.

LOS 6b: explain why forward and futures prices differ

Forward and futures contracts share similar features; however, how they are traded and the resulting cash flows mean forward and futures contracts with the same underlying asset may trade at a different price.

Causes of Differences

1. Mark-to-Market (MTM), Margining, Settlement of Gains and Losses, and Risks

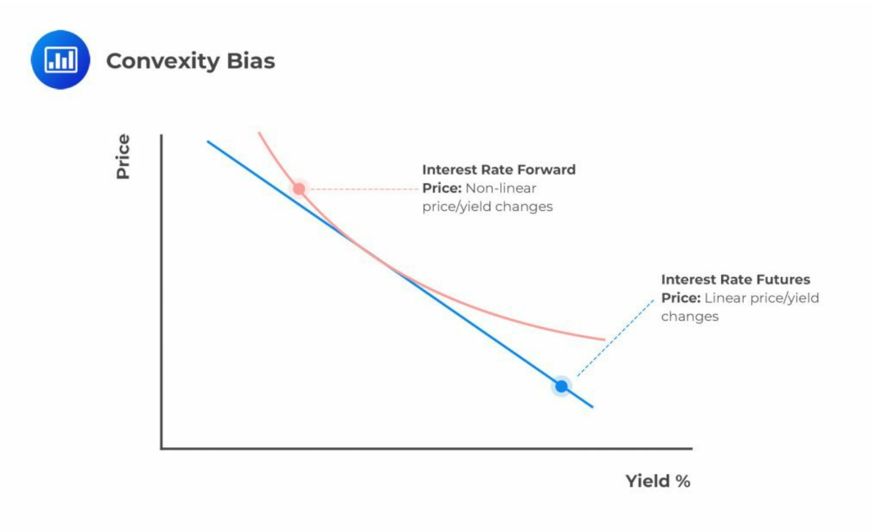
Comparable forwards and futures have symmetric payoff profiles at expiration. However, the pricing and valuation differ over the life of the comparable contracts. Remember that futures are exchange-traded derivative contracts. As such, the distinguishing features of future contracts include posting initial margin, daily mark-to-market, and settlement of gains and losses.

On the other hand, a forward contract is an OTC contract where the credit terms are privately negotiated between the counterparties, and there are no daily mark-to-market (MTM) settlements. Consequently, forward contracts are riskier than futures contracts.

One specific risk is counterparty risk, which is the risk that one party will default on the agreement. A forward contract is more prone to counterparty risk, particularly because settlement only occurs at maturity as a one-time cash settlement.

2. Convexity Bias

Convexity bias occurs when there are different price changes between interest rate futures and forward prices. Interest rate futures have a fixed linear payoff profile for a given change in basis point. On the other hand, interest rate forward (for example, FRAs) have non-linear relation with the basis point change (convexity property).



Consequently, convexity bias causes the percentage price change to be greater in absolute value when MRR decreases than when it increases for a forward contract compared to a futures contract.

Causes of Similarities

1. Relationship between Interest Rates and Futures Prices

Despite the differences in (FRAs) pricing and valuation of futures and forwards, there are instances where their prices are equal. The following assumptions must hold for the futures and forward prices to be identical:

- The interest rates must be constant.
- The futures prices and interest rates are uncorrelated.

What happens when the above assumptions do not hold? For instance, if there is a positive correlation between futures prices and interest rates, the long futures contract is more profitable than the comparable long forward contract. Rising futures prices generate futures profits that are reinvested in periods of rising interest rates. Falling futures prices, on the other hand, attract losses incurred during periods of falling interest rates.

When there is a negative correlation between the futures prices and interest rates, short futures contracts are more attractive than comparable short forward positions. This is because falling futures prices result in profits that are reinvested in periods of high-interest rates. Rising futures prices result in losses incurred during periods of falling interest rates.

2. Central Clearing on OTC Derivatives

The emergence of central clearing of derivatives has resulted in futures-like margining requirements for over-the-counter (OTC) derivative dealers. For instance, dealers are required to post cash or highly liquid securities to a central counterparty. The dealers then impose the same requirements on the derivative end-users.

As such, such a clearing structure on OTC derivatives has reduced the differences in prices between exchange-traded futures and OTC forward contracts.

Question

Which of the following statements is *most likely* true?

A. If there is a positive correlation between futures prices and interest rates, a long futures contract is more profitable than comparable long forward contracts.

B. If futures prices and interest rates are negatively correlated, short forward positions are more attractive than a comparable short futures contract.

C. Central clearing of derivatives increases the difference in futures and forward prices.

Solution

The correct answer is **A**.

When futures prices rise with interest rates, the profits from the long futures position can be reinvested during periods of high interest. On the other hand, losses incurred when futures prices fall occur during decreasing interest rates.

B is incorrect. When there is a negative correlation between the futures prices and interest rates, short futures positions are more attractive than comparable short-forward positions. This is because falling futures prices result in profits that are reinvested in periods of high-interest rates. Rising futures prices result in losses that are incurred during periods of falling interest rates.

C is incorrect. A clearing structure on OTC derivatives has reduced the differences in prices between exchange-traded futures and OTC forward contracts.

Learning Module 7: Pricing and Valuation of Interest Rate and Other Swaps

LOS 7a: describe how swap contracts are similar to but different from a series of forward contracts

Recall that a swap is a derivative contract between two counterparties to exchange a series of future cash flows. In comparison, a forward contract is also an agreement between two counterparties to exchange a single cash flow at a later date. A single-period swap can, therefore, be considered a single-forward contract.

Swaps and forward contracts are similar in that both are forward commitments with symmetric payoff profiles. Besides, in both interest swaps and forward contracts, no cash exchanges hand at initiation.

A distinguishing factor, however, is that the fixed swap rate is constant, while a series of forward contracts have different forward rates at each expiration.

Forward Rate Agreement and Interest Rate Swaps

A forward rate agreement (FRA) is a cash-settled over-the-counter (OTC) contract between two counterparties. In this contract, the buyer (long position) is borrowing a notional sum (underlying) at a fixed interest rate (the FRA rate) and for a specified period starting at an agreed-upon date.

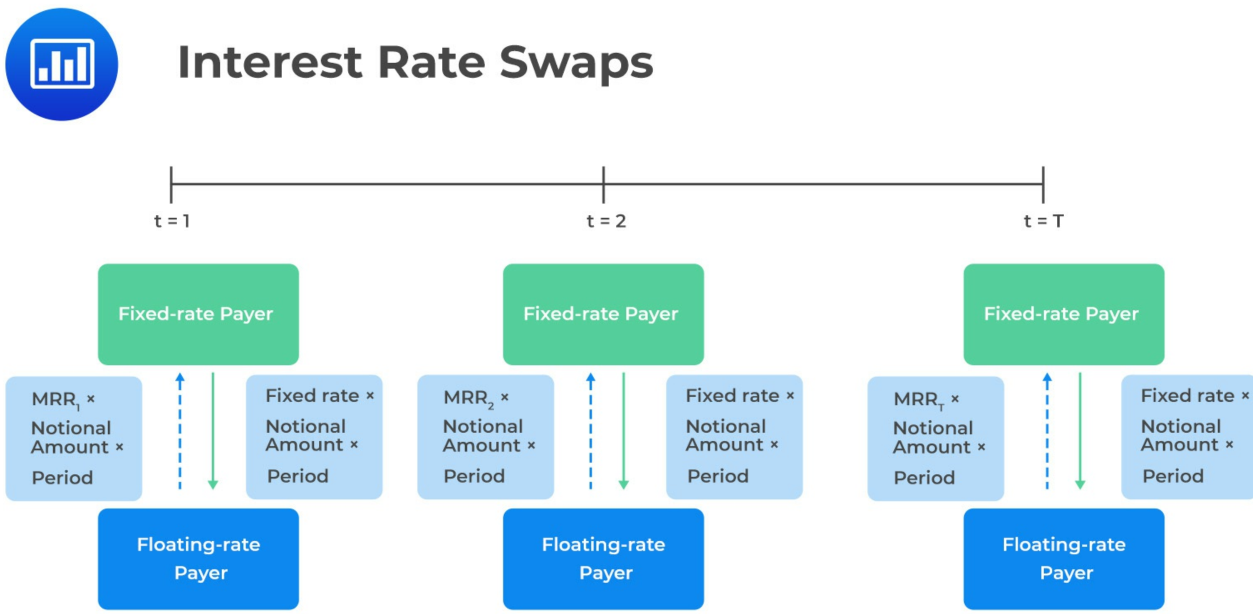
The seller deposits interest based on the market reference rate (MRR), where the MMR is established before the settlement dates, at time $t - A$.

The FRA settlement amount is a function of the difference between the forward interest rate ($IFR_{A,B-A}$) and the market reference rate (MRR_{B-A}) or the MMR for $B - A$ periods.

$$\text{Net payment} = (MRR_{B-A} - IFR_{A,B-A}) \times \text{Notional principal} \times \text{Period}$$

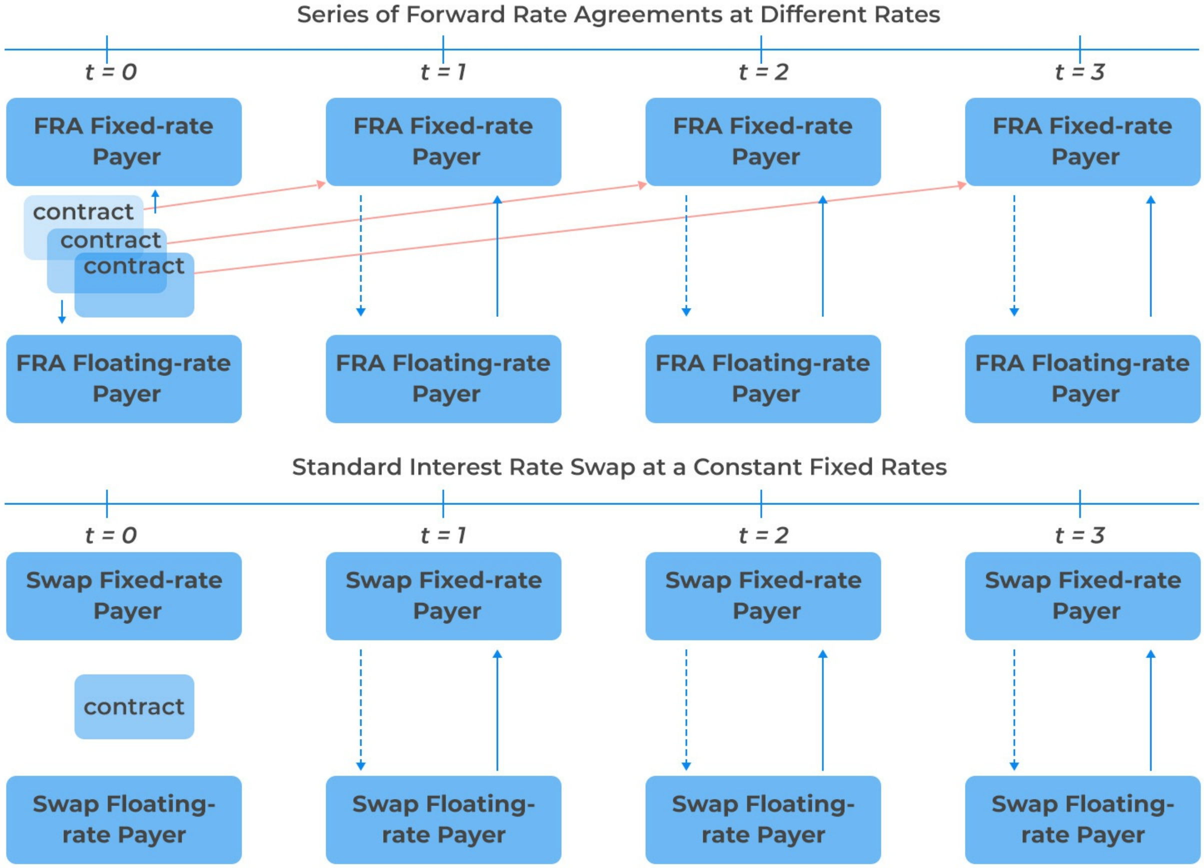
Interest Rate Swaps

In a swap contract, two parties agree to exchange a series of cash flows. In the agreement, one party pays a variable (floating) series of cash flows that will be determined by a market reference rate (MRR) that resets every period. The other party pays either (1) a variable series based on a different underlying asset or rate or (2) a fixed series.



Note that, like a single-period swap, a forward rate agreement (FRA) consists of a single cash flow. As such, a multi-period swap can be viewed as a series of forward rate agreements.

Series of Forward Rate Agreements vs. Interest Rate Swap



In single-period swaps and FRAs, the net difference between the fixed rate agreed on at inception and the market reference rate set in the future is used as the basis for determining cash settlement. Further, note that both FRAs and interest swaps have symmetric payoff profiles, no upfront cash at contract initiation, and counterparty credit exposure.

However, an FRA's single settlement is done at the beginning of the period, while in interest swap rate, periodic settlements occur at the end of the respective period. In addition, looking at a swap as a series of FRAs, we would have a different fixed rate for each future time period. In the case of an interest rate swap, the fixed rate set today would apply throughout the period of the contract.

Par Swap Rate

Note that a standard interest rate swap involves an exchange of fixed payments at a constant rate for a series of floating-rate cash represented by the implied forward rates (IFR) at time $t = 0$.

The **par swap rate** is a fixed rate that equates the present value of all future expected floating cash flows to the present value of fixed cash flows. That is,

$$\sum_{i=1}^{N} \frac{IFR}{(1+Z_i)^i} = \sum_{i=1}^{N} \frac{S_i}{(1+Z_i)^i}$$

Where:

IFR = Implied forward rates.

s_i = Par swap rate for period i.

z_i = Spot rates for period i.

Example: Solving Par Swap (Fixed) Rate

A three-year bond has the following characteristics:

Years to Maturity	Annual Coupon	PV (per 100 FV)	Zero Rates
1	1.25%	99.016	2.2565%
2	2.5%	98.634	3.2282%
3	3.0%	97.222	4.0354%

Determine the par swap (fixed) rate for a three-year contract and the fixed rate for each of the three one-year FRAs that would match the single three-year swap.

We already have the one-year forward rate at $t = 0$ i.e., $IFR_{0,1} = 2.2565\%$. We need to determine the **implied one-year forward rates (IFR)** at $t = 1$ and $t = 2$.

$$(1 + z_A)^A \times (1 + IFR_{A,B-A})^{B-A} = (1 + z_B)^B$$

Therefore, solving for $IFR_{1,1}$

$$\begin{aligned}(1+0.022565)^1 \times (1+IFR_{1,1})^1 &= (1+0.032282)^2 \\ \rightarrow IFR_{1,1} &= \frac{1.032282^2}{1.022565} - 1 \\ &= 0.042091\end{aligned}$$

and $(1+IFR_{2,1})^1$.

$$\begin{aligned}(1+0.032282)^2 \times (1+IFR_{2,1})^1 &= (1+0.040354)^3 \\ \rightarrow IFR_{2,1} &= \frac{1.040354^3}{1.032282^2} - 1 \\ &= 0.056688\end{aligned}$$

Consider the following table:

Years to Maturity	Annual Coupon	PV (per 100 FV)	Zero Rates	IFR
1	1.25%	99.016	2.2565%	2.2565%
2	2.5%	98.634	3.2282%	4.2091%
3	3.0%	97.222	4.0354%	5.6688%

In this case, the par swap rate, is the fixed rate that equates the present value of all future expected floating cash flows to the present value of fixed cash flows:

$$\sum_{i=1}^{N} \frac{IFR}{(1+Z_i)^i} = \sum_{i=1}^{N} \frac{S_i}{(1+Z_i)^i}$$

$$\begin{aligned}\rightarrow \frac{2.2565\%}{1.022565} + \frac{4.2091\%}{1.032282^2} + \frac{5.6688\%}{1.040354^3} &= \frac{s_3}{1.022565} + \frac{s_3}{1.032282^2} + \frac{s_3}{1.040354^3} \\ 0.11191 &= 2.80445 s_3 \\ \therefore s_3 &= \frac{0.11191}{2.80445} \\ &= 0.03990 \approx 3.99\%\end{aligned}$$

The three-year swap rate of 3.99% may be interpreted as a multi-period breakeven rate, or the rate at which an investor would be indifferent to:

- Paying the fixed swap rate and receiving the respective forward rates.
- Receiving the fixed swap rate and paying the respective forward rates.

Question

Which of the following *most likely* distinguishes forward rate agreements and interest rate swaps?

A. Fixed rate at each period.

B. Symmetric payoff profiles.

C. Netting of payments.

Solution

The correct answer is **A.**

Considering a swap as a series of FRAs, we would have a different fixed rate for each future time period. On the other hand, in an interest rate swap, the fixed rate set today would apply throughout the period of the contract.

Remember that interest rates are characterized by term structure, and, as such, we would expect FRAs to have fixed rates for different times to maturity.

B is incorrect. Both FRAs and interest rate swaps have symmetric payoff profiles.

C is incorrect. In single-period swaps and FRAs, the net difference between a fixed rate agreed on at inception, and an MRR (market reference rate) set in the future is used to determine cash settlement on a given notional principal over the specified time period.

LOS 7b: contrast the value and price of swaps

Remember that a swap contract involves a series of periodic settlements with a final settlement at maturity. **Swap price** (or **par swap rate)** is a periodic fixed rate that equates the present value (PV) of all future expected floating cash flows to the PV of fixed cash flows.

The swap rate is equivalent to the forward rate, $F_0(T)$; it satisfies no-arbitrage conditions. On the other hand, the current market reference rate (MRR) is the "spot" price. Therefore, from the fixed-rate payer perspective, the periodic value is given by:

$$\text{Periodic settlement value} = (\text{MRR} - S_N) \times \text{Notional amount} \times \text{Period}$$

The swap value on any settlement date is calculated as the current settlement value using the above formula plus the present value of all the remaining future swap settlements.

Like all other forward commitments, the value of a swap contract at initiation is zero.

Note that it's our assumption that MRR is set at the beginning of each interest period and has the same periodicity and day count as the swap rate. In addition, the net of fixed and floating differences is exchanged at the end of each period.

Examples: Calculating the Swap Value and Effect of Varying MRRs

FinnLay LTD has entered a 4-year interest rate swap with a financial institution with a notional amount of USD 100 million. The contract states that FinnLay signed to receive a semiannual USD fixed rate of 2.5% and, in turn, pay a semiannual market reference rate (MRR).

The MRR is expected to equal the respective implied forward rates (IFRs).

Scenario 1

If at the beginning of the sixth month, the MRR is 0.85%, the first swap settlement value from Finnlay's perspective is *closest to*:

Solution

$$
\begin{aligned}
\text{Periodic settlement value} &= (\text{MRR} - S_N) \times \text{Notional Amount} \times \text{Period} \\
&= (2.5\% - 0.85\%) \times \text{USD } 100\text{m} \times 0.5 \\
&= \text{USD } 0.825\text{m}
\end{aligned}
$$

Scenario 2

If implied forward rates **remain constant** as set at trade inception, how will this affect the MTM value from Finnlay's perspective immediately after the first settlement?

Solution

The swap price (or fixed swap rate) of 2.5% is set at the initiation of the trade, which equates to the PV of fixed versus floating payments.

If there is no change in interest rate expectations, the PV of remaining floating payments rises above the PV of fixed payments.

As such, Finnlay, as a fixed receiver, realizes an MTM loss on the swap because:

$$\sum \text{PV(Floating payments paid)} > \sum \text{PV(Fixed payments received)}$$

Scenario 2

If implied forward rates **decline** just after initiation, how will this affect the MTM value from Finnlay's perspective ?

Solution

A decrease in expected forward rates just after initiation will reduce the PV of floating payments while the fixed swap rate will remain constant.

Since FinnLay is the fixed-rate receiver, it will realize an MTM gain because:

$$\sum \text{PV(Floating payments paid)} < \sum \text{PV(Fixed payments received)}$$

Question

Invest Capital Inc has signed a three-year swap contract to receive a fixed interest rate of 2.5% on a semiannual basis and pay a 6-month USD MRR. The notional amount of the swap contract is USD 100,000.

Assume that the initial 6-month MRR sets at 0.56%, and MRR is expected to be upward sloping. The first settlement value in six months from Invest Capital is *closest* to:

A. $970.

B. $1,940.

C. $2,500.

Solution

The correct answer is **A**.

From the fixed-rate payer perspective, the periodic value is given by:

$$\begin{aligned}\text{Periodic settlement value} &= (S_N - \text{MRR}) \times \text{Notional Amount} \times \text{Period}\\ &= (2.5\% - 0.56\%) \times \text{USD } 100{,}000 \times 0.5\\ &= \$970\end{aligned}$$

B is incorrect. It is calculated as $= 2.5\% - 0.56\% \times$ USD 100,000. It omits the period in the formula.

C is incorrect. It is the amount of the fixed interest amount after six months.

Learning Module 8: Pricing and Valuation of Options

LOS 8a: Explain the exercise value, moneyness, and time value of an option

Options are derivative instruments that give the option buyer the right, but not the obligation, to buy (call option) or sell (put option) an asset from (or to) the option seller at a fixed price on or before expiration.

In other words, options are **contingent claims** that give the option buyer the right but not the obligation to transact the underlying, and the option seller is obligated to meet the obligation chosen by the buyer. The **payoff** of an option is either positive or zero. The **profit**, on the other hand, can be negative because of the option premium.

Exercise Value, Moneyness, and Time Value of an Option

An option's time value and exercise value constitute the value of an option. The **exercise value** of an option would be the value if it were immediately exercisable. On the other hand, an option's **time value** reflects the passage of time and variability of the underlying.

The **moneyness** of an option describes the relationship between the underlying price and the exercise price.

Now we look into these factors, considering the European options with no associated costs or benefits of owning an underlying asset.

Exercise Value of Options at Maturity

Call Options

Remember that in call options, the buyer has the right but not the obligation to buy the underlying. Moreover, the call option will only be exercised if the payoff is positive. Otherwise, the option expires worthless, and the option buyer incurs a loss equal to the option premium paid to the seller.

That is, the buyer would only exercise (buy the underlying) the option if (S_T>X). As such, the payoff (exercise value) to the buyer at expiration is given by:

$$c_T = \max(0, S_T - X)$$

Conversely, the exercise to the seller at expiration is:

$$-c_T = -(\max(0, S_T - X))$$

In other words, the exercise value of a call option is greater than either zero or the underlying price at expiration minus the exercise price.

Put Options

On the other hand, in a put option, the buyer has the right but not an obligation to exercise the option at expiry. Exercising the option means that the buyer sells the underlying S_T at the exercise price X at expiration. As such, the put option is only exercisable if $S_T < X$.

Therefore, the payoff to the buyer is given by:

$$p_T = \max(0, X - S_T)$$

Conversely, the payoff to the put option seller is

$$-p_T = -(\max(0, X - S_T))$$

Exercise Value of Options before Maturity

At any time before maturity ($t < T$), the investors estimate the value of options based on the underlying spot price S_t and the exercise price, X.

The exercise value of a call option is the value of an option contract at any time $t < T$, and it is calculated as spot price (S_t) minus the present value of the exercise price:

$$c_t = \max(0, S_t - X(1+r)^{-(T-t)}$$

Conversely, for the put option, the exercise value at any time $t < T$ is given by;

$$p_t = \max((0, X(1+r)^{-(T-t)} - S_t)$$

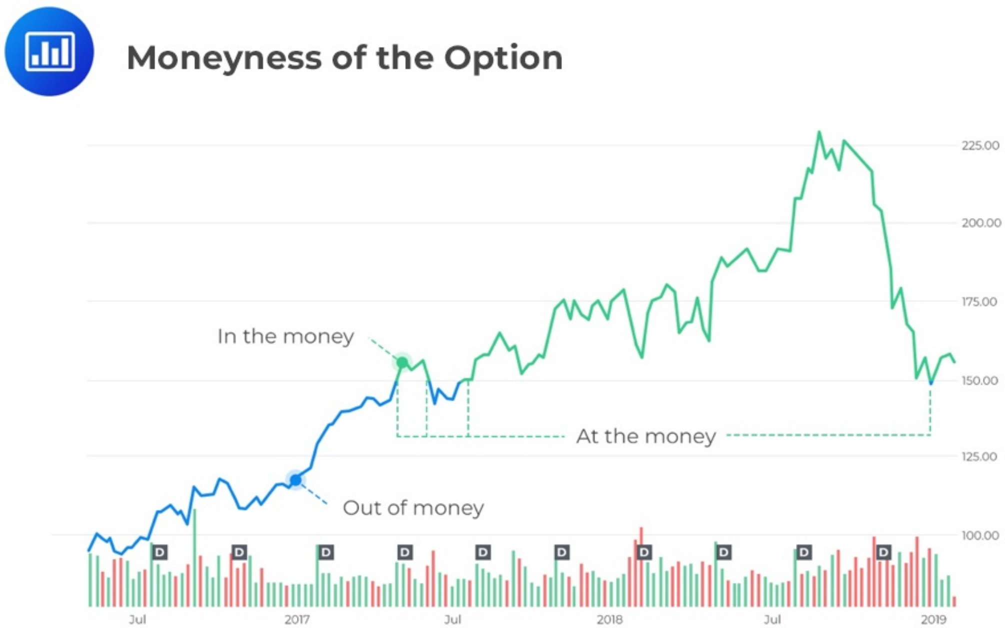
Assuming that $X = F_0(T)$ and ignoring the upfront option premium, the exercise value of a call option is equivalent to the value of the long forward commitment at time t only if the spot price is greater than the present value of X.

That is, if $S_t > PV(X)$, then:

$$S_t - PV(F_0(T)) = \max(0, S_t - PV(X))$$

Moneyness of an Option

Recall that the moneyness of an option is the relationship between the total value of an option and its exercise price. The best way to understand the concept of moneyness is via a graphical example. Below, you can see an example for a USD 150 call option. The green line represents when the option is in the money, and the blue represents times when the option is out of the money. The inverse would be true for a USD 150 put option. For both call and put options, the option is at the money when the line switches from blue to green.



In-the-Money Options

A call is said to be in the money if the underlying spot price is above the exercise price. That is, $S_t > X$.

On the other hand, a put is in the money if the spot price is less than the exercise price. That is, $S_t < X$.

In addition, an option is said to be **deep in the money** if it is highly exercisable, meaning the stock price is much higher than the exercise price.

Out-of-the-Money Options

A call is out of the money if the spot price falls below the current exercise price. That is, $S_T < X$.

On the other hand, a put is out of the money if the spot price is higher than the current exercise price $S_T > X$.

An option is said to be **deep out of the money** if it is unlikely to be exercised, for example, if the stock price is USD 1 and the call option's strike price is USD 50.

At-the-Money Options

A call or a put is called at the money if the option's exercise price equals the current underlying spot price. That is, $S_t = X$.

The moneyness of an option can be summarized in the table below:

Moneyness	Call Options	Put Options
In the Money (ITM)	$S_t > X$	$S_t < X$
At the Money (ATM)	$S_T = X$	$S_T = X$
Out of the Money (OTM)	$S_t < X$	$S_t > X$

Example: Exercise Value of an Option

Consider a 2-year call option with an exercise price of USD 100 and a risk-free rate of 2.31%. If, in six months, the underlying spot price is USD 98, the exercise value is *closest* to:

Solution

$$\begin{aligned} c_t &= \max(0, S_t - X(1+r)^{-(T-t)}) \\ &= \max(0, 98 - 100(1.0231)^{-1.5}) \\ &= \max(0, 98 - 96.63) \\ &= \text{USD } 1.37 \end{aligned}$$

Time Value of an Option

European options are only exercised at the expiration date. However, they can be purchased or sold before maturity at a price that captures the option's future expected payoff.

Denote the price by c_t for the call option and p_t for a put option. The time value of an option is defined as the difference between the current option price and the option's current payoff (or exercise value).

For a call option, the time value is given by:

$$\text{Time value} = c_t - \max(0, S_t - X(1+r)^{-(T-t)})$$

We can rearrange this so that:

$$c_t = \max(0, S_t - X(1+r)^{-(T-t)}) + \text{Time value}$$

For a put option, the time value is given by:

$$p_t = \max(0, X(1+r)^{-(T-t)} - S_t) + \text{Time value}$$

From the formulas above, it is easy to see that the current option price equals the exercise value plus the time value for both call and put options:

$$c_t = \text{Exercise value} + \text{Time value}$$

$$p_t = \text{Exercise value} + \text{Time value}$$

Example: Time Value of an Option

Consider a 2-year call option with an exercise price of USD 100 and a risk-free rate of 2.31%. If, in six months, the spot price of the underlying is USD 98 and the price of the option is USD 1.88, the option's time value is *closest* to:

Solution

$$\begin{aligned} c_t &= \text{Exercise value} + \text{Time value} \\ \text{Time value} &= c_t - \text{Exercise value} \\ &= c_t - \max(0, S_t - X(1+r)^{T-t}) \\ &= 1.88 - 1.37 \\ &= \text{USD } 0.51 \end{aligned}$$

Question

A European put option on an underlying stock has four months to maturity. The option's exercise price is USD 60. At option's maturity, the underlying price is USD 53. The underlying has no associated cost of carry.

If the risk-free rate is 1.5% and the current option price is USD 3, the time value of the option is *closest to*:

A. $0.

B. -$3.703.

C. $3.00.

Solution

The correct answer is **B**.

The time value of a put option is given by:

$$\begin{aligned}
p_t &= \text{Exercise value} + \text{Time value} \\
\text{Time value} &= p_t - \text{Exercise value} \\
&= p_t - \max(0, X(1+r)^{-(T-t)} - S_t) \\
&= 3 - \max((0, 60(1.015)^{-\frac{4}{12}} - 53) \\
&= 3 - \max(0, 6.703) \\
&= 3 - 6.703 \\
&= -3.703
\end{aligned}$$

A is incorrect. It is the payoff of the put option, four months to maturity:

$$\begin{aligned}
\text{Payoff} &= \max(0, X(1+r)^{-(T-t)} - S_t) \\
&= \max(0, 53(1.015)^{-\frac{4}{12}} - 60) \\
&= 0
\end{aligned}$$

C is incorrect.Calculates the payoff of the put option as:

$$
\begin{aligned}
\text{Time value} &= p_t - \max(0, X(1+r)^{-(T-t)} - S_t) \\
&= 3 - \max(0.53(1.015)^{-\frac{4}{12}} - 60) \\
&= 3 - \max(0, -7.2624) \\
&= 3 + 0 \\
&= 3.00
\end{aligned}
$$

LOS 8b: Contrast the use of arbitrage and replication concepts in pricing forward commitments and contingent claims

Recall that arbitrage opportunities occur if the law of one price does not hold. The no-arbitrage conditions in options are based on the payoff at maturity.

Unlike forward commitments with symmetric profiles (as presented earlier), contingent claims have asymmetric payoff profiles. That is:

$$c_T = \max(0, S_T - X)$$
$$p_T = \max(0, X - S_T)$$

Moreover, in contrast to forward commitments with an initial value of zero at initiation, the option buyer pays the seller a **premium** c_0) for call options and p_0 for a put options. Profits at maturity are:

$$\Pi_{call} = \max(0, S_T - X) - c_0$$
$$\Pi_{put} = \max(0, X - S_T) - p_0$$

An option is only exercised when it is in the money. As such, this condition calls for upper and lower no-arbitrage price bounds at any time t.

Upper and Lower Arbitrage Bounds

Call Option

A call option is exercisable if the underlying price exceeds the exercise price. That is $S_t > X$. As such, the lower bound of a call price is the underlying price minus the present value of the exercise price or zero, whichever is greater.

$$\text{Lower bound} = \max(0, S_t - X(1 + r)^{-(T-t)})$$

A call buyer will not pay more than the underlying price for the right to buy the underlying. As such, the upper bound is the current underlying price.

$$\text{Upper bound} = S_t$$

Generally, the no-arbitrage bounds of a call option are stated as follows:

$$\max(0, S_t - X(1+r)^{-(T-t)} < c_t \leq S_t)$$

Put Options

A call option buyer exercises a put option only if $S_T < X$. As such, the upper bound on the put value is thus the exercise price.

$$\text{Upper bound} = X$$

The lower bound is the present value of the exercise price minus the spot price or zero, whichever is greater:

$$\text{Lower bound} = \max(0, X(1+r)^{-(T-t)} - S_t)$$

Generally, the no-arbitrage bounds of a put option are stated as follows:

$$\max(0, X(1+r)^{-(T-t)} - S_t) < p_t \leq X$$

Example: No-arbitrage Bounds of a Call Option

Consider a 3-year call option with an exercise price of USD 100 and a risk-free rate of 1.5%. If, after six months, the spot price of the underlying is USD 105, the no-arbitrage upper and lower bounds are *closest to*:

Solution

For a call option,

$$\begin{aligned}
\text{Lower bound} &= \max(0, S_t - X(1+r)^{-(T-t)}) \\
&= \max(0, 105 - 100(1.015)^{-2.5}) \\
&= \text{USD } 8.65 \\
\text{Upper bound} &= S_t = \text{USD } 105
\end{aligned}$$

Replication in Contingent Claims

Note that replication refers to a strategy in which a derivative's cash flow stream may be recreated using a combination of long or short positions in an underlying asset and borrowing or lending cash.

Replication mirrors or offsets a derivative position, given that the law of one price holds and arbitrage does not exist. A trader can take opposing positions in a derivative and the underlying, creating a default risk-free hedge portfolio and replicating the payoff to a risk-free asset.

Replicating Call Options

Replication of a call option at the contract initiation involves borrowing at a risk-free rate, r, and then utilizing the proceeds to buy the underlying asset at a price of S_0.

At the expiration date $t = T$, there exist two replication outcomes:

- If $S_T > X$, exercise the option: Sell the underlying at S_T and use the proceeds to repay the risk-free loan.
- If $S_T < X$, no exercise: No settlement is needed.

If the exercise of the option is certain, we will borrow $X(1 + r)^{-T}$ just like in forwards. However, the exercisability of the option is not certain. As such, a proportion of $X(1 + r)^{-T}$ is borrowed depending on the likelihood of exercise at time T and linked to the moneyness of an option.

The **non-linear nature of option payoff** requires replicating transactions to be adjusted based on the likelihood of exercise.

Replicating Put Options

Replication of a put option at the contract initiation involves selling the underlying short at a price of S_0 and lending the proceeds at the risk-free rate, r.

At the expiration date ($t = T$), there exist two replication outcomes:

- If $S_T < X$, exercise the option: Buy the underlying at S_T from the proceeds of the risk-free loan.
- If $S_T > X$, no exercise: No settlement is needed.

As with call options, a proportion of $X(1+r)^{-T}$ is borrowed depending on the likelihood of exercise at time T and linked to the moneyness of an option.

Question

A 6-month put option on an underlying stock with no associated costs or benefits has an exercise price of $50. The underlying price at the contract inception is $47, and the risk-free rate is 1.5%. After three months, the underlying stock price is $45.75.

The lower bound of the put option price is *closest* to:

A. $4.06.

B. $50.

C. $45.75.

Solution

The correct answer is **A**.

The lower bound of a put option is given by:

$$\begin{aligned}\text{Lower bound} &= \max(0, X(1+r)^{-(T-t)} - S_t) \\ &= \max(0, 50(1.015)^{-(0.5-0.25)} - 45.75) \\ &= \max(0, 4.064) \\ &= \$4.064\end{aligned}$$

LOS 8c: Identify the factors that determine the value of an option and describe how each factor affects the value of an option

The factors that affect the value of an option include the value of the underlying, exercise price, time to maturity, risk-free rate, volatility, and income or cost associated with the underlying.

Value of the Underlying

The value of the underlying has a direct impact on the right to exercise an option. For a call option, it is exercisable if $S_T > X$. As such, the value of the call option (and long forward) appreciates when the **spot price** of the underlying increases.

In contrast, the put option (and short forward) appreciates when the spot price of the underlying declines. Recall that the put option is in the money if $S_T < X$.

Exercise Price

The exercise price determines whether an option buyer will exercise the option at the expiration. Remember that the payoff of a call option at maturity is $\max(0, S_T - X)$. Intuitively, a lower exercise price will increase both the likelihood of exercise and settlement value if it is in the money.

For the put option, the exercise price is the upper bound of the option price. Moreover, the payoff of a put option is $\max(0, X - S_T)$. As such, a high exercise price increases the value of the put option.

Time to Expiration

The time value of an option represents the likelihood that favorable changes to the underlying price will increase the profitability of the exercise. For both call and put options, a longer time to maturity increases the likelihood of the option finishing in the money, thus increasing the option's value

Risk-Free Rate

A risk-free rate can be seen as the opportunity cost of holding an asset. A risk-free rate is used in the no-arbitrage valuation of derivatives. Note that the value of a call option at any time before maturity $(t < T)$ is given by:

$$c_t = \max(0, S_t - X(1+r)^{-(T-t)})$$

It is easy to see that a higher risk-free rate increases the value of the call option. This is because a higher risk-free rate lowers the present value of the exercise price, provided the option is in the money. For a put option, its value at any time before maturity $(t < T)$ is given by:

$$p_t = \max(0, X(1+r)^{-(T-t)} - S_t)$$

Intuitively, a higher risk-free rate decreases the exercise value of a put option due to the same explanation in the call option.

The Volatility of the Underlying

Volatility measures the expected dispersion of future movements of an underlying asset. Higher volatility of the underlying asset increases the chances of call and put options finishing in the money without affecting the downside case – the option expires worthless. For instance, as volatility increases, a broader possibility of underlying prices increases the time value of an option and the likelihood of being in the money.

In contrast, lower volatility decreases the time value of both put and call options.

Income or Cost Associated with Owning Underlying Asset

Income (or other non-cash benefits) accrue to the underlying asset owner, not the derivative owner. In other words, the present value of the income or benefits is subtracted from the underlying price. As such, income decreases the value of a call option and increases the value of a put option.

If the asset owner incurs costs (in addition to opportunity cost), compensation is done to cover the added costs. As such, the present value of the costs is added to the underlying price. Therefore, cost increases the value of the call option and decreases the value of the put option.

The table below summarizes the factors that affect the value of an option.

Factor	Value of European Call option	Value of European Put option
Value of the Underlying	Directly proportional	Inversely proportional
Exercise price	Inversely proportional (as the exercise price increases, value decreases)	Directly proportional (as exercise price increases, value increases)
Time to Maturity	Directly proportional	Directly proportional
Risk-free rate	Directly proportional	Inversely proportional
Volatility	Directly proportional	Directly proportional
Benefits	Inversely proportional	Directly proportional
Costs	Directly proportional	Inversely proportional

Question

Which of the following is *most likely to have* the same effect on the value of a call option?

A. High risk-free rate and negative cost of carry.

B. Low exercise price and positive cost of carry.

C. Longer time to maturity and low volatility.

Solution

The correct answer is **A**.

Both a high risk-free rate and low cost of carry increase the value of a call option.

The risk-free rate increases the value of the call option because a higher risk-free rate lowers the present value of the exercise price, provided the option is in the money.

Recall that cost of carry is the net of the costs and benefits associated with owning an underlying asset for a period. Therefore, the negative cost of carry implies that the cost associated with the underlying is higher than the benefits. The present value of the costs is added to the underlying price. Therefore, cost increases the value of the call option and decreases the value of the put option.

B is incorrect. A low exercise price will increase both the likelihood of exercise and settlement value if it is in the money.

A positive cost of carry implies that the present value of the benefits associated with the underlying is higher than the present value of the costs. The present value of the income or benefits is subtracted from the underlying price. As such, income or other non-cash decreases the value of a call option and increases the value of a put option.

C is incorrect. The time value of an option represents the likelihood that favorable changes to the underlying price will increase the profitability of the exercise.

Therefore, a longer time to maturity for a call option increases the option’s value.

Lower volatility decreases the time value of both put and call options.

<u>**Learning Module 9: Option Replication Using Put-Call Parity**</u>

LOS 9a: explain put–call parity for European options

Put-call parity is a no-arbitrage concept. It involves a combination of cash and derivative instruments in a portfolio. Put-call parity allows pricing and valuation of these positions without directly modeling them using non-arbitrage conditions.

Deriving Put-Call Parity

Consider an investor whose main objective is to benefit from an increase in the underlying value and hedge an investment against a decrease in underlying value. Consider the following portfolios:

Portfolio A

At time $t = 0$, an investor buys a call option at a price of c_0 on an underlying with an exercise price of X and a risk-free bond that is redeemable at X at time $t = T$. Intuitively, assuming the call option expires at time t=T, the cost of this strategy is

$$c_0 + X(1 + r)^{-T}$$

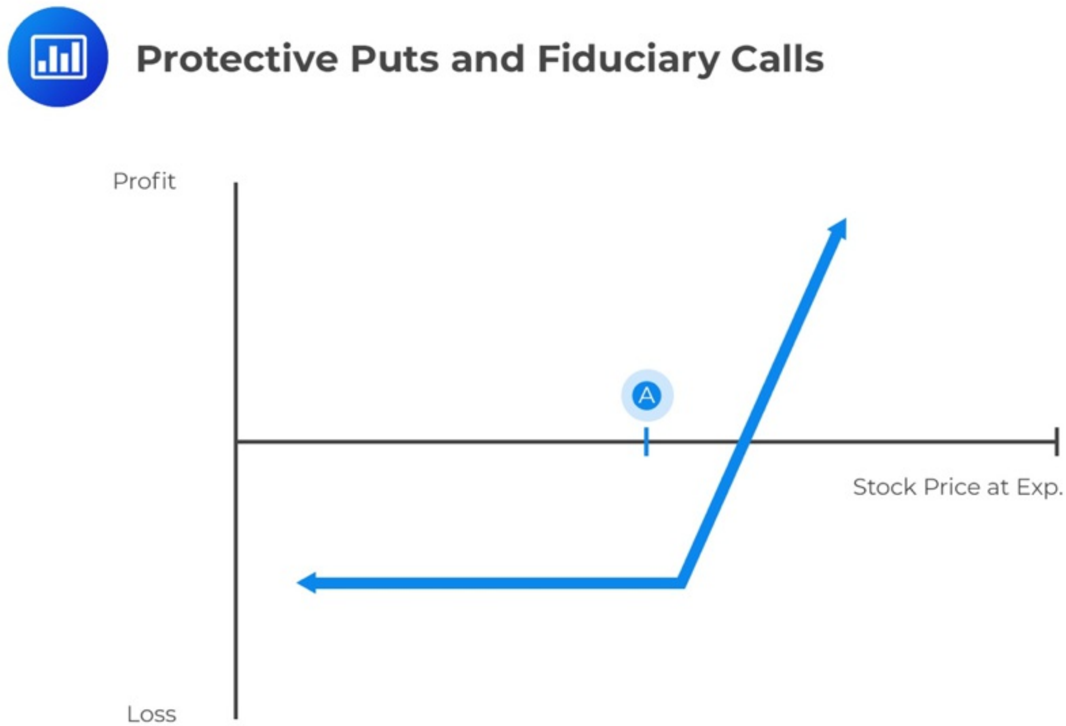
In this portfolio, the investor buys a call option with a positive payoff if the underlying price exceeds the exercise price ($S_T > X$) and invests cash in a risk-free bond. This strategy is called the **fiduciary call**.

Portfolio B

At time $t = 0$, an investor buys an underlying at a price of S_0 and a put option on the underlying price of p_0 whose exercise price is X at time $t = T$. Intuitively, the cost of this strategy is

$$p_0 + S_0$$

The strategy applied in portfolio B is called **protective put**. Protective put involves holding an asset and buying a put on the same asset.



Both portfolios allow the investor to benefit from the rise in underlying price without exposure to a decrease below the exercise price. Moreover, portfolios A and B have identical profiles. Based on the no-arbitrage condition, assets with similar future payoff profiles must trade at the same price, ignoring associated transaction costs. Consider the following table:

Portfolio Position	Put exercised $S_T < X$	No Exercise $S_T = X$	Call Exercised $S_T > X$
Fiduciary Call:			
Call Option	0	0	$S_T - X$
Risk-free Asset	X	X	X
Total:	X	$X(= S_T)$	S_T
Protective Put:			
Underlying Asset	S_T	S_T	S_T
Put option	$X - S_T$	0	0
Total:	X	$S_T(= X)$	S_T

Therefore, since portfolios A and B have identical payoffs at time $t = T$, the costs of these portfolios must be similar at time $t = 0$. For this reason, the put-call parity equation:

$$S_0 + p_0 = c_0 + X(1 + r)^{-T}$$

Where:

S_0 = Price of the underlying asset. p_0 = Put premium. c_0 = Call option premium. X = Exercise price. r = Risk-free rate. T = Time to expiration.

Put-call parity holds for European options that have similar exercise prices and expiration times. These similarities ensure a no-arbitrage relationship between the put option, call option, the underlying asset, and risk-free asset prices. Put-call parity implies that at time $t = 0$, the price of the long underlying asset plus the long put must be equal to the price of the long call option plus the risk-free asset.

Example: Put-Call Parity

Consider European put and call options, where both have an exercise price of $50 and expire in 3 months. The underlying asset is priced at $52 and makes no cash payments during the life of the options.

If the put is selling for $3.80 and the risk-free rate is 4.5%, the price of the call option is *closest* to:

Solution

The put-call parity is given by:

$$S_0 + p_0 = c_0 + X(1 + r)^{-T}$$

We need to rearrange the formula to make the subject of the formula so that:

$$\begin{aligned} c_0 &= S_0 + p_0 - X(1 + r)^{-T} \\ &= 52 + 3.80 - 50(1.045)^{-0.25} \\ &= \$6.35 \end{aligned}$$

Option Replication Using Put-Call Parity

We can rearrange the put-call parity equation to solve for the put option premium, p_0:

$$p_0 = c_0 + X(1+r)^{-T} - S_0$$

The right side of this equation is referred to as a **synthetic put**. It consists of a long call, a short position in the underlying, and a long position in the risk-free bond.

We can make another re-arrangement to solve for a long call, c_0:

$$c_0 = p_0 + S_0 - X(1+r)^{-T}$$

The right side of this equation is equivalent to a call option and is referred to as a **synthetic call**. It consists of a long put, a long position in the underlying asset, and a short position in the risk-free bond.

Also, we can further rearrange the put-call parity as follows:

$$S_0 - c_0 = X(1+r)^{-T} - p_0$$

The right-hand side of the above equation is called the **covered call position**. Intuitively, a covered call is equivalent to a long risk-free bond and short put option.

In summary, synthetic relationships with options occur by replicating a one-part position under put-call parity. Study the following table.

Position	Underlying (S_0)	Risk-free Bond ($(1+r)^{-T}$)	Call Option (c_0)	Put Option (p_0)
Underlying (S_0)	−	Long	Long	Short
Risk-free bond ($\frac{x}{(1+r)^T}$)	Long	−	Short	Long
Call option (c_0)	Long	Short	−	Long
Put Option (p_0)	Short	Long	Long	−

If the put-call parity does not hold, an arbitrage opportunity exists. The arbitrage opportunity can be exploited by selling the most expensive portfolio and purchasing the cheaper one.

Example: Arbitrage Opportunity

A European call option with a strike price of $25 sells at $7. The price of a European put option with the same strike price is also $7. If the underlying stock sells for $28, and the one-year risk-free rate is 4%, determine if there is an arbitrage opportunity.

Solution

The put-call parity equation:

$$\begin{gathered} p_0 + s_0 ? c_0 + X(1+r)^{-T} \\ 7 + 28 ? 7 + 25(1.04)^{-1} \\ 35 \neq 31.0385 \end{gathered}$$

To exploit the opportunity, we need to:

- Sell the right side (**Protective put**) for $35.
- Buy the left side (**fiduciary call**) for $31.0385.

We get a cash inflow of $\$35 - \$31.0385 = \$3.9615$. Thus, the strategy provides cash inflow ($3.9615) today and no cash outflow at expiration.

Question

European put and call options have an exercise price of $50 and expire in three months. The underlying asset is priced at $52 and makes no cash payments during the option's life. The risk-free rate is 4.5%, and the put is selling for $3.80. According to the put-call parity, the price of the call option should be *closest* to:

A. $5.25.

B. $6.35.

C. $7.12.

Solution

The correct answer is **B**.

The put-call parity is given by:

$$S_0 + p_0 = c_0 + X(1 + r)^{-T}$$

Where:

S_0 =Price of the underlying asset.

p_0 = Put premium.

c_0= Call option premium.

X = Exercise price.

r= Risk-free rate.

T = Time to expiration.

Making c_0 the subject, we have:

$$
\begin{aligned}
c_0 &= S_0 + p_0 - X(1 + r)^{-T} \\
&= 52 + 3.80 - 50(1.045)^{-0.25} \\
&= 6.35
\end{aligned}
$$

LOS 9b: explain put–call forward parity for European options

The put-call forward parity extends the put-call parity to include the forward contracts. To get the put-call forward parity, we substitute the present value of the forward price, $F_0(T)$, for the underlying price:

$$F_0(T)(1+r)^{-T}+p_0=c_0+X(1+r)^{-T}$$

Deriving Put-Call Forward Parity

Consider an investor whose main objective is to benefit from an increase in underlying value and hedge against a decrease in underlying value. Consider the following portfolios:

Portfolio A

At time $t = 0$, an investor buys a forward contract and a risk-free bond whose face value is equal to the forward price $F_0(T)$. The investor then puts an option on the underlying at a price of p_0 whose exercise price is X at $t = T$. The cost of this strategy is:

$$F_0(T)(1+r)^{-T}+p_0$$

Portfolio B

At time $t = 0$, an investor buys a call option at a price of c_0 on the same underlying exercise price of X and a risk-free bond redeemable at a price of X at time $t = T$. The cost of this transaction is:

$$c_0+X(1+r)^{-T}$$

Portfolio A is called **synthetic protective put**. Compared to a synthetic put, a synthetic protective put replaces the underlying cash position with a synthetic position using forward purchase and a risk-free bond.

Portfolio B is the same fiduciary call as in the put-call parity seen previously.

Cash flows at time $t = T$ for the synthetic protective put and the fiduciary call are shown in the following table:

Portfolio Position	Put exercised $S_T < X$	No Exercise $S_T = X$	Call Exercised $S_T > X$
Fiduciary Call:			
Purchased Call Option	0	0	$S_T - X$
Risk-free Asset	X	X	X
Total:	X	$X (= S_T)$	S_T
Synthetic Protective Put:			
Purchased Put at p_0	$X - S_T$	0	0
Purchased Forward Contract	$S_T - F_0(T)$	$S_T - F_0(T)$	$S_T - F_0(T)$
Risk-free bond are currently priced as $F_0(T)(1 + r)^{-T}$	$F_0(T)$	$F_0(T)$	$F_0(T)$
Total:	X	$S_T (= X)$	S_T

Since portfolios A and B have identical payoffs at time $t = T$, the costs of these portfolios must be identical at time $t = 0$. Therefore, based on no-arbitrage conditions, the put-call forward parity is given by:

$$F_0(T)(1 + r)^{-T} + p_0 = c_0 + X(1 + r)^{-T}$$

Example: Put-Call Forward Parity

Capital Investments would like to buy a 6-month put option on a company's shares, whose current price is \$195 per share. The exercise price of the put options is \$190.00 per share.

The 6-month call option on the same shares trades at \$64 per share with the same exercise price of \$190.00. Using the put-call forward parity and assuming a 1.5% risk-free rate, the price of the put option is *closest* to:

Solution

Using the put-call forward parity

$$F_0(T)(1 + r)^{-T} + p_0 = c_0 + X(1 + r)^{-T}$$

Making p_0 the subject of the formula, we get:

$$p_0 = c_0 + (X - F_0(T))(1 + r)^{-T}$$

$

We need to solve for $F_0(T)$, which, if you recall, is given by:

$$\begin{aligned} F_0(T) &= S_0(1 + r)^T \\ &= 195(1.015)^{0.5} \\ &= \$196.4571 \end{aligned}$$

As such,

$$\begin{aligned} p_0 &= c_0 + (X - F_0(T))(1.015)^{-T} \\ &= 64 + (190 - 196.4571)(1.015)^{-0.5} \\ &= \$57.59 \end{aligned}$$

Option Put-Call Parity Applications: Firm Value

The put-call parity relationship can be used to define a firm's value based on equity holders' and debt holders' interests.

As a rule of thumb, at time $t = 0$, a company's market value, V_0, is equivalent to the present value of its outstanding debt obligations, $PV(D)$, and equity, E_0, where the borrowed funds are in zero-coupon debt with a face value of D.

In an equation, we can express this relationship as:

$$V_0 = E_0 + PV(D)$$

In the event of debt maturity at $t = T$, the assets and debts of the company will be split between debtholders and shareholders, with two possible outcomes based on the company's value at that given time:

Solvency ($V_T > D$)

Recall that solvency refers to a company's ability to meet its financial obligations and long-term debt. If at time T, a firm's value (V_T) is greater than the face value of the debt, ($V_T > D$) the firm is solvent, and thus able to return capital to both the shareholders and the debtholders.

Debt holders come first when distributing capital returns. As such, they receive the debt repayments (D) in full. On the other hand, the shareholders receive what remains. That is $E_T = V_T - D$.

In summary, we've established that shareholders benefit if a company can meet its debt obligation and maintain solvency. On the other hand, debt holders benefit when a company is solvent and hence meets its debt obligations.

Insolvency ($V_T < D$)

Insolvency refers to a company's inability to meet its financial obligations and long-term debts. This occurs if, at debt maturity (T), a company's value is less than the debt's face value, $V_T < D$.

When a firm is insolvent, the shareholders receive the residual, which is equal to zero ($E_T = 0$), and the debtholders are owed more than the firm's total assets. As such, the debtholders receive $V_T < D$ to cover the debt of D at time T.

Payoff in Terms of Options

Note that shareholders retain the unlimited upside potential in solvency and limited downside potential in insolvency. On the other hand, the debtholders are limited to receiving debt repayment in the case of solvency and principal and interest in the case of insolvency.

Intuitively, the payoff profiles can be mathematically represented as follows:

- At time $t = T$, the payoff of the shareholders can be expressed as $\max(0, D - V_T)$.
- On the other hand, the debt holder's payoff is expressed as $\min(V_T, D)$.

In terms of options, the payoff profiles can be expressed as follows:

- **Shareholders' payoff:** They hold a long position in the underlying firm's assets value.

For instance, assume that we have bought a put option on the firm value, V_T, with the exercise price of D. The payoff is $\max(0, D - V_T)$.

- **Debtholders' payoff:** The hold a long position in the risk-free bond, valued at D, and have sold a put option to the shareholders on the firm value, V_T, with an exercise price of D.

Remember the put-call parity relationship:

$$S_0 + p_0 = c_0 + PV(X)$$

If we replace the underlying asset, (S_0), for the company's value at time 0, (V_0), and further replace the risk-free bond, (X), with debt, \(\left(D\right)), the equation becomes:

$$V_0 + p_0 = c_0 + PV(D)$$

We can also rearrange the formula to solve for the value of the company, (V_0):

$$V_0 = c_0 + PV(D) - p_0$$

From the above results, the shareholders have a payoff equivalent to that of a call option (c_0) on the firm's value. On the other hand, the debtholders hold a position of $(D) - p_0$, which is the risk-free debt plus a short position in a put option.

This put option may be seen as a **credit spread** on a company's debt or the premium above the risk-free rate a company must pay debtholders to bear insolvency risk. The value of the put option to shareholders rises as the probability of insolvency grows.

Question

Which of the following *best* describes the replication of a risk-free bond under the put-call parity?

A. Long underlying, short call option, and long put option.

B. Long underlying, short risk-free bond, and long put option.

C. Short underlying, long risk-free bond, and long call option

Solution

The correct answer is **A.**

Recall that the put-call parity relationship may be expressed as:

$$\begin{aligned} c_0 + X(1+r)^{-T} &= p_0 + s_0 \\ \Rightarrow X(1+r)^{-T} &= p_0 + s_0 - c_0 \end{aligned}$$

The risk-free bond replicating individual positions under put call parity is a long underlying, short call option and long put option.

B is incorrect: Call option individual replication position equals long underlying, short risk-free bond, and long put option.

$$c_0 = p_0 + s_0 - X(1+r)^{-T}$$

C is incorrect: Put option position equals short underlying, long risk-free bond, and long call option.

$$p_0 = c_0 + X(1+r)^{-T} - s_0$$

Learning Module 10: Valuing a Derivative Using a One-Period Binomial Model

LOS 10b: describe the concept of risk neutrality in derivatives pricing

Remember that the value of an option is not affected by the real-world probabilities of the underlying price increments or decrements but rather by the expected volatilities (R^u and R^d), which are required to price an option.

We can compute the value of a call option today by discounting its expected value at expiration at the current risk-free rate, as summarized in the equation below:

$$c_0 = \frac{\pi c_1^u + (1 - \pi)c_1^d}{(1 + r)^T}$$

Similarly, the value of the put option is given by:

$$p_0 = \frac{p_1^u + (1 - \pi)p_1^d}{(1 + r)^T}$$

The **risk-neutral probability** (denoted by π) is defined as the computed probability used in binomial option pricing that equates the discounted weighted sum of the expected values of the underlying to the option's current price. It is calculated using the risk-free rate and the assumed up-and-down gross returns of the underlying, as follows:

$$\pi = \frac{(1 + r) - R^d}{R^u - R^d}$$

Risk-neutral pricing is the process of determining the risk-neutral probability (which is used to calculate the present value of future cash flows) using only the expected volatilities, i.e., (R^u and R^d), and the risk-free rate.

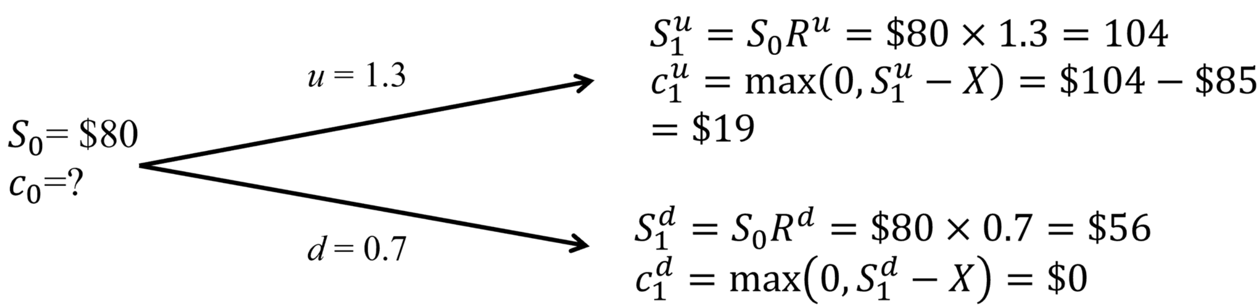
Example: Risk-neutral Probability

A company is considering selling a one-year call option on a non-dividend paying stock whose current price is $80. The exercise price of the call option is $85, and the risk-free interest is 4%.

If the stock price is expected to go up or down by 30%, what is the selling price of the option? (Using risk-neutral pricing).

Solution

Consider the following diagram:



Risk neutral probability of an upward move is given by:

$$\pi = \frac{(1 + r) - R^d}{R^u - R^d} = \frac{1.04 - 0.7}{1.3 - 0.7} = 0.56667$$

Intuitively, the risk-neutral probability of a downward move is given by:

$$1 - \pi = 1 - 0.56667 = 0.43333$$

We need:

$$c_0 = \frac{[c_1^u + (1 - \pi) c_1^d]}{1 + r} = \frac{0.56667 \times 19 + 0.43333 \times 0}{1.04} = \$10.35$$

Question

A European call option that expires in one year has an exercise price of GBP 70. The spot price of the underlying asset is GBP 70. Suppose that the underlying price is expected to increase or decrease by 20% within the next year, assuming a risk-free interest rate of 5%. The no-arbitrage price of a put option on the underlying asset (with similar exercise price and time to maturity) using the binomial model is *closest* to:

A. GBP 10.00.

B. GBP 14.00.

C. GBP 24.04.

Solution

Step 1: Determine the call option's value at maturity.

$$S_1^u = \frac{120}{100} \times 70 = 84$$
$$S_1^d = \frac{80}{100} \times 70 = 56$$
$$c_1^u = \max\ (0,\ S_1^u - X) = \max\ (0,\ 84 - 70) = 14$$
$$c_1^d = \max\ (0,\ S_1^d - X) = \max\ (0,\ 56 - 70) = 0$$

Step 2: Determining *h* (the edge ratio).

$$h = \frac{c_1^u - c_1^d}{S_1^u - S_1^d} = \frac{14 - 0}{84 - 56} = 0.5$$

Step 3: Determine the portfolio value if the price of the underlying increases or decreases.

$$V_1{}^u = hS_1^u - c_1^u = (0.5 \times 84) - 14 = 28$$
$$V_1{}^d = hS_1^d - C_1^d = (0.5 \times 56) - 0 = 28$$

Step 4: **Determining V_0.**

$$V_0 = \frac{V_1}{(1+r)^t} = \frac{28}{(1+0.05)^1} = 26.67$$

Note: $V_1 = V_1^d = V_1^u$.

Step 5: **Determining p_0 and c_0.**

We need,

$$c_0 = \frac{[c_1^u + (1-\pi)\, c_1^d]}{1+r}$$

where

$$\begin{aligned}\pi &= \frac{(1+r) - R^d}{R^u - R^d}\\ &= \frac{(1.05) - 0.8}{1.2 - 0.8}\\ &= 0.625\end{aligned}$$

So,

$$\begin{aligned}c_0 &= \frac{[c_1^u + (1-\pi)\, c_1^d]}{1+r}\\ &= \frac{[14 + (1-0.625)\, 0]}{1.05}\\ &= 13.33\end{aligned}$$

To find p_0 we need to use put-call parity:

$$\begin{aligned}p_0 &= c_0 - S_0 + X(1+r)^{-T}\\ &= 13.33 - 70 + 70(1+0.05)^{-1} = 10.00\end{aligned}$$

LOS 10a: explain how to value a derivative using a one-period binomial model

The law of arbitrage dictates that the value of any two assets (or portfolio of assets) whose payoffs are identical in all possible future scenarios at a given time must also be identical today.

Unlike forward commitments that offer symmetric payoffs at a pre-determined price in the future, contingent claims offer asymmetric payoffs. For this reason, their valuation is a challenge. The binomial model can be used to model the payoffs of contingent claims.

One-Period Binomial Model

The idea behind the binomial model is that at maturity, an asset's spot price, S_0, can either increase to S_1^u or decrease to S_1^d. We do not need to know the asset's future price in advance since it depends on a random variable's outcome. The asset price movements, from S_0 to either S_1^u or S_1^d, can be seen as the outcome of a Bernoulli trial.

Let us denote q as the probability of an increase in the asset's price. Because of the presence of only two possibilities, i.e., an increase or a decrease in the asset's price, we can denote the probability of a decrease in its price as $1 - q$, such that the two probabilities will add up to 1.

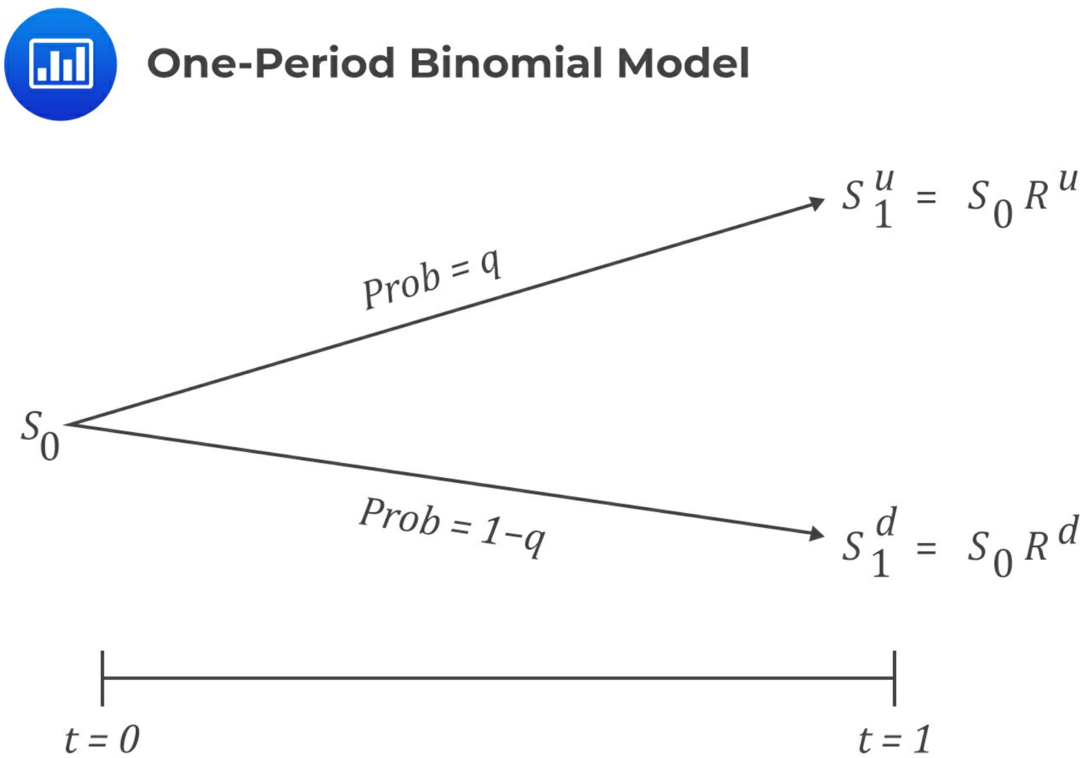
The gross return when the asset price increases will be:

$$R^u = \frac{S_1^u}{S_0} > 1$$

When the asset price decreases, the gross return will be:

$$R^d = \frac{S_1^d}{S_0} < 1$$

The difference between S_1^u (or $R^u S_o$) and S_1^d (or $R^d S_0$) is the spread of possible future price outcomes.



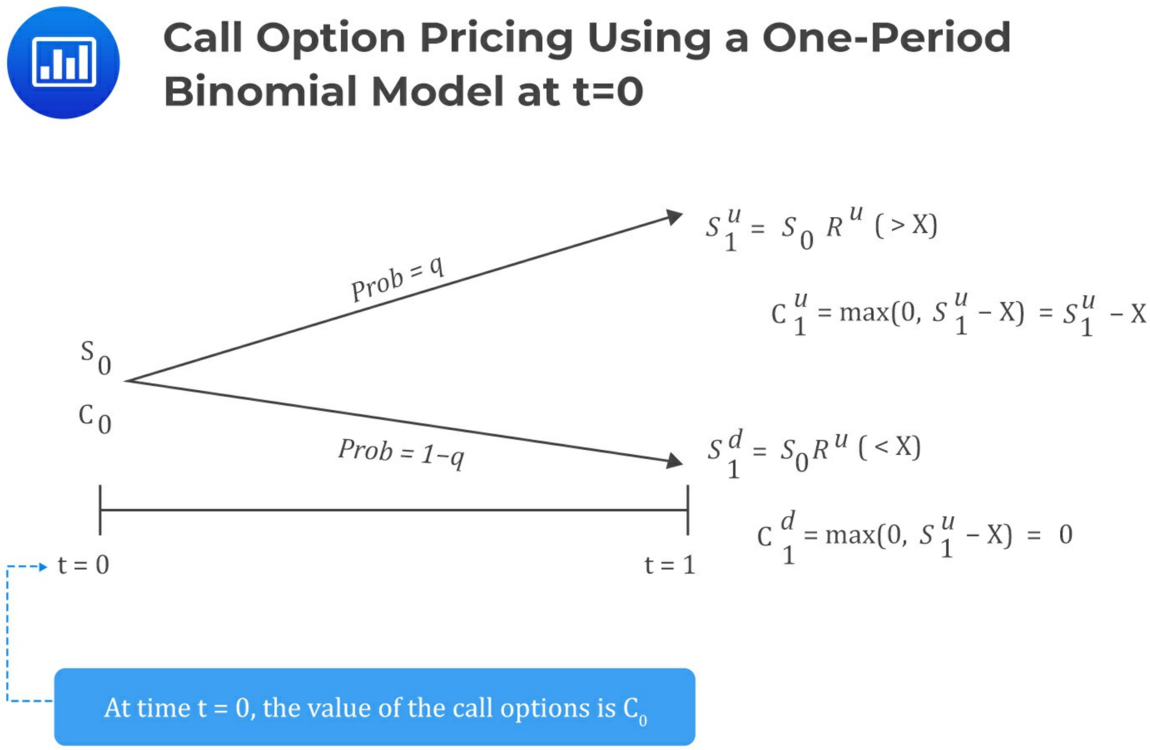
Pricing a European Call Option using a One-Period Binomial Model

Consider a one-year call option with an underlying price of S_0 and an exercise price of X. Also, assume that $S_1^d < X < S_1^u$ and that the one-period binomial model is equivalent to the time of expiration of one year.

The one-period binomial model gives the underlying asset values in one year, where the option value is defined as a function of the underlying value.

At time *t=0*:

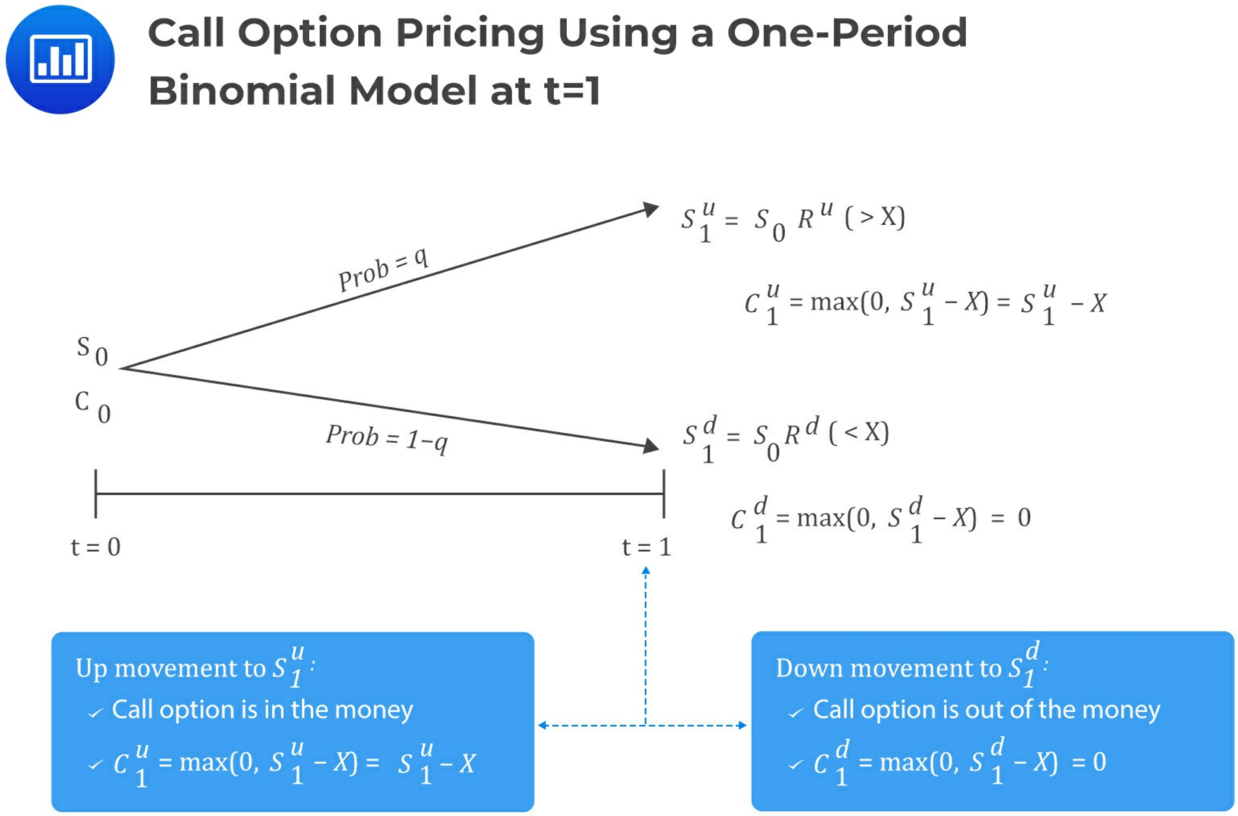
The value of the call option is c_0. Note that this value is unknown and needs to be calculated.



At time *t=1*:

After one year, the option expires. At this time, the value of the option will either be $c_1^u = \max(0, S_1^u - X)$ if the underlying price rises to S_1^u or $c_1^d = \max(0, S_1^u - X)$ if the underlying price falls to S_1^d.

Intuitively, for the up movement, the call option is in the money, and for the down direction, the option is out of the money.



Determining the Value of c_0

The value of c_0 is determined by applying replication and no-arbitrage pricing. Replication implies that the value of the option and its underlying asset in any future scenario may be used to construct a risk-free portfolio.

With that in mind, assume at time $t = 0$, we sell a call option for a price of c_0 and buy h units of the underlying asset. Also, let the value of the portfolio be V so that its value at $t = 0$ is:

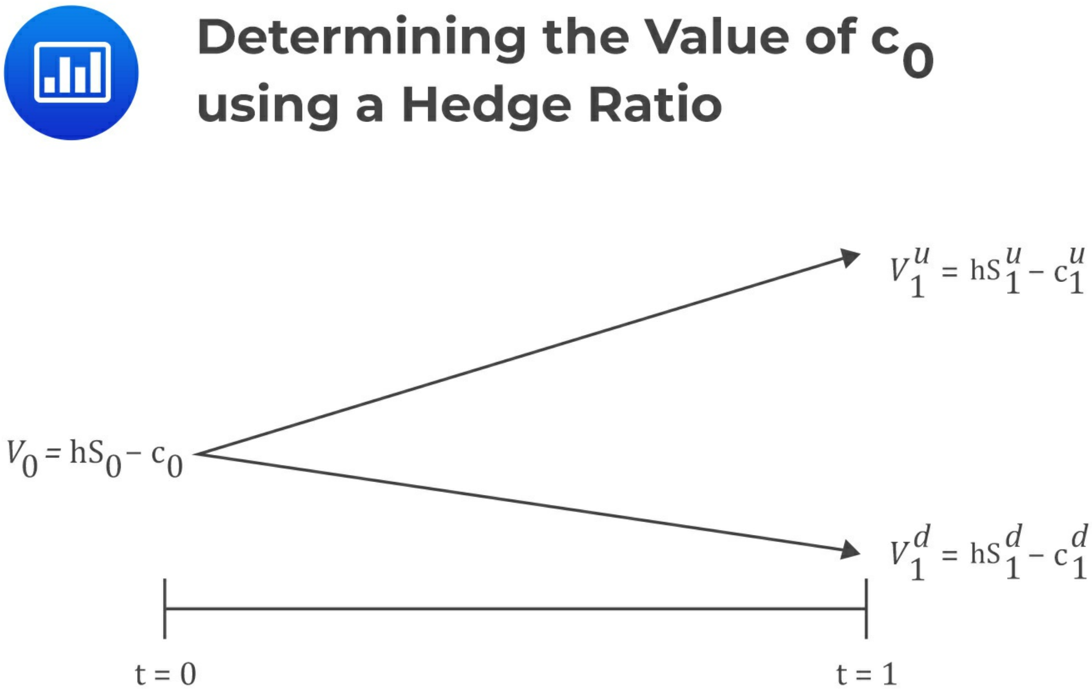
$$V_0 = hS_0 - c_0$$

The value of the portfolio, if the underlying price moves up, is given by:

$$V_1^u = hS_1^u - c_1^u = h \times R^u \times S_0 - \max(0, S_1^u - X)$$

And for the down movement:

$$V_1^d = hS_1^d - c_1^d = h \times R^d \times S_0 - \max(0, S_1^d - X)$$



Assuming no-arbitrage condition, note that we have established two portfolios with identical payoff profiles at time $t = 1$. As such, we need to find the value of h such that:

$$V_1^u = V_1^d$$

Therefore,

$$\Rightarrow hS_1^u - c_1^u = hS_1^d - c_1^d$$

Making h the subject of the formula:

$$h = \frac{c_1^u - c_1^d}{S_1^u - S_1^d}$$

The value h is called the hedge ratio. The hedge ratio is a proportion of the underlying that will offset the risk associated with an option.

Since $V_1^u = V_1^d$, we can draw two conclusions:

- We can utilize either of the two portfolios to value the option.
- The return $\frac{V_1^u}{V_0} = \frac{V_1^d}{V_0} = 1 + r$.

To avoid arbitrage, the portfolio value at $t = 1$, ($V_1 = V_1^u = V_1^d$), must be discounted using a risk-free rate so that:

$$V_0 = V_1(1 + r)^{-1}$$

However, recall that $V_0 = hS_0 - c_0$:

$$\Rightarrow hS_0 - c_0 = V_1(1 + r)^{-1}$$

Making c_0 the subject of the formula:

$$c_0 = hS_0 - V_1(1 + r)^{-1}$$

Example: Pricing Call Option Using One Period Binomial Model

A European call option that expires in one year has an exercise price of $70 and an underlying spot price of $60. Use a one-period binomial model to estimate the call option price if the underlying's spot price is expected to change by 25% in one year. Assume that the risk-free annual rate is 5%.

Solution

Step 1: Determine the Call Option's Value at Maturity *t=1*

At maturity, the value can either be c_1^u if the price of the underlying goes up or c_1^d if the price of the underlying goes down.

The spot price at maturity, if the price goes up by 25%, will be:

$$S_1^u = \frac{125}{100} \times 60 = 75$$

and the gross return will be:

$$R^u = \frac{S_1^u}{S_o} = \frac{75}{60} = 1.25$$

If the price goes down by 25%, the spot price at maturity will be:

$$S_1^d = \frac{75}{100} \times 60 = 45$$

and the gross return will be:

$$R_d = \frac{S_1^d}{S_o} = \frac{45}{60} = 0.75$$

If the underlying's price goes up (the call option expires in the money):

$$c_1^u = \max\ (0,\ S_1^u - X) = \max\ (0,\ 75 - 70) = 5$$

If the underlying's price goes down, and the call option expires out of the money:

$$c_1^d = \max\ (0,\ S_1^d - X) = \max\ (0,\ 45 - 70) = 0$$

Step 2: Determining *h*, the Hedge Ratio

The hedge ratio is given by

$$\begin{aligned} h &= \frac{c_1^u - c_1^d}{S_1^u - S_1^d} \\ &= \frac{5 - 0}{75 - 45} = 0.167 \end{aligned}$$

The hedge ratio of 0.167 implies that we either need to buy 0.167 units of the underlying for every call option we sell or sell 6 call options for each underlying asset to equate the portfolio values at maturity $(t = 1)$. Therefore, the portfolio values when the price of the underlying increases and decreases respectively is:

$$V_1^u = (0.167 \times 75) - 5 = 7.5$$
$$V_1^d = (0.167 \times 45) - 0 = 7.5$$

The portfolio values are the same, implying that either of the portfolios can be used to value the derivative.

Step 4: **Determining** V_0

We can obtain V_0 by discounting V_1 at the risk-free discount rate. Remember that $V_1 = V_1^u = V_1^d$:

$$V_o = 7.5(1 + 0.05)^{-1} = 7.14$$

Step 5: **Determining** c_0

Recall that:

$$c_0 = hS_0 - V_1(1 + r)^{-1} = hS_o - V_0$$

Therefore,

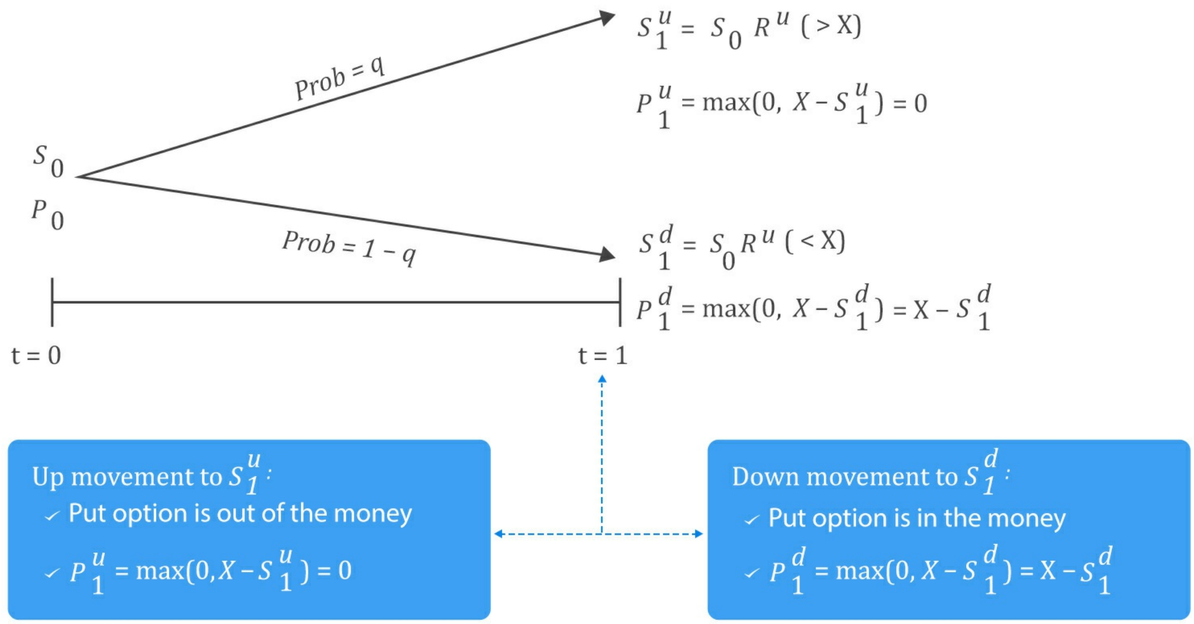
$$c_o = hS_o - V_0 = (0.167 \times 60) - 7.14 = 2.88$$

Note: The hedging approach can be used to value many derivatives, not just call options, provided the derivative's value depends on the underlying asset's value at contract maturity, i.e., $t = 1$

Pricing a European Put Option One-period Binomial Model

For put options, the same explanations we gave under the call option apply, albeit with a different replication strategy. Consider the following diagram:

Put Option Pricing Using a One-Period Binomial Model at t=1



Under the put option, the hedge ratio is given by

$$h = \frac{p_1^u - p_1^d}{S_1^d - S_1^u}$$

Note that the formula remains the same as in the call option, except for the change of notations. Replication in pricing put option using a one-period binomial model involves buying the put option and h units of the underlying so that:

$$V_0 = p_0 + hS_0$$

Therefore,

$$p_0 = V_0 - hS_0$$

Question

Consider a one-year put option on a non-dividend paying stock with an exercise price of $50. The current stock price is $45. The stock price is expected to go up or down by 20%. Calculate the non-arbitrage price of the put option if the risk-free rate of return is 4%.

A. 0

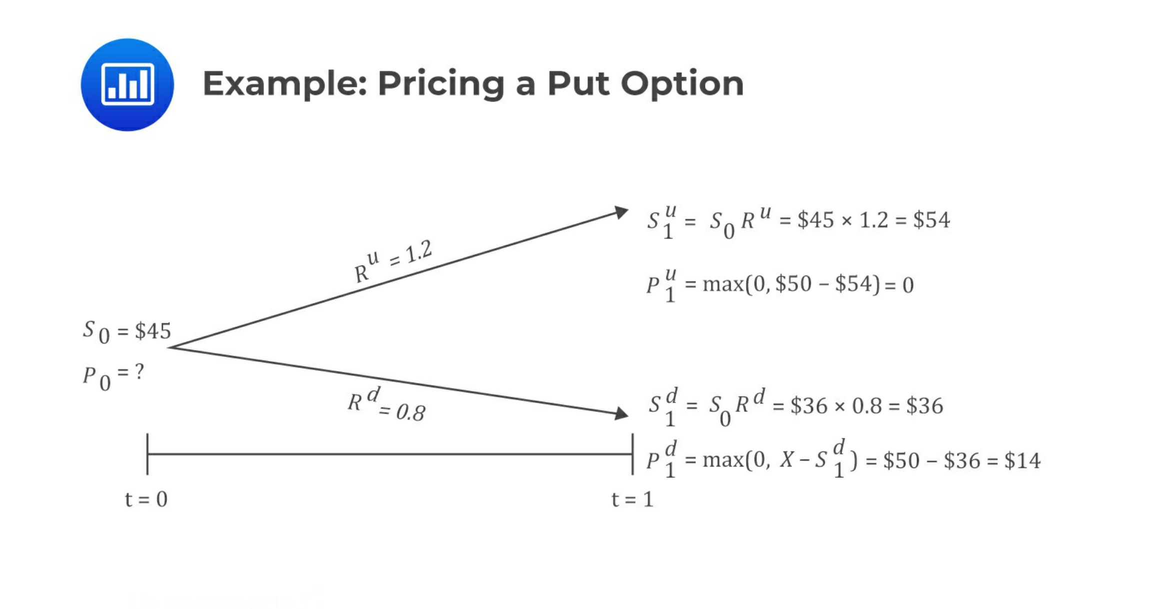
B. $5.38

C. $14.00

Solution

The correct answer is **B.**

Consider the following diagram:



From the above results, the hedge ratio is given by:

$$h = \frac{p_1^u - p_1^d}{S_1^u - S_1^d} = \frac{0 - 14}{54 - 36} = -0.7778$$

We need to calculate $V_1 = V_1^u = V_1^d$, which are:

$$V_1^u = hS_1^u + p_1^u = 0.7778 \times 54 + 0 = \$42.00$$
$$V_1^d = hS_1^d + p_1^d = 0.7778 \times 36 + 14 = \$42.00$$

Next, we can either use V_1^u or V_1^d to calculate the value of V_0:

$$V_0 = V_1(1 + r)^{-1} = 42(1.04)^{-1} = \$40.38$$

Therefore, the put option value is:

$$p_0 = V_0 - hS_0 = 40.38 - 0.7778 \times 45 = \$5.38$$