Learning Module 5: Portfolio Mathematics

LOS 5a: calculate and interpret the expected value, variance, standard deviation, covariances, and correlations of portfolio returns

A portfolio is a collection of investments a company, mutual fund, or individual investor holds. A portfolio consists of assets such as stocks, bonds, or cash equivalents. Financial professionals usually manage a portfolio.

Portfolio Expected Return

To calculate the portfolio's expected return, you take the expected returns of each security in the portfolio. Then, you multiply each security's expected return by its proportion in the portfolio and add them up. The formula below helps you find the portfolio's expected return:

$$E(R_p) = w_1 E(R_1) + w_2 E(R_2) + \cdots w_n E(R_n)$$

Where:

$w_1, w_2, \ldots, w_n$ = Weights (market value of asset/market value of the portfolio) attached to assets $1, 2, \ldots, n$.
$R_1, R_2, \ldots, R_n$ = Expected returns for assets $1, 2, \ldots, n$.

Example: Portfolio Expected Return

Assume we have a simple portfolio of two mutual funds, one invested in bonds and the other invested in stocks. Let us further assume that we expect a stock return of 8% and a bond return of 6%, and our allocation is equal in both funds. The expected return would be calculated as follows:

$$E(Rp) = (0.5 \times 0.08) + (0.5 \times 0.06) = 0.07 \text{ or } 7\%$$

Portfolio Variance

The variance of a portfolio's return is a function of the individual asset covariances as well as the covariance between each of them.

Consider a portfolio with three assets: A, B, and C. The portfolio variance is given by:

$$\begin{aligned}\text{Portfolio Variance} \\ &= W_A^2\sigma^2(R_A) + W_B^2\sigma^2(R_B) + W_C^2\sigma^2(R_C) + 2(W_A)(W_B)\text{Cov}(R_A, R_B) \\ &+ 2(W_A)(W_C)\text{Cov}(R_A, R_C) + 2(W_B)(W_C)\text{Cov}(R_B, R_C)\end{aligned}$$

If we have two assets, A and B, then:

$$\text{Portfolio Variance} = W_A^2\sigma^2(R_A) + W_B^2\sigma^2(R_B) + 2(W_A)(W_B)\text{Cov}(R_A, R_B)$$

Where:

W_A = Weight of assets A in the portfolio.

W_B = Weight of assets B in the portfolio.

$\sigma^2(R_A)$ = Variance of the returns on assets A.

$\sigma^2(R_B)$ = Variance of the returns on assets B.

Portfolio variance is a measure of risk. The higher the variance, the higher the risk. Investors usually reduce the portfolio variance by choosing assets with low or negative covariance, e.g., stocks and bonds.

Portfolio Standard Deviation

Portfolio standard deviation is simply the square root of the portfolio variance. It is a measure of the riskiness of a portfolio.

Considering a portfolio with two assets, A and B, the portfolio standard deviation is given by:

$$\text{Standard deviation} = \sqrt{(W_A^2\sigma^2(R_A) + W_B^2\sigma^2(R_B) + 2(W_A)(W_B)\text{Cov}(R_A, R_B)}$$

Covariance

Covariance is a measure of the degree of co-movement between two random variables. The general formula used to calculate the covariance between two random variables, X and Y is:

$$Cov(X, Y) = \sigma(X, Y) = E[(X - E[X])(Y - E[Y])$$

Where:

$Cov(X, Y)$ = Covariance of X and Y.

$E[X]$ = Expected value of the random variable X.

$E[Y]$ = Expected values of the random variable Y.

This formula calculates the population covariance. It does this by taking the probability-weighted average of the cross-products of the random variables' deviations from their expected values for every possible outcome.

Sample Covariance

The sample covariance between two variables, X and Y, based on a sample data of size n is:

$$Cov(X, Y) = \sum_{i=1}^{n} \frac{(X_i - \bar{X})(Y_i - \bar{Y})}{n - 1}$$

Where:

$\bar{X}$ = Sample mean of X.

$\bar{Y}$ = Sample mean of Y.

X_i and Y_i = i-th data points of X and Y, respectively.

The covariance between two random variables can be positive, negative, or zero.

- A positive number indicates co-movement. The variables tend to move in the same direction.

- A value of zero indicates no relationship.

- A negative value shows that the variables move in opposite directions.

Covariance Matrix

A covariance matrix displays a complete list of covariances between assets needed to calculate the portfolio variance. Consider a portfolio with three assets A, B, and C. The covariance matrix is as follows:

Asset	A	B	C
A	$Cov(R_A, R_A)$	$Cov(R_A, R_B)$	$Cov(R_A, R_C)$
B	$Cov(R_B, R_A)$	$Cov(R_B, R_B)$	$Cov(R_B, R_C)$
C	$Cov(R_C, R_A)$	$Cov(R_C, R_B)$	$Cov(R_C, R_C)$

The off-diagonal (bolded) terms represent variances since, for example:

$$Cov(R_A, R_A) = \rho(A, A)\sigma_A\sigma_A = 1?\sigma_A^2 = \sigma_A^2$$

As such, the table above transforms:

Asset	A	B	C
A	σ_A^2	$Cov(R_A, R_B)$	$Cov(R_A, R_C)$
B	$Cov(R_B, R_A)$	σ_B^2	$Cov(R_B, R_C)$
C	$Cov(R_C, R_A)$	$Cov(R_C, R_B)$	σ_C^2

Intuitively, a three-asset portfolio would have $3 \times 3 = 9$ entries of covariances. However, we do not count the off-diagonal terms since they contain the individual variances of the assets. As such, we have $6(= 9 - 3)$ covariances.

Note that:

$$\begin{aligned} Cov(R_B, R_A) &= Cov(R_A, R_B) \\ Cov(R_A, R_C) &= Cov(R_A, R_C) \\ Cov(R_C, R_B) &= Cov(R_B, R_C) \end{aligned}$$

Therefore, there are $\frac{6}{2} = 3$ distinct covariance terms in the above covariance matrix.

In general, if we have n securities in a portfolio, there are $\frac{n(n-1)}{2}$ distinct covariances and n variances to estimate.

Correlation

Correlation is the covariance ratio between two random variables and the product of their two standard deviations. The correlation formula for random variables X and Y is:

$$\begin{aligned}\text{Correlation }(X,Y) = \text{Corr}(X,Y) = \rho(X,Y) \\ &= \frac{\text{Cov}(X,Y)}{\text{Standard deviation}(X) \times \text{Standard deviation}(Y)} \\ &= \frac{\text{Cov}(X,Y)}{\sigma_X \sigma_Y}\end{aligned}$$

Correlation measures the strength of the linear relationship between two variables. While the covariance can take on any value between negative infinity and positive infinity, the correlation is always between -1 and +1:

- +1 indicates a perfect linear relationship (i.e., the two variables move in the same direction with equal unit changes).
- Zero indicates no linear relationship at all.
- -1 indicates a perfect inverse relationship, i.e., a unit change in one means that the other will have a unit change in the opposite direction.

Example: Calculating Correlation Coefficient from the Covariance Matrix #1

Harrison is a portfolio manager who oversees three assets: A, B, and C. The covariance matrix of these assets is shown below:

Asset	A	B	C
A	0.04	0.02	0.01
B	0.02	0.05	0.015
C	0.01	0.015	0.09

Using this information, what is the correlation coefficient between assets B and C?

Solution

Note:

$$\text{Correlation }(B, C) = \frac{\text{Cov}(B, C)}{\sigma_B \sigma_C}$$
$$= \frac{0.015}{\sqrt{0.05 \times 0.09}} = 0.224$$

Example: Calculating the Correlation Coefficient #2

We expect a 15% chance that ABC Corp's stock returns for the next year will be 6%. There's a 60% probability that they will be 8% and a 25% probability of a 10% return. The expected return is 8.2%, and the standard deviation is 1.249%.

We also anticipate that the same probabilities and states are associated with a 4%, 5%, and 5.5% return for XYZ Corp. The expected value of returns is then 4.975%, and the standard deviation is 0.46%.

To calculate the covariance and the correlation between ABC and XYZ returns, then:

$$\begin{aligned}\text{Cov}(R_{ABC}, R_{XYZ}) &= 0.15(0.06 - 0.082)(0.04 - 0.04975)\\ &+ 0.6(0.08 - 0.082)(0.05 - 0.04975)\\ &+ 0.25(0.10 - 0.082)(0.055 - 0.04975)\\ &= 0.0000561\end{aligned}$$

$$\text{Correlation}(Ri, Rj) = \frac{\text{Covariance}(R_{ABC}, R_{XYZ})}{\text{Standard deviation}(RABC) \times \text{Standard deviation}(RXYZ)}$$

Therefore:

$$\text{Correlation} = \frac{0.0000561}{(0.01249 \times 0.0046)} = 0.976$$

The correlation between the returns of the two companies is very strong (almost +1), and the returns move linearly in the same direction.

Example: Calculating Correlation Coefficient #3

An analyst studied five years of historical data to examine how changes in Central Bank interest rates affect the country's inflation rate. The covariance between the interest rate and inflation rate is -0.00075. The standard deviation of the interest rate is 5.5%, and the inflation rate is 12%. Now, let's calculate and interpret the correlation between these two variables.

Solution

$$\text{Correlation}_{\text{Interest rate, Inflation}} = \frac{\text{Covariance}_{\text{Interest Rate, Inflation}}}{\text{Standard deviation}_{\text{Interest rate}} \times \text{Standard deviation}_{\text{Inflation}}}$$

$$\text{Correlation}_{\text{Interest rate, Inflation}} = \frac{-0.00075}{(0.055 \times 0.12)} = -0.11364$$

A correlation of -0.11364 indicates a negative correlation between the interest rate and the inflation rate.

Note that if we consider, say, assets A and B, then:

$$\text{Corr}(A, B) = \rho(A, B) = \frac{\text{Cov}(A, B)}{\sigma_A \sigma_B}$$
$$\Rightarrow \text{Cov}(A, B) = \sigma_A \sigma_B \rho(A, B)$$

Consequently, in the formula for calculating portfolio variance, consisting of two assets, A and B, we substitute for $\text{Cov}(A, B)$ so that:

$$\text{Portfolio Variance} = W_A^2 \sigma^2(R_A) + W_B^2 \sigma^2(R_B) + 2(W_A)(W_B)\sigma_A \sigma_B \rho(A, B)$$

Question

Assume that we have investments in two companies, ABC and XYZ. For ABC, there's a 15% chance of a 6% return, a 60% chance of an 8% return, and a 25% chance of a 10% return. The expected return for ABC is 8.2%, and the standard deviation is 1.249%. For XYZ, there are similar probabilities of 4%, 5%, and 5.5% returns. The expected return for XYZ is 4.975%, and the standard deviation is 0.46%.

Assuming equal weights, the portfolio standard deviation is *closest to*:

A. 0.0000561.

B. 0.00007234.

C. 0.00851.

The correct answer is C.

$$\text{Portfolio Variance} = W_A^2\sigma^2(R_A) + W_B^2\sigma^2(R_B) + 2(W_A)(W_B)\text{Cov}(R_A, R_B)$$

First, we must calculate the covariance between the two stocks:

$$\begin{aligned}\text{Cov}(R_{ABC}, R_{XYZ}) &= 0.15(0.06 - 0.082)(0.04 - 0.04975)\\ &+ 0.6(0.08 - 0.082)(0.05 - 0.04975)\\ &+ 0.25(0.10 - 0.082)(0.055 - 0.04975)\\ &= 0.0000561\end{aligned}$$

Since we already have the weight and the standard deviation of each asset, we can proceed and calculate the portfolio variance:

$$\begin{aligned}\text{Portfolio variance} &= 0.5^2 \times 0.01249^2 + 0.5^2 \times 0.0046^2\\ &+ 2 \times 0.5 \times 0.5 \times 0.0000561\\ &= 0.00007234\end{aligned}$$

Therefore, the standard deviation is:

$$\sqrt{0.00007234} = 0.00851$$

LOS 5b: Calculate and interpret the covariance and correlation of portfolio returns using a joint probability function for returns

Historical covariance or other techniques, such as market model regression with historical return data, can help us forecast return covariance and correlation. We use the joint probability function of the random variables for this estimation.

The probability that values of the two random variables X and Y will occur simultaneously is given by the joint probability function of X and Y, denoted as $P(X, Y)$. For instance, $P(X = 3, Y = 4)$ represents the likelihood that X and Y will be equal to 3 and 4, respectively.

Covariance can be defined as a probability-weighted average of the cross-products of each random variable's deviation from its own expected value. That is

$$\begin{aligned} \text{Cov}_{(}X_iY_i) &= E[(X_i - \bar{X})(Y_j - \bar{Y})] \\ &= \sum_i \sum_j P(X = x_i, Y = y_j)(X_i - \bar{X})(Y_j - \bar{Y}) \end{aligned}$$

This formula calculates the covariance between random variables X and Y, such as portfolio returns.

To find it, we take the sum of the products of the deviations of X and Y from their expected values for all possible outcomes.

Each product is weighted by the probability of that specific outcome occurring.

Independence and Correlation

Two random variables, X and Y, are independent if $P(X, Y) = P(X) \cdot P(Y)$. That is, X and Y are independent. We find the product of independent probability to calculate joint probability.

The independence property is stronger than correlation because the correlation coefficient addresses linear relationships.

If random variables X and Y are uncorrelated (also holds for independent random variables), then:

$$E(XY) = E(X) \cdot E(Y)$$

Example: Calculating the Covariance #1

Suppose we wish to find the variance of each asset and the covariance between the returns of ABC and XYZ, given that the amount invested in each company is $1,000.

This table is used to calculate the expected returns:

	Strong Economy	Normal Economy	Weak Economy
Probability	15%	60%	25%
ABC Returns	40%	20%	0%
XYZ Returns	20%	15%	4%

Solution

For us to find the covariance, we must calculate the expected return of each asset as well as their variances. The assets' weights are:

$$W_{ABC} = \frac{1000}{2000} = 0.5$$
$$W_{XYZ} = \frac{1000}{2000} = 0.5$$

Next, we should calculate the individual expected returns:

$$E(R_{ABC}) = 0.15 \times 0.40 + 0.60 \times 0.2 + 0.25 \times 0.00 = 0.18$$
$$E(R_{XYZ}) = 0.15 \times 0.2 + 0.60 \times 0.15 + 0.25 \times 0.04 = 0.13$$

Finally, we can compute the covariance between the returns of the two assets:

$$\begin{aligned} Cov(R_{ABC,XYZ}) &= 0.15(0.40 - 0.18)(0.20 - 0.13) \\ &+ 0.6(0.20 - 0.18)(0.15 - 0.13) \\ &+ 0.25(0.00 - 0.18)(0.04 - 0.13) \\ &= 0.0066 \end{aligned}$$

Example: Calculating the Covariance #2

A portfolio manager is considering the following two possible economic growth of a country and the joint variability of returns on two stocks in a portfolio:

Economic Growth	< 4%	> 4%
Probability	40%	60%
Return of Stock A	2.3%	8%
Return of Stock B	6.5%	3%

What is the covariance between the return of Stock A and Stock B?

Solution

$$\text{Expected return of Stock A} = (40\% \times 2.3\%) + (60\% \times 8\%) = 5.72\%$$
$$\text{Expected return of Stock B} = (40\% \times 6.5\%) + (60\% \times 3\%) = 4.40\%$$

Note: For the rest of the calculation, your curriculum sometimes ditches the percentage signs so that 4.40% becomes simply 4.40.

The deviations of returns at the economic growth of

$$< 4\% = (2.3 - 5.72) \times (6.5 - 4.40) = ?7.182$$

The deviations of returns at the economic growth of

$$> 4\% = (8 - 5.72) \times (3 - 4.40) = ?3.192$$

The covariance of returns between stock A and stock B is computed as follows:

$$\text{Cov}(R_{A,B}) = (-7.182 \times 0.40) + (-3.192 \times 0.60) = -4.788$$

Since covariance is negative, the two returns show some co-movement in opposite signs.

Question

The following table represents the estimated returns for two motor vehicle production brands – TY and Ford, in 3 industrial environments: strong (50% probability), average (30% probability), and weak (20% probability).

	TY Returns +6%	TY Returns +3%	Y Returns −1%
Ford Sales + 10%	Strong(0.5)		
Ford Sales + 4%		Average(0.3)	
Ford Sales − 4%			Weak(0.2)

Given the above joint probability function, the covariance between TY and Ford returns is *closest to*:

A. 0.054.

B. 0.1542.

C. 0.1442.

Solution

The correct answer is C.

First, we must start by calculating the expected return for each brand:

The expected return for TY:

$$\begin{aligned} &= (0.5 \times 6\%) + (0.3 \times 3\%) + (0.2 \times (-1\%)) \\ &= 3\% + 0.9\% - 0.2\% = 3.7\% \end{aligned}$$

The expected return for Ford:

$$\begin{aligned} &= (0.5 \times 10\%) + (0.3 \times 4\%) + (0.2 \times (-4\%)) \\ &= 5\% + 1.2\% - 0.8\% = 5.4\% \end{aligned}$$

Next, we can now compute the covariance:

$$
\begin{aligned}
\text{Covariance} &= 0.5(6\% - 3.7\%)(10\% - 5.4\%) \\
&+ 0.3(3\% - 3.7\%)(4\% – 5.4\%) \\
&+ 0.2(-1\% - 3.7\%)(-4\% – 5.4\%) \\
&= 5.29\% + 0.294\% + 8.836\% \\
&= 0.1442
\end{aligned}
$$

The covariance is positive. This means that the returns for the two brands show some co-movement in the same direction.

In real life, this scenario is highly likely because the companies belong to the same industry. As a result, they share similar systematic risks.

LOS 5c: define shortfall risk, calculate the safety-first ratio, and identify an optimal portfolio using Roy's safety-first criterion

Modern Portfolio Theory (MPT) evaluates investment options based on mean return and return variance. This approach is applicable when investors are risk-averse, meaning they seek to maximize their expected satisfaction or utility from their investments.

Mean-return analysis holds under two assumptions:

i. Returns follow a normal distribution.
ii. Investors have quadratic utility functions, a mathematical model representing the balance between risk and return.

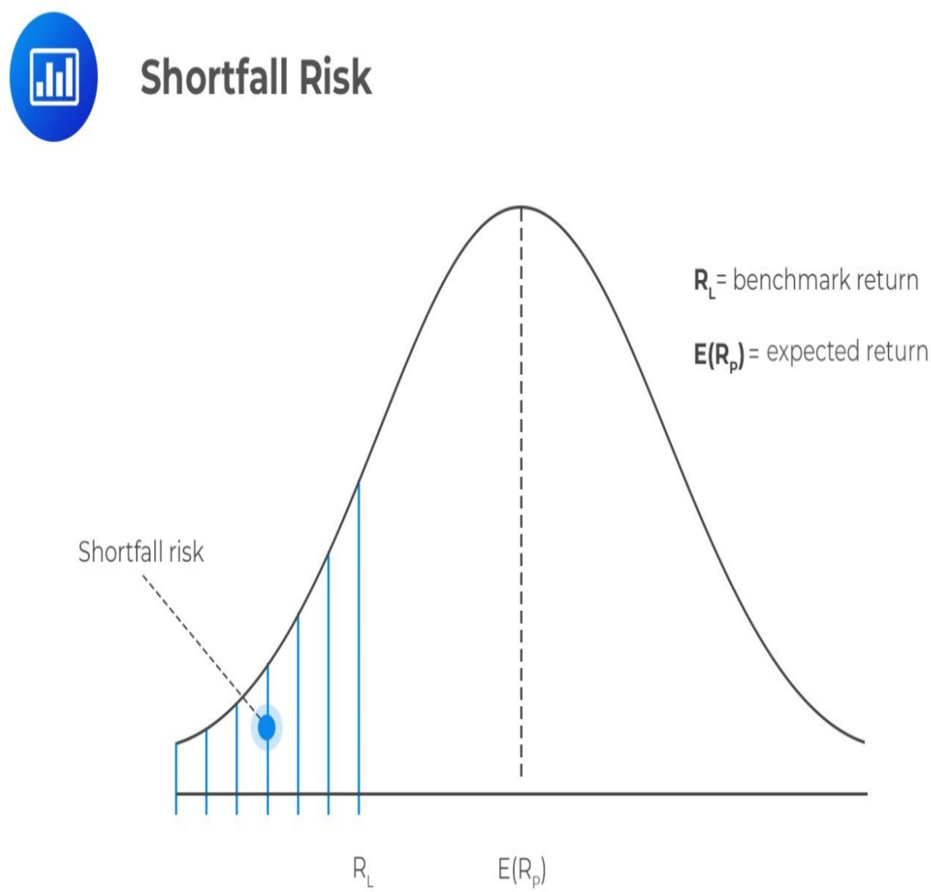
The mean-variance analysis can be reasonably accurate even if the two assumptions aren't entirely met. Professionals prefer using observable data, such as returns. The assumption that returns roughly follow a normal distribution has played a crucial role in applying MPT.

Mean-variance analysis only considers risk symmetrically. This implies that standard deviation reflects variability above and below the mean. An alternative strategy is focusing on downside risk. One such method is safety-first rules.

Before we dive into safety-first rules, we discuss Shortfall risk.

Shortfall Risk

Shortfall risk refers to the probability that a portfolio will not exceed the minimum (benchmark) return an investor sets. In other words, it is the risk that a portfolio will fall short of the level of return considered acceptable by an investor. As such, shortfall risks are downside risks. While a shortfall risk focuses on the downside economic risk, the standard deviation measures the overall volatility of a financial asset.



An illustration of shortfall risk

Safety-First Ratio

Roy's safety-first criterion states that the optimal portfolio is the one that minimizes the probability that a portfolio return, denoted by R_P, may fall below the threshold level of return, R_L. The optimal portfolio minimizes $P(R_P < R_L)$

As such, if returns are distributed normally, the optimal portfolio is the one with the highest safety-first ratio defined as:

$$\text{SFRatio} = \frac{E(R_P) - R_L}{\sigma_P}$$

The numerator, $E(R_P - R_L)$, represents the distance from the mean return to the threshold level, i.e., it measures the excess return over and above the threshold level of return per unit risk.

Intuitively, if the returns are normally distributed, the safety-first optimal portfolio maximizes the SFRatio.

Given a portfolio SFRatio, the probability that its return will be less than R_L is:

$$P(R_P < R_L) = N(-\text{SFRatio})$$

The safety-first optimal portfolio has the lowest $P(R_P < R_L)$.

Example: Safety-first Ratio

An investor sets a minimum threshold of 3%. There are three portfolios from which he is to choose one. The expected return and the standard deviation for each portfolio are given below:

	Portfolio A	Portfolio B	Portfolio C
Expected return	5%	10%	20%
Standard deviation	15%	20%	25%

What is the optimal portfolio for the investor?

Solution

Compute the safety-first ratio for each of the three portfolios and then compare them.

For portfolio A:

$$\text{SFRatio}_A = \frac{5 - 3}{15} = 0.1333$$

Similarly, for portfolio B:

$$\text{SFRatio}_B = \frac{10 - 3}{20} = 0.35$$

Lastly:

$$\text{SFRatio}_C = \frac{20 - 3}{25} = 0.68$$

The optimal portfolio should maximize the safety-first ratio. Comparing the three ratios, it is easy to notice that the safety-first ratio for portfolio C is the highest. Therefore, the investor should choose portfolio C.

Question

The returns on a fund are distributed normally. At the end of year t, the fund has a value of \$100,000. At the end of year $t+1$, the fund manager wishes to withdraw \$10,000 for further funding but is reluctant to tap into the \$100,000. There are two investment options:

	Portfolio A	Portfolio B
Expected return	14%	13%
Standard deviation	17%	20%

Which portfolio is preferable for the manager?

A. Portfolio A.

B. Portfolio B.

C. The manager is indifferent to the two portfolios.

Solution

The correct answer is A.

First, you should calculate the threshold return from the information given. Since there should be no tapping into the fund, the threshold return is:

$$\frac{10,000}{100,000} = 10\% \text{ or } 0.1$$

You should then calculate the safety-first ratio for each portfolio:

$$\text{SFRatio}_A = \frac{14-10}{17} = 0.24$$
$$\text{SFRatio}_B = \frac{13-10}{20} = 0.15$$

Portfolio A has the highest safety-first ratio. This is the reason it is the most desirable.

You can also go a step further and calculate $P(R_P < R_L)$. To do this, you would have to negate each safety-first ratio and then find the CDF of the standard normal distribution for the resulting value. That is,

$$
\begin{aligned}
P(R_P < R_L) &= N(-\text{SFRatio}) \\
N(-0.24) &= 1 - N(0.24) \\
&= 1 - 0.5948 = 0.4052 \\
N(-0.15) &= 1 - N(0.15) \\
&= 1 - 0.5596 = 0.4404
\end{aligned}
$$

$(-\text{where SFRatio is the z-value})$

For portfolio A, there is approximately a 40% probability of obtaining a return below the threshold return. For portfolio B, this probability rises to 44%. Therefore, we choose the option for which the chance of not exceeding the benchmark return is lowest – portfolio A.