Learning Module 10: Valuing a Derivative Using a One-Period Binomial Model

LOS 10b: describe the concept of risk neutrality in derivatives pricing

Remember that the value of an option is not affected by the real-world probabilities of the underlying price increments or decrements but rather by the expected volatilities (R^u and R^d), which are required to price an option.

We can compute the value of a call option today by discounting its expected value at expiration at the current risk-free rate, as summarized in the equation below:

$$c_0 = \frac{\pi c_1^u + (1 - \pi)c_1^d}{(1 + r)^T}$$

Similarly, the value of the put option is given by:

$$p_0 = \frac{p_1^u + (1 - \pi)p_1^d}{(1 + r)^T}$$

The **risk-neutral probability** (denoted by π) is defined as the computed probability used in binomial option pricing that equates the discounted weighted sum of the expected values of the underlying to the option's current price. It is calculated using the risk-free rate and the assumed up-and-down gross returns of the underlying, as follows:

$$\pi = \frac{(1 + r) - R^d}{R^u - R^d}$$

Risk-neutral pricing is the process of determining the risk-neutral probability (which is used to calculate the present value of future cash flows) using only the expected volatilities, i.e., (R^u and R^d), and the risk-free rate.

Example: Risk-neutral Probability
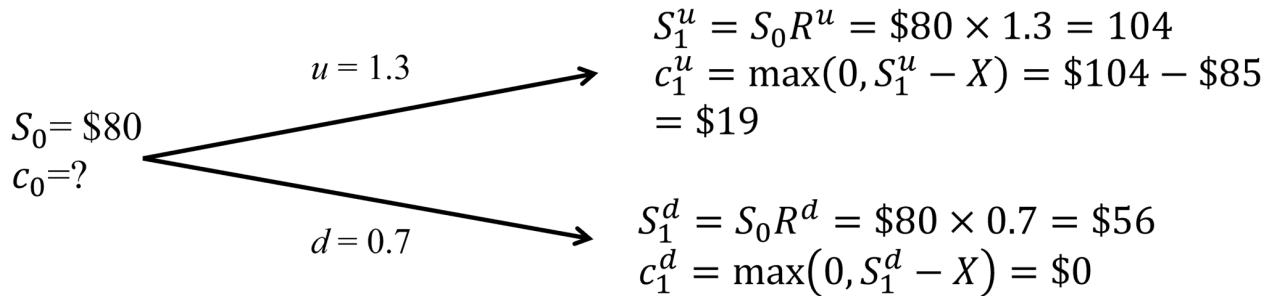
A company is considering selling a one-year call option on a non-dividend paying stock whose

current price is $80. The exercise price of the call option is $85, and the risk-free interest is 4%.

If the stock price is expected to go up or down by 30%, what is the selling price of the option? (Using risk-neutral pricing).

Solution

Consider the following diagram:

$$S_1^u = S_0 R^u = \$80 \times 1.3 = 104$$
$$c_1^u = \max(0, S_1^u - X) = \$104 - \$85 = \$19$$

$u = 1.3$

$S_0 = \$80$
$c_0 = ?$

$d = 0.7$

$$S_1^d = S_0 R^d = \$80 \times 0.7 = \$56$$
$$c_1^d = \max(0, S_1^d - X) = \$0$$

Risk neutral probability of an upward move is given by:

$$\Pi = \frac{(1 + r) - R^d}{R^u - R^d} = \frac{1.04 - 0.7}{1.3 - 0.7} = 0.56667$$

Intuitively, the risk-neutral probability of a downward move is given by:

$$1 - \Pi = 1 - 0.56667 = 0.43333$$

We need:

$$c_0 = \frac{[c_1^u + (1 - \Pi)c_1^d]}{1 + r} = \frac{0.56667 \times 19 + 0.43333 \times 0}{1.04} = \$10.35$$

Question

A European call option that expires in one year has an exercise price of GBP 70. The spot price of the underlying asset is GBP 70. Suppose that the underlying price is expected to increase or decrease by 20% within the next year, assuming a risk-free interest rate of 5%. The no-arbitrage price of a put option on the underlying asset (with similar exercise price and time to maturity) using the binomial model is *closest* to:

A. GBP 10.00.

B. GBP 14.00.

C. GBP 24.04.

Solution

Step 1: **Determine the call option's value at maturity.**

$$S_1^u = \frac{120}{100} \times 70 = 84$$
$$S_1^d = \frac{80}{100} \times 70 = 56$$
$$c_1^u = \max \ (0, \ S_1^u - X) = \max \ (0, \ 84 - 70) = 14$$
$$c_1^d = \max \ (0, \ S_1^d - X) = \max \ (0, \ 56 - 70) = 0$$

Step 2: **Determining h (the edge ratio).**

$$h = \frac{c_1^u - c_1^d}{S_1^u - S_1^d} = \frac{14 - 0}{84 - 56} = 0.5$$

Step 3: **Determine the portfolio value if the price of the underlying increases or decreases.**

$$V_1^u = \ hS_1^u - c_1^u = (0.5 \times 84) - 14 = 28$$
$$V_1^d = hS_1^d - C_1^d = (0.5 \times 56) - 0 = 28$$

Step 4: **Determining** V_0.

$$V_0 = \frac{V_1}{(1+r)^t} = \frac{28}{(1+0.05)^1} = 26.67$$

Note: $V_1 = V_1^d = V_1^u$.

Step 5: **Determining** p_0 **and** c_0.

We need,

$$c_0 = \frac{[c_1^u + (1-\pi)\, c_1^d]}{1+r}$$

where

$$\pi = \frac{(1+r) - R^d}{R^u - R^d}$$
$$= \frac{(1.05) - 0.8}{1.2 - 0.8}$$
$$= 0.625$$

So,

$$c_0 = \frac{[c_1^u + (1-\pi)\, c_1^d]}{1+r}$$
$$= \frac{[14 + (1-0.625)\, 0]}{1.05}$$
$$= 13.33$$

To find p_0 we need to use put-call parity:

$$p_0 = c_0 - S_0 + X(1+r)^{-T}$$
$$= 13.33 - 70 + 70(1+0.05)^{-1} = 10.00$$

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LOS 10a: explain how to value a derivative using a one-period binomial model

The law of arbitrage dictates that the value of any two assets (or portfolio of assets) whose payoffs are identical in all possible future scenarios at a given time must also be identical today.

Unlike forward commitments that offer symmetric payoffs at a pre-determined price in the future, contingent claims offer asymmetric payoffs. For this reason, their valuation is a challenge. The binomial model can be used to model the payoffs of contingent claims.

One-Period Binomial Model

The idea behind the binomial model is that at maturity, an asset's spot price, S_0, can either increase to S_1^u or decrease to S_1^d. We do not need to know the asset's future price in advance since it depends on a random variable's outcome. The asset price movements, from S_0 to either S_1^u or S_1^d, can be seen as the outcome of a Bernoulli trial.

Let us denote q as the probability of an increase in the asset's price. Because of the presence of only two possibilities, i.e., an increase or a decrease in the asset's price, we can denote the probability of a decrease in its price as $1 - q$, such that the two probabilities will add up to 1.

The gross return when the asset price increases will be:
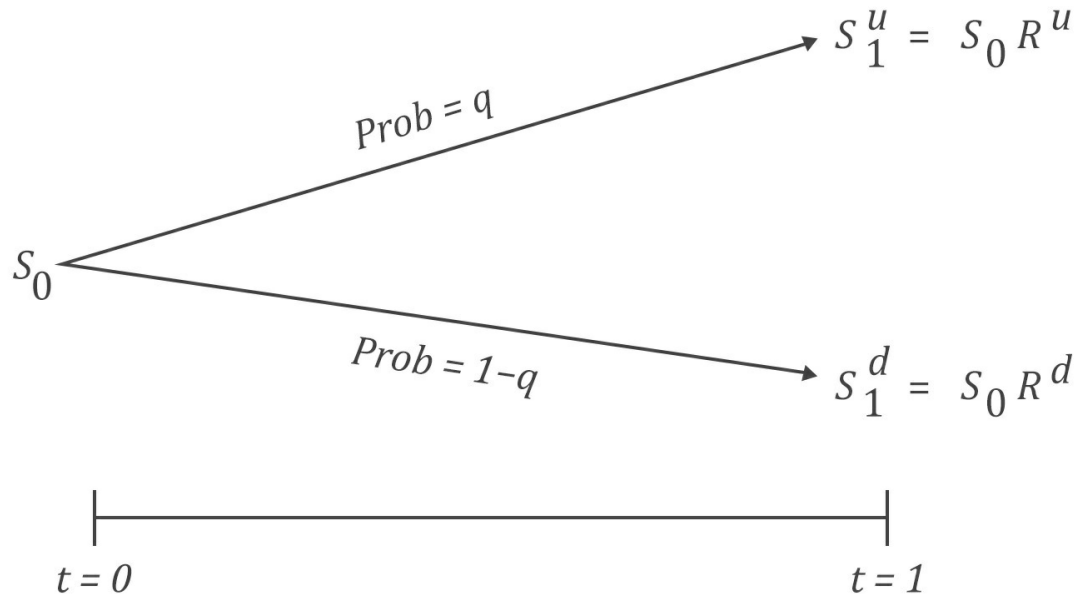
$$R^u = \frac{S_1^u}{S_0} > 1$$

When the asset price decreases, the gross return will be:

$$R^d = \frac{S_1^d}{S_0} < 1$$

The difference between S_1^u (or $R^u S_0$) and S_1^d (or $R^d S_0$) is the spread of possible future price outcomes.

One-Period Binomial Model

$$S_1^u = S_0 R^u$$

$$Prob = q$$

$$S_0$$

$$Prob = 1-q$$

$$S_1^d = S_0 R^d$$

$$t = 0 \qquad\qquad t = 1$$

Pricing a European Call Option using a One-Period Binomial Model

Consider a one-year call option with an underlying price of S_0 and an exercise price of X. Also, assume that $S_1^d < X < S_1^u$ and that the one-period binomial model is equivalent to the time of expiration of one year.
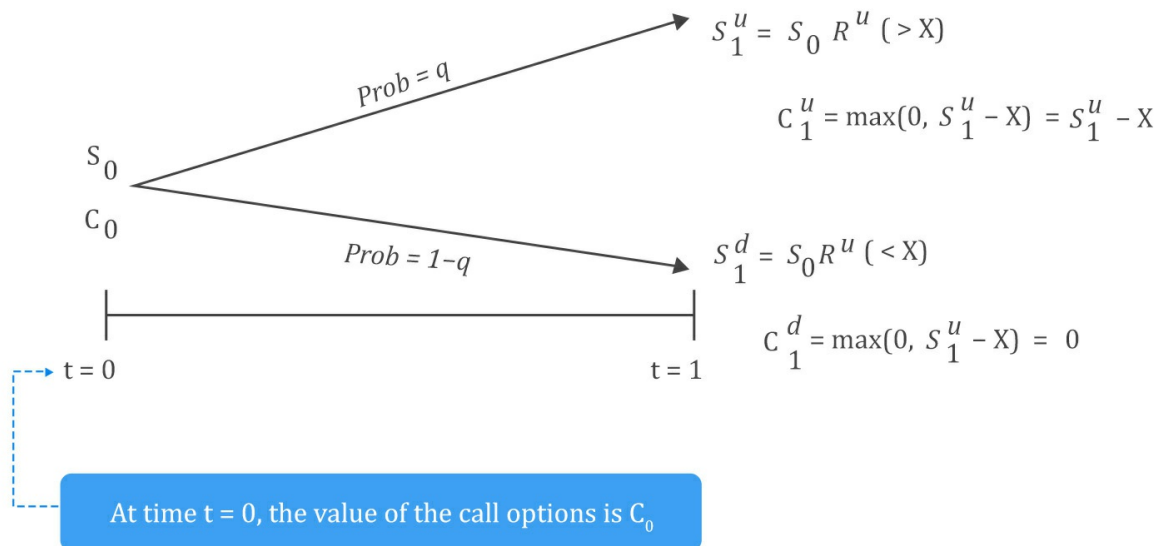
The one-period binomial model gives the underlying asset values in one year, where the option value is defined as a function of the underlying value.

At time $t=0$:

The value of the call option is c_0. Note that this value is unknown and needs to be calculated.

Call Option Pricing Using a One-Period Binomial Model at t=0

$$S_1^u = S_0 R^u \; (> X)$$

$$C_1^u = \max(0, S_1^u - X) = S_1^u - X$$

$$S_0$$
$$C_0$$

Prob = q

Prob = 1−q

$$S_1^d = S_0 R^u \; (< X)$$

$$C_1^d = \max(0, S_1^u - X) = 0$$

t = 0   t = 1

At time t = 0, the value of the call options is C_0
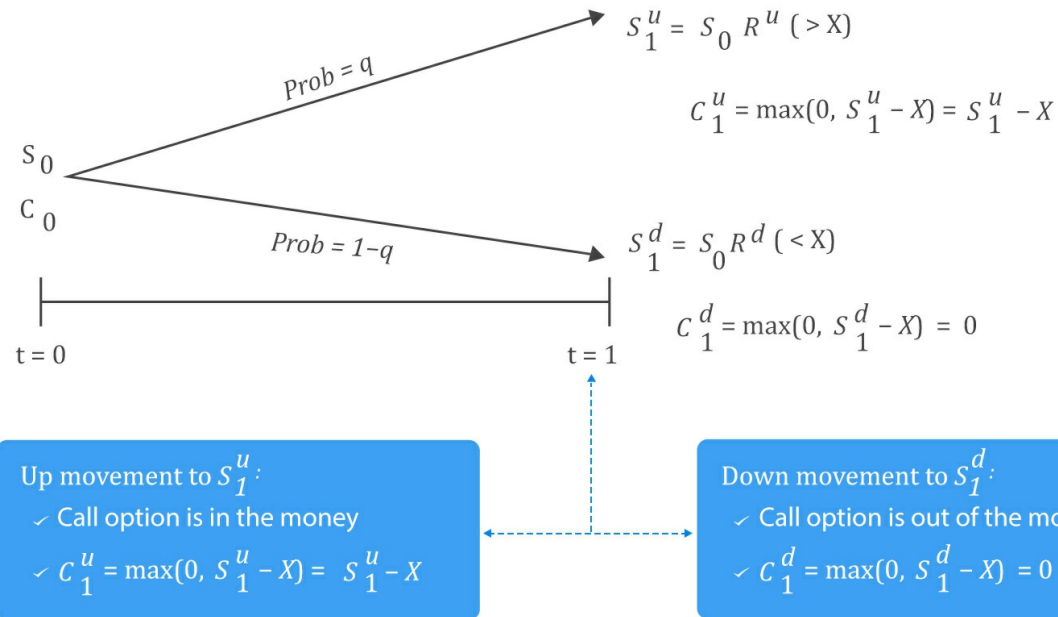
At time $t=1$:

After one year, the option expires. At this time, the value of the option will either be $c_1^u = \max(0, S_1^u - X)$ if the underlying price rises to S_1^u or $c_1^d = \max(0, S_1^u - X)$ if the underlying price falls to S_1^d.

Intuitively, for the up movement, the call option is in the money, and for the down direction, the option is out of the money.

$$S_1^u = S_0 R^u \ (> X)$$

$$C_1^u = \max(0, S_1^u - X) = S_1^u - X$$

Prob = q

$$S_0$$
$$C_0$$

Prob = 1-q

$$S_1^d = S_0 R^d \ (< X)$$

$$C_1^d = \max(0, S_1^d - X) = 0$$

t = 0   t = 1

Up movement to S_1^u:
- ✓ Call option is in the money
- ✓ $C_1^u = \max(0, S_1^u - X) = S_1^u - X$

Down movement to S_1^d:
- ✓ Call option is out of the money
- ✓ $C_1^d = \max(0, S_1^d - X) = 0$

Determining the Value of c_0

The value of c_0 is determined by applying replication and no-arbitrage pricing. Replication implies that the value of the option and its underlying asset in any future scenario may be used to construct a risk-free portfolio.

With that in mind, assume at time $t = 0$, we sell a call option for a price of c_0 and buy h units of the underlying asset. Also, let the value of the portfolio be V so that its value at $t = 0$ is:

$$V_0 = hS_0 - c_0$$

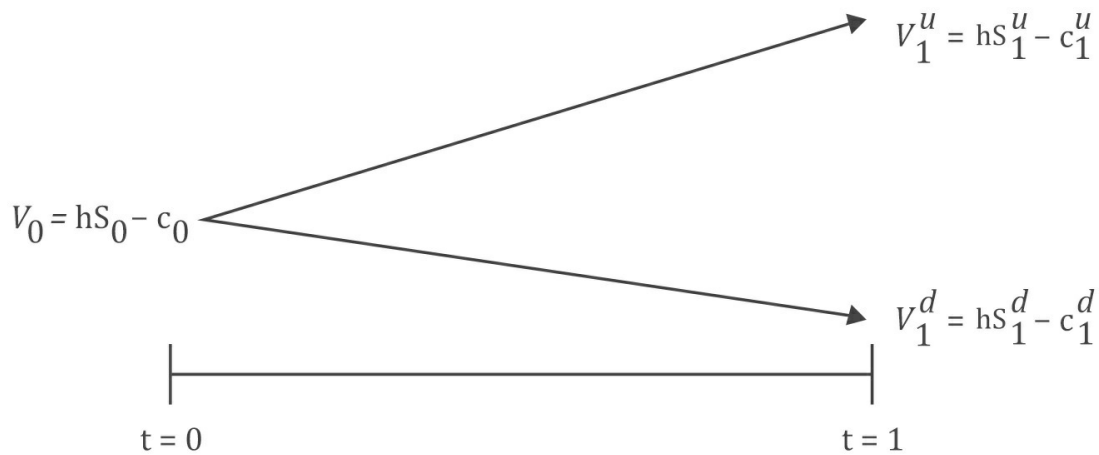The value of the portfolio, if the underlying price moves up, is given by:

$$V_1^u = hS_1^u - c_1^u = h \times R^u \times S_0 - \max(0, S_1^u - X)$$

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And for the down movement:

$$V_1^d = hS_1^d - c_1^d = h \times R^d \times S_0 - \max(0, S_1^d - X)$$

Determining the Value of c_0 using a Hedge Ratio

Assuming no-arbitrage condition, note that we have established two portfolios with identical payoff profiles at time $t = 1$. As such, we need to find the value of h such that:

$$V_1^u = V_1^d$$

Therefore,

$$\Rightarrow hS_1^u - c_1^u = hS_1^d - c_1^d$$

Making h the subject of the formula:

$$h = \frac{c_1^u - c_1^d}{S_1^u - S_1^d}$$

The value h is called the hedge ratio. The hedge ratio is a proportion of the underlying that will offset the risk associated with an option.

Since $V_1^u = V_1^d$, we can draw two conclusions:

- We can utilize either of the two portfolios to value the option.

- The return $\frac{V_1^u}{V_0} = \frac{V_1^d}{V_0} = 1 + r$.

To avoid arbitrage, the portfolio value at $t = 1$, $(V_1 = V_1^u = V_1^d)$, must be discounted using a risk-free rate so that:

$$V_0 = V_1(1 + r)^{-1}$$

However, recall that $V_0 = hS_0 - c_0$:

$$\Rightarrow hS_0 - c_0 = V_1(1 + r)^{-1}$$

Making c_0 the subject of the formula:

$$c_0 = hS_0 - V_1(1 + r)^{-1}$$

Example: Pricing Call Option Using One Period Binomial Model

A European call option that expires in one year has an exercise price of $70 and an underlying spot price of $60. Use a one-period binomial model to estimate the call option price if the underlying's spot price is expected to change by 25% in one year. Assume that the risk-free annual rate is 5%.

Solution

Step 1: Determine the Call Option's Value at Maturity $t=1$

At maturity, the value can either be c_1^u if the price of the underlying goes up or c_1^d if the price of the underlying goes down.

The spot price at maturity, if the price goes up by 25%, will be:

$$S_1^u = \frac{125}{100} \times 60 = 75$$

and the gross return will be:

$$R^u = \frac{S_1^u}{S_o} = \frac{75}{60} = 1.25$$

If the price goes down by 25%, the spot price at maturity will be:

$$S_1^d = \frac{75}{100} \times 60 = 45$$

and the gross return will be:

$$R_d = \frac{S_1^d}{S_o} = \frac{45}{60} = 0.75$$

If the underlying's price goes up (the call option expires in the money):

$$c_1^u = \max\ (0,\ S_1^u - X) = \max\ (0,\ 75 - 70) = 5$$

If the underlying's price goes down, and the call option expires out of the money:

$$c_1^d = \max\ (0,\ S_1^d - X) = \max\ (0,\ 45 - 70) = 0$$

Step 2: Determining h, the Hedge Ratio

The hedge ratio is given by

$$h = \frac{c_1^u - c_1^d}{S_1^u - S_1^d}$$
$$= \frac{5 - 0}{75 - 45} = 0.167$$

The hedge ratio of 0.167 implies that we either need to buy 0.167 units of the underlying for every call option we sell or sell 6 call options for each underlying asset to equate the portfolio values at maturity ($t = 1$). Therefore, the portfolio values when the price of the underlying

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increases and decreases respectively is:

$$V_1^u = (0.167 \times 75) - 5 = 7.5$$
$$V_1^d = (0.167 \times 45) - 0 = 7.5$$

The portfolio values are the same, implying that either of the portfolios can be used to value the derivative.

Step 4: **Determining** V_0

We can obtain V_0 by discounting V_1 at the risk-free discount rate. Remember that $V_1 = V_1^u = V_1^d$:

$$V_o = 7.5(1 + 0.05)^{-1} = 7.14$$

Step 5: **Determining** c_0

Recall that:

$$c_0 = hS_0 - V_1(1 + r)^{-1} = hS_o - V_0$$

Therefore,

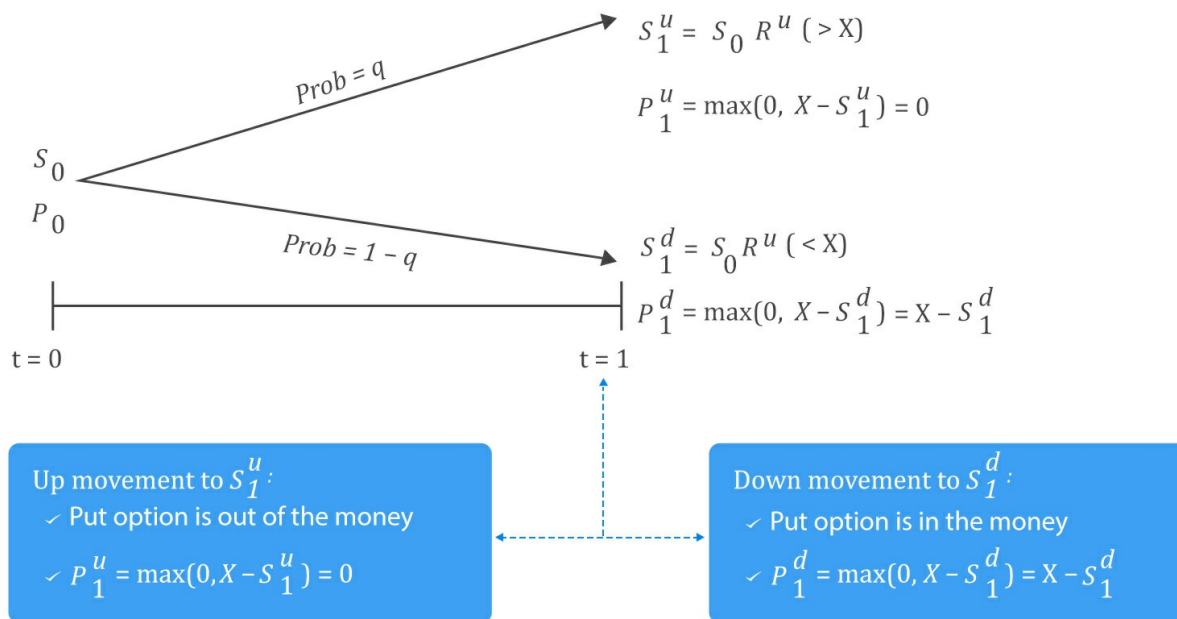$$c_o = hS_o - V_0 = (0.167 \times 60) - 7.14 = 2.88$$

Note: The hedging approach can be used to value many derivatives, not just call options, provided the derivative's value depends on the underlying asset's value at contract maturity, i.e., $t = 1$

Pricing a European Put Option One-period Binomial Model

For put options, the same explanations we gave under the call option apply, albeit with a different replication strategy. Consider the following diagram:

Put Option Pricing Using a One-Period Binomial Model at t=1

Under the put option, the hedge ratio is given by

$$h = \frac{p_1^u - p_1^d}{S_1^d - S_1^u}$$

Note that the formula remains the same as in the call option, except for the change of notations. Replication in pricing put option using a one-period binomial model involves buying the put option and h units of the underlying so that:

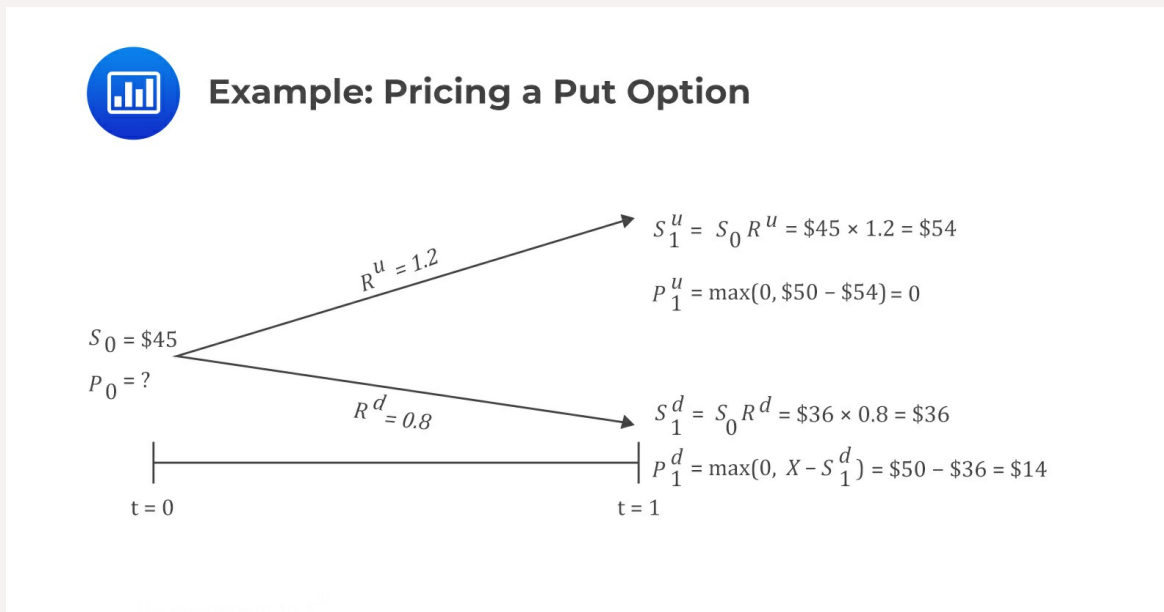$$V_0 = p_0 + hS_0$$

Therefore,

$$p_0 = V_0 - hS_0$$

Question

Consider a one-year put option on a non-dividend paying stock with an exercise price of $50. The current stock price is $45. The stock price is expected to go up or down by 20%. Calculate the non-arbitrage price of the put option if the risk-free rate of return is 4%.

A. 0

B. $5.38

C. $14.00

Solution

The correct answer is **B.**

Consider the following diagram:

Example: Pricing a Put Option

$S_1^u = S_0 R^u = \$45 \times 1.2 = \$54$

$P_1^u = \max(0, \$50 - \$54) = 0$

$R^u = 1.2$

$S_0 = \$45$

$P_0 = ?$

$R^d = 0.8$

$S_1^d = S_0 R^d = \$36 \times 0.8 = \$36$

$P_1^d = \max(0, X - S_1^d) = \$50 - \$36 = \$14$

t = 0

t = 1

From the above results, the hedge ratio is given by:

$$h = \frac{p_1^u - p_1^d}{S_1^u - S_1^d} = \frac{0 - 14}{54 - 36} = -0.7778$$

We need to calculate $V_1 = V_1^u = V_1^d$, which are:

$$V_1^u = hS_1^u + p_1^u = 0.7778 \times 54 + 0 = \$42.00$$
$$V_1^d = hS_1^d + p_1^d = 0.7778 \times 36 + 14 = \$42.00$$

Next, we can either use V_1^u or V_1^d to calculate the value of V_0:

$$V_0 = V_1(1+r)^{-1} = 42(1.04)^{-1} = \$40.38$$

Therefore, the put option value is:

$$p_0 = V_0 - hS_0 = 40.38 - 0.7778 \times 45 = \$5.38$$