Zusammenfassung Quantum Computing

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1 Basic

1.1 QBits

- **state of a single QB**: $|s\rangle = a_0|0\rangle + a_1|1\rangle = a_0 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + a_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a_0 \\ a_1 \end{bmatrix}$

- **state of two QBs**: $|s\rangle = a_0|00\rangle + a_1|01\rangle + a_2|10\rangle + a_3|11\rangle$

- **basis vectors**: $|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$
- $P(|0\rangle) = |a_0|^2$, $P(|1\rangle) = |a_1|^2$
- $\sum_{i=0}^{2^n-1} |a_n| = 1$ (for n-qubit system)
- **tensor product**: $|00\rangle = |0\rangle \bigotimes |0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \bigotimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ 0 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$
- **entanglement**: non-serperable state (can not be written as the product of qubits, the qubits are statistical dependent)
  Example: $|\psi\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$

1.2 Complex Numbers

- $z = a + ib$
  $= r \cdot e^{i\varphi}$
  $= r \cdot (cos\varphi + i \cdot sin\varphi)$
  with $a, b \in \mathbb{R}$ and $i^2 = -1$
- **conjugate**: $\bar{z} = a - bi$

1.3 Matrices

- **Transpose**: $A^\top$ swap rows and cols
- **Conjugate**: A^* each entry is the conjugate
- **Adjunct**: $A^\dagger$ transpose + conjugate
- **Unitary**: $UU^\dagger = UU^{-1} = I = U^\dagger U$ adjunct is also the inverse
- Note: every unitary operator can be written as its eigenbases

2 Gates

- every gate is reversible (as gates are unitary matrices)
- **Hardamard Gate**:

- $H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$
- $H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$
- applying H splits the probabilities in $\frac{1}{2}$ for each (simulate coinflip)
- H is self inverse as it is unitary
- **recursive definition for Hardamard**:
  $H^{\bigotimes n} = H \bigotimes H^{\bigotimes n-1} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \bigotimes H^{\bigotimes n-1} = \frac{1}{\sqrt{2}} \begin{bmatrix} H^{\bigotimes n-1} & H^{\bigotimes n-1} \\ H^{\bigotimes n-1} & -H^{\bigotimes n-1} \end{bmatrix}$
  $H^{\bigotimes 1} = H$

- **Pauli Gates**:
  - **Pauli-X**: Swaps $|0\rangle$ and $|1\rangle$, $X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
  - **Pauli-Y**: Swaps amplitudes, (adds phase ?), negates amplitudes of $|1\rangle$
    $Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$
  - **Pauli-Z**: Negates amplitudes of $|1\rangle$ $Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

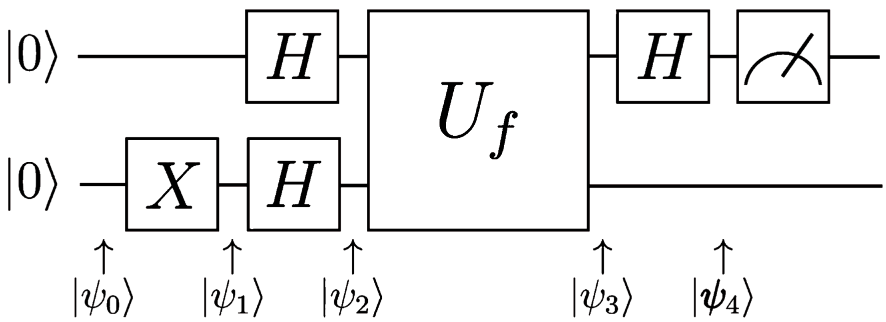
- **CNot**:
  - negates the target if the controller is active
  - permutation matrix
  - $CNOT = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

2.1 Phase Kickback

- U: one qubit unitary gate
- $|\phi\rangle$: some base state
- applying U to $|\psi\rangle$ yields $e^{i\phi}|\psi\rangle$
- the global phase factor of a quantum state is not measurable (symmetry)
- using **ancilla** qubits the global phase can be turned intro a relative phase which is measurable

3 Deutsch Algorithm

- function $f : \{0,1\} \rightarrow \{0,1\}$ that is either balanced or constant
- classical: compute f on every input
- quantum: one call of f is needed
- quantum oracle:
  - $U_f : |x\rangle|y\rangle \xrightarrow{U_f} |x\rangle|y \bigoplus f(x)\rangle$
  - $|x\rangle$ input to function
  - $|y\rangle$ qubit to write function result to
  - $|y \bigoplus f(x)\rangle$, the XOR ensures that the oracle is reversible (as each image has a unique preimage)
  - initializing $y = |0\rangle$ we only get the function value $|x\rangle|f(x)\rangle$ as $0 \bigoplus x = x$
  - initializing $y = |-\rangle$ we get phase kickback to $|x\rangle|-\rangle \xrightarrow{U_f} (-1)^{f(x)}|x\rangle|-\rangle$ (a phase is applied to the input qubit)
    $\begin{cases} |x\rangle|-\rangle & f(x) = 0 \\ -|x\rangle|-\rangle & f(x) = 1 \end{cases}$
  - Note: this is called a phase oracle ($U_f|x\rangle|-\rangle = (-1)^{f(x)}|x\rangle|-\rangle$)



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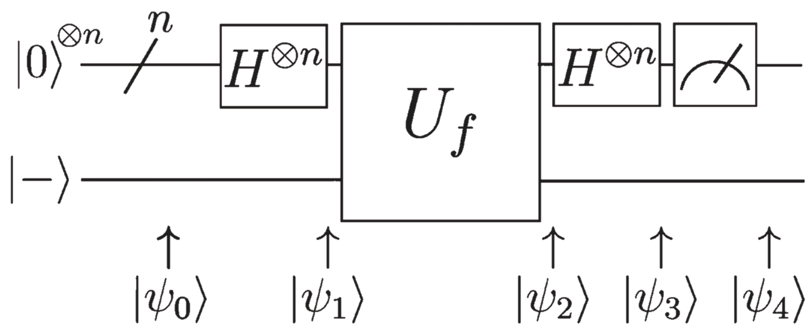
  - $|\psi_0\rangle = |00\rangle$
  - $|\psi_1\rangle = |01\rangle$
  - $|\psi_2\rangle = |+-\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle|-\rangle + |1\rangle|-\rangle)$

- $|\psi_3\rangle = U_f \frac{1}{\sqrt{2}}(|0\rangle|-\rangle + |1\rangle|-\rangle) = \frac{1}{\sqrt{2}}(U_f|0\rangle|-\rangle + U_f|1\rangle|-\rangle) \overset{\text{phase oracle}}{=} \frac{1}{\sqrt{2}}((-1)^{f(0)}|0\rangle|-\rangle + (-1)^{f(1)}|1\rangle|-\rangle)$
  ($|-\rangle$ can be omitted as its not needed)
- case $f(0) = f(1)$: $\begin{cases} |\psi_3\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), & f(0) = f(1) = 0 \\ |\psi_3\rangle = -\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), & f(0) = f(1) = 1 \end{cases}$
  $|\psi_3\rangle = \pm\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = \pm|+\rangle$
- case $f(0) \neq f(1)$: $\begin{cases} |\psi_3\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle), & f(0) = 0 \wedge f(1) = 1 \\ |\psi_3\rangle = -\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle), & f(0) = 1 \wedge f(1) = 0 \end{cases}$
  $|\psi_3\rangle = \pm\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) = \pm|-\rangle$
- $|\psi_4\rangle = \begin{cases} \pm|0\rangle, & f(0) = f(1) \\ \pm\,|1\rangle, & f(0) \neq f(1) \end{cases}$

* measuring 0 iff function is constant and 1 iff function is balanced

4 Deutsch-Jozsa Algorithm

* generalized version of Deutsch Algorithm to n qubits
* $f : \{0,1\}^n \rightarrow \{0,1\}$
* f is constant iff $\forall x, f(x) = c$
* f is balanced iff $|\{x \mid f(x) = 0\}| = |\{x \mid f(x) = 1\}|$
* classical: $2^{n-1} + 1$ function calls (input half the possible inputs)
* quantum: one call of f in needed (exponential speed up)

- 

  - $|\psi_0\rangle = |00...0\rangle|-\rangle = |0\rangle^{\otimes n}|-\rangle$ (we can get the $|-\rangle$ by $H|1\rangle$)
  - $|\psi_1\rangle = H^{\otimes n}|0\rangle^{\otimes n}|-\rangle = \frac{1}{\sqrt{2^n}}\sum_{x\in\{0,1\}^n}|x\rangle|-\rangle$ (uniform distribution)
  - $|\psi_2\rangle = \frac{1}{\sqrt{2^n}}\sum_{x\in\{0,1\}^n}U_f|x\rangle|-\rangle \overset{\text{phase oracle}}{=} \frac{1}{\sqrt{2^n}}\sum_{x\in\{0,1\}^n}(-1)^{f(x)}|x\rangle|-\rangle$ ($|-\rangle$ can be omitted)
  - Note: $H^{\otimes n}|x\rangle = \frac{1}{\sqrt{2^n}}\sum_{z\in\{0,1\}^n}(-1)^{x\cdot z}|z\rangle,\ (*)$
  - $|\psi_3\rangle = \frac{1}{\sqrt{2^n}}\sum_{x\in\{0,1\}^n}(-1)^{f(x)}H^{\otimes n}|x\rangle|-\rangle$
    $\overset{(*)}{=} \frac{1}{\sqrt{2^n}}\sum_{x\in\{0,1\}^n}(-1)^{f(x)}\frac{1}{\sqrt{2^n}}\sum_{z\in\{0,1\}^n}(-1)^{x\cdot z}|z\rangle$
    $= \frac{1}{2^n}\sum_{x\in\{0,1\}^n}\sum_{z\in\{0,1\}^n}(-1)^{f(x)+x\cdot z}|z\rangle$
  - consider the amplitude of $|0\rangle^{\otimes n}$ is $\frac{1}{2^n}\sum_{x\in\{0,1\}^n}(-1)^{f(x)}$ case f constant:
    $\begin{cases} \frac{1}{2^n}\sum_{x\in\{0,1\}^n}(-1)^0 = \frac{1}{2^n}2^n = 1, & f(x)=0 \\ \frac{1}{2^n}\sum_{x\in\{0,1\}^n}(-1)^1 = \frac{1}{2^n}(-2^n) = -1, & f(x)=1 \end{cases}$
  - hence if f is constant the probability of measuring all zeros is 1
  - hence if f is balanced half of the sum is 1 and half is -1 hence the probability of measuring all zeros is 0
  - $\rightarrow$ measure and iff we get 000...0 then $f(x)$ is constant else balanced

5 Grovers Algorithm

- Problem: given an unstructured database, find an element $\hat{x}$ within this database

- $f : \{0,1\}^n \to \{0,1\}$ with $f(\hat{x}) = 1$, $f(\neg\hat{x}) = 0$ $\qquad (x, \hat{x} \in \{0,1\}^n)$
- classical: $\frac{N+1}{2} \in O(N)$
- quantum: $\frac{\pi}{4}\sqrt{N} \in O(\sqrt{N})$
- Steps:
  1. generate uniform distribution on all elements:
     - $H^{\bigotimes n}|000...0\rangle = \dfrac{1}{\sqrt{N}} \displaystyle\sum_{x=0}^{N-1} |x\rangle$
     - $|s\rangle = H^{\bigotimes n+1}|000...0\rangle|1\rangle = |0\rangle^{\bigotimes n}|-\rangle$
  2. Grover Iteration:
     (a) Negate the amplitude of $\hat{x}$ (Oracle $\hat{U}_f$)
     - $\hat{U}_f|s\rangle = \dfrac{1}{\sqrt{2^n}} \displaystyle\sum_{x\in\{0,1\}^n} (-1)^{f(x)} H^{\bigotimes n}|x\rangle|-\rangle$ (as in Deutsch-Josza)

     (b) Mirror/Reflect/Diffuse all amplitudes a at the mean value m(Diffusion $\hat{D}$)
     - $a := 2 \cdot m - a$
     - mirroring as a quantum state: $\displaystyle\sum_{i=0}^{N-1} a_i|i\rangle$ $(*)$
     - mean value of amplitudes: $\displaystyle\sum_{j=0}^{N-1} \frac{a_j}{N}$ $(**)$
     - combining $(*)$ and $(**)$ we get: $|s\rangle = \displaystyle\sum_{i=0}^{N-1} \left(2 \cdot \sum_{j=0}^{N-1} \frac{a_j}{N} - a_i\right)|i\rangle$
     - $D_N = \begin{bmatrix} -1 + \frac{2}{N} & \dots & \frac{2}{N} \\ \vdots & -1 + \frac{2}{N} & \vdots \\ \frac{2}{N} & \dots & -1 + \frac{2}{N} \end{bmatrix}$
     - Note: D_N can be expressed as a local operation ($leq3$ bits involved)
  3. Measure $|x\rangle$ and return it off $\hat{x} > c$ (c is some constant)
- The number of grover iterations T is capped by $(2T + 1)\frac{1}{\sqrt{N}}$ since every iteration rotates by $\frac{2}{\sqrt{N}} \Rightarrow T = \frac{\pi}{4}\sqrt{4}$
- doing more than the necessary number of iteration degrades the result

6 Simons Algorithms

Todo

7 Shors Algorithm

Goal: Find prime factor $z \cdot r = n$ for some $n \in \mathbb{N}$ that is not a prime power

1. Make a random guess $a \in \{2, ..., n-1\}$
2. calculate $z := gcd(a, n)$ and return z if $z \neq 1$
   (the guess was very lucky and a prime factor was guessed, other factor is $\frac{n}{z}$)
3. calculate the period of a in $(\mathbb{Z}/n\mathbb{Z})^{\times}$: $a^p \equiv 1 \mod n$ (**Quantum**)
   (Ordnung von a innerhalb der primen Restklassengruppe)
   - choose q with $n^2 \leq q < 2n^2$
   - initialize input quantumregister with: $\frac{1}{\sqrt{q}} \sum_{p=0}^{q-1} |p\rangle|0\rangle$
     (superposition of all possible periods)
   - initialize output quantumregister with: $\frac{1}{\sqrt{q}} \sum_{p=0}^{q-1} |p\rangle|a^p \mod n\rangle$
     (superposition of the remainders of the calculation $a^p \mod n$)
   - Example $q = 16, a = 7$:
     $\frac{1}{4}(|0\rangle|1\rangle) + |2\rangle|4\rangle + |3\rangle|13\rangle +$
     $|4\rangle|1\rangle + |5\rangle|1\rangle) + |6\rangle|4\rangle + |7\rangle|13\rangle +$
     $|8\rangle|1\rangle + |9\rangle|1\rangle) + |10\rangle|4\rangle + |11\rangle|13\rangle +$
     $|12\rangle|1\rangle + |13\rangle|1\rangle) + |14\rangle|4\rangle + |15\rangle|13\rangle)$
     (Clearly the period $p = 4$ with $(1, 7, 4, 13)$
   - measuring the output/reminder register yields a superposition of all elements with the same reminder (uniformly random which exactly)
     Example: $\frac{1}{2}(|1\rangle + |5\rangle + |9\rangle + |13\rangle)|7\rangle$ (all elements with reminder 7)
   - apply QFT_n to input register to get the period p
     Example: $\frac{1}{2}(|0\rangle + |4\rangle + |8\rangle + |12\rangle)$
     (amplitudes for each element may have changed sign/phase)
   - measuring the input register yields $\{\frac{j \cdot n}{p} | j = 0, ..., 3\}$
     Example: $\{0, 4, 8, 12\}$

- calculate p using $\frac{j}{p} = \frac{y}{n}$ (y...output measurement)
  (only works if j and p have no common divisors hence measuring 0 or 8 in the Example necessitates a new run as the state is now destroyed)

4. GoTo (1) if: p is odd (we cannot calculate $\frac{p}{2} \in \mathbb{N}$)

5. calculate $z := gcd(a^{\frac{p}{2}} - 1, n)$

   - return z if $z \neq 1$ as its a factor of n

6. calculate $z := gcd(a^{\frac{p}{2}} + 1, n)$

   - if $z = n$ Goto (1), as we have a multiple of n
   - else return z

- $(a^{\frac{p}{2}} - 1) \cdot (a^{\frac{p}{2}} + 1) = a^p - 1 = k \cdot n$

- hence either the first or the second term must have a common divisor with n

Note: applying QFT to $|0\rangle$ yields a uniform superposition as a Hadamard-transformation

7.1 RSA

- asymetric encryption

- key generation:

  1. choose $p, q \in \mathbb{P}$ at random
  2. calculate $n = p \cdot q$ (this is the modulus for the public/private key)
  3. compute $\varphi(n) = (p-1)(q-1)$ (number of coprime integers)
  4. choose e s.t. $1 < e < \varphi(n)$ and e and $\varphi(n)$ are coprime (e: encryption exponent)
  5. calculate $e \cdot d \equiv 1 (mod\, \varphi(n))$ (d: decryption exponent) (calculation by adv. eucl. alg.)
  6. keys are:
     - Public: (e, n)
     - Private: (d, n)