Zusammenfassung Quantum Computing

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Inhaltsverzeichnis

1 Basic

1.1 QBits

- **state of a single QB**: $|s\rangle = a_0|0\rangle + a_1|1\rangle = a_0 \begin{bmatrix}1\\0\end{bmatrix} + a_1 \begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}a_0\\a_1\end{bmatrix}$
- **state of two QBs**: $|s\rangle = a_0|00\rangle + a_1|01\rangle + a_2|10\rangle + a_3|11\rangle$
- **basis vectors**: $|0\rangle = \begin{bmatrix}1\\0\end{bmatrix}, |1\rangle = \begin{bmatrix}0\\1\end{bmatrix}$
- $P(|0\rangle) = |a_0|^2, P(|1\rangle) = |a_1|^2$
- $\sum_{i=0}^{2^n-1} |a_n| = 1$ (for n-qubit system)

- **tensor product**: $|00\rangle = |0\rangle \bigotimes |0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \bigotimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ 0 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

- **entanglement**: non-serperable state (can not be written as the product of qubits, the qubits are statistical dependent)
  Example: $|\psi\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$

1.2 Complex Numbers

- $z = a + ib$
  $= r \cdot e^{i\varphi}$
  $= r \cdot (cos\varphi + i \cdot sin\varphi)$
  with $a, b \in \mathbb{R}$ and $i^2 = -1$

- **conjugate**: $\bar{z} = a - bi$

1.3 Matrices

- **Transpose**: $A^\top$ swap rows and cols
- **Conjugate**: A^* each entry is the conjugate
- **Adjunct**: $A^\dagger$ transpose + conjugate
- **Unitary**: $UU^\dagger = UU^{-1} = I = U^\dagger U$ adjunct is also the inverse

2 Gates

- every gate is reversible (as gates are unitary matrices)
- **Hardamard Gate**:
  - $H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$
  - $H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$
  - applying H splits the probabilities in $\frac{1}{2}$ for each (simulate coinflip)
  - H is self inverse as it is unitary

- **recursive definition for Hardamard**:
  $H^{\bigotimes n} = H \bigotimes H^{\bigotimes n-1} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \bigotimes H^{\bigotimes n-1} = \frac{1}{\sqrt{2}} \begin{bmatrix} H^{\bigotimes n-1} & H^{\bigotimes n-1} \\ H^{\bigotimes n-1} & -H^{\bigotimes n-1} \end{bmatrix}$
  $H^{\bigotimes 1} = H$

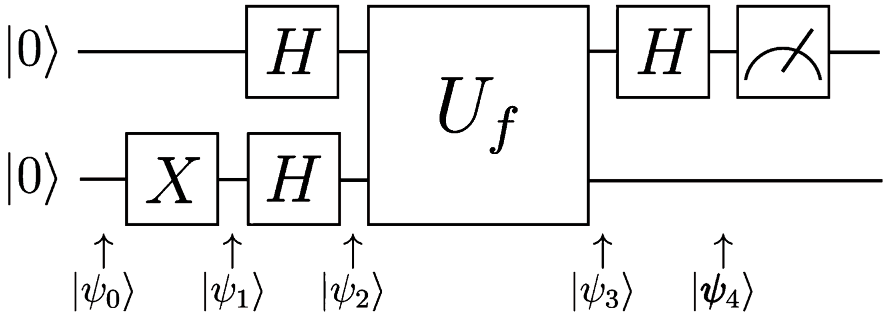
- **Pauli Gates**:
  - **Pauli-X**: Swaps $|0\rangle$ and $|1\rangle$, $X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
  - **Pauli-Y**: Swaps amplitudes, (adds phase ?), negates amplitudes of $|1\rangle$
    $Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$
  - **Pauli-Z**: Negates amplitudes of $|1\rangle$ $Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

- **CNot**:
  - negates the target if the controller is active
  - permutation matrix
  - $CNOT = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

3 Deutsch Algorithm

- function $f : \{0, 1\} \rightarrow \{0, 1\}$ that is either balanced or constant
- classical: compute f on every input
- quantum: one call of f is needed
- quantum oracle:
  - $U_f : |x\rangle|y\rangle \xrightarrow{U_f} |x\rangle|y \bigoplus f(x)\rangle$
  - $|x\rangle$ input to function
  - $|y\rangle$ qubit to write function result to
  - $|y \bigoplus f(x)\rangle$, the XOR ensures that the oracle is reversible (as each image has a unique preimage)
  - initializing $y = |0\rangle$ we only get the function value $|x\rangle|f(x)\rangle$ as $0 \bigoplus x = x$

- initializing $y = |-\rangle$ we get phase kickback to $|x\rangle|-\rangle \xrightarrow{U_f} (-1)^{f(x)}|x\rangle|-\rangle$ (a phase is applied to the input qubit)
  $\begin{cases} |x\rangle|-\rangle & f(x) = 0 \\ -|x\rangle|-\rangle & f(x) = 1 \end{cases}$
- Note: this is called a phase oracle ($U_f|x\rangle|-\rangle = (-1)^{f(x)}|x\rangle|-\rangle$)



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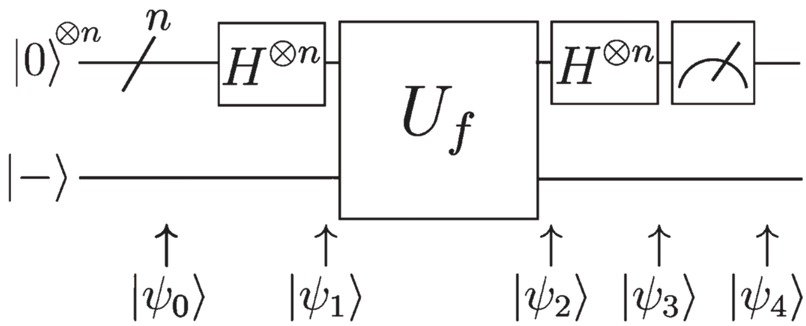
  - $|\psi_0\rangle = |00\rangle$
  - $|\psi_1\rangle = |01\rangle$
  - $|\psi_2\rangle = |+-\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle|-\rangle + |1\rangle|-\rangle)$
  - $|\psi_3\rangle = U_f\frac{1}{\sqrt{2}}(|0\rangle|-\rangle + |1\rangle|-\rangle) = \frac{1}{\sqrt{2}}(U_f|0\rangle|-\rangle + U_f|1\rangle|-\rangle) \overset{\text{phase oracle}}{=} \frac{1}{\sqrt{2}}((-1)^{f(0)}|0\rangle|-\rangle + (-1)^{f(1)}|1\rangle|-\rangle)$ ($|-\rangle$ can be omitted as its not needed)
  - case $f(0) = f(1)$: $\begin{cases} |\psi_3\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), & f(0) = f(1) = 0 \\ |\psi_3\rangle = -\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), & f(0) = f(1) = 1 \end{cases}$
    $|\psi_3\rangle = \pm\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = \pm|+\rangle$
  - case $f(0) \neq f(1)$: $\begin{cases} |\psi_3\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle), & f(0) = 0 \wedge f(1) = 1 \\ |\psi_3\rangle = -\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle), & f(0) = 1 \wedge f(1) = 0 \end{cases}$
    $|\psi_3\rangle = \pm\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) = \pm|-\rangle$
  - $|\psi_4\rangle = \begin{cases} \pm|0\rangle, & f(0) = f(1) \\ \pm\,|1\rangle, & f(0) \neq f(1) \end{cases}$

- measuring 0 iff function is constant and 1 iff function is balanced

4 Deutsch-Jozsa Algorithm

- generalized version of Deutsch Algorithm to n qubits
- $f : \{0,1\}^n \to \{0,1\}$
- f is constant iff $\forall x, f(x) = c$
- f is balanced iff $|\{x \mid f(x) = 0\}| = |\{x \mid f(x) = 1\}|$
- classical: $2^{n-1} + 1$ function calls (input half the possible inputs)
- quantum: one call of f in needed (exponential speed up)



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  - $|\psi_0\rangle = |00...0\rangle|-\rangle = |0\rangle^{\bigotimes n}|-\rangle$ (we can get the $|-\rangle$ by $H|1\rangle$)
  - $|\psi_1\rangle = H^{\bigotimes n}|0\rangle^{\bigotimes n}|-\rangle = \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} |x\rangle|-\rangle$ (uniform distribution)
  - $|\psi_2\rangle = \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} U_f|x\rangle|-\rangle \overset{\text{phase oracle}}{=} \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} (-1)^{f(x)}|x\rangle|-\rangle$
  - Note: $H^{\bigotimes n}|x\rangle = \frac{1}{\sqrt{2^n}} \sum_{z \in \{0,1\}^n} (-1)^{x \cdot z}|z\rangle, \ (*)$
  - $|\psi_3\rangle = \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} (-1)^{f(x)} H^{\bigotimes n}|x\rangle|-\rangle$
  $\overset{(*)}{=} \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} (-1)^{f(x)} \frac{1}{\sqrt{2^n}} \sum_{z \in \{0,1\}^n} (-1)^{x \cdot z}|z\rangle$
  $= \frac{1}{2^n} \sum_{x \in \{0,1\}^n} \sum_{z \in \{0,1\}^n} (-1)^{f(x) + x \cdot z}|z\rangle$

- consider the amplitude of $|0\rangle^{\otimes n}$ is $\dfrac{1}{2^n}\displaystyle\sum_{x\in\{0,1\}^n}(-1)^{f(x)}$ case f constant:
  $\begin{cases} \frac{1}{2^n}\sum_{x\in\{0,1\}^n}(-1)^0 = \frac{1}{2^n}2^n = 1, & f(x)=0 \\ \frac{1}{2^n}\sum_{x\in\{0,1\}^n}(-1)^1 = \frac{1}{2^n}(-2^n) = -1, & f(x)=1 \end{cases}$
- hence if f is constant the probability of measuring all zeros is 1
- hence if f is balanced half of the sum is 1 and half is -1 hence the probability of measuring all zeros is 0
- $\rightarrow$ measure and iff we get 000...0 then $f(x)$ is constant else balanced