Project 1: Numerical Methods for the Linear advection equation

Name – Swadesh Suman

McGill ID – 261097252

Objective statement – To solve the 1D Linear advection equation using the following schemes – Upwind Scheme, Lax Scheme, Lax – Wendroff Scheme, Leap-Frog Scheme, and MacCormack Scheme.

Given Initial condition - $u = 0.5(1+ \tanh(250(x-20)))$, for x between 0 & 40.

Problem 1 – Graphical comparison of the exact stationary solution with the solution of all numerical schemes at dt = 1 & dt = 0.5.

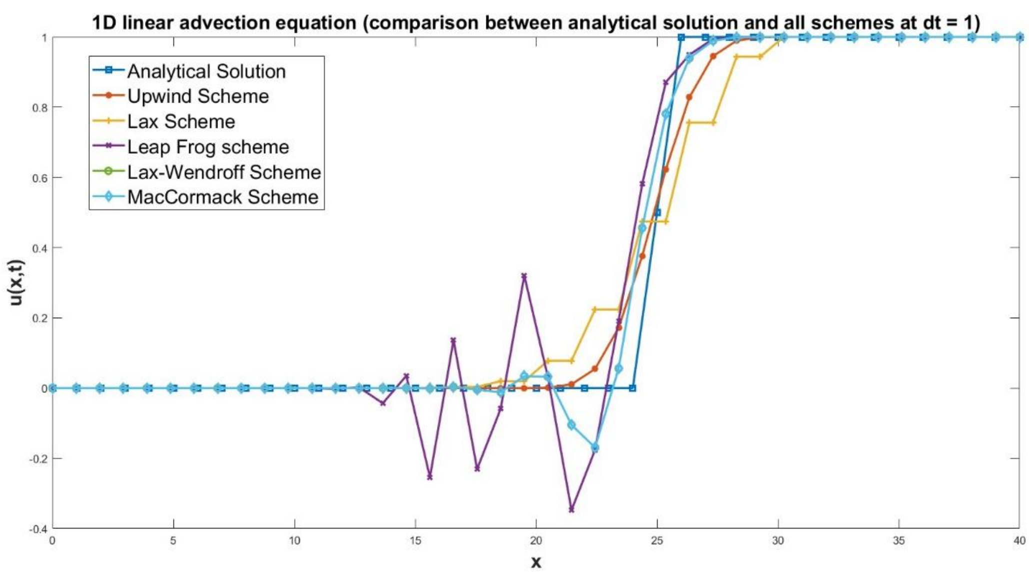

Fig. – 1

Fig. 1 & 2 shows the graphical comparison of the results between the analytical solution and the solution of all the five numerical schemes at dt = 1 & dt = 0.5, respectively.

Discussion -

- It can be seen in fig. that upwind schemes show the dissipative nature in both the upstream and downstream of the signal. Along with the upwind scheme, the Lax and Leap Frog scheme also shows the dissipative nature in both the upstream and downstream of the signal.

- The other schemes, like Lax Wendroff and MacCormack, show the dissipative nature downstream of the signal but the dispersion-type nature upstream of the signal. The results of Lax-Wendroff and MacCormack are exactly similar at all the locations.
- So, in the case of the upwind scheme, Lax and Leaf frog, the dissipative nature dominates, whereas, in the case of Lex Wendroff and Maccormack, the dispersion nature dominates.
- Comparing the results at different time scales, it can be seen that at dt = 1, the results converge faster than at dt = 0.5. This effect is seen in the results of all the numerical schemes.
- The results indicate that the upwind scheme performs better for signal representation. In contrast, the other schemes like Lex Wendroff and McCormack represent signal better downstream, but upstream show high dispersion nature.

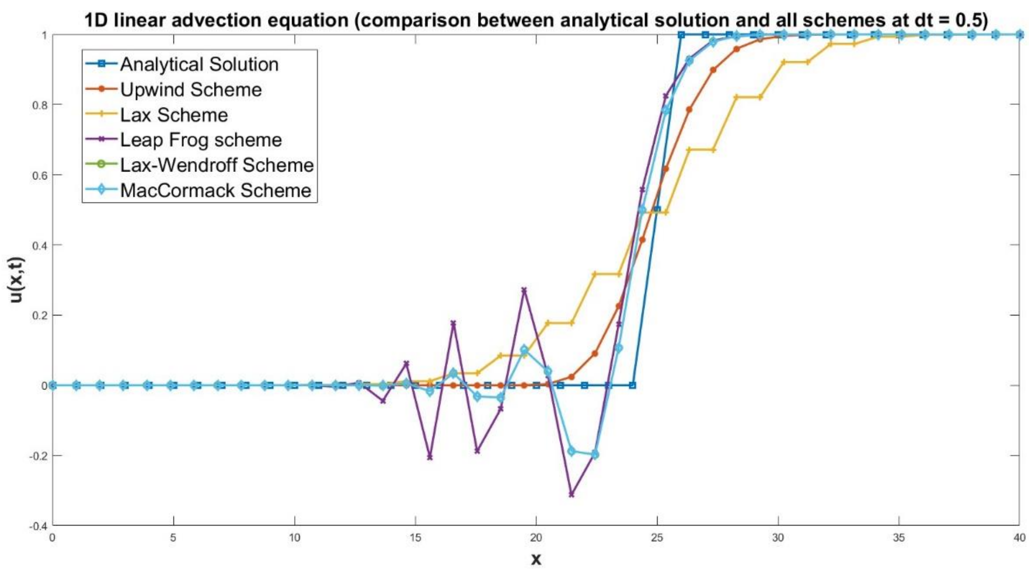

Fig. - 2

<u>Problem 2 – To perform the grid study in space and time using the Upwind and Lax Wendroff schemes.</u>

For performing the great study in space and time, an Initial 41 grid point mesh is selected. Figure 3 and figure 4 show the grid study in space and the grid study in time using the Upwind scheme, respectively. For performing the grid study in space, the time scale is chosen as 0.5; for the study in time, the length scale is selected as dx = 1.

Upwind Scheme

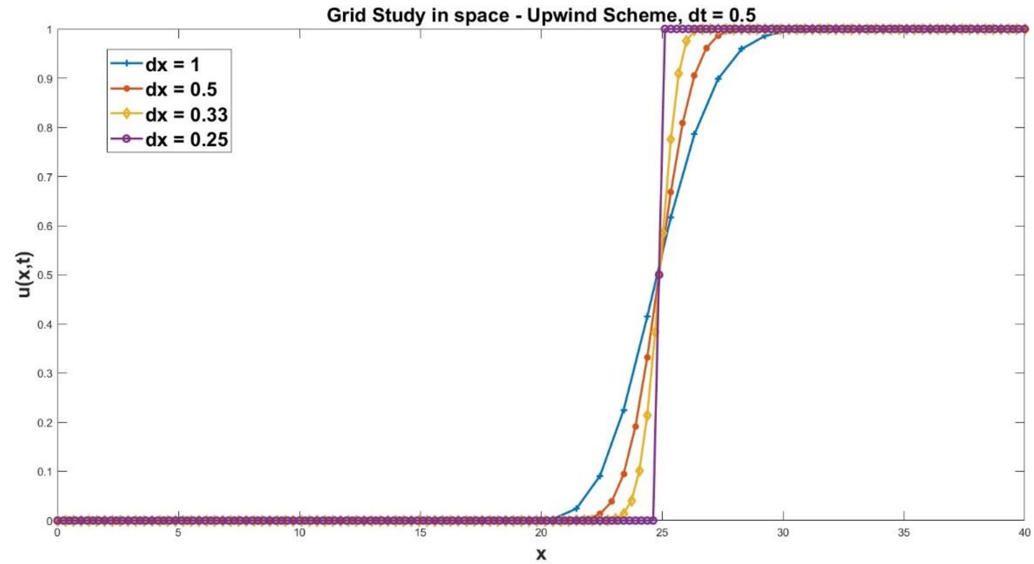

Fig. – 3

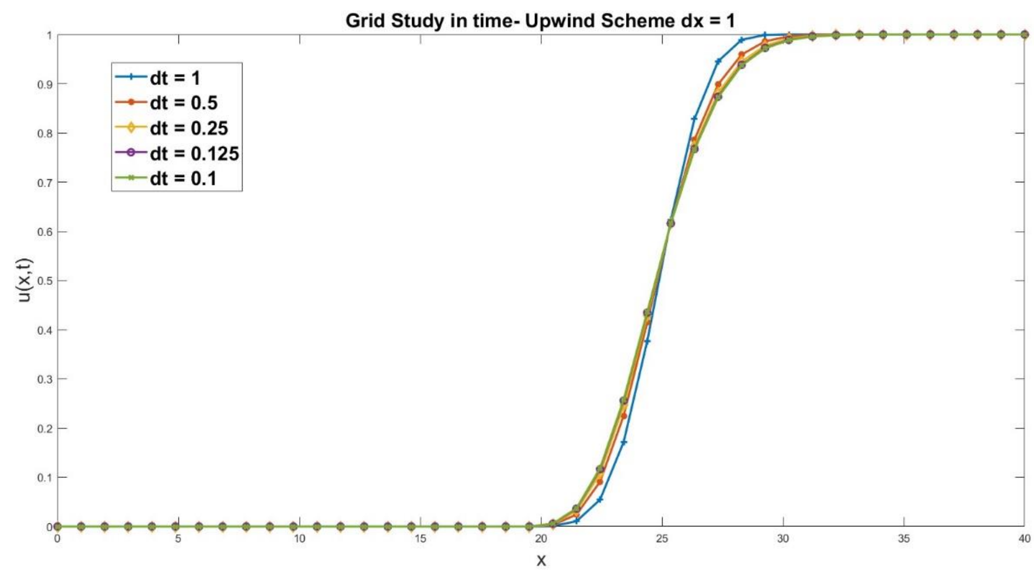

Fig. – 4

Discussion –

- From fig. 3, it can be seen that the results of the Upwind scheme continue to improve as we refine the mesh starting from dx = 1 till dx = 0.25. Further refinement from dx = 0.25 will result in an unstable upwind scheme, which will be explained in the next question.
- As the mesh is refined from dx = 1, the dissipative nature of the upwind scheme diminishes, and as seen at dx = 0.25, it almost vanishes.

- Fig 4 shows the grid study in time. There is a negligible effect of changing the time scales on the scheme's results. However, the dissipation nature of the upwind scheme increases slightly.

Lax- Wendroff Scheme

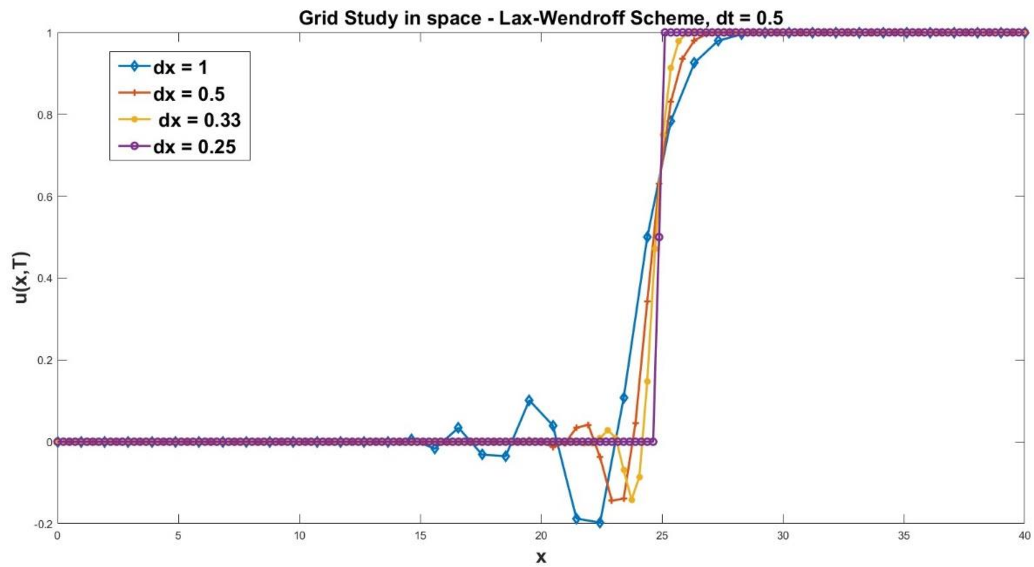

Fig. - 5

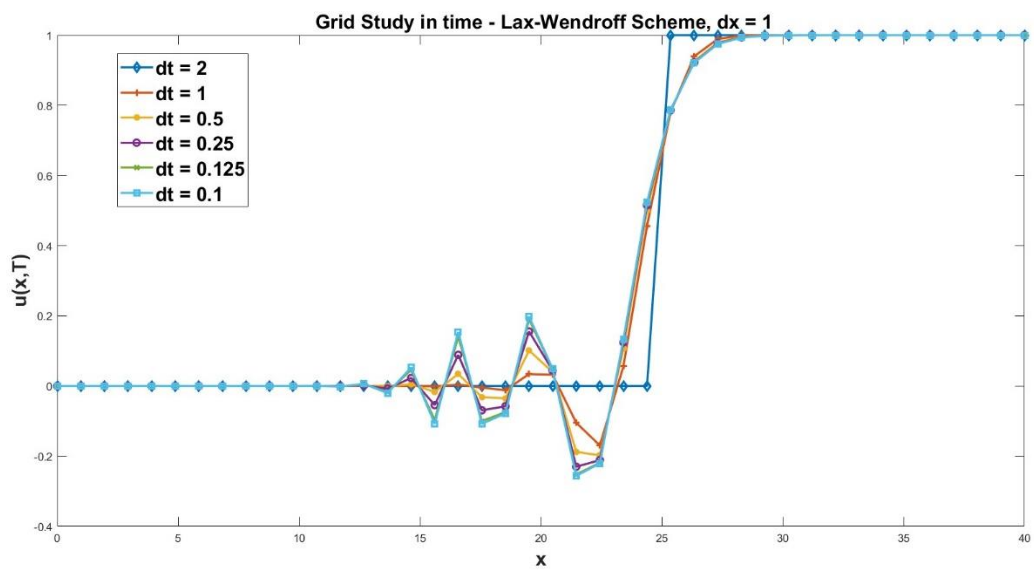

Fig. - 6

Discussion –

- Fig 5 & 6 show the spatial and temporal grid study for the Lax-Wendroff scheme.
- From fig. 5, it can be seen that the results of the Lax-Wendroff scheme continue to improve as we refine the mesh starting from dx = 1 till dx = 0.25. Further refinement from dx = 0.25 will result in an unstable Lax-Wendroff scheme, which will be explained later in the report.

- As the mesh is refined from dx = 1, the dispersive nature of the scheme diminishes, and as seen at dx = 0.25, it almost vanishes.
- In fig – 6, it can be seen that at dt = 2, which is at the upper limit of the stability of the scheme, the dispersive nature is absent. But as the temporal grid size is decreased, the dispersive and dissipation nature of the scheme continues to increase in the upstream and downstream regions of the signal, respectively, till dt = 0.1.

Problem 3 – To demonstrate the order of the accuracy of the scheme on a plot where the y-axis is the log of the error and the x-axis is dx.

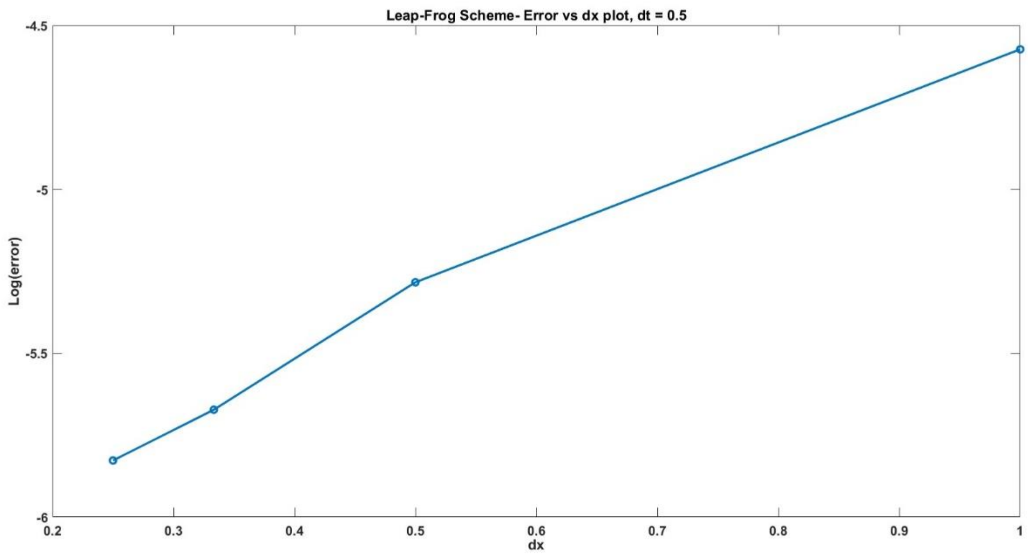

Fig. - 7

Discussion –

- Fig. 7 shows the plot of the error vs. dx, keeping the dt fixed at 0.5. This plot is used to verify the order of accuracy of the scheme. πWhere U_{exact} = Analytical solution of the equation

  U = solution obtained from Numerical scheme

- The Leap-Frog scheme is second-order accurate in space. The approximate slope of the line shown in the graph is 1.73.

Problem 4 – To demonstrate the stability conditions of the two schemes – 1. Upwind Scheme 2. Lax-Wendroff Scheme

The stability criteria for the Upwind and Lax – Wendroff are shown on the next page.

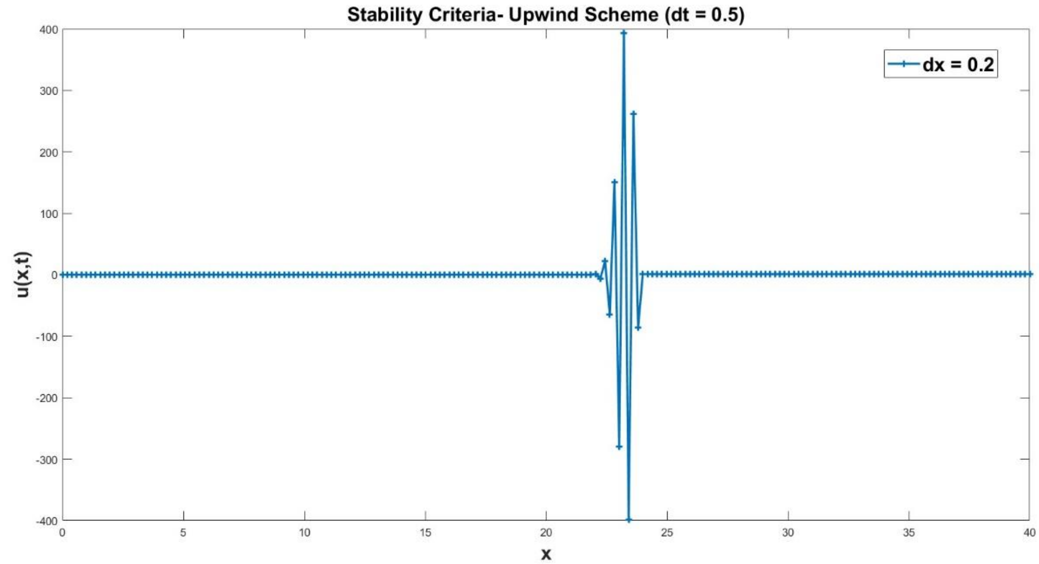

Fig. - 8

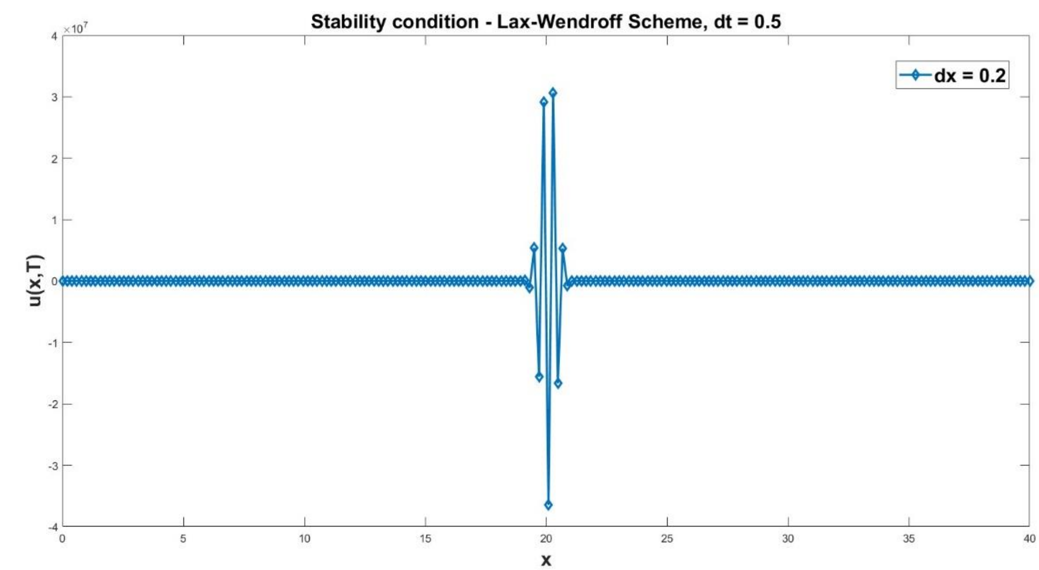

Fig. - 9

Discussion –

- Fig 8 shows the stability condition for the upwind scheme. According to the stability condition of the scheme, 0 <= v <= 1, where v(C*dt)/dx, c – advection velocity, dt – temporal grid size, dx – spatial grid size.
- It can be from fig. 3 that the upwind scheme remains stable as v remains in the range 0 to 1. But when the value of dx is reduced further from 0.25 to 0.2, the value of v goes above 1, and the scheme becomes unstable, as shown in fig. 8.

- Fig 9 shows the stability condition for the Lax-Wendroff scheme. According to the stability condition of the scheme, dt<=dx/C, where C – advection velocity, dt – temporal grid size, and dx – spatial grid size.
- Fig 5 shows the lax-Wendroff scheme remains stable till it meets the stability criteria, but as dx is further decreased from 0.25 to 0.2, the scheme becomes unstable, as shown in fig. 9.