Chapter 2 (Part 1): Bayesian Decision Theory (Sections 2.1-2.2)

Bayesian Decision Theory – Continuous Features…

- Let $\{\omega 1, \omega 2, \ldots, \omega c\}$ be the set of c states of nature (or "categories")
- Let $\{\alpha 1, \alpha 2, \ldots, \alpha a\}$ be the set of possible actions
- Let $\lambda(\alpha i \mid \omega j)$ be the loss incurred for taking action αi when the true state of nature is ωj

Bayesian Decision Theory – Continuous Features…

- Overall risk

- R = Sum of all $\underbrace{R(\alpha i \mid x)}_{\text{Conditional risk}}$ for i = 1,…,a

- Minimizing R ⟺ Minimizing $R(\alpha i \mid x)$ for i = 1,…, a

- $$R(\alpha_i \mid x) = \sum_{j=1}^{j=c} \lambda(\alpha_i \mid \omega_j) P(\omega_j \mid x)$$ for i = 1,…,a

Bayesian Decision Theory – Continuous Features…

- Select the action αi for which $R(\alpha i \mid x)$ is minimum

$$\Downarrow$$

- R is minimum and is called the Bayes risk = best performance that can be achieved!

Two-category Classification

- α_1 : deciding ω_1
- α_2 : deciding ω_2
- $\lambda_{ij} = \lambda(\alpha_i \mid \omega_j)$
- Loss incurred for deciding α_i when the true state of nature is ω_j
- Conditional risk:
  - $R(\alpha_1 \mid x) = \lambda_{11}P(\omega_1 \mid x) + \lambda_{12}P(\omega_2 \mid x)$
  - $R(\alpha_2 \mid x) = \lambda_{21}P(\omega_1 \mid x) + \lambda_{22}P(\omega_2 \mid x)$

Two-category Classification

- Our rule is the following:

  if $R(\alpha 1 \mid x) < R(\alpha 2 \mid x)$

- Action $\alpha 1$: "decide $\omega 1$" is taken
- This results in the equivalent rule :
- Decide $\omega 1$ if: $(\lambda 21 - \lambda 11)\, P(x \mid \omega 1)\, P(\omega 1) >$

  $(\lambda 12 - \lambda 22)\, P(x \mid \omega 2)\, P(\omega 2)$

- and decide $\omega 2$ otherwise

Likelihood Ratio

- The preceding rule is equivalent to the following rule:

$$if\ \frac{P(x\,|\,\omega_1)}{P(x\,|\,\omega_2)} > \frac{\lambda_{12} - \lambda_{22}}{\lambda_{21} - \lambda_{11}} \cdot \frac{P(\omega_2)}{P(\omega_1)}$$

- Then take action $\alpha 1$ (decide $\omega 1$)
- Otherwise take action $\alpha 2$ (decide $\omega 2$)

Likelihood Ratio…

- Optimal decision property

"If the likelihood ratio exceeds a threshold value independent of the input pattern x, we can take optimal actions"

Example: Checking on a course

- A student needs to make a decision which courses to take, based only on first lecture's impression
- From student's previous experience:

Quality of the course	good	fair	bad
Probability (prior)	0.2	0.4	0.4

- These are **prior probabilities**.

Example is taken from webcourse.cs.technion.ac.il/236875/Spring2007/ho/WCFiles/Tutorial2.ppt

Example: Checking on a course

- **The student also knows the class-conditionals:**

$Pr(x/w_j)$	good	fair	bad
Interesting lecture	0.8	0.5	0.1
Boring lecture	0.2	0.5	0.9

- **The loss function is given by the matrix**

$l(a_i/w_j)$	good course	fair course	bad course
Taking the course	0	5	10
Not taking the course	20	5	0

Example: Checking on a course

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Example: Checking on a course

- We can take decisions according to two rules:
  - Posterior probability
  - Risk minimization

Example: Checking on a course

- Given that the first class was interesting, the student wants to make an optimal decision.
  - Pr(bad|interesting)
  - Pr(fair|interesting)
  - Pr(good|interesting)

  - Pr(interesting|bad)
  - Pr(interesting|fair)
  - Pr(interesting|good)

  - Pr(interesting)

Example: Checking on a course

- The probability to get the "interesting lecture"(x= interesting):
- Pr(interesting)= Pr(interesting|good course)* Pr(good course)

  + Pr(interesting|fair course)* Pr(fair course)

  + Pr(interesting|bad course)* Pr(bad course)

  =0.8*0.2+0.5*0.4+0.1*0.4=0.4
- Consequently, Pr(boring)=1-0.4=0.6
- Suppose the lecture was interesting. Then we want to compute the **posterior** probabilities of each one of the 3 possible "states of nature".

Example: Checking on a course

$$\Pr(\text{good course}|\text{interesting lecture}) = \frac{\Pr(\text{interesting}|\text{good})\Pr(\text{good})}{\Pr(\text{interesting})} = \frac{0.8*0.2}{0.4} = 0.4$$

$$\Pr(\text{fair}|\text{interesting}) = \frac{\Pr(\text{interesting}|\text{fair})\Pr(\text{fair})}{\Pr(\text{interesting})} = \frac{0.5*0.4}{0.4} = 0.5$$

- We can get Pr(bad|interesting) = 0.1 either by the same method, or by noting that it complements the above two to 1.
- Now, we have all we need to make an intelligent decision about an optimal action

Example: Checking on a course

- The student needs to minimize the conditional risk

$$R(\alpha_i \mid x) = \sum_{j=1}^{c} \lambda(\alpha_i \mid \omega_j) P(\omega_j \mid x)$$

R(taking|interesting) =

Pr(good|interesting) λ(taking good course) + Pr(fair|interesting) λ(taking fair course) + Pr(bad|interesting) λ(taking bad course)

= 0.4*0+0.5*5+0.1*10 = 3.5

Example: Checking on a course

$$R(\alpha_i \mid x) = \sum_{j=1}^{c} \lambda(\alpha_i \mid \omega_j) P(\omega_j \mid x)$$

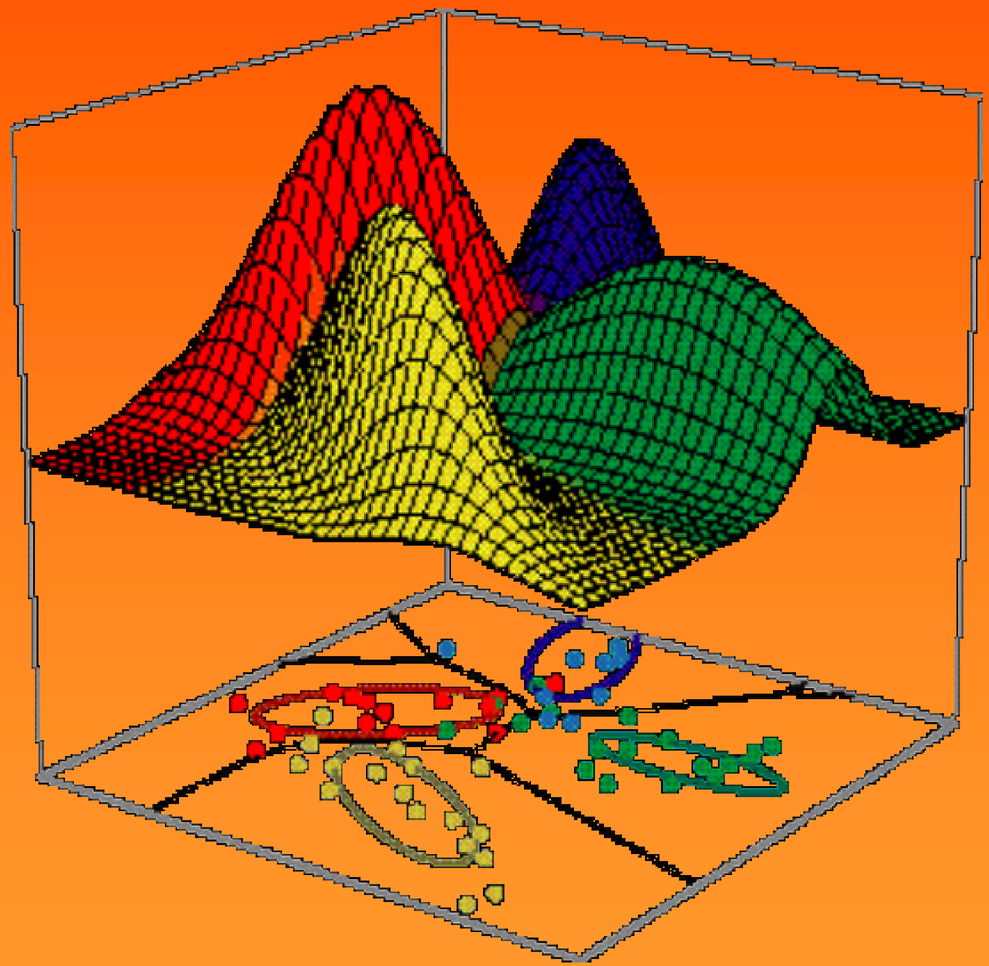
R(not taking|interesting) =

Pr(good|interesting)λ(not taking good course) +
Pr(fair|interesting)λ(not taking fair course) +
Pr(bad|interesting)λ(not taking bad course)

= 0.4*20+0.5*5+0.1*0 = 10.5

Constructing an optimal decision function

- So, if the first lecture was interesting, the student will minimize the conditional risk <u>by taking the course</u>.
- In order to construct the full decision function, we need to define the risk minimization action for the case of boring lecture, as well.



Pattern Classification

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