MA3219 Homework 3

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Question 1

Question 2

Show that a set $A \subseteq \mathbb{N}$ is accepted by some Turing machine $\iff$ it is a domain of some partial recursive function f.

Solution

Proof. $\underline{(\implies)}$: Suppose the Turing machine that accepts A has a code e. Then the universal function gives us that $\Phi(e, x)$ is the partial recursive function that is identical to the Turing machine. By s-m-n theorem, we have φ_e as the function whose domain is A.

$\underline{(\impliedby)}$: Let the partial recursive function f that has domain A be given. We want to construct a Turing machine M_f such that it mimics f. We already know how to encode as a Turing machine:

1. Succ, Zero and Projection
2. Composition and Recursion
3. Minimization (via setting up registers)

So we need only decompose down f to these building blocks of functions and we can have a Turing machine M_f that exactly mimics f. In particular, M_f accepts A. $\square$

Question 3

Let A and B be r.e. subsets of $\mathbb{N}$. Then prove that

1. both $A \cup B$ and $A \cap B$ are r.e..

   Proof. (Machine version) Suppose M_A, M_B are Turing machines that accept A, B respectively.

   For $A \cap B$, $M_A M_B$ (run one after another, changing halting states of M_A to the initial states of M_B) works.

   For $A \cup B$, we run M_A, M_B in parallel:

```
loop
    Carry out one step of M_A.
    If M_A halts, halt and return the result of M_A.
    Carry out one step of M_B.
    If M_B halts, halt and return the result of M_B.
end loop
```

$\square$

Proof. (Function version) Let φ_x, φ_y have domains A, B respectively.

For a domain of $A \cap B$, $\varphi_x \circ \varphi_y$ works. For a domain of $A \cup B$, consider

$$f(2k) = \varphi_x(k)$$
$$f(2k+1) = \varphi_y(k)$$

for any $k \in \mathbb{N}$. □

2. If $\alpha : \mathbb{N} \to \mathbb{N}$ is parial recursive, then both $\alpha^{-1}[A]$ and $\alpha[A]$ are r.e.. Moreover if A and α are recursive then $\alpha^{-1}[A]$ is recursive.

   (I don't know how to do the machine version, because I don't know how to define the "inverse" of a Turing machine. Could you give some hint please?)

   Proof. Suppose A is r.e., α is partial recursive. We want to show

   (a) $\alpha[A]$ is r.e.:
   Find φ_e such that $\varphi_e[\mathbb{N}] = A$. Then

   $$\alpha[A] = \alpha \circ \varphi_e[\mathbb{N}]$$

   gives us $\alpha[A]$ as a range of a partial recursive function $\alpha \circ \varphi_e$, therefore $\alpha[A]$ is r.e..

   (b) $\alpha^{-1}[A]$ is r.e.: Consider the partial characteristic function of $\alpha^{-1}[A]$ as f.
   The set is r.e. $\iff$ partial characteristic function is partial recursive. Then observe f:

   $$\begin{aligned} f(x) &= \begin{cases} 1 & \text{if } x \in \alpha^{-1}[A] \iff \alpha(x) \in A \\ \uparrow & \text{otherwise} \end{cases} \\ \iff f &= \chi_{A_P} \circ \alpha \end{aligned}$$

   which is a composition of two partial recursive functions. Hence the partial characteristic function of $\alpha^{-1}[A]$ is partial recursive and hence the set is r.e..

   The case where A, α are recursive is analogous to the proof in (b). □

Question 4

Solution

I am not sure how to solve using recursive functions. Below is a proof using Turing machines:

Proof. The idea is to run both machines in parallel:

```
loop
    Carry out one step of M_A.
    If M_A halts, halt and return the result of M_A.
    Carry out one step of M_B.
    If M_B halts, halt and return the result of M_B.
end loop
```

□

Question 5

Show that the class of recursive predicates is closed under bounded quantification.

Solution

Proof. Define *recursive predicates* as predicates whose characteristic function is recursive. Then we note that bounded $\forall$ is equivalent to multiplication of the results of individual characteristic functions; whereas bounded $\exists$ is equivalent to bounded addition. Since the quantifications are bounded, the formula are finite. Then recursive predicates can thus be combined in a primitive recursive manner, via addition and multiplication. $\square$

Question 6

Show that $A \subsetneq \mathbb{N}$ is recursive and infinite $\iff$ $\exists f : \mathbb{N} \to \mathbb{N}$ recursive and strictly increasing, and $A = f[\mathbb{N}]$.

Solution

Proof. $\underline{(\impliedby)}$: Let f be given. Then f is clearly infinite, suppose otherwise the maximum is $f(i) = k$, then $f(i+1)$ is defined (total) and $f(i+1) > k = f(i)$ (strictly increasing). To show that A is recursive, we define the characteristic function using f:

$$\chi_A(x) = \begin{cases} 1 & \text{if } \exists n \leq x(f(n) = x) \\ 0 & \text{otherwise.} \end{cases}$$

The search is bounded and f is recursive, thus χ_A is recursive as well.

$\underline{(\implies)}$: Let A and its characteristic function χ_A be given. We need only enumerate the set A by searching its recursive characteristic function:

$$f(0) = \min_k \{\chi_A(k) = 1\}$$
$$f(n+1) = \min_k \{\chi_A(k) = 1 \wedge k > f(n)\}.$$

Then f is by definition strictly increasing, since A is infinite. $\square$