MA2202S Cheat Sheet

1 Normal Subgroups

1. $N \triangleleft G$ is equivalent to for all $g \in G$:
   (a) $gng^{-1} \in N$ for all $n \in N$.
   (b) $gNg^{-1} \subseteq N$
   (c) $gNg^{-1} = N$
   (d) $gN = Ng$
2. $N \triangleleft G, K \leq G$. Then $N \cap K \triangleleft K$.
3. $K \triangleleft H \leq G$. Then if $N \triangleleft G$, $NK \triangleleft NH$.

2 Cyclic Groups

1. $|G| = n$ is cyclic is iff:
   (a) There is a **unique** subgroup of order d for every positive divisor d of n.
   (b) $\gcd(\varphi(n), n) = 1$.
2. $|G| = n$ is cyclic if:
   (a) $|G| = pq$ and $p > q, p \not\equiv 1(\mod q)$.
3. $\mathrm{Aut}(C_n) \cong C_{\phi(n)}$

3 Homomorphism

1. The kernel of a homomorphism is a normal subgroup.

4 Simple Groups

1. For $n \geq 5, A_n$is simple.
2. If $|G| = n, G$ simple, then
   (a) $|G| \mid k!$ for any subgroup of index k. Furthermore, if not $k = |G| = 2$, then $|G| \mid k!/2$.

5 Abelian Groups

1. A group G is Abelian iff
   (a) $G/Z(G)$ is cyclic.
   (b) All Sylow subgrops are normal and Abelian.

6 X-groups, X-Composition series

1. G is a X**-group** if X acts on G such that $x \cdot (g_1 g_2) = (x \cdot g_1)(x \cdot g_2)$.
   (a) $H \leq G$ is a X**-subgroup** if $x \cdot H \subseteq H$.
   (b) $H \triangleleft G$ is X**-normal** if $x \cdot H \subseteq H$.
   (c) $H \leq G$ is X**-simple** if it has no X-subgroups. H might not be simple.
2. If $X = \mathrm{Inn}(X)$, then
   (a) $H \triangleleft G \iff H \triangleleft_X G$
   (b) H simple in $G \iff H$ is X-simple in G
   (c) G/N simple $\iff G/N$ is X-simple.
3. G has composition series $\iff G/H$ and H has composition series.
4. (Zassenhaus' Lemma) $A_1 \triangleleft A_2$,$B_1 \triangleleft B_2$ are X-subgroups of G, then
$$\frac{A_1(A_2 \cap B_2)}{A_1(A_2 \cap B_1)} \cong \frac{B_1(A_2 \cap B_2)}{B_1(A_1 \cap B_2)}$$
5. (Dedekind Modular Law) $AB \cap C = A(B \cap C)$.
6. $G = H \times K, H, K$ simple. How many composition/chief series?
   (a) If K is non-Abelian, then $1 \triangleleft H \triangleleft G$ and $1 \triangleleft K \triangleleft G$ are the only chief/composition series.
   (b) If both are Abelian ($\implies C_p$): if $C_p \times C_q$ then same as above, otherwise $C_p \times C_p$ then there are $p+1$ normal subgroups.
7. Isomorphic $\implies$ isomorphic X-composition series. Converse is false.

7 Nilpotent and Soluble Groups

1. $[H, N] \leq N$.
2. $H, K \triangleleft G \implies [H, K] \triangleleft G$ and $[H, K] \leq H \cap K$.
3. Nilpotent group iff:
   (a) Any maximal subgroup must be normal.
   (b) Cannot have self-normalizing subgroups.
   (c) All Sylow p-subgroups are normal.
   (d) Is a direct product of all its Sylow subgroups.
   (e) $\gamma_n = 1$, where $\gamma_n = [\gamma_{n-1}, G]$.
   (f) $\zeta_n = G$, where $\zeta_i/\zeta_{i-1} = Z(G/\zeta_{i-1})$. (pre-image of center of quotient group)

8 p-groups

1. For each divisor, there is a normal subgroup of that order.
2. Every chief and composition factor are $\cong C_p$, composition length = power.
3. Every chief series is a composition series but converse is false.