# MA3201 Homework 1

Tan Yee Jian (A0190190L)

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## Problem 1

Given $\alpha = 3(1,2) - 5(2,3) + 14(1,2,3),\ \beta = 6(1) + 2(2,3) - 7(1,3,2)$, then

$$\begin{aligned}
\alpha + \beta &= 6(1) + 3(1,2) - 3(2,3) - 7(1,3,2) + 14(1,2,3) \\
2\alpha + 3\beta &= -18(1) + 6(1,2) - 16(2,3) + 21(1,3,2) + 28(1,2,3) \\
\alpha\beta &= -108(1) + 81(1,2) - 21(1,3) - 30(2,3) + 90(1,2,3) \\
\alpha^2 &= 34(1) - 70(1,2) - 28(1,3) + 42(2,3) - 15(1,2,3) + 181(1,3,2)
\end{aligned}$$

## Problem 2

1. Since $0 \neq 1$, then $n > 0$. Given that $x^n = x \cdot x^{n-1} = 0$, we have either $n = 1$ or $n > 1$.

   If $n = 1$, then $x \cdot x^{n-1} = x \cdot 1 = 0 \implies x = 0$.

   Otherwise if $n > 1$, then $x \cdot x^{n-1} = 0$ and both not zero implies $x, x^{n-1}$ are zero divisors. $\square$

2. $$\begin{aligned}
(rx)^n &= (rx)(rx)\dots(rx) \\
&= rxrx\cdots rx && \text{(associativity)} \\
&= r^n \cdot x^n && \text{($R$ is commutative)} \\
&= r^n \cdot 0 = 0
\end{aligned}$$

   thus $rx$ is a nilpotent for all $r \in R$.

3. $(1+x)(1 - x + \ldots + (-1)^{n-1}x^{n-1}) = 1 \pm x^n = 1 \pm 0 = 1$. I claim that $x \neq -1$ (therefore $1 + x \neq 0$ will be a unit). Assume the contrary that $x = -1$, then $x^n = \pm 1\ \forall n \in \mathbb{Z}_{>0}$, a contradiction to the nilpotency. Therefore, $(1 + x)$ is a unit.

4. Note that by (2), $(u^{-1}x)$ is also nilpotent, thus by (3), $(1 + u^{-1}x)$ is a unit. Then since the product of units will be a unit (Remark 1.8.2), $u \cdot (1 + u^{-1}x) = u + x$ is also a unit.

## Problem 3

1. Write $\phi(0) = \phi(1 + (1)(-1)) = \phi(1) + \phi(1)\phi(-1) = \phi(1)(1 + \phi(-1)) = 0$.

   If $\phi(1) \neq 0$ and $\phi(-1) \neq -1$, then $\phi(-1)$(and $1 + \phi(-1)$) is a zero divisor.

   A few observations:

   (a) $\phi(1) = 0 \implies \phi$ is the zero map, since then $\forall r \in R, \phi(r) = \phi(1 \cdot r) = 0 \cdot \phi(r) = 0$.

   (b) $\phi(-1) = -1$ implies $\phi(1) = \phi(-1)\phi(-1) = (-1)(-1) = 1$.

   Thus we must have $\phi(1)$ is a zero divisor.

   In the case where $S$ is an integral domain, then $\phi(1)(1 + \phi(-1)) = 0$ forces either factor to be 0. Since by observation 1, $\phi(1) = 0$ leads to $\phi$ being the zero map, we must have the second factor as 0. By the second observation, $\phi(1) = 1$ as desired.

2. I claim that the induced map is the original map restricted to $R^*$. I will verify that units in $R$ are mapped to units in $S$, ie. the new codomain $S^*$ is correct. That is the only thing we need to do since the homomorphic property of the restricted map is already given by the original map.

   Let $ab = ba = 1 = cd = dc$ for some $a, b, c, d \in R$. Then indeed, $\phi(a)\phi(b) = \phi(ab) = \phi(1) = 1$ we have units mapped to units.

3. Consider the product ring $R \times S$ of two rings $R, S$ both with $1 \neq 0$, then the embedding $\phi : R \to R \times S, r \mapsto (r, 0)$ maps $1_r$ to $(1_R, 0) \neq 1_{R\times S} = (1_R, 1_S)$.

# Problem 4

1. We take for granted the fact that $I + J$ is an ideal (proved in appendix), and just show that

   (a) $I, J$ are contained in $I + J$.

   *Proof.*
   $$I = \{i + 0|i \in I\} \subseteq \{i + j|i \in I, j \in J\} = I + J$$
   and the proof is symmetric for $J \subseteq I + J$. □

   (b) $I + J$ is the smallest ideal containing $I$ and $J$. In other words, we show for any ideal $K$ containing $I, J$, $I + J \subseteq K$.

   *Proof.* Let $K$ be given. Then for any $i \in I, j \in J$, $i \in K$ and $j \in K$. Since $K$ is an Abelian group wrt addition, $i + j \in K$ by closure property of the group $K$. Therefore $I + J \subseteq K$ for any ideal $K$ containing $I$ and $J$. □

2. We first show $IJ$ is an ideal.

   *Proof.* $IJ$ is a subring of $R$ since for any $\sum_{k=1}^n i_k j_k, \sum_{k=1}^m x_k y_k \in IJ$ where $i_k, x_k \in I$ and $j_k, y_k$ for all $k = 1, \ldots, \max(m, n)$,

   (a) $0 \in IJ \neq \emptyset$, and since $-x_k y_k = (-x_k)y_k$ can be rewritten as $i_{n+k}j_{n+k}$,
   $$\sum_{k=1}^{n} i_k j_k - \sum_{k=1}^{m} x_k y_k = \sum_{k=1}^{n} i_k j_k + \sum_{k=1}^{m} -x_k y_k = \sum_{k=1}^{n+m} i_k j_k \in IJ$$
   Thus $IJ$ is a subgroup of $R$ by the one-step subgroup test.

   (b) For products of any two elements from $IJ$, its fully expanded form must have terms of the form
   $$i_1 j_1 i_2 j_2 = [i_1(j_1 i_2)]j_2 = [i_1 r]j_2 = i_3 j_2$$
   for some $i_1, i_2, i_3 \in I, j_1, j_2 \in J$ since $I$ is an ideal. Therefore the product is a sum of finite terms of the form $i_k j_k$, which implies $IJ$ is closed under multiplication.

   Thus $I+J$ is a subring of $R$. To verify that $I+J$ is an ideal, note that For any $r \in R, i \in I, j \in J$,
   $$r(ij) = (ri)j \in IJ \ni i(jr) = (ij)r$$
   since $I, J$ are ideals. Thus $IJ$ is an ideal. □

   Now we show that $IJ$ is contained in $I \cap J$.

   *Proof.*
   $$\begin{aligned} IJ &= \{\Sigma ij|i \in I, j \in J\} \subseteq \{ir|i \in I, r \in R\} \subseteq I \\ IJ &= \{\Sigma ij|i \in I, j \in J\} \subseteq \{rj|j \in I, r \in R\} \subseteq J \\ \therefore IJ &\subseteq I \cap J \end{aligned}$$
   □

3. Since $n\mathbb{Z}$ are ideals for any $n \in \mathbb{N}$, we have $2\mathbb{Z} \cap 4\mathbb{Z} = 4\mathbb{Z} \neq (2\mathbb{Z})(4\mathbb{Z}) = 8\mathbb{Z}$.
4. We first need that $I \cap J$ is an ideal.

   *Proof.* We know that $I \cap J$ is a subgroup of $R$. For any $k, l \in I \cap J$, $I \ni k, l \in J$, therefore $kl \in I \wedge kl \in J$ ($I, J$ are rings) implies $kl \in I \cap J$. Thus $I \cap J$ is closed under multiplication. □

   We just need to show that $I \cap J \subseteq IJ$ when $R$ is commutative and $I + J = R$.

   *Proof.* Let $k \in I \cap J$. Since $I + J = R$, take any $r \in R$ and let $r = i + j \in I + J$.

$$\begin{aligned} I \cap J &= R(I \cap J) && (I \cap J \text{ is an ideal}) \\ &= (I + J)(I \cap J) && (R = I + J) \\ &= [I(I \cap J)] + [J(I \cap J)] && (\text{distributive}) \\ &\subseteq IJ + JI && (I \supseteq I \cap J \subseteq J) \\ &= IJ + IJ = IJ && (R \text{ is commutative}) \end{aligned}$$

□

# Appendix

**Lemma.** *Given a ring $R$ with $1 \neq 0$, and two ideals $I + J$, then $I + J$ is also an ideal.*

*Proof.* $I + J$ is a subring of $R$ since for any $i_1 + j_1, i_2 + j_2 \in I + J$,

1. $0 \in I + J \neq \emptyset$, and
$$i_1 + j_1 - (i_2 + j_2) = (i_1 - i_2) + (j_1 - j_2)$$
   Thus $I + J$ is a subgroup of $R$ by the one-step subgroup test.
2. $$(i_1 - i_2) \cdot (j_1 - j_2) = [i_1(j_1 + j_2) + i_2 j_1] + i_1 j_2$$
   where the first term is the sum of two elements from $I$ postmultiplied, and the latter an element from $J$ premultiplied. Thus their sum belongs to $I + J$.

Thus $I + J$ is a subring of $R$. To verify that $I + J$ is an ideal, note that any $r \in R, i \in I, j \in J$,

$$r(i + j) = ri + rj \in I + J \ni ir + jr = (i + j)r$$

since $I, J$ are ideals. Thus $I + J$ is an ideal. □