Week 10 Assignment

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March 27, 2020

1. *Proof.* Given an open covering $\mathcal{G}$ of $E = \{x_0\} \cup \bigcup_{n=1}^{\infty} K_n$, we want to find a finite subcover.

   First, x_0 must belong to some open cover $G \in \mathcal{G}$. Then since G is open, there exists $r > 0$ such that $B[x_0, r] \subseteq G$. I claim that G contains infinitely many K_n, that is, there is a $N \in \mathbb{N}$ such that for all $n > N$, $G \supseteq \bigcup_{n=N}^{\infty} K_n$.

   To prove the claim, we use the r as given above (such that $B[x_0, r] \subseteq G$). We have $N_1, N_2 \in \mathbb{N}$ such that

   $$\forall x \in K_n, n \geq N_1, \rho(x, x_0) < r/2$$

   due to the convergence of $(x_n)_{n=1}^{\infty}$. Also,

   $$\forall x \in K_n, n \geq N_2, K_n < r/2$$

   due to the convergence of the diameters. Set $N = \max\{N_1, N_2\}$, then we have

   $$\begin{aligned} \forall x \in K_n, n \geq N, \rho(x, x_0) &\leq \rho(x, x_n) + \rho(x_n, x_0) \\ &< K_n + r/2 \\ &< r/2 + r/2 \\ &< r \\ \therefore x \in B[x_0, r] \end{aligned}$$

   which implies $K_n, K_{n+1}, \dots \subseteq B[x_0, r] \subseteq G \implies \{x_0\} \cup \bigcup_{n=N}^{\infty} K_n \subseteq G$ as desired.

   Now, we only have finitely many compact sets $K_1, \dots, K_{N-1}$ left. Clearly $\mathcal{G}$ must also be an open covering for each of them, which compactness gives us finite subcovers for each $K_1, \dots, K_{N_1}$. The union of all these subcovers with G as found above is an open covering for E, and furthermore is a finite union of finite sets, which must be a finite set too. Then we have found our finite subcover given an arbitrary open covering $\mathcal{G}$, and thus E satisfies the Heine-Borel property and is compact. $\square$

2. *Proof.* $\bigcup_{x \in M} B[x, r_x]$ must be an open covering of M (trivially because it definitely contains all the points in M) where r_x is such that $f(B[x, r_x])$ is a finite set.

   Since M is compact, we use the Heine-Borel property: given the open covering $\bigcup_{x \in M} B[x, r_x]$, there must be a finite subcover, i.e. $x_1, \ldots, x_n$ such that

$$\begin{aligned} M &\subseteq \bigcup_{i=1}^{n} B[x_i, r_{x_i}] \\ \therefore f(M) &\subseteq f(\bigcup_{i=1}^{n} B[x_i, r_{x_i}]) \\ &= \bigcup_{i=1}^{n} f(B[x_i, r_{x_i}]) \end{aligned}$$

   which $\bigcup\limits_{i=1}^{n} f(B[x_i, r_{x_i}])$ is a finite union of finite sets, and therefore its subset, $f(M)$ must also be finite. $\square$

3. *Proof.* Since our domain is a closed set, which means for every $x \in \overline{E}$, there is a sequence that converges to x. I claim that $\widetilde{f} : \overline{E} \to \overline{F}$ such that $\widetilde{f}(x) = \lim_{n \to \infty} f(x_n)$, where $(x_n)_{n=1}^{\infty}$ is a sequence in M converging to x, is the function we desired. But we need to check a few things, namely that 1. $\widetilde{f}$ is well-defined, 2. $\widetilde{f}(x) = f(x)$ given $x \in E$, and lastly, 3. $\widetilde{f}$ is continuous.

   (a) <u>Proof: $\widetilde{f}$ is well-defined.</u>
   Choose two sequences in M that converges to x, $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$. We want to show that

$$\lim_{n \to \infty} x_n = \lim_{n \to \infty} y_n = \widetilde{f}(x)$$

   .

Consider the sequence $(x_1, y_1, x_2, y_2, \dots)$. We claim that since the two sequences forming it are convergent hence Cauchy, the new sequence is also Cauchy. If so, then the sequence $(f(x_1), f(y_1), f(x_2), f(y_2), \dots)$ is also Cauchy since f preserves Cauchy sequences. Since N is complete, then $(f(x_1), f(y_1), f(x_2), f(y_2), \dots)$ converges to some value in N. Then its subsequences, $(f(x_n))_{n=1}^{\infty}$ and $(f(x_n))_{n=1}^{\infty}$ must both converge to the same value, which is $\widetilde{f}(x)$ by definition, which gives us $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = \widetilde{f}(x)$ as desired. This shows that $\widetilde{f}$ is well-defined.

Now we will fill in the proof for the claim that $(x_1, y_1, x_2, y_2, \dots)$ is Cauchy. Since $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ are both convergent, then given any $\epsilon > 0$, there exists $N_1, N_2 \in \mathbb{N}$ such that

$$\forall n > N_1, \rho(x_n, x) < \frac{\epsilon}{2}$$

and

$$\forall n > N_2, \rho(y_n, x) < \frac{\epsilon}{2}$$

.

Therefore, letting $N = \max\{N_1, N_2\}$, given any $\epsilon > 0$, given $p, q > N$, we note that

$$\begin{aligned}
\rho(x_p, y_q) &\leq \rho(x_p, x) + \rho(x, y_q) < \frac{\epsilon}{2} \\
\rho(x_p, x_q) &\leq \rho(x_p, x) + \rho(x, x_q) < \frac{\epsilon}{2} \\
\rho(y_p, y_q) &\leq \rho(y_p, x) + \rho(x, y_q) < \frac{\epsilon}{2}
\end{aligned}$$

which shows that any two terms from x_N onwards is at most ϵ apart with each other, hence $x_1, y_1, x_2, y_2, \dots$ is Cauchy.

(b) Proof: $\widetilde{f}(x) = f(x)$ for all $x \in E$

Using well-definedness we shown above, it is sufficient to examine any sequence in $\overline{E}$ that converges to x. Namely, the sequence $(x, x, \dots)$ is such a convergent sequence, and thus,

$$\forall x \in E, \widetilde{f}(x) = \lim_{n\to\infty} f(x) = f(x)$$

.

(c) Proof: $\widetilde{f}$ is continuous.

We prove this by contradiction. Suppose $\widetilde{f}$ is not continuous at $a \in \overline{E}$. Then $\exists \epsilon > 0$ such that $\forall \delta > 0, \rho(x,a) < \delta \implies \tau(\widetilde{f}(x), \widetilde{f}(a)) \geq \epsilon$.

For that given ϵ, we choose the sequence $(x_n)_{n=1}^{\infty}$ from E such that

$$\rho(x_k, a) < \frac{1}{k}$$

. We claim that $(x_n)_{n=1}^{\infty}$ converges to a. To show that, let $\epsilon' > 0$ be given. Then we can choose N such that $\frac{1}{N} < \epsilon'$, thus for any $n \geq N$,

$$\rho(x_n, a) < \frac{1}{n} \leq \frac{1}{N} < \epsilon'$$

, which shows that $(x_n)_{n=1}^{\infty}$ converges to a.

Therefore $(x_n)_{n=1}^{\infty}$ is also Cauchy. The image of this sequence under f, $(f(x_n))_{n=1}^{\infty}$ is Cauchy since f preserves Cauchy sequences. Then under completeness of $\langle N, \tau \rangle$, $(f(x_n))_{n=1}^{\infty}$ converges to $\widetilde{f}(a)$. However, by definition of convergence of sequences, this implies that for any $\epsilon > 0$, $\exists N \in \mathbb{N}$ such that $\forall n > N$,

$$\tau(f(x_n), \widetilde{f}(a)) = \tau(\widetilde{f}(x_n), \widetilde{f}(a)) < \epsilon$$

, a contradiction. Therefore $\widetilde{f}$ is continuous.

$\square$