MA2108S Homework 6

Tan Yee Jian

April 3, 2020

1. *Proof.* For the sake of contradiction, assume that

$$\forall d > 0, \rho(x, y) < d \text{ for all } x \in E \text{ and all } y \in F$$

.

Then let

$$\begin{aligned}\rho(x_1, y_1) &< 1 \\ \rho(x_2, y_2) &< \frac{1}{2} \\ &\cdots \\ \rho(x_n, y_n) &< \frac{1}{n}\end{aligned}$$

Since E is compact, then the sequence $(x_n)_{n=1}^{\infty}$ has a convergent subsequence, $(x_{n_k})_{k=1}^{\infty}$, converging to a certain $x \in E$. We claim that $(y_{n_k})_{k=1}^{\infty}$ is also convergent to x. If this is true, then since F is closed, we have also $x \in F$. This then implies that

$$x \in (E \cap F) \iff (E \cap F) \neq \emptyset$$

, a contradiction.

To prove that $(y_{n_k})_{k=1}^{\infty}$ is convergent: let ϵ be given. Then since $(x_{n_k})_{k=1}^{\infty}$ is convergent,

$$\exists N_1 \in \mathbb{N} \text{ such that } \rho(x_{n_k}, x) < \frac{\epsilon}{2} \text{ for any } n_k > N_1$$

.

Also, we can choose N_2 such that $\frac{1}{N_2} < \frac{\epsilon}{2} \iff N_2 > \frac{2}{\epsilon}$, which implies

$$\rho(x_{n_k}, y_{n_k}) < \frac{1}{n_k} \leq \frac{1}{N_2} < \frac{\epsilon}{2}$$

.

Taking $N = \max\{N_1, N_2\}$, we have for any $n \geq N$,

$$\rho(y_{n_k}, x) \leq \rho(y_{n_k}, x_{n_k}) + \rho(x_{n_k}, x) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

which shows that $\lim_{k \to \infty} y_{n_k} = x$. □

2. (a) *Proof.* For any $\epsilon > 0$, we can choose $\delta < \frac{\epsilon}{C}$, and thus for any $x, y \in M_1$,

$$\begin{aligned} \rho_1(x,y) &< \delta < \frac{\epsilon}{C} \implies \\ \rho_2(f(x), f(y)) &\leq C\rho_1(x,y) \\ &< C(\frac{\epsilon}{C}) \\ &= \epsilon \end{aligned}$$

F is uniformly continuous. □

(b) *Proof.* We use the fact that g is uniformly continuous: take any $\epsilon > 0$, (for example $\epsilon = 1$), then there is a $\delta > 0$ such that

$$|x - y| < \delta \implies |g(x) - g(y)| < \epsilon$$

.

I choose $r = \delta$, then we try to find C. If $|x-y| \geq r$, then we can select $(n-1)$ "jumping points" from x to y in intervals of $\delta/2$, such that between two consecutive points, it can satisfy the uniform continuity. That is, without loss of generality, assume $x < y$, then we select $x, (x + \frac{\delta}{2}), \ldots, (x + (n-1)\frac{\delta}{2}), y$, where $(n-1)\frac{\delta}{2} < |x - y|$ and $\frac{n\delta}{2} \geq |x - y|$. By triangle inequality, we have

$$\begin{aligned} |g(x) - g(y)| \leq |g(x) - g(x + \frac{\delta}{2})| + |g(x + \frac{\delta}{2}) - g(x + \frac{2\delta}{2})| + \ldots \\ + |g(x + \frac{(n-1)\delta}{2}) - g(y)| \end{aligned}$$

Since each of the terms are at most $\frac{\delta}{2}$ apart, and we have

$$(n-1)\frac{\delta}{2} < |x - y| \implies n < \frac{2|x-y| + \delta}{\delta}$$

Thus

$$\begin{aligned} |g(x) - g(y)| &< n\epsilon \\ &< (\frac{2|x-y| + \delta}{\delta})\epsilon. \end{aligned}$$

Since $\delta = r \leq |x - y|$,

$$\begin{aligned} |g(x) - g(y)| &< (\frac{2|x-y| + \delta}{\delta})\epsilon \\ &\leq (\frac{2|x-y| + |x-y|}{\delta})\epsilon \\ &= (\frac{3\epsilon}{\delta})|x - y| \end{aligned}$$

and there we have it, $C = \frac{3\epsilon}{\delta}$. □

3. *Proof.* Yes, a uniformly convergent sequence of uniformly continuous function would converge to a uniformly continuous function, and I will show that using the "famous" $\epsilon/3$ argument.

   Let $\epsilon > 0$ be given. Then we want to show that

   $$\begin{aligned} \rho_2(f(x), f(y)) &\leq \rho_2(f(x), f_n(x)) + \rho_2(f_n(x), f_n(y)) + \rho(f_n(y), f(y)) \\ &< \epsilon \end{aligned}$$

   Using the given ϵ, the uniform convergence of $(f_n)_{n=1}^{\infty}$ gives us $N \in \mathbb{N}$ where

   $$\forall x \in M_1, \rho_2(f(x), f_n(x)) < \epsilon/3.$$

   And similarly, using the given ϵ again, the uniform continuity of $f : M_1 \to M_2$ gives us $\delta > 0$ such that

   $$\rho_1(x, y) < \delta \implies \rho_2(f_n(x), f_n(y)) < \epsilon/3.$$

   Therefore,

   $$\begin{aligned} \rho_2(f(x), f(y)) &\leq \rho_2(f(x), f_n(x)) + \rho_2(f_n(x), f_n(y)) + \rho(f_n(y), f(y)) \\ &< \epsilon/3 + \epsilon/3 + \epsilon/3 \\ &= \epsilon \end{aligned}$$

   $\therefore f$ is uniformly continuous on M_1. $\square$