UNL2210 Homework 1

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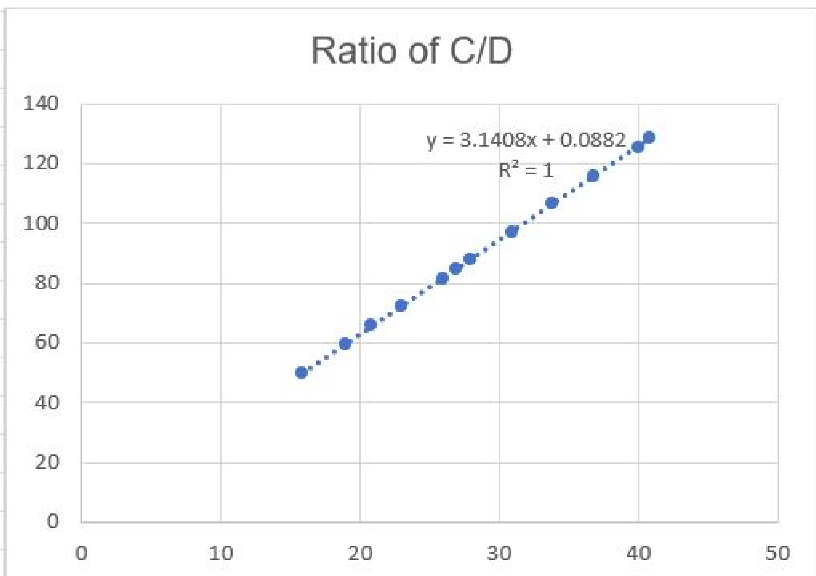
Question 7



Using Excel's LINEST function,
Gradient = 3.140807
so C/D is estimated to be 3.140807

Question 8

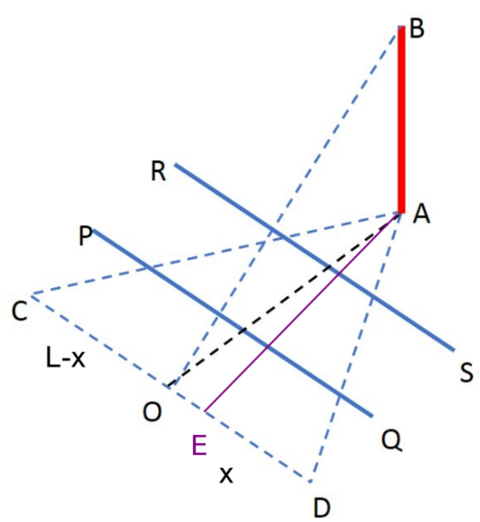
The statement that the ratio of the diameter to the circumference of any circle is a constant is a synthetic a priori statement. It is a priori as it can be proven from the axioms of Euclidean geometry and must be true for any circle.

A circle is a figure bounded by all points a constant length away from a fixed point. As such, any two circles must be similar to each other, and the ratio of their diameters must be equal to the ratio of their circumferences, ie. $C_1/C_2 = D_1/D_2$

Then $C_1/C_2 = D_1/D_2$ for any two circles, and we call this constant ratio the number π.

At the same time, the statement is also synthetic; the nature of the proposition is not encapsulated in the definition of a circle. The deductive steps of reasoning as mentioned above assume the use of Euclidean planes. Where there are non-Euclidean planes, the ratio of circumference over diameter (π) may no longer hold constant for all circles. The statement synthesises the concept of circles (geometric constructs) with the notion of a "measure" and its accompanying algebraic rules. It asserts more of the circle than what can already be found in its definition.

Question 9



Lemma.

$$AD = \frac{L \tan\beta}{(\tan\beta + \tan\alpha)\cos\alpha}$$

Proof. As the figure above, let E be the foot of perpendicular of A on CD such that AE is perpendicular to CD. Let $DE = x, CE = L - x$. Now considering the right-angled triangle AEC,

$$AE = (L - x)\tan\beta$$

and considering the right-angled triangle AED,

$$AE = x\tan\alpha$$

. Equating both, we have

$$\begin{aligned}(L - x)\tan\beta &= x\tan\alpha \\ L\tan\beta &= x\tan\beta + x\tan\alpha \\ x &= \frac{L\tan\beta}{\tan\beta + \tan\alpha}\end{aligned}$$

. Since $x/AD = \cos\alpha$, we must have

$$AD = \frac{x}{\cos\alpha} = \frac{L\tan\beta}{(\tan\beta + \tan\alpha)\cos\alpha}$$

as desired. □

Now we show

$$\begin{aligned}H &= \frac{L\tan\beta}{(\tan\beta + \tan\alpha)} \cdot \frac{\tan\gamma}{\cos\alpha} \\ R &= \frac{L\tan\beta}{(\tan\beta + \tan\alpha)} \cdot \frac{1}{\cos\alpha}\end{aligned}$$

Proof. First, notice that $H = AB$ is opposite $\angle BAD = \gamma$ in triangle $\triangle ABD$, thus

$$H = AB = AD \cdot \tan\gamma = \frac{L\tan\beta}{(\tan\beta + \tan\alpha)} \cdot \frac{\tan\gamma}{\cos\alpha}$$

and $R = AD$ is the other side in triangle $\triangle ABD$, the adjacent side of $\angle BAD$:

$$R = AD = \frac{AB}{\tan\angle BAD} = \frac{H}{\tan\gamma} = \frac{L\tan\beta}{(\tan\beta + \tan\alpha)} \cdot \frac{1}{\cos\alpha}$$

and we are done! □

Question 10

We will make a few claims with proofs first.

Claim. *Triangles $\triangle ABC$ and $\triangle ABD$ are equilateral.*

Proof. Looking at the circle with center A, we have $AB = AC = AD$ since all of them are radii of the circle. Similarly, when considering the circle with center B, we have $AB = BC = BD$. In particular,

$$AB = BC = CA \implies \triangle ABC \text{ is equilateral}$$

and

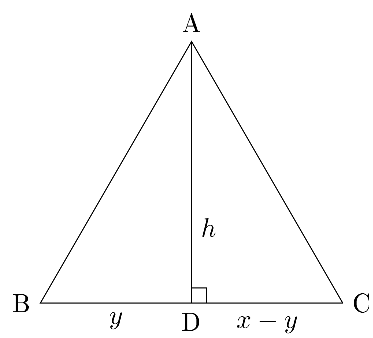
$$AB = BD = DA \implies \triangle ABD \text{ is equilateral}$$

as desired. □

Claim. *Equilateral triangles must have each interior angle equal to $60°$.*

Proof. Given any equilateral triangle $\triangle ABC$, we must have every pair of adjacent sides equal. Since the base angles of an isoceles triangle must be equal, and thus by considering different pairs of equal sides, we have all the internal angles of an equilateral triangle equal.
Using the fact that the sum of internal angles of an triangle must sum to $180°$, we have each interior angle of an equilateral triange to be $60°$. □

Claim. *Given an isoceles triangle $\triangle ABC$ where $AB = AC$, the foot of perpendicular of A on BC must bisect BC.*



Proof. Let the foot of perpendicular be D. We proceed by contradiction. Suppose otherwise, then the two sections separated by the foot of perpendicular must be of different length, i.e., $y \neq x - y$. Then we must have

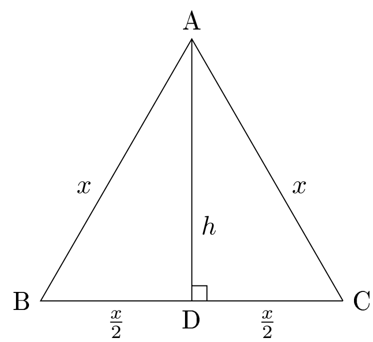
$$y \neq x - y \implies y^2 \neq (x-y)^2 \implies y^2 + h^2 \neq (x-y)^2 + h^2$$

and by the Pythagorean Theorem, $y^2 + h^2 = AB^2$ and $(x-y)^2 + h^2 = AC^2$ gives us that

$$y^2 + h^2 \neq (x-y)^2 + h^2 \implies AB^2 \neq AC^2 \implies AB \neq AC$$

, and this contradicts the assumption that fact that $AB = AC$. Therefore, we must have $AD = DC$ as desired. $\square$

Claim. *An equilateral triangle with side length x has area $\frac{\sqrt{3}}{4}x^2$.*



Proof. Similarly, let D be the foot of perpendicular of A on BC, where $\triangle ABC$ is equilateral. Our previous claim yields $BD = DC = \frac{x}{2}$ since $AB = AC$. By the Pythagorean Theorem, the length of AD, by considering either of $\triangle ABD$ or $\triangle ACD$ is

$$\sqrt{x^2 - (\frac{x}{2})^2} = \sqrt{\frac{3x^2}{4}} = \frac{\sqrt{3}x}{2}$$

Therefore,

$$\text{Area of } \triangle ABC = \frac{1}{2} \cdot BC \cdot AD = \frac{1}{2}(x)(\frac{\sqrt{3}x}{2}) = \frac{\sqrt{3}x^2}{4}$$

as desired. $\square$

Finally, with these 3 claims, we tackle the problem and show the area enclosed by the two arcs CAD and CBD is

$$2r^2(\frac{\pi}{3} - \frac{\sqrt{3}}{a}), \text{ where } a = 4$$

.

Proof. Note that the desired area is equal to the sum of area of the four sectors: ABC, BCA, ABD, BDA minus off the double-counted area, which is exactly the area of the quadilateral (rhombus) $ACBD$.

Each of the four sectors have an angle of $60°$ as seen from the equilateral triangles they contain. Since the two circles centered at A and B share a same radius r, all the sectors share the same area, that is $\frac{60°}{360°} = 1/6$ of the area of a circle with radius r. Four of these sectors contribute then to $4 \cdot (1/6) = 2/3$ the area of a circle with radius r, namely $\frac{2}{3}\pi r^2$ in total.

The area of the double-counted rhombus $ACBD$ is formed by two equilateral triangles, and therefore have an area of $\frac{\sqrt{3}r^2}{4} \times 2 = \frac{\sqrt{3}r^2}{2}$.

Taking the difference of the areas, the area enclosed by the arcs CAD and CBD is

$$\frac{2}{3}\pi r^2 - \frac{\sqrt{3}r^2}{2} = 2r^2(\frac{\pi}{3} - \frac{\sqrt{3}}{4})$$

as desired. $\square$