MA3110 Tut 1

1. ~~...~~ recall $\lim_{x\to c} f(x) = L \iff [\forall \varepsilon > 0, \exists \delta > 0$ s.t. $|x-c| < \delta \Rightarrow |f(x) - L| < \varepsilon$
$\iff -\varepsilon < f(x) - L < \varepsilon]$

$\lim_{x\to c^+} f(x) = L \iff$ —— 11 —— $c < x < c+\delta \Rightarrow |f(x) - L| < \varepsilon$.

$\lim_{x\to c^-} f(x) = L \iff$ —— 11 —— $c-\delta < x < c \Rightarrow |f(x) - L| < \varepsilon$

since $0 < |x-c| < \delta \iff 0 < x-c < \delta$ and $-\delta < x-c < 0$
$\iff c < x < c+\delta$ and $c-\delta < x < c$ result follows.

2. Basically, we want to have $0 < x < \delta \iff y > K$ for some $K \in \mathbb{R}$.
~~...~~ we suppose K is positive. otherwise, choose $K' = ~~...~~ 1-K$.
clearly letting $K = \frac{1}{\delta}$ and v.v. works. Ie,
($\Rightarrow$) $\lim_{x\to 0^+} f(x) = L$. Given a ε, we get a δ for free. Then
$x < \delta \Rightarrow \frac{1}{x} = y > K = \frac{1}{\delta}$. for the ε given.
($\Leftarrow$) $\lim_{y\to\infty} f(\frac{1}{y}) = L$. given a ε, we get a K for free. Then
$y > K$ (assume positive) > 0.
$\Rightarrow$ ~~...~~ $0 < \frac{1}{y} = x < \frac{1}{K} = \delta$. for the ε given.

3. ~~...~~ Use chain rule. First, write $r = \frac{p}{q}$ where $p \in \mathbb{Z}\setminus\{0\}$ and $q \in \mathbb{N}$.
If $p = 0$, $r = 0$ and $\frac{d}{dx} x^r = 0$.
$\therefore \frac{d}{dx} x^{\frac{p}{q}} = \frac{d}{dx} (x^{\frac{1}{q}})^p = p(x^{\frac{1}{q}})^{p-1} \frac{1}{q} x^{\frac{1}{q}-1}$
$= \frac{p}{q} x^{\frac{p-1+1-q}{q}} = \frac{p}{q} x^{\frac{p}{q}-1} = r x^{r-1}$ as ~~sho~~ desired.

4. We have $f'(c) = 0$ exists, $d := f(c)$, wts $g = f^{-1}$ is not diff'able at d.
Suppose otherwise. Then we can use chain rule:
$\frac{d}{dx} g(f(x))|_c = g'(d) f'(c) = 0$.
but $g(f(x)) = x$, $\therefore \frac{d}{dx} x|_c = 1 \neq 0$, $\rightarrow\leftarrow$. $\square$

5. ~~To apply inv. fn. thm, we need: $f(8) \neq 0$ and $f'(8)$ exists. f is ... $5x^4 + 4$ which is defined over $\mathbb{R}$. $\therefore g'(8) = ... \frac{1}{f'(1)}$~~
f is only strictly monotone on an interval, we shall restrict it to $f: [0,\infty) \to \mathbb{R}$.
$f'(x) = 5x^4 + 4$ is positive (cve. first deriv. test). and hence f is strictly monotone.
note that $f(1) = 8$, $\therefore g'(8) = \frac{1}{f'(1)} = \frac{1}{9}$. (what about other solns?)

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6. Consider $f: [0,x] \to \mathbb{R}$, $x \mapsto \sqrt{1+x}$. Then $\exists c \in (0,x)$

$$\frac{d}{dx}\sqrt{1+x}\Big|_{x=c} = \frac{\sqrt{1+x}-\sqrt{1+0}}{x-0}$$

inequalities of the form $f(x) < \text{or } a+bx$

a bound on $f(x)$.

higher powers → taylor's.

$$\frac{x}{2\sqrt{1+c}} + 1 = \sqrt{1+x}. \quad \text{since } c>0, \sqrt{1+c} > 1$$

$$\therefore \sqrt{1-x} = 1 + \frac{1}{2\sqrt{1+c}}x$$
$$< 1 + \tfrac{1}{2}x. \qquad \forall x > 0.$$

7. (a) consider $I = (-1,1)$, $c = 0 \in I$, $f(x) = x$, $g(x) = |x|$.

(b) If $f(c) \neq 0$, ie say $f(c) > 0$ then by cont. $\exists$ a neighbourhood U containing c s.t. $f(U) > 0$. On that intv, $g = f$ $\therefore$ g is diff $\Rightarrow$ f is diff

suppose $f(c) = 0$, then $g'(c) = 0$, since otherwise, by lemma 6.3.1, $\exists \delta$ s.t. for some $\cancel{t \in (c-\delta, c+\delta)}$ s.t. $g(c) < 0$.

$$g'(c) = \lim_{h\to 0} \frac{g(c+h)-g(c)}{h} = 0. \to \lim_{h\to 0}\frac{|f(c+h)|}{h} \quad \nearrow \lim_{h\to 0^-} \leq 0. \quad \searrow \lim_{h\to 0^+} \geq 0$$

$$= \lim_{h\to 0}\frac{|f(c+h)|}{h} \text{ exists and } = 0.$$

$(\forall \varepsilon > 0, \exists \delta \text{ s.t. } 0 < |h| < \delta, \left|\frac{|f(c+h)|}{h}\right| < \varepsilon$

$$\Rightarrow \left|\frac{f(c+h)}{h}\right| < \varepsilon$$

$$\therefore \lim_{h\to 0}\frac{f(c+h)-f(c)}{h} \text{ exists} = 0$$

(c) if $f(c) < 0$, $f'(c) = -g'(c)$, else $f'(c) = g'(c)$.

$\lim_{x\to c}\frac{f(x)-f(c)}{x-c} = f'(c)$

8. WTS $|f(v) - f(u) - (v-u)f'(c)| \leq \varepsilon(v-u)$ we have $\left|\frac{f(x)-f(c)}{x-c} - f'(c)\right| < \varepsilon$.

$= |f(v) - f(u) - (v-u)f'(c) - f(c) - cf'(c) + f(c) + cf'(c)|$ if $u < c < v$.

$\leq |f(v) - f(c) - (v-c)f'(c)| - |f(u) - f(c) - (u-c)f'(c)|$

$$= \left|\frac{f(v)-f(c)}{v-c} - f'(c)\right|(v-c) - \left|\frac{f(u)-f(c)}{u-c} - f'(c)\right|(c-u).$$

MA3110 Tut 2

1. WTS f diff on (a,b), $f \uparrow$ on (a,b) $\Rightarrow f'(x) \geq 0$ $\forall x \in (a,b)$.

$\forall x \in (a,b)$, $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \geq 0$.

suppose $h \to 0^+$, then $f(x+h) \geq f(x)$ $\Rightarrow$
$h \to 0^-$, then $f(x+h) \leq f(x)$

2. (i) $\lim_{x \to 0} \frac{(2x + 3x^2 \cos\frac{1}{x}) - 0}{x - 0} = 2 + \lim_{x \to 0} 3x \cos\frac{1}{x} = 2$ (by squeeze).

(ii) Let δ be given. We choose $x = \frac{1}{(2n+1)\pi} < \delta$, $n \in \mathbb{N}$.

Then ~~$f'(x) = \ldots$~~

when $x \neq 0$, $f'(x) = 2 + 6x \cos\frac{1}{x} + 3\sin(\frac{1}{x})$

at $x_n = \frac{1}{(2n - \frac{1}{2})\pi}$, $f'(x_n) = 2 + 0 - 3 = -1$. By lemma, $\exists \delta'$ s.t. $x_n + \delta' < \delta$

$\forall y \in (x_n, x_n + \delta')$ $f(x_n) > f(y)$. But x_n, y (we can force) $< \delta$.

4. f' is bdd on I: $\forall x \in I$, $|f'(x)| \leq k$ for some $k > 0$.

(if $k = 0$ the fn is constant, and the stuff we want to show is trivially true.) MVT on $[x,y]$: $\exists c \in (x,y)$,

$$|f'(c)| = \left|\frac{f(x) - f(y)}{x - y}\right| \leq k.$$

$$\Rightarrow |f(x) - f(y)| \leq k|x - y|. \quad \square$$

5. We show $\forall\, 0 < a < b \leq 1$, we have $\frac{f(b)}{b} > \frac{f(a)}{a}$ (i.e. $\frac{f(x)}{x}$ is str. inc.).

MVT on f rest. to $[0,a]$: $f'(c_1) = \frac{f(a) - 0}{a - 0} = \frac{f(a)}{a}$.

$[a,b]$: $f'(c_2) = \frac{f(b) - f(a)}{b - a}$

but $c_1 < c_2 \Rightarrow f'(c_1) < f'(c_2)$ (strict. inc)

$\Rightarrow \frac{f(a)}{a} < \frac{f(b) - f(a)}{b - a}$ since $b > a$, $a > 0$,

$\Rightarrow bf(a) \cancel{- af(a)} < af(b) \cancel{- af(a)}$

$\Rightarrow \frac{f(a)}{a} < \frac{f(b)}{b}$ (since $0 < a < b$) $\square$.

MA3110 Tut 3

1. When $n=1$, $h'(x) = \frac{2}{x^3}e^{-\frac{1}{x^2}}$ if $x \neq 0$.

~~when~~ $x=0 \Rightarrow \lim_{x\to 0} \frac{h(x)-h(0)}{x} = \lim_{x\to 0} \frac{e^{-\frac{1}{x^2}}}{x}$.

$$\lim_{x\to 0^+} \frac{e^{-1/x^2}}{x} = \lim_{y\to\infty} e^{-y^2} y$$

$$= \lim_{y\to\infty} \frac{y}{e^{y^2}}$$

$$= \lim_{y\to\infty} \frac{1}{2ye^{y^2}} = 0.$$

~~$\lim_{x\to 0^-} \frac{e^{-1/x^2}}{x} = \lim_{y\to\infty} \ldots$~~ $P_1(x) = \frac{2}{x^3}$.

Assume $n-1$. Then $h^{(n-1)}(x) = \begin{cases} e^{-\frac{1}{x^2}} P_{n-1}(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0 \end{cases}$.

$$\therefore x \neq 0 \Rightarrow h^{(n)}(x) = \frac{2}{x^3}e^{-\frac{1}{x^2}}P_{n-1}(\tfrac{1}{x}) + e^{-\frac{1}{x^2}}P'_{n-1}(\tfrac{1}{x})(-\tfrac{1}{x^2})$$

$$= e^{-\frac{1}{x^2}}\left[\frac{2}{x^3}P_{n-1}(\tfrac{1}{x}) - \frac{1}{x^2}P'_{n-1}(\tfrac{1}{x})\right].$$

where $P_n(x) =$

$$x=0 \Rightarrow \lim_{x\to 0^+} \frac{e^{-1/x^2}P_{n-1}(\frac{1}{x})}{x} = \sum_{i=0}^{k} a_k \lim_{x\to 0^+} \frac{e^{-1/x^2}}{x^{i+1}}$$

$= 0$. $\quad \hookrightarrow = 0$. $\square$

2. Given $\lim_{x\to a^+} g(x) = \lim_{x\to a^+} f(x) = 0$, $\lim_{x\to a^+} \frac{f'(x)}{g'(x)} = \infty$, WTS $\lim_{x\to a^+} \frac{f(x)}{g(x)} = \infty$.

given $M > 0$

$\lim_{x\to a^+} \frac{f'(x)}{g'(x)} = \infty \Rightarrow \exists c \in (a,b), \forall x \in (a,c)$, $\quad$ a x y c b

$\frac{f'(x)}{g'(x)} > M+1$. By Cauchy MVT on (x,y) where $a<x<y<c$,

$\frac{f(y)-f(x)}{g(y)-g(x)} = \frac{f'(u)}{g'(u)}$ for some $u \in (x,y) \subset (a,c)$ (i.e. where the derivative is good)

$\therefore$ ~~$\frac{f(y)}{g(y)}$~~ $\lim_{x\to a^+} \frac{f(y)-f(x)}{g(y)-g(x)} = \frac{f(y)}{g(y)} = \frac{f'(u)}{g'(u)} \geq M+1 > M$. $\square$

3.

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(i) $f''(c) = \lim_{h\to 0} \frac{f'(c+h) - f'(c)}{h}$

$= \lim_{h\to 0} \frac{\lim_{h_2 \to 0} \frac{f(c+h+h_2) - f(c+h)}{h_2} - f'(c)}{h^2}$

let $\lim_{h\to 0} \frac{g(h)}{h^2} = \lim_{h\to 0} \frac{f'(c+h) - f'(c-h)}{2h}$ $\int |h|$

$= \frac{1}{2}\lim_{h\to 0} \frac{f'(c+h) - f'(c)}{h} + \frac{f'(c) - f'(c-h)}{h}$

(ii) $f = \begin{cases} \frac{x^2}{2} & \text{if } x > 0 \\ -\frac{x^2}{2} & \text{if } x < 0 \\ ~~\text{undefined}~~\ 0 & x = 0. \end{cases}$

✓

Q4 show $1 + \frac{x}{2} - \frac{x^2}{8} \le \sqrt{1+x} \le 1 + \frac{x}{2}$

$\frac{d}{dx}\left(\frac{1}{2\sqrt{1+x}}\right) = \frac{1}{2}\left(-\frac{1}{2}\right)\frac{1}{\sqrt{(1+x)^3}}$

$= -\frac{1}{4}(1+x)^{-\frac{3}{2}}$.

$= -\frac{1}{4} \times \left(-\frac{3}{2}\right)(1+x)^{-\frac{5}{2}}$

$= +\frac{3}{8}(1+x)^{-5/2}$.

$\sqrt{1+x} = f(0) + \frac{f'(0)x}{1!} + \frac{f''(0)x^2}{2!}$

$= 1 + \frac{x}{2} - \frac{x^2}{8} + R_2(x)$

$R_2(x) = \frac{3}{8}(1+c)^{-5/2}$ for some $c > 0$.

$\Rightarrow (1+c)^5 \ge 1$

$\frac{3}{8} \frac{1}{\sqrt{(1+c)^5}} > 0$.

Q5. By Taylor's thm,

$f(x) = f(x_0) + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$ $/\!/ R_{n-1}$.

if n is even, $f^{(n)}(x_0) > 0$, then R_{n-1} ≥ 0 $\forall x$ around x_0, and since

< 0 a neighbourhood of.

n is odd LHS <, RHS > or v.v., neither unless $f^{(n)}(x_0) = 0$.

Q6. WTS $R_n(x) \to 0$ as $n \to \infty$.

$\lim_{n\to\infty} R_n(x) = \lim_{n\to\infty} \frac{\sin^{(n+1)}(x_0)}{(n+1)!}(x - x_0)^{n+1}$

$-\le \lim_{n\to\infty} R_n(x) \le +$ $\lim_{n\to\infty} |R_n(x)| \le \lim_{n\to\infty} \left|\frac{(x-x_0)^{n+1}}{(n+1)!}\right|$ $\frac{S_{n+1}}{S_n} = \frac{(x-x_0)}{n+2}$

$n \to \infty$, $\uparrow \to 0$

MA3110 Tut 4.

1. We have $\forall n \in \mathbb{N}$, $L(f,P_n) \le \underline{\int_a^b} f \le \overline{\int_a^b} f \le U(f,P_n)$

and as as $n \to \infty$, $\exists N$ s.t. $\forall n \ge N$,

$$A = L(f,P_n) \le \underline{\int_a^b} f \le \overline{\int_a^b} f \le U(f,P_n) = A.$$

$\therefore \int_a^b f$ is integrable, squeeze thm forces $\int_a^b f = A$. $\square$

2. (i) $L(f,P_n) = \frac{1}{n}\left(0 + \frac{1}{n^2} + \frac{2^2}{n^2} + \dots + \frac{(n-1)^2}{n^2}\right)$

$= \frac{1}{n^3}(1^2 + 2^2 + \dots + (n-1)^2)$

$= \frac{1}{n^3} \frac{(n-1)(n)(2n-1)}{6} = \frac{(n-1)(2n-1)}{6n^2} \to \frac{2}{6} = \frac{1}{3}$ as $n \to \infty$.

$U(f,P_n) = \frac{1}{n^3} \times \frac{n(n+1)(2n+1)}{6} = \frac{(n+1)(2n+1)}{6n^2} \to \frac{2}{6} = \frac{1}{3}$.

(ii) $\forall \varepsilon > 0$, $\exists N$ st. $|L(f,P_n) - \frac{1}{3}| < \frac{\varepsilon}{2}$

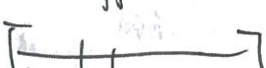
$|U(f,P_n) - \frac{1}{3}| < \frac{\varepsilon}{2}$ $\forall n > N$,

$\therefore U(f,P_n) - L(f,P_n) < \varepsilon \Rightarrow$ intgb.

intgb by Q1: $\exists$ seq. st. $U, L \to A$.

3. $f(x) = \begin{cases} 1 & \text{if rat.} \\ -1 & \text{if irrat.} \end{cases}$ is not intgb. but $|f|$ is intgb.

uppersum $= 1$, lowersum $= -1$ for any part. $\Rightarrow \overline{\int} = 1$, $\underline{\int} = -1$.

4. 

Idea is that the partition $\{0, \frac{1}{m}, \frac{1}{m}+\frac{1}{m^2}, \frac{1}{m}+\frac{2}{m^2}, \dots, \frac{1}{m}+\frac{m^2-m}{m^2}\}$ forms a sequence indexed by m.

Note that the first intv captures $\frac{1}{m}, \frac{1}{m+1}, \dots$ and the rest of the "$\frac{1}{n}$"s take place in at most $2(m-1)$ intervals.

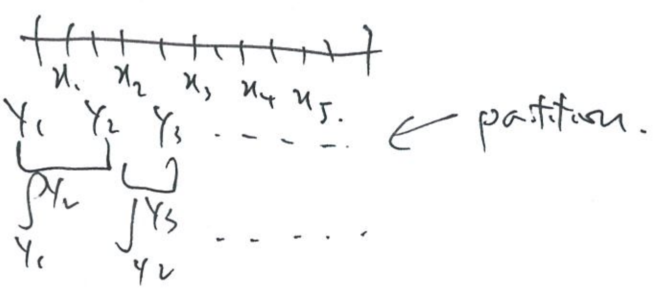
$U(f,P_m) \le \frac{1}{m}(1) + \frac{1}{m^2}(2m) = \frac{3}{m}$. $m \to \infty$, $U \to 0$.

$L(f,P_m) = 0$ clearly by density.

5. (i) Given $h(c) = k$. choose partition $\{a, c - \frac{\varepsilon}{2k}, c + \frac{\varepsilon}{2k}, b\}$, then

$U(h,P) = \frac{\varepsilon}{k}(k) = \varepsilon$, $L(h,P) = 0$.

(ii) The segments are integrable.



MA3110 Tut 5.

1. (i) ~~[illegible]~~ Monotone.

(ii) Piecewise monotone.

(iii) $h(x) = \begin{cases} \frac{-x}{x+2} & -1 \le x \le 0 \\ \frac{x}{x+2} & 0 \le x < 1 \\ x & 1 \le x \le 2 \end{cases}$ each are monotone.

2. (i) $\forall x, y \in$ ~~[illegible]~~ $[x_{i-1}, x_i]$,

$$\left|\frac{1}{h(x)} - \frac{1}{h(y)}\right| = \left|\frac{h(y)-h(x)}{h(x)h(y)}\right| \quad \because h(x)h(y) \geq c^2$$

$$\le \frac{1}{c^2}|h(y)-h(x)|$$

$$\le \frac{1}{c^2}(M_i(h,P) - m_i(h,P)) \quad \text{(Lem 7.25)}.$$

$$\therefore M_i(\tfrac{1}{h},P) - m_i(\tfrac{1}{h},P) \le \frac{1}{c^2}(M_i(h,P) - m_i(h,P)) \quad (\dagger).$$

(ii) Let ε be given. Then $\exists P$, a partition of $[a,b]$ s.t.

$$U(h,P) - L(h,P) < \frac{\varepsilon}{c^2}.$$

$$\Rightarrow U(\tfrac{1}{h},P) - L(\tfrac{1}{h},P) \le \sum_{i=1}^{|P|} (M_i(\tfrac{1}{h},P) - m_i(\tfrac{1}{h},P))\Delta x_i$$

$$\le \frac{1}{c^2}\sum_{i=1}^{|P|}(M_i(h,P) - m_i(h,P))\Delta x_i$$

$$\le \frac{1}{c^2}(U(h,P) - L(h,P))$$

$$< \frac{1}{c^2}\left(\frac{\varepsilon}{c^2}\right) = \varepsilon.$$

$\square$

3. Given $f : [a,b] \to \mathbb{R}$ itgb, $x_0 \in [a,b]$.

define $G : [a,b] \to \mathbb{R}$, by $G(x) = \int_{x_0}^{x} f$, and $\int_{x_0}^{x_0} f := 0$.

(i) WTS G is unif. ~~diff~~ cts. (transf into $F(x)$, and use Thm 7.31: f itgb $\Rightarrow F$ unif. cts.)

~~$G(x) = \int_{x_0}^{x} f$~~

$$F(x) = \int_a^x f$$

~~$= \int_a^{x_0}$~~

$$= \int_a^{x_0} f + \int_{x_0}^{x} f.$$

$$= F(x_0) + G(x).$$

$$\therefore G(x) = F(x) - F(x_0)$$

~~Let $\varepsilon > 0$ be given. Then~~ we show $|G(x) - G(y)| \le M|x-y|$ ($\Rightarrow$ unif cts)

$$|G(x) - G(y)| = |F(x) - F(x_0) - F(y) + F(x_0)|$$

$$= |F(x) - F(y)| \le M|x-y|. \text{ (by Thm 7.31)}$$

$\square$.

Therefore, G is unif. cts

MA3110 Tut 5 (2)

3. (ii) f is cts at pt $c \in [a,b]$, wts $G'(c) = f(c)$.

Strategy: ~~Fact~~ Rewrite G in F.

$\therefore G'(c) = \frac{d}{dx}(F(x) - F(x_0))\big|_{x=c}$

$= F'(c) = f(c)$ by FTC I. $\square$

4. (i) $H(x) = \int_0^{x^3} (f(t))^2 dt$. Let $g(t) = (f(t))^2$, then

$= \int_0^{x^3} g(t)\, dt$.

Let $G(x) = \int_0^x g(t) dt$, then

$H(x) = G(x^3) \Rightarrow H'(x) = 3x^2 G'(x^3)$ by chain rule.

Why can we apply chain rule? $\forall x \in [0,1]$,

1. x^3 is diffable on $[0,1]$

2. $f(t)$ is cts $\Rightarrow g(t)$ is cts on $[0,1]$

$\Rightarrow G(x)$ is diffable on $[0,1]$. (by FTC I).

$\therefore H'(x) = 3x^2 g(x^3) = 3x^2 (f(x^3))^2$

(ii) $G(x) = \int_{\frac{1}{1+x}}^1 f(\frac{1}{t}-1)\, dt = \int_{t=\frac{1}{1+x}}^{t=1} f(\frac{1}{t}-1)\, dt$. Let $y = \frac{1}{t} - 1$, $\frac{dy}{dt} = -\frac{1}{t^2}$.

$= \cancel{-\int_1^{\frac{1}{1+x}} f(\frac{1}{t}-1) dt}$. $= \int_1^0 f(y) dy$

Recall Q3: we can start at any pt. then let

$G(x) = -F(\frac{1}{1+x})$ where $F(x) = \int_1^x f(\frac{1}{t}-1) dt$

$\therefore G'(x) = \frac{1}{(1+x)^2} F'(\frac{1}{1+x})$ by chain rule. since

1. $\frac{1}{1+x}$ is diffable on $[0,1]$.

2. F is diffable on $[0,1]$ since $f(\frac{1}{t}-1)$ is cts on $[0,1]$.

$= \frac{1}{(1+x)^2} f(x)$ $\square$.

5. $\left| \sum_{k=0}^{n-1} \frac{1}{n} f(\frac{k}{n}) - \int_0^1 f(x) d(x) \right|$

$= \left| \sum_{k=0}^{n-1} \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(\frac{k}{n}) - f(x)\, dx \right|$ ($\triangle$-ineq, $|\int f| \le \int |f|$)

MA3110 Tut 5 (3)

6. (a) Consider $P \cup \{c\}$, $c \in Q \setminus P$. Then c must lie in (x_i, x_{i+1}) for some i, $\{x_0, x_1, \ldots, x_n\} = P$. Let $u_c = (c, f(c))$.

Then $\|u_i - u_{i-1}\| = \|u_i - u_c + u_c - u_{i-1}\|$
$\leq \|u_i - u_c\| + \|u_c - u_{i-1}\|$ (Δ ineq).

$\Rightarrow l(f,P) = \sum_{i=1}^{n} \|u_i - u_{i-1}\|$
$\leq \|u_1 - u_0\| + \ldots + \|u_i - u_c\| + \|u_c - u_{i-1}\| + \ldots + \|u_n - u_{n-1}\|$
$= l(f, P \cup \{c\})$.
$\leq \ldots \leq l(f, P \cup (Q \setminus P)) = l(f, Q)$. $\square$.

(c) (i) $l(f,P) = \sum_{i=1}^{n} \operatorname{sqrt}_2((x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2)$
$= \sum_{i=1}^{n} (x_i - x_{i-1}) \operatorname{sqrt}_2\left(1 + \left(\frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}\right)^2\right)$

applying MVT repeatedly for f on $[x_{i-1}, x_i]$,
$= \sum_{i=1}^{n} \Delta x_i \operatorname{sqrt}_2(1 + f'(c_i))$ for $c_i \in [x_{i-1}, x_i]$ $\forall i$

By defn of inf and sup, we have as desired.

(ii) $m(b-a) \leq L(g,P) \leq l(f,P) \leq U(g,P) \leq M(b-a)$
where $m = \inf\{f(x) \in \mathbb{R} \mid x \in [a,b]\}$.
$M = \sup$ —— " —— .

(iii) Since $l(f,P) \leq U(g,P)$ $\forall P$,
$\sup S = l(f) \leq \sup\{U(g,P) \in \mathbb{R} \mid P$ is a part $[a,b]\}$.
$= \int_a^b f$.

But g is itgble because it is bdd, and $f'(x)$ is cts $\Rightarrow \sqrt{1+g^2}$ is diff.

MA3110 Tut 6.

try to pull out, since it is the Δx. produce $\frac{k}{n}$s.

1. (i) $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{k^2}{n^3+k^3} = \lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n}\left(\frac{(\frac{k}{n})^2}{1+(\frac{k}{n})^3}\right)$

$= S(x\mapsto\frac{x^2}{1+x^3}, \{0,\frac{1}{n},\dots,\frac{n}{n}\})(\xi^{(n)})$

where $(\xi^{(n)}) = (\frac{1}{n},\frac{2}{n},\dots,1)$.

Since $n\to\infty \Rightarrow \|P\|\to 0$, Then the sum tends to the def integral

$\therefore \lim_{n\to\infty}\sum_{k=1}^{n}\frac{k^3}{n^3+k^3} = \frac{1}{3}\int_0^1\frac{3x^2}{1+x^3}dx = [\frac{1}{3}\ln|1+x^3|]_0^1 = \frac{1}{3}\ln 2 - 0 = \frac{1}{3}\ln 2$ □.

(ii) $\frac{1^k+2^k+\dots+n^k}{n^{k+1}} = \sum_{i=1}^{n}\frac{1}{n}(\frac{i}{n})^k = S(x\mapsto x^k, \{0,\frac{1}{n},\dots,\frac{n}{n}\})(\xi^{(n)})$

where $(\xi^{(n)}) = (\frac{1}{n},\dots,\frac{n}{n})$.

Since $n\to\infty$, $\|P\| = \frac{1}{n}\to 0$, the limit tends to

$\int_0^1 x^k dx = [\frac{x^{k+1}}{k+1}]_0^1 = \frac{1}{k+1}$. □

2. Claim: $\lim_{x\to\infty}f(x) = L \iff \forall\varepsilon>0, \exists M>0, x>M, x'>M \Rightarrow |f(x)-f(x')|<\varepsilon$.

($\Rightarrow$) Let $\varepsilon>0$ be given. Then $\exists M>0$ s.t. $\forall x,x'>M$, $|f(x)-L|<\varepsilon/2$.

$\therefore |f(x)-f(x')| = |f(x)-L-f(x')+L|$

$\leq |f(x)-L|+|f(x')-L|$

$< \varepsilon/2+\varepsilon/2 = \varepsilon$.

($\Leftarrow$) ~~Consider the sequence $(f(x_n))_{n=1}^{\infty}$, where $(x_n)_{n=1}^{\infty} = (a,\infty)\cap(n)_{n=1}^{\infty}$~~
~~Thus $\forall\varepsilon>0, \exists M>a \Rightarrow \exists N$ s.t. $\forall$ $n>N$,~~
~~$|f(x_n)-f(x_m)|<\varepsilon \Rightarrow (f(x_n))_{n=1}^{\infty}$ is cauchy and convergent in $\mathbb{R}$.~~

Let x_n be any sequence that tends to $+\infty$. Then $\forall M>0$, $\exists N$ such that $x_n, x_m > M$ whenever $n>N$

$\Rightarrow |f(x_n)-f(x_m)|<\varepsilon \Rightarrow (f(x_n))_{n=1}^{\infty}$ is a cauchy sequence.

By the sequential criterion, we have f is convergent in $\mathbb{R}$ and $\lim_{x\to\infty}f(x) = L$ for some $L\in\mathbb{R}$.

"for general $\int_a^\infty f$ to converge, we can check the cauchy criterion."

3. This is Q2 but replace $f(x)$ with $F(x) = \int_0^x f(t)dt$. Clear after rewriting:

$\lim_{k\to\infty}F(k) = L \iff \forall\varepsilon>0, \exists M>0$ s.t. $|F(b)-F(a)|<\varepsilon$ $\forall b>a>M$.
doesn't matter.

"for nonneg f, $\int_a^\infty f$ to converge, we just check that $\int_a^x f$ is bdd above"

4. (a) ($\Leftarrow$) Let $F(x)=\int_a^x f$. Then we know $0\le F(x)\le M$ $\forall x>a$.
F is increasing, since $\forall x\in(a,\infty)$, $F(x+\Delta x)-F(x)=F(\Delta x)\geq 0$.
Since monotone and upper bdd, by MCT, $\lim_{x\to\infty}F(x)=\sup\{F(x)\in\mathbb{R}\mid x\in(a,\infty)\}$.
Thus $\lim_{x\to\infty}F(x)$ exists.
($\Rightarrow$) $\lim_{x\to\infty}F(x)$ exists $\overset{?}{\Rightarrow}$ $F(x)\le M$ $\forall x>a$ for some M. Simply let M be the limit.

(b) $\int_a^x f\le\int_a^x g$ but $\lim_{x\to\infty}G(x)$ exists, $=L$
$\le L$
$\therefore$ F is also convergent as $x\to\infty$ by (a).

5. (a) Note that f is decreasing, so $a_k\geq f(x)\geq a_{k+1}$ $\forall x\in[a_k,a_{k+1}]$.
$\Rightarrow$ integrating from k to k+1 $a_k\geq\int_k^{k+1}f\geq a_{k+1}$
$\Rightarrow$ summing from k=1 to n-1 $\sum_{k=1}^{n-1}a_k\geq\int_1^{n-1}f\geq\sum_{k=2}^{n}a_k$.
Then $\sum a_n$ converges $\Leftrightarrow$ partial sums are bdd.
$\Leftrightarrow\int_1^n f$ is bdd $\Leftrightarrow\int_1^\infty f$ is convergent.
($\Rightarrow$) is obv, $L\geq\sum_{k=1}^{\infty}a_k\geq\int_1^\infty f$.
($\Leftarrow$) $a_1+L\geq\int_1^\infty f+a_1\geq\sum_{k=1}^{\infty}a_k$.

(b) $\sum_{n=1}^{\infty}\frac{1}{n^p}$ conv $\Leftrightarrow\int_1^\infty x^{-p}dx$ conv.
$\int_1^x t^{-p}dt=\left[\frac{t^{1-p}}{1-p}\right]_1^x=\frac{1}{1-p}(x^{1-p}-1)$
power of x, 1-p must be at most 0. But $1-p=0\Rightarrow$ divergent.
$\therefore 1-p<0\Rightarrow p>1$ □.

6. (i) Let $L_1=\int_0^\infty f$, $L_2=\int_{-\infty}^0 f$, then $\forall\varepsilon>0$, let $\left|\int_{-x}^0 f-L_1\right|<\frac{\varepsilon}{2}$ $\forall x>c_1$, $\left|\int_0^{x}f-L_2\right|<\frac{\varepsilon}{2}$ $\forall x>c_2$
~~$\left|\int_{-c}^{c}f-(L_1+L_2)\right|$~~ choose $c=\max(c_1,c_2)$, then $\forall x>c$,
$\left|\int_{-x}^{x}f-(L_1+L_2)\right|\le\left|\int_{-x}^0 f-L_1\right|+\left|\int_0^x f-L_2\right|$ since $x>c$ $\Rightarrow x>c_2,-x<-c_1$
$<\varepsilon/2+\varepsilon/2=\varepsilon$

(ii) $\int_0^\infty x\,dx$ diverges. But $\lim_{c\to\infty}\int_{-c}^{c}x\,dx=\lim_{c\to\infty}0=0$ since $x\mapsto x$ is odd.

"one of the very few computable integrals"

7. (i) $\int_1^k e^{-x}dx=[-e^{-x}]_1^k=e(1-e^{-k+1})\le e(1-e^{-1})\le e$ (ii) $e^{-x^2}<e^{-x}$ $\forall x\in[1,\infty)$.
(iii) $\int\frac{1}{x^2}$ converges since

uniform choice of K.
Method 1: Check $\|f_n - f\| \to 0$
Method 2: $\exists \varepsilon_0, \{n_k\}, x_k$ s.t. $|f_{n_k}(x_k) - f(x_k)| > \varepsilon_0$ $\forall k$
Method 3: properties

MA3110 Tut 7.

1. (i) Let $\varepsilon > 0$ be given. $\forall n \geq N > \frac{x}{\varepsilon}$ (fix x), then
$$\left|\frac{x}{x+n} - 0\right| \leq \left|\frac{x}{n}\right| = \frac{x}{n} < \frac{x}{x/\varepsilon} = \varepsilon.$$
$\lim \frac{x}{x+n} = \lim_{n\to\infty} \frac{x/n}{x/n+1} = \frac{0}{1} = 0$

(ii) $\|f_n - 0\| = \|f_n\| \leq \frac{a}{0+n} = \frac{a}{n} \to 0$ $\therefore \|f_n\| \to 0$ by squeeze. $\therefore f_n \xrightarrow{u} f$ or $[0,a]$
(method 1: $\|f_n - f\|$ or check $|f_n(x) - f(x)| \leq$ (a x-indep term) (to conclude $\forall x$))

(iii) $f_k(x) = \frac{x}{x+k} \Rightarrow f_k(k) = \frac{k}{k+k} = \frac{1}{2}$ $\therefore$ choose $\varepsilon_0 = \frac{1}{2}$, $x_k = k$, $\therefore f_n \not\xrightarrow{u} f$ on $[0,\infty)$
(This is method 2). Can use method 1: ~~[illegible]~~
$$\|f_n - f\| = \sup_{x\geq 0} \frac{x}{x+n} = 1$$

2. (i) $x \in [0,1) \Rightarrow x^n \to 0 \Rightarrow f_n(x) \to \frac{0}{1+0} = 0$.
$x = 1 \Rightarrow x^n = 1 \Rightarrow f_n(x) = \frac{1}{2}$
$x > 1 \Rightarrow x^n \to \infty \Rightarrow f_n(x) \to \frac{x^n}{x^n} = 1$.
$$\therefore g(x) = \begin{cases} 0 & x \in [0,1) \\ \frac{1}{2} & x = 1 \\ 1 & x > 1 \end{cases}$$

(ii) Let $b \in [0,1)$ be given. Then
$$\|g_n - g\|_{[0,b)} = \sup\left\{\left|\frac{x^n}{1+x^n}\right| \,\middle|\, x \in [0,b)\right\}$$
(method 1). $\leq \sup\{x^n \mid x \in [0,b)\} = b^n \to 0$ as $n \to \infty$.

(iii) Choose $\varepsilon_0 = \frac{1}{2}$, $x_k = (x^{1/k})^{\pm 1}$, then $\forall k \in \mathbb{N}$, ~~[illegible]~~
$$|f_k(x) - f(x)| = |f_k(x)| = \frac{(x^{1/k})^k}{1+(x^{1/k})^k} = \frac{1}{2} = \varepsilon_0.$$
Thus is not unif. conv. by Lemma 8.13.
(or g is disc but g_n is cts).

3. Let $\varepsilon > 0$ be given. Since $f_n \xrightarrow{u} f$ on A, then $\exists N_A$ s.t. $\forall n > N$, $\|f_n - f\|_A < \varepsilon$. Similarly since $f_n \xrightarrow{u} f$ on B, $\exists N_B$ s.t. $\forall n > N$, $\|f_n - f\|_B < \varepsilon$.
$\therefore \|f_n - f\|_{A\cup B} = \sup(\{|f_n(x) - f(x)| \mid x \in A\} \cup \{|f_n(x) - f(x)| \mid x \in B\})$
$(\sup(A\cup B) = \sup\{\sup A, \sup B\}) = \sup\{\|f_n - f\|_A, \|f_n - f\|_B\}$.
(since both $< \varepsilon$, $\sup < \varepsilon$) $< \varepsilon$ $\square$
(only for finitely many, by induction).

4. Let $\varepsilon > 0$ be given. Since $f_n \xrightarrow{u} f$, $g_n \xrightarrow{u} g$ on E, $\exists N_f, N_g$ respectively s.t. $\forall n > N_f$, $\|f_n - f\|_E < \varepsilon/2$, and $\forall n > N_g$, $\|g_n - g\|_E < \varepsilon/2$. Thus $\forall n > \max(N_1, N_2)$,
$\|(f_n + g_n) - (f+g)\|_E \leq \|f_n - f\|_E + \|g_n - g\|_E < \varepsilon/2 + \varepsilon/2 = \varepsilon$. $\square$

5. (i) Set $\varepsilon=1$, then $\exists N$ s.t. $\|f-f_N\|_E<1$.

$\Rightarrow \|f\|\le\|f-f_N\|+\|f_N\| < 1+M_N$.

(ii) Fix $\varepsilon=1$, then $\exists N$ s.t. $\|f_N-f\|_E<1$, let $M=\max\{M_i\}_{i=1}^{N}$.

$\therefore |f_n(x)|\le\|f_n\|_E\le\|f_n-f\|_E+\|f\|_E$.

$\le(M+1)+(1+M_N)$. is bdd.

6. (a) WTS $f_n(x)=x+\frac{1}{n}\xrightarrow{u} f(x)=x$.

$\|f_n-f\|=\sup\{|\frac{1}{n}|\}\to 0$ when $n\to\infty$. $\therefore f_n\xrightarrow{u} f$.

WTS $f_n^2(x)=x^2+\frac{2}{n}x+\frac{1}{n^2}\not\xrightarrow{u} f^2(x)=x^2$.

Let $\varepsilon_0=3$, $x=n$, then.

$|f_n(x)-f(x)|=|\frac{2}{n}(x)+\frac{1}{n^2}|$ $x=n$, then

$=|2+\frac{1}{n^2}|\le|2+1|=3=\varepsilon_0\neq 0$.

$\therefore f_n^2\not\xrightarrow{u} f$.

(b) By Q5, $|f_n(x)|\le M_1$ $\forall n\,\forall x$, $|g(x)|\le M_2$ $\forall x$.

Then $\|f_ng_n-fg\|\le\|f_ng_n-f_ng\|_E+\|f_ng-fg\|_E$

$=\|f_n\|\|g_n-g\|+\|g\|\|f_n-f\|$

$\forall\varepsilon>0$, let $N_1<\varepsilon/2M_1$, $N_2<\varepsilon/2M_2$ s.t.

$\forall n>N_1$, $\|g_n-g\|<\varepsilon/2M_1$ and $\forall n>N_2$, $\|f_n-f\|<\varepsilon/2M_2$.

Then $< M_1(\varepsilon/2M_1)+M_2(\varepsilon/2M_2)=\varepsilon$. $\square$

MA3110 Tut 8.

1. (a) $f(x) = 1$ ~~…~~ if $x \neq 0$, otherwise $f(0)=0$. problem

~~…~~ (b) not continuous.

(c) $|f_n(\frac{1}{n}) - f(\frac{1}{n})| = |\frac{1}{2} - 1| = \frac{1}{2}$ does not preserve cts
$\Rightarrow$ ~~does not~~ is not unif. conv.

2. (a) $f(x) = \frac{1}{3}$.

$$|f_n(x) - f(x)| = \left|\frac{3x + 3\sin nx - 3x - \sin^2 nx}{3(3x + \sin^2 nx)}\right|$$

$$\leq \left|\frac{\sin nx}{3x + \sin^2 nx}\right| + \left|\frac{\sin^2 nx}{9x + 3\sin^2 nx}\right| \quad \text{since } 0 \leq |\sin^2 nx| \leq 1, \ 0 \leq |\sin nx| \leq 1$$

$$\leq \frac{1}{3x+0} + \frac{1}{9x+0} \to 0 \text{ when } n \to \infty.$$

(b) $f(x)$ is cts hence itgb. Thus since $f_n \xrightarrow{u} f$, we have

$$\lim_{n\to\infty} \int_0^{\pi} f_n(x)\,dx = \int_0^{\pi} f(x)\,dx = \frac{\pi}{3}.$$

3. (i) $f(x) = 1$ if rational, else 0.

(ii) T4Q5(ii) states: if $f(x) = g(x)$ except for finitely many pts, then f is itgb $\Rightarrow$ g is itgb and $\int f = \int g$.

In particular, f_n differ from 0 at $\{\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \dots, \frac{1}{n}, \dots, \frac{n-1}{n}\}$ $\frac{n(n-1)}{2} + 1$ points.

However f is not itgb on $[0,1]$ since uppersum and lowersum diverges.

WTS unif conv does not preserve diff. $f_n(x) = \sqrt{x^2 + \frac{1}{n}}$ ~~…~~ $x \in (-1,1)$.

4. (i) $f(x) = |x|$.

(ii) $|f_n(x) - f(x)| = |\sqrt{x^2 + \frac{1}{n}} - \sqrt{x^2}|$
(show $f_n \xrightarrow{u} f$)

$$= \left|\frac{x^2 + \frac{1}{n} - x^2}{\sqrt{x^2+\frac{1}{n}} + \sqrt{x^2}}\right| \leq \frac{1}{n} \cdot \frac{1}{\sqrt{x^2+\frac{1}{n}} + \sqrt{x^2}} \quad \text{since } 0 \leq x^2$$

$$\leq \frac{1}{n} \cdot \frac{1}{\sqrt{0+\frac{1}{n}} + \sqrt{0}} \quad \text{since } \sqrt{x^2+\frac{1}{n}} \geq \sqrt{\frac{1}{n}}$$

$$\leq \frac{1}{\sqrt{n}} \to 0 \text{ as } n \to \infty, \ \forall x \in (-1,1).$$

(iii) x^2 is cts, $\frac{1}{n}$ is cts, $\sqrt{x}$ is cts on $[0,\infty)$ $\Rightarrow$ f_n is cts $\Rightarrow$ f_n is itgb.

Show f_n diff on $\mathbb{R}$, but f is not. $|f|$ is ~~…~~ not diff at 0. $\lim_{x\to 0^+} \frac{|x|}{x} = 1 \neq -1 = \lim_{x\to 0^-} \frac{|x|}{x}$

5. H is uniformly cts on $[-1,1]$ since cts over an ~~...~~ $[-1,1]$. Now given $\varepsilon>0$, $\exists\ \delta$ s.t. ~~...~~ $\forall x_1, x_2 \in [-1,1]$,

$$|x_1 - x_2| < \delta \Rightarrow |h(x_1) - h(x_2)| < \varepsilon.$$

$\varepsilon + 1$

$|g_n(x)| \le 1$, $g_n \xrightarrow{u} g \Rightarrow |g(x)| \le 1$ $\left(|g(x)| \le |g(x) - g_n(x)| + |g_n(x)| \right)$.

$\therefore$ given δ, $\exists N$ s.t. $\forall n \ge N$, (due to $g_n \xrightarrow{u} g$), $\forall x$,

$$|g_n(x) - g(x)| < \delta.$$

$\square$.

6. (i) $|g_n(x) - g_n(y)| = |x-y||g_n'(x_0)|$ for some $x_0 \in (x,y)$,

$\le M|x-y|$ fulfills the lipschitz cond and is unif. cts (more than that, the delta can be shared ($\forall x \ \supset \forall n \in \mathbb{N}$) g_n is equicontinuous.

(ii) Known: $g_n \to g$ pointwise on $[a,b]$.

g_n is uniformly continuous on $[a,b]$.

WTS: $g_n \xrightarrow{u} g$ on $[a,b]$.

Let ε be given. we want to find N s.t. $\forall n \ge N$, $\forall x$, $|g_n(x) - g(x)| < \varepsilon$.

1. Notice that $|g_n(x) - g(x)|$

$\le |g_n(x) - g_n(y)| + |g_n(y) - g(y)| + |g(y) - g(x)|$

we have a N for the ~~...~~ specific x, but want to find a general one.

2. For $\varepsilon/3$ to work, y has to be sufficiently close to x. Let $y \in \{x_0 = a, x_1, \dots, x_K = b\} = P$, is a partition of $[a,b]$ with norm $\|P\| <$ ~~...~~ $\min(\delta, \varepsilon/3)$. Now $\forall x \in [a,b]$, there is a friend

3. at most $\varepsilon/3$ away. ~~...~~ Take $N = \max\{N_1, \dots, N_K\}$ where N_i such that $g_n \to g$ at x_i, $|g_n(x_i) - g(x_i)| < \varepsilon/3$

4. Then $|g_n(x) - g_n(y)| < \varepsilon/3$ (by choice of y, $\|P\| < \varepsilon/3$)

δ equicts.

$|g_n(y) - g(y)| < \varepsilon/3$ ($N = \max$).

$|g(y) - g(x)| < \varepsilon/3$ ($\|P\| < \delta$ where $|y-x| < \delta \Rightarrow |g(x) - g(y)| < \varepsilon/3$)

δ gcts.

$|f'(x)| \le M$
1. MVT
2. Lipschitz.
3. Riemann sum

MA3110 Tut 9

1. (a) $\left|\frac{\cos(n\pi x^n)}{n^{3/2}}\right| \le 1/n^{3/2}$ $(\forall x)$ $\therefore \left\|\frac{\cos(n\pi x^n)}{n^{3/2}}\right\|_{\mathbb{R}} \le \frac{1}{n^{3/2}} = M_n$ $(\forall n)$.

$\sum M_n = \frac{1}{n^{3/2}}$ is a p-series with $p = 3/2 > 1$ hence is convergent.

$\Rightarrow \sum f_n = \sum_{n=1}^{\infty} \frac{\cos(n\pi x^n)}{n^{3/2}}$ is unif. conv on $\mathbb{R}$.

(b) $\left|\frac{x^n}{(\sqrt{n}+x^2)^4}\right|$ since $-1<x<1 \Rightarrow |x^n| \le 1$ and $0 \le x^2$,

$\le \frac{1}{(\sqrt{n}+0)^4} = \frac{1}{n^2} = M_n$ $\therefore \sum f_n$ is unif. conv. on $[-1,1]$.

(c) $\left|\frac{(-1)^n}{n^2+x^2}\right| \le \frac{1}{n^2+0} = M_n$.

(d) note that choose $x_n = \sqrt[n]{n!}$, then $\left|\frac{x^n}{n!} - 0\right| = 1 = \varepsilon_0$, we have

$f_n(x) = \frac{x^n}{n!}$ is not ~~conv~~ unif. conv. on $\mathbb{R}$. $\Rightarrow \sum f_n$ is not unif. conv.

nth term test $\|f_n\|_{\mathbb{R}} = \infty$.

or f_n is decreasing, $x \le 0 \Rightarrow f_n(x) \ge f_n(0) = n$ $\forall n$

2. (i) we must have $f_n \xrightarrow{u} 0$. If $x < 0$, $f_n(x) = ne^{-nx}$ is increasing with n.

If $x = 0$, $f_n(x) = n$ is increasing with n as well. ~~Suppose $r>0$ is given,~~

~~$\|ne^{-nx}\|_{[r,\infty)}$~~ Let $x>0$, we use the root test.

$\rho = \limsup_{n\to\infty} \sqrt[n]{|ne^{-nx}|} = \limsup_{n\to\infty} ne^{-x} = e^{-x} \limsup_{n\to\infty} \sqrt[n]{n} = 0$

hence $E = (0, \infty)$.

(ii) $|ne^{-nx}| \le |ne^0| = n$. $\therefore \|f_n\|_{(0,\infty)} = ne^0 = n \not\to 0$.

Hence $\sum f_n$ does not conv. unif.

(iii) claim: f is unif. cts on $[r,\infty)$ $\forall r>0$. Then since f_n are cts, we must have $f = \sum f_n$

cts on $[r,\infty)$ $\forall r>0 \Rightarrow \mathbb{R}$.

Pf: $\|ne^{-nx}\|_{[r,\infty)} = ne^{-nr}$. ~~[illegible]~~ $= M_n$.

Apply root test on $\sum M_n$, $\rho = \limsup |ne^{-nr}|^{\frac{1}{n}}$.

$= \limsup n^{\frac{1}{n}} e^{-r} = 0$.

Since $\sum M_n$ is conv, by WMT. $\sum_{n=0}^{\infty} ne^{-nx}$ is conv. on $[r,\infty)$ $\forall r>0$.

(iv) Since each f_n is integrable on $[r,\infty)$ and unif. conv. on $[r,\infty)$ $\forall r>0$,

$\int_1^2 f(x)dx = \int_1^2 \sum_{n=1}^{\infty} ne^{-nx} dx = \sum_{n=1}^{\infty} \int_1^2 ne^{-nx} dx = \sum_{n=1}^{\infty} \left[e^{-nx}\right]_1^2$

$= \sum_{n=1}^{\infty} (e^{-n} - e^{-2n}) = \frac{e^{-1}}{1-e^{-1}} - \frac{e^{-2}}{1-e^{-2}}$ since $0 < e^{-2} < e^{-1} < e^0$.

3. Claim 1: ~~[illegible]~~ $g_n(x) \geq 0 \; \forall n \forall x$.

Pf: suppose $\exists n_0, x_0$ s.t. $g_{n_0}(x_0) < 0$, then $\lim_{n_0 \to \infty} g_{n_0}(x_0) < 0$. violates unif. conv.

Claim 2: $\forall n \geq m \geq 1$, $|F_{n,m}(x) = \sum_{k=m}^{n} f_k(x)| \leq 2M$.

Pf: since $|\sum_{k=1}^{n} f_k(x)| \leq M \; \forall n \forall x$,

$|F_{nm}(x)| \leq |\sum_{k=1}^{n} f_k(x)| + |\sum_{k=1}^{m} f_k(x)| = 2M$.

Claim 3: $|\sum_{k=m}^{n} f_k(x) g_k(x)| \leq 2M g_m(x)$.

Pf: By Abel's formula, $= |F_{nm}(x) g_n(x) - \sum_{k=m}^{n-1} F_{k,m}(x)(g_{k+1}(x) - g_k(x))|$

$= |F_{nm}(x) g_n(x) + \sum_{k=m}^{n-1} F_{k,m}(x)(g_k(x) - g_{k+1})|$.

$\leq 2M |g_n(x) + g_m(x) - g_n(x)|$

$\leq 2M |g_m(x)| \leq 2M g_m(x)$.

$\therefore$ we show $\sum_{k=1}^{\infty} f_n g_n$ fulfills the cauchy criterion and is hence unif. conv.

$\Leftrightarrow \forall \varepsilon > 0, \exists K \in \mathbb{N}, n \geq m \geq K \Rightarrow$

$|\sum_{k=m}^{n} f_k g_k(x)| \leq 2M g_m(x) \overset{?}{(<)} \varepsilon$

we need only choose, since $g_m \xrightarrow{u} 0$, given $\varepsilon/2M$, sufficiently big K.

Then ~~[illegible]~~ $\forall m \geq K$, $|g_m(x) - g(x)| = |g_m(x)| < \frac{\varepsilon}{2M}$.

$= g_m(x)$. □

4. (i) let $f_n(x) = \frac{|x|}{x^2+n^2}$ then $\|f_n\|_{[-r,r]} \leq |\frac{|r|}{n^2+0^2}| = M_n$ is a p-s.r.s.

$\therefore$ By WMT, $\sum f_n(x)$ is conv. on $[-r,r] \; \forall r > 0$.

(ii) ~~[illegible]~~ f_n is cts and $\sum f_n$ is unif. conv. on $[-r,r]$

$\Rightarrow \sum f_n$ is cts on $[-r,r] \; \forall r \Rightarrow$ cts on $\mathbb{R}$.

(iii). We show ① f is conv. at $x=1$ say.

② $\sum f_n'$ is conv. at any $[a,b]$, $0 < a < b < \infty$.

Then by thm 8.3.5, we have f is diff at $(0,\infty)$ and $f' = \sum f_n'$.

Pf ① at $x=1$, $f_n(1) = \frac{1}{1+n^2} \to 0$.

② $f_n'(x) = \frac{d}{dx}\left(\frac{x}{x^2+n^2}\right) = \frac{(x^2+n^2) - 2x(x)}{(x^2+n^2)^2} = \frac{n^2-x^2}{(x^2+n^2)^2}$. on $[a,b]$

$\therefore \|f_n'\|_{[a,b]} \leq \frac{n^2-b^2}{(n^2+a^2)^2} \leq \frac{n^2-0}{(n^2+0)^2} = \frac{1}{n^2}$ $\therefore$ By WMT, $f_n'(x)$ is conv.

(iv) $\lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x \to 0} \frac{|x|}{x} \left(\sum_{n=1}^{\infty} \frac{1}{x^2+n^2}\right)$ we show $g(x) = \sum_{n=1}^{\infty} g_n(x) = \frac{1}{x^2+n^2}$ is conv., and

diverges ??? cts of $g_n(x)$ gives us ~~[illegible]~~ $\lim_{x\to 0} g(x) = g(0)$

$\|\frac{1}{x^2+n^2}\|_{\mathbb{R}} \leq \frac{1}{n^2}$ is conv. by WMT. Then $g(0) = \sum_{n=1}^{\infty} \frac{1}{0^2+n^2} > 0$. $\therefore$ diverges, $f'(0)$ does not exist. □

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1. (a) $\left|\frac{(n+1)!}{(n+1)^2 e^{2n+2}} \frac{n^2 e^{2n}}{(n!)}\right| = \left|\frac{n^2}{(n+1)e^2}\right|$ $\therefore \rho = \lim_{n\to\infty} \left|\frac{n^2}{(n+1)e^2}\right| = \infty$.

$\therefore R=0$. When $x = x_0 = 0$, $\sum_{n=1}^{\infty} \frac{n!}{n^2 e^{2n}} x^n = 0$. $\therefore \{0\}$.

(b) $\rho = \lim_{n\to\infty} \left|\frac{(n+1)^{n+1}}{[(n+1)!]^2} \times \frac{(n!)^2}{n^n}\right| = \lim_{n\to\infty} \left|\left(\frac{n+1}{n}\right)^n \times \frac{n+1}{(n+1)^2}\right| = 0$.

~~Use C-H formula:~~ $\cancel{\rho = \limsup \left|\frac{n}{\sqrt[n]{(n!)^2}}\right|}$ $\therefore R = \infty$, $\mathbb{R}$.

(c) C-H formula: $\rho = \limsup |1-(-1)^n 2| = 3$ $\therefore R = \frac{1}{3}$.

At $x = \frac{1}{3}$, $\cancel{\left|\left(\frac{1+(-1)^n 2}{3}\right)^n \times \left(\frac{3}{1+(-1)^{n-1}2}\right)^{n-1}\right| = \frac{1}{3}}$

$2n$th term $=1$. At $x = -\frac{1}{3}$, $(2n+1)$th term $=1$ $\therefore (-\frac{1}{3}, \frac{1}{3})$

(d) $\rho = \lim_{n\to\infty} \left|\frac{3^{n+1}}{(n+1)^2+n+1} \times \frac{n^2+n}{3^n}\right| = \left|\frac{3(n^2+n)}{n^2+3n+2}\right| = 3$ $\therefore R = \frac{1}{3}$.

At $x = x_0 - R = -1 - \frac{1}{3} = -4/3$,

$\sum_{n=1}^{\infty} \frac{3^n}{n^2+n}\left(-\frac{1}{3}\right)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+n}$ converges by the AST.

At $x = x_0 + R = -1 + \frac{1}{3} = -2/3$,

$\sum_{n=1}^{\infty} \frac{3^n}{n^2+n}\left(\frac{1}{3}\right)^n = \sum_{n=1}^{\infty} \frac{1}{n^2+n}$ since $\frac{1}{n^2+n} < \frac{1}{n^2}$, converges by comp. test

$\therefore [-4/3, -2/3]$.

2. (i) Method 1: Missing terms: use C-H.

$\rho = \limsup \left|\frac{1}{\sqrt[n]{n}}\right| = 1$ $\therefore R = 1$.

Method 2: Ratio test on this ~~sr~~:

$\rho = \lim_{n\to\infty} \left|\frac{x^{2n+3}}{2n+3} \times \frac{2n+1}{x^{2n+1}}\right| = \lim_{n\to\infty} \left|x^2\left(\frac{2n+1}{2n+3}\right)\right| = x^2$ $\therefore$ converges if $x^2 < 1$ $\iff |x| < 1$.

Endpts: $\sum_{n=0}^{\infty} \frac{1}{2n+1}$ ~~diverges by ...~~ diverges by comparison test.

$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$ conv. by AST.

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(2) (ii) Known that $f(x)$ cnv. on $(-1,1)$.

$$f'(x)=\sum_{n=0}^{\infty}\frac{d}{dx}\left(\frac{x^{2n+1}}{2n+1}\right)=\sum_{n=0}^{\infty}\frac{(x^2)^n}{1}=\frac{1}{1-x^2}$$

(iii) $f(x)-f(0)=\int_0^x f'(t)dt$ by FTC II. Since $f(0)=0$

$$\therefore f(x)=\int_0^x \frac{dt}{1-t^2}$$

$$=\int_0^x \frac{1-t+1+t}{2(1-t^2)}dt=\frac{1}{2}\int_0^x \frac{1}{1+t}+\frac{1}{1-t}dt.$$

$$=\frac{1}{2}\left[\ln|1+t|\right]_0^x-\frac{1}{2}\left[\ln|1-t|\right]_0^x.$$

$$=\frac{1}{2}\ln|1+x|-\frac{1}{2}\ln|1-x|=\frac{1}{2}\ln\left|\frac{1+x}{1-x}\right|.$$

note $\rho=1$. (why?)

3. (a) $$\sum_{n=1}^{\infty}nx^n=x\sum_{n=1}^{\infty}nx^{n-1}=x\sum_{n=1}^{\infty}\frac{d}{dx}(x^n)=x\frac{d}{dx}\left(\frac{1}{1-x}\right)=\frac{x}{(1-x)^2}$$

(b). $$\sum_{n=1}^{\infty}n^2x^n=\cancel{\sum_{n=1}^{\infty}n(nx^n)}=x\sum_{n=1}^{\infty}n^2x^{n-1}.$$

$$=x\sum_{n=1}^{\infty}\frac{d}{dx}(nx^n)=x\frac{d}{dx}\left(\sum_{n=1}^{\infty}nx^n\right)=x\frac{d}{dx}\left(\frac{x}{(1-x)^2}\right).$$

$$=x\left[\frac{(1-x)^2+\overset{(x2)(1-x)}{\cancel{(2x)}}(x)}{(1-x)^4}\right]=\frac{x(1-x)[(1-x)+2x]}{(1-x)^4}=\frac{x(1+x)}{(1-x)^3}.$$

(c) $$\sum_{n=1}^{\infty}\frac{n}{n+1}x^n=\frac{1}{x}\sum_{n=1}^{\infty}\frac{n}{n+1}x^{n+1}=\frac{1}{x}\sum_{n=1}^{\infty}\left[\int_0^x nt^n dt-0\right]$$

$$=\frac{1}{x}\int_0^x\left(\sum_{n=1}^{\infty}nt^n\right)dt.$$

$$=\frac{1}{x}\int_0^x\frac{t}{(1-t)^2}dt$$

$$=\frac{1}{x}\int_0^x\frac{1-(1-t)}{(1-t)^2}dt=\frac{1}{x}\int_0^x\frac{1}{(1-t)^2}-\frac{1}{(1-t)}dt.=\frac{1}{x}\left[\frac{1}{1-t}\right]_0^x+\frac{1}{x}\left[\ln(1-t)\right]_0^x$$

$$=\frac{1}{x}\left[\frac{1}{1-x}-1+\ln(1-x)\right].$$

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4. It suffices to show $R_n = \frac{1}{(n+1)!} f^{(n+1)}(c_n) x^{n+1} \to 0$ for $c_n \in (x, 0)$.

$\lim_{n\to\infty} |R_n| = \lim_{n\to\infty} \left| \frac{1}{(n+1)!} f^{(n+1)}(c_n) x^{n+1} \right| \le \lim_{n\to\infty} \frac{B r^{n+1}}{(n+1)!} = 0$.

$\therefore$ By squeeze.

5. (i) On $(-R_1, R_1)$, $a_n x^n \to 0$ $\therefore |a_n r^n| \le M_1$ for some M_1.

$\therefore$ Let $M = \max(1, M_1)$.

(ii) We do induction on n. $n=0$ is obvious.

$$|b_n| = \left|\sum_{k=1}^{n} a_k b_{n-k}\right| \le \sum_{k=1}^{n} |a_k b_{n-k}|.$$

$$\le \sum_{k=1}^{n} \frac{M}{r^k} \times \frac{2^{n-k} M^{n-k}}{r^{n-k}} \quad \text{since } 1^{k-1} \le M^{k-1},$$

$$\le \frac{M^n}{r^n} \sum_{k=1}^{n} 2^{n-k} = \frac{M^n}{r^n}(2^{n-1} + \dots + 2^0) \le \frac{M^n 2^n}{r^n}$$

(iii) $\rho = \lim_{n\to\infty} \left| \frac{(2M)^{n+1}}{r^{n+1}} \times \cancel{\frac{r^n}{(2M)^n}} \right| = \frac{2M}{r}$ $\therefore R = \frac{1}{\rho} = \frac{r}{2M}$ $r>0, M\ge 1$.

(iv) We have Merten's Thm: since f, g conv. on $(-\delta, \delta)$ where $\delta = \min(R, R_1)$ $\therefore$ on $(-\delta, \delta)$

$f(x) g(x) = \sum_{n=0}^{\infty} c_n x^n$ where c_n is the Cauchy pdt:

$$c_n = a_0 b_n + \sum_{k=1}^{n} a_k b_{n-k} = b_n - b_n = 0. \quad \forall n \ge 1.$$

$\therefore f(x) g(x) = 1 + 0 + 0 + \dots = 1$ $\therefore$ $f(x) = \frac{1}{g(x)}$ $\forall x \in (-\delta, \delta)$.

4

MA3110 Tut 11

1. (i) Recall that $\sin x$ strictly $\uparrow$ on $[0,\frac{\pi}{2}]$. (deriv. is cos, which > 0).
and $\cos x \to 0$ as $x \to \frac{\pi}{2}^-$. WTS given $M>0$, $\exists \delta>0$ s.t. $\frac{\pi}{2}-\delta<x<\frac{\pi}{2}$
$\Rightarrow \tan x > M$.
Let M be given. Let $\frac{\pi}{4}<x_0<\frac{\pi}{2} \Rightarrow$ ~~...~~ $\frac{1}{\sqrt{2}}<\sin x_0<1$.
~~Choose δ. Given ε = 1/√2~~
we want δ s.t. $\tan x = \frac{\sin x}{\cos x}$ where $\sin x > \sin x_0 = \frac{1}{\sqrt{2}}$, $\cos x < \frac{1}{\sqrt{2}M}$
$> \frac{1}{\sqrt{2}} \times \sqrt{2}M = M$.

Let $\varepsilon = \frac{1}{\sqrt{2}M}$. $\exists \delta'$ s.t. $\frac{\pi}{2}-\delta'<x<\frac{\pi}{2} \Rightarrow \cos x < \varepsilon = \frac{1}{\sqrt{2}M}$.
Now let $\delta = \min\{\frac{1}{2}, \delta'\}$. $\square$.

(ii) Since $\tan x$ is strictly increasing on $(-\frac{\pi}{2}, \frac{\pi}{2})$, its inverse, $\tan^{-1}x$ is also strictly $\uparrow$.
WTS $\forall \varepsilon>0$, $\exists M>0$ s.t. $x>M \Rightarrow |\tan^{-1}x - \frac{\pi}{2}|<\varepsilon$.
~~Since~~, since $\tan^{-1}x$ is continuous, $\tan^{-1}x < \frac{\pi}{2}$.
$\therefore$ WTS $x>M \Rightarrow \frac{\pi}{2}-\varepsilon < \tan^{-1}x$.
Let ε be given. then Let M be $\tan(\frac{\pi}{2}-\varepsilon)$, we have
$\tan^{-1}(x) > \tan^{-1}(M) > \tan^{-1}(\tan(\frac{\pi}{2}-\varepsilon))$
$= \frac{\pi}{2}-\varepsilon$. $\square$.

2. (i) $\rho = \lim_{n\to\infty}\left|\frac{(n)(n+1)}{(n+1)(n+2)}\right| = 1 \Rightarrow R=1$.

(ii) It suffices to show that ~~f is converges uniformly~~ $\sum_{n=1}^{\infty}\frac{(1)^n}{n(n+1)}$ converges, which Abel's thm implies $f(x)$ is uniformly convergent on $[0,1]$, in particular, continuous ~~on~~ at $x=1$.
$\sum_{n=1}^{\infty}\frac{(1)^n}{n(n+1)} = \lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k(k+1)} = \lim_{n\to\infty}(1-\frac{1}{n+1}) = 1$.
By continuity, $\lim_{x\to1^-}f(x) = f(1) = \sum_{n=1}^{\infty}\frac{(x)^n}{n(n+1)}\Big|_{x=1} = 1$. $\square$.

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3. (i) $\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty}(-x^2)^n = \sum_{n=0}^{\infty}(-1)^n(x)^{2n}$ when $|x|<1$. ~~since geom. srs converge~~

$\therefore \tan^{-1}(x)-\tan^{-1}(0) = \int_0^x \frac{d}{dt}\tan^{-1}(t)\,dt = \sum_{n=0}^{\infty}\int_0^x (-1)^n (t)^{2n}\,dt$

$\tan^{-1}(x) = \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}(x)^{2n+1}$, $|x|<1$.

(ii) Check that $x=\pm 1 \Rightarrow \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}$ is convergent by AST.

Thus by Abel's thm, we have $\lim_{x\to 1}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}(x)^{2n+1}$

$= \tan^{-1}(1) = \frac{\pi}{4}$.

$\therefore \pi = 4\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}$.

4. (i) Ratio test: $\rho = \lim_{n\to\infty}\left|\frac{x^{2n+2}}{3^{n+1}(n+2)}\times\frac{(n+1)3^n}{x^{2n}}\right|$

$= \lim_{n\to\infty}\left|\frac{x^2(n+1)}{3(n+2)}\right| = \frac{x^2}{3}$. To converge, $\rho<1$

$\Rightarrow |x|<\sqrt{3}$.

At $x=\pm\sqrt{3}$, $\sum_{n=0}^{\infty}\frac{x^{2n}}{3^n(n+1)} = \sum_{n=0}^{\infty}\frac{1}{n+1}$ is the harm. srs, diverges.

$\therefore E = (-\sqrt{3}, +\sqrt{3})$.

(ii) Let $f(x) = \sum_{n=0}^{\infty}\frac{x^{2n}}{3^n(n+1)}$. Then consider $x^2 f(x)$:

$\frac{d}{dx}(x^2 f(x)) = \frac{d}{dx}\sum_{n=0}^{\infty}\frac{x^{2n+2}}{3^n(n+1)} = \sum_{n=0}^{\infty}\frac{2\cancel{(n+1)}}{3^n\cancel{(n+1)}}x^{2n+1}$.

$= 2x\sum_{n=0}^{\infty}\left(\frac{x^2}{3}\right)^n$.

$= 2x\times\frac{1}{1-\frac{x^2}{3}} = \frac{6x}{3-x^2}$.

$\therefore x^2 f(x) = \int_0^x \frac{6t}{3-t^2}\,dt$. by FTC II.

$= -3\int_0^x\frac{-2t}{3-t^2}\,dt = \left[-3\ln(3-t^2)\right]_0^x = -3\left[\ln\left(\frac{3-x^2}{3}\right)\right]$

$\therefore f(x) = -\frac{3}{x^2}\ln\left(\frac{3-x^2}{3}\right)$. $|x|<\sqrt{3}$.

$\square$

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5. since $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ conv. abs, $\sum_{n=0}^{\infty} |a_n|$ and $\sum_{n=0}^{\infty} |b_n|$ conv. abs.

denote $c_n = \sum_{k=0}^{n} a_k b_{n-k}$, & $d_k = \sum_{k=0}^{n} |a_k||b_{n-k}|$.

note that $|c_n| = \left| \sum_{k=0}^{n} a_k b_{n-k} \right| \leq \sum_{k=0}^{\infty} |a_k||b_{n-k}|$ by Δ ineq.

$= d_n$.

By comparison test, since $\sum_{n=0}^{\infty} d_n$ converges (by Merten's thm), $\sum_{n=0}^{\infty} |c_n|$ also converges. Thus $\sum_{n=0}^{\infty} c_n$ converges absolutely.

6. (i) $\rho = \lim_{n\to\infty} \left| \left(\cancel{\frac{\alpha \ldots (\alpha-n)(\alpha-n+1)}{(n+1)!}} \times \cancel{\frac{n!}{\alpha \ldots (\alpha-n+1)}} \right) \right|$

$= \lim_{n\to\infty} \left| \frac{\alpha - n}{n+1} \right| = \lim_{n\to\infty} \left| \frac{\frac{\alpha}{n} - 1}{1 + \frac{1}{n}} \right| = 1 \quad \therefore R = \frac{1}{\rho} = 1.$

(ii) $f'(x) = \sum_{n=0}^{\infty} \binom{\alpha}{n} \frac{d}{dx} x^n = \sum_{n=1}^{\infty} n \binom{\alpha}{n} x^{n-1}$ $\quad = 1 + \binom{\alpha}{1}x + \binom{\alpha}{2}x^2 + \ldots$ $\quad \frac{d}{dx} \to 1\binom{\alpha}{1} + 2\binom{\alpha}{2}x + \ldots$

$(x+1) f'(x) = \sum_{n=1}^{\infty} n \binom{\alpha}{n} x^n + \sum_{n=1}^{\infty} n \binom{\alpha}{n} x^{n-1}$ sub $n = n+1$.

add $0\binom{\alpha}{0}x^0$. $= \sum_{n=0}^{\infty} n \binom{\alpha}{n} x^n + \sum_{n=0}^{\infty} (n+1) \binom{\alpha}{n+1} x^n$.

$= \sum_{n=0}^{\infty} x^n \left(n\binom{\alpha}{n} + (n+1)\binom{\alpha}{n+1} \right)$.

Claim: $n\binom{\alpha}{n} + (n+1)\binom{\alpha}{n+1} = \alpha \binom{\alpha}{n}$.

verify: $\frac{\alpha \ldots (\alpha-n+1)}{(n-1)!} + \frac{\alpha \ldots (\alpha-n)}{n!} = \frac{\alpha \ldots (\alpha-n+1)}{(n-1)!} \left[1 + \frac{\alpha-n}{n} \right]$

$= \frac{\alpha \ldots (\alpha-n+1)}{(n-1)!} \frac{\alpha}{n} = \alpha \binom{\alpha}{n}$. ✓

$= \alpha \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n = \alpha f(x) \quad \forall |x| < 1$.

(iii) $h'(x) = -\alpha (1+x)^{-\alpha-1} f(x) + (1+x)^{-\alpha} \left(\frac{\alpha f(x)}{1+x} \right)$

$= f(x)(1+x)^{-\alpha-1}(-\alpha+\alpha) = 0. \quad \forall x \in (-1,1)$.

(iv) Then h is constant. $h(x) = h(0) = 1 f(0) = 1 \quad \forall x \in (-1,1)$.

$\therefore f(x) = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n = (1+x)^{\alpha}$.