MA3110 Homework 2

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Problem 1

Let $f : (a, b) \to \mathbb{R}$ be differentiable on (a, b) and let $c \in (a, b)$. Prove that if the limit $\lim_{x\to c} f'(x) = L$ exists, then $f'(c) = L$.

Solution

As per the hint, we note that the difference quotient has numerator and denominator both tend to 0 as x approaches c and f is differentiable on (a, b):

$$\begin{aligned} f'(c) &= \lim_{x\to c} \frac{f(x) - f(c)}{x - c} \\ &= \lim_{x\to c} \frac{f'(x) - 0}{1 - 0} && \text{(L'hopital's Rule)} \\ &= \lim_{x\to c} f'(x) = L && \text{(assumption).} \end{aligned}$$

Problem 2

The fucntion $g : [-1, 1] \to \mathbb{R}$ is such that g''' exists on $[-1, 1], g(0) = g(1) = 0, g(1) = 1$ and $g'(0) = 0$.

1. Show that there exists $c \in (0, 1)$ such that $\frac{g''(0)}{2!} + \frac{g'''(c)}{3!} = 1$.

   Proof. Apply Taylor's Theorem to g on $[0, 1]$, and let $x = 1, x_0 = 0$, we have

   $$g(1) = g(0) + g'(0)x + \frac{1}{2}g''(0)x^2 + \frac{1}{6}g'''(c)x^3 \tag{1}$$

   $$1 = 0 + 0x + \frac{1}{2}g''(0)(1)^2 + \frac{1}{6}g'''(c)(1)^3 \tag{2}$$

   $$\therefore \frac{1}{2}g''(0) + \frac{1}{6}g'''(c) = 1 \tag{3}$$

   for some $c \in (0, 1)$ as desired. □

2. Show that there exists $d \in (-1, 1)$ such that $g'''(d) \geq 3$.

   Proof. Apply Taylor's Theorem to g on $[-1, 0]$, and let $x = -1, x_0 = 0$, we have

   $$g(-1) = g(0) + g'(0)x + \frac{1}{2}g''(0)x^2 + \frac{1}{6}g'''(c')x^3 \tag{4}$$

   $$0 = 0 + 0x + \frac{1}{2}g''(0)(-1)^2 + \frac{1}{6}g'''(c')(-1)^3 \tag{5}$$

   $$\therefore \frac{1}{2}g''(0) - \frac{1}{6}g'''(c') = 0 \tag{6}$$

   for some $c' \in (-1, 0)$. Taking (3)-(6), we have

   $$\begin{aligned} 1 &= g'''(c') + g'''(c) \\ 6 &= g'''(c') + g'''(c) \end{aligned}$$

   Without loss of generality, we must have $g'''(c) \geq g'''(c')$ or $g'''(c) \leq g'''(c')$. In any case, one of them must be at least 3. Therefore, let it be $d \in (-1, 1)$ and we are done. □

Problem 3

Let $f(x) = (1+3x)^{2/3},\ x > -1/3$.

1. Find the values of $f'(0), f''(0)$ and $f'''(0)$.

   Proof.

$$\begin{aligned} f'(x) &= (2/3)(1+3x)^{-1/3}(3) = 2(1+3x)^{-1/3}, & f'(0) &= 2(1) = 2. \\ f''(x) &= (2)(-1/3)(1+3x)^{-4/3}(3) = -2(1+3x)^{-4/3}, & f''(0) &= -2(1) = -2. \\ f'''(x) &= (-2)(-4/3)(1+3x)^{-7/3}(3) = 8(1+3x)^{-7/3}, & f'''(0) &= 8(1) = 8. \end{aligned}$$

$\square$

2. Use Taylor's Theorem to prove that for $x > -1/3,\ f(x) \le 1 + 2x - x^2 + 4x^3/3$.

   Proof. Applying Taylor's theorem to f with $x_0 = 0$,

$$\begin{aligned} f(x) &= f(0) + f'(0)x + \frac{1}{2}f''(0)x^2 + \frac{1}{6}f''(0)x^3 + R_3(x) \\ &= 1 + 2x + \frac{1}{2}(-2)x^2 + \frac{1}{6}8x^3 + R_3(x) \\ &= 1 + 2x - x^2 + \frac{4}{3}x^3 + R_3(x) \end{aligned}$$

Where $R_3(x) = \frac{1}{24}f^{(4)}(c)x^4$ for some $c > -1/3$, and

$$f^{(4)}(x) = (8)(-7/3)(1+3x)^{-10/3}(3) = -56(1+3x)^{-10/3}.$$

Since $x > -1/3 \implies (1+3x)^{-10/3} > 0$, and x^4 is always non-negative, therefore the only negative coefficient forces $R_3(x) \le 0$. We thus have

$$\begin{aligned} f(x) &= 1 + 2x - x^2 + \frac{4}{3}x^3 + R_3(x) \\ &\le 1 + 2x - x^2 + \frac{4}{3}x^3 & (R_3 \le 0). \end{aligned}$$

$\square$

Problem 4

Let $h : [0,2] \to \mathbb{R}$ be defined by

$$h(x) = \begin{cases} 4x, & x \text{ is rational} \\ 4, & x \text{ is irrational} \end{cases}$$

and let $P = \{0, 1/2, 1, 3/2, 2\}$. Find the upper sum $U(h,P)$ of h with respect to the partition P.

Solution

Proof.

$$\begin{aligned}
U(H,P) =& \sum_{i=1}^{4}(x_i - x_{i-1})M_{i-1} && \text{where } M_{i-1} = \sup f(x) | x \in [x_{i-1}, x_i] \\
=&(\frac{1}{2} - 0)(\sup\{4, 4(\frac{1}{2}), 4(0)\}) \\
&+ (1 - \frac{1}{2})(\sup\{4, 4(1), 4(\frac{1}{2})\}) \\
&+ (\frac{3}{2} - 1)(\sup\{4, 4(\frac{3}{2}), 4(1)\}) \\
&+ (2 - \frac{3}{2})(\sup\{4, 4(2), 4(\frac{3}{2})\}) \\
=&\frac{1}{2}[4 + 4 + 6 + 8] = 2 + 2 + 3 + 4 = 11.
\end{aligned}$$

$\square$