1 The Real Number System

1. *Definition*: a **field** is the 5-tuple $\langle \mathbb{F}, +, \cdot, e, u\rangle$, where $\mathbb{F}$ is a set containing at least the elements e and u, where $e \neq u$, and satisfies: For any $a, b, c \in \mathbb{F}$,
   (a) (commutative add) $a + b = b + a$
   (b) (associative add) $(a + b) + c = a + (b + c)$
   (c) (additive identity) $a + e = a$
   (d) (additive inverse) $\forall a, \exists b \in \mathbb{F}$ such that $a + b = e$.
   (e) (commutative multiply) $a \cdot b = b \cdot a$
   (f) (associative multiply) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
   (g) (multiplicative identity) $a \cdot u = a$
   (h) (multiplicative inverse) $\forall a, \exists b \in \mathbb{F}$ such that $a \cdot b = u$.
   (i) (distributive) $\forall a, b, c \in \mathbb{F}, a \cdot (b + c) = a \cdot b + a \cdot c$
2. *Example*: $\mathbb{Q}, \mathbb{N}, \mathbb{Z}, \mathbb{R}, \mathbb{C}, \mathbb{R} - \mathbb{Q}$ are fields.
3. *Definiton*: A field $\mathbb{F}$ is **ordered** if $\exists P \subseteq \mathbb{F}$ such that $\forall a, b \in P$,
   (a) $a + b \in P$
   (b) $a \cdot b \in P$
   (c) (trichotomy) either
      i. $a \in P$
      ii. $a = e$, or
      iii. $-a \in P$
   (d) $e \notin P$.
4. *Theorem*: $a \in P \implies -a \notin P$.
5. *Definiton*: if a subset of an ordered field, $A \subseteq \mathbb{F}$ contains an element a such that $\forall x \in \mathbb{F}, a \leq (\geq) x$, then $\mathbb{F}$ is **bounded below (above)**. Such a is called an **lower (upper) bound** of A.
6. *Definition*: if $\emptyset \neq A \subseteq \mathbb{F}$ is bounded above (below), an element b is the **least upper (greatest lower) bound** if
   (a) b is an upper(lower) bound of A and
   (b) $\forall c \in \mathbb{F}$ where c is an upper(lower) bound of A, $b \geq c (b \leq c)$.

   , denoted by $\sup A (\inf A)$ respectively.
7. *Definition*: An ordered field $\mathbb{F}$ is **(order) complete** if it has the **least upper bound property**: $\forall \emptyset \neq A \subseteq \mathbb{F}$, if A is bounded above, A has a least upper bound.
8. *Example*: $\mathbb{R}$ is order complete, but $\mathbb{Q}$ is not.

2 Sequences of Real Numbers

1. *Definition*: A **sequence** in a set S is a function, $f : \mathbb{N} \to S$, where we denote $f(n) = s_n$ for all $n \in \mathbb{N}$, and the sequence as $(s_n)_{n=1}^{\infty}$.
2. *Definiton*: given a sequence $f : \mathbb{N} \to S$, a **subsequence** of f is a sequence of the form $f \circ g : \mathbb{N} \to S$, where $g : \mathbb{N} \to \mathbb{N}$ is strictly increasing. We write $((f \circ g)(n))_{n=1}^{\infty} = (s_{g(n)})_{n=1}^{\infty}$.
3. *Analogy*: We have a real sequence $(s_n)_{n=1}^{\infty}$. We claim that <u>L is the limit</u>. An opponent then issues a <u>challenge ϵ</u>. We need to be able to to any ϵ given with a <u>N</u> such that all our terms after N,$(s_N, s_{N+1}, s_{N+2}, \dots)$ are all at most ϵ from L.
4. *Definiton*: if $\lim_{n\to\infty} s_n = L$ holds then we say $(s_n)_{n=1}^{\infty}$ is **convergent**. Conversely, $(s_n)_{n=1}^{\infty}$ is **convergent** if there exists an $L \in \mathbb{R}$ such that $(s_n)_{n=1}^{\infty}$ converges to L.
5. *Definition*: a sequence $(s_n)_{n=1}^{\infty}$ **diverges to** $\infty(-\infty)$ if $\forall M \in \mathbb{R}, \exists N \in \mathbb{N}$ such than $s_n \geq M (s_n \leq M)$ for any $n \geq N$.
6. *Proposition*: sequence is convergent $\implies$ **any subsequence** of that sequence is convergent.
7. *Definition*: (properties of sequences) A sequence $(s_n)_{n=1}^{\infty}$ is
   (a) **bounded** if $\exists M \in \mathbb{R}, M > 0$ such that $|s_n| \leq M$ for any $n > N$.
   (b) **nondecreasing** if $s_n \leq s_{n+1} \quad \forall n \in \mathbb{N}$.
   (c) **nonincreasing** if $s_n \geq s_{n+1} \quad \forall n \in \mathbb{N}$.
   (d) **monotone** if it is either nondecreasing or nonincreasing.
8. *Proposition*: bounded and nondecreasing(nonincreasing) $\implies$ convergent to its supremum (infimum).
9. *Definition*: $\lim_{n\to\infty} s_n = e(x)$. $e(x + y) = e(x) + e(y), e(0) = 1$.
10. *Theorem*: Every real sequence has a monotone subsequence. Therefore, every bounded sequence has a convergent subsequence.
11. *Proposition*: If $c > 1$, then $\lim_{n\to\infty} c_{1/n} = 1$.
12. *Proposition*: A convergent sequence of non-negative numbers converge to a nonnegative number. Similarly, if all values of a sequence are greater than k, its limit is greater than k too.
13. *Theorem*: suppose $\lim_{n\to\infty} s_n = L \in \mathbb{R}$, $\lim_{n\to\infty} t_n = M \in \mathbb{R}$, and $C \in \mathbb{R}$. Then
    (a) $\lim_{n\to\infty}(s_n + Ct_n) = L + CM$.
    (b) $\lim_{n\to\infty}(s_n t_n) = LM$.
    (c) if $M \neq 0$,then $\lim_{n\to\infty} 1/t_n = 1/M$.
14. *Definition*: Let $(s_n)_{n=1}^{\infty}$ be a real sequence. Then define **limit superior**

$$\limsup_{n\to\infty} = \begin{cases} \infty & \text{if } (s_n)_{n=1}^{\infty} \text{ is not bounded above} \\ \lim_{n\to\infty} M_n & \text{if } (M_n)_{n=1}^{\infty} \text{ is bounded below} \\ -\infty & \text{if } (M_n)_{n=1}^{\infty} \text{ is not bounded below} \end{cases}$$

    where $(M_k)_{n=1}^{\infty} = \sup\{s_k, s_{k+1}, \dots\}$, and define **limit inferior**

$$\liminf_{n\to\infty} = \begin{cases} \infty & \text{if } (s_n)_{n=1}^{\infty} \text{ is not bounded below} \\ \lim_{n\to\infty} M_n & \text{if } (M_n)_{n=1}^{\infty} \text{ is bounded above} \\ -\infty & \text{if } (M_n)_{n=1}^{\infty} \text{ is not bounded above} \end{cases}$$

    where $(M_k)_{n=1}^{\infty} = \inf\{s_k, s_{k+1}, \dots\}$.
15. *Proposition*: $\limsup_{n\to\infty} s_n = L, \limsup_{n\to\infty} t_n = M$, where $L, M \in \mathbb{R}$, and the sequences are bounded, $\implies \limsup_{n\to\infty}(s_n + t_n) \leq L + M$.
16. *Proposition*: for any sequence, $\liminf \leq \limsup$.

17. *Proposition*: for any bounded sequence, $\lim_{n\to\infty} s_n = L \iff \liminf_{n\to\infty} s_n = \limsup_{n\to\infty} s_n = L$.

18. *Theorem*: given a bounded real sequence, there exist subsequences that converge to the *limsup* and *liminf* respectively. *Any* convergent subsequence converges to at most the *limsup*, and at least the *liminf*. That is, for any subsequence $(s_{n_k})_{k=1}^{\infty}$, $\liminf_{n\to\infty} s_n \leq \lim_{k\to\infty} s_{n_k} \leq \limsup_{n\to\infty} s_n$.

19. *Definition*: A sequence $(s_n)_{n=1}^{\infty}$ is **Cauchy** if $\forall \epsilon > 0, \exists N \in \mathbb{N}$ such that $\forall n, m \geq N, |s_n - s_m| < \epsilon$.

20. *Theorem*: for real sequences, convergence $\iff$ Cauchy $\implies$ bounded.

21. *Nested Inverval Theorem*: Given $I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots$ are closed bounded intervals such that $\lim_{n\to\infty} \operatorname{diam} I_n = 0$. Then $\bigcap_{n=1}^{\infty} I_n$ contains exactly one point.

22. *Theorem*: There is no onto map $f : \mathbb{N} \to [0,1]$. In other words, $[0,1]$ is uncountable.

23. *Definition*: Given a real sequence $(s_n)_{n=1}^{\infty}$, define $\sigma_n = (s_1 + s_2 + \cdots + s_n)/n, \forall n \in \mathbb{N}$. We say $(s_n)_{n=1}^{\infty}$ is $(C,1)$ summable to $L \in \mathbb{R}$ if $\lim_{n\to\infty} \sigma_n = L$.

24. *Theorem (regularity)*: If a real sequence converges to L, then it is $(C,1)$ summable to L.

3 Series of Real Numbers

1. *Definition*: Given an **infinite series** $\sum_{n=1}^{\infty} a_n$, define $s_k = a_1 + a_2 + \cdots + a_k = \sum_{n=1}^{k} a_n, k = 1,2,3,\dots$. Then $(s_k)_{k=1}^{\infty}$ is the sequence of **partial sums** of $\sum_{k=1}^{\infty} a_k$.

2. *Definition*: The infinite series $\sum_{n=1}^{\infty} a_n$ **converges** to L if the partial sums $(s_k)_{k=1}^{\infty}$ converges to L. If $(s_k)_{k=1}^{\infty}$ diverges, then we say $\sum_{n=1}^{\infty} a_n$ also **diverges**.

3. *Proposition*: If $\sum_{n=1}^{\infty} a_n$ converges $\implies \lim_{n\to\infty} a_n = 0$. (The converse need not be true, see $\sum_{n=1}^{\infty} \frac{1}{n}$).

4. *Proposition*: If $a_n \geq 0$, then $\sum_{n=1}^{\infty} a_n$ converges $\iff$ the sequence of partial sums, $(s_n)_{n=1}^{\infty}$ is bounded above.

5. *Definition*: An **alternating series** is a series of the form $\sum_{n=1}^{\infty} (-1)^n a_n$ or $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ where $a_n \geq 0$.

6. *Proposition*: If $\sum_{n=1}^{\infty} a_n = L, \sum_{n=1}^{\infty} b_n = M, c \in \mathbb{R}$, then $\sum_{n=1}^{\infty} a_n + cb_n = L + cM$.

7. *Definition*: A series $\sum_{n=1}^{\infty} a_n$ is **absolutely convergent** if $\sum_{n=1}^{\infty} |a_n|$ is convergent. It is **conditionally convergent** if $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} |a_n|$ diverges.

8. *Theorem*: absolute convergence $\implies$ convergence.

9. *Definition*: Given series $\sum_{n=1}^{\infty} a_n$. Define

$$p_n = (a_n + |a_n|)/2, \quad q_n = (a_n - |a_n|)/2,$$

then we have properties as follows:

   (a) If $a_n \geq 0$ for all n, then $p_n = a_n, q_n = 0$.
   (b) If $a_n < 0$ for all n, then $p_n = 0, q_n = a_n$.
   (c) If $a_n < 0$ for all n, then $p_n = 0, q_n = a_n$.
   (d) If $\sum_{n=1}^{\infty} a_n$ converges absolutely $\iff$ both $\sum_{n=1}^{\infty} p_n$ and $\sum_{n=1}^{\infty} q_n$ converge.
   (e) If $\sum_{n=1}^{\infty} a_n$ converges conditionally $\implies$ both $\sum_{n=1}^{\infty} p_n$ and $\sum_{n=1}^{\infty} q_n$ diverge.

10. *Definition*: An **arrangement** of a series $\sum_{n=1}^{\infty} a_n$ is a series of the form $\sum_{n=1}^{\infty} a_{g(n)}$, where $g : \mathbb{N} \to \mathbb{N}$ is a bijection.

11. *Lemma*: If $a_n \geq 0$, and $\sum_{n=1}^{\infty} a_n$ converges to L, then any rearrangement $a_{g(n)}$ also converges to L.

12. *Theorem*: If $\sum_{n=1}^{\infty} a_n$ is absolutely convergent and $\sum_{n=1}^{\infty} a_n = L$, then any arrangement $\sum_{n=1}^{\infty} a_{g(n)}$ is absolutely convergent and $\sum_{n=1}^{\infty} a_{g(n)} = L$.

13. *Theorem*: Suppose that $\sum_{n=1}^{\infty} a_n$ is conditionally convergent, then for any $L \in \mathbb{R}$, $\exists$ a rearrangement of $\sum_{n=1}^{\infty} a_n$ that converges to L.

14. *Theorem*: Suppose both $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} a_n$ both converge absolutely. Let $c_n = \sum_{k=0}^{\infty} a_k b_{n-k}$ for all $n = 0, 1, \dots$. Then $\sum_{n=1}^{\infty} c_n$ converges absolutely and $\sum_{n=1}^{\infty} c_n = (\sum_{n=1}^{\infty} a_n)(\sum_{n=1}^{\infty} b_n)$.

15. *Theorem (Series Tests)*: Given a real series $\sum_{n=1}^{\infty} a_n$.
   (a) **Alternating Series Test**: Given an alternating series $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$, if a_n is nonincreasing and convergent to 0, then $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ converges to some $L \in \mathbb{R}$.
   Furthermore, for any $k \in \mathbb{R}$, $|\sum_{n-1}^{k} (-1)^{n+1} a_n - L| < a_{k+1}$.
   (b) **Comparison Test**: Suppose that $\exists k < \infty$ such that $\forall n \in \mathbb{N}, a_n \leq k|b_n|$. If $\sum_{n=1}^{\infty} b_n$ converges absolutely, then $\sum_{n=1}^{\infty} a_n$ converges absolutely.

(c) **Ratio Test**: $\liminf_{n\to\infty} |a_{n+1}/a_n| < 1 \implies \sum_{n=1}^{\infty} a_n$ converges absolutely. $\limsup_{n\to\infty} |a_{n+1}/a_n| > 1 \implies \sum_{n=1}^{\infty} a_n$ diverges.

(d) **Root Test**: $\limsup_{n\to\infty} \sqrt[n]{|a_n|} < 1 \implies \sum_{n=1}^{\infty} a_n$ converges absolutely. $\limsup_{n\to\infty} \sqrt[n]{|a_n|} > 1 \implies \sum_{n=1}^{\infty} a_n$ diverges.

16. *Definition*: Given series $\sum_{n=1}^{\infty} a_n$ and sequence $(b_n)_{n=1}^{\infty}$, and let $s_k = \sum_{n=1}^{k} a_n$. Then we have **summation by parts**: $\sum_{k=1}^{n} a_k b_k = s_n b_n + \sum_{k=1}^{n-1} s_k(b_k - b_{k+1})$.
    (a) **Dirichlet's Test**: $(s_k)_{k=1}^{\infty}$ bounded, $(b_k)_{k=1}^{\infty}$ is monotone and converges to 0 $\implies \sum_{n=1}^{\infty} a_n b_n$ converges.
    (b) **Abel's Test**: $(s_k)_{k=1}^{\infty}$ converges, $(b_k)_{k=1}^{\infty}$ is monotone and bounded $\implies \sum_{n=1}^{\infty} a_n b_n$ converges.

17. *Definition*: Given $\sum_{n=1}^{\infty} a_n$. If its partial sum is (C,1) summable to k, that is, $\lim_{k\to\infty} (s_1 + \cdots + s_k)/k = A$, then we say $\sum_{n=1}^{\infty} a_n$ is **(C,1) summable**, written as $\sum_{n=1}^{\infty} a_n = A \quad (C,1)$.

18. *Tauberian Theorem*: A series (C,1) summable to $L \implies$ convergent to L.

4 Limits in Metric Spaces

Convention: $\langle M, \rho\rangle, \langle M_1, \rho_1\rangle, \langle M_2, \rho_2\rangle$ are metric spaces. ρ, σ, τ are metrics.

1. *Definition*: A **metric** on a nonempty set M is a function $\rho : M \times M \to \mathbb{R}$, that satisfies for any $x, y, z \in M$:
    (a) $\rho(x, x) = 0$,
    (b) $x \neq y \implies \rho(x, y) > 0$.
    (c) (symmetry) $\rho(x, y) = \rho(y, x)$, and
    (d) (triangle inequality) $\rho(x, y) \leq \rho(x, z) + \rho(y, z)$.

    Commonly used metrics for $\mathbb{R}^n$ include:

    (a) (1-metric) $\rho_1((a_i)_{i=1}^n, (b_i)_{i=1}^n) = \sum_{i=1}^{n} |a_i - b_i|$.
    (b) (2-metric, or Euclidian metric) $\rho_2((a_i)_{i=1}^n, (b_i)_{i=1}^n) = \sqrt{\sum_{i=1}^{n} |a_i - b_i|^2}$.
    (c) (n-metric) $\rho_n((a_i)_{i=1}^n, (b_i)_{i=1}^n) = \sqrt[n]{\sum_{i=1}^{n} |a_i - b_i|^n}$.
    (d) (∞-metric) $\rho_\infty((a_i)_{i=1}^n, (b_i)_{i=1}^n) = \max\{|a_i - b_i| : 1 \leq i \leq n\}$.
    (e) (discrete-metric) given $x, y \in \langle M, \rho\rangle$,
    $$\rho_d(x, y) = \begin{cases} 0 & x \neq y, \\ 1 & x = y. \end{cases}$$

    The pair $\langle M, \rho\rangle$ is called a **metric space**.

2. *Cauchy-Schwartz Inequality*: $\forall a_i, b_i \in \mathbb{R}$, $\sum_{i=i}^{n} |a_i b_i| \leq AB$ where $A = \sqrt{\sum_{i=1}^{n} |a_i|^2}$ and $B = \sqrt{\sum_{i=1}^{n} |b_i|^2}$.

3. *Minkowski's Inequality*: $\sqrt{\sum_{i=1}^{n} |a_i + b_i|^2} \leq \sqrt{\sum_{i=1}^{n} |a_i|^2} + \sqrt{\sum_{i=1}^{n} |b_i|^2}$

4. *Definition*: $A \subseteq \langle M, \rho\rangle$. A point $a \in M$ is a **cluster point** of A if, $\forall h > 0, \exists x \in A$ such that $0 < \rho(x, a) < h$. Also known as **limit points** or **accumulation points**. A point is an **isolated point** if it is **not a cluster point**.

5. *Proposition*: (Sequential formulation of cluster points) x is a cluster point of $E \iff \exists (x_n)_{n=1}^{\infty} \in E$ that converges to x.

6. *Definition (limits)*: Given $f : M_1 \to M_2$. Suppose a is a cluster point of M_1, and $L \in M_2$. Then we say $\lim_{x\to a} f(x) = L$ if for any $\epsilon > 0, \exists\delta > 0$ such that: $0 < \rho_1(x, a) < \delta \implies \rho_2(f(x), L) < \epsilon \quad \forall x \in M$.
    (a) **Sequential characterization of limits**: $\lim_{x\to a} f(x) = L \iff$ For all sequences $(x_n)_{n=1}^{\infty}$ in M_1 that converges to a, $x_n \neq a \ \forall n \in \mathbb{N}$, $(f(x_n))_{n=1}^{\infty}$ converges to L in M_2.
    (b) **Arithmetic of limits**: Suppose $f, g : M \to \mathbb{R}$, and a is a cluster point of M. Given $\lim_{x\to a} f(x) = A$, $\lim_{x\to a} g(x) = B$, then as usual,
        i. $\lim_{x\to a} (f(x) + g(x)) = A + B$
        ii. $\lim_{x\to a} (f(x)g(x)) = AB$
        iii. $\lim_{x\to a} (f(x)/g(x)) = A/B$ if $g(x) \neq 0 \ \forall x \in M$ and $B \neq 0$.

5 Open and Closed Sets, Continuity

1. *Definition*: The **open ball** centered at $a \in M$ with radius $r > 0$ is the set $B[a, r] = \{x \in M : \rho(x, a) < r\}$.

2. *Definition*: A function $f : M_1 \to M_2$ is **continuous** at a point $a \in M_1$ if $\forall\epsilon > 0, \exists\delta > 0$ such that
    (a) $\rho_2(f(x), f(a)) < \epsilon$ for all $x \in M_1$ such that $\rho_1(x, a) < \delta$.
    (b) (Balls) $x \in B[a, \delta] \implies f(x) \in B[f(a), \epsilon]$
    (c) $B[a, \delta] \subseteq f^{-1}(B[f(a), \epsilon])$ or $f(B[a, \delta]) \subseteq B[f(a), \epsilon]$.

3. *Proposition*: Functions are always continuous at isolated points. $f : M_1 \to M_2$ is continuous at a cluster point $a \in M_1 \iff \lim_{x\to a} f(x) = f(a)$.

4. **Sequential formulation of continuity**: A function $f : M_1 \to M_2$ is continuous at $a \in M_1 \iff \forall (x_n)_{n=1}^{\infty}$ in M_1 that converges to a, $(f(x_n))_{n=1}^{\infty}$ converges to $f(a) \in M_2$

5. **Arithmetic of continuous functions**: Let $f, g : M \to \mathbb{R}$ be continuous at $a \in M$. Then $f + g, f \cdot g, f/g$ are continuous, the last case if $g(x) \neq 0 \ \forall x \in M$.

6. *Corollary*: Since $f(x) = x, f : \mathbb{R} \to \mathbb{R}$ is continuous at any $a \in \mathbb{R}$, thus any rational function with nonzero denominator is continuous.

7. *Theorem*: The composition of two continuous functions are continuous.

8. *Definition (Open Set)*: A subset $G \subseteq M$ is an **open subset in** M if $\forall x \in G$, $\exists r > 0$ such that $B[x, r] \subseteq G$. In words, every point in an open set can ball up to a certain radius and the ball will be contained in the set.

9. *Definition*: **closure** of $E = \overline{E} = E \cup \{\text{cluster points of E}\}$. A set $E \in M$ is **closed in** M if $E = \overline{E}$.
   (a) **Characterization of Closure**: $x \in \overline{E} \iff \forall r > 0, B[x, r] \cap E \neq \emptyset$.
   (b) **Consistency of closure and closed set**: for any $E \in M$, $\overline{E}$ is a closed set.

10. *Punishable by Death*: Closed $\neq$ not open, not closed $\neq$ open. $[0, 1) \subseteq \mathbb{R}$ is neither open nor closed.

11. *Proposition*: Let $E \subseteq M$. E is closed $\iff$ E' is open in M.

12. *Propositions*:
   - Any open ball in any metric space is an open set (in M).
   - In any metric space with the discrete metric, any subset of it is an open set (in M).
   - In any metric space, $\emptyset$ and M are both open and close in M.
   - The union of any number of open sets is open. The intersection of *finitely many* open sets is open.
   - The intersection of any family of closed sets is closed. The union of *finitely* many closed sets is closed.

13. *Theorem*: The following statements are equivalent:
   (a) A function $f : M_1 \to M_2$ is continuous in M_2
   (b) Whenever G is open in M_2, $f^{-1}(G)$ is open in M_1
   (c) whenever F is closed in M_1, $f(F)$ is closed in M_2.

14. *Definition*: Given $f : M_1 \to M_2$, if f is a bijection and both f, f^{-1} are continuous, then f is a **homeomorphism**. Two sets are **homeomorphic** if there is a homeomorphism from one to the other. **Homeomorphism is transitive**, since the composition of homeomorphic (continuous) functions is homeomorphic (continuous).

15. *Definition*: $E \subseteq M, E$ is **dense in M** if $\overline{E} = M$. For example, $\mathbb{Q}$ is **dense** in $\langle \mathbb{R}, \rho_e \rangle$.

16. (a) **Baire's Theorem**: Given a F_σ (union of closed) set. Then one of the sets must contain a non-empty, open interval.
   (b) *Prop*: The set of irrationals $\mathbb{R} - \mathbb{Q}$ is not a F_σ set.
   (c) *Thm*: There is no bounded continuous function $f : \mathbb{R} \to \mathbb{R}$ that is discontinous precisely at the irrational numbers.

6 1st C: Connectedness and Continuity

1. *Proposition*: (Open and closed sets in subsets) Let $A \subseteq U \subseteq M$. Then A is closed(open) in $U \iff \exists$ an open(closed) set $B \in M$ such that $A = B \cap U$.

2. *Definition*: Two formulations of **connected sets**:
   (a) A set E in M is **disconnected** if there are nonempty sets A, B so that $E = A \cup B$ and $\overline{A} \cap B = \emptyset = A \cap \overline{B}$.
   (b) Given $\emptyset \neq C \subseteq M$, C is **connected** $\iff$ $\forall$ subsets in C, only C and $\emptyset$ are both open and closed in C.

3. *Proposition (interval property)*: In $\mathbb{R}$, a set is connected $\iff$ it is an interval.

4. *Proposition*: The union of two overlapping connected sets is connected.

5. *Theorem*: Continuous functions preserve connectedness. If $f : M_1 \to M_2$ is continuous, then, if $C \subseteq M_1$ is connected, $f(C) \subseteq M_2$ is also connected.

6. *Intermediate Value Theorem*: Let $I = [a, b]$ be an interval in $\mathbb{R}$. Then $[f(a), f(b)] \subseteq f(I)$. In other words, all points between the two endpoints will be contained in the image of an interval under a continuous function.

7. *Definition*: $a, b \in A \subseteq M$. A **(continuous) path** in A from a to b is a **continuous function** $f : [0, 1] \to A$ so that $f(0) = a, f(1) = b$.

8. *Propositions*: Any open ball in $\langle \mathbb{R}^n, \rho_2 \rangle$ is path connected.

9. *Proposition*: path connected $\implies$ connected. Reverse is true for $\langle \mathbb{R}^n, \rho_2 \rangle$ but not in general.

10. *Lemma*: Suppose $a, b, c \in A \subseteq M$, and there are paths $a \to b, b \to c$. Then there exists paths $b \to a, a \to c$ in A.

7 2nd C: Total Boundedness and Completeness

Let $\langle M, \rho \rangle$, $\langle N, \tau \rangle$ and $\langle P, \sigma \rangle$, $\langle M_1, \rho_1 \rangle, \langle M_2, \rho_2 \rangle$ be metric spaces.

1. *Definition*: A subset A of M is **bounded** if there exist $x \in M$ and $0 < R < \infty$ so that $A \subseteq B[x, R]$.

2. *Definition*: A subset A of M is **totally bounded** if for any $\epsilon > 0$, there are finitely many points $x_1, \ldots, x_n$ so that $A \subseteq \bigcup_{i=1}^n B[x_i, \epsilon]$.
   (a) **Characterization**: A subset A of M is totally bounded $\iff$ every sequence in A has a Cauchy subsequence. (Lion Hunting)

3. *Remark*: If a subset A of M is **totally bounded**, we can request the center of the bounding (open) balls to be all from A.

4. *Proposition*: Totally bounded $\implies$ bounded.

5. *Proposition*: In $\langle \mathbb{N}^n, \rho_2 \rangle$, a subset is totally bounded $\iff$ bounded.

6. *Definition (complete)*: A subset A of M is **complete** if every Cauchy sequence in A converges to a point in A.

7. *Proposition*: Let $(x_k)_{k=1}^\infty$ be a sequence in $\mathbb{R}^n$. Then

(a) It is Cauchy wrt $\rho_2 \iff$ each coordinate is a Cauchy sequence in $\langle \mathbb{R}, \rho_e \rangle$.
(b) It is convergent wrt $\rho_2 \iff$ each coordinate is a convergent sequence in $\langle \mathbb{R}, \rho_e \rangle$.

8. *Proposition*: given a complete metric space M, a subset A of M is complete $\iff$ A is closed in M.

9. *Definition (diameter)*: $\operatorname{diam} A = \sup\{d(x,y) : x, y \in A\}$, the maximum distance between any 2 points in A.

10. *Nested Set Theorem*: Let M be a complete metric space. Suppose that $(A_n)_{n=1}^{\infty}$ is a sequence of bounded nonempty closed subsets of M so that

    (a) $A_1 \supseteq A_2 \supseteq \ldots$,
    (b) $\lim_{n\to\infty} \operatorname{diam} A_n = 0$

    Then there is exactly one point in $\bigcap_{n=1}^{\infty} A_n$.

11. *Definition (contraction)*: A function $T : M \to M$ is a **contraction** if there exists **contraction constant** $0 < C < 1$ so that $\rho(T(x), T(y)) \leq C\rho(x,y)$ for all $x, y \in M$.

12. *Banach Fixed Point Theorem (aka Contraction Mapping Principle)*: M is a complete metric space. If $T : M \to M$ is a contraction with constant C. Then T has a unique fixed point, i.e., there is a unique $x \in M$ so that $T(x) = x$. Furthermore, take any $x_0 \in M$ and define $x_n = T(x_{n-1})$ for any $n \in \mathbb{N}$, then the sequence converges to the fixed point x and $\rho(x_n, x) \leq \frac{C^n}{1-C}\rho(x_0, x_1)$ for all $n \in \mathbb{N}$.

13. *Baire's Theorem*: Let M be a complete metric space. Assume that $M = \bigcup_{n=1}^{\infty} F_n$, where each F_n is a closed set in M. Then there exists $n_0 \in \mathbb{N}$ so that F_{n_0} contains a nonempty open ball $B[x,r]$.

14. *Definition (isometry)*: Let $\langle M, \rho \rangle$ and $\langle N, \tau \rangle$ be metric spaces. A function $f : M \to N$ is an **isometry** if $\tau(f(x), f(y)) = \rho(x,y)$ for all $x, y \in M$.

15. *Theorem*: Let $\langle M, \rho \rangle$ be a metric space. There is a pair (N, i), where $\langle N, \tau \rangle$ is a **complete** metric space, where $i : M \to N$ is an isometry, and $i(M)$ is dense in N. That is, $\overline{i(M)} = N$. (N, i) is a **completion** of M.

16. *Theorem (completion is unique up to isometry)*: Let (N, i) and (P, j) be two completions of a metric space $\langle M, \rho \rangle$. Then there is an bijective isometry $\pi : N \to P$ so that $\pi \circ i = j$, and π^{-1} is an isometry too so that $\pi^{-1} \circ j = i$.

17. *Proposition*: Let $f : M_1 \rangle M_2$ be an isometry (not necessarily onto) and $\langle N_i, \tau_i \rangle$ be the completion of $\langle M_i, \rho_i \rangle$, $i = 1, 2$. Then there is a unique continuous function $\widetilde{f} : N_1 \to N_2$ that extends f, i.e., $\widetilde{f}(x) = f(x)$ for all $x \in M_1 \subseteq N_1$. Moreover, the extension $\widetilde{f}$ is an isometry.

8 3rd C: Compactness

1. *Definition (compact)*: $E \subseteq M$ is **compact** if E is both **complete** and **totally bounded**.

2. *Proposition*: In $\langle \mathbb{R}^n, \rho_2 \rangle$, a subset E is **compact** $\iff$ **closed and bounded**

3. *Definition (open covering)*: Let E be a subset of a metric space $\langle M, \rho \rangle$. A family $\mathcal{G}$ of sets is an **open covering** of E if

    (a) Each $G \in \mathcal{G}$ is an open set in M.
    (b) E is covered by $\bigcup\{G : G \in \mathcal{G}\}$.

4. *Definition (Heine-Borel property)*: (Every open cover of E has a finite subcover.) A subset E of a metric space has the **Heine-Borel property** if for every open covering $\mathcal{G}$ of E, there are finitely many $G_1, \ldots, G_n \in \mathcal{G}$ so that $E \subseteq G_1 \cup \cdots \cup G_n$.

5. *Theorem (multiple characterization of compactness)*: Let $E \subseteq \langle M, \rho \rangle$. The following are equivalent:

    (a) E is compact (i.e., totally bounded and complete).
    (b) **(Sequential compactness)** Every sequence in E has a convergent subsequence (to a point in E).
    (c) **(Bolzano-Weierstrass property, the weakest)**: Every infinite subset of E has a cluster point in E.
    (d) **(Heine-Borel property)**: Every open cover of E has a finite subcover.

6. *Lebesque covering Lemma*: Given a compact subset E in a metric space $\langle M, \rho \rangle$ and let $\mathcal{G}$ be an open cover of E. Then there exist a **Lebesgue's Number** $r > 0$, so that $\forall x \in E, \exists G \in \mathcal{G}$ (depending on x) so that $B[x,r] \subseteq G$.

7. *Proposition:* compact $\implies$ closed. A closed subset in a compact set is compact.

8. *Theorem*: (Continuity preserves compactness).

9. ***Extreme Value Theorem***: Given $f : M \to \mathbb{R}$ continuous, and $E \in M$ a compact set, then $\exists c, d \in E$ such that $f(c) \leq f(x) \leq f(d)\ \forall x \in E$.

10. *Theorem*: $f : M_1 \to M_2$ is continuous and bijective. If M_1 is compact, then f^{-1} is continuous.

11. *Definition*: $f : M_1 \to M_2$ is **uniformly continuous** on M_1 if $\forall \epsilon > 0, \exists \delta > 0$ such that we can use the same δ for any $x, y \in M_1$ where $\rho_1(x,y) < \delta \implies \rho_2(f(x), f(y))$ or equivalently, $y \in B[x, \delta] \implies f(y) \in B[f(x), \epsilon]$. **uniform continuity $\implies$ (pointwise) continuity**.

12. *Theorem*: Continuous function on a compact set is uniformly continuous.

13. *Theorem*: Given $E \subseteq M_1$ and M_2 is complete, a uniformly continuous function $f : E \to M_2$ can be extended to a continuous function on $\overline{E}$, $\tilde{f} : \overline{E} \to M_2$, defining $\tilde{f}(x) = \lim\limits_{n\to\infty} f(x_n)$ for some $(x_n)_{n=1}^{\infty} \to x$.

9 Sequences and Series of Functions

1. *Definition*: Given $(f_n)_{n=1}^{\infty}$ where $f_n : M_1 \to M_2$, $(f_n)_{n=1}^{\infty}$ **converges pointwise** to a function $f : M_1 \to M_2$ if $\forall x \in M_1$, $(f_n(x))_{n=1}^{\infty} \to f(x) \in M_2$. That is, given a sequence of function, take a $x \in M_1$, subbing it into every function in $(f_n)_{n=1}^{\infty}$ yields a sequence in M_2 which converges to $f(x)$.

2. *Definition*: $(f_n)_{n=1}^{\infty}$ **converges uniformly** to $f : M_1 \to M_2$ if

    (a) $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ such that $\rho_2(f_n(x), f(x)) < \epsilon$ for all $x \in M_1$ whenever $n \geq N$. **Given epsilon, the N can be shared for all $x \in M_1$.**

(b) $\iff \lim_{n\to\inf} \sup\{\rho_2(f(x), f_n(x)) : x \in M_1\} = 0$

3. *Definition*: $(f_n)_{n=1}^{\infty}$ is **uniformly Cauchy** on M_1 if given $\epsilon > 0$, $\exists N \in \mathbb{N}$ such that $\rho_2(f_m(x), f_n(x)) < \epsilon\ \forall x \in M_1$ whenever $m, n \geq N$.

4. *Theorem*: **(Uniform continuity preserves (uniform) continuity)** A uniform continuous sequence of (uniformly) continuous functions converges to a (uniformly) continuous function. The limit is not necessarily continuous if convergence is pointwise. Note that a sequence of functions is continuity preserving $\implies$ uniform continuity is **NOT** true.

5. **Dini's Theorem**: M is a compact metric space. A sequence of pointwise convergent functions $f_n : M \to \mathbb{R}$ and is non-decreasing (i.e., $\forall x \in M, f_1(x) \leq f_2(x) \leq \cdots \leq f(x)) \implies (f_n)_{n=1}^{\infty}$ converges uniformly (to the same limit).

6. *Definition (Series of functions)*: $\forall k \in \mathbb{N}$, we have $f_k : M \to \mathbb{R}$. Define the **nth partial sum** $s_n(x) = \sum_{k=1}^{n} f_k(x)$, the function that sums through the f_ks at a given x. The **infinite series** $\sum_{k=1}^{\infty} f_k(x)$ converges pointwise (uniformly) on M to a function $f : M \to \mathbb{R}$ if the sequence of partial sums, $(s_n)_{n=1}^{\infty}$ converges pointwise (uniformly) to f on M.

7. *Theorem*: **(Weierstrass M-test)** Suppose that $\sum_{k=1}^{\infty} M_k$ is a convergent numerical series, $M_k \geq 0$ (non-negative terms) so that $|f_k(x)| \leq M_k\ \forall x \in M$. Then $\sum_{k=1}^{\infty} f_k$ is uniformly convergent on M.

8. *Definition*: Let $a_k, b \in \mathbb{R}$ for $k \geq 0$. A series of the form $\sum_{k=1}^{\infty} a_k(x-b)^k$ is called the **power series**.

9. *Proposition*: The power series converges **absolutely and uniformly** on any closed interval in $(b-R, b+R)$, where $R = (\limsup_{k\to\infty} |a_k|^{1/k})^{-1}$ is the **radius of convergence**, and define $\frac{1}{\infty} = 0, \frac{1}{0} = \infty$. In $\mathbb{R} - [b-R, b+R]$, power series diverges. At the boundaries, the behavior is undefined. Furthermore, $f : (b-R, b+R) \to \mathbb{R}, f(x) = \sum_{k=1}^{\infty} a_k(x-b)^k$ is continuous on the domain.

10. *Definition*: Given $b \in \mathbb{R}$, a real-valued function f is **(real) analytic at** b if
    (a) f is defined on an open interval I containing b.
    (b) $\exists$ power series $\sum_{k=1}^{\infty} a_k(x-b)^k$ such that $f(x) = \sum_{k=1}^{\infty} a_k(x-b)^k\ \forall x \in I$.

11. *Proposition*: Suppose that $f(x) = \sum_{k=1}^{\infty} a_k(x-b)^k$ has radius of convergence $R > 0$. Then f is analytic at any $u \in (b-R, b+R)$.

12. *Proposition*: **(Strong uniqueness property)** Let I be an open interval and $f, g : I \to \mathbb{R}$ be analytic functions on I. If $\exists\ \emptyset \neq J \subseteq I$ such that $f(x) = g(x)\ \forall x \in J$, then $f(x) = g(x)\ \forall x \in I$. In words, if two analytic functions are equal in an sub-interval, they are equal throughout.

10 The metric space $C(M_1, M_2)$

In this section, $C(M_1, M_2)$ represents the set of all continuous functions from M_1 to M_2, where $\langle M_1, \rho_1 \rangle$ is always **compact**. The metric space is given the **uniform metric**, $\rho(f, g) = \max\{\rho_2(f(x), g(x)) : x \in M_1\}$.

1. *Proposition*: Let $(f_n)_{n=1}^{\infty}$ be a sequence in $C(M_1, M_2)$. Then
   (a) $(f_n)_{n=1}^{\infty}$ converges to f in $C(M_1, M_2) \iff (f_n)_{n=1}^{\infty}$ converges uniformly to f in M_1.
   (b) $(f_n)_{n=1}^{\infty}$ is Cauchy in $C(M_1, M_2) \iff (f_n)_{n=1}^{\infty}$ is uniformly Cauchy in M_1.

   Therefore ρ is called the **uniform metric**.

2. *Theorem*: $C(M_1, M_2)$ is complete (with the uniform metric) $\iff M_2$ is complete.

3. *Theorem*: Let $(F_n)_{n=1}^{\infty}$ be a sequence of closed sets in $C[0,1]$, the set of all continuous functions whose domain is $[0,1] \subseteq \mathbb{R}$. If $C[0,1] = \bigcup_{n=1}^{\infty} F_n$, then $\exists n_0$ such that F_{n_0} contains a nonempty open ball $B[f, r]$ for some $f \in C[0,1]$ and some $r > 0$.

4. *Application*: An application of the theorem above, by defining $F_n = \{f : f \in C[0,1], \exists x \in [0,1], \forall y \in [0,1], |y - x| < \frac{1}{n} \implies |f(x) - f(y)| \leq n|y - x|\}$. We will find a function that is continuous on $[0,1]$ but nowhere differentiable.