The (fairly) Reasonable Effectiveness of Mathematics - a Mathematical Viewpoint

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Abstract

Mathematics has been found an immensely useful tool in science, and subsequently a irreplaceable method to understand our nature through the lens of science. This "unreasonable" effectiveness [3] has been the topic of discussion for a long time. I propose a few conditions which account for the effectiveness of mathematics - suitable axioms that arose from nature, human's innate ability to do logical deduction, and the availability of suitable mathematical notation. This precise notation, coupled with the ability of humans to abstract and compose mathematical structures with one another, has allowed new knowledge to be discovered quickly.

We also look at the Mathematical Universe Hypothesis (MUH)'s claim of the universe is mathematical[2], and why we do not understand it yet is due to the baggage such as metaphors, constructed structures that obscure the underlying mathematical nature.

What makes something effective? It implies a tool - problem relationship: if using a tool makes the problem significantly easier to solve, then we say the tool is effective. Just like a hammer is much more effective in driving a nail into wood than say, a knife, we could either conclude how the structure of a hammer, its precisely weighted head which swings and creates a huge moment that makes

What makes something effective? The evaluation of effectiveness implies a problem-solution relationship: if using a particular solution or a tool makes the given problem significantly easier to solve, then we say the tool is effective. For example, a hammer is much more effective in driving a nail into wood than say, a knife. However, what makes the hammer so effective? In this case, the hammer is precisely designed to drive nails, therefore it is as effective as we can possibly get.

But this is not the case for mathematics as a tool to understand reality - mathematics was never designed to explain or mimic nature. Mathematicians do not limit their conceptions to the real world or the observable reality; whether abstract structures such as topological spaces or symmetric groups have a mirror image in the real world is not a concern for Mathematics. In spite of that, mathematics is surprisingly effective in understanding the real world. It is almost as if we pick up a tool that was never designed for driving nails, but it happens to do the job perfectly, or in the words of Wigner, "unreasonably effective"[3]. He gave an analogy for this situation: we are given a bunch of keys and many doors, which are not related at first glance. However as every door opens effortlessly after a few tries, we can't help but to think that the keys might be designed perfectly for the doors.

In this essay, I will be discussing why mathematics is an effective tool for humans to discover truths in nature, from three points: one, the mathematical language is precise, concise and universal and welcomes contribution and development in mathematics from around the world; two, mathematics acts as a collection of results of deductive logic which science and other fields can use easily and save a lot of time; three, mathematics has most of its axioms rooted in our perceived reality, therefore producing results that can explain our reality well. We will also look at the Mathematical Universe Hypothesis (MUH) raised by Max Tegmark, which hypothesizes that the underlying universe is mathematical[2]. The reason why we do not perceive our reality as mathematical is due to the "baggage" imposed on mathematical equations, which are abstract concepts and constructs acting as shortcuts and intuitions. For example, a cup in reality is a container that is meant to hold drinks. However, the idea of a "cup" is a baggage which obscures its underlying mathematical nature. Throwing away the abstraction, a cup might be just plastic, molded in a specific shape to allow the containment of liquid, or more particularly a specific series of chemical elements bonded together in an intricate way to create its texture. One can keep stripping off "baggage" like these in our reality and finally, hypotheses Tegmark, reach an eternal, irreducible mathematical reality. We will take a look at MUH, and argue about its possibility how it induces a paradox in scientific or mathematical development.

First of all, mathematics is so successful despite its complexity, is partly due to the excellent qualities of the mathematical language. It is a concise and precise way to record the results of deductive logic, which is what mathematics is built upon. In general, languages that encode information can run into either of these two extremes: one, being extremely clear but terse, such as how computers store information in strings of 0 and 1s, or two, being as ambiguous as natural language, where there are many ways to encode information, which are up to interpretation when decoding. The formal is ideal for storage and transfer, however it sacrifices the readability and intuition of the information. Should mathematics fall within this category, it would be difficult for mathematicians to decipher information from the mathematical language, much less to internalize the language into meaningful structures and discover new theorems on existing ones. On the other hand, natural language is ambiguous at its core, and cannot reliably transfer information, despite being easier to understand. Mathematics described in natural language will likely cause more misunderstanding and impede the development of it. Fortunately, mathematics as formulated today emerges as a precise language, although restricted, but still remains understandable to the trained individuals. This is an important condition for mathematics to develop rapidly and reach advanced enough results of purely deductive logic, which is far more than what science and other fields can leverage at the moment.

Second, while science relies on inductive logic, it aims towards theories that are consistently true, thus it tends to rely heavily on mathematical models and constructs. Oftentimes, scientific theories need a complex interaction of logic between its laws, how different objects in the system interact with each other, which is difficult to grasp. However, mathematics comes to help as it provides readily available results in deductive logic which describes interactions between highly abstract mathematical objects, and can be suitably specialized to explain results in nature while promising coherence due to the a priori nature of mathematics. This means that science can save up a lot of time needed to build a model from scratch that meets the circumstances on the topic at hand. For example, the invention of complex numbers, which has seemingly no counterpart in the real world (unlike natural numbers as counting objects, or real numbers as results of measurements), is the bread and butter in quantum mechanics - you find it in Schrodinger's equations, Heisenberg's commutation relation and more.

$$i\hbar\frac{\partial}{\partial t}\Psi(\mathbf{r},t) = \hat{H}\Psi(\mathbf{r},t)$$

Schrodinger's equation.

How do we account for this unexpected effectiveness of complex numbers in Quantum Mechanics? If we think of mathematics theories as intermediate results of deductive logic, then this effectiveness can be attributed to the fact that complex numbers stored just the appropriate results in deductive logic in an abstract form, that fits the need of Quantum Mechanics theories. The whole theory of complex numbers is a crystallization of deductive logic, encapsulated in this man-made, mathematical concept called complex numbers. Although they do not correspond directly to reality as much as constructs such as natural numbers do, since they are first devised as solutions for equations like $x^2 + 1 = 0$, which is impossible in real life. This means the theory of complex numbers is more of an exercise in deductive logic rather than real-world modelling, and its intermediate results are stored in the system of mathematics as this abstract construct. Should humans not have this tool, humans would struggle with the concept of quantum mechanics and how to explain it systematically. These of available results are like the table of integrals in calculus, or table of constants in physics, which is theoretically derivable from first principles, but is much more efficient in application when was recorded down, encapsulated in easy-to-remember format for application. Mathematics is effective precisely because it provides a shortcut in deductive logic to any field that requires it.

The third reason lies in the axiomatic nature of mathematics, which I argue is inspired from our reality and partly contributes to mathematics's effectiveness in describing nature. Axioms are indisputable truths which mathematicians take for granted, and are arbitrarily defined by mathematicians themselves. They can be as imaginative as possible, as long as it has the most important quality of axioms: coherence, which is the inability to cause contradictions. In order for mathematics to be useful, axioms act as the foundation of all mathematical theories, therefore any small paradoxes in an axiomatic system could cause a crisis in mathematics. As evident from the Foundational Crisis of Mathematics in the 20th century due to the paradoxes found when formalizing mathematics with naive set theory. The most famous of all is Russell's Paradox, which one applied version goes as follows:

"A barber only shaves those who do not shave themselves. Does the barber shave himself?"

Both answers lead to a contradiction here: if the barber shaves himself, then he is impossible as a customer to his own service by his principle; if he does not, then he qualifies as one of his own customers since he does not shave himself. A set-theoretic version of the statement has caused trouble in the early formation of set theory, now the widely accepted foundation of mathematics, the Zermelo-Fraenkel (ZF) axioms, has an axiom to avoid this situation.

Since there is a need for rigorous and coherent axioms, the formulation of such axioms has to be intuitive and logical, fitting into the patterns we observe. I claim that this means that throughout history, many axioms of formal systems are inspired by reality. For example, Euclid's axioms for geometry include facts such as "a circle can be described by a centre and a radius" are probably inspired by objects in the reality, although highly idealized. Peano axioms, on the other hand, formalized natural numbers in a series of axioms such as "every natural number has a successor, which is also a natural number" which appeals directly to our understanding of "adding one more". Even in the ZF axioms, axioms such as "two sets are equal if they have the same elements" is logical and applicable to reality. These axioms are tied to our reality, which causes mathematics to generate results that are highly effective in describing reality, even though the process of mathematics are analytic a priori and self-contained. These axioms can act as suitable initial conditions for mathematics, therefore influencing our developed mathematics to be largely coherent with reality.

Conclusion: The three reasons mathematics is an effective tool in understanding nature:

1. The mathematical language is concise and precise. This allowed effective communication between mathematicians and facilitated the development of mathematics.
2. Mathematics acts as a table of pre-computed results of deductive logic, encoded in abstract mathematical objects and theories, which Physics and other domains can specialize and utilize, saving time as compared to deducing all these facts from scratch.
3. Mathematical axioms which we use are "Intuitive", hinting to its roots in the reality. Then its effectiveness in understanding the world is no longer elusive, as it is a system of knowledge built on top of nature.

Having looked at the reasons for the effectiveness of mathematics, one might not be satisfied and want to further claim that the reality is mathematics. We thus look at the Mathematical Universe Hypothesis (MUH), raised by Max Tegmark, which states that "our external physical reality is a mathematical structure". He further postulates that all theories about nature have two components: mathematical equations and "baggage", that serve as a mental model to connect what we observe and intuitively understand about the mathematical equation. Examples of baggage can be atoms, elements, cells, organs, animals, or the earth and its atmosphere - they obscure the underlying mathematical structure that can explain anything about the world. Once we strip away all this baggage, for example, reduce a cup to its subatomic particles and the energy level, or something even lower-level, we will discover the theory of everything, which is just one mathematical reality. To understand this better, consider Newtonian Mechanics, which is just the three laws of motion by Newton, expressed in mathematical equations, combined with the abstract ideas of Forces, velocity, acceleration and other "baggage". In terms of mathematical concepts, the idea of a function, which maps some input to certain fixed outputs, is a baggage on its own, since it can be represented in terms of simple things, namely just sets, in the set-theoretic formulation.

If we were to consider this hypothesis for a moment, we would realize a paradox

- the development of mathematics, or any other theory lies upon its “baggage” already. It would be hard for mathematicians to develop a theory as complex as calculus, which involves many abstract concepts (or “baggage”) such as functions, continuity, variables, and algebra, if we were to strip away the baggage. In fact, humans did not first come up with axioms of mathematics before discovering some of the biggest mathematical results. Formal axiomatic systems such as Peano Axioms did not appear for a 200 years after Newton and Leibniz discovered calculus without the need to strip away the “baggage” of mathematics. It is precisely these abstractions, or baggage, obscured away the trivial details which interfered, rather than helped with mathematical development. Even if it is possible to strip away the baggage and peek at the underlying mathematical structure, it would be too complicated for us to understand, much less utilize them.

Let us illustrate this viewpoint with an example from the modern foundation of mathematics, set theory and see how numbers can be represented as sets in reality[1].

In the beginning, let us define 0 as the empty set $\{\}$. Now suppose a set contains only the empty set, and we denote it be $1 = \{\{\}\}$. From now onwards, given a number n and its set notation, define $n + 1$ by the union of n and the set containing only n. Now we can repeat this process and create $2, 3, 4, 5, \ldots$ represented only be sets as below:

$0 = \{\}$

$1 = 0 \cup \{0\} = \{\{\}\}$

$2 = 1 \cup \{1\} = \{\{\}, \{\{\}\}\}$

$3 = 2 \cup \{2\} = \{\{\}, \{\{\}\}, \{\{\}, \{\{\}\}\}\}$

$4 = 3 \cup \{3\} = \{\{\}, \{\{\}\}, \{\{\}, \{\{\}\}\}, \{\{\}, \{\{\}\}, \{\{\}, \{\{\}\}\}\}\}$

$5 = 4 \cup \{4\} = \{\{\}, \{\{\}\}, \{\{\}, \{\{\}\}\}, \{\{\}, \{\{\}\}, \{\{\}, \{\{\}\}\}\}, \{\{\}, \{\{\}\}, \{\{\}, \{\{\}\}\}, \{\{\}, \{\{\}\}, \{\{\}, \{\{\}\}\}\}\}\}$

There is as little baggage here as possible if we deem the baseline to be the ZF Axioms, which describes sets in detail.

Now as an example, let’s look at the representation of the natural number 5 in set theory. It is unnecessarily complicated without abstraction. Representing 5 as an integer (not just a natural number), is a lot more complicated still, as it is defined as the the set of all pairs of numbers (also represented as sets) like $(5, 0), (6, 1), (7, 2), (3, -2), \ldots$ as all these pairs have a “difference” of 5. This is already an infinite structure of pairs, just enough to define a single number. Flipping the position of every single element in the pairs in 5, we will get -5, since $(0, 5), (1, 6), (2, 7), \ldots$ etc all have a difference of -5.

Now, we all know intuitively 5 + (-5) = 0. If you have 5 dollars, and you then lose 5, you will have nothing left. However, how do we begin to make sense of these infinite structures of pairs? How can they combine to get 0? What is this "combine" operation? Will the result of the "combine" operation always return me a number? These are hardly workable already to derive some everyday, intuitive facts we have about mathematics, although it is already proven by mathematicians to be true. This gets crazier as we further extend 5 to a rational number (it is the infinite collection of integer pairs whose quotient is 5), or a real number (too complicated for the scope of this essay), which are essentially infinite structures just like above, but rational numbers are defined upon the structure of integers, and the reals upon the rationals. Baggages like the idea of 5 concretizes the idea of such mathematical objects and make it useful for mathematical or scientific advancements. How one can carry out any meaningful calculation with such cumbersome structures is a miracle, much less develop theory as advanced as quantum mechanics when we have mathematics and its baggage stripped. Granted, the underlying reality might be mathematical, but to reach the state where we can uncover this through by stripping away baggage, humans are going to have a really difficult time understanding concepts intuitively as we do now.

Coincidentally, this is what Tegmark explained in his paper about MUH: such "baggage" is needed for intuition and shortcuts, which is the invariant in gaining further understanding and developing new theories. Similar to how scientists use mathematical models more than questioning them, mathematicians themselves use mathematical concepts as baggage more than they question them. Paradoxically, MUH's hint at a higher reality requires to have all its "baggage" stripped, which means further development in mathematics will be impeded.

Therefore we conclude that even if MUH is true, humans might never uncover a satisfying theory of everything via mathematics. Therefore mathematics is only reasonably effective.

References

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