MA3110 Homework 3

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Problem H1

Let $F(x) = \displaystyle\int_x^{x^3} \frac{\cos(t^2)}{t} dt$ for $x \geq 1$. Find $F'(x)$ for $x \geq 1$.

Solution

Proof. Let $f(x) = \dfrac{\cos(x^2)}{x}, G(x) = \displaystyle\int_1^x f(t)dt$. Then by the Fundamental Theorem of Calculus I, since f is continuous on $[1, \infty)$, we have $G'(x) = f(x)$ for any $x \in [1, \infty)$. Then $\forall x \geq 1$,

$$\begin{aligned}
F(x) &= \int_x^{x^3} \frac{\cos(t^2)}{t} dt \\
&= \int_1^{x^3} \frac{\cos(t^2)}{t} dt - \int_1^x \frac{\cos(t^2)}{t} dt \\
&= G(x^3) - G(x). \\
\therefore F'(x) &= 3x^2 G'(x^3) - G'(x) && \text{chain rule} \\
&= 3x^2 f(x^3) - f(x) && \text{FTC(I)} \\
&= \frac{3}{x}\cos(x^6) - \frac{\cos(x^2)}{x}.
\end{aligned}$$

$\square$

Problem H2

Let f and g be continuous functions on $[a, b]$ and let $H : [a, b] \to \mathbb{R}$ be defined by

$$H(x) = \left(\int_a^x f(t)dt\right)\left(\int_x^b g(t)dt\right) \quad \text{for all } x \in [a, b].$$

Prove that there exists $c \in (a, b)$ such that

$$g(c)\int_a^c f(x)dx = f(c)\int_c^b g(x)dx.$$

Solution

Proof. We notice that

$$H(a) = \left(\int_a^a f(t)dt\right)\left(\int_a^b g(t)dt\right) = 0 = \left(\int_a^b f(t)dt\right)\left(\int_b^b g(t)dt\right) = H(b).$$

Let $F(x) = \displaystyle\int_a^x f$ and $G(x) = \displaystyle\int_b^x g$. We rewrite H into products of F and G:

$$\begin{aligned}
H(x) &= \left(\int_a^b f(t)dt\right)\left(\int_x^b g(t)dt\right) \\
&= (F(x))(-G(x))
\end{aligned}$$

Since f and g are continuous, F and G are differentiable on $[a, b]$ by Tutorial 6 Question 3, and $F' = f, G' = g$ on (a, b). Thus by Rolle's theorem, there exists $c \in (a, b)$ such that $H'(c) = 0$. Since F and G are differentiable on (a, b), we can use the product rule to differentiate H:

$$\begin{aligned} H'(c) = 0 &= F'(c)(-G(c)) + F(c)(-G'(c)) \\ 0 &= f(c) \int_c^b g(t)dt + \left(\int_a^c f(t)dt \right) (-g(c)) \end{aligned}$$

Therefore

$$f(c) \int_c^b g(t)dt = g(c) \int_a^c f(t)dt, \quad \text{for some } c \in (a, b).$$

□

Problem H3

Suppose that f is continuous on $[a, b]$. Prove that there exists $c \in (a, b)$ such that

$$\int_a^b f = f(c)(b - a).$$

Solution

Proof. Define

$$F(x) = \int_a^x f, \quad x \in [a, b].$$

We note that f is continous on $[a, b]$ implies that F is differentiable on $[a, b]$. Therefore we can apply Mean Value Theorem to F on the interval $[a, b]$: there exists some $c \in (a, b)$ such that

$$\begin{aligned} F'(c) &= \frac{F(b) - F(a)}{b - a} && \text{Mean Value Theorem} \\ \therefore f(c) &= \frac{1}{b - a} \left(\int_a^b f - \int_a^a f \right) && \text{FTC(I), definiton} \\ &= \frac{1}{b - a} \left(\int_a^b f \right) \end{aligned}$$

Moving terms, we have

$$\int_a^b f = f(c)(b - a).$$

□

Problem H4

Using the Riemann integral of a suitable chosen function, find the limit

$$\lim_{n \to \infty} \frac{1}{n^2} \sum_{k=1}^n k \sin\left(\frac{\pi k^2}{n^2} \right).$$

Solution

Proof. Let

$$f(x) = x \sin(\pi x^2), \; P_n = \{0, \frac{1}{n}, \dots, \frac{n}{n}\}, \; \xi^{(n)} = P_n \setminus \{0\}.$$

Then we can rewrite the summation into a Riemann sum:

$$\begin{aligned}
&\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n}\cdot\frac{k}{n}\sin\left(\pi\left(\frac{k}{n}\right)^2\right)\\
=&\lim_{n\to\infty}S(f,P_n)(\xi^{(n)})\\
=&\int_0^1 f && n\to\infty \implies \|P_n\|\to 0,\ \text{Corollary 7.4.3}\\
=&\frac{1}{2\pi}\int_0^1 2\pi x\sin(\pi x^2)dx\\
=&\frac{1}{2\pi}[-\cos(\pi\cdot 1^2)+\cos(\pi\cdot 0^2)] = \frac{1+1}{2\pi} = \frac{1}{\pi}.
\end{aligned}$$

□

Problem H5

In each part, determine if the improper integral converges. Justify your answers.

Part H5(i)

$$\int_1^\infty \frac{\sin^2 x}{x^2}dx.$$

Solution

Proof. Recall that if

$$f(x) = \frac{\sin^2 x}{x^2}$$

is non-negative over $[1,\infty)$, then the improper integral converges $\iff$ $F(x)=\int_1^x f$ is bounded for all $x \geq 1$. First notice that for any $x \geq 1$, we have

$$0 \leq (\sin x)^2 \leq 1 \implies 0 \leq f(x) = \frac{\sin^2 x}{x^2} \leq \frac{1}{x^2}.$$

Which shows f is nonnegative on $[1,\infty)$. Now we only need to show that $F(x)$ is bounded above. By Theorem 7.2.6(iii), we have

$$\begin{aligned}
\frac{\sin^2 x}{x^2} \leq \frac{1}{x^2} \implies \int_1^t \frac{\sin^2 x}{x^2} &\leq \int_1^t \frac{1}{x^2}\\
&= \left(-\frac{1}{t}+\frac{1}{1}\right)\\
&= 1-\frac{1}{t}\\
&< 1 && \forall t\in[1,\infty).
\end{aligned}$$

Hence the improper integral converges. □

Part H5(ii)

$$\int_0^1 \frac{x}{1-x^2}dx.$$

Solution

Proof. Note that

$$f(x) = \frac{x}{1-x^2}$$

is unbounded as $x \to 1$. Let us rewrite the improper integral as

$$\begin{aligned}
\int_0^1 \frac{x}{1-x^2} dx &= \lim_{t\to 1^-} -\frac{1}{2}\int_0^t \frac{-2x}{1-x^2} dx \\
&= -\frac{1}{2}\lim_{t\to 1^-} \left[\ln(1-x^2)\right]_{x=0}^{x=t} \\
&= -\frac{1}{2}\lim_{t\to 1^-} \left(\ln(1-t^2) - \ln(1)\right) && \text{then let } t' = (1-t^2) \to 0, \\
&= -\frac{1}{2}\lim_{t'\to 0^+} \ln(t') \\
&= \infty.
\end{aligned}$$

Therefore the improper integral diverges. □