Week 8 Assignment

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1. To show f is not 1-1, we will show that we can find x, x' in $[a, c]$ where

$$x \neq x' \quad \text{but} \quad f(x) = f(x')$$

which implies f is not 1-1.

Given that $f(b) > \max\{f(a), f(c)\}$, we will show only the case of $f(a) > f(c)$.

If $f(a) = f(c)$, then $x = a, x' = c$ where $a < b < c \implies a \neq c$ is what we wanted. The case of $f(a) < f(c)$ can be proved similarly as the proof given below.

Assuming we have $f(b) > f(a) > f(c)$. Since f is continuous, and $[a, b)$ is an interval in $\mathbb{R}$, intermediate value theorem gives us

$$[f(a), f(b)) \subseteq f([a, b))$$

and we can thus choose

$$x \in [a, b) \quad \text{such that} \quad f(x) \in [f(a), f(b))$$

.

Now, since $f(c) < f(a)$ by assumption, we must have

$$[f(a), f(b)) \subseteq [f(c), f(b))$$

Thus, we can also find a x' such that

$$x' \in (b, c] \quad \text{such that} \quad f(x') = f(x) \in [f(a), f(b)) \subseteq [f(c), f(b))$$

since the intermediate value theorem guarantees that all the values in $[f(c), f(b))$ must be also present in the image of $(b, c]$, $[f(c), f(b))$.

Thus we have found our x, x', and result follows. $\square$

2. (a) Note that $x \in \{x\}$. x is connected since its only possible subsets are either $\emptyset$ or itself. Thus it is impossible to find a subset S such that $\emptyset \neq S \neq \{x\}$ (much less that S is both open and close). $\square$

   (b)
$$x \sim y \implies \exists E \subseteq M, \quad x, y \in E \implies y \sim x$$
$\square$

   (c) Since $x \sim y$, $y \sim z$, let $\{x, y\} \subseteq E$ and $\{y, z\} \subseteq F$ such that E, F are both connected and are subsets of $\langle M, \rho \rangle$.
Since $y \in E, y \in F \implies E \cap F \neq \emptyset$, by theorem, $E \cup F$ must be also connected and thus $\{x, y, z\} \subseteq (E \cup F)$.
$$\therefore x \sim z$$
$\square$

3. We have shown in class that an open, connected set in $\langle \mathbb{R}^n, \rho_2 \rangle$ must be path connected. Therefore, for any $f(a), f(b)$ in $f(G)$, where a, b are in G, there exists a path g from a to b in G, such that $g : [0, 1] \to G$ is continuous and $g(0) = a, g(1) = b$.

We claim that $f \circ g$ is a path from $f(a) to f(b)$. First, by continuity in metric spaces, since g is continuous in $[0, 1]$, and f is continuous in G, then $f \circ g : [0, 1] \to f(G)$ is continuous.

Furthermore,
$$f \circ g(0) = f(g(0)) = f(a)$$
, and
$$f \circ g(1) = f(g(1)) = f(b)$$

Which shows $f \circ g$ is indeed a path from $f(a)$ to $f(b)$ in $f(G)$. Since we can always find a path for any two points, $f(a)$ and $f(b)$ in $f(G)$, therefore $f(G)$ is connected. $\square$