MA3201 Homework 5

Tan Yee Jian (A0190190L)

March 26, 2021

Question 1

Part 1(i)

Solution

Claim. *If e is central idempotent, so is $(1-e)$.*

Proof of claim.

$$\begin{aligned}(1-e)^2 &= 1-e-e+e^2 = 1-e\\ (1-e)r &= 1r-er = r-re = r(1-e).\end{aligned}$$

□

Proof. With the claim above, it suffices to show for any central idempotent $e \in R$, eM is a submodule of M. Let e be given, then we show the submodule criterion: for any $m_1, m_2 \in M, r \in R$,

$$\begin{aligned}em_1 + r\cdot(em_2) &= em_1 + (re)m_2\\ &= em_1 + (er)m_2 && (er = re)\\ &= em_1 + e\cdot(rm_2)\\ &= e(m_1 + rm_2)\\ &\in eM && (RM \subset M).\end{aligned}$$

Since $e \cdot 0 = 0 \in eM \neq \emptyset$, eM is a submodule of M. □

Part 1(ii)

Proof. We first show eM and $(1-e)M$ have trivial intersection. Suppose $x = em = (1-e)m'$ for some $m, m' \in M$, then

$$\begin{aligned}e(x) &= e(em) = em = x\\ \text{but } e(x) &= e(1-e)m' = em' - em' = 0\end{aligned}$$

Combining, we have that any $x \in eM \cap (1-e)M$ must be 0, hence the intersection is trivial. Consider the map

$$\phi : M \to eM \oplus (1-e)M,\ \ m \mapsto (em, (1-e)m).$$

We show this is an isomorphism. Let e be central idempotent, $m_1, m_2 \in M, r \in R$.

Homomorphism:

$$\begin{aligned}\phi(m_1 + rm_2) &= (e(m_1 + rm_2), (1-e)(m_1 + rm_2))\\ &= (em_1 + e(rm_2), (1-e)m_1 + (1-e)(rm_2))\\ &= (em_1, (1-e)m_1) + ((re)m_2, [r(1-e)]m_2) && \text{direct sum, central idempotence}\\ &= \phi(m_1) + r(em_2, (1-e)m_2) && \text{direct sum of modules}\\ &= \phi(m_1) + r\phi(m_2).\end{aligned}$$

Injectivity:

$$\ker\phi = \{m \in M | em = 0 = (1-e)m \in M\}$$

For any $m \in \ker\phi$, $em = m - em \implies m = 0$. Therefore the kernel is trivial.

Surjectivity: Let $(em, (1-e)m') \in em \oplus (1-e)M$ be given. Note that $e(1-e) = e - e^2 = 0 = (1-e)e$. Then

$$\begin{aligned}&\phi(em + (1-e)m') \\ =&(e^2m + e(1-e)m', (1-e)em + (1-e)^2m') \\ =&(em, (1-e)m').\end{aligned}$$

Therefore $M \cong eM \oplus (1-e)M$. □

Question 2

Proof. Denote the map as ϕ. We check for homomorphism and its kernel.

R-module homomorphism: We check $\phi(m + rn) = \phi(m) + r\phi(n)$ for any $m, m' \in M, r \in R$. For simplicity, we write

$$(m + A_1M,\ m + A_2M, \ldots,\ m + A_nM) \text{ as } (m + A_iM).$$

Then

$$\begin{aligned}\phi(m + rm') &= (m + rm' + A_iM) \\ &= (m + A_iM) + (rm' + A_iM) \\ &= \phi(m) + r \cdot (m' + A_iM) && (A_i \text{ is an ideal for all } i\ ) \\ &= \phi(m) + r \cdot \phi(m').\end{aligned}$$

On the other hand, the kernel:

$$\begin{aligned}\ker\phi &= \{M | m + A_iM = A_iM, i = 1, \ldots, n\} \\ &= \{m \in M | \bigwedge_{i=1}^{n} (m \in A_iM)\} \\ &= A_1M \cap A_2M \cap \cdots \cap A_nM. && (\forall i,\ A_iM \subset M)\end{aligned}$$

□

Question 3

Let $\phi : M \to M/A_1M \oplus M/A_2M \oplus \ldots \oplus M/A_n$ be the map as defined in the previous question. We only show the $n = 2$ case, and by properties of direct sum (in particular, cartesian product), we can repeatedly apply this process till the n desired and show what the question asked for, as long as we have that $A_1A_2 \ldots A_{n-1} + A_n = R$.

Claim 1. *Given $A_i + A_j = R$ for any $i \neq j$, then $A_1A_2 \ldots A_{n-1} + A_n = R$.*

Proof of Claim 1. It then sufficies to just show that $A_1 \ldots A_{n-1} + A_n \ni 1$. Let $x_k \in A_k$ be such that $x_k + x_n = 1$. Then,

$$\begin{aligned}1 &= (x_1 + x_n)(x_2 + x_n) \ldots (x_{n-1} + x_n) \\ &\in x_1x_2 \ldots x_{n-1} + A_n \\ &\in A_1A_2 \ldots A_{n-1} + A_i\end{aligned}$$

as desired. □

Claim 2. *Given $A_1 + A_2 = R$, $(A_1A_2)M = A_1M \cap A_2M$.*

Proof of Claim 2. Recall in Homework 1, we showed if two ideals I, J of a ring R such that $I+J=R$, then $IJ = I \cap J$. Therefore,

$$\begin{aligned} A_1A_2M &= (A_1 \cap A_2)M \\ &= \{(\sum_{\text{finite}} rm) \in M | r \in A_1 \wedge r \in A_2\} \\ &= \{(\sum_{\text{finite}} rm) \in M | r \in A_1\} \cap \{(\sum_{\text{finite}} rm) \in M | r \in A_2\} \\ &= A_1M \cap A_2M. \end{aligned}$$

□

Claim 3. *$\phi : M \to M/A_1M \oplus M/A_2M$ is surjective, given A_1, A_2 are ideals such that $A_1 + A_2 = R$.*

Proof of Claim 3. It suffices to show that $(0,1), (1,0)$ are in the image of ϕ. Since $A_1 + A_2 = R$, let $a_1 + a_2 = 1 \implies a_1m + a_2m = m$ for any $m \in M$, where $a_1 \in A_1,\ a_2 \in A_2$. Then

$$\begin{aligned} \phi(a_1m) &= (a_1m + A_1M,\ a_1m + A_2M) \\ &= (A_1M,\ 1 \cdot m - a_2m + A_2M) && (a_1m = m - a_2m) \\ &= (0,1) \in M/A_1M \oplus M/A_2M \\ \phi(a_2m) &= (a_2m + A_1M,\ a_2m + A_2M) \\ &= (1 \cdot m - a_1m + A_1M,\ A_2M) && (a_2m = m - a_1m) \\ &= (1,0) \in M/A_1M \oplus M/A_2M. \end{aligned}$$

Thus ϕ is surjective. □

Finally, by the first isomorphism theorem,

$$\begin{aligned} M/A_1M \oplus M/A_2M &\cong M/\ker\phi && (\ker\phi \text{ is surjective, Claim 3}) \\ &= M/(A_1M \cap A_2M) && (\text{Q2}) \\ &= M/(A_1A_2)M && (\text{Claim 2}). \end{aligned}$$

Together with the condition to repeat in Claim 1, we have the results desired.

Question 4

Proof. $\underline{\implies:}$ Let M be an Artininan R-module, N a submodule of M. Let $N_1 \supset N_2 \supset \dots$ be a descending chain of submodules of N. This is a descending chain of submodules of M too so it stabilizes, therefore N is Artinian. By the 4th Isomorphism Theorem, any descending chain of M/N can be written as $M_1/N \supset M_2/N \supset \dots$, where $M_1 \supset M_2 \supset \dots$ is a descending chain in M. Since the descending chain M_i stabilizes, so must M_i/N, and therefore M/N is also Artinian.

$\underline{\impliedby:}$

Take any submodule N of M, and we know N and N/M are Artinian. Then given any descending chain in M,

$$M_1 \supset M_2 \supset \dots \tag{1}$$

consider the chains

$$M_1 \cap N \supset M_2 \cap N \supset \dots \tag{2}$$

and

$$M_1 + N \supset M_2 + N \supset \dots \tag{3}$$

in N and M/N respectively. We will show that (1) must stabilize. Since N and M/N are Artinian, let n be large enough such that both (2) and (3) stabilize at the n-th term, ie. $M_n \cap N = M_{n+1} \cap N = \dots$ and $M_n + N = M_{n+1} + N = \dots$. We want to show that (1) must stabilize at n too, in other words, $M_n = M_{n+1} = \dots$. It suffices to just show $M_n \subset M_{n+1}$ since we already have the other inclusion.

Let m_n be from M_n. Then

$$\begin{aligned} & m_n + N \in M_n + N = M_{n+1} + N \\ \implies & m_n = m_{n+1} + n && \text{for some } m_{n+1} \in M_{n+1} \text{ and } n \in N. \end{aligned}$$

Now, $M_n \supset M_{n+1} \implies n = m_n - m_{n+1} \in M_n$. Therefore,

$$\begin{aligned} &m_n - m_{n+1} \in M_n \cap N = M_{n+1} \cap N \\ \implies &m_n - m_{n+1} \in M_{n+1} \\ \implies &m_n \in M_{n+1} \end{aligned}$$

Therefore, $M_n \subset M_{n+1}$ as desired. Since any descending chain of submodules in M stablilizes, M is Artinian. □

Question 5

Proof. We consider the ring $F[G]$ as a **left**-$F[G]$ module. Then $F[G]$ is naturally a F-module, hence a F-vector space, by considering the restricted action of $F \cdot 1_G$, where 1_G is the identity element in the group G.

Now since G is finite, any element in $F[G]$ can be expressed by a n-tuple, where $n = |G|$, hence it is a n-dimensional vector space. Any proper inclusion of submodules (subspaces) of $F[G]$, say $M_1 \subset M_2$ must have an increase of dimension, and thus any ascending/descending chain of submodules must stabilize due to finite dimensions. In particular, any ascending/descending chain of **left** ideals in $F[G]$ (thus $F[G]$-submodules) must stabilize.

The proof is identical considering $F[G]$ as a **right** $F[G]$-module, therefore we will have that any ascending/descending chain of **right** ideals must stabilize as well. We can then conclude that any ascending/descending chain of (**two-sided**) ideals of $F[G]$ must also stabilize, thus $F[G]$ must be both Artinian and Noetherian as a ring. □