Week 9 Assignment

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1. *Proof.* I claim that $\overline{E}$ must be totally bounded. Proof follows.

   If E is closed, then $E = \overline{E}$ and we are done. Otherwise, given $\epsilon > 0$,

$$\exists x_1 \dots x_n \quad \text{such that} \quad \bigcup_{i=1}^{n} B[x_i, \epsilon/2] \supseteq A$$

   then a cluster point $x \in \overline{E} - E$ can be as close to E as possible. Specifically, with ϵ given as above,

$$B[x, \epsilon/2] \cap E \neq \emptyset$$

   Let y be a point in the intersection. Since $y \in E$, $\exists x_k$ such that

$$\rho(y, x_k) < \epsilon/2$$

   Therefore

$$\rho(x, x_k) \leq \rho(x, y) + \rho(y, x_k) < \epsilon/2 + \epsilon/2 = \epsilon$$

   which implies

$$x \in B[x_k, \epsilon] \subseteq \bigcup_{i=1}^{n} B[x_i, \epsilon] \supseteq A$$

   Therefore $\overline{E}$ is totally bounded. □

2. *Proof.* As per the hint, we show by contradiction. Assume that $\langle M, \rho \rangle$ has the Nested Set Property but *not* complete.

   Then there must exist a Cauchy sequence, $(x_n)_{n=1}^{\infty}$ that is not convergent.

   Since $(x_n)_{n=1}^{\infty}$ is Cauchy, given $\epsilon = \frac{1}{k}$, $k = 1, 2, \dots$ there exist $N_k \in \mathbb{N}$ such that $\forall p, q > N_k$,

$$\rho(x_p, x_q) < \frac{1}{k}$$

And we can choose N_k such that $N_1 \leq N_2 \leq \dots$. Let *below is added*

$$F_k = \{x_n : n \geq N_k\}$$

which implies we have the sets $F_1 \supseteq F_2 \supseteq \dots$ as desired, as we define F_k to be further "tails" of the sequence with decreasing diameter $\frac{1}{k}$. *added ends*

We claim the sequence of sets we generated, $(F_k)_{k=1}^{\infty}$ are all closed, since if the sets (containing just members of the sequence) have cluster points, it implies that the sequence will converge to the cluster points as per definition. So in lack of the cluster points, the sets are closed.

Then we again notice that the diameter of each set, $\text{diam}F_k = 1/k$ converges as per the nested set theorem:

$$lim_{n\to\infty}\text{diam}F_n = lim_{n\to\infty}\frac{1}{k} = 0$$

and by the Nested Set Theorem, we have $\bigcap_{n=1}^{\infty} F_n \neq \emptyset$. Let the point at the intersection be x, and we claim that it is the limit of this Cauchy sequence.

Let $\epsilon > 0$ be given. We can choose $k \in \mathbb{N}$ such that $k > \frac{1}{\epsilon}$. Then $\forall n > N_k \iff x_n \in F_k$,

$$\rho(x, x_n) \leq \text{diam}F_k = \frac{1}{k} < \epsilon$$

And thus it shows that $(x_n)_{n=1}^{\infty}$ is convergent (to x). A contradiction. $\square$

3. *Proof.* If there is such a $y \in M$ such that $f(y) = \inf\{f(z) : z \in M\}$, then we are done, since $f(y) \leq f(x)$ by definition. Otherwise, if there is no such y, let $w = \inf\{f(z) \in M\}$. Since no $y \in M$ attains w via f, we define $(x_n)_{n=1}^{\infty}$ such that

$$w < f(x_k) \leq w + \frac{1}{k}$$

for all $k = 1, 2, \dots$. We claim that $(x_n)_{n=1}^{\infty} \subseteq M$ is Cauchy. Since M is complete, it must also be convergent, to say $x \in M$. Furthermore, since f is continuous, $(f(x_n))_{n=1}^{\infty}$ will also converge to $f(x)$. Since we have

$$w < f(x_k) \leq w + \frac{1}{k}$$

Taking limits as k tends to ∞, we have $w = f(x)$, which is a contradiction. Therefore $y = x$ and $f(y) \leq f(x)$ for all $x \in M$ as desired.

Proof of claim that $(x_n)_{n=1}^{\infty}$ is Cauchy:

Let $\epsilon > 0$ be given. Choose $N \in \mathbb{N}$ such that $\frac{1}{N} > \epsilon$. Then for all $i, j > N$ (*I wrote i, n previously*),

$$w < f(x_i), f(x_j) \leq w + \frac{1}{N}$$

which from the inequality in the question, let $a = x_i$ and $b = x_j$, there is a point $x \in M$ so that *added missing bracket below*

$$\rho(x_i, x_j) + f(x) \leq \frac{1}{2}(f(x_i) + f(x_j)) \leq \max\{f(x_i), f(x_j)\}$$

Note that $\max\{f(x_i), f(x_j)\} < w + 1/N$ and $w < f(x)$ since w is the infimum, we have

$$\rho(x_i, x_j) < w + 1/N - w = 1/N < \epsilon$$

and therefore $(x_n)_{n=1}^{\infty}$ is Cauchy. $\square$