MA3110 chap 6

- (seq. limits) $\lim_{x\to a} f(x) = L \iff$ if any seq (x_n) where $x_n \neq a \; \forall n$, and $\lim_{n\to\infty} f(x_n) = L$

- (div. criterion) $\lim_{x\to a} f(x)$ does not exist iff. (given $x_n \neq a$, $y_n \neq a$)
  1. $x_n \to a$ but $\lim f(x_n)$ diverges. | 2. $x_n \to a$, $y_n \to a$, but $\lim f(x_n) \neq f(y_n)$
  3. $x_n \to a$ but $\lim f(x_n) \neq L$.
- differentiable 1. a point: $\lim_{x\to a} \frac{f(x)-f(a)}{x-a}$ exists on an open intv cont. a.
  2. open intv: on every pt.
  3. closen intv: — " — + left/right at endpts.
- limit exists $\neq$ continuous.
- ~~cont~~ ctly diff^{able}: f' is cont. - diff $\Rightarrow$ cts. (at a pt, therefore open etc)
- caratheodory: $f'(c)$ exists $\iff \exists \varphi: I \to \mathbb{R}$, φ cts at c and $f(x)-f(c) = \varphi(x)(x-c)$ $\forall x \in I$
  $f: I \to \mathbb{R}$, $c \in I$.
- $\boxed{\text{chain rule}}$: g, f, range of $f \subseteq$ dom of g, f diff. at c, g diff. at $f(c)$,
  $\Rightarrow gf$ diff. at $f(c)$, $(gf)'(a) = g'(f(a)) \cdot f'(a)$.
- $\boxed{\text{Inv fn thm}}$: f is strictly monotone & cts on I. $g = f^{-1}$. f diffable at c, $f'(c) \neq 0 \Rightarrow g$ diff^{able} on $f(c)$, $g'(f(c)) = \frac{1}{f'(c)}$.
- Lemma: let $f: (a,b) \to \mathbb{R}$, $f'(c)$ exists for some $c \in (a,b)$.
  then if $f'(c) > 0 \Rightarrow \exists \delta$. st. $\forall x \in (c-\delta, c)$, $f(x) < f(c)$.
  $\forall x \in (c, c+\delta)$, $f(c) < f(x)$.

  if $f'(c) < 0 \Rightarrow \exists \delta$. st. $\forall x \in (c-\delta, c)$, $f(x) > f(c)$
  $\forall x \in (c, c+\delta)$, $f(c) > f(x)$.

  $\therefore$ if $f'(c) > 0$, $\exists$ a neighbourhood, LHS < $f(c)$, RHS > $f(c)$.
  compare $\lim_{x\to c} f(x) > 0$, then a neighbourhood $f(x) > 0$.
- Interior ext: c interior of I, $f: I \to \mathbb{R}$ diff. at c, if rel. ext at c, then $f'(c) = 0$.
  converse is false, and if non-diff, undetermined.
- let $f: [a,b] \to \mathbb{R}$ cont, and diff on (a,b). then
  Rolles: $f(a) = f(b) \Rightarrow \exists c \in (a,b)$, $f'(c) = 0$.
  MVT: $\exists c \in (a,b)$ $f'(c) = \frac{f(b)-f(a)}{b-a}$
  Cauchy MVT: $g: [a,b] \to \mathbb{R}$, g diff on (a,b), $g'(x) \neq 0$, then $\exists c \in (a,b)$ $\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$

MA3110 Midterm Cheatsheet

Divergent criteria: $\lim_{x\to a} f(x) \not\to L$ if $x_n \to a$ but $f(x_n) \not\to a$.
or $f(x_n)$ diverges.

Diffable: f differentiable at a if ① f defined on ~~an open~~ $(a-\delta, a+\delta)$.
② ~~∄~~ $f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$ exists.
at an intv (a,b) if diff at x $\forall x \in (a,b)$.

Caratheodory Thm ($\Leftrightarrow$ diff) $f: I \to \mathbb{R}$, let $c \in I$. $f'(c)$ exists.
$\Leftrightarrow \exists \varphi: I \to \mathbb{R}$ s.t. $\varphi: I \to \mathbb{R}$ ~~cont~~ cts at c,
$f(x) - f(c) = \varphi(c)(x-c)$.

Chain Rule $f: J \to \mathbb{R}$, $g: I \to \mathbb{R}$, $f(J) \subseteq I$. $J \xrightarrow{f} I \xrightarrow{g} \mathbb{R}$.
Let. $a \in J$. If f diff at a, g diff at $f(a)$, then ~~&~~ $g \circ f$ diff at a,
and $(g\circ f)'(a) = g'(f(a)) f'(a)$.

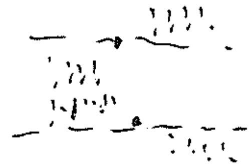
Continuous Inverse: suppose $f: I \to \mathbb{R}$ is monotone & cts, then $f^{-1}: J = f(I) \to \mathbb{R}$
is also cts and monotone. (f cts $\Rightarrow f^{-1}$ cts provided exists)

Inverse Fn Thm: ~~[illegible]~~ (f diff $\to f^{-1}$ diff provided exists). f diff at c, and $f'(c) \neq 0$,
then f^{-1} is diff at $f(c)$! $f^{-1\,\prime}(f(c)) = \frac{1}{f'(c)}$. (simply by chain rule)
~~[illegible]~~ (extra than CR).

Abs max/min: $f: I \to \mathbb{R}$ $x_0 \in I$ is abs max/min if $f(x_0) \geq (\leq)\, f(x)$ $\forall x \in I$.
Rel max/min: exists an open intv around x_0 s.t.

Continuous locality: if f cts at a, $f(a) \neq 0 \Rightarrow \exists \delta$, $(a-\delta, a+\delta) \ni x$, $f(x) \neq 0$.
an intv where it is nonzero.
$>/< 0 \Rightarrow$ $>/<$ zero.

Derivative locality at a pt: if $f'(a) > 0$, then $\exists$ intv LHS $< f(a) <$ RHS.
(insufficient for $>/<$). <0 RHS $< f(a) <$ LHS



Recall Fermat's Thm: If Abs ext is diff then deriv $= 0$. [false] (critical pts ($f'(x)=0$ or undefined) & end pts are where abs ext occur.)

Rolle's: ~~[illegible]~~ $f: [a,b] \to \mathbb{R}$, ~~d~~ cts, $f(a) = f(b)$ ~~[illegible]~~ $\Rightarrow \exists c \in (a,b)$, $f'(c) = 0$.

MVT: $f: [a,b] \to \mathbb{R}$, $\exists c \in (a,b)$, $f'(c) = \frac{f(b)-f(a)}{b-a}$

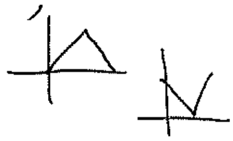
Cauchy MVT: $f: [a,b] \to \mathbb{R}$, $g: [a,b] \to \mathbb{R}$, $\exists c \in (a,b)$, $\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$

MA3110 Midterm Cheatsheet (2)

≥ 0 Positive deriv at an intv $\Leftrightarrow$ increasing on that intv. ☆ $\boxed{\text{equivalent}}$.
≤ 0 negative —— " —— $\Leftrightarrow$ decreasing —— " —— .

Pos/neg deriv. at a pt $\Rightarrow$ LHS ($>$/$<$) RHS for an open ball ~~neighborhood~~.

First deriv test : $f: [a,b] \to \mathbb{R}$ is cts, diff at (a,b) (possibly $\setminus \{c\}$),
then LHS $f' \leq 0$, RHS $f' \geq 0 \Rightarrow c$ is a rel max.
≥ 0 $\leq 0 \Rightarrow$ min.

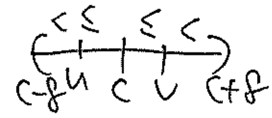


Second deriv test $f: I \to \mathbb{R}$, diff on I, $f''(c)$ exists.
$f'(c) = 0 \wedge f''(c) > / < 0 \Rightarrow$ rel min / rel max.
(strict!)

Taylor's Thm : $f: [a,b] \to \mathbb{R}$, $f \in C^n([a,b])$, $f^{(n+1)}$ exists on (a,b).
Then $\forall x_0 \in [a,b]$, $\forall x \in [a,b]$, $\exists c \in (x, x_0)$,
$f(x) = f(x_0) + f'(x_0)/1\,(x-x_0) + \dots + f^{(n+1)}(x)/(n+1)!\,(x-x_0)^{n+1}\ (=R_n)$.
Rolle's $\to$ MVT $\to$ Taylor in generality $\uparrow$.

== end of diffn lec ==

Straddle Lemma : $f: I \to \mathbb{R}$ diff at $c \in I$. Then $\forall \varepsilon > 0, \exists \delta > 0$ s.t.



$\Rightarrow |(f(v) - f(u)) - (v-u)f'(c)| \leq \varepsilon(v-u)$.

IVT of Derivatives : f diff on $[a,b]$, $f'(a) < f'(b)$. f' is not necessarily cts but
$\forall k$, $f'(a) < k < f'(b)$, $\exists c \in (a,b)$ s.t. $f'(c) = k$.

Lipschitz condition ($\Rightarrow$ uniform cts) $|f(x) - f(y)| \leq K|x-y|$

Bdd derivatives are uniformly continuous. ☆

Carathedory's fn recall $f'(c)$ exists $\Leftrightarrow \exists \varphi: I \to \mathbb{R}$ s.t. $f(x) - f(c) = \varphi(c)(x-c)$.
Then $\varphi(x) = \begin{cases} \frac{f(x) - f(a)}{x-a} & x \neq a \\ & x = a \end{cases}$.

L'Hopital's Rule if ~~$\lim f(x)/g(x)$~~ both $\lim f(x) = \lim g(x) = \infty$ or 0, then
$\lim \frac{f'}{g'} = \lim \frac{f}{g}$ ~~if~~ the former exists.
regardless

Qn: what if we diff at a point until it is not 0? can we deduce whether it is ~~a / at that p~~
a relative max/min?

Ans : if $f^{(1)}(x_0) = f^{(2)}(x_0) = \dots = f^{(n-1)}(x_0) = 0$, $f^{(n)}(x_0) \neq 0$, then only if n is even
$f^{(n)}(x_0) > 0 \Rightarrow$ rel min
$< 0 \Rightarrow$ max.

MA3110 Midterm Cheatsheet (3)

upper sum (wrt fn f, part P) $= \sum$ sup of each subintv $\times \Delta x_i = \sum M_i \Delta x_i$

lower sum (wrt f, P) $= \sum$ inf —— " —— $= \sum m_i \Delta x_i$.

lower/upper integral: sup/inf of lower/upper sum over all partitions.

integrable if lower int = upper int $\iff U(f,P) - L(f,P) < \varepsilon \ \forall \varepsilon$. (Riemann Integrability Criteria).

why must (Riemann-) integrable fns be bdd? Because defined as inf/sup, and

$m(a-b) \le L(f,P) \le U(f,P) \le M(a-b)$

inf across $[a,b]$ sup across int.

$\underline{\int_a^b} f \le \overline{\int_a^b} f$.

Refining partitions improve estimates:

$L(f,P) \le L(f,P\cup Q) \le U(f,P\cup Q) \le U(f,P)$.

cts $\vee$ monotone $\Downarrow$ integrable $\Downarrow$ bdd.

Properties of $\int_a^b f$ the R-integral: assuming integrability,

1. $f \ge 0$, then $\int_a^b f \ge 0$.
2. (integral of const mult, sums of fns) are equal to the const mult and sums of integrals. (Linearity), is a linear trans.
3. $f \le g \Rightarrow \int_a^b f \le \int_a^b g$.
4. f itgb $\Rightarrow |f|$ itgb, converse is false. $\left|\int_a^b f\right| \le \int_a^b |f|$.
5. fg is itgb if both itgb.

Lemma: $L(f,P) + L(g,P) \le L(f+g,P) \le U(f+g,P) \le U(f,P) + U(g,P)$.

You can combine and split intvs and preserve itgbility

f itgb on $[a,b]$, $[b,c] \iff$ f itgb on $[a,c]$. $\forall b \in (a,c)$,

Length of curve: If f' is cts, then length of curve $l(f) = \int_a^b \sqrt{1+(f'(x))^2}\,dx$.

R-integral is u-cts: Let $f:[a,b]\to\mathbb{R}$ be itgb on $[a,b]$. Then $\int_a^x f = F(x)$ is uniformly cts (in fact, lipschitz cts). $|F(x)-F(y)| \le M|x-y| \ \forall x,y \in [a,b]$.

FTC I: ① f is itgb on $[a,b]$. ② f is cts at $c \in [a,b]$. $F(x) = \int_a^x f$ then $F'(c) = \frac{d}{dx}\int_a^x f \Big|_{x=c} = f(c)$.

Remark ① if f is cts at $[a,b]$, then $F'(x) = f(x) \ \forall x \in [a,b]$.
② g diffable on $[a,b]$, then $G(x) = \int_0^{g(x)} f \Rightarrow G'(x) = F'(g(x))g'(x) = f(g(x))g'($

MA3110 Finals Cheatsheet (5).

FTC II (Cauchy) ① g is diff'able on $[a,b]$. ② g' is integrable on $[a,b]$. Then
$$\int_a^x g' = \int_a^x \frac{d}{dt}(g(t))\,dt = g(x) - g(a).$$

Integration by parts: functions $u, v : [a,b] \to \mathbb{R}$ diff'able, u', v' integrable on $[a,b]$, then
$$\int_a^b uv' = u(b)v(b) - u(a)v(a) - \int_a^b vu'.$$

Substitution rule ① $\phi : [a,b] \to \mathbb{R}$, ϕ' exists & itgb on $[a,b]$, $[a,b] \xrightarrow{\phi} \phi([a,b]) \subseteq I \xrightarrow{f} \mathbb{R}$.
② $f : I \to \mathbb{R}$ cts on I, $\phi([a,b]) \subseteq I$, then
$$\int_a^b f(\phi(t))\,\phi'(t)\,dt = \int_{\phi(a)}^{\phi(b)} f(x)\,dx.$$

Taylor's Thm with integral R_n. Let f be $(n+1)$-times diff'able ($f, \dots, f^{(n+1)}$ exist) on $[a, x]$, additionally $f^{(n+1)}$ is itgb on $[a,x]$. Then
$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$
$$= \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + \frac{1}{n!}\int_a^x f^{(n+1)}(t)(x-t)^n\,dt.$$
$= R_n(x)$.

Small $\|P\|$ forces upper/lower sum to approach upper/lower integral $f : [a,b] \to \mathbb{R}$ bdd. (s.t. $U(f,P)$, $L(f,P)$ exist).
$\forall \varepsilon > 0, \exists \delta > 0$ s.t. $\|P\| < \delta \Rightarrow$ $U(f,P) < \overline{\int_a^b} f + \varepsilon$, $L(f,P) > \underline{\int_a^b} f + \varepsilon$.

Riemann-sum $f : [a,b] \to \mathbb{R}$ bdd. $P = \{x_0 = a, \dots, x_n = b\}$ a partition of $[a,b]$,
$\xi_i \in [x_{i-1}, x_i]$, $\xi = (\xi_i)_{i=1}^n$
$$\therefore S(f,P)(\xi) = \sum_{i=0}^{n} f(\xi_i)\Delta x_i.$$

Integrability: $f : [a,b] \to \mathbb{R}$ bdd. f is integrable on $[a,b]$ & $\int_a^b f = A$
$$\iff \lim_{\|P\| \to 0} S(f,P)(\xi) = A \iff \overline{\int_a^b} f - \underline{\int_a^b} f < \varepsilon \quad \forall \varepsilon > 0.$$

Sequence of sums: Given a sequence of partitions $P_n = \{x_0^{(n)}, \dots, x_{m_n}^{(n)}\}$ with $\xi^{(n)} = (\xi_1^{(n)}, \dots, \xi_{m_n}^{(n)})$
and $\|P_n\| \to 0$. Then the sequence of R-sums $y_n := S(f, P_n)(\xi^{(n)})$ must tend to the integral $\int_a^b f$.

Improper Integral when ① f is unbounded or ② the interval is unbounded.
then $\int_a^b f = L$ if 1. $\lim_{x \to b^-} \int_a^x f$ exists.
2. $\lim_{x \to a^+} \int_x^b f$ exists.
otherwise diverges [f is defined on $[a,b)$ or $(a,b]$, & integrable for $[a,x]$ or $[x,b]$]

$\int_a^\infty f = L$ if $\lim_{x \to \infty} \int_a^x f$ exists $= L$
$\int_{-\infty}^a f = L$ if $\lim_{x \to -\infty} \int_x^b f$ exists $= L$.
$\int_{-\infty}^\infty f$ converges if both above ✓ converges, $= \int_{-\infty}^a f + \int_a^\infty f$.

MA3110 Finals Cheatsheet (2):

Continuity of derivatives: Let $f:[a,b]\to\mathbb{R}$ be diff'able on $[a,b]$. Then $\lim_{x\to c} f'(x)$ exists, we automatically have $f'(c)=\lim_{x\to c} f'(x)$. [impossible $\lim_{x\to c} f'(x) = L \neq f'(c)$].

Differ by finite # of pts: If $f(x)=g(x)$ on an intv except for a finite # of pts, then $\int_a^b f = \int_a^b g$ if f,g are intg.

Reciprocal is integrable if itgb fn is lower bdd: if itgb $h:[a,b]\to\mathbb{R}$, and $h(x)\geq c$ $\forall x\in[a,b]$ then $\frac{1}{h}$ is itgb on $[a,b]$.

Length of a curve: Case 1: $f\in C'[a,b]$. thus $\int_a^b \sqrt{1+(f'(x))^2}\,dx = l(f)$.

Case 2: Take any partition P of $[a,b]$. Then

$$l(f,P)=\sum_{i=1}^{n}\|\hat{u}_i-\hat{u}_{i-1}\| \text{ where } u_i=(x_i,f(x_i)) \text{ and } x_i\in P.$$
$$=\sum_{i=1}^{n} \text{sqrt}\left((x_i-x_{i-1})^2+(f(x_i)-f(x_{i-1}))^2\right)$$

then $\|P\|\to 0$, $l(f,P)=l(f) = \sup\{l(f,P) \mid P\}$.

Convergence of Improper integrals: $f:[a,\infty)\to\mathbb{R}$.

① $\int_a^\infty f(x)dx$ conv. $\iff \forall\varepsilon>0, \exists M>0$ s.t. $\left|\int_a^b f\right|<\varepsilon \quad \forall b>a>M$.

② if $f(x)\geq 0$, $\int_a^\infty f$ conv $\iff \int_a^x f\leq M \quad \forall x\in[a,\infty)$ (partial integral is bdd inf.).

③ if $0\leq f(x)\leq g(x)$, $\int_a^\infty g$ conv. $\Rightarrow \int_a^\infty g\leq M \Rightarrow \int_a^x f\leq\int_a^x g\leq M$
$\Rightarrow \int_a^\infty f$ is conv..

(1,2,3) can be extended to $f:(-\infty,b]\to\mathbb{R}$,

④ $\int_{-\infty}^{\infty} f$ conv $\Rightarrow \lim_{c\to\infty}\int_{-c}^{c} f$ exists, $=\int_{-\infty}^{\infty} f$ but converse is false.

Integral Test: a_n is +ve decreasing, $f(n)=a_n$, then $\sum_{n=1}^{\infty} a_n$ conv $=L \iff \int_1^\infty f$ conv. $=L$.

$\Rightarrow$ p-srs converges if $\frac{1}{n^p}$, $p>1$. If $p\leq 1$, diverges.

MVT for integrals: $f:[a,b]\to\mathbb{R}$ is cts, then $\exists c\in(a,b)$, $\int_a^b f = f(c)(a-b)$.

Some thm: f,g cts on $[a,b]$ then $\exists c\in(a,b)$, $g(c)\int_a^x f = f(c)\int_x^b g\,dx$.

Improper integral conv. $\int_a^b f(x)$ $\iff$ $\exists g(x)$, $f(x)\leq g(x)$ $\forall x$, $\int_a^b g(x)$ converges.

MA3110 Finals Cheatsheet (3)

Seq. of fns: (f_n) defined on E $\forall n \in \mathbb{N}$. $\Rightarrow$ (f_n) is a seq of fns on E.

Ptwise conv: $f_n \xrightarrow{\text{ptwise}} f$ on E if $\forall x \in E$, $f_n(x) \to f(x)$ as a sequence in $\mathbb{R}$.
(i.e. $\forall \varepsilon > 0$, $\exists K \in \mathbb{N}$, $n > K \Rightarrow |f_n(x) - f(x)| < \varepsilon$)

Unif conv: $f_n \xrightarrow{u} f$ on E if $\forall \varepsilon > 0, \exists N \in \mathbb{N}$, $\forall n \geq N$, $|f_n(x) - f(x)| < \varepsilon$ $\forall x \in E$
or $\Leftrightarrow \|f_n - f\|_E < \varepsilon$
$= \sup\{|f_n(x) - f(x)| \mid x \in E\}$

	ptwise conv. $f_n \to f$	unif conv. $f_n \xrightarrow{u} f$
f_n are cts on E	×	f is cts on E
f_n are itgb on E.	×	f is itgb on E, $\lim_{n\to\infty} \int_E f_n = \int_E \lim_{n\to\infty} f_n = \int_E f$
$f_n(x_0) \Rightarrow f(x_0)$, $f_n' \xrightarrow{u} g$	×	~~f~~ Automatic + $f_n \xrightarrow{u} f$ and $g = f'$.

non-unif-conv. test for seq fns
① $\|f_n - f\| \not\to 0$.
② $\exists (f_{n_k}), (x_k)$, $|f_{n_k}(x_k) - f(x_k)| \geq \varepsilon_0$.
③ f_n cts/itgb but f is not.

Srs. of fns $\sum_{n=1}^{\infty} f_n = \lim_{n\to\infty} S_n(x)$, $S_n = \sum_{k=1}^{n} f_k$. $S_n \to S$ ptwise $\Rightarrow \sum_{n=1}^{\infty} f_n$ conv. ptwise
$S_n \xrightarrow{u} S$ $\Rightarrow \sum_{n=1}^{\infty} f_n$ conv. unif.

Conv. criteria 1. $S_n(x)$ fulfills Cauchy criterion
$\Rightarrow \forall \varepsilon > 0, \exists N$ s.t. $\|S_m - S_n\| < \varepsilon$
$\Rightarrow \|f_{m+1} + \dots + f_n\| < \varepsilon$.

2. Weierstrass M-test $\|f_n\|_E \leq M_n$ $\forall n \in \mathbb{N}$, $\sum_{n=1}^{\infty} M_n$ converges
$\Rightarrow \sum_{n=1}^{\infty} f_n$ converges on E.

Srs. of fns are seq. of fns If $\sum f_n$ converges unif, then ① f_n cts $\to \sum f_n = f$ is cts
② f_n itgb $\Rightarrow f$ is itgb, $\int_a^b \sum_{n=1}^{\infty} f_n = \sum_{n=1}^{\infty} \int_a^b f_n = \int_a^b f$.
③ $\sum_{n=1}^{\infty} f_n(x_0)$ converges, $\sum_{n=1}^{\infty} f_n' \xrightarrow{u} g$, $\Rightarrow \sum f_n$ conv. u. to f, $f' = g$.

Weierstrass fn: $\exists f: \mathbb{R} \to \mathbb{R}$ that is cts everywhere but diff'able nowhere.

Unif. conv. preserve bdd and implies unif. bdd $|f_n(x)| \leq M_n$ $\forall x \in E$ $\wedge$ $f_n \xrightarrow{u} f$ $\Rightarrow$ f is bdd on E, and $|f_n(x)| \leq M$ $\forall x \in E, \forall n \in \mathbb{N}$.

Some sufficient conds. for unif. conv. ① $g_n \xrightarrow{u} g$, h is cts on $[a,b]$, then $q_n = h g_n \xrightarrow{u} q = hg$.

② Known that if $|g_n'(x)| \leq M$ $\forall x \in E$, then it is Lipschitz continuous hence unif. cts.
Any (g_n) unif cts on $[a,b]$ that converge ptwise to g, must $g_n \xrightarrow{u} g$ on $[a,b]$.
(unif. cts + ptwise conv. $\Rightarrow$ unif. conv.)

MA3110 Finals cheatsheet (4)

Unif conv. on A and B $\Rightarrow$ unif. conv on $A \cup B$.

f_n, g_n unif. conv. $\Rightarrow$ $f_n + g_n$ unif. conv. to $f+g$.

$\Rightarrow$ ($f_n g_n$ conv. unif. to fg.) if ($(f_n), (g_n)$ are seq. of bdd fns. that $f_n \xrightarrow{u} f$, $g_n \xrightarrow{u} f$.)

Criterion for $\sum f_n \overset{u}{\not\to} F$. ① ~~[illegible]~~ $\|f_n\| \not\to 0$.
② if F given, check cts/integ. of F and f_n.

Dirichlet's Test ① Partial sum of f_n is unif. bdd: $\left|\sum_{k=1}^{n} f_k(x)\right| \leq M$ $\forall n \in \mathbb{N}, \forall x \in E$.

② $g_n \xrightarrow{u} 0$ on E ③ $(g_n(x))$ decreasing for every fixed $x \in E$.

Then $\sum_{n=1}^{\infty} f_n g_n$ (composition) is unif. conv. on E.

Power srs always converge with center x_0. If $\sum_{n=0}^{\infty} a_n (x-x_0)^n$ conv, then.

conv. at $x = x_1$ $\Rightarrow$ conv at $|x-x_0| < |x_1 - x_0|$ $\forall x$.

conv. at $x = x_2$ $\Rightarrow$ div at $|x-x_0| > |x_2 - x_0|$ $\forall x$.

Defn: Radius of convergence R st. $\forall x \in (x_0 - R, x_0 + R)$, $\sum_{n=0}^{\infty} a_n (x-x_0)^n$ conv.

$R = \frac{1}{\rho}$, $\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ or $\limsup |a_{n+1}|^{\frac{1}{n}}$.

$\forall x \in (x_0 - R, x_0 + R) \Rightarrow \sum_{n=0}^{\infty} a_n (x-x_0)^n$ conv. abs.

Also conv ~~abs~~ uniformly on the set $E \supseteq (x_0 - R, x_0 + R)$.

Formula for kth term: $a_k = \frac{f^{(k)}(x_0)}{k!}$ $\forall k = 0, 1, \ldots$ $\Rightarrow$ $f^{(k)}(x_0) = k! a_k$.

Abel's Formula $(b_n), (c_n)$ real sequences. Define $B_{n,m} = \sum_{k=m}^{n} b_k$.

Then $\sum_{k=m}^{n} b_k c_k = B_{n,m} c_n - \sum_{k=m}^{n-1} B_{k,m} (c_{k+1} - c_k)$

Abel's Thm If $\sum a_n (x-x_0)^n$ conv. on $x_0 + R$, $\Rightarrow$ unif. cts on $[x_0, x_0+R]$
($\Rightarrow$ unif. cts on $[x_0 - R, x_0 + R]$).

$x_0 - R \Rightarrow$ —— " —— $[x_0 - R, x_0]$.

Taylor's srs is to equal fn $\Leftarrow \lim_{n\to\infty} R_n(x) = \lim_{n\to\infty} \frac{f^{(n+1)}(c_n)}{(n+1)!} (x-x_0)^{n+1} = 0$.

Defn: Analytic fn if ① inf. differentiable on $[a,b]$, ② Taylor srs converges on a neighborhood

Merten's Thm $\sum a_n$ conv. abs, $\sum b_n$ conv. (abs. resp.), then Cauchy pdt conv. (abs. resp.)

$\sum c_n = (\sum a_n)(\sum b_n)$, $c_n = \sum_{k=0}^{n} a_k b_{n-k}$.

~~*~~ <u>const mult, sum & pdt of power srs</u> Let $f(x)=\sum_{n=0}^{\infty} a_n(x-x_0)^n$, $|x-x_0|<R_1$,
$g(x)=\sum_{n=0}^{\infty} b_n(x-x_0)^n$, $|x-x_0|<R_2$.

① Lincomb:
$\alpha f(x)+\beta g(x)=\sum_{n=0}^{\infty}(\alpha a_n+\beta b_n)(x-x_0)^n \quad \forall\ |x-x_0|<\min\{R_1,R_2\}$.

② Pdt: $f(x)g(x)=\sum_{n=0}^{\infty}c_n(x-x_0)^n \ \forall\ |x-x_0|<\min\{R_1,R_2\}$, where $c_n=\sum_{k=0}^{n}a_kb_{n-k}$.

<u>Dirichlet's Test</u> : $f_n, g_n: E\to\mathbb{R}$. If

① $\left|\sum_{k=1}^{n} f_k(x)\right|\le M\ \forall n\in\mathbb{N},\ \forall x\in E$ uniformly bdd,

② $g_n\to 0$ uniformly on E, ③ $\forall x\in E$, $(g_n(x))$ is decreasing.

$\Longrightarrow \sum_{n=1}^{\infty} f_ng_n$ (composition) converges unif on E.

<u>Reciprocal ~~Inverse~~ of power srs</u> Let $f(x)=\sum_{n=0}^{\infty}a_nx^n$, $a_0=1$, $R_1>0$.

Then ~~f(x)~~ $\frac{1}{f(x)}=g(x)=\sum_{n=0}^{\infty}b_nx^n$, where $b_0=1$, $b_n=-\sum_{k=1}^{n}a_kb_{n-k}$. $\forall n\ge 1$.

<u>Binomial srs</u> $\forall \alpha\in\mathbb{R}$, $(1+x)^\alpha=\sum_{n=0}^{\infty}\binom{\alpha}{n}x^n \ \forall x\in(-1,1)$ where

$$\binom{\alpha}{n}=\begin{cases}1 & \text{if } n=0\\ \frac{\alpha(\alpha-1)\dots(\alpha-n+1)}{n(n-1)\dots(1)} & \text{otherwise}\end{cases}.$$

<u>Some sums</u> $\sum_{n=1}^{\infty}nx^{n-1}=\frac{1}{(1-x)^2}$ Chap 9 P9.

$\sum_{n=2}^{\infty}n(n-1)x^{n-2}=\frac{2}{(1-x)^3}$. $\quad\forall x\in(-1,1)$.