Q1. (a) (i) Let $f(x) = \cancel{x}\sin(\sqrt{x})$, $F(x) = \int_0^x \cancel{t}\sin(\sqrt{t})\,dt$,

Then $G(x) = F(x^2)$.

~~Explain~~ By FTC I, F is differentiable on $[0,\infty)$, thus by chain rule, since $x \mapsto x^2$ is differentiable on $[0,\infty)$,

$G'(x) = 2xF'(x^2)$ and FTC I gives $F'(c) = f(c)\ \forall c \in [0,\infty)$,

$= 2xf(x^2) = 2x^2\sin(x)$. $\forall x \in [0,\infty)$. □.

(ii) Since both numerator & denominator $\to 0$, and both are differentiable, by L'Hopital's rule,

$$\lim_{x\to 0^+} \frac{\int_0^{x^2} \cancel{t}\sin(\sqrt{t})\,dt}{x^5} = \lim_{x\to 0^+} \frac{2\cancel{x^2}\sin(x)}{5x^{\cancel{4}\,2}}$$

$$= \frac{2}{5}\lim_{x\to 0^+} \frac{\cos(x)}{2x}$$ by L'Hopital's rule.

$$= \infty.$$ □

(b) $$\lim_{n\to\infty} n^2 \sum_{k=1}^{n} \frac{k}{(n^2+k^2)^2} = \lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{1}{n}\right)\left(\frac{k}{n}\right)\frac{n^4}{(n^2+k^2)^2}$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{1}{n}\right)\left(\frac{k}{n}\right)\frac{1}{(1+(\frac{k}{n})^2)^2}$$

Consider $f(x) = \frac{x}{(1+x^2)^2}$, $x \in [0,1]$, $P = \{0, \frac{1}{n}, \dots, \frac{n}{n}\}$,

$\xi^{(k)} = k/n$, $= \lim_{n\to\infty} S(f,P)(\xi)$

since $n\to\infty \Rightarrow \frac{1}{n} = \|P\| \to 0$, $= \lim_{\|P\|\to 0} S(f,P)(\xi)$

$$= \int_0^1 \frac{x}{(1+x^2)}\,dx.$$

$$= -\frac{1}{2}\left[\frac{1}{1+x^2}\right]_0^1 = -\frac{1}{2}\left(\frac{1}{2}-1\right) = \frac{1}{4}$$ □

Q2. (i) $\sum_{k=1}^{n} g(\frac{k}{n}) - n\int_0^1 g(x)\,dx$

$= n\left(\sum_{k=1}^{n}(\frac{1}{n})g(\frac{k}{n}) - \sum_{k=1}^{n}\int_{\frac{k-1}{n}}^{\frac{k}{n}} g(x)\,dx\right)$

$= n\sum_{k=1}^{n}\int_{\frac{k-1}{n}}^{k/n} g(\frac{k}{n}) - g(x)\,dx.$

Given $\frac{k-1}{n} \le x \le \frac{k}{n}$, since g is differentiable on $[0,1] \supseteq [\frac{k}{n}, \frac{k+1}{n}]$ for any $k = 0, \dots, n-1$, by mean value theorem,

$\frac{g(\frac{k}{n}) - g(x)}{\frac{k}{n} - x} = g'(x_k)$ for some $x_k \in (x, \frac{k}{n})$.

$= n\sum_{k=1}^{n}\int_{\frac{k-1}{n}}^{k/n} g'(x_k)(\frac{k}{n} - x)\,dx.$ Since $g'(x_k) \le M_k$, $\forall k = 1, \dots, n$,

$= n\sum_{k=1}^{n} M_k \int_{\frac{k-1}{n}}^{k/n} \frac{k}{n} - x\,dx$

$= n\sum_{k=1}^{n} M_k \left[\frac{kx}{n} - \frac{x^2}{2}\right]_{\frac{k-1}{n}}^{\frac{k}{n}}.$

$= n\sum_{k=1}^{n} M_k \left(\frac{k^2}{n^2} - \frac{1}{2}(\frac{k^2}{n^2}) - \frac{k(k-1)}{n^2} + \frac{(k-1)^2}{2n^2}\right)$

$= n\sum_{k=1}^{n} \frac{M_k}{2n^2}(\cancel{2k^2 - k^2 - 2k^2 + 2k + k^2 - 2k} + 1)$

$= \frac{1}{2n}\sum_{k=1}^{n} M_k.$ $\square$.

(ii) Consider $\lim_{n\to\infty}\left\{\sum_{k=1}^{n} g(\frac{k}{n}) - n\int_0^1 g(x)\,dx\right\} =$ ~~...~~.

$= \lim_{n\to\infty} (\frac{1}{2})\sum_{k=1}^{n} M_k(\frac{1}{n})$

$= \frac{1}{2}\lim_{\|P\|\to 0} U(g', P = \{0, \frac{1}{n}, \dots, \frac{n}{n}\})$

$= \frac{1}{2}\int_0^1 g' = \frac{g(1) - g(0)}{2}$ by FTC II. $\square$.

Q3. (i) Fix x. Then $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{n(e^x)^2}{1+n(e^x)}$

$$= \lim_{n\to\infty} \frac{(e^x)^2}{\frac{1}{n}+e^x} = e^x.$$

$\therefore f(x) = e^x$, $x \in \mathbb{R}$. $\square$

(ii) $|f_n(x) - f(x)| = \left|\frac{ne^{2x}}{1+ne^x} - e^x\right| = \frac{|ne^{2x} - e^x - ne^{2x}|}{1+ne^{x}}$

$$= \frac{e^x}{1+ne^{x}} \text{ since } e^x > 0 \ \forall x \in \mathbb{R},$$

$$= \frac{1}{e^{-x}+n}.$$

$\therefore \|f_n - f\|_{\mathbb{R}} = \frac{1}{n}$ $\therefore \lim_{n\to\infty} \|f_n - f\|_{\mathbb{R}} = 0 \Rightarrow f_n \xrightarrow{u} f$ on $\mathbb{R}$. $\square$.

(iii) $\lim_{n\to\infty} \int_0^1 \frac{ne^{2x}}{1+ne^x} dx = \int_0^1 \lim_{n\to\infty} \frac{ne^{2x}}{1+ne^x} dx$ by uniform convergence.

$$= \int_0^1 e^x dx.$$

$$= e - 1$$

$\square$.

Q4. Idea: We claim that the delta δ from the uniform continuity of f on $[a,b]$ works using the 3-ε argument for sufficiently large N. For the rest (finitely many) f_n, we just use the unif cts.

Let $\varepsilon > 0$ be given. Then

① $\exists\, \delta_0$ s.t. $|x-y| < \delta_0 \Rightarrow |f(x)-f(y)| < \varepsilon/3$. ($f$ is unif. cts $[a,b]$).

② $\exists\, N_1$ s.t. $\forall n \geq N_1, \Rightarrow |f_n(x)-f(x)| < \varepsilon/3 \quad \forall x \in [a,b]$

($f_n \xrightarrow{u} f$ on $[a,b]$)

~~② ∃ N₁ s.t. ∀ n ≥ N₁, ⇒ |f_n|~~.

③ $\exists\, \delta_1, \delta_2, \dots, \delta_{N_1-1}$ s.t. $\forall i = 1, \dots, N_1 - 1$

$|x-y| < \delta_i \Rightarrow |f_i(x) - f_i(y)| < \varepsilon/3$. ($f_i$ are all unif. continuous on $[a,b]$.

ok for $\geq N$

Note that we choose $\delta = \max\{\delta_0, \dots, \delta_{N_1-1}\}$, then

Case 1: $n \geq N_1$ $|f_n(x) - f_n(y)|$ if $n \geq N_1$

$\leq |f_n(x) - f(x)| + |f(x) - f(y)| + |f(y) - f_n(y)|$

$< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon$.

Case 2: $n = 1, \dots, N_1 - 1$, then by ③, δ is large enough for each f_n's uniform continuity.

$\therefore$ The result follows.

$\square$

Q5. (i) Let n be given. Then

$$f_n'(x) = \frac{d}{dx} \ln\left(1+\frac{x}{n(n+1)}\right)$$

$$= \frac{\frac{1}{n(n+1)}}{1+\frac{x}{n(n+1)}} = \frac{1}{n(n+1)+x} \quad \forall x \in [0,\infty). \quad \square.$$

(ii) $\|f_n'\|_{[0,\infty)} = \frac{1}{n(n+1)} \leq \frac{1}{n^2}$. $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent as a p-series with $p=2>1$, thus by Weierstrass M-test, $\sum_{n=1}^{\infty} f_n'$ is uniformly convergent on $[0,\infty)$.

$\square$

(iii) Let $a>0$ be given. Then

$$\sum_{n=1}^{\infty} f_n(0) = \sum_{n=1}^{\infty} \ln\left(1+\frac{0}{n(n+1)}\right) = 0 \text{ is convergent.}$$

Together with (ii), we have $\sum_{n=1}^{\infty} f_n$ is uniformly convergent on $[0,a]$.

$\square$

(iv) By (iii), we must have $f' = \sum_{n=1}^{\infty} f_n'$ on $[0,a]$.

By taking a union, we must have f is differentiable on

$$\bigcup_{\substack{a \in \mathbb{R},\\ a>0}} [0,a] = [0,\infty), \text{ and } f' = \sum_{n=1}^{\infty} f_n' \text{ on } [0,\infty).$$

Thus $f'(0) = \sum_{n=1}^{\infty} f_n'(0)$

$$= \sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$

$$= \lim_{m\to\infty} \sum_{n=1}^{m} \left(\frac{1}{n} - \frac{1}{n+1}\right).$$

$$= \lim_{m\to\infty} \left(1 - \frac{1}{m+1}\right)$$

$$= 1 \quad \square.$$

Q6. (a)(i) Use ratio test directly,

$$\rho = \lim_{n\to\infty}\left| \frac{\cancel{(-1)^n(-1)}(n+1)\cancel{x^{2n-1}}x^2}{\cancel{4^n}\times 4} \times \frac{\cancel{4^n}}{\cancel{(-1)^n x^{2n-1}}(n)} \right|$$

$$= \lim_{n\to\infty} x^2 \left|\frac{n+1}{4(n)}\right| = \frac{x^2}{4}. \text{ to converge, } \rho < 1$$

$$\Rightarrow \frac{x^2}{4} < 1 \Rightarrow |x| < 2 \;.\; \therefore R = 2.$$

At $R=2$, $\sum_{n=1}^{\infty} \frac{(-1)^n (n)}{\cancel{4^n}} \frac{\cancel{4^n}}{2} = \sum_{n=1}^{\infty} \frac{(-1)^n (n)}{2}$ diverges by the nth term test, ie

$$\lim_{n\to\infty} \frac{(-1)^n n}{2} \neq 0.$$

$R = -2 \Rightarrow \sum_{n=1}^{\infty} \frac{n}{2}$ clearly diverges as well.

$\therefore I = (-2, 2).$

(ii) Let $f(x) = \sum_{n=1}^{\infty} \frac{(-1)^n n}{4^n} x^{2n-1}$

$$x f(x) = \sum_{n=1}^{\infty} \frac{(-1)^n n}{4^n} x^{2n}$$

$$\therefore \frac{d}{dx}(x f(x)) = \sum_{n=1}^{\infty} \frac{2(-1)^n}{4^n} x^{2n-1}$$

$$= \sum_{n=1}^{\infty} \left(\frac{2}{x}\right)\left(-\frac{x^2}{4}\right)^n.$$

$$= \frac{2}{x}\left(\frac{1}{1+x^2/4}\right) = \frac{8}{4x+x^3} = \frac{2(x^2+4)-2x(x)}{x(x^2+4)} = \frac{2}{x} - \frac{2x}{x^2+4}.$$

$$\therefore x f(x) = \int_0^x \frac{2}{t} - \frac{2t}{t^2+4}\,dt. = \lim_{k\to 0^+} \int_k^x \frac{2}{t}$$

$$= \left[\ln t - \ln(t^2+4)\right]_0^x = \ln x - \ln(x^2+4)$$

Q6 (5). since $R=1$,

$$\lim_{x\to 1^-}\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \lim_{x\to 1^-} a_n x^n \text{ since } x\in(-1,1).$$
$$= \sum_{n=0}^{\infty} a_n. = L$$

□