MA3201 Finals Tan Yee Jian A0190190L. Q1P1

Q1. (1) ~~False~~. True.

(2) False.

(3) False.

Q2. Consider the map $\phi : \mathbb{Z}[\frac{1+\sqrt{5}}{2}] \to R$,

$$\phi : (a+b\tfrac{1+\sqrt{5}}{2}) \longmapsto \begin{bmatrix} a & b \\ b & a+b \end{bmatrix}$$

As a shorthand, we write $x = \frac{1+\sqrt{5}}{2}$, and note $x^2 = \frac{6+2\sqrt{5}}{4} = 1+x$.

Now we verify ϕ is ~~an isomor~~ a bijective ring homomorphism.

Ring hom: WTS $\phi((a+bx)+(c+dx)(e+fx))$

$$= \phi(a+bx) + \phi(c+dx)\phi(e+fx)$$
$$= \begin{bmatrix} a & b \\ b & a+b \end{bmatrix} + \begin{bmatrix} c & d \\ d & c+d \end{bmatrix}\begin{bmatrix} e & f \\ f & e+f \end{bmatrix}.$$

LHS: $\phi(a+\underline{bx}+ce+\underline{cfx}+\underline{dex}+dfx^2)$

$$= \phi(\underline{a+ce+df} + (b+cf+de+df)x)$$ (underline for clarity).

RHS: $$\begin{bmatrix} a & b \\ b & a+b \end{bmatrix} + \begin{bmatrix} ce+df & cf+de+df \\ de+cf+df & df+ce+cf+de+df \end{bmatrix}$$

$$= \begin{bmatrix} a+ce+df & b+cf+de+df \\ b+de+cf+df & (a+ce+df)+(b+cf+de+df) \end{bmatrix}$$

$= LHS$.

Injectivity: suppose $f(~~a~~a+bx) = f(a'+b'x)$.

then $\begin{bmatrix} a & b \\ b & a+b \end{bmatrix} = \begin{bmatrix} a' & b' \\ b' & a'+b' \end{bmatrix} \Rightarrow a=a', b=b'$

$\Rightarrow a+bx = a'+b'x$ as desired.

Surjectivity: Pretty much by defn, since for R, the entries $\begin{bmatrix} a & b \\ b & a+b \end{bmatrix}$ can be given by $a+bx$.

$\square$.

Q3. Recall the nilradical is the intersection of all prime ideals. Since it is maximal, and any prime ideal is either maximal or contained in a maximal ideal, it follows that the nilradical must be the unique prime ideal, and also the unique max ideal.

Now if a is a zero div, ie $ab=0$, for $b\neq 0$, and $a\neq 0$, then (a) is ~~properly~~ contained in some maximal ideal, which in this case must be the nilradical. Then

$a^n = 0$ for some $n>0$, ~~since $a(a^{n-1}) = ab = 0$~~

~~we must have let n be the smallest possible s.t. $a^n = 0$, then $a^{n-1}\neq 0$~~, . Thus any zero div must be nilradical. $\square$.

Q4. (1) The contrapositive is obvious:

If $ab \notin D$ ($\Rightarrow ab \in P$), then $a \notin D$ or $b \notin D$ ($\Rightarrow a \in P$ or $b \in P$) is the defn of prime ideal P.

(2) From HW2 Q2.2, it suffices to show the set of all nonunits is an ideal. I claim that

① The set of all nonunits $= D^{-1}(P)$.

② $D^{-1}(P)$ is an ideal.

Then we must have $D^{-1}(R)$ a local ring.

<u>Pf of ①</u>: WTS $\{x \in D^{-1}(R) \mid x \text{ not a unit}\} = D^{-1}(P)$.

($\subseteq$): ~~Suppose otherwise. Some elm~~ we show the contrapositive: say $\frac{x}{y} \in D^{-1}(R)$ and $x \notin P$, then $x \in D$ by defn, and $\frac{y}{x} \in D^{-1}(R)$, where $\frac{x}{y} \cdot \frac{y}{x} = 1$, which shows x/y is a unit.

($\supseteq$): Let $p/d \in D^{-1}(P)$. Suppose $\frac{p}{d} \times \frac{x}{y} = 1$ for some $\frac{x}{y} \in D^{-1}(R)$, $px = dy \Rightarrow dy \in P$. ~~By defn~~, since P is a prime ideal, $d \in P$ or $y \in P$, a contradiction.

<u>Pf of ②</u>: $\forall \frac{x}{d_1} \in D^{-1}(R)$, $\frac{p}{d_2} \in D^{-1}(P)$, $x \in R$,

$$\frac{x}{d_1} \cdot \frac{p}{d_2} = \frac{xp}{d_1 d_2} \in D^{-1}(P)$$ since P is an ideal.

$\therefore D^{-1}(P)$ is an ideal. $\square$.

Q5 We directly apply Eisenstein's criterion:

$\mathbb{Z}$ is an integral domain,

$P = (3)$ is a prime ideal,

$f(x)$ is monic, $3 \mid 30, 15, 6, -120$ but

$9 \nmid 120 \Rightarrow 120 \notin (3)^2 = (9)$

Thus $f(x)$ is irreducible in $\mathbb{Z}[x]$. $\square$

MA3201 Finals A0195190L Tan Yee Jian Q6P1.

Q6. Consider a $\mathbb{C}[x]$ ~~vector~~ -module where x acts as A:

$V \cong \operatorname{coker}(xI-A)$. We perform row/col actions on $xI-A$:

$$xI-A = \begin{bmatrix} x+2 & -1 & -4 \\ 5 & x-2 & -5 \\ 1 & -1 & x-3 \end{bmatrix} \xrightarrow{C_1 \leftrightarrow C_2} \begin{bmatrix} -1 & x+2 & -4 \\ x-2 & 5 & -5 \\ -1 & 1 & x-3 \end{bmatrix}$$

$$\xrightarrow[R_3 - R_1]{R_2+(x-2)R_1} \begin{bmatrix} -1 & x+2 & -4 \\ 0 & x^2-4+5 & -4x+8-5 \\ 0 & -(x+1) & x+1 \end{bmatrix}$$

$$\xrightarrow[C_3-4C_1]{C_2+(x-2)C_1} \begin{bmatrix} -1 & 0 & 0 \\ 0 & x^2+1 & 3-4x \\ 0 & -(x+1) & (x+1) \end{bmatrix}$$

$$\xrightarrow{R_2+(x-1)R_3} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & x^2-4x+2 \\ 0 & -(x+1) & x+1 \end{bmatrix}$$

$$\xrightarrow{R_3+\frac{1}{2}(x+1)R_2} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & x^2-4x+2 \\ 0 & 0 & (x+1)+\frac{1}{2}(x+1)(x^2-4x+2) \end{bmatrix}$$

$$\xrightarrow{C_3-\frac{1}{2}(x^2-4x+2)} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & \frac{1}{2}(x+1)(x-2)^2 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} \cancel{-}1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & (x+1)(x-2)^2 \end{bmatrix}$$

$\therefore V \cong \mathbb{C}[x]/(x+1)(x-2)^2 \cong \mathbb{C}[x]/(x-2) \oplus \mathbb{C}[x]/(x+1)(x-2)$ $=(x^2-x-2)$

$\therefore$ RCF of A is $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix}$. $\cong \mathbb{C}[x]/(x+1) \oplus \mathbb{C}[x]/(x-2)^2$.

$\therefore$ JCF of A is $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$.

$\square$.