MA3110 Homework 1

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H1

1. When $x > 0$, $f'(x) = 6x + \frac{4}{x} > 0$. Then by exercise (i) on page 14 of Chapter 6 notes, we have f is increasing on $(0, \infty)$. $\square$

2. f is monotone by (i) on its domain, and is a linear combination of functions differentiable on $(0, \infty)$, namely $1, x^2, \ln x$. Thus $g(x) = f^{-1}x$ is well-defined. Since $f(1) = 2{+}3(1)^2{+}4(\ln 1) = 5$ and $f'(1) = 6(1) + 4/(1) = 10 \neq 0$, by the inverse function theorem, $g'(5) = \frac{1}{f'(1)} = \frac{1}{10}$. $\square$

H2

1. When $x \neq 0$,

$$\begin{aligned} f'(x) &= e^x + 2x\cos(\frac{1}{2x}) + x^2(-\sin(\frac{1}{2x}))(-\frac{1}{2x^2}) \\ &= e^x + 2x\cos\frac{1}{2x} + \frac{1}{2}\sin\frac{1}{2x} \end{aligned}$$

And for the derivative of f at 0,

$$\lim_{x\to 0}\frac{f(x)-f(0)}{x-0} = \lim_{x\to 0}\frac{e^x + x^2\cos\frac{1}{2x}}{x} = \lim_{x\to 0}\frac{e^x - 1}{x} + \lim_{x\to 0} x\cos(\frac{1}{2x}).$$

Using L'Hopital's rule, $\lim_{x\to 0}\frac{e^x-1}{x} = \lim_{x\to 0} e^x = 1$, and using Squeeze Theorem,

$$-1 \leq \cos\frac{1}{2x} \leq 1 \implies -x \leq x\cos\frac{1}{2x} \leq -x,$$

and taking limits when $x \to 0$, we have the limit as 0 for the second term. Thus,

$$\lim_{x\to 0}\frac{f(x)-f(0)}{x-0} = 1 + 0 = 1.$$

$$\therefore f'(x) = \begin{cases} e^x + 2x\cos\frac{1}{2x} + \frac{1}{2}\sin\frac{1}{2x} & x \neq 0 \\ 1 & x = 0 \end{cases}$$

2. We just check whether the limit $\lim_{x\to 0} f(x)$ exists, since when $x \neq 0$, f' is clearly continuous.

$$\lim_{x\to 0} e^x + 2x\cos\frac{1}{2x} + \frac{1}{2}\sin\frac{1}{2x} = 1 + \frac{1}{2}\lim_{x\to 0}\sin\frac{1}{2x}$$

But the limit is divergent. To see this, consider the sequence $(x_n = \frac{1}{4n\pi})_{n=1}^{\infty}$ and $(y_n = \frac{1}{(4n+2)\pi})_{n=1}^{\infty}$. Both $x_n, y_n \to 0$, but

$$\lim_{n\to\infty}\sin\frac{1}{2x} = \lim_{n\to\infty}\sin(2n\pi) = 0 \neq \lim_{n\to\infty}\sin\frac{1}{2y_n} = \lim_{n\to\infty}\sin((2n+1)\pi) = 1.$$

Therefore, $f \notin C^1(\mathbb{R})$.

$\square$

H3

Let $f(t) = (1+t)^n, t > -1, n = 2, 3, \ldots$, and we consider two cases.

Case 1: $x \in (0, \infty)$.

Apply Mean Value Theorem to f on $[0, x]$, since f is clearly differentiable on $\mathbb{R}$. Then we must have a $c \in (0, \infty)$, where

$$\begin{aligned} f'(c) &= \frac{f(x) - f(0)}{x - 0} \\ n(1+c)^{n-1} &= \frac{(1+x)^n - 1}{x}. \end{aligned}$$

Since $1 + c > 1 \implies (1+c)^{n-1} > 1$, we must have

$$\begin{aligned} (1+x)^n &= 1 + nx(1+c)^{n-1} \\ &> 1 + nx(1). \end{aligned}$$

Case 2: $x \in (-1, 0)$.

Apply Mean Value Theorem to f on $[x, 0]$, since f is clearly differentiable on $\mathbb{R}$. Then we must have a $c \in (-1, 0)$, where

$$\begin{aligned} f'(c) &= \frac{f(x) - f(0)}{x - 0} \\ n(1+c)^{n-1} &= \frac{(1+x)^n - 1}{x}. \end{aligned}$$

Since $1 + c < 1 \implies (1+c)^{n-1} < 1 \implies nx(1+c)^{n-1} > nx$ (since $nx < 0$), we must have

$$\begin{aligned} (1+x)^n &= 1 + nx(1+c)^{n-1} \\ &> 1 + nx(1). \end{aligned}$$

$\square$

H4

Let $g(x) = (f(x))^2 - x^2$ as per the hint. Since $x \mapsto x^2$ is differentiable on $\mathbb{R}$ and f is differentiable on $[a, b] \subset \mathbb{R}$, we have g is continuous on $[a, b]$ and differentiable on (a, b). Note that

$$g(b) = (f(b))^2 - b^2 = (f(a))^2 - a^2 = g(a),$$

and thus by Rolle's Theorem,

$$\exists c \in (a, b), g'(c) = 2f'(c)f(c) - 2c = 0 \implies f'(c)f(c) = c.$$

$\square$