MA3201 Homework 4

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Problem 1

Let R be a ring with $1 \neq 0$. Let M be a R-module. We define

$$\operatorname{Tor}(M) = \{m \in M | rm = 0, \text{ for some } 0 \neq r \in R\}.$$

Problem 1.1

Prove that $\operatorname{Tor}(M)$ is a submodule of M if R is an integral domain.

Solution

Proof. We show by the submodule criterion. $\operatorname{Tor}(M) \ni 0 = 1 \cdot 0$ is not empty.

For any $r \in R, m_1, m_2 \in M$,

$$\begin{aligned} r_1 r_2(m_1 + rm_2) &= r_2(r_1 m_1) + rr_1(r_2 m_2) && R \text{ is commutative, } M \text{ is a } R\text{-module} \\ &= r_2 \cdot 0 + rr_1 \cdot 0 \\ &= 0 + 0 = 0 \\ &\implies (m_1 + rm_2) \in \operatorname{Tor}(M). \end{aligned}$$

$\therefore$ $\operatorname{Tor}(M)$ satisties the submodule criteiron and is a submodule of M. $\square$

Problem 1.2

Give a counterexample of the statement above for general R.

Solution

Proof. Consider $R = \mathbb{Z}/10\mathbb{Z}$ as a R-module by left multiplication. Then $2 \cdot 5 = 5 \cdot 2 = 0$ implies $\{2, 5\} \subseteq \operatorname{Tor}(M)$, but $2 + 5 = 7 \notin \operatorname{Tor}(M)$. $\square$

Problem 2

Let I be a right ideal of R. Let M be a left R-module. We define

$$N = \{m \in M | rm = 0 \ \forall r \in I\}.$$

Prove that N is a R-submodule of M.

Solution

Proof. We show by the submodule criterion. Since $\forall r \in I \subset R, r \cdot 0 = 0 \implies 0 \in N \neq \emptyset$.

Also for any $r \in R, i \in I, m, m' \in N$,

$$\begin{aligned} i(m + rm') &= im + i(rm') \\ &= 0 + i'm' && I \text{ is an ideal} \implies ir = i' \text{ for some } i' \in R \\ &= 0 + 0 = 0 \\ &\implies (m + rm') \in N. \end{aligned}$$

Therefore N satisfies the submodule criterion and hence is a submodule of M. $\square$

Problem 3

Let R be a ring with $1 \neq 0$. Let M, N be (left) R-modules. Consider the abelian group $\mathrm{Hom}_{\mathbb{Z}}(M, N)$ of $\mathbb{Z}$-module homomorphisms from M to N.

Problem 3.1

For any $r \in R$, we define

$$(rf)(x) = r(f(x)), \text{ for } f \in \mathrm{Hom}_{\mathbb{Z}}(M, N), x \in M,$$

where the right hand side is the action of r on N. Prove that $\mathrm{Hom}_{\mathbb{Z}}(M, N)$ is a left R-module with the action defined above.

Solution

Proof. For any $x \in M, r \in R, f, g \in \mathrm{Hom}_{\mathbb{Z}}(M, N)$,

1. Show $r(f+g) = rf + rg$. $\forall x \in M$,

$$\begin{aligned}
(r(f+g))x &= r((f+g)x) && \text{definition} \\
&= r(fx + gx) && \mathrm{Hom}_{\mathbb{Z}}(M,N) \text{ ring distribution} \\
&= (rf + rg)(x) && \text{definition.}
\end{aligned}$$

2. Show $(rs)f = r(sf)$. $\forall x \in M$,

$$\begin{aligned}
((rs)f)(x) &= (rs)f(x) && \text{definition} \\
&= r(sf(x)) && N \text{ is a } R\text{-module} \\
&= (r(sf))x && \text{definition.}
\end{aligned}$$

3. Show $(r+s)f = rf + sf$. $\forall x \in M$,

$$\begin{aligned}
((r+s)f)(x) &= (r+s)(f(x)) && \text{definition} \\
&= rf(x) + sf(x) && N \text{ is a } R\text{-module} \\
&= (rf + sf)x && \text{definition.}
\end{aligned}$$

4. Show $1 \cdot f = f$. $\forall x \in M$,

$$\begin{aligned}
(1 \cdot f)(x) &= 1(f(x)) && \text{definition} \\
&= f(x) && N \text{ is a } R\text{-module.}
\end{aligned}$$

Result follows. $\square$

Problem 3.2

For any $r \in R$, we define

$$(fr)(x) = f(rx), \text{ for } f \in \mathrm{Hom}_{\mathbb{Z}}(M, N), x \in M,$$

where the right hand side is the action of r on N. Prove that $\mathrm{Hom}_{\mathbb{Z}}(M, N)$ is a right R-module with the action defined above.

Solution

Proof. For any $x \in M, r \in R, f, g \in \text{Hom}_{\mathbb{Z}}(M, N)$,

1. Show $(f+g)r = fr + gr$. $\forall x \in M$,

$$\begin{aligned} ((f+g)r)x &= (f+g)(rx) && \text{definition} \\ &= f(rx) + g(rx) && \text{Hom}_{\mathbb{Z}}(M,N) \text{ ring distribution} \\ &= (fr+gr)(x) && \text{definition.} \end{aligned}$$

2. Show $f(rs) = (fr)s$. $\forall x \in M$,

$$\begin{aligned} (f(rs))(x) &= f((rs)x) && \text{definition} \\ &= f(r(sx)) && M \text{ is a } R\text{-module} \\ &= (fr)(sx) && \text{definition.} \end{aligned}$$

3. Show $f(r+s) = fr + fs$. $\forall x \in M$,

$$\begin{aligned} (f(r+s))(x) &= f((r+s)x) && \text{definition} \\ &= f(rx + sx)) && M \text{ is a } R\text{-module} \\ &= (fr+fs)x && \text{definition.} \end{aligned}$$

4. Show $f \cdot 1 = f$. $\forall x \in M$,

$$\begin{aligned} (f \cdot 1)(x) &= f(1 \cdot x) && \text{definition} \\ &= f(x) && M \text{ is a } R\text{-module.} \end{aligned}$$

Result follows. $\square$

Problem 3.3

Let $f \in \text{Hom}_{\mathbb{Z}}(M, N)$. Prove that $f \in \text{Hom}_R(M, N)$ if and only if $rf = fr$ for any $r \in R$ with the actions defined above.

Solution

$(\implies)$: Let $f \in \text{Hom}_R(M, N)$ be given. Then we know, in general, for any $r \in R, m, m' \in M$,

$$f(m + rm') = f(m) + rf(m')$$

since f is a R-module homomorphism. In particular, let $m = 0$, then

$$f(rm') = rf(m') \ \forall m' \in M$$

as desired.
$(\impliedby)$: We show that $\forall r \in R, f \in \text{Hom}_{\mathbb{Z}}(M, N),\ m, m' \in M$ we have the property that $f(m+rm') = f(m) + rf(m')$ which implies f is a R-module map.

$$\begin{aligned} f(m + rm') &= f(m) + f(rm') && f \in \text{Hom}_{\mathbb{Z}}(M,N) \\ &= f(m) + rf(m') && \text{assumption.} \end{aligned}$$

Result follows.

Problem 4

Let M be a R-module for a commutative ring R. Prove that the map

$$M \to \operatorname{Hom}_R(R, M), \quad m \mapsto (f : R \to M, r \mapsto rm)$$

is an isomorphism of R-modules, where the R-module structure of $\operatorname{Hom}_R(R, M)$ is given by Q3(1).

Solution

Proof. We call the map above ϕ, and show that ϕ is a R-module homomorphism that is both injective and surjective.

R-module Homomorphism:

$$\begin{aligned}
\phi(m + rm')(s) &= s(m + rm') && \text{definition} \\
&= sm + s(rm') && s \in R, M \text{ is } R\text{-module} \\
&= sm + r(sm') && M \text{ is } R\text{-module, } R \text{ commutative} \\
&= \phi(m)s + r(\phi(m')s) \\
&= \phi(m)s + (r\phi(m'))s && \text{Q3(1).}
\end{aligned}$$

Injectivity:

$$\begin{aligned}
\ker(\phi) &= \{m \in M | (r \mapsto rm) \text{ is the zero map}\} \\
&= \{m \in M | rm = 0_M \ \forall r \in R\}
\end{aligned}$$

I claim that $\ker(\phi) = 0$. Otherwise, suppose any $0 \neq x \ni M, x \in \ker(\phi) \implies r \cdot x = 0$ for any $r \in R$. However, $1_R \cdot x = x \neq 0_M$, a contradiction. Therefore the kernel is trivial and ϕ is injective.

Surjectivity: For any $f \in \operatorname{Hom}_R(R, M)$, we have

$$f(r) = f(r \cdot 1) = r \cdot f(1).$$

Then let $f(1) = m \in M$, we have $f(r) = rm$, so $f = (r \mapsto rm) = \phi(m)$. Thus ϕ is surjective. □

Problem 5

Let R be commutative. Prove that $R \cong \operatorname{End}_R(R)$ as rings, where R is a R-module via left multiplication.

Solution

From Q4, we already know that $R \cong \operatorname{Hom}_R(R, R) = \operatorname{End}_R(R)$ as modules, hence they are isomorphic as abelian groups. We just need to show that multiplication is preserved in this map.

Consider the same map $\phi : R \to \operatorname{End}_R(R), r \mapsto (r_1 \mapsto rr_1)$. We show that $\phi(rs)x = [\phi(r)\phi(s)]x$ for any $r, s, x \in R$.

$$\begin{aligned}
\phi(rs)(x) &= rsx \\
&= r(sx) \\
&= \phi(r)(\phi(s)(x)) \\
&= [\phi(r)\phi(s)](x) && \text{multiplication is composition in } \operatorname{End}_R(R).
\end{aligned}$$

Since this R-module isomorphism preserves multiplication, it is also a ring isomorphism.

Now we can conclude the stronger result that $R^{op} \cong \operatorname{End}_R(R)$, by noticing that if R is commutative, then $R^{op} \cong R$ as rings via the canonical map $\varphi : r^{op} \mapsto r$. We already know this map is bijective (set-theoretically isomorphic), and addition is automatically preserved by definition, now the multiplication is preserved by commutativity of R:

$$a^{op}b^{op} = (ba)^{op} = (ab)^{op} \text{ (commutativity)} \implies \varphi(a^{op}b^{op}) = ab$$

for any $a, b \in R$. Therefore, we have $R^{op} \cong R \cong \operatorname{End}_R(R)$ by the isomorphism $\varphi \circ \phi$.