1 Lecture 1 - The Reals

1. *Definition*: a **field** is the 5-tuple $\langle \mathbb{F}, +, \cdot, e, u \rangle$, where $\mathbb{F}$ is a set containing at least the elements e and u, where $e \neq u$, and satisfies: For any $a, b, c \in \mathbb{F}$,
   (a) (commutative add) $a + b = b + a$
   (b) (associative add) $(a + b) + c = a + (b + c)$
   (c) (additive identity) $a + e = a$
   (d) (additive inverse) $\forall a, \exists b \in \mathbb{F}$ such that $a + b = e$.
   (e) (commutative multiply) $a \cdot b = b \cdot a$
   (f) (associative multiply) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
   (g) (multiplicative identity) $a \cdot u = a$
   (h) (multiplicative inverse) $\forall a, \exists b \in \mathbb{F}$ such that $a \cdot b = u$.
   (i) (distributive) $\forall a, b, c \in \mathbb{F}, a \cdot (b + c) = a \cdot b + a \cdot c$
2. *Example*: $\mathbb{Q}, \mathbb{N}, \mathbb{Z}, \mathbb{R}, \mathbb{C}, \mathbb{R} - \mathbb{Q}$ are fields.
3. *Definiton*: A field $\mathbb{F}$ is **ordered** if $\exists P \subseteq \mathbb{F}$ such that $\forall a, b \in P$,
   (a) $a + b \in P$
   (b) $a \cdot b \in P$
   (c) (trichotomy) either
      i. $a \in P$
      ii. $a = e$, or
      iii. $-a \in P$
   (d) $e \notin P$.
4. *Theorem*: $a \in P \implies -a \notin P$.
5. *Definiton*: if a subset of an ordered field, $A \subseteq \mathbb{F}$ contains an element a such that $\forall x \in \mathbb{F}, a \leq (\geq) x$, then $\mathbb{F}$ is **bounded below (above)**. Such a is called an **lower (upper) bound** of A.
6. *Definition*: if $\emptyset \neq A \subseteq \mathbb{F}$ is bounded above (below), an element b is the **least upper (greatest lower) bound** if
   (a) b is an upper(lower) bound of A and
   (b) $\forall c \in \mathbb{F}$ where c is an upper(lower) bound of A, $b \geq c (b \leq c)$.

   , denoted by $\sup A (\inf A)$ respectively.
7. *Definition*: An ordered field $\mathbb{F}$ is **(order) complete** if it has the **least upper bound property**: $\forall \emptyset \neq A \subseteq \mathbb{F}$, if A is bounded above, A has a least upper bound.
8. *Example*: $\mathbb{R}$ is order complete, but $\mathbb{Q}$ is not.