

1 The Real Number System

1. *Definition:* a **field** is the 5-tuple $\langle \mathbb{F}, +, \cdot, e, u \rangle$, where \mathbb{F} is a set containing at least the elements e and u , where $e \neq u$, and satisfies: For any $a, b, c \in \mathbb{F}$,

- (a) (commutative add) $a + b = b + a$
- (b) (associative add) $(a + b) + c = a + (b + c)$
- (c) (additive identity) $a + e = a$
- (d) (additive inverse) $\forall a, \exists b \in \mathbb{F}$ such that $a + b = e$.
- (e) (commutative multiply) $a \cdot b = b \cdot a$
- (f) (associative multiply) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
- (g) (multiplicative identity) $a \cdot u = a$
- (h) (multiplicative inverse) $\forall a, \exists b \in \mathbb{F}$ such that $a \cdot b = u$.
- (i) (distributive) $\forall a, b, c \in \mathbb{F}, a \cdot (b + c) = a \cdot b + a \cdot c$

2. *Example:* $\mathbb{Q}, \mathbb{N}, \mathbb{Z}, \mathbb{R}, \mathbb{C}, \mathbb{R} - \mathbb{Q}$ are fields.

3. *Definition:* A field \mathbb{F} is **ordered** if $\exists P \subseteq \mathbb{F}$ such that $\forall a, b \in P$,

- (a) $a + b \in P$
- (b) $a \cdot b \in P$
- (c) (trichotomy) either
 - i. $a \in P$
 - ii. $a = e$, or
 - iii. $-a \in P$
- (d) $e \notin P$.

4. *Theorem:* $a \in P \implies -a \notin P$.

5. *Definition:* if a subset of an ordered field, $A \subseteq \mathbb{F}$ contains an element a such that $\forall x \in \mathbb{F}, a \leq (\geq) x$, then \mathbb{F} is **bounded below (above)**. Such a is called an **lower (upper) bound** of A .

6. *Definition:* if $\emptyset \neq A \subseteq \mathbb{F}$ is bounded above (below), an element b is the **least upper (greatest lower) bound** if

- (a) b is an upper(lower) bound of A and
- (b) $\forall c \in \mathbb{F}$ where c is an upper(lower) bound of A , $b \geq c (b \leq c)$.

, denoted by $\sup A (\inf A)$ respectively.

7. *Definition:* An ordered field \mathbb{F} is **(order) complete** if it has the **least upper bound property**: $\forall \emptyset \neq A \subseteq \mathbb{F}$, if A is bounded above, A has a least upper bound.

8. *Example:* \mathbb{R} is order complete, but \mathbb{Q} is not.

2 Sequences of Real Numbers

1. *Definition:* A **sequence** in a set S is a function, $f : \mathbb{N} \rightarrow S$, where we denote $f(n) = s_n$ for all $n \in \mathbb{N}$, and the sequence as $(s_n)_{n=1}^{\infty}$.

2. *Definition:* given a sequence $f : \mathbb{N} \rightarrow S$, a **subsequence** of f is a sequence of the form $f \circ g : \mathbb{N} \rightarrow S$, where $g : \mathbb{N} \rightarrow \mathbb{N}$ is strictly increasing. We write $((f \circ g)(n))_{n=1}^{\infty} = (s_{g(n)})_{n=1}^{\infty}$.

3. *Analogy:* We have a real sequence $(s_n)_{n=1}^{\infty}$. We claim that L is the limit. An opponent then issues a challenge ϵ . We need to be able to to any ϵ given with a N such that all our terms after $N, (s_N, s_{N+1}, s_{N+2}, \dots)$ are all at most ϵ from L .

4. *Definition:* if $\lim_{n \rightarrow \infty} s_n = L$ holds then we say $(s_n)_{n=1}^{\infty}$ is **convergent**. Conversely, $(s_n)_{n=1}^{\infty}$ is **convergent** if there exists an $L \in \mathbb{R}$ such that $(s_n)_{n=1}^{\infty}$ converges to L .

5. *Definition:* a sequence $(s_n)_{n=1}^{\infty}$ **diverges to $\infty (-\infty)$** if $\forall M \in \mathbb{R}, \exists N \in \mathbb{N}$ such that $s_n \geq M (s_n \leq -M)$ for any $n \geq N$.

6. *Proposition:* sequence is convergent \implies **any subsequence** of that sequence is convergent.

7. *Definition:* (properties of sequences) A sequence $(s_n)_{n=1}^{\infty}$ is

- (a) **bounded** if $\exists M \in \mathbb{R}, M > 0$ such that $|s_n| \leq M$ for any $n > N$.
- (b) **nondecreasing** if $s_n \leq s_{n+1} \quad \forall n \in \mathbb{N}$.
- (c) **nonincreasing** if $s_n \geq s_{n+1} \quad \forall n \in \mathbb{N}$.
- (d) **monotone** if it is either nondecreasing or nonincreasing.

8. *Proposition:* bounded and nondecreasing(nonincreasing) \implies convergent to its supremum (infimum).

9. *Definition:* $\lim_{n \rightarrow \infty} s_n = e(x)$. $e(x + y) = e(x) + e(y), e(0) = 1$.

10. *Theorem:* Every real sequence has a monotone subsequence. Therefore, every bounded sequence has a convergent subsequence.

11. *Proposition:* If $c > 1$, then $\lim_{n \rightarrow \infty} c_{1/n} = 1$.

12. *Proposition:* A convergent sequence of non-negative numbers converge to a nonnegative number. Similarly, if all values of a sequence are greater than k , its limit is greater than k too.

13. *Theorem:* suppose $\lim_{n \rightarrow \infty} s_n = L \in \mathbb{R}, \lim_{n \rightarrow \infty} t_n = M \in \mathbb{R}$, and $C \in \mathbb{R}$. Then

- (a) $\lim_{n \rightarrow \infty} (s_n + Ct_n) = L + CM$.
- (b) $\lim_{n \rightarrow \infty} (s_n t_n) = LM$.
- (c) if $M \neq 0$, then $\lim_{n \rightarrow \infty} 1/t_n = 1/M$.

14. *Definition:* Let $(s_n)_{n=1}^{\infty}$ be a real sequence. Then define **limit superior**

$$\limsup_{n \rightarrow \infty} = \begin{cases} \infty & \text{if } (s_n)_{n=1}^{\infty} \text{ is not bounded above} \\ \lim_{n \rightarrow \infty} M_n & \text{if } (M_n)_{n=1}^{\infty} \text{ is bounded below} \\ -\infty & \text{if } (M_n)_{n=1}^{\infty} \text{ is not bounded below} \end{cases}$$

where $(M_k)_{n=1}^{\infty} = \sup\{s_k, s_{k+1}, \dots\}$, and define **limit inferior**

$$\liminf_{n \rightarrow \infty} = \begin{cases} \infty & \text{if } (s_n)_{n=1}^{\infty} \text{ is not bounded below} \\ \lim_{n \rightarrow \infty} M_n & \text{if } (M_n)_{n=1}^{\infty} \text{ is bounded above} \\ -\infty & \text{if } (M_n)_{n=1}^{\infty} \text{ is not bounded above} \end{cases}$$

where $(M_k)_{n=1}^{\infty} = \inf\{s_k, s_{k+1}, \dots\}$.

15. *Proposition:* $\limsup_{n \rightarrow \infty} s_n = L, \limsup_{n \rightarrow \infty} t_n = M$, where $L, M \in \mathbb{R}$, and the sequences are bounded, $\implies \limsup_{n \rightarrow \infty} (s_n + t_n) \leq L + M$.

16. *Proposition:* for any sequence, $\liminf \leq \limsup$.

17. *Proposition:* for any bounded sequence, $\lim_{n \rightarrow \infty} s_n = L \iff \liminf_{n \rightarrow \infty} s_n = \limsup_{n \rightarrow \infty} s_n = L$.
18. *Theorem:* given a bounded real sequence, there exist subsequences that converge to the *limsup* and *liminf* respectively. Any convergent subsequence converges to at most the *limsup*, and at least the *liminf*. That is, for any subsequence $(s_{n_k})_{k=1}^{\infty}$, $\liminf_{n \rightarrow \infty} s_n \leq \lim_{k \rightarrow \infty} s_{n_k} \leq \limsup_{n \rightarrow \infty} s_n$.
19. *Definition:* A sequence $(s_n)_{n=1}^{\infty}$ is **Cauchy** if $\forall \epsilon > 0, \exists N \in \mathbb{N}$ such that $\forall n, m \geq N, |s_n - s_m| < \epsilon$.
20. *Theorem:* for real sequences, convergence \iff Cauchy \implies bounded.
21. *Nested Interval Theorem:* Given $I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots$ are closed bounded intervals such that $\lim_{n \rightarrow \infty} \text{diam } I_n = 0$. Then $\bigcap_{n=1}^{\infty} I_n$ contains exactly one point.
22. *Theorem:* There is no onto map $f : \mathbb{N} \rightarrow [0, 1]$. In other words, $[0, 1]$ is uncountable.
23. *Definition:* Given a real sequence $(s_n)_{n=1}^{\infty}$, define $\sigma_n = (s_1 + s_2 + \dots + s_n)/n, \forall n \in \mathbb{N}$. We say $(s_n)_{n=1}^{\infty}$ is $(C, 1)$ summable to $L \in \mathbb{R}$ if $\lim_{n \rightarrow \infty} \sigma_n = L$.
24. *Theorem (regularity):* If a real sequence converges to L , then it is $(C, 1)$ summable to L .

3 Series of Real Numbers

1. *Definition:* Given an **infinite series** $\sum_{n=1}^{\infty} a_n$, define $s_k = a_1 + a_2 + \dots + a_k = \sum_{n=1}^k a_n, k = 1, 2, 3, \dots$. Then $(s_k)_{k=1}^{\infty}$ is the sequence of **partial sums** of $\sum_{k=1}^{\infty} a_k$.
2. *Definition:* The infinite series $\sum_{n=1}^{\infty} a_n$ **converges** to L if the partial sums $(s_k)_{k=1}^{\infty}$ converges to L . If $(s_k)_{k=1}^{\infty}$ diverges, then we say $\sum_{n=1}^{\infty} a_n$ also **diverges**.

3. *Proposition:* If $\sum_{n=1}^{\infty} a_n$ converges $\implies \lim_{n \rightarrow \infty} a_n = 0$.
(The converse need not be true, see $\sum_{n=1}^{\infty} \frac{1}{n}$).

4. *Proposition:* If $a_n \geq 0$, then $\sum_{n=1}^{\infty} a_n$ converges \iff the sequence of partial sums, $(s_n)_{n=1}^{\infty}$ is bounded above.

5. *Definition:* An **alternating series** is a series of the form $\sum_{n=1}^{\infty} (-1)^n a_n$ or $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ where $a_n \geq 0$.

6. *Proposition:* If $\sum_{n=1}^{\infty} a_n = L, \sum_{n=1}^{\infty} b_n = M, c \in \mathbb{R}$, then $\sum_{n=1}^{\infty} a_n + cb_n = L + cM$.

7. *Definition:* A series $\sum_{n=1}^{\infty} a_n$ is **absolutely convergent** if $\sum_{n=1}^{\infty} |a_n|$ is convergent. It is **conditionally convergent** if $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} |a_n|$ diverges.

8. *Theorem:* absolute convergence \implies convergence.

9. *Definition:* Given series $\sum_{n=1}^{\infty} a_n$. Define

$$p_n = (a_n + |a_n|)/2, \quad q_n = (a_n - |a_n|)/2,$$

then we have properties as follows:

- (a) If $a_n \geq 0$ for all n , then $p_n = a_n, q_n = 0$.
 (b) If $a_n < 0$ for all n , then $p_n = 0, q_n = a_n$.
 (c) If $a_n < 0$ for all n , then $p_n = 0, q_n = a_n$.
 (d) If $\sum_{n=1}^{\infty} a_n$ converges absolutely \iff both $\sum_{n=1}^{\infty} p_n$ and $\sum_{n=1}^{\infty} q_n$ converge.
 (e) If $\sum_{n=1}^{\infty} a_n$ converges conditionally \implies both $\sum_{n=1}^{\infty} p_n$ and $\sum_{n=1}^{\infty} q_n$ diverge.

10. *Definition:* An **arrangement** of a series $\sum_{n=1}^{\infty} a_n$ is a series of the form $\sum_{n=1}^{\infty} a_{g(n)}$, where $g : \mathbb{N} \rightarrow \mathbb{N}$ is a bijection.

11. *Lemma:* If $a_n \geq 0$, and $\sum_{n=1}^{\infty} a_n$ converges to L , then any rearrangement $a_{g(n)}$ also converges to L .

12. *Theorem:* If $\sum_{n=1}^{\infty} a_n$ is absolutely convergent and $\sum_{n=1}^{\infty} a_n = L$, then any arrangement $\sum_{n=1}^{\infty} a_{g(n)}$ is absolutely convergent and $\sum_{n=1}^{\infty} a_{g(n)} = L$.

13. *Theorem:* Suppose that $\sum_{n=1}^{\infty} a_n$ is conditionally convergent, then for any $L \in \mathbb{R}, \exists$ a rearrangement of $\sum_{n=1}^{\infty} a_n$ that converges to L .

14. *Theorem:* Suppose both $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} a_n$ both converge absolutely. Let $c_n = \sum_{k=0}^{\infty} a_k b_{n-k}$ for all $n = 0, 1, \dots$. Then $\sum_{n=1}^{\infty} c_n$ converges absolutely and $\sum_{n=1}^{\infty} c_n = (\sum_{n=1}^{\infty} a_n)(\sum_{n=1}^{\infty} b_n)$.

15. *Theorem (Series Tests):* Given a real series $\sum_{n=1}^{\infty} a_n$.

- (a) **Alternating Series Test:** Given an alternating series $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$, if a_n is nonincreasing and convergent to 0, then $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ converges to some $L \in \mathbb{R}$. Furthermore, for any $k \in \mathbb{R}, |\sum_{n=1}^k (-1)^{n+1} a_n - L| < a_{k+1}$.
 (b) **Comparison Test:** Suppose that $\exists k < \infty$ such that $\forall n \in \mathbb{N}, a_n \leq k|b_n|$. If $\sum_{n=1}^{\infty} b_n$ converges absolutely, then $\sum_{n=1}^{\infty} a_n$ converges absolutely.

(c) **Ratio Test:**

$\liminf_{n \rightarrow \infty} |a_{n+1}/a_n| < 1 \implies \sum_{n=1}^{\infty} a_n$ converges absolutely. $\limsup_{n \rightarrow \infty} |a_{n+1}/a_n| > 1 \implies \sum_{n=1}^{\infty} a_n$ diverges.

(d) **Root Test:**

$\limsup_{n \rightarrow \infty} \sqrt[n]{|a_n|} < 1 \implies \sum_{n=1}^{\infty} a_n$ converges absolutely. $\limsup_{n \rightarrow \infty} \sqrt[n]{|a_n|} > 1 \implies \sum_{n=1}^{\infty} a_n$ diverges.

16. **Definition:** Given series $\sum_{n=1}^{\infty} a_n$ and sequence $(b_n)_{n=1}^{\infty}$,

and let $s_k = \sum_{n=1}^k a_n$. Then we have **summation by parts:**

$$\sum_{k=1}^n a_k b_k = s_n b_n + \sum_{k=1}^{n-1} s_k (b_k - b_{k+1}).$$

(a) **Dirichlet's Test:** $(s_k)_{k=1}^{\infty}$ bounded, $(b_k)_{k=1}^{\infty}$ is monotone and converges to 0 $\implies \sum_{n=1}^{\infty} a_n b_n$ converges.

(b) **Abel's Test:** $(s_k)_{k=1}^{\infty}$ converges, $(b_k)_{k=1}^{\infty}$ is monotone and bounded $\implies \sum_{n=1}^{\infty} a_n b_n$ converges.

17. **Definition:** Given $\sum_{n=1}^{\infty} a_n$. If its partial sum is (C,1) summable to k , that is, $\lim_{k \rightarrow \infty} (s_1 + \dots + s_k)/k = A$, then we say $\sum_{n=1}^{\infty} a_n$ is **(C,1) summable**, written as $\sum_{n=1}^{\infty} a_n = A$ (C,1).

18. **Tauberian Theorem:** A series (C,1) summable to $L \implies$ convergent to L .

4 Limits in Metric Spaces

Convention: $\langle M, \rho \rangle, \langle M_1, \rho_1 \rangle, \langle M_2, \rho_2 \rangle$ are metric spaces. ρ, σ, τ are metrics.

1. **Definition:** A **metric** on a nonempty set M is a function $\rho : M \times M \rightarrow \mathbb{R}$, that satisfies for any $x, y, z \in M$:

(a) $\rho(x, x) = 0$,

(b) $x \neq y \implies \rho(x, y) > 0$.

(c) (symmetry) $\rho(x, y) = \rho(y, x)$, and

(d) (triangle inequality) $\rho(x, y) \leq \rho(x, z) + \rho(y, z)$.

Commonly used metrics for \mathbb{R}^n include:

(a) (1-metric) $\rho_1((a_i)_{i=1}^n, (b_i)_{i=1}^n) = \sum_{i=1}^n |a_i - b_i|$.

(b) (2-metric, or Euclidian metric)

$$\rho_2((a_i)_{i=1}^n, (b_i)_{i=1}^n) = \sqrt{\sum_{i=1}^n |a_i - b_i|^2}.$$

(c) (n-metric) $\rho_n((a_i)_{i=1}^n, (b_i)_{i=1}^n) = \sqrt[n]{\sum_{i=1}^n |a_i - b_i|^n}$.

(d) (∞ -metric)

$$\rho_{\infty}((a_i)_{i=1}^n, (b_i)_{i=1}^n) = \max\{|a_i - b_i| : 1 \leq i \leq n\}.$$

(e) (discrete-metric) given $x, y \in \langle M, \rho \rangle$,

$$\rho_d(x, y) = \begin{cases} 0 & x = y, \\ 1 & x \neq y. \end{cases}$$

The pair $\langle M, \rho \rangle$ is called a **metric space**.

2. **Cauchy-Schwartz Inequality:** $\forall a_i, b_i \in \mathbb{R}, \sum_{i=1}^n |a_i b_i| \leq AB$ where $A = \sqrt{\sum_{i=1}^n |a_i|^2}$ and $B = \sqrt{\sum_{i=1}^n |b_i|^2}$.

3. **Minkowski's Inequality:**

$$\sqrt{\sum_{i=1}^n |a_i + b_i|^2} \leq \sqrt{\sum_{i=1}^n |a_i|^2} + \sqrt{\sum_{i=1}^n |b_i|^2}$$

4. **Definition:** $A \subseteq \langle M, \rho \rangle$. A point $a \in M$ is a **cluster point** of A if, $\forall h > 0, \exists x \in A$ such that $0 < \rho(x, a) < h$. Also known as **limit points** or **accumulation points**. A point is an **isolated point** if it is **not a cluster point**.

5. **Proposition:** (Sequential formulation of cluster points) x is a cluster point of $E \iff \exists (x_n)_{n=1}^{\infty} \in E$ that converges to x .

6. **Definition (limits):** Given $f : M_1 \rightarrow M_2$. Suppose a is a cluster point of M_1 , and $L \in M_2$. Then we say $\lim_{x \rightarrow a} f(x) = L$ if for any $\epsilon > 0, \exists \delta > 0$ such that: $0 < \rho_1(x, a) < \delta \implies \rho_2(f(x), L) < \epsilon \quad \forall x \in M$.

(a) **Sequential characterization of limits:**

$\lim_{x \rightarrow a} f(x) = L \iff$ For all sequences $(x_n)_{n=1}^{\infty}$ in M_1 that converges to $a, x_n \neq a \quad \forall n \in \mathbb{N}, (f(x_n))_{n=1}^{\infty}$ converges to L in M_2 .

(b) **Arithmetic of limits:** Suppose $f, g : M \rightarrow \mathbb{R}$, and a is a cluster point of M . Given $\lim_{x \rightarrow a} f(x) = A, \lim_{x \rightarrow a} g(x) = B$, then as usual,

i. $\lim_{x \rightarrow a} (f(x) + g(x)) = A + B$

ii. $\lim_{x \rightarrow a} (f(x)g(x)) = AB$

iii. $\lim_{x \rightarrow a} (f(x)/g(x)) = A/B$ if $g(x) \neq 0 \quad \forall x \in M$ and $B \neq 0$.

5 Open and Closed Sets, Continuity

1. **Definition:** The **open ball** centered at $a \in M$ with radius $r > 0$ is the set $B[a, r] = \{x \in M : \rho(x, a) < r\}$.

2. **Definition:** A function $f : M_1 \rightarrow M_2$ is **continuous** at a point $a \in M_1$ if $\forall \epsilon > 0, \exists \delta > 0$ such that

(a) $\rho_2(f(x), f(a)) < \epsilon$ for all $x \in M_1$ such that $\rho_1(x, a) < \delta$.

(b) (Balls) $x \in B[a, \delta] \implies f(x) \in B[f(a), \epsilon]$

(c) $B[a, \delta] \subseteq f^{-1}(B[f(a), \epsilon])$ or $f(B[a, \delta]) \subseteq B[f(a), \epsilon]$.

3. **Proposition:** Functions are always continuous at isolated points. $f : M_1 \rightarrow M_2$ is continuous at a cluster point $a \in M_1 \iff \lim_{x \rightarrow a} f(x) = f(a)$.

4. **Sequential formulation of continuity:** A function $f : M_1 \rightarrow M_2$ is continuous at $a \in M_1 \iff \forall (x_n)_{n=1}^{\infty}$ in M_1 that converges to $a, (f(x_n))_{n=1}^{\infty}$ converges to $f(a) \in M_2$

5. **Arithmetic of continuous functions:** Let $f, g : M \rightarrow \mathbb{R}$ be continuous at $a \in M$. Then $f + g, f \cdot g, f/g$ are continuous, the last case if $g(x) \neq 0 \quad \forall x \in M$.

6. *Corollary:* Since $f(x) = x, f : \mathbb{R} \rightarrow \mathbb{R}$ is continuous at any $a \in \mathbb{R}$, thus any rational function with nonzero denominator is continuous.
7. *Theorem:* The composition of two continuous functions are continuous.
8. *Definition (Open Set):* A subset $G \subseteq M$ is an **open subset in M** if $\forall x \in G, \exists r > 0$ such that $B[x, r] \subseteq G$. In words, every point in an open set can ball up to a certain radius and the ball will be contained in the set.
9. *Definition:* **closure** of $E = \bar{E} = E \cup \{\text{cluster points of } E\}$. A set $E \in M$ is **closed in M** if $E = \bar{E}$.
 - (a) **Characterization of Closure:** $x \in \bar{E} \iff \forall r > 0, B[x, r] \cap E \neq \emptyset$.
 - (b) **Consistency of closure and closed set:** for any $E \in M, \bar{E}$ is a closed set.
10. *Punishable by Death:* Closed \neq not open, not closed \neq open. $[0, 1] \subseteq \mathbb{R}$ is neither open nor closed.
11. *Proposition:* Let $E \subseteq M$. E is closed $\iff E'$ is open in M .
12. *Propositions:*
 - Any open ball in any metric space is an open set (in M).
 - In any metric space with the discrete metric, any subset of it is an open set (in M).
 - In any metric space, \emptyset and M are both open and close in M .
 - The union of any number of open sets is open. The intersection of *finitely many* open sets is open.
 - The intersection of any family of closed sets is closed. The union of *finitely many* closed sets is closed.
13. *Theorem:* The following statements are equivalent:
 - (a) A function $f : M_1 \rightarrow M_2$ is continuous in M_2
 - (b) Whenever G is open in $M_2, f^{-1}(G)$ is open in M_1

(c) whenever F is closed in $M_1, f(F)$ is closed in M_2 .

14. *Definition:* Given $f : M_1 \rightarrow M_2$, if f is a bijection and both f, f^{-1} are continuous, then f is a **homeomorphism**. Two sets are **homeomorphic** if there is a homeomorphism from one to the other. **Homeomorphism is transitive**, since the composition of homeomorphic (continuous) functions is homeomorphic (continuous).
15. *Definition:* $E \subseteq M, E$ is **dense in M** if $\bar{E} = M$. For example, \mathbb{Q} is **dense** in (\mathbb{R}, ρ_e) .
16. Baire's Theorem:
 - (a) *Prop:* Given $f : \mathbb{R} \rightarrow \mathbb{R}$ nondecreasing, set

$$J_f(a) = \inf\{f(x) : x > a\} - \sup\{f(x) : x < a\},$$
 the “jump” function at point a of the nondecreasing function. $J_f(a) \geq 0, J_f(a) = 0 \iff f$ is continuous at a .
 - (b) *Lemma:* If a_1, \dots, a_n are points in an interval (u, v) , then $J_f(a_1) + \dots + J_f(a_n) \leq f(v) - f(u)$.
 - (c) *Defn:* A set is **countable** if it is either finite or $\exists f : \mathbb{N} \rightarrow S$ a bijection.
 - (d) *Prop:* A countable union of countable sets is countable.
 - (e) *Thm:* Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a monotone function. Then the set of discontinuities of f is countable.
 - (f) *Defn:* (Oscillation) Given $f : \mathbb{R} \rightarrow \mathbb{R}$ a bounded function. For any point $a \in \mathbb{R}$, define

$$\omega[f, a] = \inf_{\epsilon > 0} \{\sup\{f(x) : |x-a| < \epsilon\} - \inf\{f(x) : |x-a| < \epsilon\}\}$$
 - (g) *Prop:* $\omega[f, a] \geq 0 \forall a \in \mathbb{R}. f$ is continuous at $a \iff \omega[f, a] = 0$. f is discontinuous at $a \iff \omega[f, a] > 0$.
 - (h) *Cr1:* Let D be the set of discontinuities of a bounded function $f : \mathbb{R} \rightarrow \mathbb{R}$. Then D is the union of a sequence of closed sets, namely $D = \bigcup_{n=1}^{\infty} E_{1/n}$. D is called a F_σ set. (“F” stands for closed, σ stands for union).

- (i) Finally, **Baire's Theorem:** Given a F_σ (union of closed) set. Then one of the sets must contain a non-empty, open interval.
- (j) *Prop:* The set of irrationals $\mathbb{R} - \mathbb{Q}$ is not a F_σ set.
- (k) *Thm:* There is no bounded continuous function $f : \mathbb{R} \rightarrow \mathbb{R}$ that is discontinuous precisely at the irrational numbers.

6 1st C: Connectedness and Continuity

1. *Proposition:* (Open and closed sets in subsets) Let $A \subseteq U \subseteq M$. Then A is closed(open) in $U \iff \exists$ an open(closed) set $B \in M$ such that $A = B \cap U$.
2. *Definition:* Two reformulations of **connected sets**:
 - (a) A set E in M is **disconnected** if there are nonempty sets A, B so that $E = A \cup B$ and $\bar{A} \cap B = \emptyset = A \cap \bar{B}$.
 - (b) Given $\emptyset \neq C \subseteq M, C$ is **connected** $\iff \forall$ subsets in C , only C and \emptyset are both open and closed in C .
3. *Proposition (interval property):* In \mathbb{R} , a set is connected \iff it is an interval.
4. *Proposition:* The union of two overlapping connected sets is connected.
5. *Theorem:* Continuous functions preserve connectedness. If $f : M_1 \rightarrow M_2$ is continuous, then, if $C \subseteq M_1$ is connected, $f(C) \subseteq M_2$ is also connected.
6. *Intermediate Value Theorem:* Let $I = [a, b]$ be an interval in \mathbb{R} . Then $[f(a), f(b)] \subseteq f(I)$. In other words, all points between the two endpoints will be contained in the image of an interval under a continuous function.
7. *Definition:* $a, b \in A \subseteq M$. A **(continuous) path** in A from a to b is a **continuous function** $f : [0, 1] \rightarrow A$ so that $f(0) = a, f(1) = b$.
8. *Propositions:* Any open ball in (\mathbb{R}^n, ρ_2) is path connected.

9. *Proposition:* path connected \implies connected. Reverse is true for $\langle \mathbb{R}^n, \rho_2 \rangle$ but not in general.
10. *Lemma:* Suppose $a, b, c \in A \subseteq M$, and there are paths $a \rightarrow b, b \rightarrow c$. Then there exists paths $b \rightarrow a, a \rightarrow c$ in A .

7 2nd C: Total Boundedness and Completeness

Let $\langle M, \rho \rangle, \langle N, \tau \rangle$ and $\langle P, \sigma \rangle$ be metric spaces.

1. *Definition:* A subset A of M is bounded if there exist $x \in M$ and $0 < R < \infty$ so that $A \subseteq B[x, R]$.
2. *Definition:* A subset A of M is **totally bounded** if for any $\epsilon > 0$, there are finitely many points x_1, \dots, x_n so that $A \subseteq \bigcup_{i=1}^n B[x_i, \epsilon]$.
3. *Remark:* If a subset A of M is **totally bounded**, we can request the center of the bounding (open) balls to be all from A .
4. *Proposition:* Totally bounded \implies bounded.
5. *Proposition:* In $\langle \mathbb{N}^n, \rho_2 \rangle$, a subset is totally bounded \iff bounded.
6. *Theorem:* A subset A of M is totally bounded \iff every sequence in A has a Cauchy subsequence. (Lion Hunting)
7. *Definition (complete):* A subset A of M is **complete** if every Cauchy sequence in A converges to a point in A .
8. *Proposition:* Let $(x_k)_{k=1}^\infty$ be a sequence in \mathbb{R}^n . Then
 - (a) It is Cauchy wrt $\rho_2 \iff$ each coordinate is a Cauchy sequence in $\langle \mathbb{R}, \rho_e \rangle$.
 - (b) It is convergent wrt $\rho_2 \iff$ each coordinate is a convergent sequence in $\langle \mathbb{R}, \rho_e \rangle$.
9. *Proposition:* given a complete metric space M , a subset A of M is complete $\iff A$ is closed in M .
10. *Definition (diameter):* $\text{diam } A = \sup\{d(x, y) : x, y \in A\}$, the maximum distance between any 2 points in A .

11. *Nested Set Theorem:* Let M be a complete metric space. Suppose that $(A_n)_{n=1}^\infty$ is a sequence of bounded nonempty closed subsets of M so that

- (a) $A_1 \supseteq A_2 \supseteq \dots$,
- (b) $\lim_{n \rightarrow \infty} \text{diam } A_n = 0$

Then there is exactly one point in $\bigcap_{n=1}^\infty A_n$.

12. *Definition (contraction):* A function $T : M \rightarrow M$ is a **contraction** if there exists **contraction constant** $0 < C < 1$ so that $\rho(T(x), T(y)) \leq C\rho(x, y)$ for all $x, y \in M$.
13. *Banach Fixed Point Theorem (aka Contraction Mapping Principle):* M is a complete metric space. If $T : M \rightarrow M$ is a contraction with constant C . Then T has a unique fixed point, i.e., there is a unique $x \in M$ so that $T(x) = x$. Furthermore, take any $x_0 \in M$ and define $x_n = T(x_{n-1})$ for any $n \in \mathbb{N}$, then the sequence converges to the fixed point x and $\rho(x_n, x) \leq \frac{C^n}{1-C} \rho(x_0, x_1)$ for all $n \in \mathbb{N}$.
14. *Baire's Theorem:* Let M be a complete metric space. Assume that $M = \bigcup_{n=1}^\infty F_n$, where each F_n is a closed set in M . Then there exists $n_0 \in \mathbb{N}$ so that F_{n_0} contains a nonempty open ball $B[x, r]$.
15. *Definition (isometry):* Let $\langle M, \rho \rangle$ and $\langle N, \tau \rangle$ be metric spaces. A function $f : M \rightarrow N$ is an **isometry** if $\tau(f(x), f(y)) = \rho(x, y)$ for all $x, y \in M$.
16. *Theorem:* Let $\langle M, \rho \rangle$ be a metric space. There is a pair (N, i) , where $\langle N, \tau \rangle$ is a **complete** metric space, where $i : M \rightarrow N$ is an isometry, and $i(M)$ is dense in N . That is, $\overline{i(M)} = N$. (N, i) is a **completion** of M .
17. *Theorem (completion is unique up to isometry):* Let (N, i) and (P, j) be two completions of a metric space $\langle M, \rho \rangle$. Then there is a bijective isometry $\pi : N \rightarrow P$ so that $\pi \circ i = j$, and π^{-1} is an isometry too so that $\pi^{-1} \circ j = i$.
18. *Proposition:* Let $\langle M_i, \rho_i \rangle, i = 1, 2$ be metric spaces and let $f : M_1 \rightarrow M_2$ be an isometry (not necessarily onto). Let $\langle N_i, \tau_i \rangle$ be the completion of $\langle M_i, \rho_i \rangle, i = 1, 2$. There is a unique continuous function $\hat{f} : N_1 \rightarrow N_2$ that extends f , i.e., $\hat{f}(x) = f(x)$ for all $x \in M_1 \subseteq N_1$. Moreover, the extension \hat{f} is an isometry.

8 3rd C: Compactness

1. *Definition (compact):* $E \subseteq M$ is **compact** if E is both **complete** and **totally bounded**.
2. *Proposition:* In $\langle \mathbb{R}^n, \rho_2 \rangle$, a subset E is **compact** \iff **closed and bounded**
 - (a) Each $G \in \mathcal{G}$ is an open set in M .
 - (b) E is covered by $\bigcup\{G : G \in \mathcal{G}\}$.
3. *Definition (open covering):* Let E be a subset of a metric space $\langle M, \rho \rangle$. A family \mathcal{G} of sets is an **open covering** of E if
 - (a) Each $G \in \mathcal{G}$ is an open set in M .
 - (b) E is covered by $\bigcup\{G : G \in \mathcal{G}\}$.
4. *Definition (Heine-Borel property):* (Every open cover of E has a finite subcover.) A subset E of a metric space has the **Heine-Borel property** if for every open covering \mathcal{G} of E , there are finitely many $G_1, \dots, G_n \in \mathcal{G}$ so that $E \subseteq G_1 \cup \dots \cup G_n$.
5. *Theorem (multiple characterization of compactness):* Let $E \subseteq \langle M, \rho \rangle$. The following are equivalent:
 - (a) E is compact (i.e., totally bounded and complete).
 - (b) **(Sequential compactness)** Every sequence in E has a convergent subsequence (to a point in E).
 - (c) **(Bolzano-Weierstrass property, the weak-est):** Every infinite subset of E has a cluster point in E .
 - (d) **(Heine-Borel property):** Every open cover of E has a finite subcover.
6. *Lebesgue covering Lemma:* Given a compact subset E in a metric space $\langle M, \rho \rangle$ and let \mathcal{G} be an open cover of E . Then there exist a **Lebesgue's Number** $r > 0$, so that $\forall x \in E, \exists G \in \mathcal{G}$ (depending on x) so that $B[x, r] \subseteq G$.
7. *Proposition:* compact \implies closed. A closed subset in a compact set is compact.
8. *Theorem:* (Continuity preserves compactness).

9. **Extreme Value Theorem:** Given $f : M \rightarrow \mathbb{R}$ continuous, and $E \in M$ a compact set, then $\exists c, d \in E$ such that $f(c) \leq f(x) \leq f(d) \forall x \in E$.
10. **Theorem:** $f : M_1 \rightarrow M_2$ is continuous and bijective. If M_1 is compact, then f^{-1} is continuous.
11. **Definition:** $f : M_1 \rightarrow M_2$ is **uniformly continuous** on M_1 if $\forall \epsilon > 0, \exists \delta > 0$ such that we can use the same δ for any $x, y \in M_1$ where $\rho_1(x, y) < \delta \implies \rho_2(f(x), f(y)) < \epsilon$ or equivalently, $y \in B[x, \delta] \implies f(y) \in B[f(x), \epsilon]$. **uniform continuity \implies (pointwise) continuity.**
12. **Theorem:** Continuous function on a compact set is uniformly continuous.
13. **Theorem:** Given $E \subseteq M_1$, a uniformly continuous function $f : E \rightarrow M_2$ can be extended to a continuous function on \overline{E} , $\tilde{f} : \overline{E} \rightarrow M_2$, defining $\tilde{f}(x) = \lim_{n \rightarrow \infty} f(x_n)$ for some $(x_n)_{n=1}^\infty \rightarrow x$.

9 Sequences and Series of Functions

1. **Definition:** Given $(f_n)_{n=1}^\infty$ where $f_n : M_1 \rightarrow M_2$, $(f_n)_{n=1}^\infty$ **converges pointwise** to a function $f : M_1 \rightarrow M_2$ if $\forall x \in M_1, (f_n(x))_{n=1}^\infty \rightarrow f(x) \in M_2$. That is, given a sequence of function, take a $x \in M_1$, subbing it into every function in $((f_n)_{n=1}^\infty)$ yields a sequence in M_2 which converges to $f(x)$.
2. **Definition:** $(f_n)_{n=1}^\infty$ **converges uniformly** to $f : M_1 \rightarrow M_2$ if
 - (a) $\forall \epsilon > 0, \exists N \in \mathbb{N}$ such that $\rho_2(f_n(x), f(x)) < \epsilon$ for all $x \in M_1$ whenever $n \geq N$. **Given epsilon, the N can be shared for all $x \in M_1$.**
 - (b) $\iff \lim_{n \rightarrow \infty} \sup \{\rho_2(f(x), f_n(x)) : x \in M_1\} = 0$
3. **Definition:** $(f_n)_{n=1}^\infty$ is **uniformly Cauchy** on M_1 if given $\epsilon > 0, \exists N \in \mathbb{N}$ such that $\rho_2(f_m(x), f_n(x)) < \epsilon \forall x \in M_1$ whenever $m, n \geq N$.
4. **Theorem: (Uniform continuity preserves continuity)** A uniform continuous sequence of functions converges to a continuous function. The limit is

not necessarily continuous if convergence is pointwise. Note that a sequence of functions is continuity preserving \implies uniform continuity is **NOT** true.

5. **Dini's Theorem:** M is a compact metric space. A sequence of pointwise convergent functions $f_n : M \rightarrow \mathbb{R}$ and is non-decreasing (i.e., $\forall x \in M, f_1(x) \leq f_2(x) \leq \dots \leq f(x)$) $\implies (f_n)_{n=1}^\infty$ converges uniformly (to the same limit).
6. **Definition (Series of functions):** $\forall k \in \mathbb{N}$, we have $f_k : M \rightarrow \mathbb{R}$. Define the **nth partial sum** $s_n(x) = \sum_{k=1}^n f_k(x)$, the function that sums through the f_k s at a given x . The **infinite series** $\sum_{k=1}^\infty f_k(x)$ converges pointwise (uniformly) on M to a function $f : M \rightarrow \mathbb{R}$ if the sequence of partial sums, $(s_n)_{n=1}^\infty$ converges pointwise (uniformly) to f on M .
7. **Theorem: (Weierstrass M-test)** Suppose that $\sum_{k=1}^\infty M_k$ is a convergent numerical series, $M_k \geq 0$ (non-negative terms) so that $|f_k(x)| \leq M_k \forall x \in M$. Then $\sum_{k=1}^\infty f_k$ is uniformly convergent on M .
8. **Definition:** Let $a_k, b \in \mathbb{R}$ for $k \geq 0$. A series of the form $\sum_{k=1}^\infty a_k(x-b)^k$ is called the **power series**.
9. **Proposition:** The power series converges **absolutely and uniformly** on any closed interval in $(b-R, b+R)$, where $R = (\limsup_{k \rightarrow \infty} |a_k|^{1/k})^{-1}$ is the **radius of convergence**, and define $\frac{1}{\infty} = 0, \frac{1}{0} = \infty$. In $\mathbb{R} - [b-R, b+R]$, power series diverges. At the boundaries, the behavior is undefined. Furthermore, $f : (b-R, b+R) \rightarrow \mathbb{R}, f(x) = \sum_{k=1}^\infty a_k(x-b)^k$ is continuous on the domain.
10. **Definition:** Given $b \in \mathbb{R}$, a real-valued function f is **(real) analytic at b** if
 - (a) f is defined on an open interval I containing b .
 - (b) \exists power series $\sum_{k=1}^\infty a_k(x-b)^k$ such that $f(x) = \sum_{k=1}^\infty a_k(x-b)^k \forall x \in I$.

11. **Proposition:** Suppose that $f(x) = \sum_{k=1}^\infty a_k(x-b)^k$ has radius of convergence $R > 0$. Then f is analytic at any $u \in (b-R, b+R)$.
12. **Proposition: (Strong uniqueness property)** Let I be an open interval and $f, g : I \rightarrow \mathbb{R}$ be analytic functions on I . If $\exists \emptyset \neq J \subseteq I$ such that $f(x) = g(x) \forall x \in J$, then $f(x) = g(x) \forall x \in I$. In words, if two analytic functions are equal in an sub-interval, they are equal throughout.

10 The metric space $C(M_1, M_2)$

In this section, $C(M_1, M_2)$ represents the set of all continuous functions from M_1 to M_2 , where $\langle M_1, \rho_1 \rangle$ is always **compact**. The metric space is given the **uniform metric**, $\rho(f, g) = \max\{\rho_2(f(x), g(x)) : x \in M_1\}$.

1. **Proposition:** Let $(f_n)_{n=1}^\infty$ be a sequence in $C(M_1, M_2)$. Then
 - (a) $(f_n)_{n=1}^\infty$ converges to f in $C(M_1, M_2) \iff (f_n)_{n=1}^\infty$ converges uniformly to f in M_1 .
 - (b) $(f_n)_{n=1}^\infty$ is Cauchy in $C(M_1, M_2) \iff (f_n)_{n=1}^\infty$ is uniformly Cauchy in M_1 .

Therefore ρ is called the **uniform metric**.

2. **Theorem:** $C(M_1, M_2)$ is complete (with the uniform metric) $\iff M_2$ is complete.
3. **Theorem:** Let $(F_n)_{n=1}^\infty$ be a sequence of closed sets in $C[0, 1]$, the set of all continuous functions whose domain is $[0, 1] \subseteq \mathbb{R}$. If $C[0, 1] = \bigcup_{n=1}^\infty F_n$, then $\exists n_0$ such that F_{n_0} contains a nonempty open ball $B[f, r]$ for some $f \in C[0, 1]$ and some $r > 0$.
4. **Application:** An application of the theorem above, by defining $F_n = \{f : f \in C[0, 1], \exists x \in [0, 1], \forall y \in [0, 1], |y-x| < \frac{1}{n} \implies |f(x) - f(y)| \leq n|y-x|\}$. We will find a function that is continuous on $[0, 1]$ but nowhere differentiable.