MA2108S Week 7 Assignment

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1. Since **(b)** would imply the forward direction in **(a)**, I will first prove **(b)**.

   (b) We want to show that U is open and $U \subseteq E$.

   Since, by definition, $\overline{(E')}$ is closed, $U = \overline{(E')}'$ is open.

   Now we are left to show that $U \subseteq E$. By contradiction, for any $x \in U$, we assume that $x \notin E$. Then

   $$x \in E' \implies x \in \overline{(E')} = U'$$

   A contradiction, as x cannot be in both U and U'. Therefore $U = \overline{(E')}'$. $\square$

   (a) Let us first prove the forward direction.

   ($\implies$) Since U is an open set, there must exist an $r > 0$ such that $B[x, r] \subseteq U \subseteq$. $\square$

   ($\impliedby$) Now we are done with the forward direction, we shall prove that for any x,

   $$\exists r, B[x, r] \subseteq E \implies x \in U$$

   .

   We show the contrapositive, $x \notin U \implies \forall r, B[x, r] \nsubseteq E$.

   Then $x \in U'$. Since $U \subseteq E$ by the previous part, either $x \in (E - U)$ or $x \in E'$. The latter case is obvious since if that is true,

   $$x \in B[x, r] \notin E$$

   and clearly the ball is not fully contained by E.

   Otherwise, we must have $x \in E - U$. Since $x \in U' = \overline{(E')}$ which is a closed set, then either $x \in E'$ (which is impossible), or x is a cluster point of E'.

By the definition of cluster point,

$$\forall r > 0, \exists y \in B[x, r] \quad \text{such that} \quad y \in E'$$

.

Thus some part of the ball must always be in E', which gives the result. $\square$

(c) Given O is open, then by definition,

$$\forall x \in O, \exists r, B[x, r] \subseteq O \subseteq E$$

.

by 1(a), we have x must be in U as well, since every element of O must be contained by U, $O \subseteq U$. $\square$

2. We first show a lemma:

   Lemma: S, T are subsets of metric space $< M, \rho >$. If $S \subseteq \overline{T}$, then $\overline{S} \subseteq \overline{T}$.

   <u>Proof of Lemma:</u>

   Clearly, if S is closed, then $\overline{S} = \overline{T}$. Otherwise, then we just consider whether the cluster points of S are in $\overline{T}$. Suppose x is a cluster point of S, then by definition,

   $$B[x, r] \cap S \neq \emptyset$$

   and thus,

   $$B[x, r] \cap T \supseteq S \neq \emptyset$$

   x is a cluster point of T too, and thus is in $\overline{T}$. $\square$

   Now we shall proceed with the proof.

   ($\implies$) We first have $f(A) \subseteq \overline{f(A)}$. Then looking at preimages of both sets,

   $$A \subseteq f^{-1}(f(A)) \subseteq f^{-1}(\overline{f(A)})$$

   And since f is continuous, $\overline{f(A)}$ is closed $\implies f^{-1}(\overline{f(A)})$ is closed.

   Since we have $A \subseteq f^{-1}(\overline{f(A)})$, where RHS is a closed set, by the lemma, this implies

   $$\overline{A} \subseteq f^{-1}(\overline{f(A)})$$

And taking the images of both sets, we have

$$f(\overline{A}) \subseteq \overline{f(A)}$$

$\square$

($\Longleftarrow$) Given that

$$f(\overline{A}) \subseteq \overline{f(A)}$$

, we wish to show that if we have a closed set $V \subseteq M_2$, then $f^{-1}(V)$ is also closed.

By the assumption, we have

$$f(\overline{f^{-1}(V)}) \subseteq \overline{f(f^{-1}(V))}$$

and since f need not be injective,

$$f(\overline{f^{-1}(V)}) \subseteq \overline{f(f^{-1}(V))} \subseteq \overline{V} = V$$

since V is defined as closed.

We then have

$$f(\overline{f^{-1}(V)}) \subseteq V$$

which, considering their preimages,

$$\overline{f^{-1}(V)} \subseteq f^{-1}(V)$$

By definition, $\overline{f^{-1}(V)} = (\{\text{cluster points of V}\} \cup f^{-1}(V)) \subseteq f^{-1}(V)$, which implies $f^{-1}(V)$ contains all its cluster points. Thus $f^{-1}(V)$ is closed. $\square$

3. Set a such that $a > f(x) + \epsilon$ for some $\epsilon > 0$ and $x \in f^{-1}(-\infty, a)$ naturally.

   Then since $f^{-1}(-\infty, a)$ is open,

   $$\exists r > 0 \quad \text{such that} \quad B[x, r] \subseteq f^{-1}(-\infty, a)$$

   Since it is known that $(x_n)_{n=1}^{\infty}$ converges, then given r as defined above, there is an $N \in \mathbb{N}$ such that for all $n \geq N$,

   $$\rho(x_n, x) < r \quad \forall n \geq \mathbb{N}$$

   .

Since we now have

$$x_N, x_{N+1}, \dots \in B[x, r] \subseteq f^{-1}(-\infty, a)$$

and since f is upper-semicontinuous,

$$f(x_N), f(x_{N+1}), \dots < a$$

Consider $M_k = sup\{f(x_N), f(x_{N+1}), \dots\} < a$. Thus,

$$\lim_{k\to\infty} M_k < a \implies \limsup_{k\to\infty} f(x_n) < a = f(x) + \epsilon$$

Now, for every epsilon given, we can find an a, and thus N such that the previous statement is true. Given by Exercise 2.2 Question 2 in *Goldberg*,

$$\limsup_{k\to\infty} f(x_n) < f(x) + \epsilon \implies \limsup_{k\to\infty} f(x_n) < f(x)$$

□