MA3110 Homework 4

Tan Yee Jian (A0190190L)

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Question 1

Part 1(i)

Proof.

$$f(x) = \lim_{n\to\infty} f_n(x) = \begin{cases} \lim_{n\to\infty} x^n \cdot \lim_{n\to\infty} \cos(\frac{x}{n}) = 0 \cdot 1 = 0 & \text{if } x \in [0,1) \\ \lim_{n\to\infty} 1 \cdot \cos(\frac{1}{n}) = 1 & \text{if } x = 1. \end{cases}$$

□

Part 1(ii)

Proof. For $x \in [0,a]$ for any $0 < a < 1$,

$$\begin{aligned} |f_n(x) - f(x)| &= |x^n \cos(\frac{x}{n})| \\ &\leq a^n(1) && x \leq a. \\ \therefore \|f_n - f\|_{[0,a]} &\leq a^n \end{aligned}$$

When $n \to \infty$,

$$\|f_n - f\|_{[0,a]} = a^n \to 0 \quad (\text{since } 0 < a < 1).$$

Therefore f_n converges to f on $[0,a]$ uniformly for any $0 < a < 1$. □

Part 1(iii)

Proof. No. f_n are continuous on $[0,1]$ but f is discontinuous at $x = 1$ ($\lim_{x\to 1^-} f(x) = 0 \neq f(1) = 1$). □

Part 1(iv)

Proof.

$$\lim_{n\to\infty} \int_{1/2}^{1/3} x^n \cos(\frac{x}{n}) dx = \int_{1/2}^{1/3} \lim_{n\to\infty} f_n(x) dx = 0.$$

□

Question 2

Part 2(i)

Proof.

$$S_n(x) = \begin{cases} x & x = \frac{1}{k},\ k = 1, \dots, n \\ 0 & x \in [0,1] \setminus \{1, \frac{1}{2}, \dots, \frac{1}{n}\}. \end{cases}$$

Therefore,

$$f(x) = \lim_{n\to\infty} S_n(x) = \begin{cases} x & x = k,\ k \in \mathbb{N} \\ 0 & x \in [0,1] \setminus \{1/k \in \mathbb{R} | k \in \mathbb{N}\}. \end{cases}$$

□

Part 2(ii)

Proof.

$$\begin{aligned} |f(x) - S_n(x)| &= \begin{cases} x & x = \frac{1}{n+1}, \frac{1}{n+2}, \dots \\ 0 & x \in [0,1] \setminus \{\frac{1}{n+1}, \frac{1}{n+2}, \dots\} \end{cases} \\ \therefore \|f - S_n\|_{[0,1]} &= \sup\{0, \frac{1}{n+1}, \frac{1}{n+2}, \dots\} \\ &= \frac{1}{n+1} \\ \therefore \lim_{n\to\infty} \|f - S_n\| &= \lim_{n\to\infty} \frac{1}{n+1} = 0 \end{aligned}$$

Therefore $\sum_{n=1}^{\infty} f_n$ converges to f uniformly on $[0,1]$. □

Part 2(iii)

Proof.

$$\sum_{n=1}^{\infty} \|f_n\|_{[0,1]} = \sum_{n=1}^{\infty} \frac{1}{n}$$

Is the harmonic series (p-series with $p = 1$), hence diverges. □

Question 3

Part 3(i)

Proof. Let r be given. We use the Weierstrass M-test. Denote $f_n(x) = \frac{x}{n^2}\cos(\frac{x^2}{2n})$. Then

$$|f_n(x)| = |\frac{x}{n^2}\cos(\frac{x^2}{2n})| \leq \frac{r}{n^2} \quad \text{since } |\cos(x)| \leq 1, x \leq r.$$

Let

$$\|f_n\|_{[-r,r]} \leq \frac{r}{n^2} = M_n.$$

Then

$$\sum_{n=1}^{\infty} M_n = r\sum_{n=1}^{\infty} \frac{1}{n^2}$$

converges as it is a constant multiple of the p-series with $p = 2 > 1$. Therefore $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on $[-r, r]$ for each $r \in \mathbb{R}$ by the Weierstrass M-test. □

Part 3(ii)

Denote $f_n(x) = \frac{1}{n}\cos(\frac{x^2}{2n})$. Then

$$f_n'(x) = \frac{x}{n^2}\sin(\frac{x^2}{2n}).$$

Then $\sum_{n=1}^{\infty} f_n'(x)$ is the series in Question 3 Part (i), which is proved to converge uniformly on $[-r, r]$. We also have

$$\sum_{n=1}^{\infty} f_n(0) = \sum_{n=1}^{\infty} 0 = 0$$

that $\sum_{n=1}^{\infty} f_n(0)$ is convergent, thus by Theorem 8.3.5, $\sum_{n=1}^{\infty} f_n(x)$ is uniformly convergent on $[-r, r]$.

Part 3(iii)

Proof. Continuing from the previous part, we know that Theorem 8.3.5 gives that the series in the previous part is uniformly convergent on $[-r, r]$ for $r \in \mathbb{R}$. While we cannot say that that the series is uniformly convergent on $\mathbb{R}$, however the same theorem states that

$$f'(x) = \sum_{n=1}^{\infty} f_n'(x) = \sum_{n=1}^{\infty} \frac{x}{n^2} \cos(\frac{x^2}{2n})$$

on $[-r, r]$ for all $r \in \mathbb{R}$. Since differentiability is defined pointwise, we can combine the intervals and conclude that f is differentiable on $\mathbb{R}$.

In particular,

$$f'(0) = \sum_{n=1}^{\infty} \frac{0}{n^2} \cos(0) = 0.$$

$\square$