MA3110 Chap 6 : Differentiable Fns.

Date No.

Defn function f has limit L at point $x=a$, if
$\forall \varepsilon>0, \exists \delta=\delta(\varepsilon)>0$ s.t.
$0<|x-a|<\delta \Rightarrow |f(x)-L|<\varepsilon$.
and write $\lim_{x\to a} f(x)=L$.

Prop $\lim_{x\to a} f(x)=L \iff \forall$ sequence x_n in domain of f s.t.
$\lim_{n\to\infty} x_n = x$, and $x_n \neq a \ \forall n$, $\Rightarrow f(x_n)\to L$.
conv. to x — cannot be a.

↓ impt section.

Cor. ① if $\exists (x_n)$ s.t. $x_n\neq a \ \forall n$, $x_n\to a$ but $f(x_n)\neq L$, then $f(x)\not\to L$.
② $f(x_n)$ diverges, then " .
(divergent criteria.)

Ex. 1. show if $L>0$, $\lim_{x\to a} f(x)=L$ exists, then $\exists \delta>0$ s.t.
$0<|x-a|<\delta \Rightarrow f(x)>0$.

<u>Pf</u>: Let $\varepsilon = L/2$. then $\exists \delta$ s.t. $0<|x-a|<\delta \Rightarrow |f(x)-L|<\varepsilon = L/2$.
thus $|f(x)-L| < L/2$
$-\frac{L}{2} < f(x)-L < L/2$.
$L/2 < f(x) < 3L/2$

2. $L\neq 0$, $\lim_{x\to a} f(x)=L$, show $\exists \delta$ s.t. $0<|x-a|<\delta \Rightarrow f(x)\neq 0$.
<u>Pf</u>: similar as above $L/2$ should work. multiples of L should ~~not be 0~~. have the same sign thus not containing 0.

Defn Limit defns:

$x\to a$	$(\lvert x-a\rvert<\delta)$	$L\in\mathbb{R}$	$(\lvert f(x)-L\rvert<\varepsilon)$
$x\to a^-$	$(a-\delta<x<a)$	∞	$(x>M)$
$x\to a^+$	$(a<x<a+\delta)$	$-\infty$	$(x<M)$
$x\to\infty$	$(x>M)$		
$x\to-\infty$	$(x<M)$		

Defn: f is differentiable at $x=a$ if f is defined in some $\boxed{\text{open intvl}}$ containing a and the limit $f'(a)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$ exists. $f'(a)$ is then the derv. of f at a.
??? ← such that we have "space" for δ.

Date No.

Defn (diff. in open. intv) if f is diff at x $\forall x \in (a,b)$.
(diff in closed intv) $[a,b]$ if f diff at (a,b) and $\lim_{x\to b^-} \frac{f(x)-f(b)}{x-b}$ and $\lim_{x\to a^+} \frac{f(x)-f(a)}{x-a}$.
and <u>define</u> $f'(a) = \lim_{x\to a^+} \frac{f(x)-f(a)}{x-a}$ and v.v. for $f'(b)$.

for intervals like $[a,\infty)$, take differentiability at (a,∞) and at a.

Defn Let f be diff$^{\text{able}}$ on I. then the derivative of f is the function
$f' : I \to \mathbb{R}$, where $x \mapsto f'(x)$, $x \in I$.

Defn f is continuously differentiable on intv I if
1. f is diff$^{\text{able}}$ on I.
2. f' is continuous.

write $f \in C^1(I)$. (we have $C^2(I), \dots, C^\infty(I)$)

Eg. 1. $\frac{d}{dx}(c) = 0$ (the converse is true, i.e. $\frac{df}{dx} = 0 \Rightarrow f(x) = c \ \forall x$ but need more prep).

<u>Pf</u>: $\frac{d}{dx}c = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \frac{c-c}{h} = 0$.

2. $\frac{d}{dx}(x^n) = nx^{n-1}$ (use binomial exp). where $n \in \mathbb{N}$

Thm If f is diff$^{\text{able}}$ at a, then f is cont. at a.

<u>Pf</u>: $\lim_{x\to a} f(x) = \lim_{x\to a} f(x) - f(a) + f(a)$
$= \lim_{x\to a} \frac{f(x)-f(a)}{x-a}(x-a) + f(a)$.
$= f'(a)\cdot 0 + f(a) = f(a)$. $\square$

Remark: ① Converse is false. counterexample: $f(x) = |x|$. cont at 0 but not diff at 0.
To show non-diff$^{\text{bility}}$: ① sequential criteria. ← all elem are $1, -1$ alternating.
② LH deriv $= -1$, RH $= 1$.
② finite/countably many non-diff pts: sawtooth.
cont. everywhere, diff nowhere: weierstrass fn $f(x) = \sum_{n=0}^{\infty} a^n \cos(b^n \pi x)$
$0<a<1$, b +ve odd, $ab > 1 + \frac{3}{2}\pi$

MA3110 Chap 6 (2).

Date No.

Thm. (a) $\frac{d}{dx} f(x) \pm g(x)\Big|_{x=a} = f'(a) \pm g'(a)$

(b) $\frac{d}{dx} f(x)g(x)\Big|_{x=a} = f'(a)g(a) + g'(a)f(a)$.

(c) $\frac{d}{dx} \frac{f(x)}{g(x)}\Big|_{x=a} = \frac{f'(a)g(a) - f(a)g'(a)}{(g(a))^2}$.

Remark: the $\frac{d}{dx}x^n = nx^{n-1}$ rule can be extended to $\mathbb{Z}\setminus\{0\}$ with quotient rule.

Thm 6.23 (equiv. defn of diff^able) Caratheodory's Thm.

I intv, $f: I \to \mathbb{R}$, $c \in I$. $f'(c)$ exists $\iff \exists\, \varphi: I \to \mathbb{R}$ s.t. φ is cont at c,

$f(x) - f(c) = \varphi(x)(x-c) \quad \forall x \in I$.

In this case, $f'(c) = \varphi(c)$.

Pf: ($\Rightarrow$) $f'(c)$ exists. I claim

$\lim_{x\to c}\varphi(x) = \lim_{x\to c}\frac{f(x)-f(c)}{x-c}$ (defn of $\varphi(x)$)

$= f'(c)$ exists. φ is cont at c since $\varphi(x) = \frac{f(x)-f(c)}{x-c} \xrightarrow{\text{diff'able}} f'(c)$

($\Leftarrow$) suppose φ cont. at $x=c$. Then

$\lim_{x\to c}\varphi(x) = \varphi(c)$ ~~then~~ LHS: $\lim_{x\to c}\frac{f(x)-f(c)}{x-c}$ thus exists ($=\varphi(c)$)

Revision ① f is diff at $x=a$ if 1. f is defined in $(a-\delta, a+\delta)$ for some $\delta>0$

2. the limit $f'(a) = \lim_{x\to a}\frac{f(x)-f(a)}{x-a}$ exists

$\therefore f'(a)$ is the deriv. of ~~a~~ f at a.

② ~~"~~ on I if $f'(a)$ exists $\forall a \in I$.

$x \mapsto f'(x)$ for $x \in I$ is called the deriv. of f (on I). (a fn).

eg. $\frac{d}{dx}(c) = 0$ ← zero fn | $\frac{d}{dx}(x^n) = nx^{n-1}$ $(n \in \mathbb{Z}\setminus\{0\})$

③ Caratheodory: $f: I \to \mathbb{R}$ diff at a ~~[illegible]~~ cts at $a \in I$.

$\iff \exists\, \varphi: I \to \mathbb{R}$ s.t. $f(x) - f(a) = \varphi(x)(x-a)$.

Defn (composite fn) $f: I \to J$, $g: J \to \mathbb{R}$, $f \circ g: I \to \mathbb{R}$,
$(f \circ g)(x) = f(g(x))$.

Thm (chain rule) Let I, J be $\subseteq \mathbb{R}$, and $f(J) \subseteq I$. $f: J \to \mathbb{R}$, $g: I \to \mathbb{R}$.
If $a \in J$, f is diff. at a, g is diff at $f(a)$, then
$g \circ f$ is diff at a and $(g \circ f)'(x)|_{x=a} = g'(f(a)) f'(a)$

Pf: write $b = f(a)$. given $f'(a)$ and $g'(b)$ exist.
WTS $h'(a) = (g \circ f)'(a) = g'(b) f'(a)$.
By caratheodory, $\exists \varphi: J \to \mathbb{R}$, $\psi: I \to \mathbb{R}$, s.t.
① $f(x) - f(a) = \varphi(x)(x-a)$ $\forall x \in J$, φ cts at a, $\varphi(a) = f'(a)$.
② $g(y) - g(b) = \psi(y)(y-b)$ $\forall y \in I$, ψ cts at b, $\psi(b) = g'(b)$.

Since $f(J) \subseteq I$, then if we consider the $x \in J$, $f(x) \in I$.
②: $g(f(x)) - g(b) = \psi(f(x))(f(x) - f(a))$
sub ①: $g(f(x)) - g(f(a)) = \psi(f(x)) \varphi(x)(x-a)$
Let $x \mapsto \psi(f(x))\varphi(x)$ be α. then $= \alpha(x)(x-a)$.

$\therefore$ $h(x) - h(a) = \alpha(x)(x-a)$ where $h = g \circ f$.
if we verify α is cts at a, we are done and $h'(x)|_{x=a} = \alpha(a)$ (by cara.)
check that $\alpha = (\psi \circ f)(\varphi)$, all are cts at a.
and $h'(a) = \alpha(a) = \psi(f(a))\varphi(a) = g'(f(a)) f'(a)$. □

Eg. Let $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^2 \sin(1/x)$ if $x \neq 0$ else 0. where is f differentiable?
Pf: for $x \neq 0$, $f'(x) = x^2 \frac{d}{dx}(\sin\frac{1}{x}) + 2x \sin(\frac{1}{x})$
$= -\cos\frac{1}{x} + 2x\sin(\frac{1}{x})$. (diff^{able} everywhere but 0).
for $x = 0$, we have to use defn.
$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x} = \lim_{x \to 0} \frac{1}{x}(x^2 \sin(\frac{1}{x})) = \lim_{x \to 0} x \sin(\frac{1}{x})$.
note that $0 \leq |x \sin\frac{1}{x}| \leq |x|$ and take limits $x \to 0$. thus $= 0$. diff^{able}.

$\therefore f$ is diff^{able} on $\mathbb{R}$. But is it cts. diff^{able}?
$\iff$ Is $f \in C^1(\mathbb{R})$ $\iff$ is f' cts on $\mathbb{R}$?

MA3110 Chap 6. (3).

Eg (cont'd) if $f'(x) = 2x\sin\frac{1}{x} - \cos\frac{1}{x}$ if $x \neq 0$ else 0. cts?

clearly f' is cts on $x \neq 0$. (x, $\sin x$, $\cos x$ are cts).

at $x=0$, is it cts? check $\lim_{x\to 0} f'(x) = f'(0) = 0$?

(Recall sequential & divergent criteria for showing discont.)

choose $x_n = \frac{1}{2n\pi}$, clearly $x_n \to 0$.

but $f'(x_n) = 2x(\sin 2n\pi) - \cos(2n\pi) = -1$. constant $\neq 0$.

$\therefore \lim_{x\to 0} f'(x) \neq f'(0)$ (actually lim does not exist.)

Defn Suppose (i) I is an intv (ii) $f: I \to \mathbb{R}$ is strictly monotone (thus 1-1) and cont. on I.

Then (a) $J = f(I)$ is also an intv (by cts (?) and Int. Val. Thm).

(b). $g = f^{-1}: J \to \mathbb{R}$ exists and $g(f(x)) = x$ $\forall x \in I$.

Thm Cts Inv. Thm

Qn if f is cts, f^{-1} is cts. If f is diff, is f^{-1} diff?

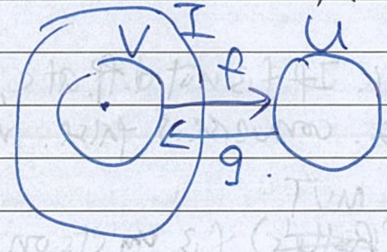
Thm (Inv. Fn Thm) f is diff. at $c \in I$ and $f'(c) \neq 0$, then f^{-1} is diff at $d := f(c)$ and $g'(d) = \frac{1}{f'(c)}$.

Pf: if f^{-1} is diff at $f(c)$ then we are done: just apply chain rule and $g(f(x)) = x \Rightarrow g'(f(c))f'(c) = 1$ (if $g = f^{-1}$ diff at $f(c)$)

we have $f'(c)$ exists $\neq 0$. By cara. thm: $\exists \varphi: I \to \mathbb{R}$, s.t.

① $f(x) - f(c) = \varphi(x)(x-c)$. $\forall x \in I$. moreover φ is cts at c, $\varphi(c) = f'(c)$

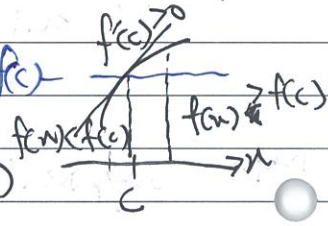
② by cty. of f at c, since $\varphi(x) = f'(c) \neq 0$, $\exists \delta$ s.t. $\forall |x-c| < \delta$, $\varphi(x) \neq 0$.



Defn
(a) ~~abs max~~ abs max: $f(x_0)$ is the max of f on I if $\forall x \in I$, $f(x_0) \geq f(x)$.
(b) rel. max: 〃 if $\exists \delta$ s.t. $\forall x \in B[x_0, \delta]$ s.t. $f(x_0) \geq f(x)$
(c) abs. min, rel. min are defined similarly.
(d) ~~rel~~ absolute extremum: abs min/max.

~~Lem 6.3.1~~ Remark: abs. ext. might not be rel. max/min if it occurs at the endpts.

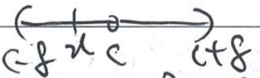
Lem 6.31 Let $f: (a,b) \to \mathbb{R}$ and $f'(c)$ exists for some $c \in (a,b)$.



If $f'(c) > 0$, $\Rightarrow \exists \delta > 0$ s.t. $f(x) < f(c)$ $\forall x \in (c-\delta, c)$
$f(x) > f(c)$ $\forall x \in (c, c+\delta)$
($f'(c)>0 \Rightarrow$ a small neighborhood: $f(x)<f(c)$ less than $(c-\delta, c)$, more than $f(c)$)

Note: f is not necessarily inc. on $(c-\delta, c+\delta)$. See Tut 2.
i.e, on $(c-\delta, c)$ not nec. inc (not implied).

Pf: (i) since $f'(c) = \lim_{x \to c} \frac{f(x)-f(c)}{x-c} > 0$ (let it be $g(x)$), $\exists \delta > 0$ s.t.
$0 < |x-c| < \delta \Rightarrow g(x) > 0$.



$x \in (c-\delta, c) \Rightarrow x < c \Rightarrow x-c<0$ but $g(x) = \frac{f(x)-f(c)}{x-c} > 0$.
$\Rightarrow f(x)-f(c) < 0$.
$x \in (c, c+\delta) \Rightarrow x > c \Rightarrow x-c>0$ but $g(x) = \frac{f(x)-f(c)}{x-c} > 0$
$\Rightarrow f(x) - f(c) > 0$. □

Thm Interior Extremum Thm: Suppose c is an interior pt of I, $f: I \to \mathbb{R}$ diff. at c. Then f has rel. ext. at c $\Rightarrow f'(c) = 0$. (converse is not true).
(interior extremum must have $f'(c)=0$ if diff. at c).
Pf: If f has rel. ext at c, then 6.31 says $f'(c)>0$ or $f'(c)<0$ cannot be the case (its left and right gives counterex. of rel ext). Thus $f'(c)=0$ □

Note 1. If f is not diff. at c, it is possible too. eg, $|x|$ has min at $x=0$.
2. converse is false. x^3 at $x=0$ has $f'(0)=0$ but neither.

Thm (MVT, ~~Rolle's~~) f is cts on $[a,b]$ and diff. on (a,b), then $\exists c \in (a,b)$ s.t.
$$f'(c) = \frac{f(b)-f(a)}{b-a}.$$
(Rolle's) if $f(a)=f(b)$, then $f'(c)=0$.

MA3110 Chap 6 (4). Date No.

<u>Pf</u> (Rolle) WTS $\exists c \in (a,b)$ s.t. $f'(c)=0$

Case 1: $f(x)=f(a)=f(b)\ \forall x\in[a,b]$. Then $\forall x\in(a,b)$, $x=c$ is a soln.

Case 2: Extreme-value thm:

$\exists\ f(x_1)\le f(x)\le f(x_2)\ \forall x\in[a,b]$.

One of them is not a or b (otherwise $f(x)$ is constant). Then the pt is not an endpt, and Interior Ext. thm gives $f'(c)=0$. where $c=x_1$ or $c=x_2$ where $f(c)\neq f(a)$ and $f(c)\neq f(b)$.

(MVT) Define $g:[a,b]\to\mathbb{R}$ by

$$g(x)=f(x)-f(a)-\frac{f(b)-f(a)}{b-a}(x-a).$$

Then 1. g is cts. on $[a,b]$ and diff. on (a,b).

2. $g(a)=g(b)=0$.

and by Rolle's thm, $g'(c)=f'(c)-\frac{f(b)-f(a)}{b-a}\cancel{(x-a)}=0$.

$$\Rightarrow f'(c)=\frac{f(b)-f(a)}{b-a}.$$

Eg. Use MVT to show $e^x\ge 1+x\ \forall x\in\mathbb{R}$.

<u>Pf</u>: Case 1: $x>0$

Then MVT: $\exists c$ s.t. $f'(c)=\frac{f(x)-f(0)}{x-0}$

$$\therefore e^c=\frac{e^x-1}{x}\Rightarrow e^x=1+e^c x>1+x.$$

Case 2: $x<0$. MVT on $[x,0]$.

Then MVT: similarly $e^x=1+e^c x$ but $e^c<1$.

$x e^c > x$.

$\therefore e^x>1+x$.

Case 3: $x=0$. $\therefore e^0=1+0=1$

Thm 6.32 If f is cts on $[a,b]$, diff. on (a,b), $f'(x)=0\ \forall x\in(a,b)\Rightarrow f$ is const. on $[a,b]$.

<u>Pf</u>: $\forall a<x\le b$ (ie $x\in(a,b]$), MVT: $\exists c$ s.t.

$$f'(c)=\frac{f(x)-f(a)}{x-a}=0\Rightarrow f(x)=f(a)\ \forall a<x\le b.\quad\square$$

Date No.

Thm 6.33 Let f be diff on (a,b).

(i) $f'(x) \geq 0 \ \forall x \in (a,b) \Rightarrow f \uparrow$ on (a,b).

(ii) $\leq 0 \Rightarrow \downarrow$.

Pf: we show (i): Let $a < x_1 < x_2 < b$. WTS $f(x_1) < f(x_2)$.
By MVT, $\exists f'(c) = \frac{f(x_2)-f(x_1)}{x_2-x_1} \geq 0$. Bottom $> 0 \Rightarrow f(x_2) \geq f(x_1)$. □.

Ex. 1. Show that if $f'(x) > 0 \ \forall x \in (a,b) \Rightarrow f$ is strictly increasing on (a,b).
The converse is false.

Pf: Let $c, d \in (a,b)$ be given. Let $x = \frac{c+d}{2} \in (a,b)$, $f'(x) > 0$.
Then $f(c) < f(x) < f(d)$ by lemma 6.31.
Converse is false: x^3 is strictly increasing on $\mathbb{R}$, but $f'(0) = 0$.

Thm. First Deriv Test: f is cts on $[a,b]$, diff on (a,b) except possibly at $c \in (a,b)$ ($f'(c)$ need not exist).

(i) If $\exists (c-\delta, c+\delta) \subseteq I$ s.t. $f'(x) \geq 0$ for $x \in (c-\delta, c)$ and
$f'(x) \leq 0$ for $x \in (c, c+\delta)$
then $f(c) \geq x \ \forall x \in (c-\delta, c+\delta)$ and $f(c)$ is hence a rel. max.

Pf: Two steps. Take a $x \in (c-\delta, c)$ and $(c, c+\delta)$, MVT on $[x, c]$ and $[c, x]$ respectively. The $f'(x)$ values force f to be bigger than its friends.

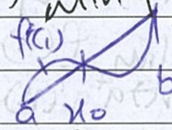
Multiple Deriv. $f^{(n)}(x)$, $C^1(I)$, $C^n(I), \dots, C^\infty(I)$.

Pf: (cont'd) on $[x, c]$, $\exists x_0$ s.t. $f'(x_0) = \frac{f(c)-f(x)}{c-x}$.
$\therefore f(c) - f(x) = f'(x_0)(c-x) \geq 0$.
$[c, x]$, $\exists x_1$ s.t. $f'(x_1) = \frac{f(x)-f(c)}{x-c}$
$\therefore f(x) - f(c) \leq 0$ □

MA3110 Chap 6 (5)

Date No.

Ex. $f:[a,b]\to\mathbb{R}$, f cts on $[a,b]$, f'' exists on (a,b), line joining $(a,f(a))$ and $(b,f(b))$ intersect at $(x_0,f(x_0))$ with f, $a<x_0<b$.
Show that $\exists\, c\in(a,b)$ s.t. $f''(c)=0$.



$f'(c_1)=f'(c_2)$
rolle's on f' gives c_3 s.t. $f''(c_3)=0$.

Thm (Second deriv test) f defined on I, f' exist on I. Suppose c is an int. pt of I s.t. $f'(c)=0$ & $f''(c)$ exists.
Then $f''(c)>0 \Rightarrow$ rel min and $f''(c)<0\Rightarrow$ rel max

<u>Pf</u>: Directly apply Lem 6.3.1 to $f''(c)$:
$f''(c)>0 \Rightarrow \exists\,\delta$, $x\in(c-\delta,c)$ then $f'(x)<f'(c)=0$.
$x\in(c,c+\delta)$ then $f'(x)>f'(c)=0$.
and we apply first deriv. test. $\Rightarrow$ $f(c)$ is a rel. min. v.v. $\square$.

Rem $f''(c)=0$ then ~~conclusion~~ inconclusive (as to whether c is an ext. (relative)).
eg: x^3 at 0 : neither, x^4 at 0 : min, $-x^4$ at 0 : max.

Thm (cauchy MVT) f,g cts on $[a,b]$, diff on (a,b), and assume that $g'(x)\neq 0\ \forall x\in(a,b)$.
Then $\exists\, c\in(a,b)$ s.t. $\dfrac{f(b)-f(a)}{g(b)-g(a)}=\dfrac{f'(c)}{g'(c)}$.

<u>Pf</u>: <u>Claim</u>: $g(b)-g(a)\neq 0$. otherwise, $g(b)=g(a)$, by Rolle's thm, $g'(x)=0$ at some $c\in(a,b)$. $\to\leftarrow$.

<u>Idea</u>: use rolles on the scalled fn. <u>cannot use MVT</u> because both c might be different.

$$h(x):=\frac{f(b)-f(a)}{g(b)-g(a)}(g(x)-g(a))-(f(x)-f(a))$$

Then $h(a)=0=h(b)$. By Rolle's, $\exists\, c\in(a,b)$ s.t. $h'(c)=0$.

$\therefore h'(c)=\dfrac{f(b)-f(a)}{g(b)-g(a)}g'(c)-f'(c)=0$. for some $c\in(a,b)$.

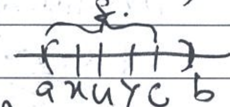
Thm (L'Hopital's Rule: 0/0 case) f, g diff. on (a,b), $g(x) \neq 0$, $g'(x) \neq 0$ $\forall x \in (a,b)$. If $\lim_{x\to a^+} f(x) = \lim_{x \to a^+} g(x) = 0$.

(i) if $\lim_{x\to a^+} \frac{f'(x)}{g'(x)} = L \in \mathbb{R}$, $\Rightarrow \lim_{x\to a^+} \frac{f(x)}{g(x)} = L$

(ii) if $\lim_{x\to a^+} \frac{f'(x)}{g'(x)} = \pm\infty \Rightarrow \lim_{x\to a^+} \frac{f(x)}{g(x)} = \pm\infty$.

Pf: (i) since $\lim_{x\to a^+} \frac{f'(x)}{g'(x)} = L$, given $\frac{\varepsilon}{2} > 0$, $\exists \delta$ s.t. $a + \delta = c$,

$\forall x \in (a,c) \Rightarrow L - \frac{\varepsilon}{2} < \frac{f'(x)}{g'(x)} < L + \frac{\varepsilon}{2}$.



Cauchy MVT: $\exists u \in (x,y)$, $\frac{f'(u)}{g'(u)} = \frac{f(y)-f(x)}{g(y)-g(x)}$.

clearly $u \in (x,y) \subseteq (a,c)$. Now then

$$L - \frac{\varepsilon}{2} < \frac{f(y)-f(x)}{g(y)-g(x)} < L + \frac{\varepsilon}{2}.$$

Take $\lim_{x\to a^+}$, noting that signs gain equality (eg. $\frac{1}{n} > 0$ but $\lim_{n\to\infty} \frac{1}{n} = 0$).

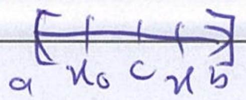
$$L - \frac{\varepsilon}{2} \leq \lim_{x\to a^+} \frac{f(y)-f(x)}{g(y)-g(x)} \leq L + \frac{\varepsilon}{2} \quad \text{but } f(x), g(x) \Longrightarrow 0.$$

$$\therefore L - \varepsilon < L + \frac{\varepsilon}{2} \leq \cancel{\lim_{x\to a^+}} \frac{f(y)}{g(y)} \leq L + \frac{\varepsilon}{2} < L + \varepsilon \quad (\text{for } y \in (a,c))$$

Ie, given any ε, we can use the same δ and have $\left|\frac{f(y)}{g(y)} - L\right| < \varepsilon$ $\forall y \in (a,c)$

$\Rightarrow \lim_{x\to a^+} \frac{f(x)}{g(x)} = L$.

MA3110 Chap 6 (6).



Date No.

Thm (Taylor's) f, $f \in C^n[a,b]$ and $f^{(n+1)}$ exists on (a,b).
If $x_0 \in [a,b]$ then $\forall x \in [a,b]$, $\exists\, c \in (x, x_0)$ s.t.

$$f(x) = \underbrace{f(x_0) + f'(x_0)(x-x_0) + f''(x_0)/2!\,(x-x_0)^2 + \dots + f^{(n)}(x_0)/n!\,(x-x_0)^n}_{\text{polynomial.}} + \underbrace{f^{(n+1)}(c)/(n+1)!\,(x-x_0)^{n+1}}_{\text{error term}}$$

Pf: Let $x \in [a,b]$ be given. we show. (and assume $x \neq x_0$)

$$f(x) = f(x_0) + \dots + f^{(n)}(x_0)/n!\,(x-x_0)^n + M(x-x_0)^{n+1}.$$

we show $M = \frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}$ for some $c \in (x, x_0)$.

Define $F: [a,b] \to \mathbb{R}$ which we will use Rolle's on, by

$$F(t) = f(t) + f'(t)(x-t) + \dots + \frac{f^{(n)}(t)}{n!}(x-t)^n + M(x-t)^{n+1}. \quad (*)$$

F is cts on $[a,b]$ (since $f \in C^n([a,b])$), diff on (a,b).

$F(x) = f(x) + f'(t)(0) + \dots + M(0) = f(x)$

$F(x_0) = f(x_0) + \dots + M(x-x_0)^{n+1} = f(x)$.

Restrict F to $[x, x_0]$, and use rolle's:

$\exists\, c \in [x, x_0]$ $F'(c) = 0$.

$$\therefore F'(c) = f'(t) + f''(t)(x-t) + f'(t)(-1) + f'''(t)(x-t)\left(\tfrac{1}{2!}\right) - f''(t)(x-t)^1 + \dots$$

$$= \frac{f^{(n+1)}(c)}{n!}(x-c)^n - M(n+1)(x-c)^n.$$

$$M = \frac{f^{(n+1)}(c)}{(n+1)!}$$

Rem. 1. $n=0$, then Taylor's Thm:

$$f(x) = f(x_0) + f'(c)(x-x_0).$$

$$\therefore f'(c) = \frac{f(x) - f(x_0)}{x - x_0}. \text{ is the MVT.}$$

2. Write $P_n(x) = f(x_0) + \dots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$. is the Taylor polynomial.
and $P_n^{(j)}(x_0) = f^{(j)}(x_0)$ for $j = 0, 1, 2, \dots, n$. (gives the deriv. at x_0 for $1 \dots n$)

Lagrange form: $R_n(x) := f(x) - P_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}$.

Date No.

Eg. 1. Show that $\cos x \geq 1 - \frac{1}{2}x^2 \quad \forall x \in \mathbb{R}$.

Let $f(x) = \cos x$, $x \in \mathbb{R}$, let $x_0 = 0$. Then by Taylor's Thm,

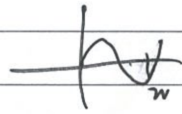
$$f(x) = f(x_0) + f'(x_0)(x - x_0) + (f''(0)/2)(x - x_0)^2 + R_2(x).$$

$$= 1 - \frac{1}{2}(x^2) + R_2(x)$$

where $R_2(x) = \frac{f'''(c)}{3!}(x)^3 = \frac{1}{6}\sin(c)\, x^3$ for some c between $0, x$.

We can just show $\frac{1}{6}\sin(c)x^3 \geq 0 \;\; \forall x \in \mathbb{R}$.

cases: $0 \leq x \leq \pi$, $-\pi \leq x \leq 0$, $|x| \geq \pi$.



case 1: $0 \leq x \leq \pi$. Then $x^3 \geq 0$, $\sin(c) \geq 0$.

case 2: $-\pi \leq x \leq 0$. Then $x^3 \leq 0$, $\sin(c) \leq 0$.

case 3: $|x| \geq \pi$.

MA3110 Chap 7: The Riemann Integral

Date No.

Defn $f:[a,b]\to\mathbb{R}$ be bounded, $P=\{x_0,\dots,x_n\}$ a partition of $[a,b]$.
For $1\le i\le n$, let $M_i=M_i(f,P)=\sup\{f(x): x\in[x_{i-1},x_i]\}$.
$m_i=m_i(f,P)=\inf\{f(x): x\in[x_{i-1},x_i]\}$.
$\Delta x_i=x_i-x_{i-1}$.

Uppersum: $U(f,P)=\sum_{i=1}^n M_i\Delta x_i$.
lower sum: $L(f,P)=\sum_{i=1}^n m_i\Delta x_i$.

Thm $f:[a,b]\to\mathbb{R}$ be bounded, P a partition of $[a,b]$,
let $m=\inf\{f(x): x\in[a,b]\}$, $M=\sup\{f(x): x\in[a,b]\}$.
then $m(b-a)\le L(f,P)\le U(f,P)\le M(b-a)$

Pf: note that ~~$\forall M_i$~~, $\{M_i\}_{i=1}^n\subset[a,b]$ $\therefore M\ge M_i\ \forall M_i$. by supremum prop
similarly with m_i.
Then $\sum M\Delta x_i\ge\sum M_i\Delta x_i=U(f,P)$.
$=M\sum\Delta x_i=M(b-a)$.

Defn $f:[a,b]\to\mathbb{R}$ be bdd. Define
upper integral: $\overline{\int_a^b} f=U(f)=\inf\{U(f,P): P \text{ partition } [a,b]\}$.
lower integral: $\underline{\int_a^b} f=L(f)=\sup\{L(f,P): P \text{ partition } [a,b]\}$.

Thm (Adding pts improve estimation) $f:[a,b]\to\mathbb{R}$ be bounded, P,Q partition $[a,b]$.
Then $L(f,P)\underset{①}{\le}L(f,P\cup Q)\underset{②}{\le}U(f,P\cup Q)\underset{③}{\le}U(f,Q)$.

Remark. given partitions $P\subseteq Q$, we say Q is a refinement of P.

Pf: ② is obvious from the defns.
①, ③: we consider the addition/removal of a pt
③: suppose $\{x_1,c,x_2\}\to\{x_1,x_2\}$ as partitions. then
sum $=\sum M_i\Delta x_i$ sum $=M\Delta x$.
but by defn $M=\sup\ge M_i$.
Since P,Q are arbitrary, $L(f,P)\le U(f,Q)$
$\Rightarrow \underline{\int_a^b} f\le\overline{\int_a^b} f$. □

Defn: (Riemann Integrable) on $[a,b]$ if $\underline{\int_a^b} f = \overline{\int_a^b} f$, write $\int_a^b f$ = the value.
define $\int_b^a f = -\int_a^b f$.

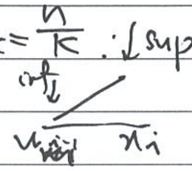
Remark R-integrable $\Rightarrow$ bounded. (note the c.p.: $\times$ bounded $\Rightarrow$ $\times$ R-int).

Eg. Show $f(x)=c$, $f:[a,b]\to\{c\}$ is integrable on its domain.
clearly for any partition P, $m_i = M_i = c \quad \forall i=1,\dots,n$.
$\therefore U(f,P) = \sum_{i=1}^{n} c(x_i - x_{i-1}) = c(b-a) = L(f,P)$.
$\therefore \int_a^b f = c(b-a)$ is R-integrable. $\square$

Eg: Dirichlet function: $g(x) = \begin{cases} 1 & x \text{ is rational} \\ 0 & \text{otherwise.} \end{cases}$ (is not R-integrable).
Pf: in any partition, take an intv (x_{i-1}, x_i).
$m_i = 0$, $M_i = 1$ for any partition, hence $U(f,P) \neq L(f,P)$ $\square$

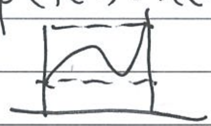
Eg. Id. fn: Let $h(x)=x$, $x\in[0,1]$. Is h integrable on $[0,1]$?

Pf: $\forall n \in \mathbb{N}$, let P_n be the partition $\{x_0, \dots, x_n\}$ where $x_k = \frac{n}{k}$. $\downarrow$ sup



Recall $U(h,P_n) = \sum M_i \Delta x_i$
$= \sum \sup\{h(x) \mid x \in [x_{i-1}, x_i]\} \Delta x_i$
$= \sum h(x_i) \Delta x_i$
$= \sum \frac{i}{n}\frac{1}{n} = \frac{1}{n^2}\frac{(n+1)n}{2} = \frac{n+1}{2n}$.
Similarly, $L(h,P_n) = \sum \frac{i-1}{n}\cdot\frac{1}{n} = \frac{1}{n^2}\frac{(n-1)n}{2} = \frac{n-1}{2n}$.
Now $L(h,P_n) = \frac{n-1}{2n} \leq \underline{\int} h \leq \overline{\int} h \leq \frac{n+1}{2n} = U(h,P_n)$
If $n\to\infty$, $\int_0^1 h = \frac{1}{2}$. $\square$

Thm Let $f:[a,b]\to\mathbb{R}$ be bdd and integrable on $[a,b]$, set
$m = \inf\{f(x) : x\in[a,b]\}$ and $M = \sup\{f(x) : x\in[a,b]\}$. Then
(i) $m(b-a) \leq \int_a^b f \leq M(b-a)$.



(ii) if $f \geq 0$, then $\int_a^b f \geq 0$.

Pf: use the property of inf/sup.

MA3110 Chap 7 (2) Date No.

Thm Riemann Integrable Criterion (RIC): f is bdd ~~and intgb~~ on $[a,b]$, ~~Then~~
Then intgb on $[a,b]$ $\iff$ $U(f,P) - L(f,P) < \varepsilon \quad \forall \varepsilon > 0$.

Pf: ($\Rightarrow$) assume f is integrable. Then $\exists\, Q, R$ partitions s.t.

$$U(f,Q) < \int_a^b f + \frac{\varepsilon}{2}$$
$$L(f,R) > \int_a^b f - \frac{\varepsilon}{2}.$$

Let $P = Q \cup R$. Then by thm 7.12,

$$L(f,R) \leq L(f,P) \leq U(f,P) \leq U(f,Q)$$

So that $U(f,P) - L(f,P) \leq U(f,Q) - L(f,R) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

($\Leftarrow$) obvious since ε is arbitrary.

Thm If $f : [a,b] \to \mathbb{R}$ is monotone on $[a,b]$, then f is integrable on $[a,b]$.

Pf: Use RIC. Note that $U(f,P) - L(f,P)$

$= \sum (M_i - m_i)\Delta x_i$ if we use an equal partition Δx,

$= \Delta x \big(\cancel{f(x_1)} - f(x_0) + \cancel{f(x_2) - f(x_1)} + \cancel{\cdots} + f(x_n) - \cancel{f(x_{n-1})}\big)$ ← it is so because of monotonicity.

$= \frac{b-a}{n}(f(b) - f(a))$ since $b-a$, $f(b)-f(a)$ are const, we cancel

$< \varepsilon$. □

Thm 7.2.6 $f, g : [a,b] \to \mathbb{R}$, integrable. $c \in \mathbb{R}$. Then

(i) cf is intgb on $[a,b]$, $\int_a^b cf = c\int_a^b f$.

(ii) $f+g$ " , $\int_a^b f+g = \int_a^b f + \int_a^b g$.

(iii) if $f(x) \leq g(x)\ \forall x \in [a,b]$, $\int_a^b f \leq \int_a^b g$.

Pf: Let $h(x) = g(x) - f(x)$. Then $h(x) \geq 0 \ \forall x \in [a,b]$,

$\therefore$ By thm, $\int_a^b h \geq 0$

$= \int_a^b g - \int_a^b f \Rightarrow \int_a^b g \geq \int_a^b f$.

~~a~~ my structure, real algebra?

(iv) $|f|$ is intgb on $[a,b]$, and $\left|\int_a^b f\right| \leq \int_a^b |f|$.

(v) fg is intgb on $[a,b]$.

Date No.

Lem (7.24)

(7.24) $f, g: [a,b] \to \mathbb{R}$ bdd, P partition of $[a,b]$, $c \in \mathbb{R}$. Then

(i) $L(cf, P) = \begin{cases} cL(f,P) & c>0 \\ cU(f,P) & c<0. \end{cases}$

(ii) $U(cf, P) = \begin{cases} cU(f,P) & c>0 \\ cL(f,P) & c<0. \end{cases}$

(iii) $L(f,P) + L(g,P) \leq L(f+g, P) \leq U(f+g, P) \leq U(f,P) + U(g,P)$.

Thm (7.26)

Pf: (i) WTS cf intgb, $\int_a^b cf = c\int_a^b f$.

Suppose $c>0$, then

$\overline{\int_a^b} cf = \inf_P U(cf, P)$ (defn).

$= \inf_P cU(f,P)$ (Lem 7.24)

$= c \inf_P U(f,P)$ (defn of inf, $c>0$).

$= \cancel{\ldots} = c\overline{\int_a^b} f = c\int_a^b$ ($= c\underline{\int_a^b} f$ by a symmetric prf.)

Suppose $c<0$, then ~~lower integral = C · up. intg~~.

$\overline{\int_a^b} cf = U(cf) = cL(f) = c\underline{\int_a^b} f = c\,\cancel{\ldots}\,U(f) = L(cf)$. $\square$

(ii) WTS $f+g$ intgb, $\int_a^b f+g = \int_a^b f + \int_a^b g$

we want to have.

$\overline{\int_a^b} f+g \overset{①}{\leq} \overline{\int_a^b} f + \overline{\int_a^b} g = \underline{\int_a^b} f + \underline{\int_a^b} g \overset{②}{\leq} \underline{\int_a^b} f+g$.

$\geq$

Let $\varepsilon > 0$ be given. $\exists P, Q$ partitions, (existence of P, Q due to $\overline{\int_a^b}$ being the inf. then adding $\frac{\varepsilon}{2}$ gives $\exists P$ lower than inf)

②: $U(f,P) < \overline{\int_a^b} f + \frac{\varepsilon}{2}$, $U(g,Q) < \overline{\int_a^b} g + \frac{\varepsilon}{2}$.

Let $R = P \cup Q$. Then by Thm 7.12,

$U(f+g, R) \underset{(7.24)}{\leq} U(f,R) + U(g,R) \underset{(7.12)}{\leq} U(f,P) + U(g,Q)$

$< U(f) + U(g) + \varepsilon$. $\forall \varepsilon > 0$.

MA3110 chap 7 (3).

Pf (cont'd) since $U(f+g, R)$ is an overestimate and $\overline{\int_a^b} f+g = U(f+g)$ is the inf,

$$\overline{\int_a^b} f+g \leq U(f+g, R) < \overline{\int_a^b} f + \overline{\int_a^b} g + \varepsilon$$

Since ε is arbitrary, we have $\overline{\int_a^b} f+g \leq \overline{\int_a^b} f + \overline{\int_a^b} g$.

(i): Similarly, $\underline{\int_a^b} f+g \geq \underline{\int_a^b} f + \underline{\int_a^b} g$

$\therefore$ equality forced since f, g integrable. $\therefore U(f+g) = L(f, g)$ $\square$

Pf: f intgb on $[a,b] \Rightarrow |f|$ intgb on $[a,b]$, $|\int_a^b f| \leq \int_a^b |f|$

Converse is false. $f = \begin{cases} 1 & \text{rat} \\ -1 & \text{irrat} \end{cases}$.

1. RIC gives us that $\forall \varepsilon$, $\exists$ partition P, $U(f, P) - L(f, P) < \varepsilon$.
2. Want to show $U(|f|, P) - L(|f|, P) \leq U(f, P) - L(f, P) < \varepsilon$. which will give itgb ility of $|f|$ by RIC.

3. Lemma: let $\emptyset \neq S \subseteq \mathbb{R}$ be bdd. If $\exists k > 0$ s.t. $|s - t| \leq k$ $\forall s, t \in S$, $\sup S - \inf S \leq k$.

   Pf: obvious, just note $\sup S \geq \inf S$.
4. Using the lemma: take a section $[x_i, x_{i+1}]$ from P. Then let $S = \{|f(x)| \mid x \in [x_i, x_{i+1}]\}$.

   By Lemma, $\sup S = M_i(|f|, P) - \inf S = m_i(|f|, P) \geq$ (?)

   4.1 Note that $\forall u, v \in [x_i, x_{i+1}]$,
   $$||f(u)| - |f(v)|| \leq |f(u) - f(v)| \leq M_i(f, P) - m_i(f, P) := k.$$

   4.2 $\therefore \sup S - \inf S \leq k = M_i(f, P) - m_i(f, P)$
   $M_i(|f|, P) - m_i(|f|, P)$.

   4.3 which is wanted in step (2) if we sum up till partition exhausts

   ~~5. Thus intgb~~ since LHS: $U(|f|, P) - L(|f|, P) \leq U(f, P) - L(f, P)$.
5. By RIC, $|f|$ is intgb. on $[a, b]$.
6. The inequality is due to $-|f(x)| \leq f(x) \leq |f(x)|$ in general.
7. Integrate 6, $-\int_a^b |f| \leq \int_a^b f \leq \int_a^b |f| \Rightarrow |\int_a^b f| \leq \int_a^b |f|$. $\square$

Thm $f:[a,b]\to\mathbb{R}$ itgb on $[a,c]$, $[c,b]$. Then f itgb on $[a,b]$ and

$$\int_a^b f = \int_a^c f + \int_c^b f.$$

Pf: We prove RIC for $[a,b]$. By RIC on $[a,c]$, $[c,b]$, we have partitions P, Q (on $[a,c]$ and $[c,b]$ resp.) s.t.

$$U(f,P) < \overline{\int_a^c} f + \frac{\varepsilon}{2}, \quad U(f,Q) < \overline{\int_c^b} f + \frac{\varepsilon}{2}.$$

Note that $R = P \cup Q$ is a partition on $[a,b]$, and

$$\overline{\int_a^b} f \underset{(\inf)}{\le} U(f,R) \underset{\text{dfn of } U}{=} U(f,P) + U(f,Q) = \int_a^c f + \int_c^b f + \varepsilon.$$

Similarly, we have $\underline{\int_a^b} f \ge \int_a^c f + \int_c^b f$.

Thm. If f is itgb on $[a,b]$, then $\forall$ midpt $c \in (a,b)$, f is itgb on $[a,c]$ and $[c,b]$.

Pf: Since f is itgb on $[a,b]$. Let $\varepsilon > 0$ be given, we can find a partition P s.t.

$$U(f,P) - L(f,P) < \varepsilon.$$

Now refine $Q = P \cup \{c\}$. Then better estimates: (thm 7.12)

$$U(f,Q) - L(f,Q) \le U(f,P) - L(f,P) < \varepsilon.$$

Let $R = Q \cap [a,c]$. Then R is a partition of $[a,c]$, and $R \subseteq Q$. Thus

$$U(f,R) - L(f,R) \overset{?}{\le} U(f,Q) - L(f,Q) < \varepsilon.$$

$\Rightarrow$ f is itgb on $[a,c]$. Similarly with $[c,b]$.

MA3110 Chap 7 (4) Date No.

Eg We look at some intgb fns. Recall that 1. cnt. or 2. monotone ⇒ intgb.
Also sums, constant mults are integrable.

1. polynomials (cnt).
2. The $[x]$ greatest integer value fn. intgb (monotone).
3. $|[x]|$
4. $f_3(x) = x^2[x] + \frac{1}{x^2+1}$ ← cts. products, sum
5. $f_4 : [-1,2] \to \mathbb{R}$, $f_4(x) = \begin{cases} |x| & \text{if } |x| \le 1 \\ 2 & \text{if } 1 < x \le 2 \end{cases}$ ← cts ← monotone. combine domain.

Rem. (Length of a curve) Intuition: partition, sum up secant length.

Let f be cts on $[a,b]$. Given partition P, define.

$$l(f,P) = \sum_{i=1}^{n} \sqrt{(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2}$$

= length of curve by polygonal line. Always under-estimate.

$\therefore l(f) = \sup\{l(f,P) : P \text{ is a partition of } [a,b]\}$.

possible to be an unbdd set!

UTS: $f \in C^1([a,b])$ ($\Leftrightarrow f'$ exists, f' cts on $[a,b]$)

$$\Rightarrow l(f) = \int_a^b \sqrt{1 + (f'(x))^2}\, dx$$

Defn $f : [a,b] \to \mathbb{R}$ intgb on $[a,b]$. Define the "intg fn"
$F(x) = \int^x f$ and $F(a) = \int_a^a f = 0$.
called the indefinite integral.

Motivation Integration is motivated by: 1. Rigorous area under curve.
2. Find new (nice) fns.

Thm The indefinite integral F of f is unif. cts on $[a,b]$.

Date No.

<u>pf</u> Let $M > 0$ be s.t. $|f(x)| \leq M \ \forall x \in [a,b]$.
For $x, y \in [a,b]$, let $x < y$,

$$|F(y) - F(x)| = \left|\int_a^y f - \int_a^x \right|$$
$$= \left|\int_x^y f + \int_a^x f - \int_a^x f\right|$$
$$= \left|\int_x^y f\right| \leq \int_x^y |f| \leq \int_x^y M = M(y-x)$$

Some show the Lipschitz condition $\{$ ~~...~~ $\Rightarrow$ unif cts $\}$.
Let $\varepsilon > 0$ be given. Then ~~...~~ $\exists \delta$ ~~...~~ $= \varepsilon/M$ s.t.
$|F(x) - F(y)| = \dots \leq M|x-y| < M\delta = \varepsilon$.

Thm (FTC1) Let f be intgb at $[a,b]$, and let
$$F(x) = \int_a^x f, \ x \in [a,b]$$
If f cts ~~on~~ at some $c \in [a,b] \Rightarrow F$ is diff at c, ~~then~~ and
$$F'(c) = f(c).$$

<u>pf</u>: f is cts at $c \Rightarrow \forall \varepsilon > 0, \exists \delta > 0$ s.t. $\forall t \in (c-\delta, c+\delta)$,
$$|f(t) - f(c)| < \tfrac{\varepsilon}{2}$$
$$\Rightarrow f(c) - \varepsilon/2 < f(t) < f(c) + \varepsilon/2$$

$c-\delta \quad x \quad y \quad c \quad c+\delta$

fix any x on the lower half of the open intv, namely $x \in (c-\delta, c)$
Then $\forall y \in [x, c] \subseteq (c-\delta, c+\delta)$, then by cond above,
$f(c) - \varepsilon/2 < f(y) < f(c) + \varepsilon/2$ (just copied, using y)
Integrate over $[x, c]$:
$$\int_x^c (f(c) - \varepsilon/2) \leq \int_x^c f \leq \int_x^c (f(c) + \varepsilon/2) \quad \text{cst fns.}$$
$$\therefore (f(c) - \varepsilon/2)(c-x) \leq F(c) - F(x) \leq (f(c) + \varepsilon/2)(c-x).$$
$$\therefore \left|\frac{F(c) - F(x)}{c-x} - f(c)\right| \leq \frac{\varepsilon}{2} < \varepsilon.$$
$$\Rightarrow \lim_{x \to c^-} \frac{F(x) - F(c)}{x-c} = f(c). \text{ Similarly, } \lim_{x \to c^+} \frac{F(x) - F(c)}{x-c} = f(c)$$

Since F is diff at c, $F'(c) = f(c)$.

MA3110 Chp 7 (4).   Date   No.

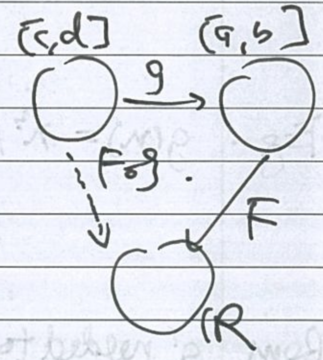
Cor 7.33 If f is cts on $[a,b]$, then F is diff on $[a,b]$ and
$$F'(x)=f(x)$$

cts fn $\xrightarrow[\text{"upgrade"}]{\text{integration}}$ diff fns.

Eg. $f(x)=\sqrt{5+x^3}$ is cts on $[0,\infty)$.
Let $F(x)=\int_0^x f=\int_0^x \sqrt{5+t^3}\,dt \quad x\in[0,\infty)$
By cor 7.33, F is diff on $[0,\infty)$
and $F'(x)=f(x)=\sqrt{5+x^3}$

Rem f cts on $[a,b]$, then the indef intg is the antiderivative of f? No in general.

Cor 7.34 f cont on $[a,b]$, g diff on $[c,d]$, $g([c,d])\subseteq[a,b]$. Then the fn
$$G(x)=\int_a^{g(x)} f \quad x\in[c,d]$$
is diff. on $[c,d]$, and
$$G'(x)=f(g(x))\,g'(x).$$



Pf: $F:[a,b]\to\mathbb{R}$, $F(x)=\int_a^x f \quad x\in[a,b]$.
(FTC1) $\Rightarrow F'(x)=f(x) \quad \forall\ x\in[a,b]$

Let us compose g, F. Consider $F\circ g:[c,d]\to\mathbb{R}$
$$(F\circ g)(x)=F(g(x))$$
$$=\int_a^{g(x)} f$$
$$=G(x).$$
$\therefore\ G=F\circ g$.
By the chain rule, F, g diff on domains and properly contained, $F\circ g$ is diff. and $G'(x)=(F\circ g)'(x)=F'(g(x))\,g'(x)$.

Eg $G(x)=\int_0^{x^2}\sqrt{5+t^3}\,dt$.
Let $f(x)=\sqrt{5+x^3} \quad x\in[0,\infty)$
$g(x)=x^2 \quad x\in[0,\infty)$
$F(x)=\int_0^x f \quad x\in[0,\infty)$

By FTC(I), $F'(x)=f(x)\ \forall x\in[0,\infty)$
$G(x)=(F\circ g)(x)=\int_0^{x^2}\sqrt{1+t^3}\,dt$.
$\therefore G'(x)=F'(g(x))\,g'(x)$
$=\sqrt{5+g(x)^3}\,2x$.
$=2x\sqrt{5+g(x)^3}$.

Date No.

Thm (FTC 2) Let g be diff on $[a,b]$, assume that g' is cts on $[a,b]$.
Then
$$\int_a^b g' = g(b) - g(a).$$

Rem: FTC 1: $F'(x) = f(x)$, where $F(x) = \int_a^x f$.
(derivative of antiderivative).
FTC 2: $\int_a^b g' = g(b) - g(a)$.
(antiderivative of derivative).

Pf: (FTC 2) g' is cts $\Rightarrow$ g' is itgb.
Let $F(x) = \int_a^x g' \quad \forall x \in [a,b]$.
By FTC 1, $F' = g'$
$\Rightarrow (F-g)' = 0$ on $[a,b]$.
$\Rightarrow F - g = c$ for some constant $c \in \mathbb{R}$.
$\Rightarrow F(x) = g(x) + c \quad \forall x \in [a,b]$.
$$\int_a^b g' = F(b) - 0$$
$$= F(b) - F(a)$$
$$= g(b) + c - g(a) - c = g(b) - g(a)$$ □

Eg. $g(x) = x^3/3$. $g'(x) = x^2$ is cts on $[0,1]$. By FTC II:
$$\int_0^1 g' = \int_0^1 x^2 dx = g(1) - g(0) = \tfrac{1}{3} - 0 = \tfrac{1}{3}.$$

Rem g' needed to be cts is unnecessary, but easy to prove using FTC 1.

Thm (candy FTC) Let g diff on $[a,b]$, g' ~~diff~~ itgb on $[a,b]$. Then
$$\int_a^b g' = g(b) - g(a)$$

Pf: Let P be a partition of $[a,b]$, $\{x_0, x_1, \ldots, x_n\}$.
Then for every partition (x_{i-1}, x_i) $\forall i = 1, \ldots, n$, By MVT on g
$\exists c_i \in [x_{i-1}, x_i)$ s.t. $\dfrac{g(x_i) - g(x_{i-1})}{x_i - x_{i-1}} = g'(c_i)$.
$$\Rightarrow g'(c_i)\Delta x = g(x_i) - g(x_{i-1})$$

MA3110 W5L1

Date No.

Pf (Cauchy FTC, cont'd).

This is a telescoping sum on RHS. Summing up,

$$\sum g'(c_i)\Delta x_i = g(x_n) - g(x_0) = g(b) - g(a).$$

~~But g' is integrable~~, its upper & lower sum ~~are~~ must bound. i.e.

$$\sum m_i \Delta x_i \leq \sum g'(c_i)\Delta x_i = g(b)-g(a) \leq \sum M_i \Delta x_i .$$

Since P is arbitrary, g is integrable $\Rightarrow \forall$ partition P,

$$L(g', P) \leq g(b) - g(a) \leq U(g', P) \quad \text{(in general)}$$

$\Rightarrow$ (itgb) $\underline{\int_a^b} g' \leq g(b) - g(a) \leq \overline{\int_a^b} g'$ and forces equality.

Recap Indef. integral $F(x) = \int_a^x f$, $x \in [a,b]$. where f is intgb.

$F(x)$ is ~~an~~ unif. cont, and differentiable (if f is cont).

— Every cont. f is a derivative (since its antideriv is diff.able).

Eg. $\lim_{x\to 0} \frac{1}{x}\int_x^{x^3} \underbrace{\sqrt{4+2t^2+3t^4}}_{=f(t)}\,dt$.

since f is cts, it is itgb. Consider

$F(x) = \int_0^x f$ $x \in \mathbb{R}$, ~~then by FTC I~~, ~~$F'(x) = f(x)$ $\forall x \in \mathbb{R}$~~.

$$\therefore \lim_{x\to 0}\frac{1}{x}\int_x^{x^3} f(t)\,dt = \lim_{x\to 0}\frac{1}{x}\left(\int_0^{x^3} f(t)\,dt - \int_0^x f(t)\,dt\right)$$

$$= \lim_{x\to 0}\frac{1}{x}\left(F(x^3) - F(x)\right)$$

Since conts, $\lim_{x\to 0} F(x^3) - F(x) = F(0) - F(0) = 0$

By L'Hopital, $= \lim_{x\to 0} \frac{d}{dx}\left(F(x^3) - F(x)\right)$

$$= \lim_{x\to 0} 3x^2 F'(x^3) - F'(x).$$

$$= \lim_{x\to 0} 3x^2 f(x^3) - f(x).$$

$$= 0 - \sqrt{4} = -2$$

Date No.

Eg. Let f cts $[a,b]$ ~~...~~, $a>0$.

$G:[a,b]\to\mathbb{R}$ be defined by

$$G(x)=\frac{1}{x}\int_a^x f(t)\,dt \quad \forall x\in[a,b].$$

(i) WTS G is diff on $[a,b]$ and $x^2G'(x)=xf(x)-\int_a^x f(t)\,dt \quad \forall x\in[a,b]$

<u>Pf</u>: Let $\int_a^x f=F(x)$, and F is diff. on $[a,b]$.

$\therefore\ G(x)=\frac{F(x)}{x}$ is diff.

$\therefore\ G'(x)=\frac{F'(x)x+F(x)}{x^2}$

$\Rightarrow x^2G'(x)=xf(x)-\int_a^x f.$

(ii) we try to use one of MVT, Rolle's, or Taylor's.

Given $F(b)=0$ and $F(a)=\int_a^a f=0$.

$\therefore\ G(a)=G(b)=\frac{0}{a}$ or $\frac{0}{b}=0$, Thus by rolles, we have

$\exists\, c\in(a,b),\quad G'(c)=0.$

$f(c)=0.$

Recap. FTC II: $\int_a^b g'=g(b)-g(a)$ if g' is cts.

Cauchy FTC: only need g' is integrable.

(*) Remark 1. Possible for derivative to be $\boxed{\text{not}}$ integrable. (when the deri. is not bdd).

$$g(x)=\begin{cases}x^2\sin(\frac{1}{x^2}) & x\neq 0\\ 0 & x=0\end{cases}\quad \text{then } g'(x) \text{ not itgb on } [-1,1].$$

$$g'(x)=\begin{cases}2x\sin(\frac{1}{x^2})-\frac{2}{x}\cos(\frac{1}{x^2}) & x\neq 0\\ 0 & x=0.\end{cases}$$

is not bounded. $\to\infty$ when $x\to 0$.

eg. take a sequence, $\frac{1}{x^2}=2n\pi$. $\Rightarrow x=\frac{1}{\sqrt{2n\pi}}$.

2.

MAB110 W5L2.

Thm 7.37 (Int. by parts) $u,v:[a,b]\to\mathbb{R}$ diff, u',v' itgb on $[a,b]$, then

$$\int_a^b uv' = u(b)v(b) - u(a)v(a) - \int_a^b vu'.$$

__Pf__: Let $g(x) = u(x)v(x)$, $x\in[a,b]$.
Then $g' = u'v + v'u$. must be itgb since u, v, u', v' itgb.
By Cauchy FTC $\int_a^b uv' = \int_a^b g' = g(b) - g(a)$.

$$\therefore \int_a^b u'v + \int_a^b v'u = u(b)v(b) - u(a)v(a) \qquad \square.$$

Thm 7.38 (Int. by sub) Suppose $\phi:[a,b]\to\mathbb{R}$, s.t. ϕ' exists, itgb on $[a,b]$.
$f: I\to\mathbb{R}$ cts on I containing $\phi([a,b])$, then

$$\int_a^b f(\phi(t))\,\phi'(t)\,dt = \int_{\phi(a)}^{\phi(b)} f(x)\,dx.$$

__Pf__: Since we can use any basept in I, the dom. of f as a basept,
let $F(x) = \int_{\phi(a)}^{x} f$, $x\in I$.
$\therefore F'(x) = f(x)$ $\forall x\in I$ by FTC! $\therefore$ is diff

$$\begin{tikzcd} {[a,b]} \arrow[r, "\phi"] & I \arrow[d, "F \text{ is diff}"] \\ & \mathbb{R} \end{tikzcd} \qquad g = F\circ\phi$$

We want to find a g' to use in
CFTC: $\int g' = g(b) - g(a)$ where
g is itgb.

say $g = F\circ\phi$. then $g'(x) = (F\circ\phi)(x)$

$$= F'(\phi(x))\,\phi'(x) = \overset{\text{cts}}{f}(\overset{\text{cts}}{\phi(x)})\,\overset{\text{itgb}}{\phi'(x)}$$

is itgb. By Cauchy FTC,

$$\int_a^b g'(t)\,dt = F\circ\phi(b) - F\circ\phi(a)$$
$$= \int_{\phi(a)}^{\phi(b)} f(t)\,dt. \qquad \square$$

Thm 7.39. Let f be a fn, $f, \ldots, f^{(n+1)}$ exist on $[a,x]$ and $f^{(n+1)}$ is itgb on $[a,x]$. Then

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + \underbrace{\frac{1}{n!}\int_a^x f^{(n+1)}(t)(x-t)^n\,dt}_{R_n(x)}.$$

Pf: Note that $R_n(x) = \int_a^x u'v$ where $u = f^{(n)}(t)$, $v = \frac{1}{n!}(x-t)^n$.

By parts,

$$\begin{aligned}
&= uv\Big|_a^x - \int_a^x uv' \\
&= f^{(n)}(x)\underbrace{\frac{(x-x)^n}{n!}}_{=0} - f^{(n)}(a)\frac{(x-a)^n}{n!} - \frac{1}{n!}\int_a^x f^{(n)}(t)\frac{d}{dt}(x-t)^n\,dt \\
&= -f^{(n)}(a)\frac{(x-a)^n}{n!} + \frac{1}{(n-1)!}\int_a^x f^{(n)}(t)(x-t)^{n-1} \\
&= \quad -\text{''} \quad + R_{n-1}(x) \\
&= -\sum_{k=1}^{n} f^{(k)}(a)\frac{(x-a)^k}{k!} + R_0(x) \qquad \text{itgb by assum.} \\
&= -\sum_{k=1}^{n} f^{(k)}(a)\frac{(x-a)^k}{k!} + \frac{1}{0!}\int_a^x f^{(1)}(t)(x-t)^0\,dt. \\
&\qquad\qquad = f(x) - f(a) \text{ by FTC1}
\end{aligned}$$

Moving terms, we have as desired.

$f(x) = \sum \ldots + R_n(x)$. □

Defn $P = \{x_0, \ldots, x_n\}$ partition of $[a,b]$, $\Delta x_i = x_i - x_{i-1}$, $\forall\, 1 \le i \le n$.
$\|P\| := \max\{\Delta x_i : 1 \le i \le n\}$. (max dist).

Rem. If $P \subseteq Q$ then $\|Q\| \le \|P\|$.

Qn: We have $\overline{\int_a^b} f + \varepsilon > U(f,P)$ for some P, but what is P?
Ans: $\|P\|$ is small.

Lem 7.41 Let $[a,b] \to \mathbb{R}$, f be a bdd fn. Then $\forall\, \varepsilon > 0$, $\exists\, \delta > 0$ s.t. for any partition P of $[a,b]$,

$$\|P\| < \delta \Rightarrow \begin{cases} U(f,P) < \overline{\int_a^b} f + \varepsilon \\ L(f,P) > \underline{\int_a^b} f - \varepsilon. \end{cases}$$

MA3110 W5 L2 (2).

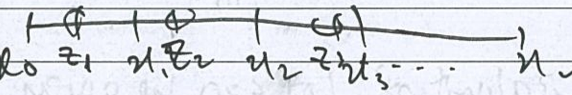
Date No.

Defn. (Riemann Sum) Let $f: [a,b] \to \mathbb{R}$ be <u>bounded</u>.
Let $P = \{x_0, \dots, x_n\}$ be a partition of $[a,b]$, let $z_i \in [x_{i-1}, x_i]$ $\forall 1 \le i \le n$

Define the sum $S(f,P)(z_i) = \sum_{i=1}^{n} f(z_i)(x_i - x_{i-1}) = \sum_{i=1}^{n} f(z_i)\Delta x_i$.

ie, <u>a random pt from the intv</u>

as the Riemann sum.



Defn if $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $\forall$ partition P with norm at most δ, then

$$|S(f,P)(z) - A| < \varepsilon, \quad (\text{ie the sum} \to A)$$

we say A is the $\boxed{\text{limit}}$ of Riemann sums as $\|P\| \to 0$, and write

$$\lim_{\|P\| \to 0} S(f,P)(z) = A =: \int_a^b f. \leftarrow \text{define.}$$

Recap. Riemann sum: $\sum \Delta x_i f(\xi_i)$ where ξ_i is any pt in $[x_{i-1}, x_i]$.
(upper: supremum, lower: infimum).

A is the limit of the Riemann sum if $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t.

$\|P\| = $ largest subintv length $< \delta$

$\implies |S(f,P)(\xi_i) - A| < \varepsilon$.

Riemann intgb if $\exists A$ s.t. $\lim_{\|P\| \to 0} S(f,P)(\xi_i) = A$.

MA3110 W6 L1

Date No.

Lem. 7.41 $f:[a,b]\to\mathbb{R}$ be bdd. $\forall \varepsilon>0, \exists \delta<0$ s.t. $\forall$ partition P of $[a,b]$.

$$\|P\|<\delta \Rightarrow \begin{cases} U(f,P) < \overline{\int_a^b} f+\varepsilon. \\ L(f,P) > \underline{\int_a^b} f-\varepsilon. \end{cases}$$

Ie, $\lim_{\|P\|\to 0} U(f,P) = \overline{\int_a^b} f$, $\lim_{\|P\|\to 0} L(f,P) = \underline{\int_a^b} f$.

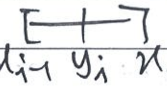
Pf: This is a limit pf. Let $\varepsilon>0$ be given. Since $\overline{\int_a^b} f$ is the sup,
$\exists Q = \{y_0, y_1, \dots, y_N\}$. s.t.

$$U(f,Q) < \overline{\int_a^b} f + \frac{\varepsilon}{2}$$ ← no longer ~~sth.~~ least.

Let $\eta = \min_{1\le i\le N} \Delta y_i$, $M = \sup_{x\in[a,b]} f(x)$.

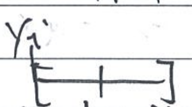
Choose $0<\delta<\min(\eta, \quad)$,
we claim that any part. P, $\|P\|<\delta$, then $U(f,P) < \overline{\int_a^b} f+\varepsilon$.
Consider the n subintvs of P: $[x_0,x_1], \dots, [x_{n-1},x_n]$.
where each length $<\delta\le\eta$, the smallest intv in Q. (why?)
Then any $[x_{i-1},x_i]$ cannot contain more than one pt in Q,

Divide these intv into 2 classes.

(P1) 

$(x_{i-1},x_i)\cap Q = \{y_j\}$.
for some $1\le j\le N-1$.

then $|P1| \le N-1$ (except y_0, y_N).

(P2) 

$(x_{i-1},x_i)\cap Q=\emptyset$.

Consider $R=P\cup Q$. There are two types of intvs again:
(R1): arises from P1. $|R1| \le 2(N-1)$.
(R2): arises from P2.

$$0\le U(f,P) - U(f,R) = \Big(\sum_{P1}+\sum_{P2}\Big) - \Big(\sum_{R1}+\sum_{R2}\Big) = \sum_{P1} - \sum_{R1}.$$
← better refined. (equal)

$$\le \Big|\sum_{P1}\Big| + \Big|\sum_{R1}\Big|$$

MA3110 W6L1 (2).

~~Let~~ any term in P_1:

$|M_i(f,P)\,\Delta x_i| \le M\|P\| < M\delta$, $\#(P_1) \le N-1$

$\Rightarrow |\sum_{P_1}| \le (N-1)M\delta$.

$|M_j(f,R)\,\Delta z_j| \le M\|R\| < M\delta$, $\#(R_1) \le 2(N-1)$.

$\Rightarrow |\sum_{R_1}| \le 2(N-1)M\delta$.

$\therefore 0 \le U(f,P) - U(f,Q) \le 3(N-1)M\delta \underset{\text{Let}}{<} \frac{\varepsilon}{2}$.

$\Rightarrow U(f,P) = (U(f,P) - U(f,Q)) + U(f,Q)$.

$< \frac{\varepsilon}{2} + \overline{\int_a^b} f + \frac{\varepsilon}{2}$. □.

Thm 7.42 $f:[a,b] \to \mathbb{R}$ bdd. f is itgb on $[a,b]$, $\int_a^b f = A \iff \lim_{\|P\|\to 0} S(f,P)(\xi) = A$.

<u>Pf</u>: ($\Rightarrow$) Let $\varepsilon > 0$. By lemma, we have. $\exists \delta > 0$ s.t. $\forall P$, $\|P\| < \delta$, then

$\underline{\int_a^b} f - \varepsilon \le L(f,P) \le U(f,P) < \overline{\int_a^b} f + \varepsilon$

but itgb $\Rightarrow$ uppersum = lowersum = A

and $L(f,P) \le S(f,P)(\xi) \le U(f,P)$

$\therefore |S(f,P) - A| < \varepsilon$.

($\Leftarrow$) Let $\varepsilon > 0$, then $\exists \delta > 0$ s.t. $\forall P$, $\|P\| < \delta$,

$A - \varepsilon < S(f,P)(\xi_\varepsilon) < A + \varepsilon$.

Claim: we can choose ξ s.t.

$A + \varepsilon > S(f,P)(\xi) \underset{(?)}{>} U(f,P) - \varepsilon \ge \overline{\int_a^b} f - \varepsilon$.

Then $\overline{\int_a^b} f \le A + 2\varepsilon$. Let $\varepsilon \to 0$, then $\overline{\int_a^b} f \le A$.

restricting the lowersum, we force them to be $= A$.

<u>Pf of claim</u>: how to choose a ξ s.t. $S(f,P)(\xi) > U(f,P) - \varepsilon$?

(intuition: $U(f,P)$ is the sum of all supremums. loosening the bound, we can find an arbitrary set of points ξ.)

Let $P = \{x_0, \dots, x_n\}$. $\forall 1 \le i \le n$, choose ξ_i in $[x_{i-1}, x_i]$ s.t.

$f(\xi_i) > M_i(f,P) - \frac{\varepsilon}{b-a}$.

when summed up, gives $-\varepsilon$.

□

Date No.

Cor 7.4.3 Let $f:[a,b]\to\mathbb{R}$ be int'ble on $[a,b]$. $\forall n\in\mathbb{N}$, let

$P_n=\{x_0^{(n)}, x_1^{(n)}, \dots, x_m^{(n)}\}$

be a part. of $[a,b]$, and $\xi^{(n)}$ be the partition that. $\xi_i^{(n)}\in[x_{i-1}^{(n)}, x_i^{(n)}]$

Then the sequence of R-sum, $y_n := S(f,P_n)(\xi^{(n)})$, $n\in\mathbb{N}$, will tend to the integral as $n\to\infty$. if $\lim_{n\to\infty}\|P_n\|=0$.

Sequential formulation: if we have diam-decreasing seq. of partitions $(\to 0)$, then the R-sum seq should tend to the integral.

Eg. Find the limit $\lim_{n\to\infty}\left(\frac{1}{n+1}+\dots+\frac{1}{2n}\right)$

Pf: Let $f(x)=\frac{1}{x}$. Let $y_n=\frac{1}{n+1}+\dots+\frac{1}{2n}$ $\to g(\frac{1}{n})$ $\to g(\frac{n}{n})$

$g(x)=\frac{1}{1+x}$. $\quad = \frac{1}{n}\left(1/(1+\frac{1}{n})+\dots+1/(1+\frac{n}{n})\right)$

Δx_n

$\therefore$ We have a natural partition $P_n=\{1, 1+\frac{1}{n}, \dots, 1+\frac{n}{n}=2\}$.

and $\xi^{(n)} := (1+\frac{1}{n}, 1+\frac{2}{n}, \dots, 1+\frac{n}{n})$. $\uparrow$ then $P_n=\{0,\frac{1}{n},\frac{2}{n},\dots,\frac{n}{n}=1\}$

$\therefore y_n=\left[f(\xi_1^{(n)})+\dots+f(\xi_n^{(n)})\right]\Delta\xi^{(n)}$.

$= S(f,P)(\xi^{(n)})$

and $\|P\|=\frac{1}{n}\to 0$ as $n\to\infty$.

$=\frac{1}{n}\to 0$.

$\therefore$ By cor 7.4.3, $\lim_{n\to\infty} y_n=\lim_{n\to\infty} S(f,P)(\xi^{(n)})$

$=\int_1^2 f=\int_1^2\frac{1}{x}\,dx=\ln|x|\Big|_1^2=\ln 2$.

$\therefore \quad =\int_0^1\frac{1}{1+x}\,dx=\ln(1+x)\Big|_0^1=\ln 2$.

MA3110 W6L2

Date No.

Improper integral	Proper integral
over unbounded intv.	over $[a,b]$
unbounded.	bdd over $[a,b]$ (so sup/inf are defined).

Defn. Suppose f defined on $[a,b)$, f itgb on $[a,c]$ $\forall c \in (a,b)$. (if itgb on $[a,b]$ just take Riemann)

$a \quad c \quad b$

If $L = \lim_{c \to b^-} \int_a^c f(x)\,dx$ exists,

we say the improper integral $\int_a^b f(x)\,dx$ converges and define

$$\int_a^b f(x)\,dx :\overset{\text{def}}{=} L.$$

Otherwise, we say the improper interval diverges.

If f is itgb on $[a,b)$, then $F(x) = \int_a^x f$ is cts

$\therefore F(c) = \lim_{x \to c} F(x) = L$. (they coincide).

Similarly, if f is defined on $(a,b]$, itgb on $[c,b]$ $\forall c \in (a,b)$, then

$$L = \lim_{c \to a^+} \int_c^b f(x)\,dx =: \int_a^b f(x)\,dx.$$

Eg. Let $f(x) = x^{-1/3}$, $x \in [0,1]$.

Pf: unbounded on $[0,1]$ $\Rightarrow$ not integrable on $[0,1]$.

But $\forall c \in (0,1)$, f is integrable on $[c,1]$.

$$\int_c^1 x^{-1/3}\,dx = \frac{3}{2}x^{2/3}\Big|_c^1 = \frac{3}{2}(1 - c^{2/3}).$$

Take limits: $\lim_{x \to 0^+} \frac{3}{2}(1 - c^{2/3}) = \frac{3}{2}$ exists. (as an improper integral, not Riemann integral)

$$\therefore \int_0^1 x^{-1/3}\,dx = \frac{3}{2}.$$

Rem if the function is x^{-2}, still integrable on $[c,1]$:

$$\int_c^1 x^{-2}\,dx = -\frac{1}{2}\int_c^1 x^{-1}\,dx = -\frac{1}{2}\left(1 - \frac{1}{c}\right).$$

but $\lim_{c \to 0^+} -\frac{1}{2}\left(1 - \frac{1}{c}\right) = \infty$.

Date [a,b] No.

Defn: (~~[illegible]~~ integral to inf). f defined on $[a,\infty)$, intgb. on $[a,b]$ $\forall b\in(a,\infty)$,

Then $\int_a^{\infty} f(x)dx = \lim_{b\to\infty}\int_a^b f(x)dx$. if lim exists.

~~" "~~ $(-\infty,a]$, " $[b,a]$ $\forall b\in(-\infty,a)$

$\int_{-\infty}^{a} f(x)dx = \lim_{b\to-\infty}\int_b^a f(x)dx$ if lim exists.

Eg. Does $\int_1^{\infty}\frac{1}{x^2}dx$ converge?

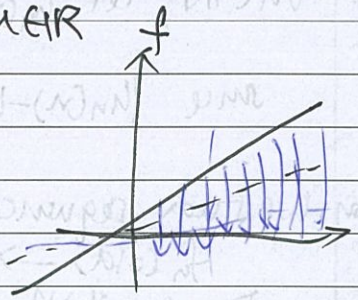
$\lim_{a\to\infty}\int_1^a\frac{1}{x^2} = \lim_{a\to\infty}(1-\frac{1}{a}) = 1$ converges.

Defn If both improper integrals $\int_0^{\infty} f$ and $\int_{-\infty}^{0} f$ converge, then

$\int_{-\infty}^{\infty} f$ converge and $\int_{-\infty}^{\infty} f = \int_0^{\infty} f + \int_{-\infty}^{0} f$.

MA3110 W6 L2 (2) Chap 8: Seq & Srs of Fns.

Eg. $\forall n \in \mathbb{N}$, let $f_n : \mathbb{R} \to \mathbb{R}$ be $f_n(x) = \frac{x}{n}$ $\forall x \in \mathbb{R}$



$\therefore$ $(f_n) = (x, \frac{x}{2}, \frac{x}{3}, \ldots\ldots, \frac{x}{n}, \ldots)$

is a sequence of functions on $\mathbb{R}$.

Then $(f_n(0)) = (0, 0, \ldots\ldots)$

$(f_n(1)) = (1, \frac{1}{2}, \ldots\ldots)$

A collection of sequences (of nums) indexed by the pts in the domain.

Defn Suppose $(f_n(x))$ converges for all x. "Record" the values by defining:

$$f(x) := \lim_{n\to\infty} f_n(x) \quad \forall x \in E.$$

Then we say (f_n) converges ~~to f~~ ptwise on E, and write

$f_n \to f$ pointwise on E.

$\Leftrightarrow \forall x \in E, \forall \varepsilon > 0, \exists K = K(\varepsilon, x)$ st. $n \geq K \Rightarrow |f_n(x) - f(x)| < \varepsilon$.

dep on ε and x.

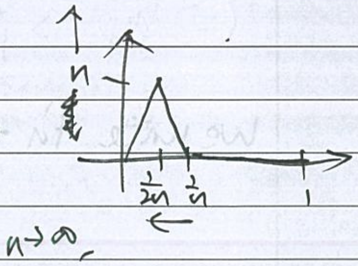
Eg. $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{x}{n} = 0$ $\forall x \in \mathbb{R}$.

Eg. $f_n(x) = x^n$, $x \in [0,1]$.

$\lim_{n\to\infty} f_n(x) = \begin{cases} 1 & x = 1 \\ 0 & \text{otherwise} \end{cases}$

Eg.

$$g_n(x) = \begin{cases} 2n^2x & 0 \leq x \leq \frac{1}{2n} \quad g(\frac{1}{2n}) = n \\ 2n - 2n^2x & \frac{1}{2n} \leq x \leq \frac{1}{n} \quad g(\frac{1}{n}) = 0 \\ 0 & \frac{1}{n} \leq x \leq 1 \end{cases}$$



Look at any $g_n(x)$. ~~...~~ WTS $g_n(x) \to 0$.

At $x = 0$, $g_n(0) = 0$.

At $0 < x \leq 1$, $g_n(x) = 0$ if we force $\frac{1}{n}$ take as small as possible. ~~Choose~~

Let x be given, choose $K > \frac{1}{x}$ (Archemedean prop).

Then $\frac{1}{n} \leq \frac{1}{K} < x \Rightarrow g_n(x) = 0$ by defn. $\therefore g_n(x) \to 0$.

$\therefore g_n \to g = 0$ ptwise on $[0,1]$.

Eg. $\forall n \in \mathbb{N}$, let $h_n(x) = \frac{1}{\sqrt{n}} \sin nx$, $x \in \mathbb{R}$, then $h_n \to 0$.

since $|h_n(x) - h(x)| = |\frac{1}{\sqrt{n}} \sin nx| \le \frac{1}{\sqrt{n}} \to 0$.

Motivation We study sequences of fns to find new functions.

f_n (old) $\longrightarrow f$ (new)

Eg $(1+\frac{x}{n})^n \to e^x$

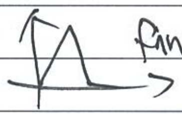
Qns. 1. Does ptwise convergence preserve continuity?

2. integrability?

2-1 If so, does $\int g_n \to \int g$?

3. Does the derivative sequence converge as well?

Ans 1. NO. $f_n(x) = x^n$, $x \in [0,1]$.

2.1 NO.  finite integral $\to$  zero integral

$= \frac{1}{n} \times n \times \frac{1}{2} = \frac{1}{2}$

3. NO. $h_n(x) = \frac{1}{\sqrt{n}} \sin nx \to 0$, but

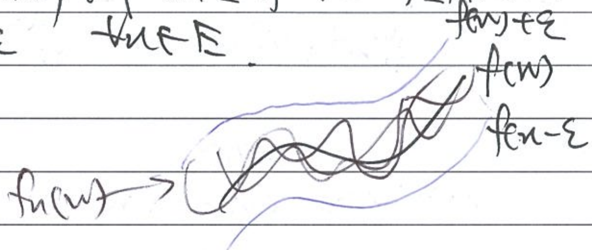
$h_n'(x) = \frac{n}{\sqrt{n}} \cos nx \not\to 0$.

Defn. A sequence (f_n) of fns converges uniformly to f on E if $\forall \varepsilon > 0, \exists K \in \mathbb{N}$ s.t.

$n \ge K \Rightarrow |f_n(x) - f(x)| < \varepsilon \quad \forall x \in E$.

($\Rightarrow$ implies ptwise converges).

We write $f_n \xrightarrow{u} f$ on E.



MA3110 W7L1

Date No.

Recall sequence of fns: $(f_n)_{n=1}^{\infty}$, $f_n: E \to \mathbb{R}$

ptwise convergence: $\forall x \in E$, $(f_n(x))_{n=1}^{\infty}$ converges to $f(x)$.
$\Rightarrow (f_n)_{n=1}^{\infty}$ converges ptwise to f.

unif convergence: Given ε, $\exists M$ s.t. $\forall n > M$, $\forall x \in E$,
$|f_n(x) - f(x)| < \varepsilon$.

Eg. $\forall n \in \mathbb{N}$, let $h_n(x) = \frac{1}{\sqrt{n}} \sin nx$, let $h(x) = 0$.

Claim: $h_n \to h$ uniformly on $\mathbb{R}$.

Let $\varepsilon > 0$ be given. Then ~~th~~ choose $K > \frac{1}{\varepsilon^2}$.

we have $n \geq K \Rightarrow |h_n(x) - h(x)| < \frac{1}{\sqrt{n}} < \frac{1}{\sqrt{K}} < \varepsilon$ $\square$.

Defn. Let $E \subseteq \mathbb{R}$, and $\varphi: E \to \mathbb{R}$ a bdd fn. The uniform norm of φ on E is defined as $\|\varphi\|_E := \sup\{|\varphi(x)| : x \in E\}$.

bounds $|\varphi(x)| \leq \|\varphi\|_E$ $\forall x \in E$.

Note that this is a norm in the vector space of all fns $E \to \mathbb{R}$, since it fulfills:

1. $\|\varphi\|_E \geq 0$ and $\|\varphi\|_E = 0 \Rightarrow \varphi = 0$.
2. $\|\lambda \varphi\|_E = |\lambda| \, \|\varphi\|_E$ $\forall \lambda \in \mathbb{R}$
3. Δ-ineq: $\|\varphi_1 + \varphi_2\| \leq \|\varphi_1\|_E + \|\varphi_2\|_E$

simply by abs value.

Lem 8.11 A seqn of fns $f_n \xrightarrow{u} f$ ~~unif~~ on E $\Longleftrightarrow$ $\|f_n - f\|_E \to 0$.

(seqn of real num).

Pf: $(\Rightarrow)$ $f_n \xrightarrow{u} f$. Therefore $\forall \varepsilon > 0$, $\exists K$ s.t. $\forall n > K$, $\forall x \in E$,
$|f_n(x) - f(x)| < \varepsilon/2$

$\therefore \forall n > K$, $\|f_n - f\|_E = \sup\{|f_n(x) - f(x)| : x \in E\}$.
$\leq \varepsilon/2 < \varepsilon$.

$(\Leftarrow)$ $\|f_n - f\|_E \to 0$. Let $\varepsilon > 0$, $\exists K$ s.t. $\forall n > K$,
$\|f_n - f\| < \varepsilon \Rightarrow |f_n(x) - f(x)| \underset{(\sup)}{\leq} \|f_n - f\|_E < \varepsilon$ $\forall x \in E$

$\Rightarrow f_n \xrightarrow{u} f$. $\square$

Eg. $h_n(x) = \frac{1}{\sqrt{n}} \sin nx$, $h(x) = 0$.

$\|h_n - h\| = \|h_n\| = \frac{1}{\sqrt{n}} \to 0$ when $n \to \infty$. (by Lem 8.11).

Thm 8.12 Cauchy Criterion for Uniform Convergence

Given f_n, f_n conv. unif. on $E \iff \forall \varepsilon > 0, \exists K \in \mathbb{N}, \forall n > K$,
$\|f_n - f_m\|_E < \varepsilon \quad \forall n, m \geq K$.

Pf: ($\Rightarrow$) $f_n \xrightarrow{u} f$ on E. $\forall \varepsilon > 0$, $\|f_n - f\|_E < \varepsilon/2 \quad \forall n > K \in \mathbb{N}$ for some K

$\therefore \|f_n - f_m\|_E \leq \|f_n - f\|_E + \|f - f_m\|_E$
$< \varepsilon/2 + \varepsilon/2 = \varepsilon$.

($\Leftarrow$) Let $\varepsilon > 0$, $\exists K \in \mathbb{N}, \forall n > K$, $\|f_n - f_m\|_E < \varepsilon/2 \quad \forall n, m \geq K$.

Fix $x \in E$. Then $|f_n(x) - f_m(x)| \leq \|f_n - f_m\|_E < \varepsilon/2$.

$\therefore f_n(x)$ is a cauchy sequence in $\mathbb{R}$.

Since $\mathbb{R}$ is complete, $f_n(x) \to f(x)$. (define f).

$\Rightarrow f_n \to f$ ptwise.

$\therefore \lim_{m \to \infty} |f_n(x) - f_m(x)| \leq \lim_{m \to \infty} \varepsilon/2 \quad \forall x \in E, \forall n > K$.

$\Rightarrow \quad = |f_n(x) - f| \quad < \varepsilon$ □

Cor. 8.13 $f_n \overset{u}{\not\to} f \iff \exists \varepsilon_0 > 0, \forall k \in \mathbb{N}, \exists n_0 \geq k, \exists x \in E$ s.t.
$|f_{n_0}(x) - f(x)| \geq \varepsilon_0$

Eg Show $f_n(x) = x^n$, $E = [0,1]$, $f = \begin{cases} 1 & x=1 \\ 0 & \text{otherwise} \end{cases}$. Show $f_n \overset{u}{\not\to} f$.

Pf(1): Check that $\|f_n - f\| \not\to 0$.

$= \sup\{|f_n(x) - f| : x \in [0,1]\}$.

$= \sup\{\{x^n : x \in [0,1)\} \cup \{0\}\} = 1 \not\to 0$.

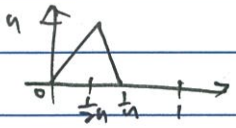
(2): $f_k(x) = x^k \Rightarrow f_k(1/2^{1/k}) = \frac{1}{2}$.

$|f_k(\frac{1}{2^{1/k}}) - f(1/2^{1/k})| = |\frac{1}{2} - 0| = \frac{1}{2} \quad \forall k \in \mathbb{N}$.

Take $\varepsilon_0 = \frac{1}{2}$, $x_k = 1/2^{1/k}$ and $n_k = k$.

By Lem 8.13, $f_n \not\to f$ on $[0,1]$.

MA3110 W8L1

Eg. Recall  . It does not uniformly converge to 0. Show it using methods 1 (check unif norm of $f_n - f$) & (find 1 ε_0)

Method 1: $\|f_n - \overset{0}{f}\|_{[0,1]} = \sup\{f_n(x)\} = n \not\to 0$.

Method 2: $|f_n(\frac{1}{2n}) - f(\frac{1}{2n})| = n \geq 1 = \varepsilon_0$.

Eg. $f_n(x) = x^n(1-x^n)$ $x \in [0,1]$, let $f(x) = 0$.

(i) Show that $f_n \to f$ ptwise.

Pf: $\forall x \in [0,1]$ $\lim_{n\to\infty} |f_n(x)| = \lim_{n\to\infty} |x^n(1-x^n)| < \lim_{n\to\infty} |x^n| = 0$ if $x \neq 1$

but if $x=1$, $(1-x^n) = 0 \Rightarrow \lim = 0$.

(ii) Does $f_n \xrightarrow{u} f$ on $[0,1]$?

Pf: No. Method 1:

$x^n(1-x^n) = x^n - x^{2n}$
$= -(x^n - \frac{1}{2})^2 + \frac{1}{4}$

$\|f_n - f\|_{[0,1]} = \|f_n\|_{[0,1]} = \sup\{x^n(1-x^n) : x \in [0,1]\}$.

$= \frac{1}{4} \not\to 0$.

Method 2: $x_k = \frac{1}{2^{1/k}} \in [0,1]$.

$|f_k(x_k) - f(x_k)| = \frac{1}{4} = \varepsilon_0$ $\forall k$.

(iii) Does $f_n \xrightarrow{u} f$ on $[0,r]$ for $0<r<1$?

$|f_n(x) - f(x)| = |x^n(1-x^n)| = |x^n - x^{2n}|$

$\leq |x^n| - |x^{2n}|$ $\forall x \in [0,r]$.

$\leq r^n + r^{2n}$ where

$\therefore \|f_n - f\|_{[0,r]} \leq r^n + r^{2n} \to 0$ $\therefore$ Yes, by Lemma 8.11. □

Thm 8.21: (unif conv. preserves continuity. If $f_n \xrightarrow{u} f$ on I, and each f_n is cts at $x_0 \in I \Rightarrow f$ is cts at x_0.

Pf: Let $\varepsilon > 0$. $f_n \xrightarrow{u} f$ on I, therefore $\exists K \in \mathbb{N}$ s.t. $n \geq K \Rightarrow$

$|f_n(x) - f(x)| < \frac{\varepsilon}{3}$ $\forall x \in I$.

Take the delta for continuity on f_K, $|x - x_0| < \delta \Rightarrow |f_K(x) - f_K(x_0)| < \frac{\varepsilon}{3}$.

Then $|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| \leq |f(x) - f_K(x)| + |f_K(x) - f_K(x_0)| + |f_K(x_0) - f(x_0)|$

$< \underbrace{\varepsilon/3}_{\text{by unif. cts}} + \underbrace{\varepsilon/3}_{\text{ctnuity}} + \underbrace{\varepsilon/3}_{\text{unif cts}} = \varepsilon$. □

Rem. 1. We now know $\lim_{x\to x_0} f(x) = f(x_0)$

$\iff \lim_{x\to x_0}\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty}(\lim_{x\to x_0} x_0)$

Thus switching of limits is not always true.

2. Another method for showing $f_n \not\xrightarrow{u} f$: f_n are cts but f is not.

Thm 8.23: (Uniform convergence preserves integral). Let $f_n \xrightarrow{u} f$ on $[a,b]$ and f_n are integrable. Then

(i) f is itgb on $[a,b]$.

(ii) $\forall x_0 \in [a,b]$, $F_n(x) = \int_{x_0}^{x} f_n(t)\,dt \xrightarrow{u}$

$F(x) = \int_{x_0}^{x} f(t)\,dt$. on $[a,b]$.

$\therefore \lim_{n\to\infty} \int_{x_0}^{x} f_n(t)\,dt = \int_{x_0}^{x} \lim_{n\to\infty} f_n(t)\,dt$.

In particular, $\lim_{n\to\infty} \int_a^b f_n(t)\,dt = \int_a^b f(t)\,dt$.

Pf. (i) Let $\varepsilon_n = \|f_n - f\|_{[a,b]}$, $n\in\mathbb{N}$. Since $f_n \xrightarrow{u} f$, $\varepsilon_n \to 0$.

$\therefore |f_n(x) - f(x)| \le \|f_n - f\| = \varepsilon_n$. $\forall x\in[a,b]$

$f_n(x) - \varepsilon_n \le f(x) \le f_n(x) + \varepsilon_n$. $\forall x\in[a,b]$.

Taking integrals, $\int_a^b f_n - \varepsilon_n(b-a) = \underline{\int_a^b}(f_n - \varepsilon_n) \le \underline{\int_a^b} f \le \overline{\int_a^b} f \le \overline{\int_a^b}(f_n + \varepsilon_n) = \int_a^b f_n + \varepsilon(b-a)$.

Then RIC: $\overline{\int_a^b} f - \underline{\int_a^b} f \le \cancel{\int_a^b f_n} + \varepsilon_n(b-a) - \cancel{\int_a^b f_n} + \varepsilon_n(b-a)$

$= 2\varepsilon_n(b-a) \to 0$.

(ii) Prove $F_n(x) = \int_{x_0}^{x} f_n \xrightarrow{u} F(x) = \int_{x_0}^{x} f$ on $[a,b]$.

$|F_n(x) - F(x)| = |\int_{x_0}^{x} f_n - \int_{x_0}^{x} f| = |\int_{x_0}^{x} f_n - f|$

$\le \int_{x_0}^{x} |f_n - f|$

$\le \int_{x_0}^{x} \varepsilon_n = \varepsilon_n(x - x_0)$

Eg. Find $\lim_{n\to\infty} \int_0^{\pi/4} x^n \sin nx\,dx$.

Consider $f_n(x) = x^n \sin nx$, $x\in[0,\frac{\pi}{4}]$. Then claim $f_n \xrightarrow{u} f = 0$:

$\|f_n - f\| = (\frac{\pi}{4})^n \to 0 \Rightarrow \lim_{n\to\infty} \int_0^{\pi/4} x^n \sin nx\,dx = \int_0^{\pi/4} 0\,dx = 0$.

MA3110 W8L2

Recall: 8.1: unif conv $\Rightarrow$ ptwise conv.

8.2: 8.21/2 $f_n \xrightarrow{u} f$, f_n cts $\Rightarrow f$ cts. (preserve cts)

8.23 $f_n \xrightarrow{u} f$. If f_n itgb, then f_n is itgb, and $\lim_{n\to\infty} \int_{x_0}^{x} f_n = \int_{x_0}^{x} \lim_{n\to\infty} f_n = \int_{x_0}^{x} f$. (preserve integral).

Question: Does unif conv. preserve differentiability? No.

derivative? No.

EG. $h_n(x) = \frac{1}{\sqrt{n}} \sin nx$, $h(x) = 0$.

Thm 8.24. (f_n) diffable on $[a,b]$. Suppose.

(i) $(f_n(x_0))$ converges for some $x_0 \in [a,b]$ (just 1 pt) $\lim_{n\to\infty} f_n(x_0) = L$

(ii) (f_n') converges unif. $f_n' \xrightarrow{u} G$ on $[a,b]$.

Then ① $(f_n) \xrightarrow{u} f$ for some function f.

② f is differentiable, and $\lim_{n\to\infty} f_n'(x) = f'(x)$ $\forall x \in [a,b]$.

(ie $G = f'$) interchange

IOW, $\lim_{n\to\infty} \frac{d}{dx} f_n(x) = \frac{d}{dx} \lim_{n\to\infty} f_n(x) = \frac{d}{dx} f(x)$.

Pf: (Full proof on Bartle pg. 250).

Pf (assuming f_n' is cts on $[a,b]$).

Given $\lim_{n\to\infty} f_n(x_0) = L$ for some $x_0 \in [a,b]$, $L \in \mathbb{R}$,

$f_n' \xrightarrow{u} G$ on $[a,b]$.

Then by FTC II, $\int_{x_0}^{x} f_n' = f_n(x) - f_n(x_0)$.

$\therefore f_n(x) = \int_{x_0}^{x} f_n' + f_n(x_0)$. ①

Now $f_n' \xrightarrow{u} G$, f_n' is cts $\Rightarrow G$ is cts, then (in particular itgb)

$\int_{x_0}^{x} f_n' \xrightarrow{u} \int_{x_0}^{x} G$

by ①, $\Rightarrow \lim_{n\to\infty} f_n(x) = \int_{x_0}^{x} G + L \overset{\text{def}}{=} f$.

We check f is diffable: sum of diffable fns (G is cts $\Rightarrow \int G$ is diff).

and furthermore, $\frac{d}{dx}\left(\lim_{n\to\infty} f_n(x)\right) = \lim_{n\to\infty} f_n'(x)$

$= G$ as desired. $\square$.

Review: Infinite series: Given $(a_n)_{n=1}^{\infty}$, construct $s_n = \sum_{k=1}^{n} a_k$,

Then $(s_n)_{n=1}^{\infty}$ is a sequence $= \sum_{n=1}^{\infty} a_n$. (notation).

$\therefore$ If $s_n \to s$, $\sum_{n=1}^{\infty} a_n$ converges.

Fact: $\sum_{n=1}^{\infty} a_n$ converges $\Rightarrow a_n \to 0$. (Converse is false)

The contrapositive is called the (nth term test).

Cor. of Cauchy criterion: $\sum_{n=1}^{\infty} a_n$ converges

$\Leftrightarrow \forall \varepsilon > 0, \exists N, \forall n \geq m \geq N$,

$|s_n - s_m| = |a_{m+1} + \dots + a_n| < \varepsilon$.

Defn: Positive series $\sum_{n=1}^{\infty} a_n$ where $a_n \geq 0 \ \forall n$.

Eg: p-series $(p > 0)$ $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges $\Leftrightarrow p > 1$ (by integral test).

Defn: (comparison test). If $0 \leq a_n \leq b_n \ \forall n \geq k$ for some k.

Then $\sum_{n=1}^{\infty} b_n$ conv $\Rightarrow \sum_{n=1}^{\infty} a_n$ conv.

Defn: If $\sum_{n=1}^{\infty} |a_n|$ converges, then we say (the original srs) converges absolutely.

Fact: Absolute convergence $\Rightarrow$ convergence.

8.3 Infinite series of functions.

Defn: (f_n) seq. of fns, we say $\sum_{n=1}^{\infty} f_n$ is an inf srs of fns.

nth partial sum: $\sum_{i=1}^{n} f_i(x) = s_n(x)$.

$\sum_{n=1}^{\infty} f_n$ conv. ptwise (unif) $\Leftrightarrow s_n$ conv. ptwise (unif).

$\sum_{n=1}^{\infty} f_n$ conv. abs $\Leftrightarrow \sum_{n=1}^{\infty} |f_n|$ conv. pointwise.

MA3110 W8L2 (2).

Thm (Cauchy criterion for unif. conv. of srs of fns).

$\sum_{n=1}^{\infty} f_n$ ~~conv~~ conv. unif on E $\Leftrightarrow$ $\forall \varepsilon > 0$, $\exists K \in \mathbb{N}$ s.t. $n > m \geq K$

$\Rightarrow \|f_{m+1} + f_{m+2} + \ldots + f_n\|_E < \varepsilon$.

$\Rightarrow \sup\{|f_{m+1}(x) + \ldots + f_n(x)| \mid x \in E\} < \varepsilon$.

$\Rightarrow |f_{m+1}(x) + \ldots + f_n(x)| < \varepsilon \; \forall x \in E$

Cor 8.3.1 $\sum_{n=1}^{\infty} f_n$ conv. unif. on E, then $f_n \to 0$ unif on E.

Pf: $\forall \varepsilon > 0$, $\exists K$ such that by the cauchy criterion, $\forall n > m \geq K$,

$\|f_{m+1} + \ldots + f_n\|_E < \varepsilon$.

Take $m = n-1$, $n > K \Rightarrow \|f_n\|_E < \varepsilon$

$\Rightarrow \|f_n - 0\| < \varepsilon \Rightarrow f_n \xrightarrow{u} 0$.

Thm (Weierstrass M-test) (f_n) on E, $(M_n) \subset \mathbb{R}^+$, such that

$\|f_n\|_E \leq M_n \; \forall n \in \mathbb{N}$.

If $\sum_{n=1}^{\infty} M_n$ conv, $\Rightarrow$ $\sum_{n=1}^{\infty} f_n$ conv. uniformly on E.

Pf: Suppose $\sum_{n=1}^{\infty} M_n$ is conv. Then cauchy criterion: $\forall \varepsilon > 0$,

$\exists K \in \mathbb{N}$, $\forall m > n \geq K$, $M_{m+1} + \ldots + M_n < \varepsilon$.

$\therefore \|f_{m+1} + \ldots + f_n\|_E \leq \|f_{m+1}\|_E + \ldots + \|f_n\|_E$

$\leq M_{m+1} + \ldots + M_n < \varepsilon$.

$\therefore \sum_{n=1}^{\infty} f_n$ converges uniformly.

Eg. Consider $\sum_{n=1}^{\infty} \frac{\sin nx}{n^2}$. Converge unif?

Pf: Let $M_n = \frac{1}{n^2}$. $\|f_n(x)\| = \left|\frac{\sin nx}{n^2}\right| \leq \frac{1}{n^2}$.

Applying the Weierstrass M-test with $M_n = \frac{1}{n^2}$, since this is a p-srs,

$p = 2 > 1 \Rightarrow \sum_{n=1}^{\infty} M_n$ is convergent

$\Rightarrow \sum_{n=1}^{\infty} \frac{\sin nx}{n^2}$ conv. uniformly.

Then $f(x) = \sum_{n=1}^{\infty} \frac{\sin nx}{n^2}$ is a new function that is cts (partial sums/fn are cts)

MA3110 W9L1

Recall:
- f_n cts $\xrightarrow{u} f \Rightarrow f$ cts
- f_n itgb $\xrightarrow{u} f \Rightarrow f$ itgb, $\lim \int f_n = \int f$.
- If 1. $\exists x_0 \in I$, $f_n(x_0) \to f(x_0)$,
  2. $f_n' \xrightarrow{u} G$ for some ~~g~~ $G \in I$,

  Then $f_n \xrightarrow[\infty]{u} f$ for some f, and $f' = G$.
- Srs of fns: $\sum_{n=1}^{\infty} f_n$ conv ptwise (unif) $\Leftrightarrow S_n = \sum_{k=1}^{n} f_k$ conv. ptwise (unif).
- Cauchy Criterion:

  $\sum_{n=1}^{\infty} f_n$ conv. unif on $E \Leftrightarrow \forall \varepsilon > 0, \exists N \in \mathbb{N}$ s.t. $n > m \geq N \Rightarrow \|f_{m+1} + \dots + f_n\|_E < \varepsilon$.
- Cor 8.31: $\sum_{n=1}^{\infty} f_n$ conv. unif on $E \Rightarrow f_n \xrightarrow{u} 0$ on E.
- nth term test $\sum_{n=1}^{\infty} f_n$ diverges on $E \Leftarrow f_n \not\xrightarrow{u} 0$
- Weierstrass M-test: If $\|f_n\| < M_n \; \forall n$, $\sum_{n=1}^{\infty} M_n$ conv.

  $\Rightarrow \sum_{n=1}^{\infty} f_n$ conv. unif on E. (we get for free that $\sum \|f_n\|_E$ converges)
- converse is false: $\sum_{n=1}^{\infty} f_n$ conv. uniform on $E \not\Rightarrow \sum_{n=1}^{\infty} \|f_n\|_E$ conv.

  Consider $f_n(x) = \begin{cases} \frac{1}{n} & x = \frac{1}{n} \text{ (1 ptonly)} \\ 0 & \text{otherwise} \end{cases}$. Then ① $\sum_{n=1}^{\infty} f_n$ conv. unif on $[0,1]$

  ② $\sum_{n=1}^{\infty} \|f_n\|_{[0,1]}$ diverges.

  $f_n \xrightarrow{u} f$ ~~#~~ ~~except~~ $x = \frac{1}{n}$ $n \in \mathbb{N}$.

Eg. (Geometric series) consider $\sum_{n=0}^{\infty} x^n$ on $(-1,1)$.

(i) Show that $\sum_{n=0}^{\infty} x^n$ is ptwise convergent.

partial sum $S_n(x) = \sum_{k=0}^{n} x^k = x^0 + x^1 + \dots + x^n = \frac{x^{n+1} - 1}{x - 1} = \frac{1 - x^{n+1}}{1 - x}$

$\lim_{n\to\infty} S_n(x) = \lim_{n\to\infty} \frac{1 - x^{n+1}}{1 - x} = \frac{1}{1-x}$

$\therefore S_n(x) \to S(x) = \frac{1}{1-x}$ pointwise.

$\therefore \sum_{k=1}^{\infty} x^n = \frac{1}{1-x}$ pointwise.

(ii) Show that $\forall r \in [0,1)$, we have $\sum_{n=0}^{\infty} x^n$ conv. unif on $[-r, r]$.

$\|x^n\| \leq r^n := M_n$. $\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + \dots = \frac{1}{1-r}$ converges.

$\therefore \sum_{n=0}^{\infty} x^n \xrightarrow{u}$ on $[-r, r]$.

(iii) Is $\sum_{n=0}^{\infty} x^n$ conv. unif. on $[-1,1]$?

$\|x^n\|_{[-1,1]} = 1 \not\xrightarrow{u} 0$ on $[-1,1]$. $\therefore$ no.

MA3110 W9L1 (2)

Thm 8.32 If $\sum_{n=0}^{\infty} f_n$ conv. unif to f on I and each f_n is cts at $x_0 \in I$, then f is cts at x_0.

Pf: Simply notice that $S_n = \sum_{k=1}^{n} f_k$ is a sum of cts fns and is thus cts.

Cor 8.33 If $\sum_{n=0}^{\infty} f_n$ is unif. conv. to f on I, f_n is cts over I, then f is cts over I.

Thm 8.34. $\sum_{n=0}^{\infty} f_n \xrightarrow{u} f$ on I, then if f_n is itgb on I, f is itgb on I, and the integral of limit = lim of integral.

Pf: This also relies on the fact that sum of itgb fns is itgb, and sum of integrals = integrals of sum.

Eg. For every $r>0$, the srs $f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!}$ conv. unif on $[-r, r]$.
(the limit is actually $f(x) = e^{-x^2}$)

Pf: $|f_n(x)| = \frac{|x^{2n}|}{n!} \leq \frac{r^{2n}}{n!} \quad \forall x \in [-r, r]$.
$\therefore \|f_n\|_{[-r,r]} \leq \frac{r^{2n}}{n!} \quad \forall n$.
$\sum_{n=0}^{\infty} \frac{r^{2n}}{n!}$ conv. by the Ratio Test:

$$\left| \frac{r^{2n+2}/(n+1)!}{r^{2n}/n!} \right| = \frac{r^2}{n+1} \to 0 < 1$$

$\therefore$ By WMT, the given srs conv. unif on $[-r, r]$.

Qn: can $F(x) = \int_0^x f$ be represented by a srs?

$$\therefore F(x) = \int_0^x \left(\sum_{n=0}^{\infty} (-1)^n \frac{t^{2n}}{n!} \right) dt$$

$$= \sum_{n=0}^{\infty} \int_0^x (-1)^n \frac{t^{2n}}{n!} dt$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{x^{2n+1}}{2n+1}$$

Thm 8.35. If $\sum_{n=0}^{\infty} f_n(x_0) \to f(x_0)$ and the deriv. $\sum_{n=0} f_n' \to G$,
then $\sum_{n=0} f_n \to f$ unif and $f' = G$.

Pf: similarly, due to diff being a linear trans.

Eg: is $\sum_{n=1}^{\infty} f_n(x) = \sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n}} \cos\left(\frac{x}{n}\right)$ unif conv on $[-r, r]$, $r > 0$?

Pf: $|f_n(x)| = \frac{1}{\sqrt{n}}$ does not converge since $p < 1$.
cannot use weierstrass M-test.
we use Thm 8.35:
- choose $x_0 = 0$, then $\sum_{n=1}^{\infty} f_n(0) \left(= \frac{(-1)^n}{\sqrt{n}}\right)$ is convergent by the AST.
- $\sum_{n=0} f_n'(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^{3/2}} \sin\frac{x}{n}$.

and $|f_n'(x)| \le \frac{1}{n^{3/2}}$ ∴ $\sum f_n'(x)$ converges by the p-test.

∴ f_n' converges by WMT.
∴ f_n is unif convergent. (Thm 8.35).

Thm 8.36. There exist a function $\mathbb{R} \to \mathbb{R}$ that is continuous everywhere but differentiable nowhere.

Pf: A known cts but not diff fn is $\phi(x) = |x|$.
Make it periodic: $\phi(x+2) = x$. show $\forall x, y \in \mathbb{R}$,
$|\phi(x) - \phi(y)| \le |x - y|$ $\forall x, y \in \mathbb{R}$.

Let $f_n(x) = \left(\frac{3}{4}\right)^n \phi(4^n x)$ $\forall x \in \mathbb{R}$. f_n is cts on $\mathbb{R}$.
Consider $f = \sum_{n=0}^{\infty} f_n$.
$|f_n(x)| = \left|\left(\frac{3}{4}\right)^n \phi(4^n x)\right| \le \left(\frac{3}{4}\right)^n$ $\forall x \in \mathbb{R}$.
∴ $\sum_{n=1}^{\infty} f_n \xrightarrow{u} f$ by WMT. Note that f is cts.

we show f is however, not diff anywhere.
$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ if it exists. define a sequence:
$\forall m \in \mathbb{N}$, $\left(4^m a - \frac{1}{2},\ 4^m a + \frac{1}{2}\right)$ contains at most 1 integer.

Pf (Weierstrass's fn) (cont'd).

$4^m a - \frac{1}{2}$ $\quad 4^m a \quad$ $4^m a + \frac{1}{2}$

define $h_m = \pm \frac{4^{-m}}{2}$ s.t. there is no integer between $4^m a$ and $4^m a + (4^m h_m) = \pm\frac{1}{2}$

$h_m = \left(\frac{\pm 4^{-m}}{2}\right)$.

Then $h_m \to 0$. Let $g_m = \dfrac{f(a+h_m) - f(a)}{h_m}$, $m \in \mathbb{N}$.

We claim g_m diverges, hence witnessing that $f'(a)$ does not exist $\Rightarrow f$ is not differentiable at a.

Claim: g_m is divergent.

$f(a+h_m) - f(a) = \sum_{n=0}^{\infty} (f_n(a+h_m) - f_n(a))$ (defn of $f \xleftarrow{u} \sum f_n$).

$$= \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n \left(\phi(4^n a + 4^n h_m) - \phi(4^n a)\right)$$

$4^n h_m = \pm\frac{4^{n-m}}{2}$ is an even integer whenever $n > m$.

But $\phi(x+2k) = \phi(x)$ is periodic. Then $\phi(4^n a + 4^n h_m) = \phi(4^n a)$.

$\Rightarrow \forall n > m$, the sum is 0.

$$= \sum_{n=0}^{m} \left(\frac{3}{4}\right)^n \left(\phi(4^n a + 4^n h_m) - \phi(4^n a)\right)$$

$$|f(a+h_m) - f(a)| \geq \left(\frac{3}{4}\right)^m |\phi(4^m a + 4^m h_m) - \phi(4^m a)| - \left|\sum_{n=0}^{m-1} \phi(4^n a + 4^n h_m) - \phi(4^n a)\right|$$ (△ ineq).

simplify

move in using △ ineq.

Recall $|\phi(x) - \phi(y)| \leq |x-y| \Rightarrow |\phi(4^n a + 4^n h_m) - \phi(4^n a)| \leq |4^n h_m|$.

Simplifying $|\phi(4^m a + 4^m h_m) - \phi(4^m a)|$: we have $= 4^m h_m$ (why?)

$$\therefore \geq \left(\frac{3}{4}\right)^m 4^m h_m - \sum_{n=0}^{m-1} \left(\frac{3}{4}\right)^n 4^n |h_m|$$

$$= |h_m| \left(3^m - \sum_{n=0}^{m-1} 3^n\right) = \frac{3^m + 1}{2} \to \infty \quad \square$$

MA3110 W9L2

Power series

Defn: A series of the form $\sum_{n=0}^{\infty} a_n(x-\overset{\text{center}}{x_0})^n = a_0 + a_1(x-x_0) + \ldots$ with const $a_0, \ldots$ is called a <u>power series in $(x-x_0)$</u>.

$$\sum_{n=0}^{\infty} a_n(x-x_0)^n = \sum_{n=0}^{\infty} f_n.$$

Qn: On what set does a power series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ converge?

Eg. It converges at least at $x = x_0$, " $= a_0$

Recall: Ratio Test. Suppose $\sum_{n=1}^{\infty} a_n$, $a_n \neq 0$.
~~then~~ and $\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ exists.

(i) $\rho < 1 \Rightarrow$ conv. abs.

(ii) $\rho > 1 \Rightarrow$ diverges (including $\rho = \infty$)

(iii) no conclusion if $\rho = 1$.

Eg. Consider the power srs $\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} n!\, x^n$

- center $= x_0 = 0$, thus convergent at $x = 0$.
- for $x \neq 0$, $\rho = \lim_{n\to\infty} \left|\frac{(n+1)!}{n!}\right| = \infty \Rightarrow \sum_{n=0}^{\infty} n!\,x^n$ diverges by Ratio Test.

$\therefore$ converges only at $x = 0$

Eg. Consider the power srs $\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$\therefore \rho = \lim_{n\to\infty} \left| \frac{x^{n+1}/(n+1)!}{x^n/(n!)} \right| = \lim_{n\to\infty} \frac{|x|}{n+1} \to 0 \Rightarrow$ Converges everywhere by R.T.

Eg. Consider the geom. srs $\sum_{n=0}^{\infty} x^n$ ($\to \frac{1}{1-x}$ for $|x| < 1$, diverges $|x| \geq 1$)

Thm 9.1.1 $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ given a power series:

(i) converges at $x = x_1 \Rightarrow$ abs conv. for all values of x, $|x-x_0| < |x_1 - x_0|$

(ii) diverges at $x = x_2 \Rightarrow$ diverges for any x, $|x-x_0| > |x_2 - x_0|$

MA3110 W9L2.(2)

Pf (i) Suppose $\sum_{n=0}^{\infty} a_n(x_1-x_0)$ conv. WTS $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ conv. abs for $|x-x_0|<|x_1-x_0|$

$\therefore$ We have $a_n(x_1-x_0) \to 0$.

$\Rightarrow$ bdd $\Rightarrow \exists M>0$, $|a_n(x_1-x_0)^n| \leq M$.

By comparison test, $|a_n(x-x_0)^n|$

$= |a_n(x_1-x_0)^n| \times \left|\frac{(x-x_0)^n}{(x_1-x_0)^n}\right|$

$= |a_n(x_1-x_0)^n| \left|\frac{x-x_0}{x_1-x_0}\right|^n$ Let $r = \frac{x-x_0}{x_1-x_0}$.

$\leq Mr^n$ and by $|x-x_0|<|x_1-x_0| \Rightarrow |r| = \left|\frac{x-x_0}{x_1-x_0}\right| < 1$

$\sum_{n=0}^{\infty} Mr^n = M\sum_{n=0}^{\infty} r^n = \frac{M}{1-r}$ converges.

$\Rightarrow$ By comparison test, $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ conv. absolutely. □

(ii) Suppose $\sum_{n=0}^{\infty}(x_2-x_0)^n$ ~~converges~~ diverges. WTS x s.t. $|x-x_0|>|x_2-x_0|$.

If $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ is convergent, then by (i), $\sum_{n=0}^{\infty} a_n(x_2-x_0)^n$ will converge. Hence $\sum_{n=0}^{\infty}(x_2-x_0)^n$ diverges $\forall x$ s.t. $|x-x_0|>|x_2-x_0|$.

Defn: Radius of conv $R = \sup\{|x-x_0| \mid \sum a_n(x-x_0)^n \text{ conv.}\} = \sup S$.

Thm 9.1.2 ~~a~~ power sr $\sum_{n=0}^{\infty} a_n(x-x_0)^n$

1. converges absolutely $\forall x \in (x_0-R, x_0+R)$,
2. diverges for all $(-\infty, x_0-R)$, (x_0+R, ∞).

(says nothing about endpts.)

Pf (1): Let x be given, then $|x-x_0|<R \Rightarrow \exists |x_1-x_0| \in S$, $|x-x_0|<|x_1-x_0|$

$\Rightarrow$ convergent at x by thm 8.1.1.

(2) By contradiction, if $x \notin [x_0-R, x_0+R]$ converges $\Rightarrow x \in S$ and.

$\Rightarrow |x-x_0|>R$ ($\to\leftarrow$).

Thm 9.13 Suppose $a_n \neq 0 \ \forall n$.

(i) If $\ell = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$ exists, radius of conv $\sum_{n=0}^{\infty} a_n (x-x_0)^n$

$$R = \begin{cases} \frac{1}{\ell} & \text{if } \ell > 0 \\ \infty & \text{if } \ell = 0. \end{cases}$$

(ii) If $\ell = \infty$, $R = 0$.

Pf: for $x \neq x_0$, apply ratio test to $\sum_{n=0}^{\infty} a_n (x-x_0)^n$:

$$\lim_{n\to\infty} \left| \frac{a_{n+1}(x-x_0)^{n+1}}{a_n (x-x_0)^n} \right| = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| |x-x_0| = |x-x_0| \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$
$$= |x-x_0| \ell$$

By ratio test, converges $\Leftrightarrow$ $|x-x_0| < \frac{1}{\ell}$.

Eg. $\sum_{n=1}^{\infty} a_n (x-x_0)^n = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (x-1)^n$.

$$\ell = \lim_{n\to\infty} \left| (-1) \frac{(n)}{(n+1)} \right| = \lim_{n\to\infty} \left| \frac{n}{n+1} \right| = 1$$

$\therefore R = \frac{1}{\ell} = 1 \Rightarrow$ conv. abs on $(x_0 - 1, x_0 + 1) = (0, 2)$.

At $x = 0$, $\sum -\frac{1}{n}$ diverges, $x = 2$, $\sum \frac{1}{n}(-1)^{n+1}$ converges.

MA3110 W10 L1

Recap: Power Series $\sum_{n=0}^{\infty} a_n(x-x_0)^n = a_0 + a_1(x-x_0) + \dots$

$= \sum_{n=0}^{\infty} f_n(x)$, $f_n(x) = a_n(x-x_0)^n$.

$\boxed{\text{When does it converge?}}$

Collect the convergent points $E = \{x \in \mathbb{R} \mid \sum_{n=0}^{\infty} a_n(x-x_0)^n \text{ converges}\}$.

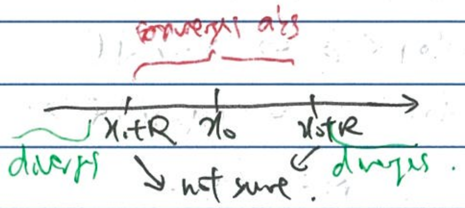
Then - srs converge pointwise on E.

- E has a fixed form: is either $E = \{x_0\}$ or an interval centered at x_0.
- Let R = radius of convergence, then either.

① $R=0 \Rightarrow E = \{x_0\}$.

② $R=\infty \Rightarrow E = \mathbb{R}$

③ ~~R≠~~ $0 < R < \infty$



Thus the interval E

can be open, closed, half open/closed → 4 possibilities.

$\boxed{\text{How to calculate R?}}$

If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \ell$ exists, then $R = \frac{1}{\ell}$.

Defn (limsup for bdd sequence) (b_n) is bdd sequence. If there is a subsequence (b_{n_k}) and $(b_{n_k}) \to b$, then b is the cluster/limit pt of (b_n).

Why bdd? Bolzano-Weierstrass property: any bounded sequence has a conv. subseq.

Let $C(b_n)$ be the set of cluster pts of (b_n). Then define limit superior

$\limsup b_n := \sup C(b_n)$ $(= \max(b_n))$

Eg 1. If (b_n) is conv, $(b_n) \to b$, then any subseq $(b_{n_k}) \to b$. Thus $C(b_n) = b$

$\Rightarrow \limsup b_n = \liminf b_n = b$

2. Let $b_n = (-1)^n + \frac{1}{n}$, $n \in \mathbb{N}$. Then $b_{2k} = 1 + \frac{1}{2k} \to 1$, $b_{2k+1} = -1 + \frac{1}{2k+1} \Rightarrow -1$.

$\therefore C(b_n) = \{1, -1\} \Rightarrow \limsup b_n = 1$.

Thm (Root Test) $\ell = \limsup |a_n|^{1/n}$.

(i) $\ell < 1 \Rightarrow \sum_{n=0}^{\infty} a_n$ conv. abs.

(ii) $\ell > 1 \Rightarrow \sum_{n=0}^{\infty} a_n$ diverges

(iii) $\ell = 1 \Rightarrow$ no conclusion.

weaker version:
If $\ell = \lim_{n\to\infty} |a_n|^{1/n}$ exists, then (i), (ii), (iii) holds.

Thm (Cauchy-Hadamard) $\sum_{n=0}^{\infty} a_n(x-x_0)^n$, and let $\rho = \limsup |a_n|^{1/n}$.
Then radius of convergence $R = \begin{cases} 0 & \text{if } \rho = \infty \\ 1/\rho & \text{if } 0<\rho<\infty \\ \infty & \text{if } \rho = 0. \end{cases}$

Pf: Apply the root test to $\sum_{n=0}^{\infty} a_n (x-x_0)^n$: positive constant.

$\limsup |a_n(x-x_0)^n|^{1/n} = |x-x_0| \limsup |a_n|^{1/n}$.

$= |x-x_0| \rho = \begin{cases} 0 & \text{if } \rho=0 \Rightarrow \text{abs. conv } \forall x \in \mathbb{R}. \\ \infty & \text{if } \rho = \infty \end{cases}$

Assume $0<\rho<\infty$. Then

$|x-x_0| < \frac{1}{\rho} \Rightarrow |x-x_0|\rho < 1 \Rightarrow \sum_{n=0}^{\infty} a_n(x-x_0)^n$ conv. abs.

$|x-x_0| > 1/\rho \Rightarrow |x-x_0|\rho > 1 \Rightarrow$ diverge.

$\therefore R = 1/\rho$.

Eg. consider $\sum_{n=0}^{\infty} (\sin \frac{n\pi}{4}) x^n$.

There are some zero a_n and thus $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \rho$ ~~dn~~ does not exist.

Use Cauchy-Hadamard:

$(|a_n|^{1/n}) = |\sin \frac{n\pi}{4}|^{1/n}$.

Some constant subsequences: include $a_{n_{4k+1}} = \{1, 1, \dots\}$

and is the limsup. (any clusterpt of (a_n) cannot exceed 1).

$\therefore R = 1/1 = 1 \Rightarrow$ converges abs on $(-1,1)$.

Check end pts: at $x=1$, does not converge. (since $a_n \not\to 0$).

$x=-1$, ~~conv. by alt. ser. test~~ (also $a_n \not\to 0$)

Eg. $\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \frac{1}{2^n} x^{2n} = 1 + 0 + \frac{x^2}{2} + 0 + \dots$

1. Use Cauchy-Hadamard formula.

$(|a_n|^{1/n}) = (\frac{1}{2^n})^{\frac{1}{2n}} = \frac{1}{\sqrt{2}} \to \frac{1}{\sqrt{2}}$. $\rho = \limsup_{n} |a_n|^{1/n} = \frac{1}{\sqrt{2}}$.

$\therefore R = 1/\rho = \sqrt{2}$. conv. abs on $(-\sqrt{2}, \sqrt{2})$.

2. or use ratio test directly:

$\lim_{n\to\infty} \left| \frac{x^{2n+2}/2^{n+1}}{x^{2n}/2^n} \right| = \lim_{n\to\infty} \frac{x^2}{2} = \frac{x^2}{2}$.

By ratio test, conv. abs when $\frac{x^2}{2} < 1 \Rightarrow |x| < \sqrt{2}$.

$x^2/2 > 1 \Rightarrow |x| > \sqrt{2}$.

$\therefore R = \sqrt{2}$, conv. at $(-\sqrt{2}, \sqrt{2})$.

MA3110 W10L1 (2)

Eg 2 (cont'd) At endpts:

$\sum_{n=0}^{\infty} x^{2n}/2^n$ when $x=\pm\sqrt{2}$, $x^{2n}=2^n \Rightarrow \sum_{n=0}^{\infty} 1$ diverges. □

<u>Properties of power series.</u>

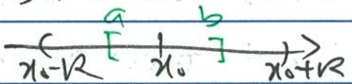
Qn. We know $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ converges on $(x_0-R,\ x_0+R)$ for some R.

Define $f:(x_0-R, x_0+R)\to\mathbb{R}$, $f=\sum_{n=0}^{\infty} a_n(x-x_0)^n$.

Is f continuous? intgb? differentiable? (note the fns in $\sum f_n$ are all 3.)

Thm. 9.21 Suppose $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ has a nonzero radius of conv. R, let a,b

$x_0-R<a<b<x_0+R$

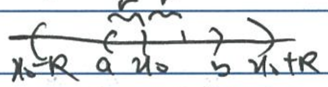


Then $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ conv. uniformly on $[a,b]$.

Q: do we have unif. conv on $(x_0-R,\ x_0+R)$?

A: NO.

Pf:



Let $r=\max(|a-x_0|, |b-x_0|)$.

We want to show unif. conv on a symmetric interval about the center

$\therefore$ let $[x_0-r,\ x_0+r]\supseteq[a,b]$.

$\sum_{n=0}^{\infty} a_n(x-x_0)^n$ conv. $\forall x\in(x_0-R,\ x_0+R)$

$\Rightarrow$ when $x=x_0+r$, $\sum_{n=0}^{\infty} a_n(x_0+r-x_0)^n=\sum_{n=0}^{\infty} a_n r^n$ conv. abs.

$\forall x\in[a,b]$ $|x-x_0|\le r$

$\Rightarrow$ $|a_n(x-x_0)^n| = |a_n|r^n$

$\Rightarrow$ $\|a_n(x-x_0)^n\|_{[a,b]}\le |a_n r^n| = M_n$ (we can then use Weierstrass M-test)

Since $\sum M_n=\sum a_n r^n$ conv. abs, $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ conv. unif on $[a,b]$.

Remark: Then f is hence cts, itgb and diff on $[a,b]$, $\forall\ x_0-R<a<b<x_0+R$.

In fact, $f\in C^{\infty}(x_0-R,\ x_0+R)$.

MA3110 W10L2

Recall ~~thm~~: Given $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ with rad. of conv $R (= \frac{1}{\rho}, \rho = \limsup |a_n|^{1/n})$,

1. then

Lem 9.22 $\lim_{n\to\infty} a_n = a > 0$ and (b_n) is bdd $\Rightarrow a \limsup b_n = \limsup a_n b_n$

Thm 9.23 (power srs is inf. diff in rad. of conv.)

$\sum_{n=0}^{\infty} a_n(x-x_0)^n$ has rad. of conv. R. Then within (x_0-R, x_0+R), we let the limit be f, then f is infinitely differentiable, where

$f'(x) = \sum_{n=1}^{\infty} n a_n (x-x_0)^{n-1}$, in ~~gen~~ general,

$f^{k}(x) = \sum_{n=k} n(n-1)\dots(n-k+1)(x-x_0)^k$

Pf Recall H-C test: $R = \frac{1}{\rho}$, $\rho = \limsup |a_n|^{1/n}$.

We want to show $\frac{d}{dx}\sum_{n=0}^{\infty} a_n(x-x_0)^n \overset{?}{=} \sum_{n=0}^{\infty} \frac{d}{dx}(a_n(x-x_0)^n) = \sum_{n=0}^{\infty} n a_n (x-x_0)^{n-1}$

RHS is also a power srs. By srs of functions, this is true if $f_n'(x) = n a_n (x-x_0)^{n-1}$ converges on the same interval (x_0-R, x_0+R).

<u>Claim:</u> $\sum f_n$ and $\sum f_n'$ have the same R.

Denote the rad. of conv. of $\sum f'$ by R'. Using HC test,

$R' = \frac{1}{\rho'}$, where $\rho' = \limsup |n a_n|^{1/n}$

$= \limsup (n^{1/n}) |a_n|^{1/n} = \limsup |a_n|^{1/n} = \rho$

$\Rightarrow R' = R$. □

Now we conclude $\sum_{n=1}^{\infty} f_n' = \sum_{n=1}^{\infty} n a_n (x-x_0)^{n-1}$ conv. unif. on any closed interval contained by (x_0-R, x_0+R), same as $\sum f_n$.

Thm 8.35 gives us that $\sum f_n$ and $\sum f_n'$ are unif conv on (x_0-R, x_0+R)

$\Rightarrow \left(\sum_{n=0}^{\infty} a_n(x-x_0)^n\right)' = \sum_{n=0}^{\infty} f_n'(x)$ (term-by-term diff.) □.

MA3110 W10L2 (2).

Eg. (Geometric series) Let $\frac{1}{1-x} = f(x) = \sum_{n=0}^{\infty} x^n$, $x \in (-1,1)$

Note that the domain of $\frac{1}{1-x}$ is $\mathbb{R} \setminus \{1\}$, but

intv of conv. of $\sum_{n=0}^{\infty} x^n$ is $(-1,1)$.

1. We have $f'(x) = \frac{1}{(1-x)^2} = \frac{d}{dx} \sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} n x^{n-1}$

$\therefore \sum_{n=1}^{\infty} n x^{n-1} = \frac{1}{(1-x)^2}$ (we found another sum which we can write.)

Rem: Power series and derived (as in derivative) series has the same R, they may converge on diff sets.

Eg. $f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2}$

① By Thm 9.13: $\rho = \lim_{n\to\infty} \frac{1/(n+1)^2}{1/n^2} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^2 = 1 \Rightarrow R = 1/1 = 1$.

Check end pts: $\sum_{n=1}^{\infty} \frac{(1)^n}{n^2}$ is a p-series, since $p>1$ it converges.

$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ is an alt-srs. since $\frac{1}{n^2} \to 0$, convergent.

$\therefore$ convergent at $[-1,1]$.

② $f'(x) = \sum_{n=1}^{\infty} \frac{x^{n-1}}{n}$.

$\rho = \lim_{n\to\infty} \frac{1/(n+1)}{1/n} = \lim_{n\to\infty} \frac{n}{n+1} = 1 \Rightarrow R = 1$

Check end pts: $\sum_{n=1}^{\infty} \frac{(1)^{n-1}}{n}$ is the harmonic srs and diverges.

$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ is an alt. srs, conv. by the AST $\therefore [-1,1)$.

③ $f''(x) = \sum_{n=1}^{\infty} \frac{(n-1)}{n} x^{n-2}$ $\forall x \in (-1,1)$

When $x=1$, $\sum_{n=1}^{\infty} 1-\frac{1}{n}$ diverges since $1-\frac{1}{n} \to 1$ as $n\to\infty$.

$x=-1$, $\sum_{n=1}^{\infty} (-1)^{n-2}(1-\frac{1}{n})$ also diverges. $\therefore (-1,1)$.

Cor 9.24. $f(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n$ $\forall x \in (x_0 - r, x_0 + r)$. ← int of the ROC.

for some $r>0$, then $a_k = \frac{f^{(k)}(x_0)}{k!}$ $\forall k \in \mathbb{N} \cup \{0\}$.

Pf: Let R be the roc of $f(x)$. Then $r \le R$.

$\therefore$ By thm 9.23, $f^{(k)}(x) = \sum_{n=k}^{\infty} n(n-1)\dots(n-k+1) a_n (x-x_0)^{n-k}$ $\forall x \in (x_0-r, x_0+r)$

$= k! a_k (x-x_0)^0 + \frac{(k+1)!}{1} a_{k+1} (x-x_0)^1 + \dots$

$\therefore f^{(k)}(x_0) = k! a_k + 0$.

$\Rightarrow a_k = \frac{f^{(k)}(x_0)}{k!}$ □

Cor 9.25 (uniqueness of power srs).
If $\sum_{n=0}^{\infty} a_n(x-x_0)^n = \sum_{n=0}^{\infty} b_n(x-x_0)^n \quad \forall x \in (x_0-r, x_0+r)$, then
$a_k = \frac{f^k(x_0)}{k!} = b_k \quad \forall k = 0, \dots \Rightarrow$ they are the same srs.

Eg. Consider Geom. srs: $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad |x|<1$.
Replace x with $-x$, $\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n (x)^n$, $|-x| = |x| < 1$.
x^2, $\frac{1}{1-x^2} = \sum_{n=0}^{\infty} x^{2n}$, $|x^2| < 1 \Leftrightarrow |x| < 1$
$-x^2$, $\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}$, $|-x^2| < 1 \Leftrightarrow |x| < 1$
(these representations are unique)

Ex: $f(x) = \frac{1}{1+x^2}$ what is $f^{(100)}(0)$?
$\Rightarrow f(x) = \sum_{n=0}^{\infty} (x^2)^n = \sum_{n=0}^{\infty} x^{2n} = 1 + 0 + x^2 + 0 + x^4 + \dots$
By cor 9.24. $f^{(100)}(0) = k! a_k = 100!$

Cor 9.26: Given $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ with RoC $= R$, then $\forall x_0 - R < a < b < x_0 + R$,
$$\int_a^b \sum_{n=0}^{\infty} a_n(x-x_0)^n \, dx = \sum_{n=0}^{\infty} \int_a^b a_n(x-x_0)^n \, dx \quad \text{(term-by-term)}.$$

skip ahead and define it this way.
will be defined as e as power srs.

Eg. $\ln(1+x) = \int_0^x \frac{dt}{1+t}$ is there a power srs representation?
$\frac{1}{1+t} = \sum_{n=0}^{\infty} (-1)^n t^n \Rightarrow \ln(1+x) = \int_0^x \left(\sum_{n=0}^{\infty} (-1)^n t^n\right) dt$
$= \sum_{n=0}^{\infty} \int_0^x (-1)^n t^n \, dt$
$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1} \quad x \in (-1, 1)$.
when $x = -1$, $\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} (-1)^{n+1} = -\sum_{n=0}^{\infty} \frac{1}{n+1}$ diverges.
$x = 1$ $\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$ convergent by AST.
Note that $\frac{1}{1+t}$ conv. at $(-1, 1)$, but $\int \frac{1}{1+t}$ conv. at $(-1, 1]$
$\rightarrow$ one extra pt.
Now $\ln(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1} \quad \forall x \in (-1, 1)$
since RHS is conv at $(-1, 1]$, then can we say holds at $x = 1$?
Ans: Yes, but need more thms.

MA3110 W11L1

Eg. Given $\sum_{n=0}^{\infty} \frac{x^{3n}}{n+1}$, find ① R, ② set at which it converges, ③ closed form.
Hence evaluate $\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$.

Pf: $\sum_{n=0}^{\infty} \frac{x^{3n}}{n+1} = 1+0+0+\frac{x^3}{2}+0+0+\frac{x^6}{3}+\dots$
$= \sum_{n=0}^{\infty} a_n x^n$. Since some $a_n = 0$, we can't use
$R = \frac{1}{\rho}$, $\rho = \left|\frac{a_{n+1}}{a_n}\right|$. But we can directly apply ratio test to the srs:
Fix $x \neq 0$. Write $\sum_{n=0}^{\infty} \frac{x^{3n}}{n+1} = \sum_{n=0}^{\infty} b_n$

$\therefore \rho_b = \lim_{n\to\infty} \left|\frac{b_{n+1}}{b_n}\right| = \lim_{n\to\infty} \left|\frac{x^{3n+3}/n+2}{x^{3n}/n+1}\right| = \lim_{n\to\infty} \left|\frac{n+1}{n+2}\right| |x^3| = |x^3|$.

$\therefore$ By the ratio test, the lim. converges if $\rho = |x^3| < 1 \iff |x| < 1$
diverges if $\rho = |x^3| > 1 \iff |x| > 1$

① $\therefore R = 1$. Let us check end pts.
when $x = 1$, $\sum_{n=0}^{\infty} \frac{1}{n+1}$ diverges.
$x = -1$, $\sum_{n=0}^{\infty} \frac{(-1)^{3n}}{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$ converges by AST.

② $[-1, 1)$.

③ We technically only know closed form for geometric srs. $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, $|x| < 1$
$x^3 f(x) = \sum_{n=0}^{\infty} \frac{x^{3n+3}}{n+1}$ for $x \in (-1,1)$, diff TBT. (since convergent).
$3x^2 f(x) + x^3 f'(x) = \sum_{n=0}^{\infty} \frac{x^{3n+2}}{\cancel{n+1}}(3) = \sum_{n=0}^{\infty} (3x^2)(x^3)^n$
$= 3x^2 \frac{1}{1-x^3}$

Let $g(x) = x^3 f(x)$. Known $g'(x) = \frac{3x^2}{1-x^3}$.
$\therefore g(x) - g(0) = \int_0^x g'(t)\,dt = \int_0^x \frac{3t^2}{1-t^3} = -\ln|1-t^3|\Big|_0^x = -\ln(1-x^3)$

$\therefore g(x) = 3x^2 f(x) = -\ln(1-x^3) \Rightarrow f(x) = \frac{-\ln(1-x^3)}{3x^2}$ when $x \neq 0$.

when $x = 0$, $\sum_{n=0}^{\infty} \frac{x^{3n}}{n+1}\Big|_{x=0} = 1$
$x = -1$, By Abel's Theorem,
$f(-1) = \lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} \left(-\frac{\ln(1-x^3)}{x^3}\right) = \ln 2 = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$

Recap: Given $\sum_{n=0}^{\infty} a_n (x-x_0)^n$ have $R > 0$. Then it converges absolutely on (x_0-R, x_0+R) and uniformly $\forall [a,b] \subset (x_0-R, x_0+R)$.
If we define $f: I \to \mathbb{R}$, $f(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n$.
(un)Surprisingly, $f'(x) = \sum_{n=0}^{\infty} (a_n (x-x_0)^n)'$ is another power series having the

same radius of conv. Similarly, we can differentiate it further, and conclude ~~that~~ $f \in C^\infty(x_0-R, x_0+R) \subseteq \mathbb{R}$ manifolds.

Similarly, integrating power series yield another power series that converge with the same R.

Qn: will the endpt, if convergent, ~~represented~~ at $x_0 \pm R$, will the $\sum_{n=0}^\infty a_n (x-x_0)^n$ be unif. convergent on $[x_0-R, x_0+R]$?

Ans: Yes, by Abel's thm.

Thm. Abel's Formula $(b_n), (c_n)$ be sequences in $\mathbb{R}$. $\forall n \geq m \geq 1$, let
$$B_{n,m} = \sum_{k=m}^{n} b_k = S_n^b - S_{m-1}^b.$$ (similar to cauchy's criterion)

Then $\sum_{k=m}^{n} b_k c_k = B_{n,m} c_n - \sum_{k=m}^{n-1} (c_{k+1} - c_k)$.

Pf: fix $m \geq 1$. For $k > m$, $B_{k,m} = \sum_{j=m}^{k} b_j = b_m + \dots + b_k$.
$$B_{k-1,m} = b_m + \dots + b_{k-1}.$$
$$\Rightarrow B_{k,m} - B_{k-1,m} = b_k \quad \& \quad b_m = B_{m,m}.$$
$$\therefore \forall n \geq m \geq 1, \sum_{k=m}^{n} b_k c_k = b_m c_m + \sum_{k=m+1}^{n} (B_{k,m} - B_{k-1,m}) c_k.$$
$$= B_{m,m} c_m + \sum_{k=m+1}^{n} B_{k,m} c_k + \sum_{k=m+1}^{n} B_{k-1,m} c_k$$
Let $k = k+1$
$$= B_{n,m} c_n + \sum_{k=m}^{n-1} B_{k,m} c_k + \sum_{k=m}^{n-1} B_{k,m} c_{k+1}$$
$$= B_{n,m} c_n + \sum_{k=m}^{n-1} B_{k,m} (c_k - c_{k+1}).$$

Thm Abel's Theorem Suppose that $\sum_{n=0}^\infty a_n (x-x_0)^n$ has a finite, non 0 rad. of R. $(0 < R < \infty)$

(i) If the srs converges at $x_0 + R$ $(x_0 - R)$, then it conv. uniformly on $[x_0, x_0+R]$. $([x_0-R, x_0].)$

Pf: we use cauchy criterion to show unif. conv. on $[x_0, x_0+R]$; given $\sum_{n=0}^\infty a_n (R)^n$ is convergent. WTS given $\varepsilon > 0$, $\exists K$ s.t. $n \geq m \geq K \Rightarrow \left|\sum_{k=m}^{n} f_k(x) = a_k (x-x_0)^k\right| < \varepsilon$ $\forall x \in [x_0, x_0+R]$.

= use Abel's formula.

MA3110 W11L1 (2)

Pf (Abel's Thm, ctd). Let $x_1 \in (x_0, x_0+R]$.

$$\sum_{n=0}^{\infty} a_n(x_1-x_0)^n = \sum_{n=0}^{\infty}(a_nR^n)\left(\frac{x_1-x_0}{R}\right)^n = \sum b_n c_n$$

where $b_n = a_nR^n$, $c_n = r^n = \frac{x_1-x_0}{R} < \frac{R}{R} = 1$ also $x_1 > x_0 \Rightarrow c_n > 0$.

By Abel's formula, $\left|\sum_{k=m}^{n} a_k(x_1-x_0)^k\right| = \left|\sum_{k=m}^{n} b_k c_k\right|$

$$= \left|B_{nm}c_n + \sum_{k=m}^{n} B_{km}(c_k - c_{k+1})\right|$$

$$\leq |B_{nm}c_n| + \sum_{k=m}^{n} |B_{km}(c_k - c_{k+1})| \quad \text{since } c_k \text{ is } \downarrow.$$

$$= c_n\left|\sum_{j=m}^{n} b_j\right| + \sum_{k=m}^{n}\left[(c_k - c_{k+1})\left|\sum_{j=m}^{k} b_j\right|\right].$$

Since $\sum_{n=0}^{\infty} b_n = a_nR^n$ converges, by Cauchy criterion, $\forall \varepsilon > 0$, $\exists K \in \mathbb{N}$ s.t.

$n \geq m \geq K \Rightarrow \left|\sum_{j=m}^{n} b_j\right| < \varepsilon$.

$$\leq c_n\varepsilon + \sum_{k=m}^{n}(c_k - c_{k+1})\varepsilon = c_m\varepsilon = r^n\varepsilon \leq \varepsilon.$$

This is also valid (vacuously) at x_0. □.

Cor 9.27. Let $f(x) = \sum_{n=0}^{\infty} a_n(x-x_0)^n$, $x \in (x_0-R, x_0+R)$.

where R is the rad of conv. of the series, and $0 < R < \infty$. Then the series is continuous at x_0+R (resp. x_0-R) if f is conv. at $x = x_0+R$ (resp $x = x_0-R$).

Eg. $\frac{1}{1+x} = \sum_{n=0}^{\infty}(-1)^n x^n$. LHS is defined on $(-1,\infty)$ but

RHS — " — $(-1,1)$ (convergent).

~~Recall $\sum_{n=0}^{\infty}(-1)^n x^n$ ...~~

$\ln(1+x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{n+1}}{n+1}$, $-1 < x < 1$. but LHS defined on $(-1,\infty)$.

Note that $\left.\sum_{n=0}^{\infty}(-1)^n \frac{x^{n+1}}{n+1}\right|_{x=1}$ is convergent at $x=1$.

$\therefore$ By Cor 9.27, $\ln(1+1) = \sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}$ by continuity at $x=1$.

(in fact, $(-1,1]$).

MA3110 W11L2

Taylor & Maclaurin series

If we have a smooth (C^∞) function, we can write it as a power series with center x_0.

Defn. Suppose $f \in C^\infty(x_0 - r, x_0 + r)$, then $\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$ is called the Taylor series of f. If $x_0 = 0$, it is called the Maclaurin srs.

Remark: - Taylor srs of f does not always converge to f near x_0.

Eg: $f(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$ has $f^{(n)}(0) = 0$. Its maclaurin srs is thus 0,

$\therefore f(x) \neq \sum_{n=0}^{\infty} f^{(n)}(0) \frac{x^n}{n!} \quad \forall x \in \mathbb{R} \setminus \{0\}$.

- But a power series that converge to f has to be the Taylor srs. By Cor 9.24

Eg. Recall that integrating $\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$ the Geom srs gives us

$$\ln(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1}, \quad -1 < x < 1.$$

Let $f(x) = \ln x$, $x_0 > 0$. Find a series about x_0 which rep. f.

Pf: $f(x) = \ln(x_0 + x - x_0)$

$$= \ln\left(x_0\left(1 + \frac{x - x_0}{x_0}\right)\right)$$

$$= \ln x_0 + \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}\left(\frac{x-x_0}{x_0}\right)^{n+1}, \text{ if } -1 < \frac{x-x_0}{x_0} < 1$$

$$= \ln x_0 + \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)x_0^{n+1}}(x-x_0)^{n+1} \quad \text{if } 0 < x < 2x_0.$$

Recall Taylor's Thm: f, and $f, f', \ldots, f^{(n)}$ cts on $[a,b]$, $f^{(n+1)}$ exists on (a,b). If $x_0 \in [a,b]$ then $\forall x \in [a,b]\ \exists c_n \in (x, x_0)$ s.t.

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + \frac{f^{(n+1)}(c_n)}{(n+1)!}(x-x_0)^{n+1}.$$

Thm 9.31. Taking $n \to \infty$, $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n \quad \forall x \in (x_0 - r, x_0 + r)$

$\iff R_n \to 0$.

Eg. e^x, $\sin x$, $\cos x$ have $R_n \to 0$ as $n \to \infty$ hence the Taylor srs are convergent on $\mathbb{R}$.

MA3110 W11 L2 (2).

Eg. Let $g(x) = (1-3x^2)\cos x^2$ for $x \in \mathbb{R}$. Let $g^{(n)}$ be the nth ord. deriv. of g.

Find $g^{(2010)}(0)$, $g^{(2011)}(0)$, $g^{(2012)}(0)$.

Pf: Derive the Maclaurin series of $g = \sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} x^n$.

$\because \cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$.

$\cos(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{4n}$.

$g(x) = (1-3x^2) \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{4n}$.

$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{4n} - \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} 3x^{4n+2}$, $x \in \mathbb{R}$.

Since $2010 = 4(502)+2$,

coeff of $x^{2010} = \frac{g^{(2010)}(0)}{2010!} = \frac{(-1)^{502}}{(1004)!} 3$

$\therefore g^{(2010)}(0) = 3 \times \frac{2010!}{1004!}$

since 2011 is neither $4n$ nor $(4n+2)$, $g^{(2011)}(0) = 0$.

coeff of $x^{2012} = x^{4 \times 503}$ $\therefore g^{(2012)}(0) = \frac{2012!}{1006!}(-1)^{503} = -{}^{2012}C_{1006}$.

Defn: f is <u>analytic</u> on (a,b) if.

(i) f is infinitely diff on (a,b), and

(ii) $\forall x_0 \in (a,b)$, The Taylor srs of f about $x_0 \to f$ in a neighborhood of x_0

Lemma: If taylor series converge on (a,b) (g. centred on $x_0 \in (a,b)$) then it is analytic. (a,b).

Eg. 1. e^x, $\sin x$, $\cos x$, are analytic (rep. by Taylor srs on $\mathbb{R}$.

2. $\frac{1}{1-x}$ is analytic on $(-1,1)$.

3. Analytic $\subset C^\infty \subset C^1$ = ~~different~~ cont. diff. $\subset$ diff $\subset$ unif cts $\subset$ continuous $= C^0 \subset$ all fns.

4. $e^x = 1 + \frac{x}{1} + \frac{x^2}{2!} + \ldots$ for $x \in \mathbb{R}$.

$\therefore e^x - 1 = x\left(1 + \frac{x}{2!} + \frac{x^2}{3!} + \ldots\right)$

$\therefore \frac{e^x - 1}{x} = \sum_{n=0}^{\infty} \frac{x^n}{(n+1)!}$ for $x \neq 0$.

Since it is analytic except at $x=0$, we define $f(0) = \sum_{n=0}^{\infty} \frac{x^n}{(n+1)!} \Big|_{x=0} = 1$.

$\therefore f(x) = \begin{cases} \sum_{n=0}^{\infty} \frac{x^n}{(n+1)!} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$. then f is analytic on $\mathbb{R}$.

Qn: Can we do arithmetic on power series?

$\sum a_n + \sum b_n = \sum (a_n + b_n)$. Straightforward.

$c \sum a_n = \sum c a_n$.

How about $(\sum a_n)(\sum b_n) = \sum c_n$, c_n in terms of a_n & b_n?

Defn: The cauchy product of series $\sum_{n=0}^{\infty} a_n$, $\sum_{n=0}^{\infty} b_n$ is the series $\sum_{n=0}^{\infty} c_n$, where $\forall n \in \mathbb{N}$,

$$c_n = \sum_{k=0}^{n} a_k b_{n-k} = a_0 b_n + \dots + a_n b_0.$$

Motivation: Attaching x terms to group,

$$(\sum a_n x^n)(\sum b_n x^n) = (a_0 + a_1 x + a_2 x^2 + \dots)(b_0 + b_1 x + b_2 x^2 + \dots)$$

$$= a_0 b_0 + a_0 b_1 x + a_0 b_2 x^2 + \dots$$
$$+ a_1 b_0 x + a_1 b_1 x^2 + \dots$$
$$+ a_2 b_0 x^2 + \dots$$
$$+ \dots$$

$$= c_1 + c_2 x + c_3 x^2 + \dots$$
$$= \sum_{n=0}^{\infty} c_n x^n$$

Thm (Merten's) $\sum a_n$ converges absolutely, $\sum b_n$ converges, then $\sum c_n = (\sum a_n)(\sum b_n)$ by the cauchy product converges.

(sums are 0 to inf otherwise notated).

Pf: Let $\sum |a_n| = \alpha$, $\sum a_n = A$, $\sum b_n = B$.

write partial sums $\sum_{n=0}^{k} a_n = A_k$, $\sum_{n=0}^{k} b_n = B_k$, $\sum_{k=0}^{n} c_k = C_k$

where by defn of cauchy product, $c_k = \sum_{i+j=k} a_i b_j$

We are given that $A_n \to A$, $B_n \to B$. WTS $C_n \to AB$ as $n \to \infty$.

Note that $A_n B_n \neq C_n$ easily, $A_2 B_2$ has "power 4" term but C_2 does not.

$$C_n = c_0 + c_1 + \dots + c_n = (a_0 b_0) + (a_0 b_1 + a_1 b_0) + \dots + (a_0 b_n + \dots + a_n b_0)$$
$$= a_0 (b_0 + b_1 + \dots + b_n) + a_1 (b_0 + \dots + b_{n-1}) + \dots + a_n (b_0).$$
$$= a_0 B_n + a_1 B_{n-1} + \dots + a_n B_0$$
$$= \sum_{i+j=n} a_i B_j.$$

Let $\beta_n = B_n - B$ $\forall n$. note $\beta_n \to 0$ and $B_n = B + \beta_n$.

MA3110 W11L2 (3).

(Ct'd) $c_n = a_0(B+\beta_n)+\dots+a_n(B+\beta_0)$

$= A_n B + \underbrace{(a_0\beta_n+\dots+a_n\beta_0)}_{\gamma_n}$.

we need only to show $\gamma_n \to 0$. When n is large s.t. ( )

$|\gamma_n| = |a_0\beta_n+\dots+a_n\beta_0| \le |a_0||\beta_n|+\dots+|a_n||\beta_0|$.

$= |a_0||\beta_n|+\dots+|a_{n-k}||\beta_k|$ (up till k).

$+|a_{n-k+1}||\beta_{k-1}|+\dots+|a_n||\beta_0|$ (rest).

Since $\beta_n \to 0$, $\forall \varepsilon>0, \exists k$ s.t. $\forall j \ge k$, $|\beta_j - 0| < \varepsilon/2\alpha$.

$< \left(\sum_{i=0}^{n-k} |a_i|\right)\frac{\varepsilon}{2\alpha} + \left(\sum_{j=n-k+1}^{n} |a_j|\right) M$

for some bound M s.t. $|\beta_i| \le M \ \forall i$.

(less than $\sum a_n$) $\le \alpha(\varepsilon/2\alpha) + (\varepsilon/2M)M = \varepsilon$. □

Recap: Merten's Theorem say if one of $\sum a_n, \sum b_n$ is abs conv, the other cond. conv, then $\sum c_n$ the Cauchy product is cond. conv. (might be abs. conv.

Note: If both are cond. conv. Cauchy pdt might not converge.

(★) Eg: Let $a_n = b_n = \frac{(-1)^{n+1}}{\sqrt{n+1}}$. Then $\sum_{n=0} a_n$ and $\sum_{n=0}$ converges conditionally

(by AST. & $\sum_n |a_n|$ diverges since p-srs with $p=\frac{1}{2}\le 1$.)

$\therefore$ Cauchy pdt: $c_n = \sum_{k=0}^{n} a_k b_{n-k} = \sum_{k=0}^{n} \frac{(-1)^{(k+1)}(-1)^{(n-k+1)}}{\sqrt{k+1}\,\sqrt{n-k+1}}$

all terms are +ve or -ve

$\therefore |c_n| = \left|\sum_{k=0}^{n} \frac{(-1)^n}{\sqrt{k+1}\sqrt{n-k+1}}\right| \ge \sum_{k=0}^{n} \frac{1}{\sqrt{n+1}\sqrt{n+1}} = \frac{1}{n+1} \times n+1 = 1$.

$\therefore \sum |c_n|$ diverges. since $|c_n| \ge 1$, $c_n \not\to 0$, $\sum c_n$ diverges.

$\therefore \sum a_n, \sum b_n$ converges (cond.), and $\sum c_n$ (cauchy) ~~converges~~ diverges.

Cor. 9.41: If both srs conv. abs, then the cauchy pdt conv. abs. and $\sum c_n = (\sum a_n)(\sum b_n)$.

Pf: Tut 11.

Eg. Let $|r|<1$. $\sum_{n=1}^{\infty} r^n$ conv. abs to $\frac{1}{1-r}$. Then the Cauchy product with itself:
$$c_n = \sum_{k=0}^{n} a_k b_{n-k} = \sum_{k=0}^{n} r^k r^{n-k} = (n+1) r^n.$$
Then by Cor 9.41, $\sum_{n=0}^{\infty} (n+1) r^n = \left(\frac{1}{1-r}\right)^2$.

Thm 9.42 Let $f(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n$, $|x-x_0|<R_1$,
$g(x) = \sum_{n=0}^{\infty} b_n (x-x_0)^n$, $|x-x_0|<R_2$ (same center).
α, β be constants. Then.
(i) $\alpha f(x) + \beta g(x) = \sum_{n=0}^{\infty} (\alpha a_n + \beta b_n)(x-x_0)^n$ for $|x-x_0| < \min(R_1, R_2)$
(ii) $f(x) g(x) = \sum_{n=0}^{\infty} c_n (x-x_0)^n$, $|x-x_0| < \min(R_1, R_2)$.
where c_n is the n^{th} term of the cauchy product of $(\sum a_n, \sum b_n)$,
$$c_n = \sum_{i+j=n} a_i b_j.$$

Pf (ii) Let x be given, where $|x-x_0| < \min(R_1, R_2) = R$
Then $f(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n = \sum_{n=0}^{\infty} \alpha_n$
$g(x) = \sum_{n=0}^{\infty} b_n (x-x_0)^n = \sum_{n=0}^{\infty} \beta_n$ } conv abs, since $x \in (x_0 - R, x_0 + R)$
$\therefore$ The cauchy product converges absolutely:
$$\sum_{n=0}^{\infty} \gamma_n = \left(\sum_{n=0}^{\infty} \alpha_n\right)\left(\sum_{n=0}^{\infty} \beta_n\right) = f(x) g(x).$$
where
$$\begin{aligned}\gamma_n &= \sum_{k=0}^{n} \alpha_k \beta_{n-k} \\ &= \sum_{k=0}^{n} a_k (x-x_0)^k b_{n-k} (x-x_0)^{n-k} \\ &= (x-x_0)^n \sum_{k=0}^{n} a_k b_{n-k} \text{ cauchy pdt.} \\ &= c_n (x-x_0)^n\end{aligned}$$
$$\therefore \sum_{n=0}^{\infty} \gamma_n = \sum_{n=0}^{\infty} c_n (x-x_0)^n$$
□

Qn: Can we express $\frac{f(x)}{g(x)} = f(x) \cdot \frac{1}{g(x)}$ as a power series?
A: Yes, but $f(x_0)$ (or $g(x_0)$, rather) $\neq 0 \Rightarrow a_0 \neq 0$.
Assume $\frac{1}{f(x)}$ can be written as $\sum_{n=0}^{\infty} b_n (x-x_0)^n$
Then $\frac{1}{f(x)} = \sum_{n=0}^{\infty} b_n (x-x_0)^n$ for some b_n.
$f(x) \cdot \frac{1}{f(x)} = 1 = 1 + 0(x-x_0) + \ldots$
where $c_0 = a_0 b_0$, $c_1 = a_0 b_1 + a_1 b_0 = a_0 b_1$, ...
(Details in Tut 10).

MA3110 W12L1.

Defn: $E(x) = \sum_{n=0} \frac{x^n}{n!}$ is called the exponential function.

Thm 9.5.1 The exponential function $E(x): \mathbb{R} \to \mathbb{R}$ has the following properties:

(i) $E(0) = 1$ and $E'(x) = E(x)$ $\forall x \in \mathbb{R}$.

(ii) $E(x+y) = E(x)E(y)$. $\forall x, y \in \mathbb{R}$.

(iii) $E(x) > 0$ $\forall x \in \mathbb{R}$.

(iv) E is strictly increasing on $\mathbb{R}$.

(v) $\lim_{x\to\infty} E(x) = \infty$, $\lim_{x\to-\infty} E(x) = 0$.

Pf (i): $E(0) = 1 + 0 + 0 + \dots = 1$.

$$E'(x) = \sum_{n=0}^{\infty} \frac{d}{dx}\left(\frac{x^n}{n!}\right) = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} \frac{x^n}{n!} = E(x).$$

(ii) $E(x)E(y) = \left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right)\left(\sum_{n=0}^{\infty} \frac{y^n}{n!}\right)$.

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{x^k}{k!} \times \frac{y^{n-k}}{(n-k)!}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$ use the binom. thm,

$$= \sum_{n=0}^{\infty} \frac{1}{n!} (x+y)^n.$$

$$= E(x+y).$$

(iii) For $x > 0$, $\frac{x^n}{n!} > 0 \Rightarrow E(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} > \sum_{k=0}^{n} \frac{x^k}{k!}$ any partial sum.

In particular, $E(x) > 1 = S_1$ for any $x > 0$.

$\therefore E(x) > 0$. But $E(-x)E(x) = E(0) = 1 > 0 \Rightarrow E(-x) > 0$ $\forall x > 0$.

$\therefore E(x) > 0$ $\forall x \in \mathbb{R}$.

(iv) $E'(x) = E(x) > 0$ $\forall x \in \mathbb{R} \Rightarrow E$ is strictly $\uparrow$.

(v) For $x > 0$, $E(x) > 1 + x \to \infty$.

$$\lim_{x\to-\infty} E(x) = \lim_{x\to\infty} E(-x) = \frac{1}{\lim_{x\to\infty} E(x)} = 0.$$

Remark. $E(x) \Leftrightarrow$ (i) is uniquely determined by property (i).

Pf: By induction, $f^{(k)}(x) = \dots = f''(x) = f'(x) = f(x)$.

By Taylor's thm, $f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k + R_n(x)$,

$$R_n(x) = \frac{f^{(n+1)}(c_{n+1})}{(n+1)!}(c_{n+1})^{n+1} = \frac{f(c_{n+1})}{(n+1)!}(c_{n+1})^{n+1}$$ as $n \to \infty$, $\to 0$.

Compare with other values (defns):

(a) Euler's number $e = \lim_{n\to\infty} (1+\frac{1}{n})^n$.

(b) $e = \sum_{n=0}^{\infty} \frac{1}{n!} = E(1)$.

(c) $x \in \mathbb{R}$, $e^x := \lim_{n\to\infty} e^{r_n}$ where r_n is a rational seq. conv. to x.

Thm 9.52. We have $E(x) = e^x \; \forall x \in \mathbb{R}$.

Pf: First show $\mathbb{N}$: $\forall n \in \mathbb{N}$, $E(n) = E(1+\dots+1)$
$= e^1 e^1 \dots e^1 = e^n$.

Next show $\mathbb{Q}$: Given any $r = \frac{n}{m}$, $r \in \mathbb{Q}^+$, then

$$E(rm) = E(n) = e^n$$
$$= E(\underbrace{r+r+\dots+r}_{m}) = (E(r))^m.$$
$$\therefore E(r) = e^{n/m}.$$
$$E(-r) = \frac{1}{E(r)} = e^{-r}.$$

For $\mathbb{R}$: $E(x) = e^x$ on all rational numbers. By the density thm, (since any two cts fns that agree on rat, agree on $\mathbb{R}$)
$E(x) = e^x \; \forall x \in \mathbb{R}$.

The logarithm function

Since $E(x)$ is strictly $\uparrow$ hence injective, we have an inverse: $L: (0,\infty) \to \mathbb{R}$,
$L(E(x)) = x \; \forall x$, $E(L(y)) = y \; \forall y > 0$.

Derivative: since $\forall x \in \mathbb{R}$, $E'(x) \neq 0$, L is diff at $y = E(x) \; \forall x \in \mathbb{R}$,
$\therefore L'(y) = \frac{1}{E'(x)} = \frac{1}{E(x)} = \frac{1}{y}$.

By FTC 2: $\int_1^y \frac{1}{t}\,dt = L(y) - L(1) = L(y)$.

Defn: $L: (0,\infty) \to \mathbb{R}$ is the natural logarithm, written as $\ln$.
$\therefore e^{\ln x} = x \; \forall x > 0$,
$\ln(e^x) = x \; \forall x \in \mathbb{R}$.

MA3110 W12L1 (2).

Pf (ct'd). We know $A \subset \ker$ by computation (?). Then we have a map of abelian gps $F(V\times W)/A \twoheadrightarrow V\otimes_{\mathbb{R}} W$.

We then define a $\mathbb{R}$-module (v.s.) structure on $F(V\times W)/A$ via.

$$k\cdot(v,w) = (kv, w) = (v, kw) \quad \forall (v,w) \in F(V\times W)/A.$$

extending this linearly, this is a well-defined $\mathbb{R}$-mod structure on $F(V\times W)/A$. Also, the map is a mod-hom:

$$k\cdot(v,w) = (kv, w) \mapsto (kv)\otimes_{\mathbb{R}}(w) = k\cdot(v\otimes_{\mathbb{R}} w)$$

It remains to check $\dim_{\mathbb{R}} F(V\times W)/A \leq 9$, or to find a $\mathbb{R}$-spanning set of cardinality 9. Then the map is inj since $\dim(\mathrm{im}) = 9$.

Suppose V, W be $\mathrm{Mat}_{3\times 3}(\mathbb{R})$, then let the $\mathbb{R}$-bases of V, W: $\{v_1, v_2, v_3\}$ be a basis of V, $\{w_1, w_2, w_3\}$ be a basis of W.

Claim: $\{(v_i, w_j) \mid i, j = 1,2,3\}$ spans $F(V\times W)/A$.

eg. Let $(v_1+v_2+v_3, w_1+2w_2+3w_3) \in F(V\times W)/A$.

$$= (v_1, w_1+2w_2+3w_3) + (v_2+v_3, w_1+2w_2+3w_3)$$

$$= \cdots = (v_1, w_1) + 2(v_1, w_2) + \ldots$$

is spanned by the set claimed. □

Eg (motivation). Recall if $S \subset R$ (S is a subring with 1), then any R-mod is a S-mod (by restriction).

Qn: Given a S-mod, can we make it a (meaningful) R-mod? No, restricting the R-mod will get S-mod back?

Eg: $\mathbb{Z} \subset \mathbb{Q}$, then $M = \mathbb{Z}_2$ is a $\mathbb{Z}$-mod but M cannot be a $\mathbb{Q}$-mod, since any $\mathbb{Q}$-mod is a $\mathbb{Q}$-vs. (? 1 elem is of dim 0 is ...?)

Defn: Let $S \subset R$ be a subring of R containing 1. Let N be a S-mod. We define R-mod $R\otimes_S N$ together with an S-mod hom $\iota: N \to R\otimes_S N$ via the following univ. prop: $\forall$ R-mod M, considered as a S-mod, and any S-mod hom $\varphi: N \to M$, $\exists$ a uniq R-mod map $\bar{\varphi}: R\otimes_S N \to M$ s.t.:

$$\begin{tikzcd}
\underset{\text{S-mod}}{N} \arrow[r, "\iota"] \arrow[dr] & \underset{\text{R-mod}}{R\otimes_S N} \arrow[d, dashed, "\bar{\varphi}"] \\
& \underset{\text{R-mod}}{M}
\end{tikzcd}$$

Recall: $F(\mathbb{Z}) = \{\sum_{i\in\mathbb{Z}} a_i \underline{i}\}$ but $a_i 1 + a_i 2 \neq a_i 3$. $N \xrightarrow{i} R\otimes_S N$

use its elems as bases ← cannot reach one another.

$$\begin{tikzcd} N \arrow[r, "i"] \arrow[dr] & R\otimes_S N \arrow[d, "\bar\varphi"] \\ & M \end{tikzcd}$$

Thm: $R\otimes_S N$ exists and is unique up to isom. of R-mod.

pf: The uniqueness is standard (exercise).

We prove the existence following the previous eg/prop.

Let's start with the free ab. gp of inf. rank(?) $F(R\times N)$. Then we can define $\pi: F(R\times N) \to M$.

① start with an Ab gp

Before that, let $A \subset F(R\times N)$ to be gen. by.

② mod out minimal relations:

(1) $(r_1+r_2, n) - (r_1, n) - (r_2, n)$ (so when quotiented $= \sum$ RHS ?)

(2) $(r, n_1+n_2) - (r, n_1) - (r, n_2)$.

☆ (3) $(rs, n) - (r, s\cdot n)$ for $s\in S$. ($rs = r\cdot s$ if we consider R as a right as a right S-mod)

Then in $F(R\times N)/A$ we have $(r_1+r_2, n) = (r_1, n) + (r_2, n)$.

We claim $F(R\times N)/A$ with the embedding $N \to F(R\times N)/A$, $n \mapsto (1, n)$ is the R-mod we want. We first check the diagram:

$\forall\ \varphi: N\to M$, we have a map $F(R\times N) \to M$, $(r, n) \mapsto r\cdot\varphi(n)$.

(Q4) $\therefore$

$$\begin{tikzcd} N \arrow[r] \arrow[dr, "\varphi"] & F(R\times N) \arrow[r] & F(R\times N)/A \\ & M & \end{tikzcd}$$

$n \mapsto (1,n) \mapsto (1,n)$; $\varphi(n)$, $1\cdot\varphi(n)$.

③ check that the mod hom props are preserved (universally).

To check the map factors through $F(R\times N)/A$, we need to check A maps to 0 in M. We check (3) here:

$$(rs, n) - (r, s\cdot n) \mapsto rs\,\varphi(n) - r\,\varphi(s\cdot n) = rs(\varphi(n) - \varphi(n)) = 0.$$

It is straightfwd to check $\bar\varphi$ is the unique R-mod hom to make everything commute. For the R-mod struct on $F(R\times N)/A$, define $r\cdot(r', n) \overset{\text{def}}{=} (r\cdot r', n)$. (check ~~that~~ R-mod struct.) □

We write $r\otimes_S n$ as the elem (r, n) in $F(R, N)/A$,

$(r\otimes_S n = r\cdot(i(n))$

If S is understood, we write $\otimes_S$ as $\otimes$.

MA3110 W12L2 Date No.

Thm 9.6.1 The natural log, $\ln: (0,\infty) \to \mathbb{R}$ has the below props:

(i) $\frac{d}{dx} \ln y = \frac{1}{y}$ $\forall y > 0$.

(ii) $\ln y = \int_1^y \frac{1}{t} dt$.

(iii) $\ln(xy) = \ln x + \ln y$ $\forall x, y > 0$.

(iv) $\ln(1) = 0$, $\ln(e) = 1$.

(v) $\forall x > 0$, $\alpha \in \mathbb{R}$, $x^\alpha = e^{\alpha \ln x}$.

Pf: (iii) $e^{\ln x + \ln y} = e^{\ln x} \cdot e^{\ln y} = xy$

$\therefore \ln(e^{\ln x + \ln y}) = \ln(xy)$. $\Rightarrow \ln x + \ln y = \ln xy$. $\forall x, y > 0$.

(v) α is rational is well-defined (roots etc).

α is irrational then construct a rational sequence to it.

Case 1: $\alpha \in \mathbb{N}$, $\alpha = n$.

Then $\ln(x^n) = \ln(x \cdot x \cdot \ldots \cdot x)$

$= n \ln x$.

$\therefore e^{\ln(x^n)} = e^{n \ln x}$.

Case 2: $\alpha \in \mathbb{Q}$. Let $\alpha = \frac{p}{q}$, $\gcd(p,q) = 1$

$\ln(x^{p/q}) = \ln(x^{\frac{1}{q}} \cdots x^{\frac{1}{q}} x^{\frac{1}{q}})$

$= p \ln(x^{1/q})$

Cor. 9.6.2 (diff. rule for $\mathbb{R}$ exponents). Let $\alpha \in \mathbb{R}$, then $f(x) = x^\alpha$ $(x > 0)$ have derivative $f'(x) = \alpha x^{\alpha - 1}$ $(x > 0)$.

Pf $f'(x) = \frac{d}{dx} e^{\alpha \ln x} = e^{\alpha \ln x} \frac{d}{dx}(\alpha \ln x) = x^\alpha \frac{\alpha}{x} = \alpha x^{\alpha - 1}$ $\forall x > 0$. $\square$

The trigonometric functions:

$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$ and $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$ are conv. on $\mathbb{R}$ by ratio test.

Defn: $C(x)\ (=\cos x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$ $\forall x \in \mathbb{R}$. is the cosine fn.

$S(x)\ (=\sin x) = \sum_{n=0}^{\infty} (-1) \frac{x^{2n+1}}{(2n+1)!}$ $\forall x \in \mathbb{R}$ is the sine fn.

We then verify the properties of $C(x)$ and $S(x)$.

Thm 9.71 $C'(x) = -S(x)$ and $S'(x) = C(x)$ $\forall x \in \mathbb{R}$.

Pf $C'(x) = \frac{d}{dx}\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$ since $C(x)$ converges on $\mathbb{R}$, can diff T by T:

$= \sum_{n=0}^{\infty} \frac{d}{dx}\left((-1)^n \frac{x^{2n}}{(2n)!}\right)$

$= \sum_{n=1}^{\infty} (-1)^n \frac{x^{2n-1}}{(2n-1)!} = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{x^{2n+1}}{(2n+1)!} = -S(x)$.

$S'(x) = \frac{d}{dx}\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{d}{dx}\left((-1)^n \frac{x^{2n+1}}{(2n+1)!}\right)$

$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = C(x)$. □

Thm 9.72 $C^2(x) + S^2(x) = 1$ $\forall x \in \mathbb{R}$.

Pf Let $f(x) = C^2(x) + S^2(x)$ $\forall x \in \mathbb{R}$.

$\therefore f'(x) = -2C(x)S(x) + 2S(x)C(x) = 0$.

$\therefore f(x)$ is constant on $\mathbb{R}$.

$f(0) = C(0)^2 + S(0)^2 = 1^2 + 0^2 = 1$. $\Rightarrow f(x) = 1$ $\forall x \in \mathbb{R}$. □

Thm 9.73 $g: \mathbb{R} \to \mathbb{R}$ has the prop: $g''(x) = -g(x)$ $\forall x \in \mathbb{R}$ $\Leftrightarrow$
$g(x) = \alpha C(x) + \beta S(x)$ $\forall x \in \mathbb{R}$, where $\alpha = g(0)$ and $\beta = g'(0)$.

Pf ($\Leftarrow$) can be verified. Let $g(x) = g(0)C(x) + g'(0)S(x)$.

$\therefore g'(x) = -g(0)S(x) + g'(0)C(x)$

$g''(x) = -g(0)C(x) - g'(0)S(x) = -g(x)$.

MA3110 W12L2 (2)

Date No.

Pf (cf'd) ($\Rightarrow$). Let $h(x) = g(x) - \alpha C(x) - \beta S(x)$ where $\alpha = g(0)$, $\beta = g'(0)$.

we want to show $h(x) = 0$ $\forall x \in \mathbb{R}$.

Maclaurin srs of h: $\sum_{n=0}^{\infty} \frac{h^{(n)}(x)}{n!} x^n$.

$h(0) = g(0) - \cancel{g(0)}\alpha \overset{=1}{C(0)} - \beta \overset{=0}{S(0)} = g(0) - \alpha = 0$.

$h'(0) = \cancel{g(0)} g'(0) + \alpha S(0) - \beta C(0) = g'(0) - \beta = 0$.

$h''(x) = -h(x)$ $\forall x \in \mathbb{R}$ (because this prop holds for C, S.)

$h'''(x) = -h'(x)$. $\Rightarrow h''(0) = -h(0) = 0$

$h'''(0) = -h'(0) = 0$.

$\therefore h^{(n)}(0) = 0$ $\forall n \in \mathbb{N}$.

$\therefore$ Maclaurin srs $\sum_{n=0}^{\infty} 0 x^n \overset{?}{=} h(x)$.

By T10Q5, $|h^{n}(x)| \le M$ $\forall n \forall x$ near 0 $\Rightarrow R_n(x) \to 0$.

and we have the Maclaurin srs $= h(x)$ on $x \in \mathbb{R}$. $(=0)$ $\square$.

Thm 9.74. (sum of angles).

(a) C is even. $C(-x) = C(x)$ $\forall x \in \mathbb{R}$, and

$C(x+y) = S(x)S(y) - C(x)C(y)$. $\forall x, y \in \mathbb{R}$.

(b) S is odd. $S(-x) = -S(x)$ $\forall x \in \mathbb{R}$, and

$S(x+y) = S(x)C(y) + S(y)C(x)$. $\forall x, y \in \mathbb{R}$.

Pf: (a): Fix y. Let $f(x) = C(x+y)$ $\forall x \in \mathbb{R}$.

WTS $f''(x) = -f(x)$.

$f'(x) = -S(x+y)$ $f''(x) = -C(x+y) = -f(x)$.

By thm 9.73, $f(x) = \alpha C(x) + \beta S(x)$.

$\therefore C(x+y) = C(y)C(x) \cancel{+} - S(y)S(x)$.

By noting $\alpha = f(0)$, $\beta = f'(0)$.

(b):

Thm 9.7.5 If $x \geq 0$, then we have.

(i) $-x \le s(x) \le x$

(ii) $1-\frac{x^2}{2} \le c(x) \le 1$ need these to get ☆.

(iii) $x - x^3/6 \le s(x) \le x$.

(iv) $1 - x^2/2 \le c(x) \le 1 - x^2/2 + x^4/24$. ☆

Cannot use Taylor's thm since we want it to be true $\forall x > 0$. (why?)

Pf: (i) $c^2(x) \le c^2(x) + s^2(x) = 1 \Rightarrow |c(x)| \le 1$.

$\therefore -1 \le c(x) \le 1$

$\int_0^x -1\,dt \le \int_0^x c(t)\,dt \le \int_0^x 1\,dt$.

$-x \le s(x) - s(0) \le x$ □

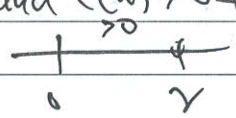
(ii) $\int_0^x (-t)\,dt \le \int_0^x s(t)\,dt \le \int_0^x t\,dt$

$-\frac{x^2}{2} \le [-c(t)]_0^x \le \frac{x^2}{2}. \Rightarrow \underbrace{-\frac{x^2}{2} \le 1 - c(x) \le \frac{x^2}{2}}$.

redundant. why? $c(x) \ge 1 - \frac{x^2}{2}$.

Lem 9.7.6 C has a root r in the interval $[\sqrt{2}, \sqrt{6-2\sqrt{3}}]$ and $c(x) > 0 \ \forall x \in [0, r)$

IOW, r is the smallest +ve root of C.



Pf. Let $p(x) = 1 - \frac{x^2}{2} \le c(x) \le 1 - \frac{x^2}{2} + \frac{x^4}{24} = q(x)$.

$p(\sqrt{2}) = 1 - \frac{2}{2} = 0$. $\Rightarrow x = \sqrt{2}$ is a root for $p(x)$.

solve $q(x) = 0$: $(x^2)^2 - (x^2) \times 12 + 24 = 0$.

MA3110 W12L2 (3) Date No.

Pf (+1d) $\frac{+12 \pm \sqrt{144-4\times 24}}{2} = x^2 \Rightarrow x^2 = 6 \pm 2\sqrt{3}$.

Note: $\sqrt{6-2\sqrt{3}} > \sqrt{6-2\sqrt{4}} = \sqrt{6-4} = \sqrt{2}$.

$\therefore$ $0 \quad a=\sqrt{2} \quad b=\sqrt{6-2\sqrt{3}}$

we have $\overset{+ve}{c(a) \geq p(a) = 0}$, $\overset{-ve}{c(b) \leq q(b) = 0}$.

By intermediate value theorem, we assume r is the smallest root between a, b.

The claim that $c(x) > 0$ is because

$c(x) > p(x) > 0 \quad \forall x \in (0, a)$.

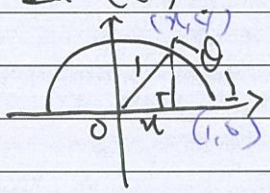
From a to r: by continuity, $c(x) > 0 \ \forall x \in (0, r)$.

Defn (since $\pi/2$ is the smallest positive root of cosine), define $\pi = 2r$.

Remark: We can check π is the smallest +ve root of $s(x)$ and $s(x), c(x)$ are periodic with period $= 2\pi$.

$\tan(x) = \frac{\sin(x)}{\cos(x)}$, etc, the rest can be defined by $s(x)$ & $c(x)$ (no need power series)

Final Qn ~~Is~~ $c(x) \overset{?}{=} \cos(x)$? that is, does $c(x)$ have geometric qualities?



$\cos(\theta) = x \Rightarrow \theta = \cos^{-1}(x)$.

θ can be found by the arclength formula; taking $y = \sqrt{1-x^2}$:

$$\theta = \int_1^x \sqrt{1+(f'(t))^2}\,dt. \quad f(t) = \sqrt{1-t^2},\ f'(t) = -\frac{t}{\sqrt{1-t^2}}$$

$$= \int_1^x \sqrt{1+\frac{t^2}{1-t^2}}\,dt. = -\int_1^x \frac{dt}{\sqrt{1-t^2}}.$$

$$\therefore \frac{d}{dx}\theta = \frac{d}{dx}\int_1^x \frac{dt}{\sqrt{1-t^2}} = -\frac{1}{\sqrt{1-x^2}} = c^{-1\prime}(x).$$

$$\therefore \frac{d}{dx}\left(\cos^{-1}(x) - c^{-1}(x)\right) = 0$$

$$\Rightarrow \cos(x) - c(x) = c(1) - \cos(1) = 0$$

$$\Rightarrow \cos(x) = c(x).$$

$\square$.

Date No.

Note

sin is strictly increasing on $[-\frac{\pi}{2}, \frac{\pi}{2}]$ (but not on $\mathbb{R}$), and $\sin(-\frac{\pi}{2}) = -1$, $\sin(\frac{\pi}{2}) = 1$ $\therefore \exists \sin^{-1} : [-1, 1] \to \mathbb{R}$.

By Inv. fn. thm,

$$\frac{d}{dy}\sin^{-1}(y) = \frac{1}{\sin'(x)} = \frac{1}{\cos(x)} = \frac{1}{\sqrt{1-\sin^2 x}} = \frac{1}{\sqrt{1-y^2}} \quad \forall y \in (-1, 1).$$

$\therefore \sin^{-1}$ is differentiable on $(-1, 1)$. (why?)

Similarly, cosine is strictly $\downarrow$ in $[0, \pi]$, and $\cos(0) = 1$, $\cos(\pi) = -1$.

$\therefore \exists \cos^{-1} : [-1, 1] \to \mathbb{R}$.

By Inv. fn. thm,

$$\frac{d}{dy}\cos^{-1}(y) = \frac{1}{\cos'(y)} = \frac{1}{-\sin x} = \frac{1}{-\sqrt{1-\cos^2 x}} = \frac{1}{-\sqrt{1-y^2}}$$

$\cos^{-1}$ is differentiable on $(-1, 1)$.

Once again, tan is strictly increasing on $[-\frac{\pi}{2}, \frac{\pi}{2}]$, $\tan(-\frac{\pi}{2}, \frac{\pi}{2}) = \mathbb{R}$.

$$\frac{d}{dy}\tan^{-1}(y) = \frac{1}{\tan'(x)} = \frac{1}{\sec^2(x)} = \frac{1}{1+\tan^2 x} = \frac{1}{1+y^2} \quad \forall y \in \mathbb{R}.$$

so $\tan^{-1}$ is diff. on $\mathbb{R}$.

verify the final eqn for $\sin x$ / $\tan x$.