

ELECTROSTATIC POTENTIAL AND CAPACITANCE

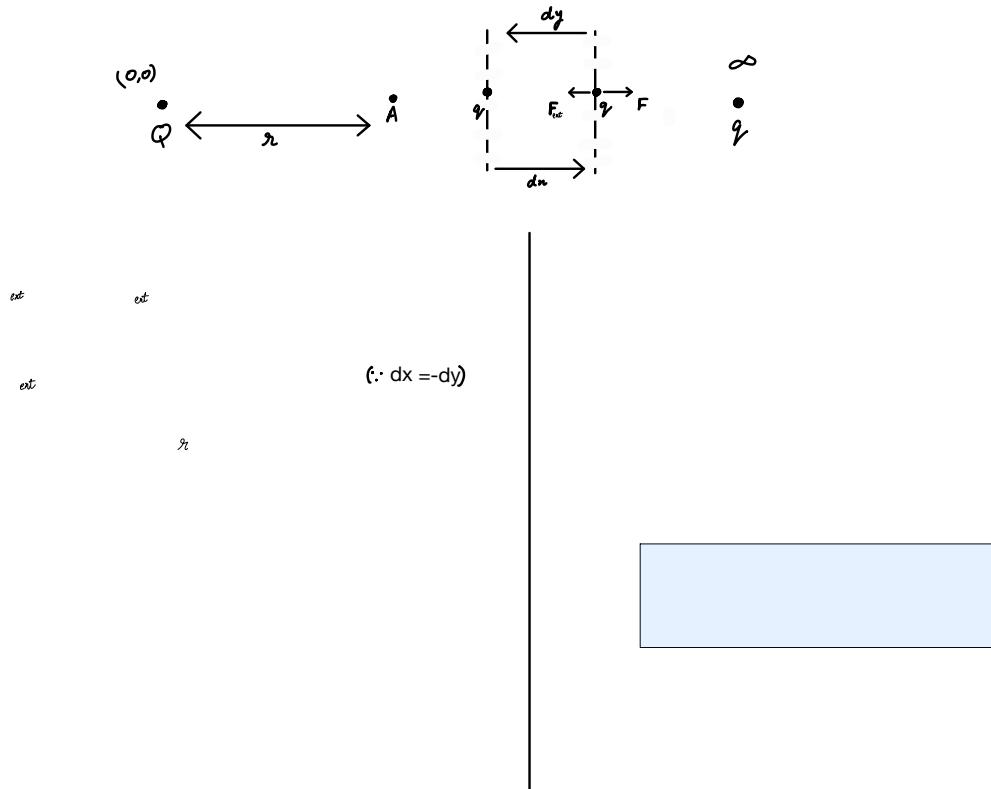


⚡ Electrostatic Potential Energy (U)(PYQ 2020,2018,2014,2010)

Electrostatic potential energy of a body at a point in an electric field can be defined as the work done by external force in moving a charge (slowly, without acceleration) from infinity to the given point, against the field. SI Unit is Joule.

Derivation (PYQ 2020)

Consider a source charge Q at origin and another charge q at infinity. An external force F_{ext} acts on q such that it is equal in magnitude but opposite in direction the Coulomb force between Q and q . So the work done by F_{ext} on q on moving it by a distance dx is-



Note: 1) We cannot calculate absolute potential energy of a body; we can only calculate the difference in potential energy of the body between two points.

2) Electrostatic force is a conservative force so work done and hence potential energy difference only depends on the initial and final positions of the object

⚡ ELECTROSTATIC POTENTIAL (V) (PYQ 2018, 2016, 2012, 2011)

Electrostatic potential at a point in an electric field is defined as the work done by external force to bring a unit positive charge (slowly, without acceleration) from infinity to the given point.(SI Unit- joule/coulomb or Volt)

By definition,

Note: We cannot calculate absolute potential at a point; we can only calculate the potential difference between two points.

*To derive expression for potential, just substitute Q_2 for 1 in the derivation for U .

⦿ **Ques:** Draw a graph showing variation of potential (V) and distance (r) (PYQ 2012)

Ans:

✓

$$V \propto \frac{1}{r}$$

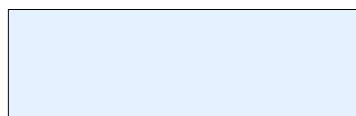
Similar PYQs

(Hint : electrostatic force is conservative)

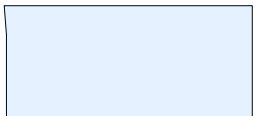
(PYQ 2018)

⚡ POTENTIAL DUE TO AN ELECTRIC DIPOLE (PYQ 2019)

Consider a dipole of dipole moment \mathbf{p} and length $2a$. Let us calculate the potential at a point O at a distance r from the center of the dipole.



1) **At axial position**



2) **At equatorial position**



Important PYQs

⌚ **Ques:** Derive an expression for the electrostatic potential at any point along the axial line of a dipole (**PYQ 2019**)

⚡ **POTENTIAL DUE TO A SYSTEM OF CHARGES**

Total potential at a point due to a system of charges is equal to the algebraic sum of potentials due to all the individual charges at that point.

Note: In the previous chapter we saw that electric field inside a charged hollow sphere is 0. So potential inside the shell is constant and is equal to its value at the surface as no work is done on moving a charge inside the shell.

Important PYQs

⌚ **Ques:** The potential at the surface of a spherical hollow metal shell with radius 6cm is 12V. What will be the potential at its center? (**PYQ 2011**)

Ans: Since the electric field inside a shell is 0, no work is done to move a charge inside the shell therefore potential at the center will be the same as the potential at its surface i.e. 12V

⚡ **EQUIPOTENTIAL SURFACES (PYQ 2020, 2019, 2014, 2013, 2010)**

Equipotential surfaces are surfaces which have a Constant value of potential at all points.

*Equipotential surfaces are always perpendicular to the electric field. (PYQ 2014)

Proof: To prove this, let us assume that electric field is not perpendicular to the equipotential surfaces. This implies that the electric field can be resolved into two components, one of which will be perpendicular to the equipotential surface and one will be parallel or along the surface.

Now, consider a charge at a point A on the surface. Since there is a component of field along the surface, when we move the charge to a point B on the surface some work will be done. This means that potential at points A and B will not be the same which is not possible as it is an equipotential surface so potential at all points on it must be the same. Therefore, by contradiction we can conclude that electric field is always perpendicular to the equipotential surface.

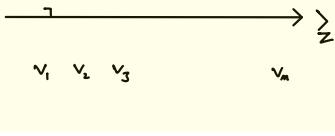
Important PYQ

Ques: Draw the equipotential surfaces due to a uniform electric field along z direction (PYQ 2019)

Ans:

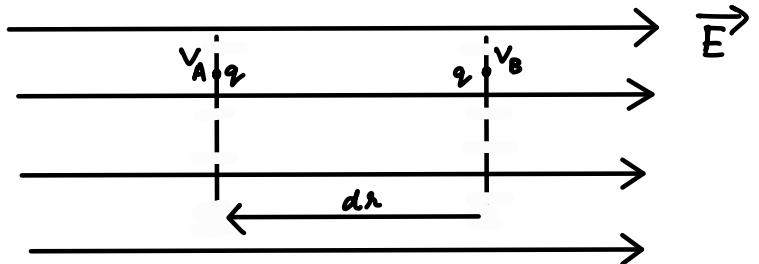
$$\vec{E} = E \hat{z}$$

$$\vec{E} = E \hat{z}$$

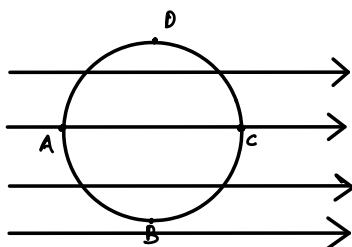
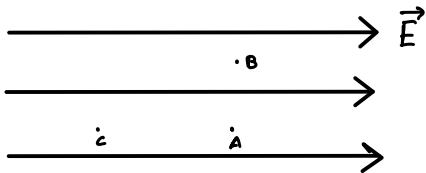


Relation between potential (V) and field (E)

Work done in moving q from B to A-

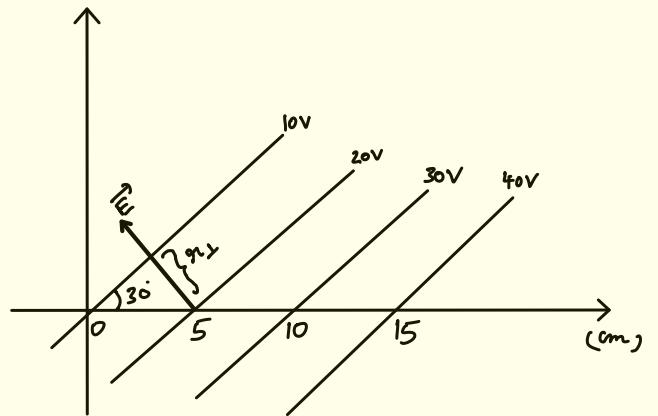


- 1) Electric field is in the direction where potential decreases the steepest.
- 2) Magnitude of electric field is given by the change in magnitude of potential per unit displacement normal to the equipotential surface at the point.



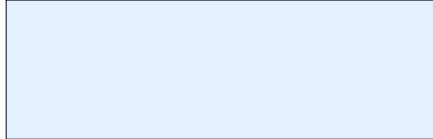
Ques: The following diagram depicts some equipotential surfaces. Calculate the magnitude and direction of electric field.

Ans: We know electric field is perpendicular to the equipotential Surfaces so let us draw a perpendicular to these surfaces.



⚡ Potential energy due to a system of charges (PYQ 2020, 2010)

Consider a charge q_1 which is brought from infinity to a point A. Since there is no electric field no work is done in this process. Now, another charge q_2 is brought from infinity to a point B at a distance r from q_1 . Since, the potential at B is $V = kq_1/r$, work is done in bringing the charge q_2 to B which will be equal to Vq_2 . Therefore, the potential energy of the system will be



Ques: Three point charges $1\mu C$, $-1\mu C$, $2\mu C$ were initially infinite distance apart. Calculate the work done in assembling these charges at the vertices of an equilateral triangle of side 10cm (PYQ 2020)

Ans: To calculate the work done, we calculate the difference in the final and Initial potential energy of the system.

The initial P.E of the system is zero as all charges were infinitely separated
The final P.E of system can be calculated as-

$$q_1 = 1\mu C$$



Note: Work done by system (W_{sys}) = -Work done by ext. force (W_{ext})

⚡ Potential energy in an external field

1. Potential energy of a single charge

If a charge q is brought from infinity to a given point in an external electric field $E(r)$, its potential energy will be equal to $q \cdot V(r)$ where $V(r)$ is the potential due to the external field at that point.

For e.g. if an electron is accelerated through a potential difference of V it will gain energy equal to eV . This unit of energy is defined as electron volt (eV) and $1\text{eV} = 1.6 \times 10^{-19} \text{ J}$.

2. Potential energy of a system of two charges in an external field (PYQ 2018)

To calculate the potential energy of a system of two charges (q_1, q_2) in

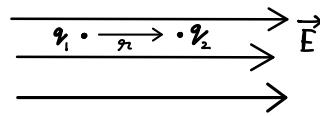
An external field, first we calculate the work done in bringing charge q_1

From infinity which will be $q_1 \cdot V(r_1)$. Then we calculate the work done in

Bringing q_2 from infinity, but this time the work done is not just against the external electric field but also against the field due to q_1 .

Work done on q_2 due to external field = $q_2 \cdot V(r_2)$

Work done on q_2 due to q_1 = kq_1q_2/r_{12}



Therefore, total potential energy of the system = $q_1 \cdot V(r_1) + q_2 \cdot V(r_2) + kq_1q_2/r_{12}$

Ques: Two charges $7\mu\text{C}$ and $-2\mu\text{C}$ are placed at $(-9\text{cm}, 0, 0)$ and $(9\text{cm}, 0, 0)$ resp. in an external electric field $E = A/r^2$ where $A = 9 \times 10^5 \text{ Cm}^{-2}$. What is the potential energy of the system?

Ans: $U = q_1 \cdot V(r_1) + q_2 \cdot V(r_2) + kq_1q_2/r_{12}$

First we need to calculate the expression for potential due to the field –

⚡ Potential energy of dipole in external field

We know that when a dipole is kept in an external electric field, it

Experiences a torque (τ) = $\mathbf{p} \times \mathbf{E}$ which causes it to rotate.

Let the dipole be at an angle θ_0 with the field initially. An external

Torque (τ_{ext}) is applied to it which is equal and opposite to the torque

Due to E . τ_{ext} rotates the dipole (very slowly and without acceleration)

Such that it now makes an angle θ_1 with the field.

Let us calculate the work done in the process-

Let the dipole rotate in the plane of the paper by an infinitesimal amount $d\theta$. Let the work done in this process be dw

This work done by the ext torque is stored in the form of potential energy of the dipole. Let $U = 0$ when $\theta = \Pi/2$



Note: We take P.E of dipole 0 at $\theta = \Pi/2$ because the work done in bringing the two charges from infinity will be equal and opposite and will cancel out in this configuration.

Important PQYs



Ques: calculate P.E of a dipole with charges $q, -q$ And length L and in stable equilibrium with field E (PYQ 2020)

Ans: stable equilibrium means $\theta = 0^\circ$

$$U = -\mathbf{p} \cdot \mathbf{E} = -qL E \cos 0^\circ = -qLE$$

Ques: A dipole of length 4cm makes an angle of 60° with the electric field and experiences a torque of $4\sqrt{3}$ Nm. Calculate P.E of dipole if it has a charge Of $\pm 8nC$ (PYQ 2014)

Ans:

⚡ Electrostatics of conductors

Metallic conductors have mobile charge carriers in the form of free electrons.

1. Inside a conductor, electrostatic field is zero (PYQ 2012)

In electrostatic situation, when there is no current inside or on the surface of the conductor, the electric field is zero inside the conductor. This is because, in static situation the free charges inside the conductor are distributed in such a way that the net electrostatic field inside a conductor is zero everywhere.

2. At the surface of a charged conductor electrostatic field must be normal to the surface at every point

If E was not normal to the surface, there would be a component of E parallel to the surface which would cause the charges on the surface to move. But since it is a static situation, that should not be the case. Therefore, electrostatic field should be normal to the surface of a conductor at all points.

3. The interior of a conductor can have no excess charge in the static situation

When a conductor is charged, all the excess charge must reside at the surface in static situation. This follows from Gauss's law. Consider any arbitrary Gaussian surface inside a conductor, since electric field is 0 inside the conductor, net flux through the surface is also 0. This implies that the net charge enclosed inside the surface is also zero.

4. Electrostatic potential is constant throughout the volume of the conductor and has the same value (as inside) on the surface

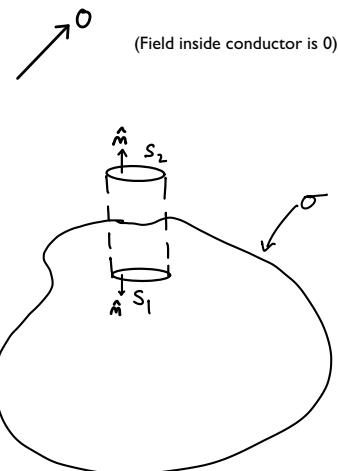
This follows from results 1 and 2. Since $E=0$ everywhere inside the conductor and has no tangential component, no work is done in moving a charged particle inside or on the surface of the conductor hence there is no potential difference between any two points inside or on the surface of a conductor.

5. Electric field at the surface of a charged conductor



Where σ is the surface charge density and \hat{m} is a unit vector normal to the surface of the conductor in the Outward direction.

Proof: consider a short cylindrical Gaussian surface that is half inside and half outside the conductor.



6. Electrostatic shielding

Whatever be the charge and field configuration outside, any cavity in a conductor remains shielded from outside electric influence i.e. the field inside the cavity is always zero.

Use- to protect sensitive instruments from outside electrical influence.

Note: when a conductor is kept in an external electric field, the charge carriers (free e⁻s) move and are redistributed in such a way that the electric field produced due to the induced charges opposes the external field such that the two fields cancel each other out completely and net electrostatic field inside the conductor is zero.

⚡ Dielectrics and polarization

Dielectrics are non-conducting substances.

Dielectric in an external field

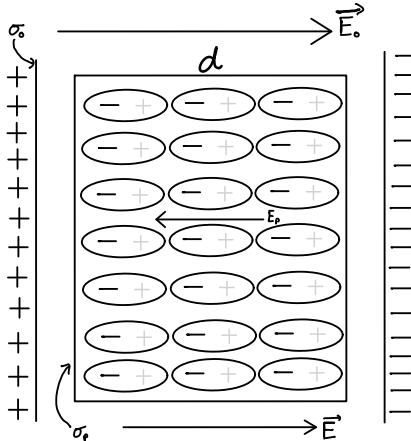
When a dielectric is kept in an external electric field, the field causes re orientation of the molecules of the dielectric such that each molecule becomes a dipole and the net result of this is the appearance of charges on the surface of the dielectric which produces a field which opposes the external field. This is called polarization of a dielectric.

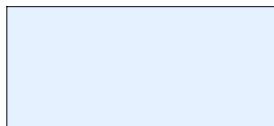
Unlike a conductor, this induced field doesn't cancel out the external field completely but can only reduce it.

The extent of polarization depends on two factors-

- i) The potential energy of the dipole in the external field trying to align the dipole along the field
- ii) The thermal energy tending to disrupt the alignment

Let E_0 be the external field and E_p be the field due to polarization of the dielectric and let E be the net field inside the dielectric





Note:

- **linear isotropic dielectrics**- dielectrics for which the induced dipole moment is the direction of the field and is proportional to the field strength
 - **polar molecules**- the +ve and –ve centers coincide | **non-polar molecules**- the +ve and –ve centers don't coincide
 - e.g. O₂, H₂
 - e.g. HCl, H₂O

Polarization (P)

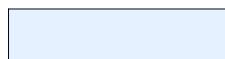
It is a vector | Polarization is defined as the induced dipole moment per unit volume. Its direction is given by the direction of the induced dipole moment.



i.e. Polarization vector is numerically equal to the induced surface charge density

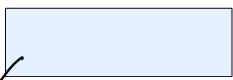
Electrical displacement (D)

It is a vector | Electrical displacement is directly related to free charge density in a dielectric medium (i.e. it replaces \mathbf{E} which is directly related to the free charge density in free space)



Since \mathbf{P} is in the same direction as \mathbf{E} , all three quantities D , \mathbf{P} , \mathbf{E} are parallel.

The ratio of magnitudes of **D** and **E** –



Which is called electric susceptibility

⚡Capacitors and Capacitance (PYQ 2020, 2019, 2018, 2017, 2016, 2015, 2014, 2013, 2011, 2010) ⚡

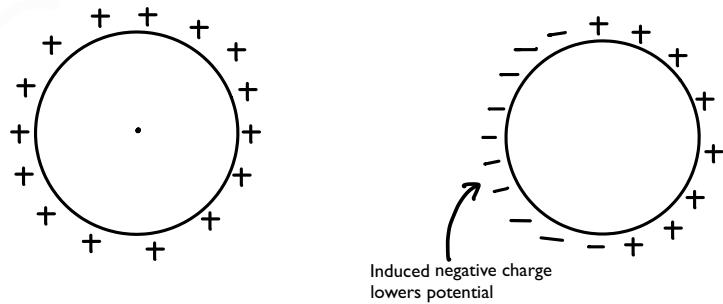
Capacitor: A system of two conductors separated by an insulator which is used to store electrical energy/ charge.

- The two conductors have charges Q and $-Q$. Q is called the charge of the capacitor.
- The total charge on capacitor = 0
- The electric field in the region between the conductors is proportional to the charge.
- The pot. Difference (V) b/w the conductors is the work done in taking a unit +ve charge from one conductor to another

Capacitance: The ratio Q/V for a capacitor is constant and is called the capacitance (C) of the capacitor. | SI unit: Farad (F)

Note: C is independent of both Q and V and only depends on the geometrical configuration (shape, size, separation) of the system of the conductors

Principle of capacitors: When a charged conductor is kept near an uncharged conductor, its capacitance increases.

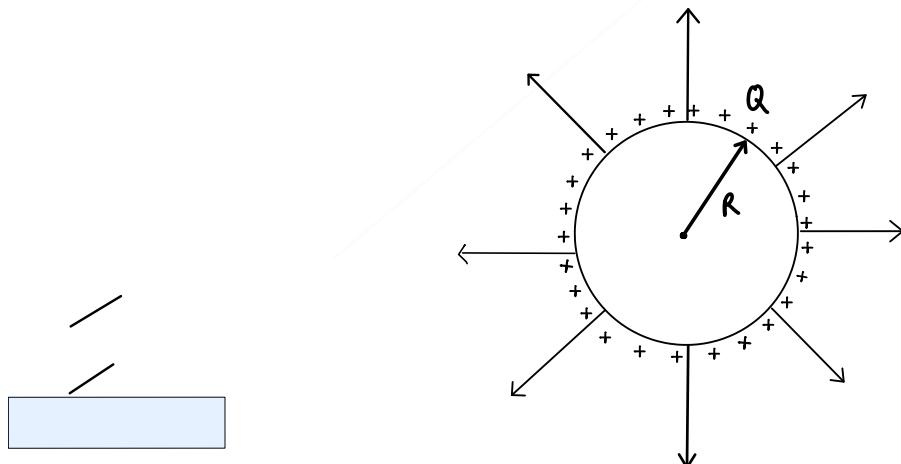


Types of capacitors

1. Spherical capacitor (NCERT back exercise Q 2.29)

Isolated spherical capacitor- it consists of a single isolated hollow metal sphere. It stores energy in the region from its surface to infinity.

Consider a hollow metal sphere with charge Q and radius R . The potential at its surface can be written as-

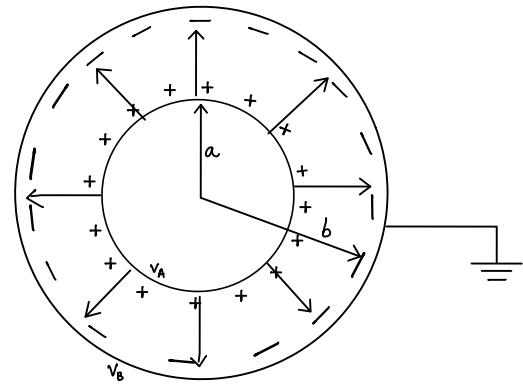


Ques:



Ans: Concentric spherical capacitor- it consists of two charged concentric spheres. Energy is stored inside the cavity b/w the spheres.

Consider two concentric spheres with radius a and b ($b>a$). Let charge on a be $+Q$ and on b be $-Q$. The potential difference between sphere a and b -

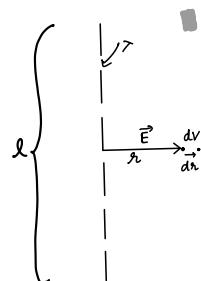


2. Cylindrical capacitor (NCERT back exercise Q 2.32)

Ques:



It consists of two concentric cylinders of radius a and b ($b>a$). Let charge density of the cylinder be σ . To calculate the potential difference b/w cylinder a and b , let us consider the field E at a point P at a distance r from the axis of a . We know that,



3. Parallel plate capacitor (PYQ 2014)

A parallel plate capacitor consists of two large plane parallel conducting plates separated by a small distance.

Consider the following setup; two conducting plates of area A with charge Q and $-Q$ respectively and surface charge density $\sigma = Q/A$ are separated by a distance d.

III

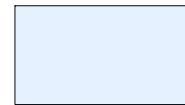
Let us calculate the Electric field in the three regions-

I

II

III

We know,



Note: Fringing effect/ End effect of field- for plates of finite area, the field lines bend outwards at the edges.

Important PYQ

Ques:

(PYQ 2013)

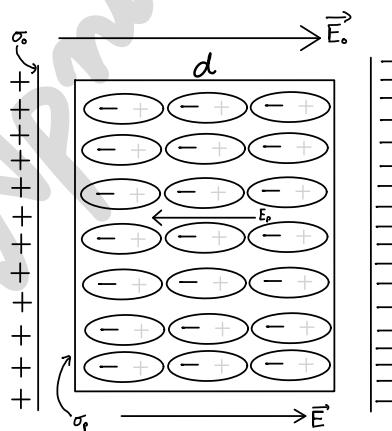
Ans:1) $C = Q/V$
 $C = 360\mu C/V; C = 120 \mu C/V - 120$
 Equating both we get,
 $V = 180V, C = 2\mu F$

$$\begin{aligned} 2) Q &= CV \\ &= 2\mu F \times 400 V \\ &= 800\mu C \end{aligned}$$

Effect of dielectric on capacitance (PYQ 2020, 2016, 2010)

A dielectric medium of dielectric constant K is inserted between the plates of a parallel plate capacitor. Earlier the field b/w the plates was $E_0 = \sigma/\epsilon_0$. But now since there is a dielectric between the plates, the electric field will be modified (as described in polarisation of dielectrics)

Case 1: when space b/w the plates are completely filled with dielectric



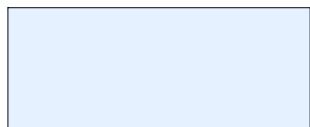
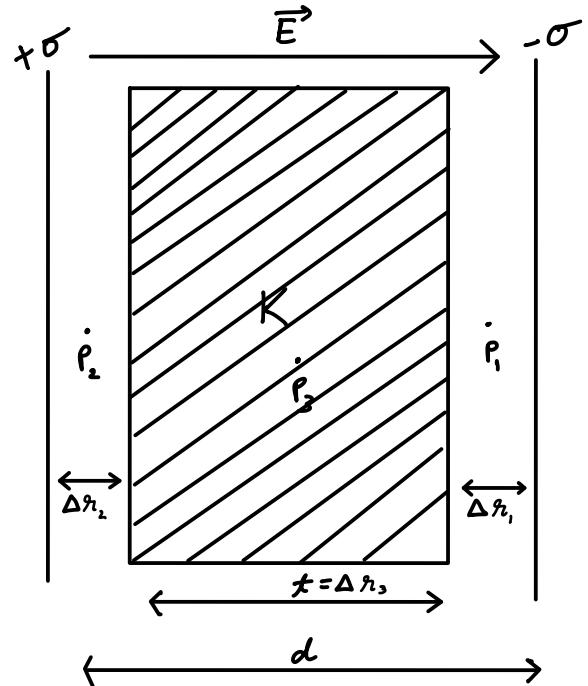
Let the new field be E -



Case 2: When space b/w the plates is partially filled with dielectric .

Consider a case where a dielectric medium of dielectric constant K and thickness t is placed b/w the plates of a capacitor of plate separation d ($d > t$)

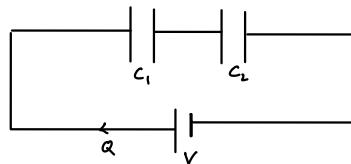
Let us calculate the electric field at the 3 points-



⚡ Combination of Capacitors

1. Series combination

Consider the two capacitors C_1 & C_2 , connected with each other and a battery of emf V (as shown in the figure), have charge Q each. Let the potential drop across C_1 & C_2 be V_1 and V_2 resp. Then we can say that the total potential drop across the combination is equal to the sum of the potential drops across C_1 & C_2 i.e. –

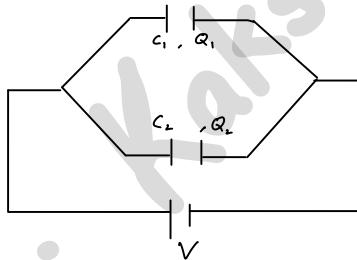


We can see the combination as a capacitor with an effective capacitance C , charge Q and potential drop V -



2. Parallel combination

Consider the two capacitors C_1 & C_2 , connected with each other and a battery of emf V (as shown in the figure). In this case, both capacitors will have the same potential difference. Let the charge on C_1 & C_2 be Q_1 & Q_2 resp. We can imagine the combination as a capacitor (C) with charge Q , which is the sum of the charges Q_1 & Q_2 -



Note: Both of these results are true for any arbitrary number (n) of capacitors.

Important PYQ

(PYQ 2020)

Case 1

Equating both

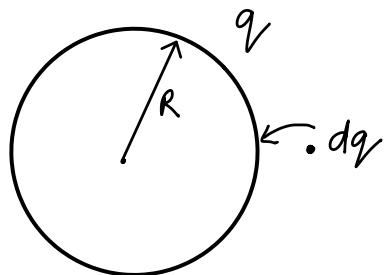
Case 2

In the second case, we can imagine the capacitor as a combination of two capacitors connected in series.

⚡ Energy stored in capacitor (U) (PYQ 2019, 2018, 2017, 2016, 2015, 2014, 2011, 2010)

Consider an uncharged isolated hollow metal sphere. To calculate the energy stored when it will be fully charged to a charge Q , we need to find the work done in charging this sphere to a total charge Q . To calculate the total work done, we will first calculate the work done in transferring an infinitesimal charge dq (from infinity) when the sphere has acquired a total charge q .

Work done (dw) = change in potential energy of charge dq = $V \cdot dq$ (where V is potential of sphere with total charge q)



This work done is stored as P.E
in the capacitor

Also,



⚡ Energy stored per unit volume/ energy density (u) (PYQ 2014)

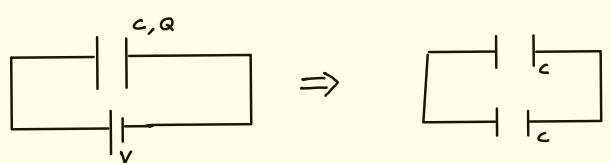
consider a parallel plate capacitor with plate area A , distance b/w plates d and surface charge density σ . The energy stored in it will be-



Important PYQs 



(PYQ 2019)



Ans: 1) Initial energy

2) Final energy

Total charge will remain conserved,

Similar PYQ 



(PYQ 2018)

Ques:

(PYQ 2017)

Ans: 1) Initial energy

For B, Q will remain constant (conservation of charge)

2) After dielectric is added

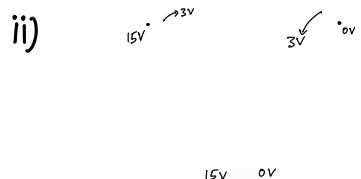
For A, V will remain const. As it remained connected to the battery

Ques:

(PYQ 2016)

Ans: i)

(Series comb.)



Similar PYQ

Ques:

(PYQ 2014)

(Hint=- electric field is conservative)

Ques:

(PYQ 2011)

Ques:

(PYQ 2010)

Ans:

1)

2) Charge will remain conserved ,

3)

SOLUTIONS FOR SIMILAR PYQS



1) PYQ 2018 soln

a. Potential at $(0,0,z)$

(mod because z can be less than or greater than a)

Potential at $(x,y,0)$

Since this point lies in the XY plane, it will be equidistant from both the charges lying on the z axis. Therefore,

2) PYQ 2018 soln

a. let P.E. Of capacitors be U_1 and U_2

b. Let charge on capacitors be Q_1 and Q_2

(Total charge remains conserved)

$(Q=CV)$

b. Similarly, both these points lie on the x axis and hence will be equidistant from the two charges and potential at both points will be zero. So,

c. Since electrostatic force is a conservative force, work done by it depends only on the initial and final position of the charge and not the path followed

c. When the two capacitors are connected in parallel, redistribution of charge takes place between them and some energy is lost as heat during this process.

3. PYQ 2014 soln

Electrostatic field is conservative in nature i.e. the work done by it depends only on the initial and final position of the charge and not the path followed. Since the charge is moved in a closed loop abcd i.e. initial and final position are same, the net work done in the process is zero.

4. PYQ 2011 soln

- a. Let capacitance of each be C
We know that net capacitance in series is-

Therefore, net capacitance in parallel-

- b. Let energy in series and parallel be U_1 and U_2 resp

Sahyadri Classes

Mathematics

Time : 1.00 hr

XII - A Div

Marks : 50

Vector Algebra

- 1) The volume of the parallelopiped, whose edges are represented by $-12\hat{i} + \alpha\hat{k}$, $3\hat{j} - \hat{k}$, $2\hat{i} + \hat{j} - 15\hat{k}$, is 546, then α is :
- A) 3 B) 2 C) -3 D) -2.
- 2) $(\vec{a} \times \vec{b})^2$ is equal to :
- A) $\vec{a}^2 \vec{b}^2 - (\vec{a} \cdot \vec{b})^2$ B) $\vec{a}^2 \vec{b}^2 - (\vec{a} \cdot \vec{b})$
 C) $(\vec{a} \cdot \vec{b})^2$ D) $a^2 b^2$
- 3) If $\vec{a} = 4\hat{i} + 6\hat{j}$ and $\vec{b} = 3\hat{j} + 4\hat{k}$, then the vector form of the component of \vec{a} along \vec{b} is :
- A) $\frac{18}{10\sqrt{13}}(3\hat{j} + 4\hat{k})$ B) $\frac{18}{25}(3\hat{j} + 4\hat{k})$
 C) $\frac{18}{\sqrt{113}}(3\hat{j} + 4\hat{k})$ D) $3\hat{j} + 4\hat{k}$
- 4) The vectors $\hat{i} + 2\hat{j} + 3\hat{k}$, $\lambda\hat{i} + 4\hat{j} + 7\hat{k}$, $-3\hat{i} - 2\hat{j} - 5\hat{k}$ are collinear if λ is :
- A) 3 B) 4 C) 5 D) 6.
- 5) If \vec{a} and \vec{b} are two vectors such that $\vec{a} \cdot \vec{b} = 0$ and $\vec{a} \times \vec{b} = 0$, then :
- A) \vec{a} is parallel to \vec{b}
 B) $\vec{a} \perp \vec{b}$
 C) either \vec{a} or \vec{b} is a null vector
 D) None of these.
- 6) The projection of the vector $\hat{i} - 2\hat{j} + \hat{k}$ on the vector $4\hat{i} - 4\hat{j} + 7\hat{k}$ is:
- A) $\frac{5}{19}\sqrt{5}$ B) $2\frac{1}{9}$ C) $\frac{9}{19}$ D) $\frac{1}{19}\sqrt{6}$
- 7) If the vectors $2\hat{i} - \hat{j} + \lambda\hat{k}$, $\hat{i} - \hat{j} + 2\hat{k}$ and $3\hat{i} - 2\hat{j} + \hat{k}$ are coplanar, then the value of λ is :
- A) -1 B) -2 C) -3 D) -4.
- 8) $\vec{a} \cdot (\vec{a} \times \vec{b}) =$
- A) $\vec{a} \cdot \vec{b}$ B) $a^2 b$ C) 0 D) $a^2 + ab$
- 9) The points with position vectors $60\hat{i} + 3\hat{j}$, $40\hat{i} - 8\hat{j}$ and $a\hat{i} - 52\hat{j}$ are collinear if :
- A) $a = -40$ B) $a = 40$
 C) $a = 20$ D) None of these.
- 10) Let \vec{a} , \vec{b} , \vec{c} be three non-coplanar vectors and \vec{p} , \vec{q} , \vec{r} be vectors defined by the relations: $\vec{p} = \frac{\vec{b} \times \vec{c}}{[\vec{a} \vec{b} \vec{c}]}$, $\vec{q} = \frac{\vec{c} \times \vec{a}}{[\vec{a} \vec{b} \vec{c}]}$, $\vec{r} = \frac{\vec{a} \times \vec{b}}{[\vec{a} \vec{b} \vec{c}]}$, then the value of the expression : $(\vec{a} + \vec{b}) \cdot \vec{p} + (\vec{b} + \vec{c}) \cdot \vec{q} + (\vec{c} + \vec{a}) \cdot \vec{r}$ is equal to:
- A) 0 B) 1 C) 2 D) 3.
- 11) The scalar $\vec{A} \cdot (\vec{B} + \vec{C}) \times (\vec{A} + \vec{B} + \vec{C})$ is :
- A) 0
 B) $[\vec{A} \vec{B} \vec{C}] + [\vec{B} \vec{C} \vec{A}]$
 C) $2[\vec{A} \vec{B} \vec{C}]$
 D) $[\vec{A} \vec{B} \vec{C}]$.
- 12) If the vectors $a\hat{i} + \hat{j} + \hat{k}$, $\hat{i} + b\hat{j} + \hat{k}$ and $\hat{i} + \hat{j} + c\hat{k}$ ($a \neq 1$, $b \neq 1$, $c \neq 1$) are coplanar, then the value of $\frac{1}{1-a} + \frac{1}{1-b} + \frac{1}{1-c}$ is:
- A) 1 B) 2
 C) 0 D) None of these.
- 13) If the sum of two unit vectors is a unit vector, then the angle between them is equal to :
- A) $\pi/6$ B) $\pi/3$ C) $\pi/2$ D) $2\pi/3$.
- 14) If \vec{a} , \vec{b} and \vec{c} are unit coplanar vectors, then the scalar triple product : $[2\vec{a} - \vec{b}, 2\vec{b} - \vec{c}, 2\vec{c} - \vec{a}]$ =
- A) 0 B) 1 C) $-\sqrt{3}$ D) $\sqrt{3}$
- 15) Given two vectors $\hat{i} - \hat{j}$ and $\hat{i} + 2\hat{j}$, the unit vector coplanar with the two vectors and perpendicular to first is:
- A) $\frac{1}{\sqrt{2}}(\hat{i} + \hat{j})$ B) $\frac{1}{\sqrt{5}}(2\hat{i} + \hat{j})$
 C) $\pm\frac{1}{\sqrt{2}}(\hat{i} + \hat{k})$ D) None of these.
- 16) If $\vec{a} = 3\hat{i} - 5\hat{j}$ and $\vec{b} = 6\hat{i} + 3\hat{j}$ are two vectors and \vec{c} a vector such that $\vec{c} = \vec{a} \times \vec{b}$, then $|\vec{a}| : |\vec{b}| : |\vec{c}|$ =
- A) $\sqrt{34} : \sqrt{45} : \sqrt{39}$ B) $\sqrt{34} : \sqrt{45} : 39$
 C) 34:39:45 D) 39:35:34.

- 17) Let $\vec{a} = \hat{i} + \hat{j} + \hat{k}$, $\vec{b} = \hat{i} - \hat{j} + 2\hat{k}$ and $\vec{c} = x\hat{i} + (x-2)\hat{j} - \hat{k}$. If the vector \vec{c} lies in the plane of \vec{a} and \vec{b} , then x equals:
- A) 1 B) -4 C) -2 D) 0.
- 18) Given two vectors $\vec{a} = 2\hat{i} - 3\hat{j} + 6\hat{k}$, $\vec{b} = -2\hat{i} + 2\hat{j} - \hat{k}$ and $p = \frac{\text{projection of } \vec{b} \text{ on } \vec{a}}{\text{projection of } \vec{a} \text{ on } \vec{b}}$, then the value of p is:
- A) 3/7 B) 7/3 C) 3 D) 7.
- 19) If the vectors $\vec{a} = \hat{i} - \hat{j} + 2\hat{k}$, $\vec{b} = 2\hat{i} + 4\hat{j} + \hat{k}$ and $\vec{c} = \lambda\hat{i} + \hat{j} + \mu\hat{k}$ are mutually orthogonal, then (λ, μ) is:
- A) (-3, 2) B) (2, -3) C) (-2, 3) D) (3, -2).
- 20) If $\vec{a} = \frac{1}{\sqrt{10}}(3\hat{i} + \hat{k})$ and $\vec{b} = \frac{1}{7}(2\hat{i} + 3\hat{j} - 6\hat{k})$, then the value of $(2\vec{a} - \vec{b}) \cdot [(\vec{a} \times \vec{b}) \times (\vec{a} + 2\vec{b})]$ is:
- A) -5 B) -3 C) 5 D) 3
- 21) If the vectors $\vec{AB} = 3\hat{i} + 4\hat{k}$ and $\vec{AC} = 5\hat{i} - 2\hat{j} + 4\hat{k}$ the sides of a triangle ABC, then the length of the median through A is :
- A) $\sqrt{72}$ B) $\sqrt{33}$ C) $\sqrt{45}$ D) $\sqrt{18}$
- 22) $[\vec{a} \times \vec{b} \quad \vec{b} \times \vec{c} \quad \vec{c} \times \vec{a}] = \lambda [\vec{a} \quad \vec{b} \quad \vec{c}]^2$, then λ is equal to :
- A) 3 B) 0 C) 1 D) 2
- 23) If one side of a square be represented by the vector $3\hat{i} + 4\hat{j} + 5\hat{k}$, then the area of the square is
- A) 12 B) 13 C) 25 D) 50
- 24) If C is the middle point of AB and P is any point outside AB, then
- A) $\vec{PA} + \vec{PB} = \vec{PC}$
 B) $\vec{PA} + \vec{PB} = 2\vec{PC}$
 C) $\vec{PA} + \vec{PB} + \vec{PC} = 0$
 D) $\vec{PA} + \vec{PB} + 2\vec{PC} = 0$
- 25) The value of b such that scalar product of the vectors $(\hat{i} + \hat{j} + \hat{k})$ with the unit vector parallel to the sum of the vectors $(2\hat{i} + 4\hat{j} - 5\hat{k})$ and $(b\hat{i} + 2\hat{j} + 3\hat{k})$ is 1, is
- A) -2 B) -1 C) 0 D) 1
- 26) A vector whose modulus is $\sqrt{51}$ and makes the same angle with $a = \frac{\hat{i} - 2\hat{j} + 2\hat{k}}{3}$, $b = \frac{-4\hat{i} - 3\hat{k}}{5}$ and $c = \hat{j}$, will be
- A) $5\hat{i} + 5\hat{j} + \hat{k}$ B) $5\hat{i} + \hat{j} - 5\hat{k}$
 C) $5\hat{i} + \hat{j} + 5\hat{k}$ D) $\pm(5\hat{i} - \hat{j} - 5\hat{k})$
- 27) If in a right angled triangle ABC, the hypotenuse $\vec{AB} = p$, then $\vec{AB} \cdot \vec{AC} + \vec{BC} \cdot \vec{BA} + \vec{CA} \cdot \vec{CB}$ is equal to
- A) $2p^2$ B) $\frac{p^2}{2}$
 C) p^2 D) None of these
- 28) A, B, C, D are any four points, then $\vec{AB} \cdot \vec{CD} + \vec{BC} \cdot \vec{AD} + \vec{CA} \cdot \vec{BD} =$
- A) $2\vec{AB} \cdot \vec{BC} \cdot \vec{CD}$ B) $\vec{AB} + \vec{BC} + \vec{CD}$
 C) $5\sqrt{3}$ D) 0
- 29) If $|a| = 3$, $|b| = 4$ then a value of λ for which $a + \lambda b$ is perpendicular to $a - \lambda b$ is
- A) $\frac{9}{16}$ B) $\frac{3}{4}$ C) $\frac{3}{2}$ D) $\frac{4}{3}$
- 30) If a and b are unit vectors and $a - b$ is also a unit vector, then the angle between a and b is
- A) $\frac{\pi}{4}$ B) $\frac{\pi}{3}$ C) $\frac{\pi}{2}$ D) $\frac{2\pi}{3}$
- 31) If a, b and c are unit vectors such that $a + b - c = 0$, then the angle between a and b is
- A) $\pi/6$ B) $\pi/3$ C) $\pi/2$ D) $2\pi/3$
- 32) If the sum of two unit vectors is a unit vector, then the magnitude of their difference is
- A) $\sqrt{2}$ B) $\sqrt{3}$ C) $\frac{1}{\sqrt{3}}$ D) 1
- 33) If a and b are two unit vectors such that $a + 2b$ and $5a - 4b$ are perpendicular to each other, then the angle between a and b is
- A) 45° B) 60°
 C) $\cos^{-1}\left(\frac{1}{3}\right)$ D) $\cos^{-1}\left(\frac{2}{7}\right)$
- 34) If $a \neq 0$, $b \neq 0$ and $|a + b| = |a - b|$, then the vectors a and b are
- A) Parallel to each other
 B) Perpendicular to each other
 C) Inclined at an angle of 60°
 D) Neither perpendicular nor parallel
- 35) $(a - b) \times (a + b) =$
- A) $2(a \times b)$ B) $a \times b$
 C) $a^2 - b^2$ D) None of these
- 36) $(2a + 3b) \times (5a + 7b) =$

- A) $\mathbf{a} \times \mathbf{b}$ B) $\mathbf{b} \times \mathbf{a}$
 C) $\mathbf{a} + \mathbf{b}$ D) $7\mathbf{a} + 10\mathbf{b}$
- 37) If \mathbf{a} and \mathbf{b} are two vectors such that $\mathbf{a} \cdot \mathbf{b} = 0$ and $\mathbf{a} \times \mathbf{b} = 0$, then
 A) \mathbf{a} is parallel to \mathbf{b}
 B) \mathbf{a} is perpendicular to \mathbf{b}
 C) Either \mathbf{a} or \mathbf{b} is a null vector
 D) None of these
- 38) $|\mathbf{(a} \times \mathbf{b}) \cdot \mathbf{c}| = |\mathbf{a}| |\mathbf{b}| |\mathbf{c}|$, if
 A) $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{c} = 0$
 B) $\mathbf{b} \cdot \mathbf{c} = \mathbf{c} \cdot \mathbf{a} = 0$
 C) $\mathbf{c} \cdot \mathbf{a} = \mathbf{a} \cdot \mathbf{b} = 0$
 D) $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{c} = \mathbf{c} \cdot \mathbf{a} = 0$
- 39) $|\mathbf{a} \times \mathbf{i}|^2 + |\mathbf{a} \times \mathbf{j}|^2 + |\mathbf{a} \times \mathbf{k}|^2 =$
 A) $|\mathbf{a}|^2$ B) $2|\mathbf{a}|^2$ C) $3|\mathbf{a}|^2$ D) $4|\mathbf{a}|^2$
- 40) If $(\mathbf{a} \times \mathbf{b})^2 + (\mathbf{a} \cdot \mathbf{b})^2 = 144$ and $|\mathbf{a}| = 4$, then $|\mathbf{b}| =$
 A) 16 B) 8 C) 3 D) 12
- 41) A unit vector in the plane of the vectors $2\mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{i} - \mathbf{j} + \mathbf{k}$ and orthogonal to $5\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}$ is
 A) $\frac{6\mathbf{i} - 5\mathbf{k}}{\sqrt{61}}$ B) $\frac{3\mathbf{j} - \mathbf{k}}{\sqrt{10}}$
 C) $\frac{2\mathbf{i} - 5\mathbf{j}}{\sqrt{29}}$ D) $\frac{2\mathbf{i} + \mathbf{j} - 2\mathbf{k}}{3}$
- 42) The area of the parallelogram whose diagonals are $\mathbf{a} = 3\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ and $\mathbf{b} = \mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$ is
 A) $10\sqrt{3}$ B) $5\sqrt{3}$ C) 8 D) 4
- 43) The moment of a force represented by $\vec{F} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ about the point $2\mathbf{i} - \mathbf{j} + \mathbf{k} =$
 A) $5\mathbf{i} - 5\mathbf{j} + 5\mathbf{k}$ B) $5\mathbf{i} - 5\mathbf{j} + 5\mathbf{k}$
 C) $-5\mathbf{i} + 5\mathbf{j} + 5\mathbf{k}$ D) $-5\mathbf{i} - 5\mathbf{j} + 5\mathbf{k}$
- 44) If the vectors $2\mathbf{i} - 3\mathbf{j}$, $\mathbf{i} + \mathbf{j} - \mathbf{k}$ and $3\mathbf{i} - \mathbf{k}$ form three concurrent edges of a parallelopiped, then the volume of the parallelopiped is
 A) 8 B) 10 C) 4 D) 14
- 45) If \mathbf{a} and \mathbf{b} be parallel vectors, then $[\mathbf{a} \mathbf{c} \mathbf{b}] =$
 A) 0 B) 1
 C) 2 D) None of these
- 46) If the vectors $2\mathbf{i} - \mathbf{j} + \mathbf{k}$, $\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$ and $3\mathbf{i} + \lambda\mathbf{j} + 5\mathbf{k}$ be coplanar, then
 A) -1 B) -2 C) -3 D) -4
- 47) If \mathbf{a}, \mathbf{b} and \mathbf{c} are unit coplanar vectors then the scalar triple product $[\mathbf{2a} - \mathbf{b} \mathbf{2b} - \mathbf{c} \mathbf{2c} - \mathbf{a}]$ is equal to
 A) 0 B) 1 C) $-\sqrt{3}$ D) $\sqrt{3}$
- 48) The value of $[\mathbf{a} - \mathbf{b} \mathbf{b} - \mathbf{c} \mathbf{c} - \mathbf{a}]$, where $|\mathbf{a}| = 1$, $|\mathbf{b}| = 5$ and $|\mathbf{c}| = 3$ is
 A) 0 B) 1 C) 2 D) 4
- 49) If \mathbf{u}, \mathbf{v} and \mathbf{w} are three non-coplanar vectors, then $(\mathbf{u} + \mathbf{v} - \mathbf{w}) \cdot [(\mathbf{u} - \mathbf{v}) \times (\mathbf{v} - \mathbf{w})]$ equals
 A) 0 B) $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$
 C) $\mathbf{u} \cdot (\mathbf{w} \times \mathbf{v})$ D) $3\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$
- 50) $\mathbf{i} \times (\mathbf{j} \times \mathbf{k}) =$
 A) 1 B) 0
 C) -1 D) None of these

Answer Sheet

Mathematics : Vector Algebra

Total Questions : 50

Total Marks : 50

Solution Sheet

Mathematics : Vector Algebra

Total Questions : 50

Total Marks : 50

$$1) \text{ Volume} = \begin{vmatrix} -12 & 0 & \alpha \\ 0 & 3 & -1 \\ 2 & 1 & -15 \end{vmatrix} = 546$$

$$\Rightarrow -12(-44) + 0 + \alpha(-6) = 546 \\ \Rightarrow 528 - 6\alpha = 546 \\ \Rightarrow 6\alpha = -546 + 528 = -18 \Rightarrow \alpha = -3.$$

$$2) (\vec{a} \times \vec{b})^2 = (ab \sin \theta)^2 \\ = a^2 b^2 (1 - \cos^2 \theta) \\ = a^2 b^2 - a^2 b^2 \cos^2 \theta = \vec{a}^2 \vec{b}^2 - (\vec{a} \cdot \vec{b})^2.$$

$$3) \text{ Vector form of the component of } \vec{a} \text{ along } \vec{b} \\ = \left(\frac{\vec{a} \cdot \vec{b}}{|\vec{b}|^2} \right) \vec{b} \\ = (4 \times 0 + 6 \times 3 + 0 \times 4) \frac{(3\hat{j} + 4\hat{k})}{(\sqrt{9+16})^2} \\ = \frac{18}{25} (3\hat{j} + 4\hat{k})$$

$$4) \begin{vmatrix} 1 & 2 & 3 \\ \lambda & 4 & 7 \\ -3 & -2 & -5 \end{vmatrix} = 0 \\ \Rightarrow 1.(-20+14) - 2(-5\lambda+21) + 3(-2\lambda+12) = 0 \\ \Rightarrow -6 + 10\lambda - 42 - 6\lambda + 36 = 0 \\ \Rightarrow 4\lambda = 12 \Rightarrow \lambda = 3.$$

5) $\vec{a} \cdot \vec{b} = 0$ and $\vec{a} \times \vec{b} = 0$ means either \vec{a} or \vec{b} is a null vector.
 \therefore both can't be parallel and perp.

$$6) \text{ Projection} = \frac{(1)(4) + (-2)(-4) + (1)(7)}{\sqrt{16+16+49}} \\ = \frac{4+8+7}{\sqrt{81}} = \frac{19}{9} = 2\frac{1}{9}.$$

7) Since the vectors are coplanar,

$$\therefore \begin{vmatrix} 2 & -1 & \lambda \\ 1 & -2 & 2 \\ 3 & -2 & 1 \end{vmatrix} = 0 \\ \Rightarrow 2(-1+4) + 1(1-6) + \lambda(-2+3) = 0 \\ \Rightarrow 6 - 5 + \lambda = 0 \Rightarrow \lambda = -1.$$

8) Since scalar triple product is zero if two vectors are identical.

9) Let $A(60\hat{i} + 3\hat{j})$, $B(40\hat{i} - 8\hat{j})$, $C(a\hat{i} - 52\hat{j})$ be given points.

$$\therefore \vec{AB} = 40\hat{i} - 8\hat{j} - 60\hat{i} - 3\hat{j} = -20\hat{i} - 11\hat{j}$$

$$\text{and } \vec{AC} = a\hat{i} - 52\hat{j} - 60\hat{i} - 3\hat{j}$$

$$= (a-60)\hat{i} - 55\hat{j}$$

Since A, B, C are collinear,

$$\therefore \vec{AB} = k\vec{AC}, \text{ for some scalar } k$$

$$\Rightarrow -20\hat{i} - 11\hat{j} = k[(a-60)\hat{i} - 55\hat{j}]$$

$$\Rightarrow -11 = -55k \Rightarrow k = 1/5$$

$$\text{and } -20 = k(a-60) = \frac{1}{5}(a-60)$$

$$\Rightarrow -100 = a-60$$

$$\Rightarrow a = -40.$$

$$10) (\vec{a} + \vec{b}) \cdot \vec{p} = \frac{\vec{a} \cdot \vec{b} \times \vec{c}}{[\vec{a} \vec{b} \vec{c}]} + \frac{\vec{b} \cdot (\vec{b} \times \vec{c})}{[\vec{a} \vec{b} \vec{c}]} \\ = \frac{[\vec{a} \vec{b} \vec{c}]}{[\vec{a} \vec{b} \vec{c}]} + 0 = 1.$$

Similarly $(\vec{b} + \vec{c}) \cdot \vec{q} = 1$ and $(\vec{c} + \vec{a}) \cdot \vec{r} = 1$.
 \therefore Total value = 3.

$$11) \vec{A} \cdot (\vec{B} + \vec{C}) \times (\vec{A} + \vec{B} + \vec{C}) \\ = \vec{A} \cdot (\vec{B} \times \vec{A} + \vec{B} \times \vec{C} + \vec{C} \times \vec{A} + \vec{C} \times \vec{B}) \\ = \vec{A} \cdot (\vec{B} \times \vec{A} + \vec{C} \times \vec{A}) \\ = \vec{A} \cdot \vec{B} \times \vec{A} + \vec{A} \cdot \vec{C} \times \vec{A} \\ = [\vec{A} \vec{B} \vec{A}] + [\vec{A} \vec{C} \vec{A}] \\ = 0 + 0 = 0.$$

12) Since given vectors are coplanar,

$$\therefore [a\hat{i} + \hat{j} + \hat{k}, \hat{i} + b\hat{j} + \hat{k}, \hat{i} + \hat{j} + c\hat{k}] = 0$$

$$\Rightarrow \begin{vmatrix} a & 1 & 1 \\ 1 & b & 1 \\ 1 & 1 & c \end{vmatrix} = 0 \text{ i.e. } \begin{vmatrix} a & 1-a & 1-a \\ 1 & b-1 & 0 \\ 1 & 0 & c-1 \end{vmatrix} = 0$$

[Operating $C_2 \rightarrow C_2 - C_1, C_3 \rightarrow C_3 - C_1$]

$$\Rightarrow a(b-1)(c-1) - (1-a)(c-1) - 1(1-a)(b-1) = 0$$

Dividing throughout by $(1-a)(1-b)(1-c)$, we get:

$$\frac{a}{1-a} + \frac{1}{1-b} + \frac{1}{1-c} = 0$$

$$\Rightarrow -1 + \frac{1}{1-a} + \frac{1}{1-b} + \frac{1}{1-c} = 0$$

$$\Rightarrow \frac{1}{1-a} + \frac{1}{1-b} + \frac{1}{1-c} = 1$$

13) Let \vec{a} , \vec{b} be unit vectors. Then their sum

$$\begin{aligned}\vec{a} + \vec{b} &= \\ \vec{c} &\Rightarrow (\vec{a} + \vec{b})^2 = (\vec{c})^2 \Rightarrow 1 + 1 + 2\vec{a} \cdot \vec{b} = 1 \\ \Rightarrow 2|\vec{a}||\vec{b}|\cos\theta &= -1 \\ \Rightarrow \cos\theta &= -1/2 = -\cos(\pi/3) = \cos\frac{2\pi}{3} \\ \Rightarrow \theta &= 2\pi/3.\end{aligned}$$

14) Since \vec{a} , \vec{b} , \vec{c} are unit coplanar vectors,
 $\therefore 2\vec{a} - \vec{b}$, $2\vec{b} - \vec{c}$, $2\vec{c} - \vec{a}$ are also coplanar
vectors,
 \therefore their scalar triple product is zero,

15) Let $\vec{a} = \hat{i} - \hat{j}$ and $\vec{b} = \hat{i} - 2\hat{j}$
The reqd. vector is along the vector
 $\vec{a} \times (\vec{a} \times \vec{b}) = (\vec{a} \cdot \vec{b})\vec{a} - (\vec{a} \cdot \vec{a})\vec{b}$
 $= -(\hat{i} - \hat{j}) - 2(\hat{i} + 2\hat{j}) = -3\hat{i} - 3\hat{j}$
Hence reqd. vectors are given by :
 $\pm \frac{(-3\hat{i} - 3\hat{j})}{\sqrt{9+9}} = \pm \frac{1}{\sqrt{2}}(\hat{i} + \hat{j}).$

16) Here $|\vec{a}| = \sqrt{9+25+0} = \sqrt{34}$
 $|\vec{b}| = \sqrt{9+25+0} = \sqrt{45}$
Now $|\vec{c}| = |\vec{a} \times \vec{b}|$
 $= |(3\hat{i} - 5\hat{j}) \times (6\hat{i} + 3\hat{j})|$
 $= |9\hat{k} + 30\hat{k}| = |39\hat{k}| = 39.$
Hence $|\vec{a}| : |\vec{b}| : |\vec{c}| = \sqrt{34} : \sqrt{45} : 39$

17) Here $[\vec{a} \vec{b} \vec{c}] = 0$

$$\begin{aligned}\Rightarrow \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ x & x-2 & -1 \end{vmatrix} &= 0 \\ \Rightarrow \begin{vmatrix} 1 & 0 & 0 \\ 1 & -2 & 3 \\ x & -2 & 1-x \end{vmatrix} &= 0\end{aligned}$$

[Operating $C_2 \rightarrow C_2 - C_1$ and $C_3 \rightarrow C_3 - C_2$]
 $\Rightarrow 1.(-2+2x+6) = 0 \Rightarrow 2x = -4 \Rightarrow x = -2.$

$$\begin{aligned}18) \frac{(\vec{b} \cdot \vec{a})}{|\vec{a}|} &= \frac{|\vec{b}|}{|\vec{a}|} = \frac{3}{7}. \\ \frac{|\vec{b}|}{|\vec{a}|} &= \vec{b} \cdot \vec{a} \quad [\because \vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}]\end{aligned}$$

19) \vec{a} , \vec{b} and \vec{c} are mutually orthogonal

$$\begin{aligned}\Rightarrow \vec{b} \cdot \vec{c} &= 0 \text{ and } \vec{a} \cdot \vec{c} = 0 \\ \Rightarrow (2\hat{i} + 4\hat{j} + \hat{k}) \cdot (\lambda\hat{i} + \hat{j} + \mu\hat{k}) &= 0 \\ \Rightarrow 2\lambda + 4 + \mu &= 0 \Rightarrow 2\lambda + \mu = -4 \dots(1)\end{aligned}$$

and $\Rightarrow (\hat{i} - \hat{j} + 2\hat{k}) \cdot (\lambda\hat{i} + \hat{j} + \mu\hat{k}) = 0$

$$\Rightarrow \lambda - 1 + 2\mu = 0 \Rightarrow \lambda + 2\mu = 1 \dots(2)$$

Solving (1) and (2), $\lambda = -3$ and $\mu = 2$

$$\begin{aligned}20) (2\vec{a} - \vec{b}) \cdot [(\vec{a} \times \vec{b}) \times (\vec{a} + 2\vec{b})] &= (2\vec{a} - \vec{b}) \cdot [(\vec{a} \times \vec{b}) \times \vec{a} + 2(\vec{a} \times \vec{b}) \times \vec{b}] \\ &= (2\vec{a} - \vec{b}) \cdot [(\vec{a} \cdot \vec{a})\vec{b} - (\vec{a} \cdot \vec{b})\vec{a} + 2(\vec{a} \cdot \vec{b})\vec{b} - 2(\vec{b} \cdot \vec{b})\vec{a}]\end{aligned}$$

$$\text{But } \vec{a} \cdot \vec{a} = \frac{1}{\sqrt{10}}(3\hat{i} + \hat{k}) \cdot \frac{1}{\sqrt{10}}(3\hat{i} + \hat{k})$$

$$= \frac{1}{10}[(3)(3) + (1)(1)] = \frac{10}{10} = 1$$

$$\vec{a} \cdot \vec{b} = \frac{1}{\sqrt{10}}(3\hat{i} + \hat{k}) \cdot \frac{1}{7}(2\hat{i} + 3\hat{j} - 6\hat{k})$$

$$= \frac{1}{7\sqrt{10}}[(3)(2) + 0 + (1)(-6)] = 0$$

$$\text{and } \vec{b} \cdot \vec{b} = \frac{1}{7}(2\hat{i} + 3\hat{j} - 6\hat{k}) \cdot \frac{1}{7}(2\hat{i} + 3\hat{j} - 6\hat{k})$$

$$= \frac{1}{49}[(2)(2) + (3)(3) + (-6)(-6)]$$

$$= \frac{1}{49}[4 + 9 + 36] = 1$$

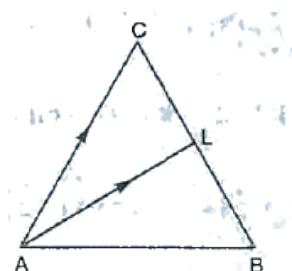
\therefore From (1), Given $= 2(\vec{a} - \vec{b})$

$$\begin{aligned}&[(1)\vec{b} - 0 + 0 - 2(1)\vec{a}] \\ &= (2\vec{a} - \vec{b}) \cdot (\vec{b} - 2\vec{a}) = 2\vec{a} \cdot \vec{b} - 4\vec{a} \cdot \vec{a} \\ &\quad - \vec{b} \cdot \vec{b} + 2\vec{b} \cdot \vec{a} \\ &= 2(0) - 4(1) - (1) + 2(0) = -4 - 1 = -5.\end{aligned}$$

21) Here $\vec{A}L = \vec{A}B + \frac{\vec{A}C}{2}$

$$= \frac{(3\hat{i} + 4\hat{k}) + (5\hat{i} - 2\hat{j} + 4\hat{k})}{2} = 4\hat{i} - \hat{j} + 4\hat{k}$$

$$\therefore |\vec{A}L| = \sqrt{16+1+16} = \sqrt{33}$$



22) As usual, we will have :

$$[\vec{a} \times \vec{b} \quad \vec{b} \times \vec{c} \quad \vec{c} \times \vec{a}] = \lambda [\vec{a} \quad \vec{b} \quad \vec{c}]^2$$

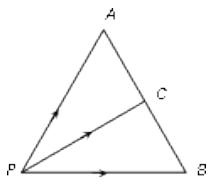
$$\text{Given : } [\vec{a} \times \vec{b} \quad \vec{b} \times \vec{c} \quad \vec{c} \times \vec{a}] = \lambda [\vec{a} \quad \vec{b} \quad \vec{c}]^2$$

Hence $\lambda = 1$.

23) $|\mathbf{a}| = \sqrt{9 + 16 + 25} = 5\sqrt{2}$

Area = $|\mathbf{a}|^2 = 25 \times 2 = 50$.

24) $\overrightarrow{PA} + \overrightarrow{PB} = (\overrightarrow{PA} + \overrightarrow{AC}) + (\overrightarrow{PB} + \overrightarrow{BC}) - (\overrightarrow{AC} + \overrightarrow{BC})$
 $= \overrightarrow{PC} + \overrightarrow{PC} - (\overrightarrow{AC} - \overrightarrow{CB}) = 2\overrightarrow{PC} - 0, (\because \overrightarrow{AC} = \overrightarrow{CB})$
 $\therefore \overrightarrow{PA} + \overrightarrow{PB} = 2\overrightarrow{PC}$.



25) Parallel vector $= (2+b)\mathbf{i} + 6\mathbf{j} - 2\mathbf{k}$

$$\text{Unit vector} = \frac{(2+b)\mathbf{i} + 6\mathbf{j} - 2\mathbf{k}}{\sqrt{b^2 + 4b + 44}}$$

$$\text{According to the condition, } 1 = \frac{(2+b) + 6 - 2}{\sqrt{b^2 + 4b + 44}}$$

$$\Rightarrow b^2 + 4b + 44 = b^2 + 12b + 36 \Rightarrow 8b = 8 \Rightarrow b = 1.$$

26) Let the required vector be $\alpha = d_1\mathbf{i} + d_2\mathbf{j} + d_3\mathbf{k}$, where $d_1^2 + d_2^2 + d_3^2 = 51$, (given)(i)

Now, each of the given vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ is a unit vector $\cos \theta = \frac{\mathbf{d} \cdot \mathbf{a}}{|\mathbf{d}| |\mathbf{a}|} = \frac{\mathbf{d} \cdot \mathbf{b}}{|\mathbf{d}| |\mathbf{b}|} = \frac{\mathbf{d} \cdot \mathbf{c}}{|\mathbf{d}| |\mathbf{c}|}$

or $\mathbf{d} \cdot \mathbf{a} = \mathbf{d} \cdot \mathbf{b} = \mathbf{d} \cdot \mathbf{c}$

$|\mathbf{d}| = \sqrt{51}$ cancels out and $|\mathbf{a}| = |\mathbf{b}| = |\mathbf{c}| = 1$

Hence,

$$\frac{1}{3}(d_1 - 2d_2 + 2d_3) = \frac{1}{5}(-4d_1 + 0d_2 - 3d_3) = d_2$$
 $\Rightarrow d_1 - 5d_2 + 2d_3 = 0 \text{ and } 4d_1 + 5d_2 + 3d_3 = 0$

On solving, $\frac{d_1}{5} = \frac{d_2}{-1} = \frac{d_3}{-5} = \lambda$ (say)

Putting d_1, d_2 and d_3 in (i), we get $\lambda = \pm 1$

Hence the required vectors are $\pm(5\mathbf{i} - \mathbf{j} - 5\mathbf{k})$.

Trick : Check it with the options.

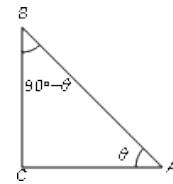
27) We have $\overrightarrow{AB} \cdot \overrightarrow{AC} + \overrightarrow{BC} \cdot \overrightarrow{BA} + \overrightarrow{CA} \cdot \overrightarrow{CB}$

$$(AB)(AC) \cos \theta + (BC)(BA) \cos(90^\circ - \theta) + 0$$

$$= AB(AC \cos \theta + BC \sin \theta) =$$

$$AB \left(\frac{(AC)^2}{AB} + \frac{(BC)^2}{AB} \right)$$

$$= AC^2 + BC^2 = AB^2 = p^2.$$



28) $\overrightarrow{AD} = \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} = \mathbf{a} + \mathbf{b} + \mathbf{c}$

$$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC} = \mathbf{a} + \mathbf{b} \text{ or } \overrightarrow{CA} = -(\mathbf{a} + \mathbf{b})$$

$$\overrightarrow{BD} = \overrightarrow{BC} + \overrightarrow{CD} = \mathbf{b} + \mathbf{c}$$

$$\text{Therefore } \overrightarrow{AB} \cdot \overrightarrow{CD} + \overrightarrow{BC} \cdot \overrightarrow{AD} + \overrightarrow{CA} \cdot \overrightarrow{BD}$$

$$= \mathbf{a} \cdot \mathbf{c} + \mathbf{b} \cdot (\mathbf{a} + \mathbf{b} + \mathbf{c}) + (-\mathbf{a} - \mathbf{b}) \cdot (\mathbf{b} + \mathbf{c})$$

=

$$\mathbf{a} \cdot \mathbf{c} + \mathbf{b} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{c} - \mathbf{a} \cdot \mathbf{b} - \mathbf{a} \cdot \mathbf{c} - \mathbf{b} \cdot \mathbf{b} - \mathbf{b} \cdot \mathbf{c} = 0$$

29) Since $\mathbf{a} + \lambda \mathbf{b}$ is perpendicular to $\mathbf{a} - \lambda \mathbf{b}$, then their product will be zero.

$$\text{So, } (\mathbf{a} + \lambda \mathbf{b}) \cdot (\mathbf{a} - \lambda \mathbf{b}) = 0 \Rightarrow |\mathbf{a}|^2 - \lambda^2 |\mathbf{b}|^2 = 0$$

$$\text{or } \lambda^2 = \frac{|\mathbf{a}|^2}{|\mathbf{b}|^2} \Rightarrow \lambda^2 = \frac{9}{16} \text{ or}$$

$$\lambda = \pm \frac{3}{4}, [\because |\mathbf{a}| = 3, |\mathbf{b}| = 4]$$

30) $(\mathbf{a} - q\mathbf{b})^2 = 1 = 2 - 2 \cos \theta \Rightarrow \theta = 60^\circ$.

31) Given condition is $\mathbf{a} + \mathbf{b} = \mathbf{c}$.

$$\text{Using dot product, } (\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b}) = \mathbf{c} \cdot \mathbf{c}$$

$$\Rightarrow \mathbf{a} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{b} + 2\mathbf{a} \cdot \mathbf{b} = \mathbf{c} \cdot \mathbf{c}$$

$$\Rightarrow |\mathbf{a}| \cdot |\mathbf{a}| \cos 0^\circ + |\mathbf{b}| \cdot |\mathbf{b}| \cos 0^\circ + 2|\mathbf{a}| \cdot |\mathbf{b}| \cos \alpha$$

$$= |\mathbf{c}| \cdot |\mathbf{c}| \cos 0^\circ, (\because |\mathbf{a}| = |\mathbf{b}| = |\mathbf{c}| = 1)$$

$$\Rightarrow 1 + 1 + 2 \cos \alpha = 1 \Rightarrow \cos \alpha = -\frac{1}{2} \Rightarrow \alpha = \frac{2\pi}{3}.$$

32) Let $|\mathbf{a}| = 1$ and $|\mathbf{b}| = 1$

Also

$$|\mathbf{a} + \mathbf{b}|^2 = 1^2 \Rightarrow 1 + 1 + 2 \cos \theta = 1 \Rightarrow \theta = 120^\circ$$

$$\therefore |\mathbf{a} - \mathbf{b}|^2 = 1 + 1 - 2 \cos \theta = 3 \Rightarrow |\mathbf{a} - \mathbf{b}| = \sqrt{3}.$$

33) $(\mathbf{a} + 2\mathbf{b}) \cdot (5\mathbf{a} - 4\mathbf{b}) = 0 \text{ or } 5\mathbf{a}^2 + 6\mathbf{a} \cdot \mathbf{b} - 8\mathbf{b}^2 = 0$

$$\text{or } 6\mathbf{a} \cdot \mathbf{b} = 3, (\because \mathbf{a}^2 = 1, \mathbf{b}^2 = 1)$$

$$\therefore \mathbf{a} \cdot \mathbf{b} = \frac{1}{2} \text{ or } |\mathbf{a}| |\mathbf{b}| \cos \theta = \frac{1}{2}$$

$$\therefore \cos \theta = \frac{1}{2}, \therefore \theta = 60^\circ.$$

34) $|\mathbf{a} + \mathbf{b}| = |\mathbf{a} - \mathbf{b}|$; Squaring both sides, we get

$$4\mathbf{a} \cdot \mathbf{b} = 0$$

$\Rightarrow \mathbf{a}$ is perpendicular to \mathbf{b} .

35) $(\mathbf{a} - \mathbf{b}) \times (\mathbf{a} + \mathbf{b}) = \mathbf{a} \times \mathbf{a} - \mathbf{b} \times \mathbf{a} + \mathbf{a} \times \mathbf{b} - \mathbf{b} \times \mathbf{b}$

$$= \mathbf{a} \times \mathbf{b} - \mathbf{b} \times \mathbf{a} = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{b} = 2(\mathbf{a} \times \mathbf{b}).$$

36) $14(\mathbf{a} \times \mathbf{b}) + 15(\mathbf{b} \times \mathbf{a}) = \mathbf{b} \times \mathbf{a}$.

37) $\mathbf{a} \cdot \mathbf{b} = 0 \Rightarrow \mathbf{a} \perp \mathbf{b}$ or $\mathbf{a} = 0$ or $\mathbf{b} = 0$
and $\mathbf{a} \times \mathbf{b} = 0 \Rightarrow \mathbf{a} \parallel \mathbf{b}$ or $\mathbf{a} = 0$ or $\mathbf{b} = 0$
Hence, either \mathbf{a} or \mathbf{b} is a null vector.

38) We have $|(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}| = |\mathbf{a}||\mathbf{b}||\mathbf{c}|$
 $\Rightarrow ||\mathbf{a}||\mathbf{b}|\sin \theta \mathbf{n} \cdot \mathbf{c}| = |\mathbf{a}||\mathbf{b}||\mathbf{c}|$
 $\Rightarrow ||\mathbf{a}||\mathbf{b}||\mathbf{c}|\sin \theta \cos \alpha| = |\mathbf{a}||\mathbf{b}||\mathbf{c}|$
 $\Rightarrow |\sin \theta||\cos \alpha| = 1 \Rightarrow \theta = \frac{\pi}{2}$ and $\alpha = 0$
 $\Rightarrow \mathbf{a} \perp \mathbf{b}$ and $\mathbf{c} \parallel \mathbf{n}$
 $\Rightarrow \mathbf{a} \perp \mathbf{b}$ and \mathbf{c} is perpendicular to both \mathbf{a} and \mathbf{b}
 $\therefore \mathbf{a}, \mathbf{b}, \mathbf{c}$ are mutually perpendicular
Hence, $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{c} = \mathbf{c} \cdot \mathbf{a} = 0$.

39) $|\mathbf{a} \times \mathbf{i}|^2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ 1 & 0 & 0 \end{vmatrix}^2,$

(Since $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$)
 $= |a_3\mathbf{j} - a_2\mathbf{k}|^2 = a_3^2 + a_2^2$
Similarly, $|\mathbf{a} \times \mathbf{j}|^2 = a_1^2 + a_3^2$ and $|\mathbf{a} \times \mathbf{k}|^2 = a_1^2 + a_2^2$
Hence the required result can be given as
 $2(a_1^2 + a_2^2 + a_3^2) = 2|\mathbf{a}|^2$.

40) We know that $(\mathbf{a} \times \mathbf{b})^2 + (\mathbf{a} \cdot \mathbf{b})^2 = |\mathbf{a}|^2|\mathbf{b}|^2$
 $\therefore 144 = 16|\mathbf{b}|^2 \Rightarrow |\mathbf{b}| = 3$.

41) Let \mathbf{a} unit vector in the plane of $2\mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{i} - \mathbf{j} + \mathbf{k}$
be $\hat{\mathbf{a}} = \alpha(2\mathbf{i} + \mathbf{j} + \mathbf{k}) + \beta(\mathbf{i} - \mathbf{j} + \mathbf{k})$
 $\hat{\mathbf{a}} = (2\alpha + \beta)\mathbf{i} + (\alpha - \beta)\mathbf{j} + (\alpha + \beta)\mathbf{k}$
As $\hat{\mathbf{a}}$ is unit vector, we have
 $\Rightarrow (2\alpha + \beta)^2 + (\alpha - \beta)^2 + (\alpha + \beta)^2 = 1$
 $\Rightarrow 6\alpha^2 + 4\alpha\beta + 3\beta^2 = 1 \dots\dots(i)$
As $\hat{\mathbf{a}}$ is orthogonal to $5\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}$, we get
 $5(2\alpha + \beta) + 2(\alpha - \beta) + 6(\alpha + \beta) = 0$
 $\Rightarrow 18\alpha + 9\beta = 0 \Rightarrow \beta = -2\alpha$
From (i), we get $6\alpha^2 - 8\alpha^2 + 12\alpha^2 = 1$
 $\Rightarrow \alpha = \pm \frac{1}{\sqrt{10}} \Rightarrow \beta = \mp \frac{2}{\sqrt{10}}$. Thus
 $\hat{\mathbf{a}} = \pm \left(\frac{3}{\sqrt{10}}\mathbf{j} - \frac{1}{\sqrt{10}}\mathbf{k} \right)$.

42) $\Delta = \frac{1}{2}|\mathbf{a} \times \mathbf{b}|$

But $\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 1 & -2 \\ 1 & -3 & 4 \end{vmatrix} = -2\mathbf{i} - 14\mathbf{j} - 10\mathbf{k}$.

Hence $\Delta = \frac{1}{2}|\mathbf{a} \times \mathbf{b}| = \frac{1}{2}\sqrt{4 + 196 + 100} = 5\sqrt{3}$.

43) $\mathbf{r} \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 1 \\ 1 & 2 & 3 \end{vmatrix} = -5\mathbf{i} - 5\mathbf{j} + 5\mathbf{k}$.

44) Here, $\overrightarrow{OA} = 2\mathbf{i} - 3\mathbf{j} = \mathbf{a}$ (say)

$\overrightarrow{OB} = \mathbf{i} + \mathbf{j} - \mathbf{k} = \mathbf{b}$ (say)

and $\overrightarrow{OC} = 3\mathbf{i} - \mathbf{k} = \mathbf{c}$ (say)

Hence volume is

$[\mathbf{a} \mathbf{b} \mathbf{c}] = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} 2 & -3 & 0 \\ 1 & 1 & -1 \\ 3 & 0 & -1 \end{vmatrix} = 4$

46) If the given vectors are coplanar, then their scalar triple product is zero.

$\begin{vmatrix} 2 & -1 & 1 \\ 1 & 2 & -3 \\ 3 & \lambda & 5 \end{vmatrix} = 0 \Rightarrow \lambda = -4$.

47) Here $[\mathbf{a} \mathbf{b} \mathbf{c}] = 0$

The given scalar triple product = $k[\mathbf{a} \mathbf{b} \mathbf{c}] = 0$.

48) $[\mathbf{a} - \mathbf{b} \mathbf{b} - \mathbf{c} \mathbf{c} - \mathbf{a}] = \{(\mathbf{a} - \mathbf{b}) \times (\mathbf{b} - \mathbf{c})\} \cdot (\mathbf{c} - \mathbf{a})$
 $= (\mathbf{a} \times \mathbf{b} - \mathbf{a} \times \mathbf{c} - \mathbf{b} \times \mathbf{b} + \mathbf{b} \times \mathbf{c}) \cdot (\mathbf{c} - \mathbf{a})$
 $= (\mathbf{a} \times \mathbf{ab} + \mathbf{ca} \times \mathbf{a} + \mathbf{b} \times \mathbf{c}) \cdot (\mathbf{c} - \mathbf{a})$
 $= (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} - (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{a} + (\mathbf{c} \times \mathbf{a}) \cdot \mathbf{c} - (\mathbf{c} \times \mathbf{a}) \cdot \mathbf{a}$
 $= (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} - (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{a} + (\mathbf{c} \times \mathbf{a}) \cdot \mathbf{c} - (\mathbf{c} \times \mathbf{a}) \cdot \mathbf{a}$
 $+ (\mathbf{b} \times \mathbf{c}) \cdot \mathbf{c} - (\mathbf{b} \times \mathbf{c}) \cdot \mathbf{a}$
 $=$
 $[\mathbf{a} \mathbf{b} \mathbf{c}] - [\mathbf{a} \mathbf{b} \mathbf{a}] + [\mathbf{c} \mathbf{a} \mathbf{c}] - [\mathbf{c} \mathbf{a} \mathbf{a}] + [\mathbf{b} \mathbf{c} \mathbf{c}] - [\mathbf{b} \mathbf{c} \mathbf{a}] = 0$

49) $(\mathbf{u} + \mathbf{v} - \mathbf{w}) \cdot (\mathbf{u} \times \mathbf{v} - \mathbf{u} \times \mathbf{w} - \mathbf{v} \times \mathbf{v} + \mathbf{v} \times \mathbf{w})$
 $= (\mathbf{u} + \mathbf{v} - \mathbf{w}) (\mathbf{u} \times \mathbf{v} - \mathbf{u} \times \mathbf{w} + \mathbf{v} \times \mathbf{w})$
 $= \frac{\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v})}{0} - \frac{\mathbf{u} \cdot (\mathbf{u} \times \mathbf{w})}{0} + \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) + \frac{\mathbf{v} \cdot (\mathbf{u} \times \mathbf{v})}{0}$
 $- \mathbf{v} \cdot (\mathbf{u} \times \mathbf{w}) + \frac{\mathbf{v} \cdot (\mathbf{v} \times \mathbf{w})}{0} - \mathbf{w} \cdot (\mathbf{u} \times \mathbf{v}) + \frac{\mathbf{w} \cdot (\mathbf{u} \times \mathbf{w})}{0}$
 $- \frac{\mathbf{w} \cdot (\mathbf{u} \times \mathbf{w})}{0} = \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) - \mathbf{v} \cdot (\mathbf{u} \times \mathbf{w}) - \mathbf{w} \cdot (\mathbf{u} \times \mathbf{v})$
 $= [\mathbf{u} \mathbf{v} \mathbf{w}] + [\mathbf{v} \mathbf{w} \mathbf{u}] - [\mathbf{w} \mathbf{u} \mathbf{v}] = \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$.

50) $\mathbf{i} \times \mathbf{j} \times \mathbf{k} = \mathbf{k} \times \mathbf{k} = 0$.
