

Intern Roster

Introduction

We have 11 interns. Let each intern be i .

We have 13 rotations. Let each rotation be j . There are also three annual leave rotations. Therefore the total is 16.

We have 54 weeks for the whole period of the roster. Let each week be k .

1a: Summing Everything

We could say that allocating each intern to a rotation (discluding leave) for a week can be expressed as:

$$\sum_{i=1}^{11} \sum_{j=1}^{13} \sum_{k=1}^{54} x_{ijk} = 1$$

There are a few problems with this model however. Lets expand some of this equation and see what we find.

$$x_{1,1,1} + \dots + x_{11,1,1} + x_{1,2,1} + \dots + x_{11,2,1} + x_{1,3,1} + \dots + x_{11,3,1} + \dots \dots + x_{1,1,2} + \dots + x_{11,1,2} + \dots \dots x_{11,13,54} = 1$$

So this equation is wildly incorrect. It is saying that for all interns for all rotations for all jobs there is only one instance where this can be true!

1b: Piece by Piece

So lets try to remedy it piece by piece. Clearly we cannot have just one long equation, we need a system of linear equations.

i For all i, Sum j

Let's start by just getting each intern into each rotation.

$$\forall i \sum_{j=1}^{13} x_{ij} = 1$$

So, what does this mean?

$$x_{1,1} + x_{1,2} + \cdots + x_{1,13} = 1$$

$$x_{2,1} + x_{2,2} + \cdots + x_{2,13} = 1$$

$$\vdots \quad \quad \quad \vdots$$

$$x_{11,1} + x_{11,2} + \cdots + x_{11,13} = 1$$

Now what does *this* mean?

It is saying that each intern will do one job exactly *once*.

This might be useful later, if say, we had this for each week.

ii For all j, Sum i

Let's look at this system the other way around.

$$\forall j \sum_{i=1}^{11} x_{ij} = 1$$

$$x_{1,1} + x_{2,1} + \cdots + x_{11,1} = 1$$

$$x_{1,2} + x_{2,2} + \cdots + x_{11,2} = 1$$

$$\vdots \quad \quad \quad \vdots$$

$$x_{1,13} + x_{2,13} + \cdots + x_{11,13} = 1$$

iii For all j, Sum i = 11

Now this system is saying the for each rotation, each intern will work it exactly once.

But actually we want every intern to work each rotation.

So we need the system:

$$\forall j \sum_{i=1}^{11} x_{ij} = 11$$

$$x_{1,1} + x_{2,1} + \cdots + x_{11,1} = 11$$

$$x_{1,2} + x_{2,2} + \cdots + x_{11,2} = 11$$

$$\vdots \quad \quad \quad \vdots$$

$$x_{1,13} + x_{2,13} + \cdots + x_{11,13} = 11$$

Of course, the result for this would be a 11x13 matrix with a 1 for every value, were this the only constraint.

2a: Constraints for Rotation Duration

Ok, different approach. We look at each rotations time. We construct our constraints around this. We will think about each rotation as having a cost. We will then seek to minimise this cost.

$$x_{1,1} = 8$$

Intern 1 spends 8 weeks at Rotation 1

$$x_{1,2} = 4$$

Intern 1 spends 4 weeks at Rotation 2

$$x_{1,3} = 4$$

Intern 1 spends 3 weeks at Rotation 3

⋮

So how do we express this is in a compacted form?

Let's look at Intern 2.

$$x_{2,1} = 8$$

Intern 2 spends 8 weeks at Rotation 1

$$x_{2,2} = 4$$

Intern 2 spends 4 weeks at Rotation 2

$$x_{2,3} = 4$$

Intern 2 spends 3 weeks at Rotation 3

So this is the same for all interns.

∴ We can express these at least as linear expressions.

Our rotation constraints are:

$$\forall i \ x_{i1} = 8, \quad (i = 1, 2, \dots, 11)$$

$$\forall i \ x_{i2} = 4, \quad (i = 1, 2, \dots, 11)$$

$$\forall i \ x_{i3} = 4, \quad (i = 1, 2, \dots, 11)$$

$$\forall i \ x_{i4} = 4, \quad (i = 1, 2, \dots, 11)$$

$$\forall i \ x_{i5} = 2, \quad (i = 1, 2, \dots, 11)$$

$$\forall i \ x_{i6} = 3, \quad (i = 1, 2, \dots, 11)$$

$$\forall i \ x_{i7} = 3, \quad (i = 1, 2, \dots, 11)$$

$$\forall i \ x_{i8} = 2, \quad (i = 1, 2, \dots, 11)$$

$$\forall i \ x_{i9} = 4, \quad (i = 1, 2, \dots, 11)$$

$$\forall i \ x_{i10} = 3, \quad (i = 1, 2, \dots, 11)$$

$$\forall i \ x_{i11} = 5, \quad (i = 1, 2, \dots, 11)$$

$$\forall i \ x_{i12} = 1, \quad (i = 1, 2, \dots, 11)$$

$$\forall i \ x_{i13} = 1, \quad (i = 1, 2, \dots, 11)$$

$$\forall i \ x_{i14} = 1, \quad (i = 1, 2, \dots, 11)$$

$$\forall i \ x_{i15} = 1, \quad (i = 1, 2, \dots, 11)$$

2b

Actually, looking back, 1b-i looks quite pertinent, however it needs a constraint for each week.

ie. For each week, each intern will do one job.

But maybe there is another way to look at it. Vicky wrote on a sheet of paper some random notes about potential equations for this problem. Lets have a look at some of these.

3a: Vicky's notes

Let $x_{jk}^i = 1$ if person i starts job j at time k .

For the first time constraint, Vicky wrote:

$$\sum_i x_{jk}^i = 2 \quad \forall j, \forall k$$

This is for rotation 1 having two interns at a time I think.

Now below that, we have something which i can't *quite* understand but it looks damn useful.

$$\sum_{\alpha=1}^8 y_{jk+\alpha}^i = 8 \quad \text{if } x_{jk}^i = 1$$

Looks to be a way to define a time period for a rotation. Very useful.

There is a "leave" constraint defined:

$$\sum_i x_{jk}^i \leq 11 y_k$$

And finally, what looks to be an objective function.

$$\max \sum_i \sum_j \sum_k C_{ij} x_{jk}^i$$

Oh actually one more that I'm not sure about:

$$\sum_k y_k = 1 \quad \text{where } y_k \text{ is boolean.}$$

So, like before, lets expand some of these equations so as to try to understand them.

3b: Rotation person capacity constraint

$$\sum_i x_{1,k}^i \leq 2 \quad \forall k$$

$$x_{1,1}^{(1)} + x_{1,1}^{(2)} + \cdots + x_{1,1}^{(11)} \leq 2$$

$$x_{1,2}^{(1)} + x_{1,2}^{(2)} + \cdots + x_{1,2}^{(11)} \leq 2$$

$$\vdots \quad \vdots \quad \vdots$$

$$x_{1,54}^{(1)} + x_{1,54}^{(2)} + \cdots + x_{1,54}^{(11)} \leq 2$$

OK this is solid - for each week, the maximum number of interns in rotation 1 will be 2.

So we can write this out for all Rotations.

$$\begin{aligned}
\sum_i x_{1,k}^i &\leq 2 & \forall k \\
\sum_i x_{2,k}^i &\leq 1 & \forall k \\
\sum_i x_{3,k}^i &\leq 1 & \forall k \\
\sum_i x_{4,k}^i &\leq 1 & \forall k \\
\sum_i x_{5,k}^i &\leq 1 & \forall k \\
\sum_i x_{6,k}^i &\leq 1 & \forall k \\
\sum_i x_{7,k}^i &\geq 0 & \forall k \\
\sum_i x_{8,k}^i &\geq 0 & \forall k \\
\sum_i x_{9,k}^i &\leq 2 & \forall k \\
\sum_i x_{10,k}^i &\geq 0 & \forall k \\
\sum_i x_{11,k}^i &\geq 0 & \forall k \\
\sum_i x_{12,k}^i &\leq 1 & \forall k \\
\sum_i x_{13,k}^i &\leq 1 & \forall k \\
\sum_i x_{14,k}^i &= 11 & \forall k \\
\sum_{i=1}^6 x_{15,k}^i &= 6 & \forall k \\
\sum_{i=7}^{11} x_{16,k}^i &= 5 & \forall k
\end{aligned}$$

And we have our constraints for rotation capacity.

3c: Rotation Duration Constraint

$$\sum_{\alpha=1}^8 y_{j,k+\alpha}^i = 8 \text{ if } x_{jk}^i = 1$$

Here, Vicky has not specified whether we should sum over i, j or k. I am thinking this is our duration constraint.

Lets expand it out and see what illuminates.

$$y_{j,k+1}^i + y_{j,k+2}^i + \cdots + y_{j,k+8}^i = 8 \text{ if } x_{jk}^i = 1$$

So say, for instance, that we have $x_{1,1}^1 = 1$

$$y_{1,2}^1 + y_{1,3}^1 + \cdots + y_{1,9}^1 = 8$$

This is saying that when an intern is rostered at a location for a week, for the following 8 weeks they are also rostered there.

This is pertinent, but not exactly correct.

A few issues to correct.

The first is that alpha should start at 0 and go to 7 if we have a goal of 8 weeks, as k will have a value already, being the first week of that rotation.

The second is we need to link x and y back together somehow, so that values for k do not ascribed multiple times.

The third is that the equations will differ for each rotation. But we can develop that system immediately.

$$\sum_{\alpha=0}^7 y_{1,k+\alpha}^i = 8 \text{ if } x_{1,k}^i = 1$$

$$\sum_{\alpha=0}^3 y_{2,k+\alpha}^i = 4 \text{ if } x_{2,k}^i = 1$$

$$\sum_{\alpha=0}^3 y_{3,k+\alpha}^i = 4 \text{ if } x_{3,k}^i = 1$$

$$\sum_{\alpha=0}^3 y_{4,k+\alpha}^i = 4 \text{ if } x_{4,k}^i = 1$$

$$\sum_{\alpha=0}^1 y_{5,k+\alpha}^i = 2 \text{ if } x_{5,k}^i = 1$$

$$\sum_{\alpha=0}^2 y_{6,k+\alpha}^i = 3 \text{ if } x_{6,k}^i = 1$$

$$\sum_{\alpha=0}^2 y_{7,k+\alpha}^i = 3 \text{ if } x_{7,k}^i = 1$$

$$\sum_{\alpha=0}^1 y_{8,k+\alpha}^i = 2 \text{ if } x_{8,k}^i = 1$$

$$\sum_{\alpha=0}^3 y_{9,k+\alpha}^i = 4 \text{ if } x_{9,k}^i = 1$$

$$\sum_{\alpha=0}^2 y_{10,k+\alpha}^i = 3 \text{ if } x_{10,k}^i = 1$$

$$\sum_{\alpha=0}^4 y_{11,k+\alpha}^i = 5 \text{ if } x_{11,k}^i = 1$$

$$y_{12,k}^i = 1 \text{ if } x_{12,k}^i = 1$$

$$y_{13,k}^i = 1 \text{ if } x_{13,k}^i = 1$$

$$y_{14,k}^i = 1 \text{ if } x_{14,k}^i = 1$$

$$y_{15,k}^i = 1 \text{ if } x_{15,k}^i = 1$$

$$y_{16,k}^i = 1 \text{ if } x_{16,k}^i = 1$$

We'll leave connected x and y for now and continue in our pursuit of understanding Vicky's equations.

3d: Leave Constraint

experimentation

$$\sum_i x_{jk}^i \leq 11y_k$$

If this is indeed the leave constraint, then we know what our j values will be. This may also be our method of connecting x and y .

$$x_{14,k}^1 + x_{14,k}^2 + \cdots + x_{14,k}^{11} \leq 11y_k$$

$$x_{15,k}^1 + x_{15,k}^2 + \cdots + x_{15,k}^{11} \leq 6y_k$$

$$x_{16,k}^1 + x_{16,k}^2 + \cdots + x_{16,k}^{11} \leq 5y_k$$

But is the constant of y_k correct?

It would seem we have neglected this part of the constraint:

$$\sum_k y_k = 1 \quad \text{where } y_k \text{ is boolean.}$$

That's a bit more sensical. But clearly here, y is different to the y in the duration constraint. So we rename it z .

Let's rewrite all this with this in mind.

revised

$$\sum_i x_{jk}^i = (K)z_k \quad \text{if } \sum_k z_k = 1$$

$$x_{14,k}^1 + x_{14,k}^2 + \cdots + x_{14,k}^{11} = 11z_k$$

$$x_{15,k}^1 + x_{15,k}^2 + \cdots + x_{15,k}^{11} = 6z_k$$

$$x_{16,k}^1 + x_{16,k}^2 + \cdots + x_{16,k}^{11} = 5z_k$$

Hmmmmmm.

So if we have leave for everyone on week 1:

$$x_{14,1}^1 + x_{14,1}^2 + \cdots + x_{14,1}^{11} = 11z_1$$

Ok, this will leave us with a system of three equations.

$$\sum_i x_{14,k}^i = 11z_k \quad \text{if} \quad \sum_k z_k = 1$$

$$\sum_i x_{15,k}^i = 6z_k \quad \text{if} \quad \sum_k z_k = 1$$

$$\sum_i x_{16,k}^i = 5z_k \quad \text{if} \quad \sum_k z_k = 1$$

3e:

$$\max \sum_i \sum_j \sum_k C_{ij} x_{jk}^i$$

Whats up with that C ?