MAST90045 - Assignment 1, 2020

The level of a dam

In this assignment we will model the changing level (or height) of water in a dam (Figure 1). The minimum level is 0 and the maximum is $h_{\max}$. The level increases when rain falls in the catchment area and decreases as a result of evaporation and use. We will ignore any loss due to leaks or seepage.

Height and volume

Volume Let $A(h)$ be the cross-sectional area of the dam at height h.

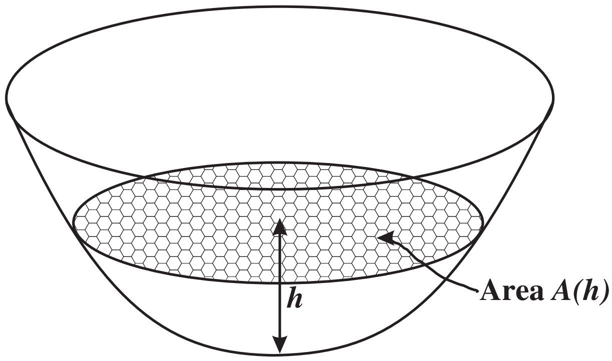

Figure 1: An idealized dam.

The volume of water contained by the dam when it is filled to level h is

$$V(h) = \int_0^h A(u)\,du.$$

Write a function `volume(h, hmax, ftn)` that returns $V(h)$ for $h \in [0, h_{\max}]$, where `hmax` is $h_{\max}$ and `ftn` is a function of a single variable which is assumed to return $A(h)$. For $h < 0$ or $h > h_{\max}$ your function should return `NA`.

Give consideration to the accuracy of any numerical algorithms you use. Discuss your choices.

Height If the current level of the dam is h and the volume of the dam changes by an amount v, then the level of the dam becomes $u = H(h, v)$ where u satisfies

$$V(u) = V(h) + v.$$

Note that if the right-hand side of this equation is $> V_{\max} = V(h_{\max})$ or < 0, then this equation has no solution. In this case we take $u = h_{\max}$ or $u = 0$, respectively.

Using a root-finding algorithm, write a function `height(h, hmax, v, ftn)` that returns $H(h, v)$, where `hmax` is $h_{\max}$ and `ftn` is a function of a single variable that is assumed to return $A(h)$. For $h < 0$ or $h > h_{\max}$ your function should return `NA`.

What is a suitable tolerance for the root-finding algorithm? Discuss your choice.

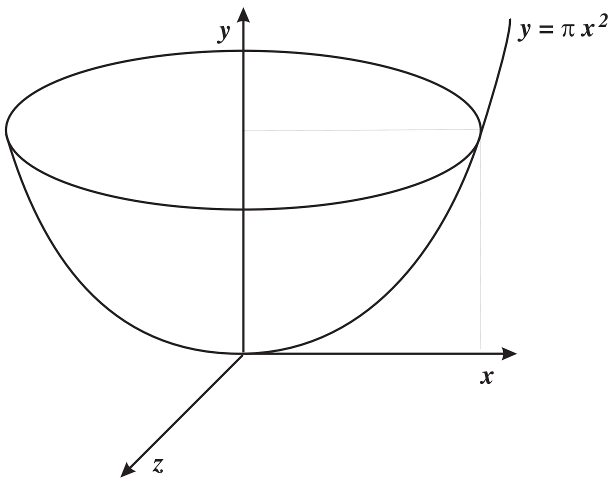

Figure 2: A schematic dam.

Test case Suppose that the dam is bowl-shaped with profile given by the equation $y = \pi x^2$. That is, the dam has the shape obtained by rotating the curve $y = \pi x^2$ about the y-axis (Figure 2).

In this case it is easily seen that, for $h \in [0, h_{\max}]$ and $v \in [-h^2/2, V_{\max} - h^2/2]$,

$$\begin{aligned} A(h) &= h; \\ V(h) &= h^2/2; \text{ and} \\ H(h, v) &= \sqrt{h^2 + 2v}. \end{aligned}$$

To test that your function `height(h, hmax, v, ftn)` works, define

```
A <- function(h) return(h)
```

then calculate `height(h, hmax = 4, v, ftn = A)` for the following values of h and v:

h	-1	0.5	0.5	3.5	3.5	3.5	5
v	1	1	-1	1	2	-1	1

Check that your numerical results agree with the formula given above.

Tracking height over time

Suppose that $h(t)$ is the level of the dam at the start of day t, and that $v(t)$ is the volume of rain falling into the catchment during day t, for $t = 1, \ldots, n$. Also let α be the volume of water taken from the dam for use per day, and let $\beta A(h(t))$ be the volume of water lost due to evaporation during day t. Then the level of water in the dam at the start of day $t + 1$ is given by

$$h(t + 1) = H(h(t), v(t) - \alpha - \beta A(h(t))).$$

Further suppose that $h_{\max} = 3$, $\alpha = 2$, $\beta = 0.1$, and $A(h)$ has the form

$$A(h) = \begin{cases} 100h^2 & \text{for } 0 \leq h \leq 2; \\ 400(h - 1) & \text{for } 2 \leq h \leq 3. \end{cases}$$

The file `catchment_b.txt` gives $v(t)$ for $n = 100$ consecutive days. Write a program that reads this file then, for a given value of $h(1)$, calculates $h(2), \ldots, h(n+1)$. Plot your output for the case $h(1) = 1$, as in the figure below.

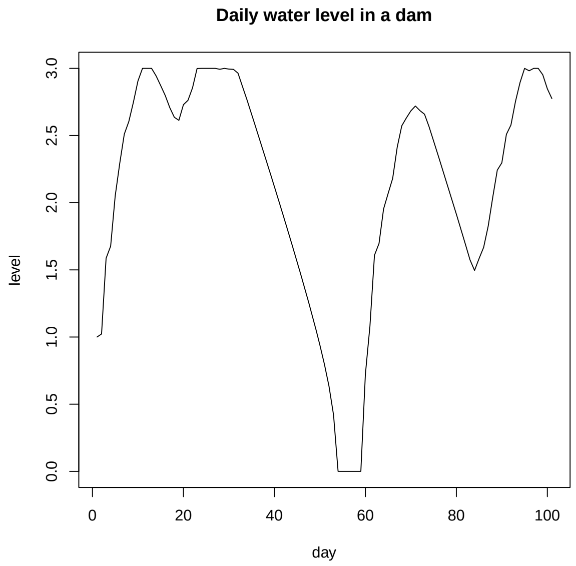

Figure 3: Simulated time trace of water level for dam, $h(1) = 1$.