Techniques in Operations Research
Assignment 1 – 2021

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Question 1

a)

b)

$$\begin{aligned} \min \quad & f(x) = -x_1 - 3x_2 \\ s.t. \quad & g_1(x) = -x_1 - x_2 + 1 \leq 0 \\ & g_2(x) = x_1 - 2 \leq 0 \\ & h(x) = 2x_1 - x_2 + 1 = 0 \end{aligned}$$

c)

The KKT conditions for the program are:

KKTa:

$$0 = \nabla f(x^*) + \nabla g(x^*)\lambda^* + \nabla h(x^*)\eta^*$$

KKTb:

$$\begin{aligned} &1) \;\; g(x^*) \leq 0 \\ &2) \;\; \lambda^* \geq 0 \\ &3) \;\; \lambda_i^* g_i(x^*) = 0, \;\; \forall i \end{aligned}$$

KKTc:

$$h(x^*) = 0$$

d)

$L(x^*, \lambda^*, \eta^*) = -x_1 - 3x_2 + \lambda_1^*(-x_1 - x_2 + 1) + \lambda_2^*(x_1 - 2) + \eta^*(2x_1 - x_2 + 1)$

KKTa:

$$\nabla L_x(x^*, \lambda^*, \eta^*) = \begin{bmatrix} -1 - \lambda_1^* + \lambda_2^* + 2\eta^* \\ -3 - \lambda_1^* - \eta^* \end{bmatrix} = 0$$

$\Rightarrow$

$$-1 - \lambda_1^* + \lambda_2^* + 2\eta^* = 0 \tag{1}$$
$$-3 - \lambda_1^* - \eta^* = 0 \tag{2}$$

KKTc:

$$2x_1^* - x_2^* + 1 = 0$$

Consider:

Both g_1 and g_2 active:

KKTb:

$$\Rightarrow -x_1^* - x_2^* + 1 = 0, \quad x_1 - 2 = 0$$

$$\Rightarrow x_1^* = 2, \Rightarrow x_2^* = -1$$

$$x^* = (2, -1)$$

Verify x^* with KKTc:

$$h(x^*) = 2(2) - (-1) + 1 = 6 \neq 0$$

Therefore x^* infeasible

g_1 active, g_2 inactive:

KKTb:

$$\Rightarrow -x_1^* - x_2^* + 1 = 0, \quad \lambda_2^* = 0$$

$$\Rightarrow x_1^* = -x_2^* + 1$$

Substitute x_1^* into KKTc:

$$\Rightarrow 2(-x_2^* + 1) - x_2^* + 1 = 0,$$

$$\Rightarrow -3x_2^* + 3 = 0$$

$$\Rightarrow x_2^* = 1 \quad \Rightarrow x_1^* = 0$$

$$x^* = (0, 1)$$

Verify x^* satisfies KKTb(1):

$$g_2(x^*) = (0) - 1 \leq 0$$

So far $x^* = (2, 5)$ is promising, but lack enough information to test for optimality, so further testing required.

g_2 active, g_1 inactive:

KKTb:

$$\Rightarrow x_1^* - 2 = 0, \quad \lambda_1^* = 0$$

$$\Rightarrow x_1^* = 2$$

Substitute x_1^* into KKTc:

$$\Rightarrow 2(-2) - x_2^* + 1 = 0,$$

$$\Rightarrow x_2^* = 5$$

$$x^* = (2, 5)$$

Verify x^* satisfies KKTb(1):

$$g_1(x^*) = -(2) - (5) + 1 \leq 0$$

So far $x^* = (2, 5)$ is promising, but lack enough information to test for optimality, so further testing required.

Both g_1 and g_2 inactive:

KKTb:

$$\Rightarrow \lambda_1^*, \lambda_2^* = 0$$

$$\Rightarrow \lambda^* = (0, 0)$$

Substitute λ^* into KKTa:

$$-1 + 2\eta^* = 0$$

$$-3 - \eta^* = 0$$

$\nexists\ \eta^*$ which satisfies KKTa, therefore problem unsolvable

Question 2

a)

The KKT conditions for the program are:

KKTa:

$$\nabla f(x^*) + \nabla g(x^*)\lambda^* + \nabla h(x^*)\eta^* = 0$$

$$\Rightarrow \nabla L_x(x^*, \lambda^*) = \begin{bmatrix} (x_1^* - 2)^3 + \lambda_1^* - \lambda_2^* \\ 4(x_2^*)^3 - \lambda_1^* + 2x_2^*\lambda_2^* \end{bmatrix} = 0$$

KKTb:

$$\begin{aligned} &1)\ \ g(x^*) \leq 0 \\ &2)\ \ \lambda^* \geq 0 \\ &3)\ \ \lambda_i^* g_i(x^*) = 0, \ \ \forall i \end{aligned}$$

Consider:

Both g_1 and g_2 active:

KKTb:

$$\Rightarrow x_1^* - x_2^* - 8 = 0, \quad -x_1 + (x_2^*)^2 + 4 = 0$$

$$\Rightarrow x_1^* = x_2^* + 8 \quad \Rightarrow \quad -x_2^* + (x_2^*)^2 - 4 = 0$$

No real solutions for x^*, therefore infeasible

g_1 active, g_2 inactive:

KKTb:

$$\Rightarrow x_1^* - x_2^* - 8 = 0, \quad \lambda_2^* = 0$$

$$\Rightarrow x_1^* = x_2^* + 8$$

Substitute x_1^* into KKTa:

$$\Rightarrow \nabla L_x(x_2^* + 8, x_2^*, \lambda_1^*, 0) = \begin{bmatrix} (x_2^* + 6)^3 + \lambda_1^* \\ 4(x_2^*)^3 - \lambda_1^* \end{bmatrix} = 0$$

Solving we get:

$$x^* = (5.681, -2.319), \quad \lambda^* = (-49.8975, 0)$$

But λ^* violates KKTb. Therefore, infeasible

g_2 active, g_1 inactive:

KKTb:

$$\Rightarrow -x_1^* + (x_2^*)^2 + 4 = 0, \quad \lambda_1^* = 0$$
$$\Rightarrow x_1^* = (x_2^*)^2 + 4$$

Substitute x_1^* into KKTa:

$$\Rightarrow \nabla L_x((x_2^*)^2 + 4, x_2^*, 0, \lambda_2^*) = \begin{bmatrix} ((x_2^*)^2 + 2)^3 - \lambda_2^* \\ 4(x_2^*)^3 + 2x_2^*\lambda_2^* \end{bmatrix} = 0$$

$$\Rightarrow ((x_2^*)^2 + 2)^3 - \lambda_2^* = 0 \tag{3}$$
$$2x_2^*(2(x_2^*)^2 + \lambda_2^*) = 0 \tag{4}$$

With (4) we get:

$$x_2^* = 0, \qquad x_2^* = \pm\sqrt{\frac{-\lambda_2^*}{2}}$$

But since we need $\lambda_2^* \geq 0$ (KKTb), we reject second solution:

$$\therefore \ \mathbf{x_2^* = 0}$$

Using (3), solve for λ_2^*:

$$((0)^2 + 2)^3 - \lambda_2^* = 0 \quad \Rightarrow \boldsymbol{\lambda_2^* = 8}$$

Using KKTb, solve for x_1^*:

$$x_1^* = (0)^2 + 4 \quad \Rightarrow \mathbf{x_1^* = 4}$$
$$\therefore \ \mathbf{x}^* = (\mathbf{4}, \mathbf{0}), \quad \lambda^* = (\mathbf{0}, \mathbf{8})$$

Verify x^* satisfies KKTb(1):

$$g_1(x^*) = (4) - (0) - 8 = -4 \leq 0$$

Therefore, all KKT conditions have been satisfied.

Both g_1 and g_2 inactive:

KKTb:

$$\Rightarrow \lambda_1^*, \lambda_2^* = 0$$

Solve for x^* using KKTa:

$$\Rightarrow \nabla L_x(x_1^*, x_2^*, 0, 0) = \begin{bmatrix} ((x_1^* - 2)^3 \\ 4(x_2^*)^3 \end{bmatrix} = 0$$

$$\Rightarrow x^* = (0, 0)$$

Verify x^* satisfies KKTb(1):

$$g_1(x^*) = (0) - (0) - 8 \leq 0$$

$$g_2(x^*) = -(0) + (0)^2 + 4 \not\leq 0$$

x^* violates KKTb, therefore infeasible.

b)

$$\nabla g_1(x) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \nabla g_2(x) = \begin{bmatrix} -1 \\ 2x_2 \end{bmatrix}$$

$$\Rightarrow \nabla g(x) = \begin{bmatrix} 1 & -1 \\ -1 & 2x_2 \end{bmatrix}$$

Inspect $x^* = (4, 0)$:

$$\nabla g(x^*) = \begin{bmatrix} 1 & -1 \\ -1 & 0 \end{bmatrix}$$

To satisfy Mangasarian-Fomovitz condition, need:

$$\nabla g(x^*)^T d < 0$$

$$\Rightarrow \begin{bmatrix} 1 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} < 0$$

$$\begin{aligned} d_1 - d_2 < 0 \quad &\Rightarrow \mathbf{d_1 < d_2} \\ -d_1 < 0 \quad &\Rightarrow \mathbf{d_1 > 0} \end{aligned}$$

Choose d:

$$d = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

At $x^* = (4, 0)$, $\exists\, d$ which satistfies the M-F Condition.

c)

To check local minimality, need:

$$d^T \nabla^2_{xx} L(x^*, \lambda^*) d > 0$$

Find Hessian of Lagrangian:

$$\nabla L_x = \begin{bmatrix} (x_1^* - 2)^3 + \lambda_1^* - \lambda_2^* \\ 4(x_2^*)^3 - \lambda_1^* + 2x_2^* \lambda_2^* \end{bmatrix} \quad \nabla^2 L_{xx} = \begin{bmatrix} 3(x_1 - 2)^2 & 0 \\ 0 & 12x_2^2 \end{bmatrix}$$

Check sufficient optimality condition at $x^* = (4, 0)$:

$$\begin{aligned} d^T \nabla^2_{xx} L(x^*, \lambda^*) d &= \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 12 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ &= 12 > 0 \end{aligned}$$

Therefore, sufficient optimality condition is satisfied.
Therefore, $(4, 0)$ is a local minimum.