MAST30013 Techniques in Operations Research
Assignment 3

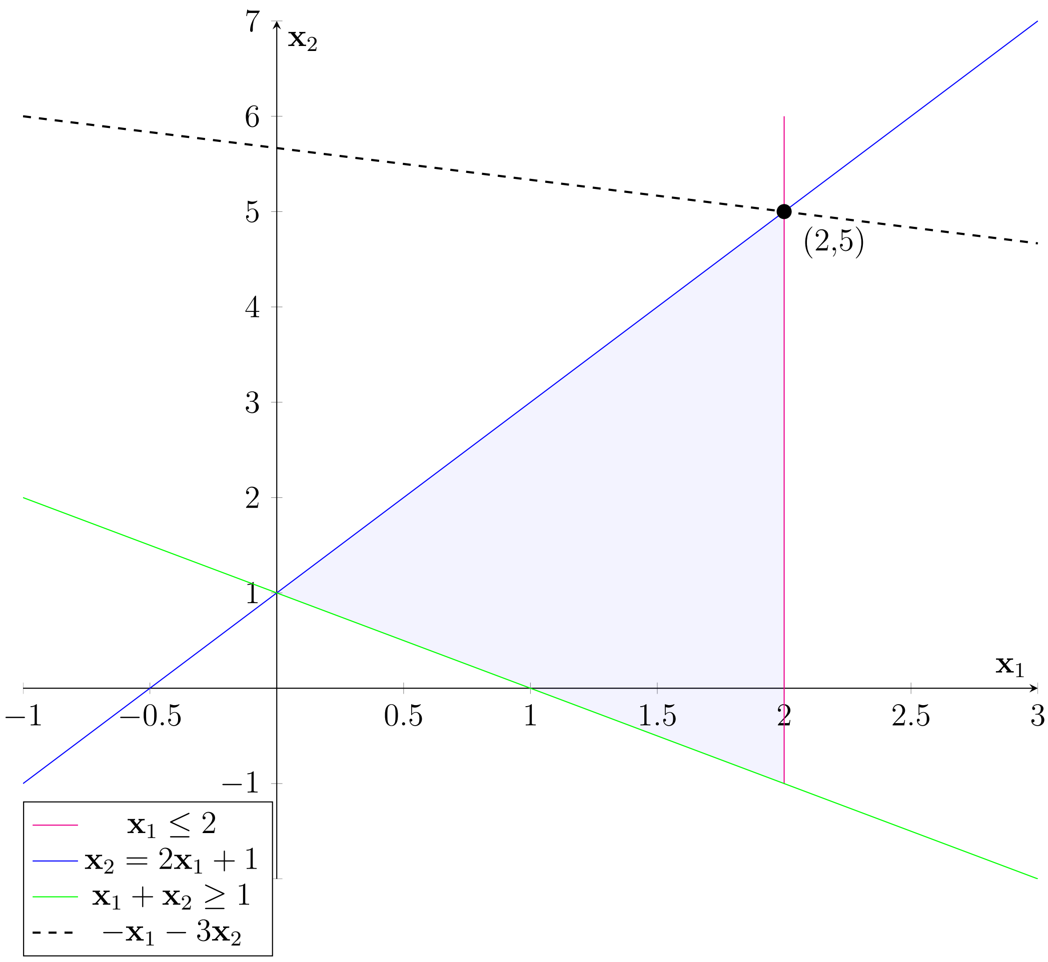
Chris Swan 370502

Due 5pm Monday, 17 May 2021

1

(a)

Graphical solution



(b)

$$
\begin{aligned}
\min \quad & -\mathbf{x}_1 - 3\mathbf{x}_2 \\
\text{s.t.} \quad & g_1(\mathbf{x}) = -\mathbf{x}_1 - \mathbf{x}_2 + 1 \leq 0 \\
& g_2(\mathbf{x}) = \mathbf{x}_1 - 2 \leq 0 \\
& h(\mathbf{x}) = 2\mathbf{x}_1 - \mathbf{x}_2 + 1 = 0
\end{aligned}
$$

The Lagrangian is:

$$L(\mathbf{x}, \lambda, \eta) = -\mathbf{x}_1 - 3\mathbf{x}_2 + \lambda_1(-\mathbf{x}_1 - \mathbf{x}_2 + 1) + \lambda_2(\mathbf{x}_1 - 2) + \eta(2\mathbf{x}_1 - \mathbf{x}_2 + 1)$$

(c)

KKTa:

$$
\begin{align}
-1 - \lambda_1 + \lambda_2 + 2\eta = 0 \tag{1}\\
-3 - \lambda_1 - \eta = 0 \tag{2}
\end{align}
$$

KKTb:

$$
\begin{align}
-\mathbf{x}_1 - \mathbf{x}_2 + 1 \leq 0 \tag{3}\\
\mathbf{x}_1 - 2 \leq 0 \tag{4}\\
\lambda_1 \geq 0 \tag{5}\\
\lambda_2 \geq 0 \tag{6}\\
\lambda_1(-\mathbf{x}_1 - \mathbf{x}_2 + 1) \tag{7}\\
\lambda_2(\mathbf{x}_1 - 2) \tag{8}
\end{align}
$$

KKTc:

$$
2\mathbf{x}_1 - \mathbf{x}_2 + 1 = 0 \tag{9}
$$

(d)

i $\lambda_1 = \lambda_2 = 0$**:**

KKTa:

$$
\begin{aligned}
(1) : -1 + 2\eta &= 0\\
\eta &= \frac{1}{2}\\
(2) : \eta &= -3
\end{aligned}
$$

Contradiction. No candidate point.

ii $\lambda_1 > 0, \lambda_2 > 0$**:**

KKTb:

$$
\begin{align}
(8) : \mathbf{x}_1 &= 2 \tag{10}\\
(10) \to (7) : -\mathbf{x}_2 &= -1 + 2 \nonumber\\
-\mathbf{x}_2 &= 1 \nonumber\\
\mathbf{x}_2 &= -1 \tag{11}
\end{align}
$$

KKTc:

$$
(11) \to (9) : 2(2) - (-1) + 1 = 6 \neq 0
$$

Contradiction. No candidate point.

iii $\lambda_1 > 0, \lambda_2 = 0$**:**

KKTb:

$$
\begin{align}
(7) : -\mathbf{x}_1 - \mathbf{x}_2 + 1 &= 0 \nonumber\\
\mathbf{x}_1 &= -\mathbf{x}_2 + 1 \tag{12}
\end{align}
$$

KKTc:

$$
\begin{align}
(12) \to (9) : 2(-\mathbf{x}_2 + 1) - \mathbf{x}_2 + 1 &= 0 \nonumber\\
-2\mathbf{x}_2 + 2 - \mathbf{x}_2 + 1 &= 0 \nonumber\\
-3\mathbf{x}_2 + 3 &= 0 \nonumber\\
\mathbf{x}_2 &= 1 \tag{13}
\end{align}
$$

KKTb:

$$(13) \rightarrow (12) : \mathbf{x}_1 = 0 \tag{14}$$

KKTa:

$$\begin{align}
(2) : -3 - \lambda_1 - \eta &= 0 \nonumber \\
\lambda_1 &= -\eta - 3 \tag{15} \\
(15) \rightarrow (1) : -1 + \eta + 3 + 2\eta &= 0 \nonumber \\
\eta &= -\frac{2}{3} \tag{16} \\
(16) \rightarrow (15) : \lambda_1 &= -\frac{7}{3} \not> 0 \nonumber
\end{align}$$

Contradiction. No candidate point.

iv $\lambda_1 = 0, \lambda_2 > 0$**:**

KKTb:

$$(8) : \mathbf{x}_1 = 2 \tag{17}$$

KKTc:

$$\begin{align}
(17) \rightarrow (9) : 2(2) - \mathbf{x}_2 + 1 &= 0 \nonumber \\
\mathbf{x}_2 &= 5 \tag{18}
\end{align}$$

KKTa:

$$\begin{align}
(2) : \eta &= -3 \tag{19} \\
(1) : -1 - \lambda_2 + 2(-3) &= 0 \nonumber \\
\lambda_2 &= 7 \tag{20}
\end{align}$$

Therefore we have the candidate point

$$(\lambda_1^\star, \lambda_2^\star, \eta^\star) = (0, 7, -3)$$
$$(\mathbf{x}_1^\star, \mathbf{x}_2^\star) = (2, 5)$$

2

(a)

$$\begin{aligned} \min \quad & \frac{(\mathbf{x}_1-2)^4}{4}+\mathbf{x}_2^4+4 \\ \text{s.t.} \quad & g_1(\mathbf{x})=\mathbf{x}_1-\mathbf{x}_2-8\leq 0 \\ & g_2(\mathbf{x})=-\mathbf{x}_1+2\mathbf{x}_2^2+4\leq 0 \end{aligned}$$

The Lagrangian is:

$$L(\mathbf{x},\lambda)=\frac{(\mathbf{x}_1-2)^4}{4}+\mathbf{x}_2^4+4+\lambda_1(\mathbf{x}_1-\mathbf{x}_2-8)+\lambda_2(-\mathbf{x}_1+2\mathbf{x}_2^2+4) \tag{21}$$

and the gradient of the Lagrangian is:

$$\nabla_{\mathbf{x}}L(\mathbf{x},\lambda)=\begin{bmatrix}(\mathbf{x}_1-2)^3+\lambda_1-\lambda_2 \\ 4\mathbf{x}_2^3-\lambda_1+2\lambda_2\mathbf{x}_2\end{bmatrix} \tag{22}$$

KKTa:

$$(\mathbf{x}_1-2)^3+\lambda_1-\lambda_2=0 \tag{23}$$
$$4\mathbf{x}_2^3-\lambda_1+2\lambda_2\mathbf{x}_2=0 \tag{24}$$

KKTb:

$$\mathbf{x}_1-\mathbf{x}_2-8\leq 0 \tag{25}$$
$$-\mathbf{x}_1+2\mathbf{x}_2^2+4\leq 0 \tag{26}$$
$$\lambda_1\geq 0 \tag{27}$$
$$\lambda_2\geq 0 \tag{28}$$
$$\lambda_1(\mathbf{x}_1-\mathbf{x}_2-8)=0 \tag{29}$$
$$\lambda_2(-\mathbf{x}_1+2\mathbf{x}_2^2+4)=0 \tag{30}$$

i $\lambda_1=\lambda_2=0$:

KKTa:

$$(23):\mathbf{x}_1=2 \tag{31}$$
$$(31)\rightarrow(23):4\mathbf{x}_2^3=0$$
$$\mathbf{x}_2=0 \tag{32}$$

KKTb:

$$(31),(32)\rightarrow(25):(2)-(0)-8\leq 0$$
$$(31),(32)\rightarrow(26):-(2)+2(0)^2+4\nleq 0 \tag{33}$$

Contradiction. No candidate point.

ii $\lambda_1>0,\lambda_2=0$:

KKTa:

$$(23):(\mathbf{x}_1-2)^3+\lambda_1=0$$
$$\lambda_1=-(\mathbf{x}_1-2)^3 \tag{34}$$
$$(24):4\mathbf{x}_2^3-\lambda_1=0$$
$$\lambda_1=4\mathbf{x}_2^3$$
$$\therefore 4\mathbf{x}_2^3=-(\mathbf{x}_1-2)^3$$
$$\mathbf{x}_2=-\frac{\mathbf{x}_1-2}{\sqrt[3]{4}} \tag{35}$$

KKTb:

$$\begin{aligned}
(35) \rightarrow (29) : \mathbf{x}_1 + \frac{\mathbf{x}_1 - 2}{\sqrt[3]{4}} - 8 = 0 \\
\mathbf{x}_1 + \frac{\mathbf{x}_1}{\sqrt[3]{4}} - \frac{2}{\sqrt[3]{4}} - 8 = 0 \\
\sqrt[3]{4}\mathbf{x}_1 + \mathbf{x}_1 &= 2 + 8\sqrt[3]{4} \\
\mathbf{x}_1(\sqrt[3]{4} + 1) &= 2 + 8\sqrt[3]{4} \\
\mathbf{x}_1 &= \frac{2 + 8\sqrt[3]{4}}{1 + \sqrt[3]{4}} \approx 5.6811 \qquad (36)
\end{aligned}$$

KKTa:

$$\begin{aligned}
(36) \rightarrow (35) : \mathbf{x}_2 &= -\frac{\frac{2+8\sqrt[3]{4}}{1+\sqrt[3]{4}} - 2}{\sqrt[3]{4}} \approx -2.3189 \\
(36) \rightarrow (34) : \lambda_1 &= -(\frac{2 + 8\sqrt[3]{4}}{1 + \sqrt[3]{4}} - 2)^3 \approx -49.8795 \not> 0
\end{aligned}$$

Contradiction. No candidate point.

iii $\lambda_1 = 0, \lambda_2 > 0$**:**

KKTa:

$$\begin{aligned}
(23) : (\mathbf{x}_1 - 2)^3 - \lambda_2 &= 0 \\
\lambda_2 &= (\mathbf{x}_1 - 2)^3 \qquad (37) \\
(24) : 4\mathbf{x}_2^3 + 2\lambda_2\mathbf{x}_2 &= 0 \\
2\mathbf{x}_2(2x_2^2 + \lambda_2) &= 0 \\
\mathbf{x}_2 = 0 \text{ and } \sqrt{\frac{-\lambda_2}{2}} \\
\therefore \mathbf{x}_2 = 0 \qquad (38) \\
\left(\sqrt{\frac{-\lambda_2}{2}} \text{ is a non-real solution as } \lambda_2 > 0.\right) \qquad (39)
\end{aligned}$$

KKTb:

$$\begin{aligned}
(30) : -\mathbf{x}_1 + 2(0)^2 + 4 &= 0 \\
\mathbf{x}_1 &= 4 \qquad (40)
\end{aligned}$$

KKTa:

$$(40) \rightarrow (37) : \lambda_2 = ((4) - 2)^3 = 8$$

Therefore we have the candidate point

$$\begin{aligned}
(\lambda_1^\star, \lambda_2^\star) &= (0, 8) \\
(\mathbf{x}_1^\star, \mathbf{x}_2^\star) &= (4, 0)
\end{aligned}$$

iv $\lambda_1, \lambda_2 > 0$**:**

KKTa:

$$\begin{aligned}
(23) : \lambda_1 &= \lambda_2 - (\mathbf{x}_1 - 2)^3 \qquad (41) \\
(41) \rightarrow (24) : 4\mathbf{x}_2^3 - (\lambda_2 - (\mathbf{x}_1 - 2)^3) + 2\lambda_2\mathbf{x}_2 &= 0 \\
\lambda_2(2\mathbf{x}_2 - 1) &= -4\mathbf{x}_2^3 - (\mathbf{x}_1 - 2)^3 \\
\lambda_2 &= \frac{-4\mathbf{x}_2^3 - (\mathbf{x}_1 - 2)^3}{(2\mathbf{x}_2 - 1)} \qquad (42)
\end{aligned}$$

KKTb:

$$\begin{aligned}(25) : \mathbf{x}_1 &= \mathbf{x}_2 + 8 && (43)\\ (43) \rightarrow (26) : -\mathbf{x}_2 - 8 + \mathbf{x}_2^2 + 4 &= 0\\ \mathbf{x}_2^2 - x_2 - 4 &= 0\\ \mathbf{x}_2 &= \frac{1 \pm \sqrt{17}}{2} && (44)\end{aligned}$$

$$(44) \rightarrow (43) : \quad \therefore \mathbf{x}^\star = \left(\frac{17+\sqrt{17}}{2}, \frac{1+\sqrt{17}}{2}\right) \text{ and } \left(\frac{17-\sqrt{17}}{2}, \frac{1-\sqrt{17}}{2}\right) \qquad (45)$$

Substituting $\mathbf{x}^\star = \left(\frac{17+\sqrt{17}}{2}, \frac{1+\sqrt{17}}{2}\right) \rightarrow (42)$:

$$\lambda_2 = \frac{-4(\frac{1+\sqrt{17}}{2})^3 - (\frac{17+\sqrt{17}}{2} - 2)^3}{\sqrt{17}} \approx -168.5124 \not> 0$$

Substituting $\mathbf{x}^\star = \left(\frac{17-\sqrt{17}}{2}, \frac{1-\sqrt{17}}{2}\right) \rightarrow (42)$:

$$\lambda_2 = \frac{-4(\frac{1-\sqrt{17}}{2})^3 - (\frac{17-\sqrt{17}}{2} - 2)^3}{-\sqrt{17}} \approx 17.5124 \qquad (46)$$

KKTa:

$$\begin{aligned}(46) \rightarrow (41) : \lambda_1 &= \lambda_2 - (\mathbf{x}_1 - 2)^3\\ \lambda_1 &= \frac{-4(\frac{1-\sqrt{17}}{2})^3 - (\frac{17-\sqrt{17}}{2} - 2)^3}{-\sqrt{17}} - \left(\frac{17-\sqrt{17}}{2} - 2\right)^3 \approx -23.1401 \not> 0\end{aligned}$$

Contradiction. No candidate point.

(b)

Using the Mangasarian-Fromovitz Constraint Qualification:

For the candidate point

$$\begin{aligned}(\lambda_1^\star, \lambda_2^\star) &= (0, 8)\\ (\mathbf{x}_1^\star, \mathbf{x}_2^\star) &= (4, 0)\end{aligned}$$

For $i \in I(\mathbf{x}^\star)$,

$$\nabla g_i(\mathbf{x}^\star)^T d < 0$$

$$\begin{aligned}\nabla g_2(\mathbf{x}^\star)^T d &< 0\\ \begin{bmatrix}-1 & 0\end{bmatrix}\begin{bmatrix}d_1\\ d_2\end{bmatrix} &< 0\\ -d_1 &< 0\\ d_1 &> 0\end{aligned}$$

We can find some $d \in \mathbb{R}^2$ that satisfies this direction.

ie.

$$d = \begin{bmatrix}1\\ 0\end{bmatrix}$$

We can see this clearly through examination of $g_2(\mathbf{x}^\star)$. Any move in the negative direction would violate this constraint.

(c)

For the candidate point

$$(\lambda_1^\star, \lambda_2^\star) = (0, 8)$$
$$(\mathbf{x}_1^\star, \mathbf{x}_2^\star) = (4, 0)$$

From (22), we can derive the Hessian of the Lagrangian with respect to $\mathbf{x}$.

$$\nabla^2_{\mathbf{x},\mathbf{x}} L(\mathbf{x}^\star, \lambda^\star) = \begin{bmatrix} 3(\mathbf{x}_1 - 2)^2 & 0 \\ 0 & 12\mathbf{x}_2^2 + 2\lambda_2 \end{bmatrix} \tag{47}$$

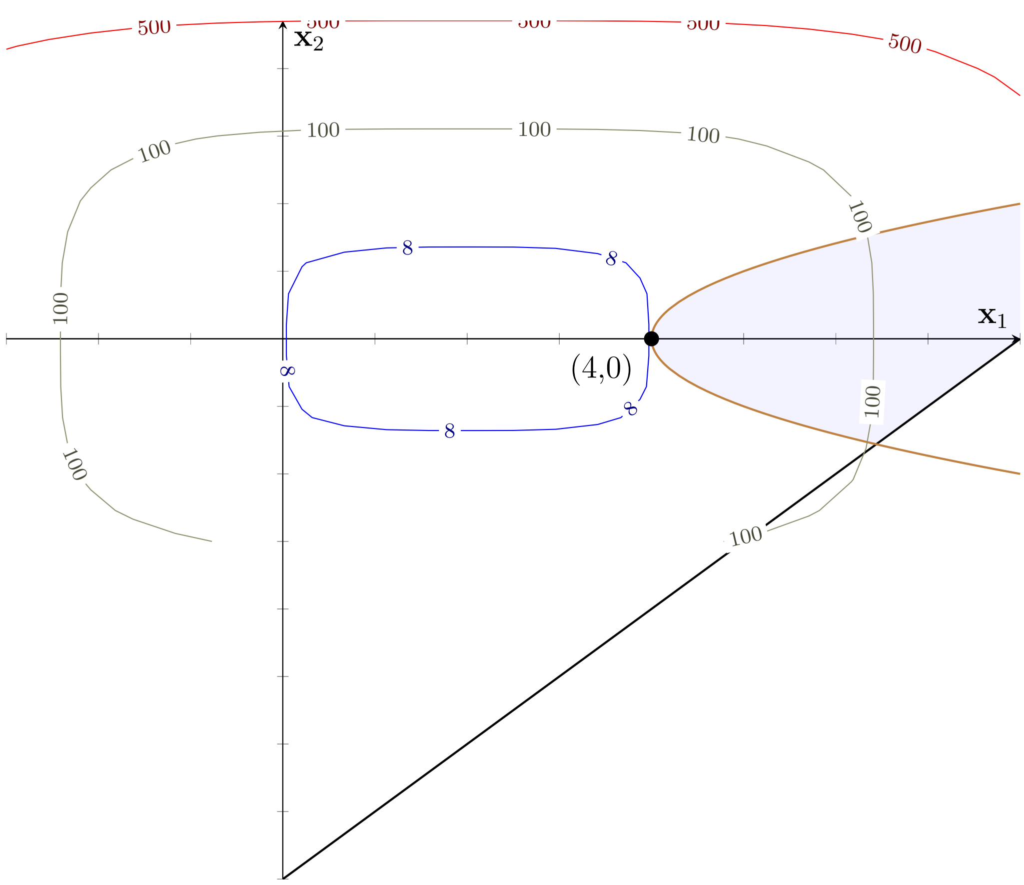
$$\begin{aligned} d^T \nabla^2_{\mathbf{x},\mathbf{x}} L((4,0)^T, (0,8)^T) d &= \begin{bmatrix} d_1 & d_2 \end{bmatrix} \begin{bmatrix} 12 & 0 \\ 0 & 16 \end{bmatrix} \begin{bmatrix} d_1 \\ d2 \end{bmatrix} \\ &= \begin{bmatrix} d_1 & d_2 \end{bmatrix} \begin{bmatrix} 12d_1 & 0 \\ 0 & 16d_2 \end{bmatrix} \\ &= 12d_1^2 + 16d_2^2 > 0 \end{aligned}$$

Therefore, we can see that the candidate point is a local minimum.

(Note: we also could have shown this simply by stating that the eigenvalues of the Hessian at the candidate point are positive (12 and 16). However, we wanted to err away from overusing λ and so this method seemed neater.)

(d)

Graphical illustration of NLP



(e)

We have by definition that a function is convex if its Hessian is positive semi-definite for all inputs $\mathbf{x} \in \mathbb{R}^n$. (We note that if the Hessian is positive definite, then the function is strongly convex).

Examining (47) we can see that it is a diagonal matrix, with the eigenvalues on the diagonal. For all inputs $\mathbf{x} \in \mathbb{R}^2, 3(x_1 - 2)^2 \geq 0$ and $12\mathbf{x}_2^2 + \lambda_2 > 0$. Therefore, the Lagrangian is convex and the objective function is convex for all $\mathbf{x}$ in the constraint set.