5. Vector integration

Vector integration:-

If $\bar{F}(t)$ & $\bar{G}(t)$ be two vector functions of a scalar variable t such that $\frac{\partial \bar{F}(t)}{\partial t} = \bar{g}(t)$ then $\bar{F}$ is known as

$$\bar{F} = \int \bar{g}(t)\,dt + c$$

Line integral:-

$\int_a^b \bar{F}\cdot d\bar{R}$ is called line integral of $\bar{F}$

Physical significance of line integral:-

The line integral gives the work done in moving a particle from one point a to another point b along a given curve.

$\oint_C \bar{F}\cdot d\bar{R}$ denotes the integration over a closed curve.

In fluid dynamics if $\bar{F}$ represents the velocity vector of a flowing fluid then $\oint_C \bar{F}\cdot d\bar{R}$ gives the circulation of $\bar{F}$ around C

Ex:- If $\bar{F} = (5xy - 6x^2)\bar{i} + (2y - 4x)\bar{j}$, evaluate $\int_C \bar{F}\cdot d\bar{r}$ along the curve 'c' in xy plane $y = x^3$ from (1,1) to (2,8)

Sol Given $\bar{F} = (5xy - 6x^2)\bar{i} + (2y - 4x)\bar{j}$

$\bar{r} = x\bar{i} + y\bar{j}$ ($\because z = 0$ on xy plane)

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Now $\bar{F}\cdot d\bar{r} = (5xy - 6x^2)dx + (2y - 4x)dy$

$\because y = x^3$

$dy = 3x^2 dx$

$\Rightarrow (5x(0x^3) - 6x^2)dx + (2x^3 - 4x)3x^2 dx$

$\Rightarrow (5x^4 - 6x^2 + 6x^5 - 12x^3)dx.$

$\cancel{\Rightarrow 20x^3 - 12x + 30x^4 - 36x^2}$

$\cancel{\bar{F}\cdot d\bar{r} = 20x^3 - 12x + 30x^4 - 36x^2}$

$\oint_C \bar{F}\cdot d\bar{r} = \int_1^2 (6x^5 + 5x^4 - 12x^3 - 6x^2)\,dx$

$= \left[\frac{6x^6}{6} + \frac{5x^5}{5} - \frac{12x^4}{4} - \frac{6x^3}{3}\right]_1^2$

$= \left[x^6 + x^5 - 3x^4 - 2x^3\right]_1^2$

$= [2^6 + 2^5 - 3(2)^4 - 2(2)^3] - [1 + 1 - 3 - 2]$

$= [64 - 32 - 48 - 16] + (3)$

$= 35$

Ex:- Using line integral compute the work done the force $\bar{F} = (2y+3)\bar{i} + xz\bar{j} + (yz - x)\bar{k}$ at the given points $(0,0,0)$ to $(2,1,1)$ along the curve $x = 2t^2$; $y = t$; $z = t^3$

Sol Given $\bar{F} = (2y+3)\bar{i} + xz\bar{j} + (yz - x)\bar{k}$

$d\bar{r} = dx\bar{i} + dy\bar{j} + dz\bar{k}$

$\bar{F}\cdot d\bar{r} = (2y+3)dx + (xz)dy + (yz - x)dz$

$= (2t+3)4t\,dt + (2t^2\cdot t^3)dt + (t\cdot t^3 - 2t^2)3t^2 dt$

On the curve $(0,0,0)$ represents $t = 0$ and $(2,1,1)$ represents $t = 1$.

The required work done $= \int \bar{F}\cdot d\bar{r}$

$$\Rightarrow \int_0^1 8t^2 + 12t + 2t^5 + 3t^6 - 6t^4 \, dt$$

$$= \left[\frac{8t^3}{3} + \frac{12t^2}{2} + \frac{2t^6}{6} + \frac{3t^7}{7} - \frac{6t^5}{5}\right]_0^1$$

$$= \frac{8}{3} + 6 + \frac{1}{3} + \frac{3}{7} - \frac{6}{5} = 9 + \frac{15-42}{35} = 9 - \frac{27}{35}$$

$$= \frac{9\times 35 - 27}{35} = \frac{288}{35}$$

$\begin{array}{r} 42 \\ -15 \\ \hline 27 \end{array}$

Ex:- Evaluate $\int_C \bar{F} \cdot d\bar{r}$ where $\bar{F} = (2xy + z^3)\bar{i} + x^2\bar{j} + 3xz^2\bar{k}$ along the st. line joining the points $(1,-2,1)$ to $(3,1,4)$ Ans:- 116

Sol The eqn of the st. line joining the 2 points is $\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$

$\therefore$ The line joining two points is

$$\frac{x-1}{2} = \frac{y+2}{3} = \frac{z-1}{3} = t$$

$\Rightarrow x = 2t+1;\ y = 3t-2;\ z = 3t+1$

$3 = 2t+1$
$2t = 2$
$\boxed{t=1}$

$\bar{F} \cdot d\bar{r} = \text{-------}$

$$\int_0^1 \bar{F} \cdot d\bar{r} = \int_0^1 \text{-------} \, dt$$
$$= 116$$

Ex:- A vector field is given by $\bar{F} = \sin y\,\bar{i} + x(1+\cos y)\bar{j}$. Evaluate integral over a circular path given by $x^2+y^2=a^2,\ z=0$

Sol Given that $\bar{F} = \sin y\,\bar{i} + x(1+\cos y)\bar{j}$

$$\bar{F} \cdot d\bar{r} = \{(\sin y)dx + x(1+\cos y)dy\}$$

The parametric eqns of circular path are $x = a\cos t$, $y = a\sin t$, where t is the angle which varies from 0 to 2π.

$\therefore \oint \bar{F} \cdot d\bar{r} = \oint (\sin y\,dx) + x(1+\cos y)dy$

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$$= \int_{t=0}^{2\pi} d\,(x \sin y) + x\, dy$$

$$= \int_{0}^{2\pi} d(a\cos t + \sin(a\sin t)) + a\cos t\,(a\cos t)\,dt$$

$$= a\cos t + \sin(a\sin t)\Big]_0^{2\pi} + a^2\int_0^{2\pi} \frac{1+\cos 2t}{2}\,dt$$

$$= 0 + \frac{a^2}{2}[t]_0^{2\pi} = \frac{a^2}{\cancel{2}} \times \cancel{2}\pi = \pi a^2$$

NMVC

Evaluation of surface integrals.

* Evaluate $\int_S \bar{F}\cdot\hat{n}\,ds$, where.

$\bar{F}=(x+y^2)\bar{i}-2x\bar{j}+2yz\bar{k}$

s is the surface of Plane $2x+y+2z=6$ in 1st Quadrant octant.

Sol Given $\bar{F}=(x+y^2)\bar{i}-2x\bar{j}+2yz\bar{k}$

surface is $2x+y+2z=6$

normal unit vector $\hat{n}$ is,

$$\hat{n}=\nabla s=2\bar{i}+\bar{j}+2\bar{k}$$

$$\hat{n}=\frac{\nabla s}{|\nabla s|}=\frac{2\bar{i}+\bar{j}+2\bar{k}}{3}$$

$$\bar{F}\cdot\hat{n}=\frac{2}{3}(x+y^2)-(2x)\frac{1}{3}+(2yz)\left(\frac{2}{3}\right)$$

$$=\frac{2x}{3}+\frac{2}{3}y^2-\frac{2x}{3}+\frac{4yz}{3}$$

$$=\frac{2y^2}{3}+\frac{4yz}{3}=(y^2+2yz)\frac{2}{3}$$

Now, $\int_S \bar{F}\cdot\hat{n}\,ds$ on xy plane $z=0$

$2x+y=6$

Here $y=0$ to $y=6-2x$

$x=0$ to $x=3$

$$\int_S \bar{F}\cdot\hat{n}\,ds=\iint_R \bar{F}\cdot\hat{n}\,\frac{dx\,dy}{|\hat{n}\cdot\bar{k}|}$$

$$= \int_0^{6-2x}\int_0^{3} \frac{2}{3}(y^2+2yz)\,dx\,dy$$

$$= \frac{2}{3}\int_0^{3}\int_0^{6-2x} y^2+\cancel{2}y\left(\frac{6-y-2x}{\cancel{2}}\right)\,dy\,dx$$

$$= \frac{2}{3}\int_0^{3}\int_0^{6-2x} y^2+6y-y^2-2xy\,dy\,dx$$

$$= \frac{2}{3}\int_0^{3}\left[\frac{y^3}{3}+\frac{6y^2}{2}-\frac{y^3}{3}-\frac{2xy^2}{2}\right]_0^{6-2x}$$

$$= \frac{2}{3}\int_0^{3} 3(6-2x)^2 - x(6-2x)^2$$

$$= \frac{2}{3}\int_0^{3} 3(36+4x^2-24x) - x(36+4x^2-24x)$$

$$= \frac{2}{3}\int_0^{3} 108+12x^2-72x-36x+4x^3-24x^3\,dx$$

$$= \frac{2}{3}\int_0^{3} -20x^3+12x^2-108x+108\,dx$$

$$= \frac{2}{3}\left[-20(3)^3+12(3)^2-108(3)+108\right.$$

$$= \frac{2}{3}\left[-20\frac{x^4}{4}+\frac{12x^3}{3}-\frac{108x^2}{2}+108x\right]_0^3$$

$$= \frac{2}{3}\left[-5x^4+4x^3-54x^2+108x\right]_0^3$$

$$= \frac{2}{3}\left[-5(3)^4+4(3)^3-54(3)^2+108(3)\right]$$

$$= 2\left[-5(3)^3+4(3)^2-54(3)+108\right]$$

$$= 2\left[-5(3)^3+4(3)^2\right.$$

$$= 81$$

* If $\bar{F} = 2y\bar{i} - z\bar{j} + x^2\bar{k}$ and S is the surface of the parabolic cylinder $y^2 = 8x$ in the 1^{st} octant bounded by planes $y=4$ & $z=6$ then evaluate $\int_S \bar{F}\cdot\hat{n}\, ds$

Sol Given $\bar{F} = 2y\bar{i} - z\bar{j} + x^2\bar{k}$

let $S = y^2 - 8x$

Now $\hat{n} = \frac{\nabla S}{|\nabla S|}$ is the unit normal

$$\hat{n} = \frac{-8\bar{i} + 2y\bar{j}}{\sqrt{64+4y^2}}$$

$$\bar{F}\cdot\hat{n} = \frac{-8}{\sqrt{64+4y^2}} \times 2y - \frac{2yz}{\sqrt{64+4y^2}}$$

$$= \frac{-8y - yz}{\sqrt{y^2+16}}$$

on yz plane, $\left.\begin{array}{l} y=0 \text{ to } 4 \\ z=0 \text{ to } 6 \end{array}\right\}$

$$\therefore \int_S \bar{F}\cdot\hat{n}\, ds = \iint_{R_{yz}} \bar{F}\cdot\hat{n} \frac{dy\,dz}{|\hat{n}\cdot\bar{i}|}$$

$$= \int_0^{6}\int_0^{4} \frac{8y+4z}{\sqrt{y^2+16}} \cdot \frac{dy\,dz}{\frac{4}{\sqrt{y^2+16}}}$$

$$= \int_0^6\int_0^4 \frac{8y+4z}{4} = \int_0^6\int_0^4 y(8+z)\, dy\, dz$$

$$= 132$$

* If the velocity vector $\bar{F} = y\bar{i} + 2\bar{j} + xz\bar{k}$ then show that flux of water through the parabolic $y = x^2$, $0 \le x \le 3$, $0 \le z \le 2$. is 69 m/sec²:

sol. $\bar{F} = y\bar{i} + 2\bar{j} + xz\bar{k}$

$$\nabla S = -2x\bar{i} + \bar{j}$$

$\hat{n}$ be unit normal to the surface S

$$\hat{n} = \frac{\nabla S}{|\nabla S|} = \frac{-2x\bar{i} + \bar{j}}{\sqrt{4x^2+1}}$$

$$\bar{F}\cdot\hat{n} = \frac{-2xy}{\sqrt{4x^2+1}} + \frac{2}{\sqrt{4x^2+1}}$$

$$\therefore \int \bar{F}\cdot\hat{n}\, dS = \iint_{R_{xz}} \bar{F}\cdot\hat{n} \frac{dx\,dz}{|\hat{n}\cdot\hat{j}|}$$

$$= \int_0^3 \int_0^2 \frac{-2xy+2}{\sqrt{4x^2+1}} \cdot \frac{dx\,dz}{\frac{1}{\sqrt{4x^2+1}}}$$

$$= \int_0^3 \int_0^2 2 - 2xy = 2\int_0^3\int_0^2 1 - xy \; dx\, dz$$

$$= 2\int_0^3 \left[x - \frac{x^2}{2}y\right]_0^2$$

$$= 4\int_0^3 (1 - x^3)\, dx$$

$$= 4\left[3 - \frac{81}{9}\right] = 69 \text{ m/sec}^2$$

Green's Theorem on a plane:

If $\phi(x,y)$ & $\psi(x,y)$ be two continuous functions of x & y having continuous partial derivatives ϕ_y & ψ_x in a region of xy plane bounded by the closed curve C. then,

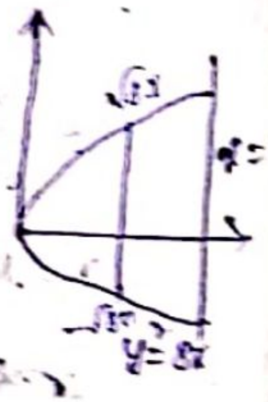
$$\oint_C [\phi\, dx + \psi\, dy] = \iint_R \left[\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y}\right] dx\, dy$$

<u>NOTE:-</u>

Green's theorem is a relation b/w line integral and double integral it useful for changing a line integral along a closed curve C in to a double integral Enclosed by that curve C.

* Evaluate $\oint_C (x^2-2xy)\,dx + (x^2y+3)\,dy$ around the boundary of Region $y^2=8x$ and $x=2$. using Green's theorem.

<u>Sol</u> given $C = \oint_C (x^2-2xy)\,dx + (x^2y+3)\,dy$



compare with $\oint_C \phi\, dx + \psi\, dy$

$\phi(x,y) = x^2-2xy$

$\psi(x,y) = x^2y+3$

$$\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y} = 2xy + 2x$$

According to Green's theorem.

$$\oint_C (x^2-2xy)\,dx + (x^2y+3)\,dy = \iint_R (2xy+2x)\,dx\,dy$$

$$= \int_0^2 \int_{-\sqrt{8x}}^{\sqrt{8x}} (2xy+2x)\,dy\,dx$$

$$= 2\int_0^2 \left[\frac{xy^2}{2} + 2xy\right]_{\sqrt{8x}}^{\sqrt{8x}} dx = 2\int_0^2 x(\sqrt{8x}+\sqrt{8x})^2 + 2x(\sqrt{8x}+\sqrt{8x})\, dx$$

$$= 2\int_0^2 x\left[(2\sqrt{8x})^2 + 2\sqrt{8}\right] dx$$

$$= 2\int_0^2 x\,(4\times 8x + 4\sqrt{2})\, dx$$

$$= 8\int_0^2 8x^2 + \sqrt{2}\,x \, dx$$

$$= 8\left[\frac{8x^3}{3} + \frac{\sqrt{2}\,x^2}{2}\right]_0^2$$

$$= 8\left[\frac{8(8)}{3} + \frac{\sqrt{2}(4)}{2}\right]$$

$$= \frac{512}{3} + 2\sqrt{2} = \frac{128}{5}$$

* ~~Evaluate~~ Apply Green's theorem to evaluate $\oint_C (2x^2-y^2)dx + (x^2+y^2)dy$; C is boundary of area enclosed by x-axis upper half of the circle $x^2+y^2=a^2$. !!

Sol Given $\oint_C (2x^2-y^2)dx + (x^2+y^2)dy$

$\phi(x,y) = 2x^2-y^2$

$\psi(x,y) = x^2+y^2$

$\frac{\partial\phi}{\partial y} = -2y \qquad \frac{\partial\psi}{\partial x} = 2x$

By green's theorem,

$$\oint_C (2x^2-y^2)dx + (x^2+y^2)dy = \iint_R (2x+2y)\,dx\,dy$$

$$= \int_{\theta=0}^{\pi}\int_{r=0}^{a} (2r\cos\theta + 2r\sin\theta)\, r\,dr\,d\theta$$

$$= 2\int_0^{\pi} (\cos\theta + \sin\theta)\left[\frac{r^3}{3}\right]_0^a d\theta$$

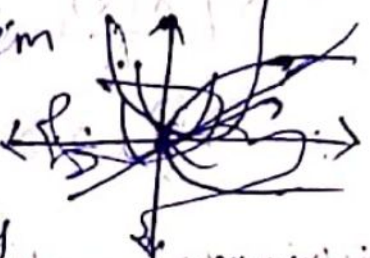
$$= \frac{2a^3}{3}\int_0^{\pi} \cos\theta + \sin\theta \, d\theta.$$

$$= \frac{2a^3}{3}\left[(-\sin\theta)+(\cos\theta)\right]_0^{\pi}$$

$$= \frac{2a^3}{3}[1+1] = \frac{4a^3}{3}$$

* Verify Green's theorem for the integral $\oint_C (xy+y^2)dx + x^2 dy$ where C is the closed curve bounded by $y=x$ & $y=x^2$.

Sol:- According to green's theorem



$$\oint_C (xy+y^2)dx + x^2 dy = \iint_R (2x - x - 2y)\, dx\, dy$$

$$= \iint_R (x-2y)\, dx\, dy \quad \text{--- (1)}$$

i) Along the arc OA, $y = x^2$

$dy = 2x\, dx$

$x = 0$ to 1

$$\therefore \int_{OA} (xy+y^2)dx + x^2 dy = \int_{x=0}^{1} (x^3+x^4)\,dx + (x^2)(2x)\,dx$$

$$= \int_0^1 (3x^3+x^4)\,dx$$

$$= \left[\frac{3x^4}{4} + \frac{x^5}{5}\right]_0^1$$

$$= \frac{3}{4}+\frac{1}{5} = \frac{15+4}{20} = \frac{19}{20}$$

(ii) Along the line AO, $y = x$

$dy = dx$

$x = 1$ to 0

$\therefore \int_{AO} (xy+y^2)\,dx + x^2\,dy = \int_1^0 3x^2\,dx = \frac{3x^3}{3}\Big]_1^0 = \frac{3(1)}{3} = -1$

$\therefore LHS = \oint_C (xy+y^2)\,dx + x^2\,dy = \int_{OA} + \int_{AO}$

$= \frac{19}{20} - 1 = \frac{-1}{20}$

$RHS = \iint_R (x-2y)\,dx\,dy$

$= \int_0^1 \int_{x^2}^{x} (x-2y)\,dy\,dx$

$= \int_0^1 \left[x(y) - \frac{2y^2}{2}\right]_{x^2}^{x} dx$

$= \int_0^1 (x^2 - 2x^4) - x^3 + x^4$

$= \frac{-1}{20}$

*Verify Green's theorem for the $\oint_C (3x^2-8y^2)\,dx + (4y-6xy)\,dy$ where C is the boundary of the region bounded by $x=0, y=0$ & $x+y=1$

Sol given $\oint_C (3x^2-8y^2)\,dx + (4y-6xy)\,dy$

$\phi(x,y) = 3x^2 - 8y^2, \frac{\partial\phi}{\partial y} = -16y$

$\psi(x,y) = 4y-6xy, \frac{\partial\psi}{\partial x} = -6y$

According to green's theorem,

$\oint_C (3x^2-8y^2)\,dx + (4y-6xy)\,dy = \iint_R (-6y+16y)\,dx\,dy$

$= \iint_R 10y\,dx\,dy$

$LHS = \oint_C (3x^2-8y^2)\,dx + (4y-6xy)\,dy = \int_{OA} + \int_{AB} + \int_{BO}$

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(i) Along line AO, $y=0$, $dy=0$, $x:0$ to 1

$$LHS = \int_0^1 3x^2 dx = 1$$

(ii) Along line AB, $x+y=1$, $x=1$ to 0

$dy = -dx$

$$LHS = \int_1^0 3x^2 - 8(1-x)^2 dx + 4(1-x)\cdot 6x(1-x)(-dx)$$

$$= \int_0^1 [3x^2 - 8 - 8x^2 + 16x + 4(x - 4x - 6x + 6x^2] dx$$

$$= \int_0^1 (-11x^2 + 26x - 12) dx$$

$$= \left[\frac{-11x^3}{3} + \frac{26x^2}{2} - 12x\right]_0^1$$

$$= -\frac{11}{3} + \frac{26}{2} - 12 = \frac{-11}{3} + 1 = \frac{-11+3}{3} = \frac{8}{3}$$

(iii) Along the line BO, $x=0$, $dx=0$, $y=1$ to 0

$$LHS = \int_1^0 4y\, dy = \left[\frac{4y^2}{2}\right]_1^0 = 2(0) - 2(1) = -2$$

$$\therefore LHS = 1 + \frac{8}{3} - 2 = \frac{8}{3} - 1 = \frac{5}{3}$$

$$\therefore \oint_C RHS = \iint_R 10y\, dx\, dy$$

$$= 10\int_0^1 \int_0^{1-x} y\, dy\, dx$$

$$= 10\int_0^1 \left[\frac{y^2}{2}\right]_0^{1-x} dx = 5\int_0^1 (1-x)^2 dx$$

$$= 5\left[\frac{(1-x)^3}{3}\right]_0^1 = \frac{5}{3}$$

Stoke's Theorem:-

Suppose $\bar{F}$ be a continuously differentiable vector function defined over an open surface S bounded by a closed curve C then $\oint_C \bar{F}\cdot d\bar{r} = \int_S \text{curl}\,\bar{F}\cdot\hat{n}\,ds$ where $\hat{n}$ is the unit normal drawn outward to the surface.

Ex:- Verify Stoke's theorem for vector field $\bar{F} = (2x-y)\bar{i} - yz^2\bar{j} - y^2z\bar{k}$ over the upper half surface of sphere $x^2+y^2+z^2=1$ bounded by its projection on xy plane.

Sol Given $\bar{F} = (2x-y)\bar{i} - yz^2\bar{j} - y^2z\bar{k}$

$$\text{curl}\,\bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x-y & -yz^2 & -y^2z \end{vmatrix}$$

$$= \bar{i}(-2yz + 2yz) - (0-0)\bar{j} + \bar{k}(-0+1)$$

$$= \bar{k}$$

$$S = x^2+y^2+z^2 = 1$$

$$\nabla S = 2x\bar{i} + 2y\bar{j} + 2z\bar{k}$$

$$\hat{n} = \frac{\nabla S}{|\nabla S|} = \frac{2x\bar{i} + 2y\bar{j} + 2z\bar{k}}{\sqrt{4x^2+4y^2+4z^2}} = x\bar{i} + y\bar{j} + z\bar{k}$$

$$\text{curl}\,\bar{F}\cdot\hat{n} = z$$

By Stoke's Theorem,

$$\oint_C \bar{F}\cdot d\bar{R} = \int_S \text{curl}\,\bar{F}\cdot\hat{n}\,ds$$

$$\oint_C (2x-y)\hat{i} - yz^2\hat{j} - y^2z\hat{k} = \int_S \hat{k}\,\frac{dx\,dy}{\hat{k}}$$

$$= \iint_R dx\,dy$$

Here on xy plane $z=0$, and on the boundary, $x = \cos\theta$, $y = \sin\theta$ } $\theta = 0$ to 2π

$$dx = -\sin\theta\, d\theta$$

$$\therefore \oint_C (2x-y)\,dx - yz^2 dy - y^2 z\,dz = LHS$$

$$= \int_0^{2\pi} (2\cos\theta - \sin\theta) - 0 - 0$$

$$= \int_0^{2\pi} -\sin 2\theta + \left(1 - \frac{\cos 2\theta}{2}\right) d\theta$$

$$= \left[\frac{\cos 2\theta}{2}\right]_0^{2\pi} + \left(\frac{\theta}{2}\right)_0^{2\pi} - \left(\frac{\sin 2\theta}{2}\right)_0^{2\pi}$$

$$= \pi$$

$$RHS = \iint_{R_{xy}} dx\,dy$$

$$= \int_0^{2\pi}\int_0^1 r\,dr\,d\theta$$

$$= \int_0^{2\pi} \left[\frac{r^2}{2}\right]_0^1 d\theta = \frac{1}{2}\int_0^{2\pi} 1\,d\theta = \frac{1}{2}(2\pi) = \pi$$

LHS = RHS.

Ex: Verify Stokes theorem for vector $\bar{F} = -y^3\bar{i} + x^3\bar{j}$ in the region $x^2+y^2 \le 1$, $z=0$,

Sol Given $\bar{F} = -y^3\bar{i} + x^3\bar{j}$

By stoke's theorem we have

$$\oint_C \bar{F}\cdot d\bar{R} = \int_S \text{curl}\,\bar{F}\cdot\hat{n}\,ds \quad \text{—}(1)$$

$$\text{curl}\,\bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^3 & x^3 & 0 \end{vmatrix}$$

$$= \bar{i}(0-0) - \bar{j}(0-0) + \bar{k}(3x^2+3y^2)$$

$$= (3x^2+3y^2)\bar{k}$$

since our surface $x^2+y^2 \le 1$, $z=0$

$\hat{n} = \bar{k}$

$\text{curl}\,\bar{F}\cdot\hat{n} = 3(x^2+y^2)$

from (1)

$$\oint_C (-y^3\bar{i} + x^3\bar{j}) = \int_S 3(x^2+y^2)\,ds$$

$$LHS = \oint_C (-y^3\bar{i} + x^3\bar{j})$$

on the boundary $\left.\begin{aligned} x &= \cos\theta \\ y &= \sin\theta \end{aligned}\right\}$ $\begin{aligned} dx &= -\sin\theta\,d\theta \\ dy &= \cos\theta\,d\theta \end{aligned}$

$\theta = 0$ to 2π

$$= \int_0^{2\pi} (-\sin^3\theta)(-\sin\theta)d\theta + (\cos^3\theta)(\cos\theta)d\theta$$

$$= \int_0^{2\pi} \sin^4\theta + \cos^4\theta\,d\theta$$

$$= \int_0^{2\pi} 1 - 2\sin^2\theta\cos^2\theta \, d\theta$$

$$= \int_0^{2\pi} 1 - \frac{2\sin^2 2\theta}{2}$$

$$= \int_0^{2\pi} 1 - \left(\frac{1-\cos 4\theta}{4}\right) d\theta$$

$$= \int_0^{2\pi} \frac{3}{4} - \frac{\cos 4\theta}{4} \, d\theta$$

$$= \frac{3}{4}(\theta)_0^{2\pi} + \left[\frac{\sin 4\theta}{16}\right]_0^{2\pi}$$

$$= \frac{3}{4_2} \times 2\pi + 0 = \frac{3\pi}{2}$$

$$RHS = \int_S (3x^2+3y^2)\,ds = \iint_R 3x^2+3y^2 \, \frac{dx\,dy}{1}$$

$$= \int_0^{2\pi}\int_0^1 3(r^2\sin^2\theta + r^2\cos^2\theta)\, r\,dr\,d\theta$$

$$= 3\int_0^{2\pi}\int_0^1 r^3\,dr\,d\theta$$

$$= 3\int_0^{2\pi} \left(\frac{r^4}{4}\right)_0^1 d\theta$$

$$= \frac{3}{4}\int_0^{2\pi} \theta = \frac{3}{4_2} \times 2\pi = \frac{3\pi}{2}$$

LHS = RHS.

Ex:- Verify stoke's theorem for the vector field $\bar{F} = (y-z+2)\bar{i} + (yz+4)\bar{j} - xz\bar{k}$ where S is the surface of the cube $x=0, y=0, z=0$ & $x=2, y=2, z=2$.

Sol By Stokes theorem,

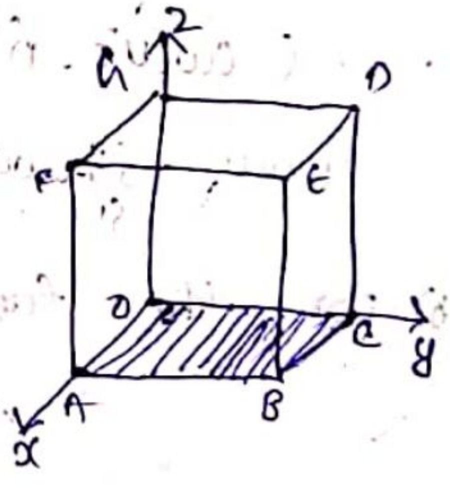
$$\oint_C \bar{F}\cdot d\bar{R} = \int_S \operatorname{curl}\bar{F}\cdot\hat{n}\,ds \quad -\ (1)$$

Now, $\operatorname{curl}\bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y-z+2 & yz+4 & -xz \end{vmatrix}$

$$= \bar{i}(0-y) - \bar{j}(-z+1) + \bar{k}(0-1)$$

$$= -y\bar{i} - \bar{j}(1-z) - \bar{k}$$

Here the open surface consists of 5 faces of the cube as shown in the diagram. The closed curve OABCO is the boundary of the open surface.



i). Along OA, $y=0, z=0$; $x = 0$ to 2

$$\int_{OA} (y-z+2)\bar{i} + (yz+4)\bar{j} - xz\bar{k}$$

$$= \int_{x=0}^{2} 2\,dx = 4$$

(ii) Along AB: $x = 2$, $z=0$ & $y = 0$ to 2

$$\int_{AB} (y-z+2)\,dx + (yz+4)\,dy - xz\,dz = \int_0^2 4\,dy$$

$$= 8$$

(iii) Along line BC, $y = 2$, $z = 0$, $x = 2$ to 0

$$\therefore \int_{BC} (y-z+2)\,dx + (yz+4)\,dy - xz\bar{k}$$

$$= \int_2^0 4\,dx = 8$$

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(iv) Along CO $x=0$ $z=0$ $y=2$ to 0

$dx=0$ $dz=0$

$$\therefore \int_{CO}(y-2z+2x)dx+(yz+4)dy-xz\,dz$$

$$=\int_2^0 4\,dy=-8$$

Here $\int_S \text{curl}\,\bar{F}\cdot\hat{n}=\int_{S1\,(ABEF)}+\int_{S2\,(BCDE)}+\int_{S3\,(OEFG)}+\int_{S4\,(OAFG)}+\int_{S5\,(OCDG)}$

a) For the phase ABEF, $\hat{n}=\hat{i}$

$(\text{curl}\,\bar{F}\cdot\hat{n})=-y$

Now, $\int_{S1}\text{curl}\,\bar{F}\cdot\hat{n}\,dS_1=\int_{y=0}\int_{z=0}(-y)\frac{dy\,dz}{(1)}=-4$

b) For the face BCDE, $\hat{n}=\hat{j}$ $ds=\frac{dx\,dz}{|\hat{n}\cdot\hat{j}|}=dxdz$

$x=0$ to 2

$z=0$ to 2

$$\therefore \int_{S2}\text{curl}\,\bar{F}\cdot\hat{n}\,ds=\int_0^2\int_0^2(z-1)\,dx\,dz$$

$$=\int_0^2(z-1)(2)\,dz$$

$$=2\left[\frac{z^2}{2}-z\right]_0^2$$

$$=2\left(\frac{4}{2}-2\right)=0$$

c) For the plane DEFG ; $\hat{n} = \hat{k}$ $ds = dx\,dy$

$x = 0$ to 2

$y = 0$ to 2

$$\therefore \int_{S3} \text{curl}\,\bar{F} \cdot \hat{n} = \int_0^2 \int_0^2 -1 \, dx\,dy$$

$$= -2[2] = -4$$

d) For the face OAFG ; $\hat{n} = -\hat{j}$ $ds = dx\,dz$

$x = 0$ to 2

$z = 0$ to 2

$$\int_{S4} \text{curl}\,\bar{F} \cdot \hat{n} = \int_0^2 \int_0^2 (1 - z)\, dx\,dz$$

$$= 2 \int_0^2 (1 - z)\, dz$$

$$= \left(z - \frac{z^2}{2}\right)_0^2 2$$

$$= \left(2 - \frac{4}{2}\right) 2 = 0$$

e) For face OCDG ; $\hat{n} = -\hat{i}$ $ds = dy\,dz$

$y = 0$ to 2

$z = 0$ to 2

$$\therefore \int_{S5} \text{curl}\,\bar{F} \cdot \hat{n} = \int_0^2 \int_0^2 y\, dy\, dz = 4$$

$$\therefore \int_S \text{curl}\,\bar{F} \cdot \hat{n} = -4 + 0 - 4 + 0 + 4 = -4$$

Any integral which is to be evaluated along volume is called "Volume integral".

It is denoted by $\iiint_V F(x,y,z)\,dV$

Gauss Divergence theorem:

If F be a continuously differentiable vector function in a region R bounded by a ~~region~~ closed surface S then $\oint_S F\cdot\hat{n}\,ds = \iiint_V \text{Div}\,F\,dV$.

Q: Evaluate double integral along S

$\iint_S x^3\,dy\,dz + x^2y\,dz\,dx + x^2z\,dx\,dy$ where S is the closed surface consisting of the cylinder $x^2+y^2=a^2$, $z=0$ & $z=b$

Sol $\iint_S x^3\,dy\,dz + x^2y\,dz\,dx + x^2z\,dx\,dy$

$\bar{F} = x^3\bar{i} + x^2y\,\bar{j} + x^2z\,\bar{k}$

Now $\text{div}\,\bar{F} = \nabla\cdot F = 3x^2 + x^2 + x^2 = 5x^2$

By Gauss divergence theorem,

$\iint_S x^3\,dy\,dz + x^2y\,dz\,dx + x^2z\,dx\,dy$

$= \iiint \text{Div}\,F\,dv$

$= \int_0^b \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_{-a}^{a} 5x^2\,dy\,dx\,dz$

$= 5\int_0^b \int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} x^2\,dy\,dx\,dz$

$$= 5\int_0^b\int_{-a}^{a} x^2 (y)_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\, dx\, dz$$

$a^2 - x^2 = t^2$

$-2x\,dx = 2t\,dt$

$$= 5\int_0^b\int_{-a}^{a} x^2 (2\sqrt{a^2-x^2})\, dx\, dz$$

$$= 20\int_0^b\int_0^a x^2\sqrt{a^2-x^2}\, dx\, dz$$

$x = a\sin\theta$

$dx = a\cos\theta\, d\theta$

$x=0 \quad \theta = 0$

$x = a \quad \theta = \pi/2$

$$= 20\int_0^b\int_0^{\pi/2} (a^2\sin^2\theta)(a\cos\theta)(a\cos\theta)\, d\theta\, dz$$

$$= 20a^4\int_0^b\int_0^{\pi/2} \sin^2\theta\cos^2\theta$$

$$= \frac{20}{4}a^4\int_0^b\int_0^{\pi/2} \sin^2 2\theta\, d\theta\, dz$$

$$= 5a^4\int_0^b\int_0^{\pi/2}\left(\frac{1-\cos 4\theta}{2}\right) d\theta\, dz$$

$$= \frac{5a^4 b}{2}\left[\theta\right]_0^{\pi/2} = \frac{5a^4 b\pi}{4}$$

Eg:- Evaluate $\iint_S \bar{F}\cdot\hat{n}\, ds$ taken over the region bounded by $x^2+y^2=4,\ z=0,\ z=3$.

$\bar{F} = 4x\bar{i} - 2y^2\bar{j} + z^2\bar{k}$ Ans:- 84π.

Verify $\iint_S \bar{F}\cdot\hat{n}\, ds = \int_{S_1} \bar{F}\cdot\hat{n}\, ds + \int_{S_2} \bar{F}\cdot\hat{n}\, ds + \int_{S_3} \bar{F}\cdot\hat{n}\, ds$

where S_1 = circular base in $z=0$

S_2 = circular top in $z=3$

S_3 = The curved surface of the cylinder

$x^2+y^2=4$.

i) Along S_1: $z=0$, $\hat{n} = -\bar{k}$

$$\therefore \bar{F}\cdot\hat{n}\, ds = \int_{S_1} z^2 \frac{dx\, dy}{(1)} = 0.$$

(ii) Along the surface S_2:

$$z=3,\ \hat{n}=\hat{k}$$

$$\int_{S_2} F\cdot\hat{n}\,ds = \iint_R (z^2)\frac{dx\,dy}{|\hat{n}\cdot\hat{k}|} = 3^2\iint_{R_{xy}} dx\,dy$$

$$=9(\pi)(2)^2 = 36\pi$$

(iii) Along the surface S_3:

$$S = x^2+y^2=4$$

$$\nabla S = 2x\bar{i}+2y\bar{j}$$

$$\hat{n} = \frac{\nabla S}{|\nabla S|} = \frac{2x\bar{i}+2y\bar{j}}{\sqrt{4x^2+4y^2}} = \frac{x\bar{i}+y\bar{j}}{2}$$

Now. $F\cdot\hat{n} = \frac{4x^2}{2} - \frac{2y^3}{2} = 2x^2-y^3$

on S_3, $ds = 2\,d\theta\,dz$

$\theta = 0$ to 2π.

$z = 0$ to 3

$$\int_{S_3} F\cdot\hat{n} = \int_{S_3}(2x^2-y^3)\,dz$$

$$= \int_0^3\int_0^{2\pi} 2(4\cos^2\theta - 8\sin^3\theta)\,d\theta$$

$$= 48\pi$$

Eg:- Compute $\int_S ax^2+by^2+cz^2\,ds$ over the surface of the sphere $x^2+y^2+z^2=1$

Sol Given $F\cdot\hat{n} = ax^2+by^2+cz^2$

let $S = x^2+y^2+z^2-1$

$$\nabla S = 2x\bar{i}+2y\bar{j}+2z\bar{k}$$

$$\frac{\nabla s'}{|\nabla s|} = \frac{2x\bar{i}+2y\bar{j}+2z\bar{k}}{\sqrt{4x^2+4y^2+4z^2}} = x\bar{i}+y\bar{j}+z\bar{k}$$

$$F\cdot\hat{n} = (\bar{F})(x\bar{i}+y\bar{j}+z\bar{k}) = ax^2+by^2+cz^2$$

$$\therefore \bar{F} = ax\bar{i}+by\bar{j}+cz\bar{k}$$

By guass divergence theorem,

$$\int_S F\cdot\hat{n}\,ds = \iiint_V \mathrm{Div}(\bar{F})\,dv$$

$$\mathrm{Div}\,\bar{F} = \iiint(a+b+c)\,dv$$

$$\int_S F\cdot\hat{n}\,ds = \iiint_V (a+b+c)\,dv$$

$$=(a+b+c)\iiint_V dv \quad (\because \text{volume of sphere})$$

$$=(a+b+c)\left(\frac{4\pi}{3}(1)^3\right)$$

$$=\frac{4\pi}{3}(a+b+c)$$

Eg:- verify guass divergence theorem for $\bar{F} = 4xz\bar{i} - y^2\bar{j} + yz\bar{k}$ taken over the cube bounded by $x=0, y=0, z=0$ & $x=a, y=a, z=a$