Swapnil Daxini (V00861672) Due: Nov 2, 2018

CSC 349A Assignment 4

Q1) a)

$$A = \begin{bmatrix} 0 & -2 & -2 & -4 \\ -1 & -1 & 1 & 0 \\ 2 & 4 & -2 & 0 \\ 1 & 1 & -1 & 0.5 \end{bmatrix}$$

We need to find the fourth column vector of A^{-1} (denote it by $x^{(4)}$), thus we need to solve the system of linear equations:

$$A x^{(4)} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Gaussian elimination with partial pivoting:

1) $\left[\begin{array}{cccc|c} 0 & -2 & -2 & -4 & 0 \\ -1 & -1 & 1 & 0 & 0 \\ 2 & 4 & -2 & 0 & 0 \\ 1 & 1 & -1 & 0.5 & 1 \end{array}\right] \xrightarrow[\text{rows 1 and 3}]{\text{Interchange}}$

2) $\left[\begin{array}{cccc|c} 2 & 4 & -2 & 0 & 0 \\ -1 & -1 & 1 & 0 & 0 \\ 0 & -2 & -2 & -4 & 0 \\ 1 & 1 & -1 & 0.5 & 1 \end{array}\right] \xrightarrow[E_2 \leftarrow E_2 - m_{21}E_1]{\text{Perform forward elimination}}$

3) $\left[\begin{array}{cccc|c} 2 & 4 & -2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & -2 & -2 & -4 & 0 \\ 1 & 1 & -1 & 0.5 & 1 \end{array}\right] \xrightarrow[E_4 \leftarrow E_4 - m_{41}E_1]{\text{Perform Forward elimination}}$

(4) $\left[\begin{array}{cccc|c} 2 & 4 & -2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & -2 & -2 & -4 & 0 \\ 0 & -1 & 0 & 0.5 & 1 \end{array}\right] \xrightarrow[E_3 \leftarrow E_3 - m_{32}E_2]{\text{Perform forward elimination}}$

5) $\left[\begin{array}{cccc|c} 2 & 4 & -2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -2 & -4 & 0 \\ 0 & -1 & 0 & 0.5 & 1 \end{array}\right] \xrightarrow[E_4 \leftarrow E_4 - m_{42}E_2]{\text{Perform forward elimination}} \left[\begin{array}{cccc|c} 2 & 4 & -2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -2 & -4 & 0 \\ 0 & 0 & 0 & 0.5 & 1 \end{array}\right]$

Obtained upper triangular form. Perform back substitution

$$x_4 = \frac{(1)}{0.5} = \underline{\underline{2}}$$

$$x_3 = \frac{(0 - (-4)(2))}{-2} = \underline{\underline{-4}}$$

$$x_2 = \frac{0}{1} = \underline{\underline{0}}$$

$$x_1 = \frac{0 - (-2)(-4) - (4)(0)}{2} = \underline{\underline{-4}}$$

$$\therefore x^{(4)} = \begin{bmatrix} -4 \\ 0 \\ -4 \\ 2 \end{bmatrix}$$ Fourth column of A^{-1}

(b) From part (a), we obtained our upper-triangular form of A to be:

$$\begin{bmatrix} 2 & 4 & -2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & -4 \\ 0 & 0 & 0 & 0.5 \end{bmatrix}$$

We know $\det(A) = \det(\text{Upper triangular form of } A)$

$$= (-1)^m a_{11} a_{22}^{(1)} a_{33}^{(2)} \cdots a_{nn}^{(n-1)}$$

where m is <u>number of row interchanges</u>

In part a), we performed one row interchange, thus $m = 1$

$$\therefore \det A = (-1)^1 \cdot (2) \cdot (1) \cdot (-2) \cdot (0.5)$$

$$= -1 \cdot -2$$

$$\underline{\det A = 2}$$

(Q2) a) Pseudocode for special case of back substitution:

$$x_1 \leftarrow \frac{b_1}{a_{1,1}}$$

$$x_2 \leftarrow \frac{b_2 - a_{2,1} \cdot x_1}{a_{2,2}}$$

for $i = 3$ to n

$$x_i \leftarrow \frac{b_i - a_{i,i-1} \cdot x_{i-1} - a_{i,i-2} \cdot x_{i-2}}{a_{i,i}}$$

(b) Outside the loop, we perform 1 subtraction, 2 divisions, 1 multiplication.

For each iteration of the loop, it performs 2 subtractions, 2 multiplications and 1 divisions.

The loop iterates $(n-3)+1$ times, thus $(n-2)$ times

Thus, we have 5 flops per iteration, and an additional 4 flops

$\therefore$ number of flops $= 5(n-2) + 4$

$= 5n - 10 + 4$

$= 5n + 6$ flops

(c) See attached code at the end of pdf.

(Q3) a) $Ax = b$:

$$\begin{bmatrix} -0.2345 & 2.107 \\ 0.1234 & -1.115 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2.345 \\ 1.001 \end{bmatrix}$$

Performing Gaussian elimination with partial pivoting using base 10, $k=4$, idealized rounding floating point arithmetic:

Find m_{21}:

$$fl(m_{21}) = fl\left(\frac{a_{21}}{a_{11}}\right) = fl\left(\frac{0.1234}{-0.2345}\right) = \underline{-0.5262}$$

We can find a_{22} and b_2 after forward elimination:

$$fl(a_{22}) = fl(a_{22} - fl(m_{21}a_{12})) = fl(a_{22} - fl(-0.5262 \cdot 2.107))$$

$$= fl(a_{22} + 1.109) = fl(-1.115 + 1.109)$$

$$\underline{fl(a_{22}) = -0.006}$$

$$fl(b_2) = fl(b_2 - fl(m_{21}b_1)) = fl(1.001 - fl(-0.5262 \cdot -2.345))$$

$$= fl(1.001 - 1.234)$$

$$\underline{fl(b_2) = -0.2330}$$

We can peform back substitution to find x_1 and x_2

$$fl(x_2) = fl\left(\frac{b_2}{a_{22}}\right) = fl(-0.2330 / -0.006)$$

$$fl(x_2) = 38.83$$

$$x_1 = \frac{b_1 - a_{12} \cdot x_2}{a_{11}}$$

Floating point calculation:

$$fl(a_{12} \cdot x_2) = fl(-2.107 \cdot 38.83) = 81.81$$
$$fl(b_1 - a_{12} \cdot x_2) = fl(-2.345 - 81.81) = -84.16$$
$$fl(x_1) = fl\left(\frac{b_1 - a_{12} \cdot x_2}{a_{11}}\right) = fl\left(\frac{-84.16}{-0.2345}\right)$$

$$fl(x_1) = 358.9$$

$$\therefore fl\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 358.9 \\ 38.83 \end{bmatrix}, \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 345.404\ldots \\ 37.329\ldots \end{bmatrix}$$

Thus, this linear system is ill-conditioned as the small pertubations caused by the floating-point arthimetic causes a relative large ($\sim 4\%$) difference in the calculated values of x_1 and x_2.

However, it should be noted that the pertubations caused by using the floating-point arthimetic does have as large as an effect caused by the pertubation of a_{22} value (as shown in question). This can be explained by the fact that the calculation of a_{22} in upper triangular form leads to a large loss of significant figures due to subtractive cancellation. Thus, small changes in a_{22} has large effect in the calculation of x_1 and x_2.

(b) See attached code and output at the end of pdf.

From the number, we can tell that A is ill-conditioned.