Swapnil Daxini (V00861672) 20/11/18

# CSC349A - Assignment 5

Q1) a) $f(x) = \sin^2(x) \quad x \in [0, 2\pi]$

| $x_i$ | $f(x_i)$ |
| --- | --- |
| $0$ | $0$ |
| $\frac{2\pi}{3}$ | $0.75$ |
| $\frac{4\pi}{3}$ | $0.75$ |
| $2\pi$ | $0$ |

In our case, $n=3$ and

$$P(x) = \sum_{i=0}^{3} L_i(x) f(x_i)$$

$$= L_0(x_0)f(x_0) + L_1(x_1)f(x_1) + L_2(x_2)f(x_2) + L_3(x_3)f(x_3)$$

We can substitute values from the table:

$$= L_0(0)\cdot 0 + L_1\left(\frac{2\pi}{3}\right)\cdot 0.75 + L_2\left(\frac{4\pi}{2}\right)\cdot 0.75 + L_3(2\pi)\cdot 0$$

$$= 0.75\left(L_1\left(\frac{2\pi}{3}\right) + L_2\left(\frac{4\pi}{3}\right)\right)$$

Thus, we only need to find $L_1(x_0)$ and $L_2(x_0)$.

$$L_i = \prod_{j=0, j\neq i}^{3} \frac{x - x_j}{x_i - x_j}, \quad \text{for } i = 0,1,2,3$$

Thus,

$$L_1 = \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} = \frac{(x-0)(x-\frac{4\pi}{3})(x-2\pi)}{(\frac{2\pi}{3}-0)(\frac{2\pi}{3}-\frac{4\pi}{3})(\frac{2\pi}{3}-2\pi)}$$

$$L_1\left(\frac{2\pi}{3}\right) = \frac{(x^2 - \frac{4\pi}{3}x)(x-2\pi)}{\frac{2\pi}{3}\cdot\left(-\frac{2\pi}{3}\right)\cdot\left(-\frac{4\pi}{3}\right)} = \frac{x^3 - 2\pi x^2 - \frac{4\pi}{3}x^2 + \frac{8\pi^2}{3}x}{\frac{16\pi^3}{27}}$$

$$L_1\left(\frac{2\pi}{3}\right) = \frac{x^3 - \frac{10\pi}{3}x^2 + \frac{8\pi^2}{3}x}{\frac{16\pi^3}{27}} = \underline{\underline{\frac{27x^3}{16\pi^3} - \frac{45}{8\pi^2}x^2 + \frac{9}{2\pi}x}}$$

$$L_2(x_2) = \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} = \frac{x\left(x-\frac{2\pi}{3}\right)(x-2\pi)}{\left(\frac{4\pi}{3}\right)\left(\frac{4\pi}{3}-\frac{2\pi}{3}\right)\left(\frac{4\pi}{3}-2\pi\right)}$$

$$= \frac{\left(x^2-\frac{2\pi}{3}x\right)(x-2\pi)}{\frac{4\pi}{3}\cdot\frac{2\pi}{3}\cdot\left(-\frac{2\pi}{3}\right)} = \frac{x^3-2\pi x^2-\frac{2\pi}{3}x^2+\frac{4\pi^2}{3}x}{-\frac{16\pi^3}{27}}$$

$$\underline{\underline{L_2\left(\tfrac{4\pi}{3}\right) = \frac{-27x^3}{16\pi^3} + \frac{9}{2\pi^2}x^2 - \frac{9}{4\pi}x}}$$

Thus $P(x) = \frac{3}{4}\left(L_1\left(\frac{2\pi}{3}\right) + L_2\left(\frac{4\pi}{3}\right)\right)$

$$= \frac{3}{4}\left(\left(\cancel{\frac{27x^3}{16\pi^3}} - \frac{45}{8\pi^2}x^2 + \frac{9}{2\pi}x\right) + \left(\cancel{-\frac{27}{16\pi^3}x^3} + \frac{9}{2\pi^2}x^2 - \frac{9}{4\pi}x\right)\right)$$

$$\underline{\underline{P(x) = \frac{3}{4}\left(-\frac{9}{8\pi^2}x^2 + \frac{9}{4\pi}x\right) = -\frac{27}{32\pi^2}x^2 + \frac{27}{16\pi}x}}$$

b) Refer to Matlab code attached at the end.

Q2

$$S(x) = \begin{cases} S_0(x), & 0 \le x \le \frac{2\pi}{3} \\ S_1(x), & \frac{2\pi}{3} \le x \le \frac{4\pi}{3} \\ S_2(x), & \frac{4\pi}{3} \le x \le 2\pi \end{cases}$$

| $x_i$ | $f(x_i)$ |
|---|---|
| 0 | 0 |
| $\frac{2\pi}{3}$ | 0.75 |
| $\frac{4\pi}{3}$ | 0.75 |
| $2\pi$ | 0 |

$$S_0(x) = a_0 + b_0x + c_0x^2 + d_0x^3$$
$$S_1(x) = a_1 + b_1\left(x-\tfrac{2\pi}{3}\right) + c_1\left(x-\tfrac{2\pi}{3}\right)^2 + d_1\left(x-\tfrac{2\pi}{3}\right)^3$$
$$S_2(x) = a_2 + b_2\left(x-\tfrac{4\pi}{3}\right) + c_2\left(x-\tfrac{4\pi}{3}\right)^2 + d_2\left(x-\tfrac{4\pi}{3}\right)^3$$

Given $S(x)$ is clamped, we have that $S'(x_0) = f'(x_0)$ and $S'(x_3) = f'(x_3)$.

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We can now look at the definition of a cubic spline interpolant to obtain our 12 equation:

(b) $S_j(x_j) = f(x_j)$ :

① $S_0(x_0) = f(x_0) \Rightarrow S_0(0) = a_0 = 0$

② $S_1(x_1) = f(x_1) \Rightarrow S_1(\frac{2\pi}{3}) = a_1 = 0.75$

③ $S_2(x_2) = f(x_2) \Rightarrow S_2(\frac{4\pi}{3}) = a_2 = 0.75$

④ $S_2(x_3) = f(x_3) \Rightarrow S_2(2\pi) = a_2 + b_2\left(\frac{2\pi}{3}\right) + c_2\left(\frac{2\pi}{3}\right)^2 + d_2\left(\frac{2\pi}{3}\right)^3 = 0$

$$\Rightarrow +\frac{2\pi}{3}b_2 + \frac{4\pi^2}{9}c_2 + \frac{8\pi^3}{27}d_2 + a_2 = 0$$

(c) $S_{j+1}(x_{j+1}) = S_j(x_{j+1})$ :

⑤ $S_1(x_1) = S_0(x_1) \Rightarrow -a_1 + a_0 + \frac{2\pi b_0}{3} + \frac{4\pi^2}{9}c_0 + \frac{8\pi^3}{27}d_0 = 0$

⑥ $S_2(x_2) = S_1(x_2) \Rightarrow a_2 = a_1 + b_1 \cdot \frac{2\pi}{3} + c_1\left(\frac{2\pi}{3}\right)^2 + d_1\left(\frac{2\pi}{3}\right)^3$

$$a_1 - a_2 + \frac{2\pi b_1}{3} + \frac{4\pi^2 c_1}{9} + \frac{8\pi^3 d_1}{27} = 0$$

(d) $S'_{j+1}(x_{j+1}) = S'_j(x_{j+1})$ :

⑦ $S'_1(x_1) = S'_0(x_1) \Rightarrow b_1 = b_0 + 2c_0\left(\frac{2\pi}{3}\right) + 3d_0\left(\frac{2\pi}{3}\right)^2$

$$b_0 - b_1 + \frac{4\pi c_0}{3} + \frac{4\pi^2 d_0}{3} = 0$$

⑧ $S'_2(x_2) = S'_1(x_2) \Rightarrow b_2 = b_1 + 2c_1\left(\frac{2\pi}{3}\right) + 3d_1\left(\frac{2\pi}{3}\right)^2$

$$b_1 - b_2 + \frac{4\pi c_1}{3} + \frac{4\pi^2 d_1}{3} = 0$$

(e) $S''_{j+1}(x_{j+1}) = S''_j(x_{j+1})$

⑨ $S''_1(x_1) = S''_0(x_1) \Rightarrow 2c_1 = 2c_0 + 6d_0\left(\frac{2\pi}{3}\right)$

$$2c_1 - 2c_0 - 4\pi d_0 = 0$$

⑩ $S''_2(x_2) = S''_1(x_2) \Rightarrow 2c_2 = 2c_1 + 6d_1\left(\frac{4\pi}{3}\right)$

$$2c_2 - 2c_1 - 8\pi d_1 = 0$$

We now apply the clamped boundary conditions:

(f) $S'(x_0) = f'(x_0)$, $S'(x_n) = f'(x_3)$

$$f'(x) = 2\sin(x)\cos(x)$$

$\therefore$ (11) $S'(x_0) = S_0'(0) = f'(0) \Rightarrow \underline{b_0 = 0}$

(12) $S'(x_n) = S_2'(2\pi) = f'(2\pi) \Rightarrow b_2 + 2c_2\left(\frac{2\pi}{3}\right) + 3d_2\left(\frac{2\pi}{3}\right)^2 = 0$

$$\underline{b_2 + \frac{4\pi c_2}{3} + \frac{4\pi^2}{3}d_2 = 0}$$

Thus our augmented matrix is:

$$
\begin{array}{cc}
 & \begin{matrix} a_0 & b_0 & c_0 & d_0 & a_1 & b_1 & c_1 & d_1 & a_2 & b_2 & c_2 & d_2 \end{matrix} \\
\begin{matrix} (1) \\ (2) \\ (3) \\ (4) \\ (5) \\ (6) \\ (7) \\ (8) \\ (9) \\ (10) \\ (11) \\ (12) \end{matrix} &
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \frac{2\pi}{3} & \frac{4\pi^2}{9} & \frac{8\pi^3}{27} \\
1 & \frac{2\pi}{3} & \frac{4\pi^2}{9} & \frac{8\pi^3}{27} & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & \frac{2\pi}{3} & \frac{4\pi^2}{9} & \frac{8\pi^3}{27} & -1 & 0 & 0 & 0 \\
0 & 1 & \frac{4\pi}{3} & \frac{4\pi^2}{3} & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & \frac{4\pi}{3} & \frac{4\pi^2}{3} & 0 & -1 & 0 & 0 \\
0 & 0 & -2 & -4\pi & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -2 & -8\pi & 0 & 0 & 2 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \frac{4\pi}{3} & \frac{4\pi^2}{3}
\end{bmatrix}
\end{array}
\begin{bmatrix} a_0 \\ b_0 \\ c_0 \\ d_0 \\ a_1 \\ b_1 \\ c_1 \\ d_1 \\ a_2 \\ b_2 \\ c_2 \\ d_2 \end{bmatrix}
=
\begin{bmatrix} 0 \\ 0.75 \\ 0.75 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

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Q3a) See attached code at the end to see Matlab code and output of the commands:

$$S(x) = \begin{cases} S_0(x) = -0.0816\,x^3 + 0.3420\,x^2 & , 0 \leq x \leq \frac{2\pi}{3} \\ S_1(x) = -0.1710\left(x-\frac{2\pi}{3}\right)^2 + 0.3581\left(x-\frac{2\pi}{3}\right) + 0.75 & , \frac{2\pi}{3} \leq x \leq \frac{4\pi}{3} \\ S_2(x) = 0.0816\left(x-\frac{4\pi}{3}\right)^3 - 0.171\left(x-\frac{4\pi}{3}\right)^2 - 0.3581\left(x-\frac{4\pi}{3}\right) + 0.75 & , \frac{4\pi}{3} \leq x \leq 2\pi \end{cases}$$

(b) See attached code at the end for plot.

# Swapnil Daxini (V00861672) Assignment 5

## Contents



## Question 1 Part b

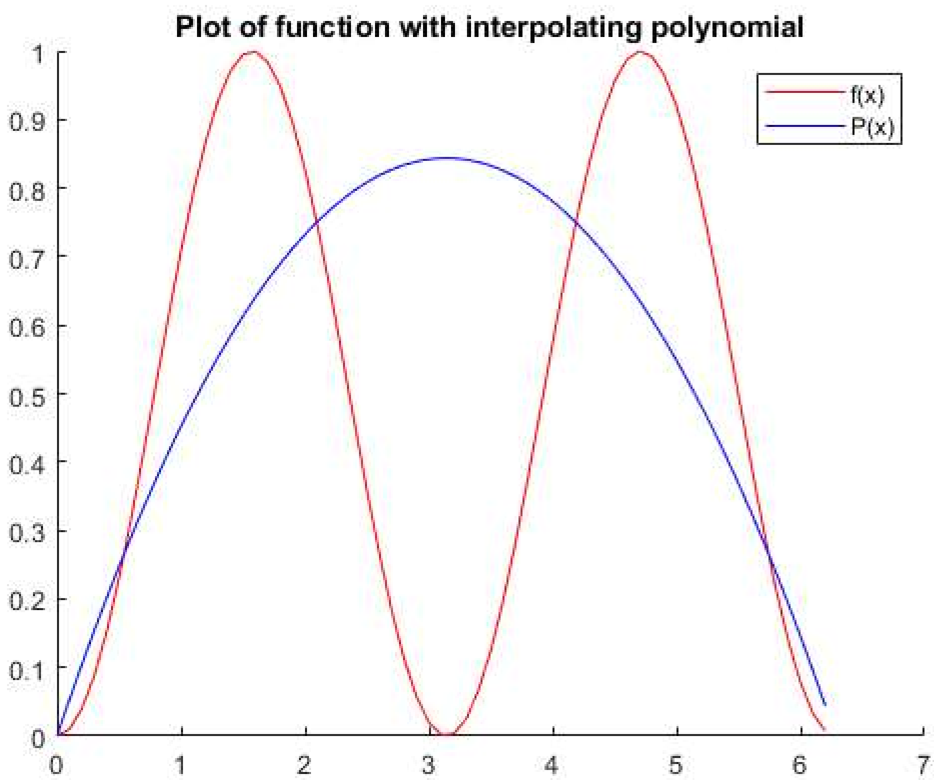
```
x = [0:0.1:2*pi];
y = sin(x).^2;

% Define interpolating polynomial
P = (-27/(32*pi^2))*x.^2+(27/(16*pi))*x;

hold on

plot(x, y, 'Red', x, P, 'Blue')
title('Plot of function with interpolating polynomial')
legend({'f(x)', 'P(x)'})

hold off
```

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# Swapnil Daxini (V00861672) Assignment 5

## Contents



## Question 3

## Part A

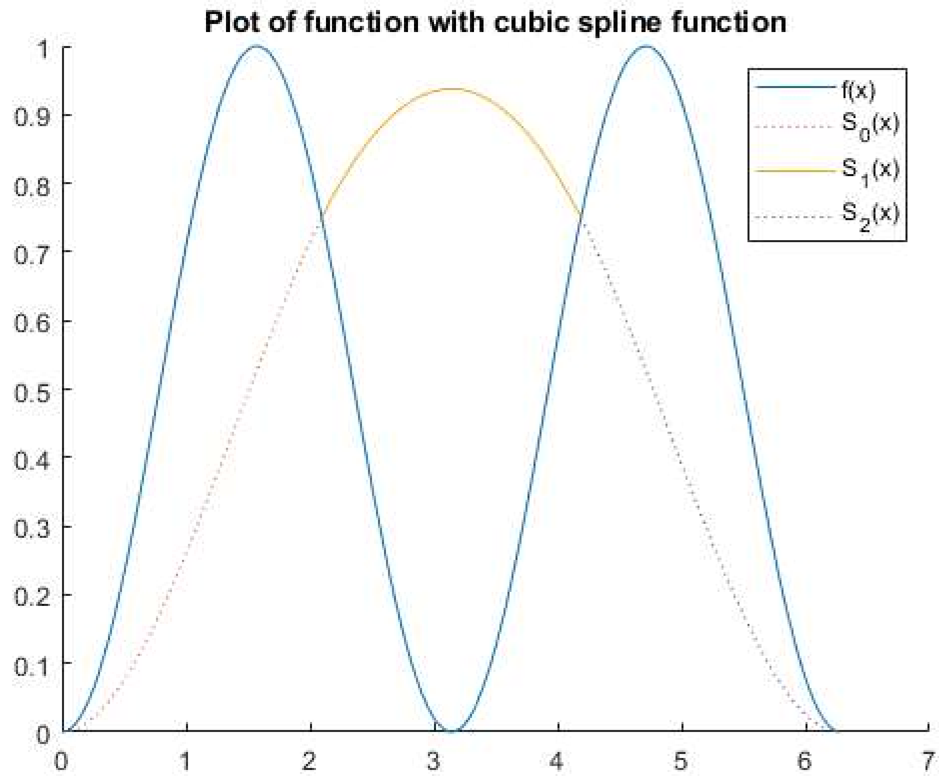
```
X = [0 2*pi/3 4*pi/3 2*pi];

% Note the first and last entries are 0 which are our clamped boundary
% conditions
Y = [0 0 0.75 0.75 0 0];

pp = spline(X, Y);

format short;

[b, c] = unmkpp( pp )
```

```
b =

         0    2.0944    4.1888    6.2832


c =

   -0.0816    0.3420         0         0
    0.0000   -0.1710    0.3581    0.7500
    0.0816   -0.1710   -0.3581    0.7500
```

## Part B

```
hold on;
x = linspace(0, 2*pi, 150);
y = sin(x).^2;
plot(x, y)

X1 = linspace(0, 2*pi/3, 50);
Y1 = c(1,1)*X1.^3 + c(1,2)*X1.^2 + c(1,3)*X1 + c(1,4);
plot(X1, Y1, ':')

X2 = linspace(2*pi/3, 4*pi/3, 50);
Y2 = c(2,1)*(X2 - 2*pi/3).^3 + c(2,2)*(X2 - 2*pi/3).^2 + c(2,3)*(X2 - 2*pi/3) + c(2,4);
plot(X2, Y2, '-')

X3 = linspace(4*pi/3, 2*pi, 50);
Y3 = c(3,1)*(X3 - 4*pi/3).^3 + c(3,2)*(X3 - 4*pi/3).^2 + c(3,3)*(X3 - 4*pi/3) + c(3,4);
plot(X3, Y3, ':')
```

```
legend({'f(x)', 'S_0(x)', 'S_1(x)', 'S_2(x)'})
title('Plot of function with cubic spline function')

hold off;
```

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