# Beta & Gamma Functions ①

Euler's First Integrals (Beta fun.) Def$^n$: a definite integral

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$ , m & n takes positive values. i.e. $m>0, n>0$

<u>Properties</u>

(i) $B(m,n) = B(n,m)$

Substitute $x = 1-y$ in def$^n$

$$B(m,n) = \int_1^0 (1-y)^{m-1} y^{n-1}(-dy)$$

$$= \int_0^1 (1-y)^{m-1} y^{n-1}\,dy$$

$$= \int_0^1 x^{n-1}(1-x)^{m-1}\,dx = B(n,m)$$

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Alternative forms → (#)(I) $B(m,n) = \int_0^\infty \dfrac{x^{m-1}}{(1+x)^{m+n}}\,dx$ [In terms of improper integra

Substitute $x = \dfrac{y}{1+y}$ in def$^n$

$$B(m,n) = \int_0^\infty \frac{y^{m-1}}{(1+y)^{m-1}} \cdot \frac{1}{(1+y)^{n-1}} \cdot \frac{1}{(1+y)^2}\,dy$$

$$= \int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}}\,dy = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$$

(II) In terms of trignometric fun

$$B(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\,d\theta$$

Sub. $x = \sin^2\theta$ in def$^n$

$$B(m,n) = \int_0^{\pi/2} (\sin^2\theta)^{m-1}(1-\sin^2\theta)^{n-1} \cdot 2\sin\theta\cos\theta\,d\theta$$

$$= 2\int_0^{\pi/2} (\sin^2\theta)^{m-1}(\cos^2\theta)^{n-1} \cdot \sin\theta\cos\theta\,d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\,d\theta$$

(iii) $B(m,n) = B(m+1,n) + B(m,n+1)$

Proof As $B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$

$\therefore \quad B(m+1,n) = \int_0^1 x^{m}(1-x)^{n-1}dx \quad -①$

$B(m,n+1) = \int_0^1 x^{m-1}(1-x)^{n}dx \quad -②$

Adding ① & ②

$B(m+1,n) + B(m,n+1)$

$= \int_0^1 x^{m}(1-x)^{n-1}dx + \int_0^1 x^{m-1}(1-x)^{n}dx$

$= \int_0^1 x^{m-1}(1-x)^{n-1}\{x+1-x\}dx$

$= \int_0^1 x^{m-1}(1-x)^{n-1}dx = B(m,n)$ Proved

(i) <u>Evaluation of $B(m,n)$ if $m$ & $n$ both are natural numbers</u>

$$B(m,n) = \frac{\lfloor m-1 \, \lfloor n-1}{\lfloor m+n-1}$$

(ii) If only $m$ is natural no., then

$$B(m,n) = \frac{\lfloor m-1}{n(n+1)(n+2)\cdots(n+m-2)(n+m-1)}$$

(iii) If only $n$ is natural no., then

$$B(m,n) = \frac{\lfloor n-1}{m(m+1)(m+2)\cdots(m+n-2)(m+n-1)}$$

## Euler's Second Integral

Complete Definition of $\Gamma n$

We know

$$\Gamma n = \int_0^\infty e^{-x} x^{n-1}\,dx, \quad n>0$$

$$\Gamma n = \frac{\Gamma(n+1)}{n}, \quad n<0$$

i.e.

(i) $\Gamma(n+1) = n!$, if $n$ is a positive integer

(ii) $\Gamma(n+1) = n\Gamma n$, if $n$ is positive real number

(iii) $\Gamma n = \frac{\Gamma(n+1)}{n}$, if $n$ is a negative fraction

(iv) Gamma is not defined for 0 and Negative integer

(v) $\Gamma\frac{1}{2} = \sqrt{\pi}$, $\Gamma 0 = \infty$, $\Gamma(1-n) = \infty$, for $n \in N$

Proof (vi) $\Gamma n\,\Gamma(1-n) = \frac{\pi}{\sin n\pi}$

(vii) $$\int_0^{\pi/2} \sin^m\theta \cos^n\theta\,d\theta = \frac{\Gamma\frac{m+1}{2}\,\Gamma\frac{n+1}{2}}{2\,\Gamma\frac{m+n+2}{2}}$$

(viii) Duplication Formula

$$\Gamma 2m = \frac{2^{2m-1}}{\sqrt{\pi}}\Gamma m\,\Gamma\left(m+\frac{1}{2}\right)$$

(ix) Modified Form of Gamma function

$$\frac{\Gamma n}{a^n} = \int_0^\infty x^{n-1} e^{-ax}\,dx.$$

[Hint: Substitute $x = az$ in def$^n$ of Gamma fun.

(x) $\Gamma(n+3) = (n+2)(n+1)n\Gamma n$

Q Prove that

(i) $\Gamma\frac{1}{2} = \sqrt{\pi}$ (ii) $\Gamma 0 = \infty$

Sol (i) Method I We know that $\Gamma n \,\Gamma 1-n = \frac{\pi}{\sin n\pi}$

Put $n = \frac{1}{2}$, we get $\Gamma\frac{1}{2}\,\Gamma\frac{1}{2} = \frac{\pi}{\sin \pi/2} = \pi$

or $\left(\Gamma\frac{1}{2}\right)^2 = \pi \Rightarrow \Gamma\frac{1}{2} = \sqrt{\pi}$

Method II We know that

$$\int_0^{\pi/2} \sin^m\theta \cos^n\theta \, d\theta = \frac{\Gamma\frac{m+1}{2}\,\Gamma\frac{n+1}{2}}{2\,\Gamma\frac{m+n+2}{2}}$$

Let $m = n = 0$, we get $\int_0^{\pi/2} d\theta = \frac{\Gamma\frac{1}{2}\,\Gamma\frac{1}{2}}{2\,\Gamma\frac{2}{2}} = \frac{1}{2}\left(\Gamma\frac{1}{2}\right)^2$

Or $\left(\Gamma\frac{1}{2}\right)^2 = 2\,[\theta]_0^{\pi/2} = 2 \times \frac{\pi}{2} = \pi \Rightarrow \Gamma\frac{1}{2} = \sqrt{\pi}$

(ii) We know that $\Gamma n = \frac{\Gamma 1+n}{n}$

Let $n \to 0$, $\Gamma 0 = \lim_{n\to 0} \frac{\Gamma 1}{n} = \lim_{n \to 0} \frac{1}{n} \to \infty$

Q Find the value of

$$\int_0^{\pi/2} \sin^2\theta \cos^5\theta \, d\theta$$

Sol We know that $\int_0^{\pi/2} \sin^m\theta \cos^n\theta \, d\theta = \frac{\Gamma\frac{m+1}{2}\,\Gamma\frac{n+1}{2}}{2\,\Gamma\frac{m+n+2}{2}}$

$$\therefore \int_0^{\pi/2} \sin^2\theta \cos^5\theta \, d\theta = \frac{\Gamma\frac{3}{2}\,\Gamma 3}{2\,\Gamma\frac{9}{2}} = \frac{\frac{1}{2}\sqrt{\pi} \times 2 \times 1}{2 \times \frac{7}{2} \times \frac{5}{2} \times \frac{3}{2} \times \frac{1}{2}\sqrt{\pi}} = \frac{\sqrt{\pi}}{\frac{105}{8}\sqrt{\pi}}$$

$$= \frac{8}{105} \text{ Ans}$$

Q Prove that $\Gamma n+1 = n\Gamma n$

Sol $\Gamma n+1 = \int_0^\infty e^{-x} x^n \, dx = \left[-e^{-x} x^n\right]_0^\infty + n\int_0^\infty e^{-x} x^{n-1} \, dx$ $\quad \Big| \because \lim_{n\to\infty} \frac{x^n}{e^x} = 0$, $\lim_{n\to 0} \frac{x^n}{e^x} = 0$

$= 0 + n\Gamma n = n\Gamma n$

(4) Relation between Beta & Gamma fun

$$B(m,n) = \frac{\Gamma m \, \Gamma n}{\Gamma m+n}, \quad (m>0,\ n>0)$$

Proof

Proof By ~~Standard Integral~~ Defⁿ

$$\frac{\Gamma n}{\cancel{a^n}} = \int_0^\infty x^{n-1} e^{-\cancel{a}x} dx$$

Put $x = t^2 \Rightarrow dx = 2t\,dt$

$$\Gamma n = \int_0^\infty t^{2(n-1)} e^{-t^2} 2t\,dt$$

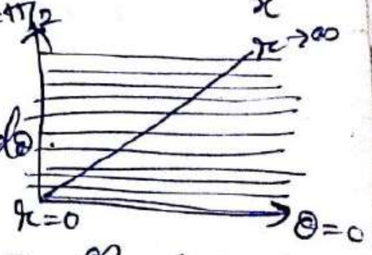
$$= 2\int_0^\infty t^{2n-1} \cdot e^{-t^2} dt$$

Then $\Gamma m \, \Gamma n = \left[2\int_0^\infty x^{2m-1} e^{-x^2} dx\right] \left[2\int_0^\infty y^{2n-1} e^{-y^2} dy\right]$

$$= 4\int_0^\infty\int_0^\infty x^{2m-1} y^{2n-1} e^{-(x^2+y^2)} dx\,dy.$$

Now, let $x = r\cos\theta$, $y = r\sin\theta \Rightarrow r = \sqrt{x^2+y^2}$

Changing into polar coor. $\theta = \tan^{-1}\frac{y}{x}$




$$\Gamma m \, \Gamma n = 4\int_{r=0}^{\infty}\int_{\theta=0}^{\pi/2} (r\cos\theta)^{2m-1} (r\sin\theta)^{2n-1} e^{-r^2} r\,dr\,d\theta$$

$$= 4\left[\int_0^{\pi/2} \sin^{2n-1}\theta \cos^{2m-1}\theta \, d\theta\right]\left[\int_0^\infty e^{-r^2} r^{2m+2n-1} dr\right]$$

using $t = r^2$

$$\Gamma m \, \Gamma n = 4 \cdot \frac{1}{2}B(m,n)\left[\frac{1}{2}\int_0^\infty e^{-t} t^{m+n-1} dt\right].$$

$$= B(m,n)\,\Gamma m+n$$

Hence the result.

**Duplication formula** $\Gamma m \, \Gamma\left(m+\tfrac{1}{2}\right) = \dfrac{\sqrt{\pi}\,\Gamma 2m}{2^{2m-1}}$

**Proof** $B(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$

Putting $n = m$

$$B(m,m) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2m-1}\theta \, d\theta$$

$$= 2\int_0^{\pi/2} (\sin\theta\cos\theta)^{2m-1} d\theta = \frac{2}{2^{2m-1}}\int_0^{\pi/2} \sin^{2m-1}2\theta \, d\theta$$

Let $2\theta = t$ $\Rightarrow \dfrac{\Gamma m\,\Gamma m}{\Gamma 2m} = \dfrac{2}{2^{2m-1}\cancel{\times 2}}\displaystyle\int_0^{\pi}(\sin t)^{2m-1}\frac{dt}{2}$

$$\frac{\Gamma m\,\Gamma m}{\Gamma 2m} = \frac{2}{2^{2m-1}}\int_0^{\pi/2}(\sin t)^{2m-1}(\cos t)^{0}\,dt \qquad \left[\because \int_0^{2a} f(x)\,dx = 2\int_0^{a} f(x)\,dx \text{ if } f(2a-x) = f(x)\right]$$

$$= \frac{2}{2^{2m-1}}\int_0^{\pi/2}(\sin t)^{2m-1}(\cos t)^{2\cdot\frac{1}{2}-1}\,dt$$

$$= \frac{1}{2^{2m-1}} B\left(m,\tfrac{1}{2}\right) = \frac{1}{2^{2m-1}}\frac{\Gamma m\,\Gamma\frac{1}{2}}{\Gamma\left(m+\frac{1}{2}\right)}$$

$$\frac{\Gamma m\,\Gamma m}{\Gamma 2m} = \frac{1}{2^{2m-1}}\frac{\Gamma m\,\sqrt{\pi}}{\Gamma\left(m+\frac{1}{2}\right)} \Rightarrow \Gamma m\,\Gamma\left(m+\tfrac{1}{2}\right) = \frac{\sqrt{\pi}\,\Gamma 2m}{2^{2m-1}}$$

Q. Find (i) $B\left(\frac{3}{2},\frac{1}{2}\right)$, (ii) $B\left(\frac{4}{3},\frac{5}{3}\right)$

Ans $\pi/2$

**Sol.** (ii) $B\left(\frac{4}{3},\frac{5}{3}\right) = \dfrac{\Gamma\frac{4}{3}\,\Gamma\frac{5}{3}}{\Gamma 3} = \dfrac{1}{2}\Gamma\left(\frac{1}{3}+1\right)\Gamma\left(\frac{2}{3}+1\right) = \dfrac{1}{2}\dfrac{1}{3}\Gamma\frac{1}{3}\,\frac{2}{3}\Gamma\frac{2}{3}$

$$= \frac{1}{9}\Gamma\frac{1}{3}\,\Gamma\left(1-\frac{1}{3}\right) = \frac{1}{9}\,\pi/\sin\frac{\pi}{3} \qquad \left[\because \Gamma n\,\Gamma(1-n) = \frac{\pi}{\sin n\pi}\right]$$

$$= \frac{1}{9}\cdot\frac{2\pi}{\sqrt{3}} = \frac{2\pi}{9\sqrt{3}}$$

Q If $B(n,3) = \frac{1}{60}$ & $n = +$integer, find $n$.

Sol $B(n,3) = \dfrac{1}{60} \Rightarrow \dfrac{\Gamma n\,\Gamma 3}{\Gamma(n+3)} = \dfrac{1}{60} \Rightarrow \dfrac{\Gamma n \cdot 2}{(n+2)(n+1)n\,\Gamma n} = \dfrac{1}{60}$

$\Rightarrow n^3 + 3n^2 + 2n = 120$ or $n^3 + 3n^2 + 2n - 120 = 0$

$n = 4, -3.5$

But $n$ is a +integer. Hence $n = 4$

Q. Express $\int_0^1 x^m(1-x^n)^p dx$ in terms of Beta fn.
Hence evaluate $\int_0^1 x^5(1-x^3)^{10}dx$.

Sol:

Let $I = \int_0^1 x^m(1-x^n)^p dx$

$x^n = u$ put $x = u^{1/n} \Rightarrow dx = \frac{1}{n} u^{1/n - 1} du$

$$I = \int_0^1 u^{m/n}(1-u)^p \frac{1}{n} u^{\frac{1}{n}-1} du$$

$$= \frac{1}{n}\int_0^1 u^{\left(\frac{m+1}{n}-1\right)}(1-u)^{(p+1)-1} du$$

$$= \frac{1}{n} B\left(\frac{m+1}{n}, p+1\right)$$

Now taking $m=5$, $n=3$ and $p=10$ in above, we get

$$\int_0^1 x^5(1-x^3)^{10}dx = \frac{1}{3} B\left(\frac{5+1}{3}, 10+1\right)$$

$$= \frac{1}{3} B(2,11)$$

$$= \frac{1}{3}\frac{\Gamma 2\, \Gamma 11}{\Gamma 13} = \frac{1}{3}\frac{\Gamma 11}{12 \times 11\, \Gamma 11}$$

$$= \frac{1}{396}$$

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Q (i) $\int_0^1 (1-x^3)^{-1/2} dx$ (ii) $\int_0^1 (1-x^3)^{1/2} dx$

Sol. (i) Let $x^3 = t \Rightarrow 3x^2 dx = dt$ or $dx = \dfrac{dt}{3t^{2/3}}$

$$\int_0^1 (1-x^3)^{-1/2} dx = \frac{1}{3}\int_0^1 t^{-2/3}(1-t)^{-1/2} dt$$

$$= \frac{1}{3}\beta\left(\frac{1}{3}, \frac{1}{2}\right)$$

$$= \frac{1}{3}\frac{\Gamma\frac{1}{3}\Gamma\frac{1}{2}}{\Gamma\left(\frac{1}{3}+\frac{1}{2}\right)} = \frac{\sqrt{\pi}}{3}\frac{\Gamma\frac{1}{3}}{\Gamma\frac{5}{6}}$$

$m-1 = -\frac{2}{3}$, $m = \frac{1}{3}$, $n-1 = -\frac{1}{2}$, $n = 1/2$

(ii) Ans $\dfrac{1}{3}\dfrac{\Gamma\frac{1}{3}\Gamma\frac{3}{2}}{\Gamma\frac{11}{6}}$

Q $\int_0^1 x^2(1-x)^3 dx = \int_0^1 x^{3-1}(1-x)^{4-1} dx = \dfrac{\Gamma 3\,\Gamma 4}{\Gamma 3+4} = \dfrac{1}{60}$

Q. $\int_0^2 \dfrac{x^2}{\sqrt{2-x}} dx$

Sol. Let $x = 2\sin^2\theta$, $dx = 4\sin\theta\cos\theta\, d\theta$

$$I = \int_0^{\pi/2} \frac{4\sin^4\theta}{\sqrt{2}\cos\theta} 4\sin\theta\cos\theta\, d\theta = \frac{16}{\sqrt{2}}\int_0^{\pi/2}\sin^5\theta\, d\theta$$

$$= \frac{16}{\sqrt{2}}\frac{\Gamma\frac{6}{2}\Gamma\frac{1}{2}}{2\Gamma\frac{7}{2}} = \frac{16}{\sqrt{2}}\frac{2\sqrt{\pi}}{2\times\frac{5}{2}\times\frac{3}{2}\times\frac{1}{2}\sqrt{\pi}} = \frac{16\times2\times2\times2}{\sqrt{2}\,15} = \frac{64\sqrt{2}}{15}$$

Q $\int_0^\infty \dfrac{dx}{1+x^4}$

Sol Let $x^2 = \tan\theta \Rightarrow 2x\,dx = \sec^2\theta\, d\theta \Rightarrow dx = \dfrac{\sec^2\theta}{2\sqrt{\tan\theta}} d\theta$

$$I = \int_0^{\pi/2} \frac{1}{(1+\tan^2\theta)}\frac{\sec^2\theta}{2\sqrt{\tan\theta}} d\theta = \frac{1}{2}\int_0^{\pi/2}\sin^{-1/2}\theta\cos^{1/2}\theta\, d\theta$$

$\because \Gamma n\,\Gamma 1-n = \dfrac{\pi}{\sin n\pi}$

$$= \frac{1}{2}\frac{\Gamma\frac{1}{4}\Gamma\frac{3}{4}}{2\Gamma 1} = \frac{1}{4}\frac{\pi}{\sin\frac{\pi}{4}} = \frac{\pi\sqrt{2}}{4} \text{ Ans}$$

10) $\int_0^1 \frac{dx}{\sqrt{x \log(\frac{1}{x})}}$ Put $x = e^{-t}$

Ans: $\sqrt{2\pi}$

11) Evaluate: $\int_0^\infty \frac{x^a}{a^x}\, dx$ -

Let $a^x = e^t$

$x \log a = t \Rightarrow dx = \frac{dt}{\log a}$

$$\therefore I = \int_0^\infty \left(\frac{t}{\log a}\right)^a \frac{1}{e^t} \frac{dt}{\log a}$$

$$= \frac{1}{(\log a)^{a+1}} \int_0^\infty e^{-t}\, t^a\, dt$$

$$= \frac{\Gamma(a+1)}{(\log a)^{a+1}}$$ ANS.

12) Evaluate $\int_0^\infty 3^{-4x^2} dx$

Let $3^{-4x^2} = e^{-t}$

$-4x^2 \log 3 = -t \log e$

$\Rightarrow 4x^2 \log 3 = t$

$\Rightarrow 8x \log 3\, dx = dt$

$\Rightarrow dx = \frac{dt}{8x \log 3}$

$= \frac{dt}{8 \log 3 \cdot \sqrt{t}} \times 2\sqrt{\log 3} = \frac{dt}{4\sqrt{\log 3}\sqrt{t}}$

$\therefore I = \int_0^\infty e^{-t} \frac{dt}{4\sqrt{\log 3}\sqrt{t}}$

$= \frac{1}{4\sqrt{\log 3}} \int_0^\infty e^{-t}\, t^{-1/2}\, dt$

$= \frac{\Gamma(\frac{1}{2})}{4\sqrt{\log 3}} = \frac{\sqrt{\pi}}{4\sqrt{\log 3}}$

Ans

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Q Prove $\int_0^1 \left(\log \frac{1}{x}\right)^{n-1} dx = \Gamma n$

Sol. Let $\log \frac{1}{x} = t \Rightarrow x = e^{-t} \Rightarrow dx = -e^{-t} dt$ $\quad \left| \begin{array}{l} x=0,\ t=\infty \\ x=1,\ t=0 \end{array} \right.$

$$\int_0^1 \left(\log \frac{1}{x}\right)^{n-1} dx = -\int_\infty^0 t^{n-1} e^{-t} dt$$

$$= \int_0^\infty e^{-t} t^{n-1} dt = \Gamma n \qquad \underline{\underline{H.P.}}$$

Q. $\int_0^{\pi/6} \cos^4 3\theta \sin^2 6\theta \, d\theta = \frac{5\pi}{192}$

Sol Let $3\theta = t \Rightarrow d\theta = \frac{dt}{3}$, when $\begin{array}{l} \theta = 0 \to t = 0 \\ \theta = \frac{\pi}{6} \to t = \frac{\pi}{2} \end{array}$

$$\int_0^{\pi/6} \cos^4 3\theta \sin^2 6\theta \, d\theta = \int_0^{\pi/2} \cos^4 t \sin^2 2t \frac{dt}{3}$$

$$= \frac{4}{3} \int_0^{\pi/2} \cos^4 t \cdot \sin^2 t \cos^2 t \, dt$$

$$= \frac{4}{3} \int_0^{\pi/2} \sin^2 t \cos^6 t \, dt = \frac{4}{3} \frac{\Gamma\frac{3}{2} \Gamma\frac{7}{2}}{2 \Gamma\frac{10}{2}}$$

$$= \frac{4}{3} \frac{\frac{1}{2} \cdot \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \pi}{2 \times 4 \times 3 \times 2} = \frac{\cancel{4}}{\cancel{3}} \cdot \frac{5 \times \cancel{3}}{\cancel{4} \times 4} \times \frac{\pi}{4 \times 4 \times 3}$$

$$= \frac{5\pi}{64 \times 3} = \frac{5\pi}{192}$$

Q $\int_0^1 x^4\left[\log\left(\frac{1}{x}\right)\right]^3 dx = I$

Sol. Let $\log\frac{1}{x} = t \Rightarrow x = e^{-t}$

$$I = -\int_\infty^0 e^{-4t}\, t^3 e^{-t} dt = +\int_0^\infty e^{-5t} t^3 dt$$

Let $5t = y \Rightarrow dt = \frac{dy}{5}$

$$I = \int_0^\infty e^{-y} \frac{y^3}{125} \cdot \frac{dy}{5} = \frac{1}{625}\int_0^\infty e^{-y} y^{4-1} dy$$

$$= \frac{1}{625}\Gamma 4 = \frac{6}{625} \text{ Ans.}$$

Q. $\int_0^1 (\log x)^5 \, dx = I$

Sol. Let $x = e^{-t} \Rightarrow dx = -e^{-t} dt$

$I = \int_\infty^0 (-t)^5 (-e^{-t}) dt = -\int_0^\infty e^{-t} t^5 dt = -\Gamma 6 = -120$

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Q Find $\Gamma\left(-\frac{5}{2}\right)$

Sol $\Gamma n = \frac{\Gamma(n+1)}{n} \Rightarrow \Gamma\left(-\frac{5}{2}\right) = \frac{\Gamma\left(-\frac{5}{2}+1\right)}{-\frac{5}{2}} = -\frac{2}{5}\Gamma\left(-\frac{3}{2}\right)$

$= -\frac{2}{5}\frac{\Gamma\left(-\frac{3}{2}+1\right)}{-\frac{3}{2}} = \frac{4}{15}\Gamma\left(-\frac{1}{2}\right) = \frac{4}{15}\frac{\Gamma\left(-\frac{1}{2}+1\right)}{-\frac{1}{2}} = -\frac{8}{15}\Gamma\left(\frac{1}{2}\right)$

$= -\frac{8}{15}\sqrt{\pi}$.

Q Given $\Gamma\left(\frac{8}{5}\right) = 0.8935$, find the value of $\Gamma\left(-\frac{12}{5}\right)$.

Sol $\Gamma n = \frac{\Gamma(n+1)}{n} \Rightarrow \Gamma\left(-\frac{12}{5}\right) = \frac{\Gamma\left(-\frac{12}{5}+1\right)}{-\frac{12}{5}} = -\frac{5}{12}\frac{\Gamma\left(-\frac{7}{5}+1\right)}{-\frac{7}{5}} = \frac{25}{84}\cdot\frac{\Gamma\left(-\frac{2}{5}+1\right)}{-\frac{2}{5}}$

$= -\frac{125}{168}\frac{\Gamma\left(\frac{3}{5}+1\right)}{\frac{3}{5}} = -\frac{625}{504}\Gamma\left(\frac{8}{5}\right) = -\frac{625}{504}(0.8935) = -1.108$

Q Evaluate $\int_0^\infty e^{-x^3}dx$.

Sol Let $x^3 = t$; $x = t^{1/3} \Rightarrow dx = \frac{1}{3}t^{-2/3}dt$

$\int_0^\infty e^{-x^3}dx = \int_0^\infty e^{-t}\frac{1}{3}t^{-2/3}dt = \frac{1}{3}\int_0^\infty e^{-t}t^{1/3-1}dt = \frac{1}{3}\Gamma\left(\frac{1}{3}\right)$.

Q Evaluate $\int_0^\infty e^{-\sqrt{x}}x^{1/4}dx$.

Sol Let $\sqrt{x} = t$, $x = t^2$, $dx = 2t\,dt$

$$\int_0^\infty e^{-\sqrt{x}}\, x^{1/4}\, dx = \int_0^\infty e^{-t}\sqrt{t}\, 2t\, dt = 2\int_0^\infty e^{-t} t^{3/2}\, dt$$

$$= 2\int_0^\infty e^{-t} t^{5/2-1}\, dt = 2\Gamma\tfrac{5}{2} = 2\cdot\tfrac{3}{2}\cdot\tfrac{1}{2}\sqrt{\pi} = \tfrac{3}{2}\sqrt{\pi}$$

Q. $\int_0^\infty (x^2+4)\, e^{-2x^2}\, dx$

Sol Let $2x^2 = t$, $x = \left(\frac{t}{2}\right)^{1/2}$, $dx = \frac{1}{2}\frac{1}{\sqrt{2}} t^{-1/2} dt = \frac{t^{-1/2}}{2\sqrt{2}} dt$

$$\int_0^\infty (x^2+4)\, e^{-2x^2}\, dx = \int_0^\infty \left(\frac{t}{2}+4\right) e^{-t}\, \frac{t^{-1/2}}{2\sqrt{2}}\, dt$$

$$= \frac{1}{4\sqrt{2}}\int_0^\infty e^{-t} t^{1/2}\, dt + \frac{2}{\sqrt{2}}\int_0^\infty e^{-t} t^{-1/2}\, dt$$

$$= \frac{1}{4\sqrt{2}}\Gamma\frac{3}{2} + \frac{2}{\sqrt{2}}\Gamma\frac{1}{2} = \frac{1}{4\sqrt{2}}\cdot\frac{1}{2}\Gamma\frac{1}{2} + \frac{2}{\sqrt{2}}\Gamma\frac{1}{2}$$

$$= \frac{1}{8\sqrt{2}}\sqrt{\pi} + \frac{2}{\sqrt{2}}\sqrt{\pi} = \frac{17\sqrt{\pi}}{8\sqrt{2}}$$

Q $\int_0^\infty \sqrt{x}\, e^{-x^2} dx \cdot \int_0^\infty \frac{e^{-x^2}}{\sqrt{x}}\, dx$

Sol $x^2 = t$, $x = t^{1/2}$, $dx = \frac{1}{2} t^{-1/2} dt$

$$\int_0^\infty \sqrt{x}\, e^{-x^2} dx \cdot \int_0^\infty \frac{e^{-x^2}}{\sqrt{x}}\, dx = \int_0^\infty t^{1/4} e^{-t}\cdot\frac{1}{2} t^{-1/2} dt \cdot \int_0^\infty \frac{e^{-t}}{t^{1/4}}\cdot\frac{1}{2} t^{-1/2} dt$$

$$= \frac{1}{4}\int_0^\infty e^{-t} t^{-1/4} dt \cdot \int_0^\infty e^{-t} t^{-3/4} dt$$

$$= \frac{1}{4}\Gamma\frac{3}{4}\Gamma\frac{1}{4} = \frac{1}{4}\Gamma\left(1-\frac{1}{4}\right)\Gamma\frac{1}{4} = \frac{1}{4}\frac{\pi}{\sin\frac{\pi}{4}} = \frac{1}{4}\cdot\pi\sqrt{2}$$

$$= \frac{\pi}{2\sqrt{2}}.$$

Q. Evaluate $\int_0^\infty \frac{x^a}{a^x}\, dx$

Sol Let $a^x = e^t$, $x\log a = t$ ; $dx = \frac{1}{\log a} dt$

$$\int_0^\infty \frac{x^a}{a^x}\, dx = \int_0^\infty \left(\frac{t}{\log a}\right)^a \cdot \frac{1}{e^t}\cdot\frac{1}{\log a}\, dt = \frac{1}{(\log a)^{a+1}}\int_0^\infty e^{-t} t^a\, dt$$

$$= \frac{1}{(\log a)^{a+1}}\int_0^\infty e^{-t} t^{(a+1)-1}\, dt = \frac{1}{(\log a)^{a+1}}\Gamma(a+1)$$

$$= \frac{\Gamma(a+1)}{(\log a)^{a+1}}$$

Q Express the following in terms of Gamma func[tion]

ⓐ $\displaystyle\int_0^1 \frac{1}{\sqrt{1-x^4}}\,dx$ ⓑ $\displaystyle\int_0^{\pi/2} \sqrt{\tan\theta}\,d\theta$ ⓒ $\displaystyle\int_0^1 \frac{x}{\sqrt{1-x^3}}\,dx$

Sol ⓐ $\displaystyle I = \int_0^1 \frac{1}{\sqrt{1-x^4}}\,dx$ Put $x = \sin^{1/2}\theta$

$\Rightarrow dx = \frac{1}{2}\sin^{-1/2}\theta\cos\theta\,d\theta$

$$I = \int_0^{\pi/2} \frac{\frac{1}{2}\sin^{-1/2}\theta\cos\theta\,d\theta}{\cos\theta} = \frac{1}{2}\int_0^{\pi/2}\sin^{-1/2}\theta\,d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2}\sin^{-1/2}\theta\cos^0\theta\,d\theta = \frac{1}{2}\,\frac{\Gamma\frac{1}{4}\,\Gamma\frac{1}{2}}{2\,\Gamma\frac{3}{4}} = \frac{\sqrt{\pi}}{4}\,\frac{\Gamma\frac{1}{4}}{\Gamma\frac{3}{4}}$$

$$\left[\because \int_0^{\pi/2}\sin^n\theta\cos^m\theta\,d\theta = \frac{\Gamma\frac{n+1}{2}\,\Gamma\frac{m+1}{2}}{2\,\Gamma\frac{m+n+2}{2}}\right.$$

ⓑ

$$I = \int_0^{\pi/2}\sqrt{\tan\theta} = \int_0^{\pi/2}\sin^{1/2}\theta\cos^{-1/2}\theta\,d\theta = \frac{\Gamma\frac{3}{4}\,\Gamma\frac{1}{4}}{2\,\Gamma 1}$$

$$= \frac{1}{2}\Gamma\frac{3}{4}\,\Gamma\frac{1}{4} = \frac{1}{2}\Gamma\frac{1}{4}\,\Gamma\left(1-\frac{1}{4}\right) = \frac{1}{2}\,\frac{\pi}{\sin\pi/4}$$

$$= \frac{1}{2}\,\frac{\pi}{1/\sqrt{2}} = \frac{\pi}{\sqrt{2}} \qquad \left[\because \Gamma n\,\Gamma(1-n) = \frac{\pi}{\sin n\pi}\right.$$

ⓒ

Put $x = \sin^{2/3}\theta \Rightarrow dx = \frac{2}{3}\sin^{-1/3}\theta\cos\theta\,d\theta$

$$I = \int_0^{\pi/2}\sin^{2/3}\theta\cos^{-1}\theta\cdot\frac{2}{3}\sin^{-1/3}\theta\cos\theta\,d\theta$$

$$= \frac{2}{3}\int_0^{\pi/2}\sin^{1/3}\theta\,d\theta = \frac{2}{3}\,\frac{\Gamma\frac{2}{3}\,\Gamma\frac{1}{2}}{2\,\Gamma\frac{7}{6}} = \frac{\sqrt{\pi}}{3}\,\frac{\Gamma 2/3}{\Gamma 7/6}$$

$$= \frac{\sqrt{\pi}}{3}\,\frac{\Gamma 2/3}{\Gamma\left(\frac{1}{6}+1\right)} = \frac{\sqrt{\pi}}{3}\,\frac{\Gamma 2/3}{\frac{1}{6}\Gamma\frac{1}{6}} = \frac{2\sqrt{\pi}\,\Gamma 2/3}{\Gamma 1/6}.$$

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$$\boxed{\Gamma(n)\,\Gamma(1-n) = \frac{\pi}{\sin n\pi} \quad ; \; 0<n<1}$$

We know that

$$B(m,n) = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx \quad ; \; m,n>0$$

$$\Rightarrow \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)} = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$$

Put $m+n=1$

$$\Rightarrow \frac{\Gamma(1-n)\,\Gamma(n)}{\Gamma(1)} = \int_0^\infty \frac{x^{n-1}}{1+x}\,dx$$

$$\Rightarrow \Gamma(n)\,\Gamma(1-n) = \frac{\pi}{\sin n\pi} \; ; \; 0 < n < 1$$

$$\left[ \because \int_0^\infty \frac{x^{n-1}}{1+x}\,dx = \frac{\pi}{\sin n\pi} \; ; \; 0 < n < 1 \text{ and } \Gamma(1) = 1 \right]$$

by residue Theorem