DOUBLE INTEGRALS

Introduction: The process of integration can be extended to functions of more than one variable. This leads us to two generalizations of the definite integral, namely multiple integrals and repeated integrals. Multiple integrals are definite integrals of functions of several variables. Double and triple integrals arise while evaluating quantities such as area, volume, mass, moments, centroids and moment of inertia find many applications in science and engineering problems.

Double and triple integrals:

Let $f(x,y)$ be a continuous and single valued function of x and y within a region R bounded by a closed curve 'c' and upon the boundary c. Let the region R be subdivided in any manner into n subregions of areas $\delta R_1, \delta R_2 \ldots \delta R_n$.

Let (x_i, y_i) be any point in the subregion of area δR_i.

Consider the sum

$$\sum_{i=1}^{n} f(x_i, y_i)\delta R_i$$

The limit of this sum as $n \to \infty$ $(i = 1, 2, \ldots)$ is defined as the double integral of $f(x,y)$ over the region R and is written as

$$\iint f(x,y)\,dA$$

$$\therefore \quad \iint_R f(x,y)\,dx\,dy = \underset{n\to\infty}{Lt} \sum_{i=1}^{n} f(x_i, y_i)\delta R_i$$

(a) Suppose the region R is described by the inequalities

$$c \leq y \leq d \text{ and } g(y) \leq x \leq h(y)$$

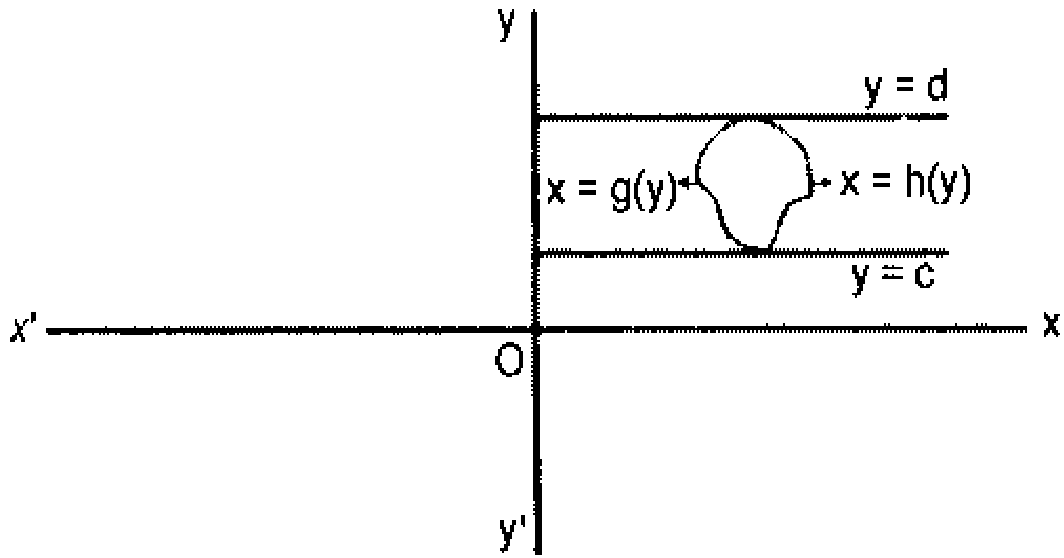


Fig. 5.5.1 (a)

Then

$$\iint\limits_R f(x,y)\,dy\,dx = \iint\limits_R f(x,y)\,dx\,dy = \int_{y=c}^{y=d} \int_{x=g(y)}^{x=h(y)} f(x,y)\,dx\,dy$$

(b) If the region R is described by inequalities

$$a \le x \le b \text{ and } g_1(x) \le y \le h_1(x)$$

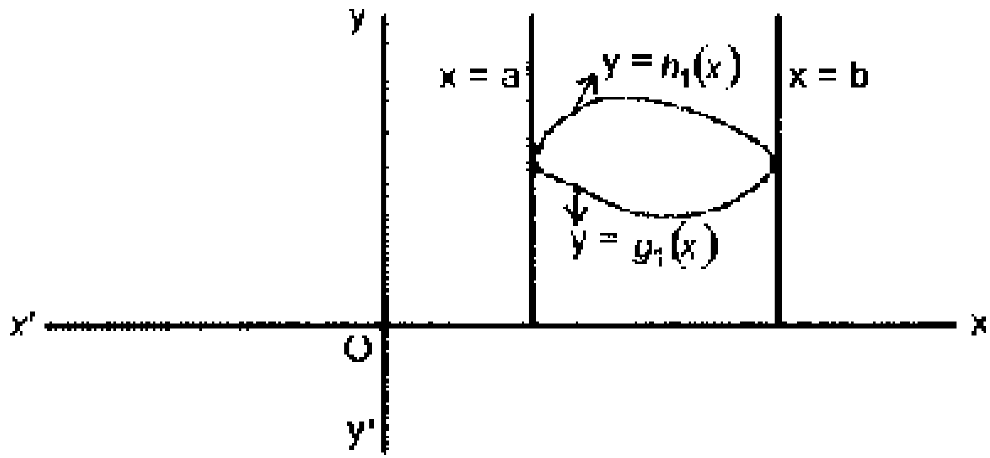


Fig. 5.5.1(b)

Then $$\iint_R f(x,y)\,dx\,dy = \iint_R f(x,y)\,dy\,dx = \int_{x=a}^{x=b}\int_{y=g_1(x)}^{y=h_1(x)} f(x,y)\,dx\,dy$$

(c) If the region R is bounded by the lines $x = a$, $x = b$, $y = c$, $y = d$ (rectangle)

Then $$\iint_R f(x,y)\,dx\,dy = \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

Note: The order of integration is immaterial for constant limits

The double integral can be interpreted and applied in a variety of ways. The following are illustrations:

I. *Volume*. If $z = f(x, y)$ is the equation of a surface, then

$$V = \iint\limits_R f(x, y)\, dx\, dy$$

gives the volume between the surface and the xy-plane, volumes above the xy-plane being counted positively and those below the xy-plane being counted negatively.

II. *Area*. If one takes $f(x, y) \equiv 1$, one obtains

$$A = \text{area of } R = \iint\limits_R dx\, dy.$$

III. *Mass*. If f is interpreted as density, that is, as mass per unit area, then

$$M = \text{mass of } R = \iint_R f(x, y)\, dx\, dy.$$

IV. *Center of mass*. If f is density, then the center of mass $(\bar{x}, \bar{y})$ of the *thin plate* represented by R is located by the equations

$$M\bar{x} = \iint_R x f(x, y)\, dx\, dy, \qquad M\bar{y} = \iint_R y f(x, y)\, dx\, dy,$$

where M is given by (4.37).

EXAMPLE Let R be the quarter-circle, $0 \le y \le \sqrt{1-x^2}$, $0 \le x \le 1$, and let $f(x, y) = x^2 + y^2$. Then one has

$$\iint_R (x^2 + y^2)\,dx\,dy = \int_0^1 \int_0^{\sqrt{1-x^2}} (x^2 + y^2)\,dy\,dx$$

$$= \int_0^1 \left(x^2\sqrt{1-x^2} + \frac{1}{3}(1-x^2)^{\frac{3}{2}} \right) dx$$

$$= \int_0^{\frac{\pi}{2}} \left(\sin^2\theta \cos^2\theta + \frac{1}{3}\cos^4\theta \right) d\theta = \frac{\pi}{8}.$$

Example:

Evaluate $\displaystyle\int_0^1 \int_1^2 \left(x^2 + y^2\right) dx\, dy$

Solution:

$$\int_0^1 \int_1^2 \left(x^2 + y^2\right) dx\, dy = \int_0^1 \left[\frac{x^3}{3} + y^2 x\right]_1^2 dy$$

$$= \int_0^1 \frac{8}{3} + 2y^2 - \frac{1}{3} - y^2\, dy$$

$$= \int_0^1 \left[y^2 + \frac{7}{3}\right] dy = \frac{y^3}{3} + \frac{7}{3} y \Big|_0^1$$

$$= \frac{1}{3} + \frac{7}{3} = \frac{8}{3}$$

Example:

Evaluate $\displaystyle\int_1^4 \int_0^{\sqrt{4-x}} xy\,dy\,dx$

Solution:

$$\int_1^4 \int_0^{\sqrt{4-x}} xy\,dy\,dx = \int_{x=1}^{4} \int_0^{y=\sqrt{4-x}} [xy\,dy]\, dx$$

$$= \int_{x=1}^{4} \int_0^{y=\sqrt{4-x}} xy\,dy\, .dx$$

$$= \int_{x=1}^{4} \left[x\frac{y^2}{2}\right]_{y=0}^{y=\sqrt{4-x}} dx$$

$$= \int_{x=1}^{4} \frac{x}{2}\left(4 - x\right) dx = \frac{4x^2}{4} - \frac{x^3}{6}\Big|_1^4 = \frac{9}{2}$$

Evaluate $\displaystyle\int_0^1 \int_{\sqrt{y}}^{2-y} x^2\,dx\,dy$

Solution:

$$\int_{y=0}^{y=1} \int_{x=\sqrt{y}}^{x=2-y} [x^2\,dx]\,dy = \int_{y=0}^{1} \left[\frac{x^3}{3}\right]_{x=\sqrt{y}}^{x=2-y} dy$$

$$= \int_{y=0}^{y=1} \frac{1}{3}(2-y)^3 - \frac{1}{3}\left(\sqrt{y}\right)^3 dy$$

$$= \frac{1}{3}\int_{y=0}^{y=1} 8 - y^3 - 12y + 6y^2 - y^{\frac{3}{2}}\,dy$$

$$= \frac{1}{3}\left[8y - \frac{y^4}{4} - \frac{12y^2}{2} + \frac{6y^3}{3} - \frac{y^{\frac{3}{2}+1}}{\frac{3}{2}+1}\right]_0^1$$

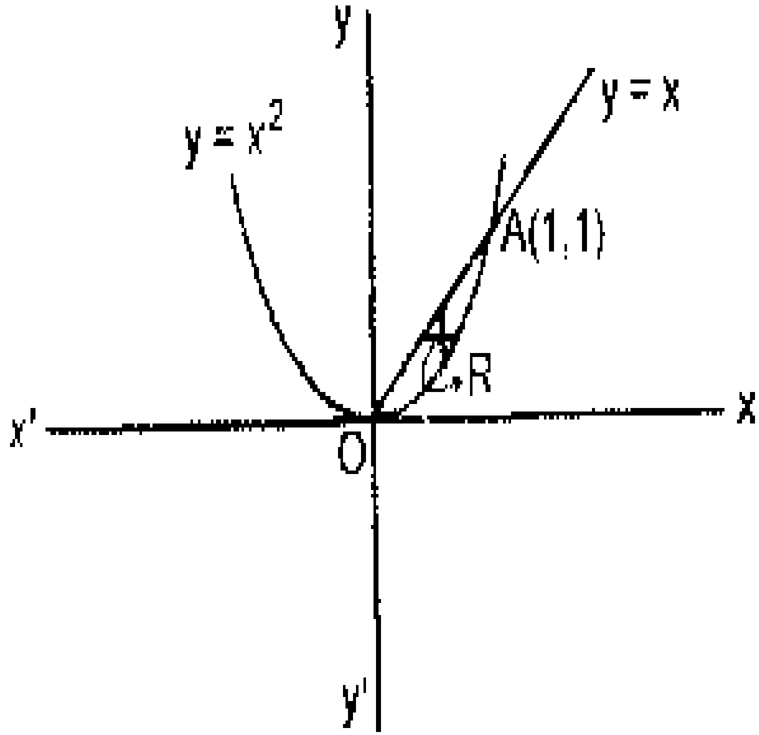
$$= \frac{1}{3}\left[8 - \frac{1}{4} - 6 + 2 - \frac{2}{5}\right]$$

$$= \frac{67}{60}$$

Example:

Find the value of $\iint xy(x+y)\,dx\,dy$ taken over the region enclosed by the curves $y = x$ and $y = x^2$.

Solution:



R is the region bounded by the curves $y = x$ and $y = x^2$

$$I = \iint_R xy(x+y)\,dx\,dy$$

$$= \int_{x=0}^{1}\left[\int_{y=x^2}^{y=x}\left(x^2y + xy^2\right)dy\right]dx$$

$$= \int_{x=0}^{1}\left[\frac{x^2y^2}{2} + \frac{xy^3}{3}\right]_{y=x^2}^{y=x} dx$$

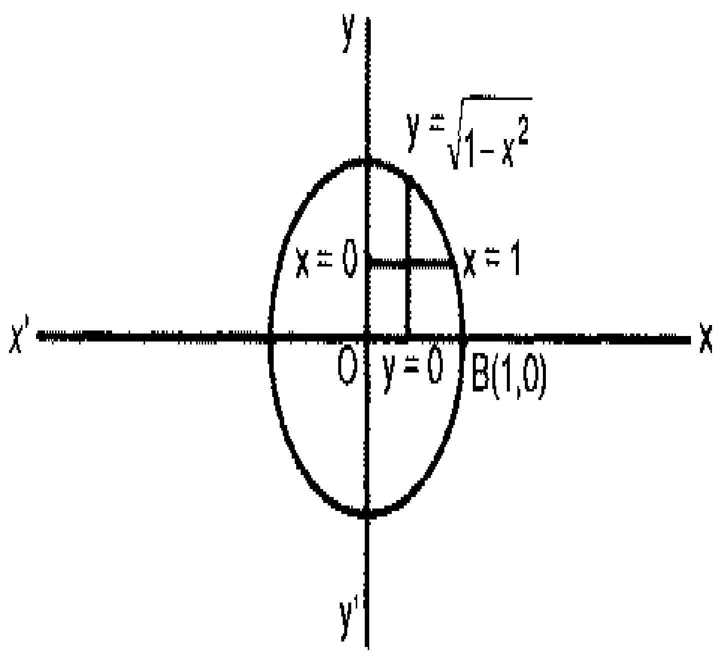
$$= \int_{x=0}^{x=1}\frac{x^4}{2} + \frac{x^4}{3} - \frac{x^6}{2} - \frac{x^7}{3}\,dx$$

$$= \frac{x^5}{10} + \frac{x^5}{15} - \frac{x^7}{14} - \frac{x^8}{24}\Big|_0^1$$

Example:

Evaluate $\iint\limits_A \frac{xy}{\sqrt{1-y^2}}dxdy$ where A is the area in the positive quadrant of the circle $x^2+y^2=1$.

Solution:



$$I=\int_{x=0}^{x=1}\int_{y=0}^{y=\sqrt{1-x^2}}\frac{xy\,dx\,dy}{\sqrt{1-y^2}}$$

$$I=\int_{x=0}^{1}x\left[\int_{y=0}^{y=\sqrt{1-x^2}}\frac{(1-y^2)^{-\frac{1}{2}}\ (-2y)dy}{-2}\right]dx$$

$$=-\frac{1}{2}\int_0^1 x\left[\frac{(1-y^2)^{\frac{1}{2}}}{\frac{1}{2}}\right]_0^{\sqrt{1-x^2}}dx$$

$$=-\int_0^1 x\left\{1\left[1-(1-x^2)\right]^{\frac{1}{2}}-1\right\}dx$$

$$=-\int_0^1(x^2-x)dx$$

$$=\left[\frac{-x^3}{3}+\frac{x^2}{2}\right]_0^1$$

$$=\frac{-1}{3}+\frac{1}{2}=\frac{1}{6}$$

Example : Evaluate $\int_0^1\int_0^{\sqrt{1+x^2}} \frac{dydx}{1+x^2+y^2}$

Solution: Since the limits of y are functions of x, the integration will first be performed w.r.t y (treating x as a constant). Thus

$$\int_0^1\int_0^{\sqrt{1+x^2}} \frac{dydx}{1+x^2+y^2} = \int_0^1\left(\int_0^{\sqrt{1+x^2}} \frac{dy}{1+x^2+y^2}\right)dx$$

$$= \int_0^1\left[\frac{1}{\sqrt{1+x^2}}\tan^{-1}\left(\frac{y}{\sqrt{1+x^2}}\right)\right]_0^{\sqrt{1+x^2}} dx$$

$$= \int_0^1\left\{\frac{1}{\sqrt{1+x^2}}\tan^{-1}(1)\right\}dx$$

$$= \int_0^{\pi/2}\frac{1}{\sqrt{1+x^2}}\tan^{-1}\tan\frac{\pi}{4}dx$$

$$= \frac{\pi}{4}\int_0^1\frac{dx}{\sqrt{1+x^2}}$$

$$= \frac{\pi}{4}\left[\log\left\{x+\sqrt{(1+x^2)}\right\}\right]_0^1 = \frac{\pi}{4}\log\left(1+\sqrt{2}\right)$$ Answer

Example : Evaluate $\int_0^1\int_0^{y^2} e^{x/y}dydx$

Solution: $\int_0^1\int_0^{y^2} e^{x/y}dydx = \int_0^1 dy\int_0^{y^2} e^{x/y}dx$

$= \int_0^1 dy\left\{ye^{x/y}\right\}_0^{y^2}$

Let $\frac{x}{y} = t$

$\Rightarrow dx = y\ dt$

$= \int_0^1 ydy\left\{e^{y^2/y} - e^0\right\}$

$= \int_0^1 ydy\left(e^y - 1\right)$

$= \int_0^1 y\left(e^y - 1\right)dy$

$= \left\{y\left(e^y - y\right)\right\}_0^1 - \int_0^1 1.\left(e^y - y\right)dy$

$= \left\{y\left(e^y - y\right)\right\}_0^1 - \left\{\left(e^y - \frac{y^2}{2}\right)\right\}_0^1$

$= \left\{1\left(e^1 - 1\right) - 0\right\} - \left\{e^1 - \frac{1}{2}\right\} - \left\{e^0 - 0\right\}$

$= e - 1 - e + \frac{1}{2} + 1$

$= \frac{1}{2}$ Answer.

CHANGING CARTESIAN INTEGRALS INTO POLAR INTEGRALS

The procedure for changing a Cartesian integral $\iint_R f(x, y)\, dx\, dy$ into a polar integral has two steps.

Step 1. Substiute $x=r\cos\theta$ and $y=r\sin\theta$, and replace $dx\, dy$ by $r\, dr\, d\theta$ in the Cartesian integral.

Step 2. Supply polar limits of integration for the boundary of R. The Cartesian integral then becomes

$$\iint_R f(x,y)\, dx\, dy = \iint_G f(r\cos\theta, r\sin\theta) r\, dr\, d\theta$$

where G denotes the region of integration in polar coordinates.

Notice that dx dy is not replaced by dr dθ but by r dr dθ

EXAMPLE

Evalaute the double integral $\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2+y^2)\, dy\, dx$ bychanging to polar coordinates.

The region of integration is bounded by

$0 \leq y \leq \sqrt{1-x^2}$ and $0 \leq x \leq 1$

$y = \sqrt{1-x^2}$ is the circle

$x^2 + y^2 = 1, \quad r = 1$

On changing into the polar coordinates, the given integral is

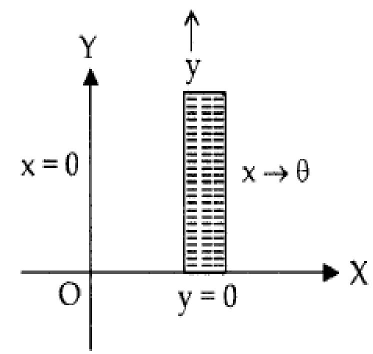
$$\int_0^{\pi/2} \int_0^1 r^3\, dr\, d\theta = \int_0^{\pi/2} \left| \frac{r^4}{4} \right|_0^1 d\theta = \int_0^{\pi/2} \frac{1}{4}\, d\theta = \left| \frac{\theta}{4} \right|_0^{\pi/2} = \frac{\pi}{8}$$

Example:

Transform the integral $\int\limits_{0}^{\infty}\int\limits_{r=0}^{\infty} e^{-\left(x^2+y^2\right)}dxdy$ to polar coordinates and evaluate it.

Solution:

The limits of x and y are both from 0 and '∞'. Therefore the region is in the first quadrant where r varies from '0' to '∞' and θ varies from '0' to $\frac{\pi}{2}$

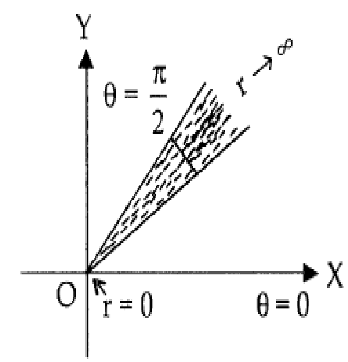


Substituting $x = r\cos\theta$, $y = r\sin\theta$, and $dxdy = rdrd\theta$.

$$\int\limits_{0}^{\infty}\int\limits_{0}^{\infty} e^{-\left(x^2+y^2\right)}dxdy = \int\limits_{\theta=0}^{\frac{\pi}{2}}\int\limits_{r=0}^{\infty} e^{-r^2}rdrd\theta$$

$$= \int\limits_{0}^{\frac{\pi}{2}}\left[\frac{e^{-r^2}}{-2}\right]_0^{\infty}d\theta = \frac{1}{2}\int\limits_{0}^{\frac{\pi}{2}}d\theta = \frac{1}{2}\left[\theta\right]_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{4}$$



Hence show that $\int_0^{\infty} e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$

Now, let

$$I = \int_0^{\infty} e^{-x^2}dx \qquad (1)$$

$$\text{Also } I = \int_0^{\infty} e^{-y^2}dy \qquad (2)$$

(by property of definite integrals)
Multiplying (1) and (2) we get

$$I^2 = \int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)}dx\,dy = \frac{\pi}{4}$$

$$\Rightarrow I = \sqrt{\left(\frac{\pi}{4}\right)} \text{ as obtained above}$$

$$\Rightarrow I = \frac{\sqrt{\pi}}{2} \qquad \text{Proved.}$$

Q1. Evaluate $I = \iint_D \frac{dx\,dy}{x^2 + y^2}$ by changing to polar coordinates, where D is the region in the first quadrant between the circles.

$x^2+y^2=a^2$ and $x^2+y^2=b^2$, $0<a<b$

Q2. Evaluate the double integral $\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2+y^2)\,dy\,dx$ by changing to polar coordinates.

The region of integration is bounded by $0 < y < \sqrt{1-x^2}$ and $0 \le x \le 1$

Example 13. *Evaluate* $\int_0^2 \int_0^{\sqrt{2x-x^2}} (x^2 + y^2)\, dy\, dx$

Solution. $\int_0^2 \int_0^{\sqrt{2x-x^2}} (x^2 + y^2)\, dy\, dx$

Limits of $\quad y = \sqrt{2x - x^2} \Rightarrow y^2 = 2x - x^2 \Rightarrow x^2 + y^2 - 2x = 0 \qquad ...(1)$

(1) represents a circle whose centre is (1, 0) and radius = 1.

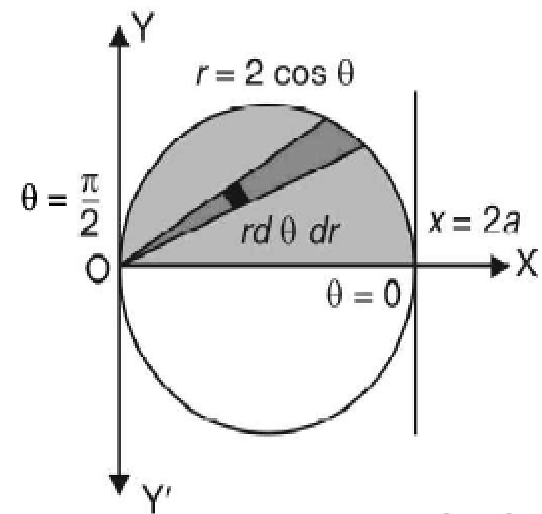
Lower limit of y is 0 *i.e.*, x-axis.

Region of integration is upper half circle.

Let us convert (1) into polar co-ordinate by putting

$$x = r\cos\theta,\ y = r\sin\theta$$

$$r^2 - 2r\cos\theta = 0 \Rightarrow r = 2\cos\theta$$



Limits of r are 0 to $2\cos\theta$

Limits of θ are 0 to $\frac{\pi}{2}$

$$\int_0^2 \int_0^{\sqrt{2x-x^2}} (x^2 + y^2)\, dy\, dx = \int_0^{\frac{\pi}{2}} \int_0^{2\cos\theta} r^2\, (r\, d\theta\, dr) = \int_0^{\frac{\pi}{2}} d\theta \int_0^{2\cos\theta} r^3\, dr = \int_0^{\frac{\pi}{2}} d\theta \left[\frac{r^4}{4}\right]_0^{2\cos\theta}$$

$$= 4\int_0^{\frac{\pi}{2}} \cos^4\theta\, d\theta = 4 \times \frac{3 \times 1 \times \pi}{4 \times 2 \times 2} = \frac{3\pi}{4} \qquad \textbf{Ans.}$$

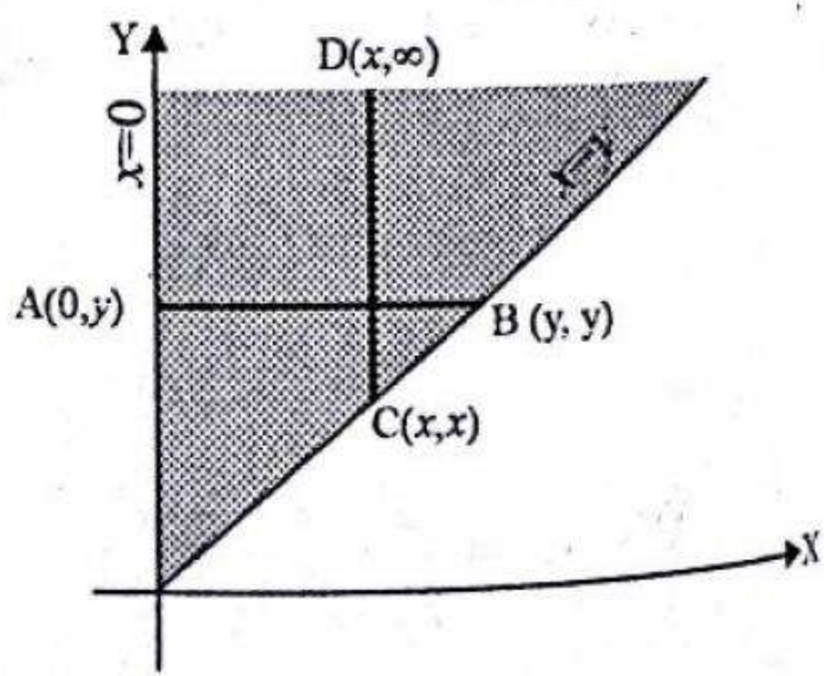
CHANGE OF ORDER OF INTEGRATION

The double integration can be integrated with respect to y first and then with respect to x or it can be integrated with respect to 'x' first and then with respect to 'y'. In the former case, the limits of integration are determined for the given region by drawing strips parallel to y-axis while in the second case by drawing strips parallel to x-axis.

Example 7.23: *Evaluate* $\iint_R \frac{e^{-y}}{y} dx\,dy$, *by choosing the order of integration suitably, given that R is the region bounded by the lines* $x = 0$, $x = y$ *and* $y = \infty$.

[Raj.Univ. 2004]

Solution: Let $I = \iint_R \frac{e^{-y}}{y} dx\,dy$



Suppose, we wish to integrate w.r.t. y first. Then, considering vertical strip

$$I = \int_0^\infty \int_x^\infty \frac{e^{-y}}{y} dy\,dx$$

In this case, we are unable to solve the integral as the inner integration cannot be performed. Now try to integrate w.r.t. x first (consider horizontal strip)

$$I = \int_0^\infty \int_0^y \frac{e^{-y}}{y} dx\, dy$$

$$= \int_0^\infty (x)_0^y \frac{e^{-y}}{y} dy$$

$$= \int_0^\infty e^{-y} dy$$

$$= -(e^{-y})_0^\infty = 1$$

Q. $\int_0^1 \int_{e^x}^{e} \frac{dx\,dy}{\log y}$. Evaluate by changing the order of integration.

Q. . Evaluate $\int_0^a \int_y^a \frac{x}{x^2+y^2}\,dx\,dy$ by changing the order of integration.

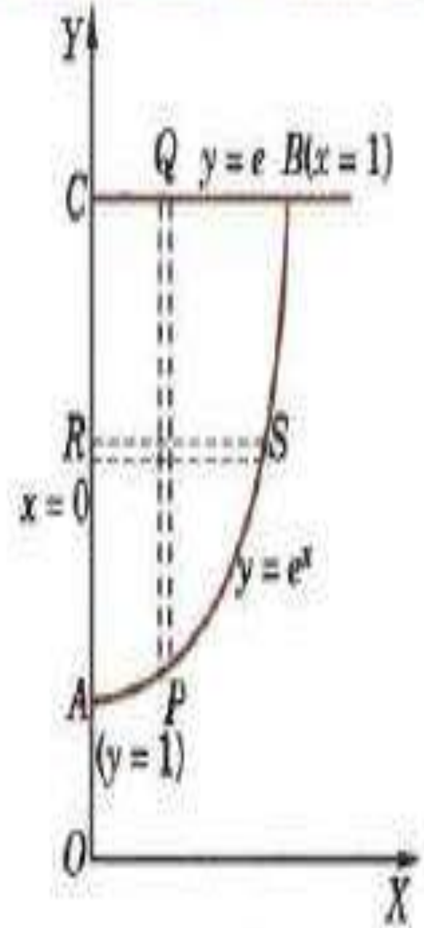
Example *Transform*

$$\int_0^a \int_0^{\sqrt{a^2-x^2}} y^2\sqrt{x^2+y^2}\,dy\,dx$$

by changing to polar coordinates and hence evaluate it.

Example 7.6. *Evaluate* $\int_0^1 \int_{e^x}^{e} dydx/\log y$ *by changing the order of integration.*

Solution. Here the integration is first w.r.t. y from P on $y = e^x$ to Q on the line $y = e$. Then the integration is w.r.t. x from $x = 0$ to $x = 1$, giving the shaded region ABC (Fig. 7.7).



On changing the order of integration, we first integrate w.r.t. x from R on $x = 0$ to S on $x = \log y$ and then w.r.t. y from $y = 1$ to $y = e$.

Thus $$\int_0^1 \int_{e^x}^{e} \frac{dydx}{\log y} = \int_1^e \int_0^{\log y} \frac{dxdy}{\log y}$$

$$= \int_1^e \frac{dy}{\log y} \Big| x \Big|_0^{\log y} = \int_1^e dy = \Big| y \Big|_1^e = e - 1.$$

. *Evaluate* $\int_0^a \int_y^a \frac{x}{x^2 + y^2}\, dx\, dy$ *by changing the order of integration.*

Solution. Here we have

$$I = \int_0^a \int_y^a \frac{x}{x^2 + y^2}\, dx\, dy$$

Here $x = a$, $x = y$, $y = 0$ and $y = a$

The area of integration is OAB.

On changing the order of integration Lower limit of $y = 0$ and upper limit is $y = x$.

Lower limit of $x = 0$ and upper limit is $x = a$.

$$I = \int_0^a x\,dx \int_0^{y=x} \frac{1}{x^2 + y^2}\, dy$$

$$= \int_0^a x\,dx \left[\frac{1}{x} \tan^{-1} \frac{y}{x}\right]_0^{y=x}$$

$$= \int_0^a \frac{x}{x}\, dx \left(\tan^{-1} \frac{x}{x} - \tan^{-1} 0\right)$$

$$= \int_0^a dx \left(\frac{\pi}{4}\right) \quad = \frac{\pi}{4}[x]_0^a = \frac{a\pi}{4} \quad \textbf{Ans.}$$

Example *Transform the integral*

$$\int_0^a \int_0^{\sqrt{a^2-x^2}} y^2\sqrt{x^2+y^2}\,dydx$$

by changing to polar coordinates and hence evaluate it.

Solution: The region of integration is

$$0 \le y \le \sqrt{a^2-x^2}, 0 \le x \le a.$$

i.e. the area bounded by the circle $x^2+y^2=a^2$, lying in the I quadrant.

Put $x = r\cos\theta, y = r\sin\theta$

$\therefore$ Equation of circle : $(r\cos\theta)^2+(r\sin\theta)^2=a^2$

$\Rightarrow \quad r = a$

and for I quadrant, θ varies from 0 to $\frac{\pi}{2}$. Thus, in polar coordinates limits of integration are

$$0 \le r \le a, 0 \le \theta \le \frac{\pi}{2}$$

$$\therefore \quad I = \int_0^{\pi/2}\int_0^a r^2 \sin^2\theta \sqrt{r^2\cos^2\theta + r^2\sin^2\theta}\, r dr\, d\theta$$

$$= \int_0^{\pi/2}\int_0^a r^4 \sin^2\theta\, dr\, d\theta$$

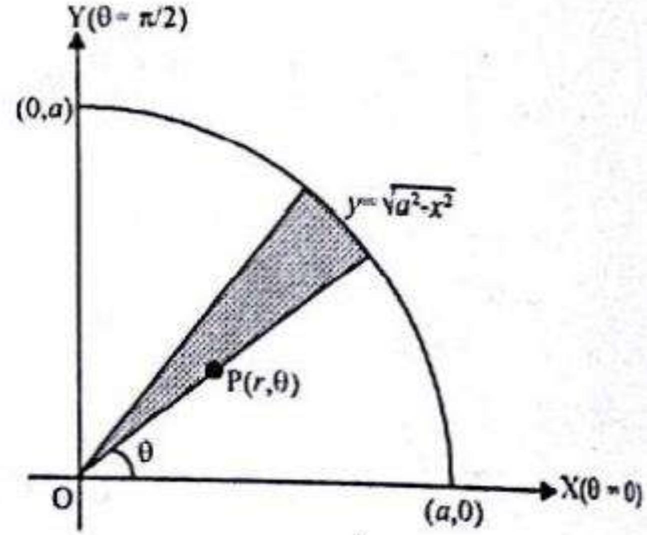
$$= \int_0^{\pi/2} \left(\frac{r^5}{5}\right)_0^a \sin^2\theta\, d\theta$$

$$= \frac{a^5}{5}\int_0^{\pi/2} \sin^2\theta\, d\theta$$

$$= \frac{a^5}{5}\int_0^{\pi/2} \frac{(1-\cos 2\theta)}{2} d\theta$$

$$= \frac{a^5}{5}\left[\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_0^{\pi/2}$$

$$= \frac{\pi a^5}{20}$$



Example : : Find the area enclosed between the parabola $y = 4x - x^2$ and the line $y = x$.

(U.P.T.U. 2008)

Solution: The given curves intersect at the points whose abscissas are given by $y = 4x - x^2$ and $y = x$, Therefore

$x = 4x - x^2$

or $3x - x^2 = 0$

$\Rightarrow x(3 - x) = 0$

$\Rightarrow x = 0, 3$

The area under consideration lies between the curves $y = x$, $y = 4x - x^2$, $x = 0$ and $x = 3$.

Hence, integrating along the vertical strip PQ first, we get the required area as

$$\text{Area} = \int_0^3 \int_{y=x}^{y=4x-x^2} dy\,dx$$

$$= \int_0^3 [y]_x^{4x-x^2}\,dx$$

$$= \int_0^3 \left(4x - x^2 - x\right)dx$$

$$= \int_0^3 \left(3x - x^2\right)dx$$

$$= \left[\frac{3}{2}x^2 - \frac{x^3}{3}\right]_0^3$$

$$= \left[\frac{3}{2}(3)^2 - \frac{(3)^3}{2}\right]$$

$$= \frac{27}{2} - 9 = \frac{9}{2} \qquad \text{Answer.}$$

Example 1. *Find the area bounded by the parabola* $y^2 = 4ax$ *and its latus rectum.*

Solution. Required area $= 2$ (area (ASL)

$$= 2\int_0^a \int_0^{2\sqrt{ax}} dy\, dx$$

$$= 2\int_0^a 2\sqrt{ax}\, dx$$

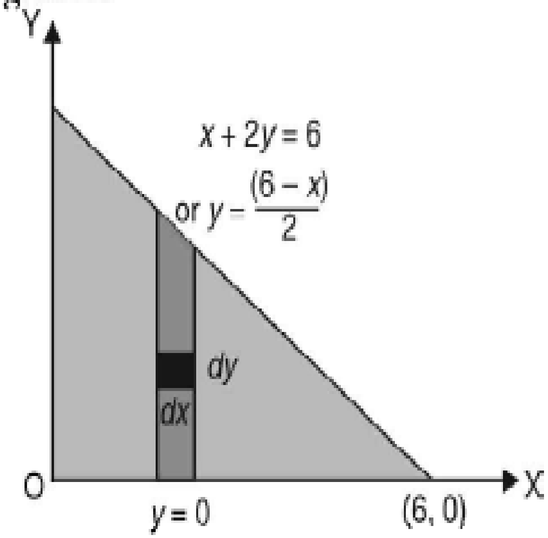
$$= 4\sqrt{a}\left(\frac{x^{3/2}}{3/2}\right)_0^a = \frac{8a^2}{3}$$

Example 2. *Find the area between the parabolas* $y^2 - 4\,ax$ *and* $x^2 - 4\,ay$.

$$= \frac{16}{3}a^2$$

Ans.

Example . *A pyramid is bounded by the three co-ordinate planes and the plane $x + 2y + 3z = 6$. Compute this volume by double integration.*



Solution. $$x + 2y + 3z = 6 \qquad ...(1)$$

$x = 0$, $y = 0$, $z = 0$ are co-ordinate planes.

The line of intersection of plane (1) and xy plane ($z = 0$) is

$$x + 2y = 6 \qquad ...(2)$$

The base of the pyramid may be taken to be the triangle bounded by x-axis, y-axis and the line (2).

An elementary area on the base is $dx\, dy$.

Consider the elementary rod standing on this area and having height z, where

$$3z = 6 - x - 2y \text{ or } z = \frac{6 - x - 2y}{3}$$

Volume of the rod = $dx\, dy$, z, Limits for z are 0 and $\frac{6 - x - 2y}{3}$.

Limits of y are 0 and $\frac{6 - x}{2}$ and limits of x are 0 and 6.

$$\text{Required volume} = \int_0^6 \int_0^{\frac{6-x}{2}} z\,dx\,dy = \int_0^6 dx \int_0^{\frac{6-x}{2}} \frac{6-x-2y}{3}\,dy$$

$$= \frac{1}{3}\int_0^6 dx \left(6x - xy - y^2\right)_0^{\frac{6-x}{2}} = \frac{1}{3}\int_0^6 \left(\frac{6(6-x)}{2} - \frac{x(6-x)}{2} - \left(\frac{6-x}{2}\right)^2\right) dx$$

$$= \frac{1}{3}\int_0^6 \left(\frac{36-6x}{2} - \frac{6x-x^2}{2} - \frac{36+x^2-12x}{4}\right) dx$$

$$= \frac{1}{12}\int_0^6 (72 - 12x - 12x + 2x^2 - 36 - x^2 + 12x)\,dx$$

$$= \frac{1}{12}\int_0^6 (x^2 - 12x + 36)\,dx = \frac{1}{12}\left[\frac{x^3}{3} - \frac{12x^2}{2} + 36x\right]_0^6$$

$$= \frac{1}{12}[72 - 216 + 216] = 6$$

Ans.