Matrices

INTRODUCTION : Today, the subject of matrices is one of the most important and powerful tool in mathematics which has found application to a large number of disciplines such as Engineering, Economics, Statistics, Atomic Physics, Chemistry, Biology, Sociology etc. Matrices are a powerful tool in modern mathematics. Matrices also play an important role in computer storage devices. The algebra and caluclus of matrices forms the basis for methods of solving systems of linear algebraic equations, for solving systems of linear differential equations and for analysis solutions of systems of nonlinear differential equations.

Determinant of a matrix : - Every square matrix A with numbers as elements has associated with it a single unique number called the determinant of A, which is written det A. if A is $n \times n$, the determinant of A is indicated by displaying the elements a_{ij} of A between vertical bars

$$\det A = \begin{vmatrix} a_{11} & a_{12} & \cdots\cdots a_{1n} \\ a_{21} & a_{22} & \cdots\cdots a_{2n} \\ \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots\cdots a_{nn} \end{vmatrix}$$

The number n is called the order of determinant A.

Non Singular and singular Matrices :

A square matrix $A = [a_{ij}]$ is said to be non singular according as $|A| \neq 0$ or $|A| = 0$

Adjoint of a matrix : Let $A = [a_{ij}]$ be a square matrix. Then the adjoint of A, denoted by adj (A), is the matrix given by adj (A) = $\left[A_{ij}\right]^T_{n\times n}$, where A_{ij} is the cofactor of a_{ij} in A, i.e. if

$$A = \begin{vmatrix} a_{11} & a_{12} & \cdots\cdots a_{1n} \\ a_{21} & a_{22} & \cdots\cdots a_{2n} \\ \cdots & \cdots & \cdots\cdots\cdots \\ a_{n1} & a_{n2} & \cdots\cdots a_{nn} \end{vmatrix}$$

Then $\text{adj}(A) = \begin{bmatrix} A_{11} & A_{12} & \ldots\ldots\ldots A_{n1} \\ A_{21} & A_{22} & \ldots\ldots\ldots A_{n2} \\ \ldots & \ldots & \ldots\ldots\ldots\ldots \\ A_{1n} & A_{2n} & \ldots\ldots\ldots A_{nn} \end{bmatrix}$

Inverse of a matrix : Let A be square matrix of order n. Then the matrix B of order n if it exists, such that

$AB = BA = I_n$

is called the inverse of A and denoted by A^{-1} we have

$A(\text{adj}A) = |A| I$

or $A\dfrac{(\text{adj}A)}{|A|} = I$, Provided $|A| \neq 0$

or $A^{-1} = \dfrac{\text{adj}A}{|A|}$, if $|A| \neq 0$

Theorem : The inverse of a matrix is unique.

Proof : Let us consider that B and C are two inverse matrices of a given matrix, say A

Then $AB = BA = I$ $\quad\therefore$ B is inverse of A

and $AC = CA = I$ $\quad\therefore$ C is inverse of A

$C(AB) = (CA)B$ $\quad$ by associative law

or $\quad CI = IB$

or $\quad C = B$

Thus, the inverse of matrix is unique.

Existence of the Inverse : Theorem: A necessary and sufficient condition for a square matrix A to possess the inverse is that $|A| \neq 0$.

(I.A.S. 1973)

Proof : The condition is necessary

Let A be an $n \times n$ matrix and let B be the inverse of A.

Then $AB = I_n$

$\therefore\ |AB| = |I_n| = I$

$\therefore\ |A|\,|B| = 1$ $\quad\therefore |AB| = |A|\,|B|$

$\therefore\ |A|$ must be different from 0.

Conversely, the condition is also sufficient.

If $|A| \neq 0$, then let us define a matrix B by the relation

$B = \dfrac{1}{|A|}(\text{adj}.A)$

Then $AB = A\left(\dfrac{1}{|A|}\text{adj}.A\right)$

$$= \frac{1}{|A|}(\text{adj}.A) = \frac{1}{|A|}|A|I_n = I_n$$

Similarly $BA = \left(\frac{1}{|A|}\text{adj}.A\right)A = \frac{1}{|A|}(\text{adj}.A)A$

$$= \frac{1}{|A|}.|A|I_n = I_n$$

Thus $AB = BA = I_n$

Hence, the matrix A is invertible and B is the inverse of A.

Reversal law for the inverse of a Product, Theorem : If A, B be two n rowed non-singular matrices, then AB is also non - singular and

$(AB)^{-1} = B^{-1} A^{-1}$

i.e. the inverse of a product is the product of the inverse taken in the reverse order.

(I.A.S. 1969, U.P.P.C.S. 1995)

Proof : Let A and B be two n rowed non singular matrices,

We have $|AB| = |A|\,|B|$

Since $|A| \neq 0$ and $|B| \neq 0$

therefore $|AB| \neq 0$. Hence the matrix AB is invertible

Let us define a matrix C by the relation $C = B^{-1}A^{-1}$

Then $C(AB) = (B^{-1} A^{-1})(AB) = B^{-1}(A^{-1}A)B$

$= B^{-1} I_n B$

$= B^{-1} B = I_n$

Also $(AB)C = (AB)(B^{-1} A^{-1}) = A(BB^{-1})A^{-1}$

$= A I_n A^{-1} = AA^{-1}$

$= I_n$

Thus $C(AB) = (AB)C = I_n$

Hence $C = B^{-1} A^{-1}$ is the inverse of AB.

Elementary Row Operations and Elementary Matrices

When we solve a system of linear algebric equations by elimination of unknown, we routinely perform three kinds of operations: Interchange of equations, multiplication of an equation by a nonzero constant and addition of a constant multiple of one equation to another equation.

When we write a homogeneous system in matrix form AX = 0, row k of A lists the coefficients in equation K of the system. The three operations on equations correspond respectively, to the interchange of two rows of A, multiplication of a row A by a constant and addition of a scalar multiple of one row of A to another row of A. We will focus on these row operations in anticipation of using them to solve the system.

DEFINITION: Let A be an n×n matrix. The three elementary row operations that can be performed on A are

1. Type I operation : interchanging two rows of A.
2. Type II operation : multiply a row of A by a non zero constant.
3. Type III operation : Add a scalar multiple of one row to another row.

The rows of A are m - vectors. In a type II operation, multiply a row by a non zero constant by multiplying this row vector by the number. That is, multiply each element of the row by that number. Similarly in a type III operation, we add a scalar multiple of one row vector to another row vector.

Inverse of Non - Singular Matrices Using Elementary Transformations :

If A is non singular matrix of order n and is reduced to the unit matrix I_n by a sequence of E - row transformations only, then the same sequence of E - row transformations applied to the unit matrix I_n gives the inverse of A (i.e.A^{-1}).

Let A be a non singular matrix of order n. It is reduced to unit matrix I_n by a finite number of E- row transformations only. Here, each E - row transformation of the matrix A is equivalent to pre - multiplications by the corresponding E - matrix. Therefore, there exist elementary matrices say, $E_1, E_2, \ldots\ldots\ldots\ldots E_r$ such that

$[E_r, E_{r-1}, \ldots\ldots\ldots\ldots E_2, E_1]\, A = I_n$

Post-multiplying both sides by A^{-1}, we obtain

$[E_r, E_{r-1}, \ldots\ldots\ldots\ldots E_2, E_1]\, A\, A^{-1} = I_n\, A^{-1}$

$\Rightarrow [E_r, E_{r-1}, \ldots\ldots\ldots\ldots E_2, E_1]\, I_n = A^{-1}$

$\because AA^{-1} = I_n$

$I_n\, A^{-1} = A^{-1}$

or $A^{-1} = [E_r, E_{r-1}, \ldots\ldots\ldots\ldots E_2, E_1]\, I_n$

Working rule to find the inverse of a non - singular matrix :

Suppose A is a non singular matrix of order n, then first we write

$A = I_n A$

Next, we apply E row transformations to a matrix A and I_nA till matrix A is reduced to I_n. Then B is equal to A^{-1}, i.e.

$B = A^{-1}$

Example 1 : Find by elementary row transformation the inverse of the matrix

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$$

(U.P.T.U. 2000, 2003)

Solution : The given matrix is

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$$

we can write the given matrix as

A= IA

Applying elementary transformations, we have

$$\begin{bmatrix}0 & 1 & 2\\1 & 2 & 3\\3 & 1 & 1\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}A$$

$$\Rightarrow\begin{bmatrix}1 & 2 & 3\\0 & 1 & 2\\3 & 1 & 1\end{bmatrix}=\begin{bmatrix}0 & 1 & 0\\1 & 0 & 0\\0 & 0 & 1\end{bmatrix}A,\ R_1 \leftrightarrow R_2$$

$$\Rightarrow\begin{bmatrix}1 & 2 & 3\\0 & 1 & 2\\0 & -5 & -8\end{bmatrix}=\begin{bmatrix}0 & 1 & 0\\1 & 0 & 0\\0 & -3 & 1\end{bmatrix}A, R_3 \to R_3 - 3R_1$$

$$\Rightarrow\begin{bmatrix}1 & 2 & 3\\0 & 1 & 2\\0 & 0 & 2\end{bmatrix}=\begin{bmatrix}0 & 1 & 0\\1 & 0 & 0\\5 & -3 & 1\end{bmatrix}A, R_3 \to R_3 + 5R_2$$

$$\Rightarrow\begin{bmatrix}1 & 2 & 3\\0 & 1 & 2\\0 & 0 & 1\end{bmatrix}=\begin{bmatrix}0 & 1 & 0\\1 & 0 & 0\\5/2 & -3/2 & 1/2\end{bmatrix}A,\ R_3/2$$

$$\Rightarrow\begin{bmatrix}1 & 2 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}=\begin{bmatrix}-15/2 & 11/2 & -3/2\\-4 & 3 & -1\\5/2 & -3/2 & 1/2\end{bmatrix}A, R_1 \to R_1 - 3R_3\, R_2 \to R_2 - 3R_3$$

$$\Rightarrow\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}=\begin{bmatrix}1/2 & -1/2 & 1/2\\-4 & 3 & -1\\5/2 & -3/2 & 1/2\end{bmatrix}A, R_1 \to R_1 - 2R_2$$

i.e. I = BA where B = A^{-1}

$$\text{or } A^{-1}=\begin{bmatrix}1/2 & -1/2 & 1/2\\-4 & 3 & -1\\5/2 & -3/2 & 1/2\end{bmatrix}\text{Answer}$$

Example 2 : Find by elementary row transformations the inverse of the matrix

$$\begin{bmatrix}3 & -3 & 4\\2 & -3 & 4\\0 & -1 & 1\end{bmatrix}$$

(U.P.T.U. 2002)

Solution : The given matrix is

$$A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$$

we can write the given matrix as

A = IA

$$\text{i.e. } \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Applying elementary transformations, we get

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A,\ R_1 \to R_1 - R_2$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} A,\ R_2 \to R_2 - 2R_1$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & -4/3 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 2/3 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A,\ R_2 / -3$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & -4/3 \\ 0 & 0 & -1/3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 2/3 & -1 & 0 \\ 2/3 & -1 & 1 \end{bmatrix} A,\ R_3 \to R_3 + R_2$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -4/3 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 2/3 & -1 & 0 \\ -2 & 3 & -3 \end{bmatrix} A,\ R_3 \to R_3 \times (-3)$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix} A,\ R_2 \to R_2 + \frac{4}{3} R_3$$

I = BA

Where $B = A^{-1}$

$$A^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix} \text{ Ans.}$$

Normal Form :

Every non - zero matrix of order m × n with rank r can be reduced by a sequence of elementary transformations to any of the following forms

1. I_r
2. $[I_r, 0]$
3. $\begin{bmatrix} I_r \\ 0 \end{bmatrix}$
4. $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$

The above forms are called normal form of A. r so obtained is a number called the rank of matrix A.

Equivalence of matrix :

Suppose matrix B of order m×n is obtained from matrix A (of the same order as B) by finite number of elementary transformations on A; then A is called equivalent to B i.e A~ B. Matrices A and B have same rank and can be expressed as B = PAQ, where P and Q are non singular matrices. If A is of order m×n, then P has order m×m and has n×n such that

B = PAQ

Working rule : Let A be a matrix of order m×n

1. we write $A = I_m A I_n$
2. Next, we transform matrix A to normal form using elementary transformations.
3. Elementary row transformation is applied simultaneously to A and I_m i.e. the prefactor matrix.
4. Elementary column operation applied to A is also applied to I_n i.e. the post factor matrix.
5. Finally, we find B= PAQ, where B is the normal form of A.

Example 3 : Reduce matrix A to its normal form, where

$$A = \begin{bmatrix} 1 & 2 & -1 & 4 \\ 2 & 4 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ -1 & -2 & 6 & -7 \end{bmatrix}$$

Hence, find the rank.

(I.A.S 2006; U.P.T.U. 2001, 2004)

Solution : The given matrix is

$$A = \begin{bmatrix} 1 & 2 & -1 & 4 \\ 2 & 4 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ -1 & -2 & 6 & -7 \end{bmatrix}$$

Applying elementary row transformation, we have

$$A = \begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 0 & 5 & -4 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 5 & -3 \end{bmatrix}, \quad \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1 \\ R_4 \to R_4 + R_1 \end{array}$$

Now, applying elementary column transformation we have

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 5 & -4 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 5 & -3 \end{bmatrix}, \quad \begin{array}{l} C_2 \to C_2 - 2C_1 \\ C_3 \to C_3 + C_1 \\ C_4 \to C_4 - 4C_1 \end{array}$$

Interchanging C_2 and C_3, we have

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 0 & -4 \\ 0 & 4 & 0 & 0 \\ 0 & 5 & 0 & -3 \end{bmatrix}, \quad C_3 \leftrightarrow C_2$$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 0 & -4 \\ 0 & 0 & 0 & 16/5 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \begin{array}{l} R_3 \to R_3 - \frac{4}{5}R_2 \\ R_4 \to R_4 - R_2 \end{array}$$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & -4 & 0 \\ 0 & 0 & 16/5 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad C_4 \leftrightarrow C_3$$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 16/5 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{array}{l} R_2 \to R_2 + 4R_4 \\ R_4 \to R_4 - \frac{5}{16} R_3 \end{array}$$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \begin{array}{l} R_2 \to \frac{1}{5} R_2 \\ R_3 \to \frac{5}{16} R_3 \end{array}$$

$$A \sim \begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}$$

which is the required normal form.

Here, we have three non zero rows. Thus the rank of matrix A is 3. Ans.

Example 4 : Find non - singular matrices P,Q so that PAQ is a normal form, where

$$A = \begin{bmatrix} 2 & 1 & -3 & -6 \\ 3 & -3 & 1 & 2 \\ 1 & 1 & 1 & 2 \end{bmatrix}$$

(U.P.T.U. 2002)

Solution : The order of A is 3×4

Total number of rows in A = 3, therefore consider unit matrix I_3.

Total number of columns in A = 4, hence, consider unit matrix I_4

$\therefore\ A_{3 \times 4} = I_3\, A\, I_4$

$$\begin{bmatrix} 2 & 1 & -3 & -6 \\ 2 & -3 & 1 & 2 \\ 1 & 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 3 & -3 & 1 & 2 \\ 2 & 1 & -3 & -6 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad R_1 \leftrightarrow R_3$$

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & -6 & -2 & -4 \\ 0 & -1 & -5 & -10 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -3 \\ 1 & 0 & -2 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{array}{l} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - 2R_1 \end{array}$$

$$\begin{bmatrix}1&0&0&0\\0&-6&-2&-4\\0&-1&-5&-10\end{bmatrix}=\begin{bmatrix}0&0&1\\0&1&-3\\1&0&-2\end{bmatrix}A\begin{bmatrix}1&-1&-1&-2\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\quad \begin{matrix}C_2\to C_2-C_1\\C_3\to C_3-C_1\\C_4\to C_4-2C_1\end{matrix}$$

$$\begin{bmatrix}1&0&0&0\\0&1&5&10\\0&6&2&4\end{bmatrix}=\begin{bmatrix}0&0&1\\-1&0&2\\0&-1&3\end{bmatrix}A\begin{bmatrix}1&-1&-1&-2\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\quad \begin{matrix}R_2\to -R_2\\R_3\to -R_3\\R_2\leftrightarrow R_3\end{matrix}$$

$$\begin{bmatrix}1&0&0&0\\0&1&5&10\\0&0&-28&-56\end{bmatrix}=\begin{bmatrix}0&0&1\\-1&0&2\\6&-1&-9\end{bmatrix}A\begin{bmatrix}1&-1&-1&-2\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\text{ by } R_3\to R_3-6R_2$$

$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&-28&-56\end{bmatrix}=\begin{bmatrix}0&0&1\\-1&0&2\\6&-1&-9\end{bmatrix}A\begin{bmatrix}1&-1&4&8\\0&1&-5&-10\\0&0&1&0\\0&0&0&1\end{bmatrix}$$

As by $C_3\to C_3-5C_2$
& $C_4\to C_4-10C_2$

$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&2\end{bmatrix}=\begin{bmatrix}0&0&1\\-1&0&2\\-\frac{6}{28}&\frac{1}{28}&\frac{9}{28}\end{bmatrix}A\begin{bmatrix}1&-1&4&8\\0&1&-5&-10\\0&0&1&0\\0&0&0&1\end{bmatrix}R_3\to -\frac{1}{28}R_3$$

$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\end{bmatrix}=\begin{bmatrix}0&0&1\\-1&0&2\\-\frac{6}{28}&\frac{1}{28}&\frac{9}{28}\end{bmatrix}A\begin{bmatrix}1&-1&4&0\\0&1&-5&0\\0&0&1&-2\\0&0&0&1\end{bmatrix}\text{ As by } C_4\to C_4-2C_3$$

$N = PAQ$

$$P=\begin{bmatrix}0&0&1\\-1&0&2\\-\frac{3}{14}&\frac{1}{28}&\frac{9}{28}\end{bmatrix},\ Q=\begin{bmatrix}1&-1&4&0\\0&1&-5&0\\0&0&1&-2\\0&0&0&1\end{bmatrix}\text{Answer}$$

Example 5 : Find the rank of the matrix

$$A = \sim \begin{bmatrix} 6 & 1 & 3 & 8 \\ 4 & 2 & 6 & -1 \\ 10 & 3 & 9 & 7 \\ 16 & 4 & 12 & 15 \end{bmatrix}$$

Solution : Sometimes to determine the rank of a matrix we need not reduce it to its normal form. Certain rows or columns can easily be seen to be linearly dependent on some of the others and hence they can be reduceds zeros by E - row or column transformations. Then we try to find some non-vanishing determinant of the highest order in the matrix, the order of which determines the rank.

We have the matrix

$$A \sim \begin{bmatrix} 6 & 1 & 3 & 8 \\ 4 & 2 & 6 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_3 \rightarrow R_3 - R_2 - R_1 \\ R_4 \rightarrow R_4 - R_3 - R_1 \end{matrix}$$

Since $\begin{vmatrix} 6 & 1 \\ 4 & 2 \end{vmatrix} = 8 \neq 0$

Therefore rank (A) = 2 Answer

Alter

The determinant of order 4 formed by this matrix

$$= \begin{bmatrix} 6 & 1 & 3 & 8 \\ 4 & 2 & 6 & -1 \\ 10 & 3 & 9 & 7 \\ 16 & 4 & 12 & 15 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 1 & 3 & 8 \\ 4 & 2 & 6 & -1 \\ 6 & 1 & 3 & 8 \\ 6 & 1 & 3 & 8 \end{bmatrix} \quad \begin{matrix} R_3 \rightarrow R_3 - R_2 \\ \& \ R_4 \rightarrow R_4 - R_3 \end{matrix}$$

$= 0 \quad \therefore R_3$ and R_4 are identical.

A minor of order 3

$$= \begin{bmatrix} 6 & 1 & 3 \\ 4 & 2 & 6 \\ 10 & 3 & 9 \end{bmatrix} = \begin{bmatrix} 6 & 1 & 3 \\ 4 & 2 & 6 \\ 6 & 1 & 3 \end{bmatrix} \text{by } R_3 \rightarrow R_3 - R_2$$

$= 0$

In similar way we can prove that all the minors of order 3 are zero.

A minor of order 2

$$\begin{vmatrix} 6 & 1 \\ 4 & 2 \end{vmatrix} = 8 \neq 0$$

Hence rank of the matrix = 2 Answer.

Example 6 : Prove that the points (x_1, y_1), (x_2, y_2), (x_3, y_3) are collinear if and only if the rank of the matrix $\begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix}$ is less than 3.

(U.P.PC.S. 1997)

Solution : Suppose the points (x_1, y_1), (x_2, y_2), (x_3, y_3) are collinear and they lie on the line whose equation is

$ax + by + c = 0$

Then

$ax_1 + by_1 + c = 0$ (i)

$ax_2 + by_2 + c = 0$ (ii)

$ax_3 + by_3 + c = 0$ (iii)

Eliminating a, b and c between (i), (ii) and (iii) we get

$$\begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix} = 0$$

Thus the rank of matrix

$$A = \begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix}$$

is less than 3.

Conversely, if the rank of the matrix A is less than 3, then

$$\begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix} = 0$$

Therefore the area of the triangle whose vertices are (x_1, y_1), (x_2, y_2), (x_3, y_3) is equal to zero. Hence three points are collinear.

Consistent system of Equations:

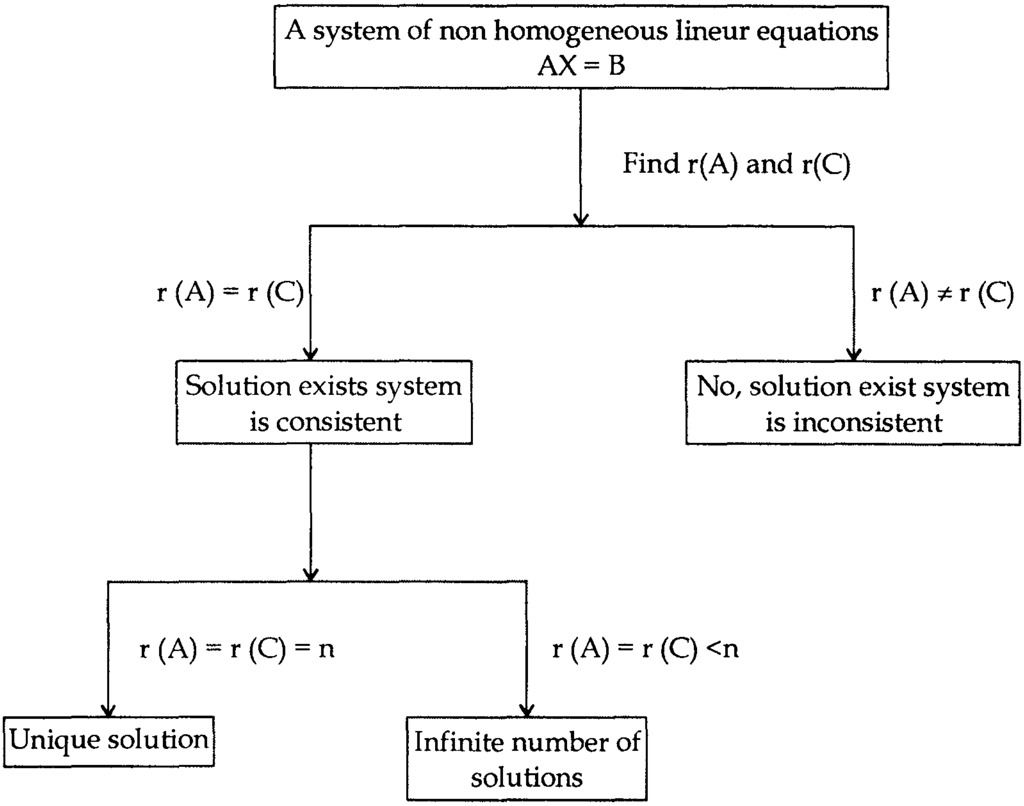
A non -homogeneous system AX = B is said to be consistent if there exists a solution. If there is no solution the system is inconsistent.

For a system of non - homogeneous linear equations AX = B (where A is the coefficient matrix) and C = [A B] is an augmented matrix :

1. If r (A) = r (C), the system is inconsistent

2. If r (A) = r (C) = n (number of unknowns) the system has a unique solution.
3. If r (A) = r (C)< n, the system has an infinite number of solutions.

The above conclusions are depicted in figure as given below



Example 7 : Using the matrix method, show that the equations $3x + 3y + 2z = 1$; $x + 2y = 4$ $10y + 3z = -2$ $2x - 3y - z = 5$ are consistent and hence obtain the solution for x, y and z

(U.P.T.U. 2000)

Solution : The given system of linear equations can be written as

AX = B i.e.

$$\begin{bmatrix} 3 & 3 & 2 \\ 1 & 2 & 0 \\ 0 & 10 & 3 \\ 2 & -3 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ -2 \\ 5 \end{bmatrix}$$

The augmented matrix is

$$C = [A : B] = \begin{bmatrix} 3 & 3 & 2 & 1 \\ 1 & 2 & 0 & 4 \\ 0 & 10 & 3 & -2 \\ 2 & -3 & -1 & 5 \end{bmatrix}$$

Applying elementary row transformations to C, we have

$$C \sim \begin{bmatrix} 1 & 2 & 0 & 4 \\ 3 & 3 & 2 & 1 \\ 0 & 10 & 3 & -2 \\ 2 & -3 & -1 & 5 \end{bmatrix}, \qquad R_1 \to R_2$$

$$\sim \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & -3 & 2 & -11 \\ 0 & 10 & 3 & -2 \\ 0 & -7 & -1 & -3 \end{bmatrix}, \qquad \begin{matrix} R_2 \to R_2 - 3R_1 \\ R_4 \to R_4 - 2R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & -3 & 2 & -11 \\ 0 & 0 & 29/3 & -116/3 \\ 0 & 0 & -17/3 & 68/3 \end{bmatrix}, \qquad \begin{matrix} R_3 \to R_3 + \frac{10}{3} R_2 \\ R_4 \to R_4 - \frac{7}{3} R_2 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & -3 & 2 & -11 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 1 & -4 \end{bmatrix}, \qquad \begin{matrix} R_3 \to \frac{3}{29} R_3 \\ R_4 \to -\frac{3}{17} R_4 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & -3 & 2 & -11 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad R_4 \to R_4 - R_3$$

Thus r(C) = r (A) = 3 hence the given system is consistant and has a unique solution

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & -3 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -11 \\ -4 \\ 0 \end{bmatrix}$$

or $z = -4$, $-3y + 2z = -11$, $x + 2y = 4$

or $z = -4$, $y = \frac{1}{3}(2z + 11) = 1$, $x = 4 - 2y = 2$

Thus, the solution is

$x = 2, y = 1, z = -4$ Answer.

Example 8 : Examine the consistency of the following system of equations and solve them if they are consistent $x_1 + 2x_2 - x_3 = 3$; $3x_1 - x_2 + 2x_3 = 1$; $2x_1 - 2x_2 + 3x_3 = 2$; $x_1 - x_2 + x_3 = -1$

(U.P.T.U. 2002)

Solution : The given system of linear equatins can be written in matrix form as A X = B i.e.

$$\begin{bmatrix} 1 & 2 & -1 \\ 3 & -1 & 2 \\ 2 & -2 & 3 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \\ -1 \end{bmatrix}$$

The augmented matrix is

C = [A : B]

$$\text{i.e. } C = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 3 & -1 & 2 & 1 \\ 2 & -2 & 3 & 2 \\ 1 & -1 & 1 & -1 \end{bmatrix}$$

Applying elementary row transformations to the augmented matrix, we obtain

$$C \sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -7 & 5 & -8 \\ 0 & -6 & 5 & -4 \\ 0 & -3 & 2 & -4 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - 2R_1 \\ R_4 \to R_4 - R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -7 & 5 & -8 \\ 0 & 0 & 5/7 & 20/7 \\ 0 & 0 & -1/7 & -4/7 \end{bmatrix} \quad \begin{matrix} R_3 \to R_3 - \frac{6}{7}R_2 \\ R_4 \to R_4 - \frac{3}{7}R_2 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -7 & 5 & -8 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} R_3 \to \frac{7}{5}R_3 \\ R_4 \to R_4 + \frac{1}{5}R_3 \end{matrix}$$

i.e. r (C) = 3 = r (A)

Hence, the system is consistent and has a unique solution, thus

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & -7 & 5 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ -8 \\ 4 \\ 0 \end{bmatrix}$$

$\Rightarrow x_1 + 2x_2 - x_3 = 3, -7x_2 + 5x_3 = -8$ and $x_3 = 4$

$\Rightarrow 7x_2 = 5x_3 + 8$

$\Rightarrow x_2 = \frac{1}{7}(5 \times 4 + 8) = 4$

and $x_1 = 3 - 2x_2 + x_3 = 3 - 8 + 4 = -1$

Thus, the solution is $x_1 = -1$ $x_2 = 4$, $x_3 = 4$ Answer

Example 9 : Examine the consistency of the following system of linear equations and hence, find the solution $4x_1 - x_2 = 12; -x_1 + 5x_2 - 2x_3 = 0; -2x_2 + 4x_3 = -8$

(U.P.T.U. 2005)

Solution : The given equations can be written in matrix form as

$$\begin{bmatrix} 4 & -1 & 0 \\ -1 & 5 & -2 \\ 0 & -2 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 12 \\ 0 \\ -8 \end{bmatrix}$$

i.e. AX = B where

$$A = \begin{bmatrix} 4 & -1 & 0 \\ -1 & 5 & -2 \\ 0 & -2 & 4 \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, B = \begin{bmatrix} 12 \\ 0 \\ -8 \end{bmatrix}$$

Now the augmented matrix C is

C = [A : B]

$$\text{i.e. } C = \begin{bmatrix} 4 & -1 & 0 & 12 \\ -1 & 5 & -2 & 0 \\ 0 & -2 & 4 & -8 \end{bmatrix}$$

Applying elementary row transfomations to matrix C to reduce it to upper triangular form we get

$$B \sim \begin{bmatrix} 1 & 14 & -6 & 12 \\ -1 & 5 & -2 & 0 \\ 0 & -2 & 4 & -8 \end{bmatrix}, \qquad R_1 \to R_1 + 3R_2$$

$$\sim \begin{bmatrix} 1 & 14 & -6 & 12 \\ 0 & 19 & -8 & 12 \\ 0 & -2 & 4 & -8 \end{bmatrix}, \qquad R_2 \to R_2 + R_1$$

$$\sim\begin{bmatrix} 1 & 14 & -6 & 12 \\ 0 & 19 & -8 & 12 \\ 0 & 0 & 60/19 & -128/19 \end{bmatrix}, \quad R_3 \to R_3 + \frac{2}{19}R_2$$

Hence, we see that ranks of A and C are 3 i.e. r (A) = 3 = r (C). The the system of linear equations is consistent and has a unique solution. Thus, the given system of linear equations is

$$\begin{bmatrix} 1 & 14 & -6 \\ 0 & 19 & -2 \\ 0 & 0 & 60/19 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 12 \\ 12 \\ -128/19 \end{bmatrix}$$

i.e. $x_1 + 14\,x_2 - 6\,x_3 = 12$

$19\,x_2 - 8x_3 = 12$

and $\frac{60}{19}x_3 = -\frac{-128}{19} \Rightarrow x_3 = \frac{-128}{60}$

i.e. $x_3 = -\frac{32}{15}$

on putting the value of x_3 in $19x_2 - 8\,x_3 = 12$

$$x_2 = \frac{1}{19}(12 + 8x_3) = \frac{1}{19}\left(12 + 8\left(\frac{-32}{15}\right)\right)$$

$$\Rightarrow x_2 = -\frac{4}{15}$$

Lastly, putting x_2, x_3 in $x_1 + 14\,x_2 - 6\,x_3 = 12$ we have

$$x_1 + 14 \times \left(\frac{-4}{15}\right) - 6\left(\frac{-32}{15}\right) = 12$$

$$\Rightarrow x_1 = \frac{44}{15}$$

Therefore, the solution is

$x_1 = \frac{44}{15},\ x_2 = \frac{-4}{15},\ x_3 = \frac{-32}{15}$ Answer

Example 10 : For what values of λ and μ, the equations $x + y + z = 6$; $x + 2y + 3z = 10$; $x + 2y + \lambda z = \mu$ have (i) no solution (ii) unique solution and (iii) infinite solutions. (I.A.S 2006, U.P.T.U. 2002)

Solution : The given system of linear equations can be written as

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \\ \mu \end{bmatrix}$$

i.e. AX = B

$$C = [A :] = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{bmatrix}$$

Applying elementary row transformations to C, we get

$$B \sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & \lambda-1 & \mu-6 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda-3 & \mu-10 \end{bmatrix}, \quad R_3 \to R_3 - R_1$$

(i) For no solution, we must have

$r(A) \neq r(C)$

i.e $\lambda - 3 = 0$ or $\lambda = 3$ and $\mu - 10 \neq 0 \Rightarrow \mu \neq 10$

(ii) for unique solution, we must have

$r(A) = r(C) = 3$

i.e. $\lambda - 3 \neq 0 \Rightarrow \lambda \neq 3$

and $\mu - 10 \neq 0 \Rightarrow \mu \neq 10$

(iii) for infinite solutions, we must have

$r(A) = r(C) < 3$

i.e. $\lambda - 3 = 0 \Rightarrow \lambda = 3$

and $\mu - 10 = 0 \Rightarrow \mu = 10$ Answer.

Solution of Homogeneous system of Linear - Equations :

A system of linear equations of the form AX = 0 is said to be homogeneous where A denotes the coefficient matrix and O denotes the null vector i.e.

$a_{11}x_1 + a_{12}x_2 + \dots\dots\dots + a_{1n}x_n = 0$

$a_{21}x_1 + a_{22}x_2 + \dots\dots\dots + a_{2n}x_n = 0$

$a_{m_1}x_1 + a_{m_2}x_2 + \dots\dots\dots + a_{mn}x_n = 0$

i.e. AX = O

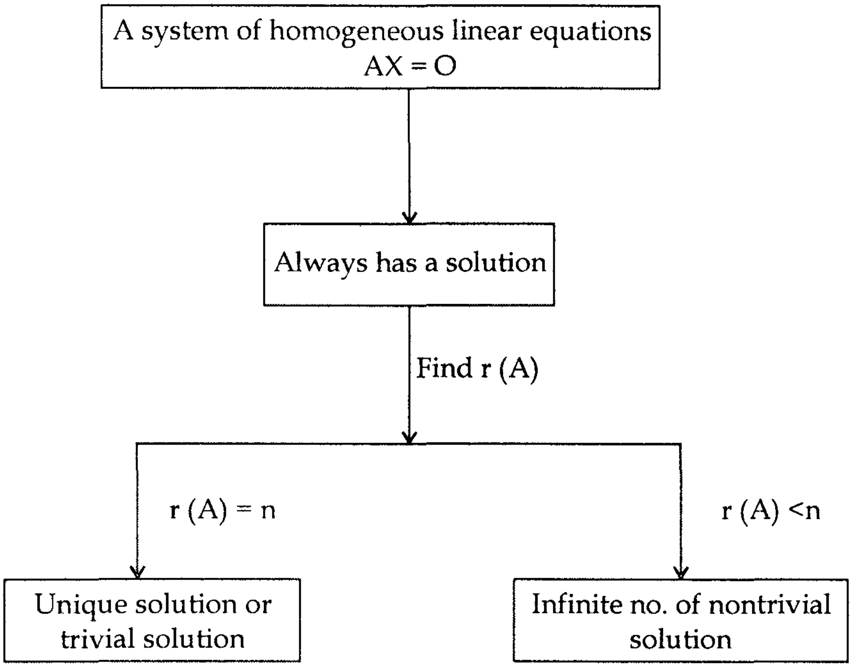
$$\text{or} \begin{bmatrix} a_{11} & a_{12}\dots\dots a_{1n} \\ a_{21} & a_{22}\dots\dots a_{21} \\ & \vdots \\ a_{m1} & a_{m2}\dots\dots a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

The above system has m equations and n unknowns. We will apply the matrix method to find the solution of the above system of linear equations. For the system $AX = O$, we see that $X = O$ is always a solution. This solution is called null solution or trivial solution. Thus a homogeneous system is always consistent. We will apply the techniques, already developed for non homogenous systems of linear equations to homogeneous linear equations.

(i) If r (A) = n (number of unknown) the system has only trivial solution.

(ii) If r (A) <n, the system has infinite number of solutions.

The figure as given below shows a flow chart which depicts the procedure for the solution of a homogenous system of linear equations.



Example 11 : Find the solution of the following homogeneous system of linear equations

$x_1 + x_2 + 2x_3 + 3x_4 = 0$; $3x_1 + 4x_2 + 7x_3 + 10x_4 = 0$

$5x_1 + 7x_2 + 11x_3 + 17x_4 = 0$; $6x_1 + 8x_2 + 13x_3 + 16x_4 = 0$

Solution : The given system of linear equations can be written in matrix form as

$$\begin{bmatrix} 1 & 1 & 2 & 3 \\ 3 & 4 & 7 & 10 \\ 5 & 7 & 11 & 17 \\ 6 & 8 & 13 & 16 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Applying elementary row transformations, we get

$$\begin{bmatrix}1 & 1 & 2 & 3\\0 & 1 & 1 & 1\\0 & 2 & 1 & 2\\0 & 2 & 1 & -2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix} \quad \begin{array}{l}R_2 \to R_2 - 3R_1\\R_3 \to R_3 - 5R_1\\R_4 \to R_4 - 6R_1\end{array}$$

$$\sim\begin{bmatrix}1 & 1 & 2 & 3\\0 & 1 & 1 & 0\\0 & 0 & -1 & 0\\0 & 0 & 0 & -4\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix} \quad \begin{array}{l}R_3 \to R_3 - 2R_2\\R_4 \to R_4 - R_3\end{array}$$

i.e. $r(A) = 4$ = number of variables

Hence the given system of homogeneous linear equations has trivial solution i.e. $x_1 = 0 = x_2 = x_3 = x_4$ Answer.

Linear Combination of vectors :

Let x_1, x_2 ....x_k be a set of k vectors in R^n. Then the linear combination of these k vectors is sum of the form $\alpha_1x_1 + \alpha_2 x_2 +$ ....$+ \alpha_k x_k$, in which α_j is a real number.

Linear Dependence and Independence Vectors :

Let x_1, x_2 ....x_k be a set of k vectors in R^n. Then the set is said to be linearly dependent if and only if one of the k vectors can be expressed as a linear combination of the remaining k vectors.

If the given set of vectors is not linearly dependent, it is said to be set of liearly independent vectors.

Example 12 : Examine for linear dependence (1, 0, 3, 1), (0, 1, -6, -1) and (0, 2, 1, 0) in R^4.

Solution : Consider the matrix equation $\alpha_1x_1 + \alpha_2 x_2 + \alpha_3 x_3 = 0$

i.e $\alpha_1(1, 0, 3, 1) + \alpha_2 (0, 1, -6, -1) + \alpha_3 (0, 2, 1, 0) =0$

$\Rightarrow (\alpha_1 + 0\alpha_2 + 0\,\alpha_3, 0\,\alpha_1 + \alpha_2 + 2\,\alpha_3, 3\,\alpha_1 - 6\,\alpha_1 - 6\,\alpha_2 + \alpha_3, \alpha_1 - \alpha_2 + 0\alpha_3) =0$

$\Rightarrow \alpha_1 =0$

$\alpha_2 + 2\alpha_3 =0$

$3\alpha_1 - 6\alpha_2 + \alpha_3 =0$

and $\alpha_1 - \alpha_2 =0$

$$\text{i.e}\begin{bmatrix}1 & 0 & 0\\0 & 1 & 2\\3 & -6 & 1\\1 & -1 & 0\end{bmatrix}\begin{bmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & -6 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{matrix} R_3 \to R_3 - 3R_1 \\ R_4 \to R_4 - R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 13 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{matrix} R_3 \to R_3 + 6R_2 \\ R_4 \to R_4 + R_2 \end{matrix}$$

i.e. $\alpha_1 = 0, \alpha_2 + 2\alpha_2 = 0 \Rightarrow \alpha_2 = 0$
$13\alpha_3 = 0 \Rightarrow \alpha_3 = 0$
i.e. $\alpha_1 = 0 = \alpha_2 = \alpha_3$
Thus, the given vectors are linearly independent. Answer.

Example 13 : Examine the following vectors for linear dependence and find the relation if it exists.
$X_1 = (1, 2, 4), X_2 = (2, -1, 3), X_3 = (0, 1, 2)$ and $X_4 = (-3, 7, 2)$

(U.P.T.U. 2002)

Solution : The linear combination of the given vectors can be written in matrix equations as
$\alpha_1 X_1 + \alpha_2 X_2 + \alpha_3 X_3 + \alpha_4 X_4 = 0$
$\Rightarrow \alpha_1 (1, 2, 4) + \alpha_2 (2, -1, 3) + \alpha_3 (0, 1, 2) + \alpha_4 (-3, 7, 2) = 0$
$\Rightarrow (\alpha_1 + 2\alpha_2 + 0\alpha_3 - 3\alpha_4, 2\alpha_1 - \alpha_2 + \alpha_3 + 7\alpha_4, 4\alpha_1 + 3\alpha_2 + 2\alpha_3 + 2\alpha_4) = 0$
$\Rightarrow \alpha_1 + 2\alpha_2 + 0\alpha_3 - 3\alpha_4 = 0$
$2\alpha_1 - \alpha_2 + \alpha_3 + 7\alpha_4 = 0$
$4\alpha_1 + 3\alpha_2 + 2\alpha_3 + 2\alpha_4 = 0$
This is a homogenous system i.e.

$$\begin{bmatrix} 1 & 2 & 0 & -3 \\ 2 & -1 & 1 & 7 \\ 4 & 3 & 2 & 2 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Applying elementary row transformations we have

$$\sim \begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & -5 & 1 & 13 \\ 0 & -5 & 2 & 14 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 4R_1 \end{matrix}$$

$$\sim\begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & -5 & 1 & 13 \\ 0 & 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, R_3 \to R_3 - R_2$$

Hence the given vectors are linearly independent

i.e. $\alpha_1 + 2\alpha_2 - 3\alpha_4 = 0$

$-5\alpha_2 + \alpha_3 + 13\alpha_4 = 0$

$\alpha_3 + \alpha_4 = 0$

putting $\alpha_4 = k$ in $\alpha_3 + \alpha_4 = 0$ we get $\alpha_3 = -k$

$-5\alpha_2 - k + 13k = 0$

i.e. $\alpha_2 = \frac{12}{5}k$ and $\alpha_1 + 2 \times \frac{12}{5}k - 3k = 0$

$\Rightarrow \alpha_1 = -\frac{9}{5}k$

Hence the given vectors are linearly dependent substituting the values of α in $\alpha_1 X_1 + \alpha_2 X_2 + \alpha_2 X_2 + \alpha_4 X_4 = 0$ we get

$$\frac{-9k}{5}X_1 + \frac{12k}{5} \times 2 - kX_3 + kX_4 = 0$$

or $9X_1 - 12X_2 + 5X_3 - 5X_4 = 0$

Characteristic Equation and Roots of a Matrix :

Let $A = [a_{ij}]$ be an $n \times n$ matrix,

(i) Characteristic matrix of A : - The matrix $A - \lambda I$ is called the characteristic matrix of A, where I is the identity matrix.

(ii) Characteristic polynomial of A : The determinant $|A - \lambda I|$ is called the Characteristic polynormial of A.

(iii) Characteristic equation of A : The equation $|A - \lambda I| = 0$ is known as the characteristic equation of A and its roots are called the characteristic roots or latent roots or eigenvalues or characteristic values or latent values or poper values of A.

The Cayley - Hamilton Theorem : Every square matrix satisfies its characteristic equation i.e if for a square matrix A of order n,

$|A - \lambda I| = (-1)^n [\lambda^n + a_1 \lambda^{n-1} + a_2 \lambda^{n-2} + \dots\dots\dots\dots + a_n]$

Then the matrix equation

$X^n + a_1 X^{n-1} + a_2 X^{n-2} + a_3 X^{n-3} + \dots\dots\dots\dots a_n I = 0$

is satisfied by $X = A$

i.e. $A^n + a_1 A^{n-1} + a_2 A^{n-2} + \dots\dots\dots\dots + a_1 I = 0$

(U.P.P.C.S. 2002; B.P.Sc 1997)

Proof : Since the element of A $-\lambda I$ are at most of the first degree in λ, the elements of Adj $(A - \lambda I)$ are ordinary polynomials in λ of degree n -1 or less.
Therefore Adj $(A - \lambda I)$ can be written as a matrix polynomial in λ, given by
Adj $(A - \lambda I) = B_0\lambda^{n-1} + B_1\lambda^{n-2} + \ldots\ldots\ldots + B_{n-2}\lambda + B_{n-1}$ where $B_0, B_1 \ldots\ldots\ldots B_{n-1}$ are matrices of the type $n \times n$ whose elements are functions of a_{ij}, s
Now $(A - \lambda I)$ adj. $(A - \lambda I) = |A - \lambda I| I$
$\therefore$ A adj A = $|A|\ I_n$
$\therefore (A - \lambda I)(B_0\lambda^{n-1} + B_1\lambda^{n-2} + \ldots\ldots\ldots + B_{n-2}\lambda + B_{n-1})$
$= (-1)^n [\lambda^n + a_1\lambda^{n-1} + \ldots\ldots\ldots + a_n] I$
Comparing coefficients of like powers of λ on both sides, we get
$- I B_0 = (-1)^n I$
$AB_0 - IB_1 = (-1)^n a_1 I$
$AB_1 - IB_2 = (-1)^n a_1 I$
....
....
$AB_{n-1} = (-1)^n a_n I$
Premultiplying these successively by $A^n, A^{n-1} \ldots\ldots\ldots I$ and adding we get
$0 = (-1)^n [A^n + a_1 A^{n-1} + a_2 A^{n-2} + \ldots\ldots\ldots + a_n I]$
Thus,
$A^n + a_1 A^{n-1} + a_2 A^{n-2} + \ldots\ldots\ldots + a_{n-1} A + a_n I = 0$ Proved.

Example 14 : Find the characteristic equation of the matrix

$$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

and verfy that it is satisfied by A and hence obtain A^{-1}.

(U.P.P.C.S1997; U.P.T.U. 2005)

Solution : We have

$$|A - \lambda I| = \begin{bmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{bmatrix}$$

$= (2 - \lambda) \{ (2 - \lambda)^2 - 1\} + 1\{-1(2 - \lambda) + 1\} + \{ 1 - (2 - \lambda)\}$
$= (2 - \lambda)(3 - 4\lambda + \lambda^2) + (\lambda - 1) + (\lambda - 1)$
$= -\lambda^3 + 6\lambda^2 - 9\lambda + 4$
we are now to verify that
$A^3 - 6A^2 + 9A - 4I = 0$ (i)
we have

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$A^2 = A \times A \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$$

$$A^3 = A^2 \times A \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix}$$

Now we can verify that $A^3 - 6A^2 + 9A - 4I$

$$= \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix} - 6 \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} + 9 \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} - 4 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Multiplying (i) by A^{-1}, we get

$A^2 - 6A + 9I - 4A^{-1} = 0$

$\therefore A^{-1} = \frac{1}{4}(A^2 - 6A + 9I)$

Now $A^2 - 6A + 9I$

$$= \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} + \begin{bmatrix} -12 & 6 & -6 \\ 6 & -12 & 6 \\ -6 & 6 & -12 \end{bmatrix} + \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{4} \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix} \quad \text{Answer.}$$

Example 15 : Use Cayley - Hamilton theroem to find the inverse of the following matrix

$$A = \begin{bmatrix} 4 & 3 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 1 \end{bmatrix}$$

and hence deduce the value $A^5 - 6A^4 + 6A^3 - 11A^2 + 2A + 3$

(U.P.T.U. 2002)

Solution : The characteristic equation of the given matrix is

$|A - \lambda I| = 0$

$$\begin{bmatrix} 4-\lambda & 3 & 1 \\ 2 & 1-\lambda & -2 \\ 1 & 2 & 1-\lambda \end{bmatrix} = 0$$

i.e. $(4 - \lambda)\,[(1 - \lambda)^2 + 4] - 3\,[2\,(1 - \lambda) + 2] + 1[4 - (1 - \lambda)] = 0$

or $\lambda^3 - 6\lambda^2 + 6\lambda - 11 = 0$

By Cayley - Hamilton theorem we have

$A^3 - 6A^2 + 6A - 11\,I = 0$

Multiplying by A^{-1}, we have

$A^{-1}A^3 - 6A^{-1}A^2 + 6A^{-1}A - 11\,A^{-1}I = 0$

or $A^2 - 6A + 6I - 11A^{-1} = 0$

$11\,A^{-1} = A^2 - 6A + 6I$

$$= \begin{bmatrix} 4 & 3 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} 4 & 3 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 1 \end{bmatrix} - 6\begin{bmatrix} 4 & 3 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 1 \end{bmatrix} + 6\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 23 & 17 & -1 \\ 8 & 3 & -2 \\ 9 & 7 & -2 \end{bmatrix} - \begin{bmatrix} 24 & 18 & 6 \\ 12 & 6 & -12 \\ 6 & 12 & 6 \end{bmatrix} + \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & -1 & -7 \\ -4 & 3 & 10 \\ 3 & -5 & -2 \end{bmatrix}$$

Therefore

$$A^{-1} = \frac{1}{11}\begin{bmatrix} 5 & -1 & -7 \\ -4 & 3 & 10 \\ 3 & -5 & -2 \end{bmatrix}$$

Now

$A^5 - 6A^4 + 6A^3 - 11A^2 + 2A + 3 = A^2\,(A^3 - 6A^2 + 6A - 11I) + 2A + 3I$

$= A^2\,(0) + 2A + 3I$

$$= 2\begin{bmatrix} 4 & 3 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 1 \end{bmatrix} + 3\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 6 & 2 \\ 4 & 2 & -4 \\ 2 & 4 & 2 \end{bmatrix} + \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 11 & 6 & 2 \\ 4 & 5 & -4 \\ 2 & 4 & 5 \end{bmatrix} \quad \text{Answer.}$$

Eigen vectors of a Matrix :

Let A be an n × n square matrix, λ be the scalar called eigen values of A and X be the non zero vectors, then they satisfy the equation

$AX = \lambda X$

or $[A - I\lambda]\, X = 0$

For known values of λ, one can calculate the eigen vectors.

Eigenvectors of matrices have following properties.

1. The eigen vector X of a matrix A is not unique.
2. If $\lambda_1, \lambda_2 \ldots\ldots\ldots\ldots \lambda_n$ be distinct eigen values of n × n matrix then the corresponding eigen vectors $X_1, X_2 \ldots\ldots\ldots\ldots\ldots\ldots X_n$ form a linearly independent set.
3. For two or more eigenvalues, it may or may not be possible to get linearly independent eigenvectors corresponding to the equal roots.
4. Two eigenvectors X_1 and X_2 are orthogonal if $X_1X_2 = 0$
5. Eigenvetors of a symmetric matrix corresponding to different eigenvalues are orthogonal.

In this section we will discuss four cases for finding eigenvectors, namely.

1. Eigen vectors of non-symmetric matrices with non - repeated eigenvalues.
2. Eigenvectors of non-symmetric matrix with repeated eigenvalues.
3. Eigenvectors of symmetric matrices with non - repeated eigenvalues.
4. Eigenvectors of symmetric matrices with repeated eigenvalues.

Example 16 : Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$$

(IAS 1994; U.P.P.C.S 2005; U.P.T.U. (C.O.) 2002)

Solution : The characteristic equation of the given matrix is

$|A - \lambda I| = 0$

$$\begin{vmatrix} 3-\lambda & 1 & 4 \\ 0 & 2-\lambda & 6 \\ 0 & 0 & 5-\lambda \end{vmatrix} = 0$$

or $(3 - \lambda)(2 - \lambda)(5 - \lambda) = 0$

$\therefore \lambda = 3, 2, 5$

Thus the eigenvalues of the given matrix are

$\lambda_1 = 2, \lambda_2 = 3, \lambda_3 = 5$

The eigenvectors of the matrix A corresponding to $\lambda = 2$ is

$[A - \lambda_1 I]\, X = 0$

i.e. $$\begin{bmatrix} 3-2 & 1 & 4 \\ 0 & 2-2 & 6 \\ 0 & 0 & 5-2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 4 \\ 0 & 0 & 6 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

i.e. $x_1 + x_2 + 4x_3 = 0$

$6x_3 = 0 \Rightarrow x_3 = 0$

$x_1 + x_2 = 0$

$\Rightarrow x_1 = -x_2 = k_1$ (say), $k_1 \neq 0$

Thus, the corresponding vector is

$$X_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k_1 \\ -k_1 \\ 0 \end{bmatrix} = k_1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

The eigenvector corresponding to eigenvalue $\lambda_2 = 3$

$[A - \lambda_2 I]\, X = 0$

$$\begin{bmatrix} 3-3 & 1 & 4 \\ 0 & 2-3 & 6 \\ 0 & 0 & 5-3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 4 \\ 0 & -1 & 6 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

i.e. $x_2 + 4x_3 = 0$

$-x_2 + 6x_3 = 0$

and $2x_3 = 0 \Rightarrow x_3 = 0$

i.e. $x_2 = 0 \qquad (\therefore x_3 = 0)$

Now let $x_1 = k_2$, we get the corresponding eigenvector as

$$X_2 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k_2 \\ 0 \\ 0 \end{bmatrix} = k_2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Again when $\lambda = 5$, the eigenvector is given by

$[A - \lambda_3 I] X = 0$

$$\begin{bmatrix} 3-5 & 1 & 4 \\ 0 & 2-5 & 6 \\ 0 & 0 & 5-5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -2 & 1 & 4 \\ 0 & -3 & 6 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

i.e. $-2x_1 + x_2 + 4x_3 = 0$

$-3x_2 + 6x_3 = 0$

or $x_2 = 2x_3 = k_3$ (say), $k_3 \neq 0$

Then

$2x_1 = x_2 + 4x_3 = k_3 + 2k_3$

$= 3k_3$

$$x_1 = \frac{3}{2}k_3$$

Thus, the corresponding vector is

$$X_3 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{3}{2}k_3 \\ k_3 \\ \frac{1}{2}k_3 \end{bmatrix} = \frac{1}{2}k_3 \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

Example 17 : Find the characteristic equation of the matrix

$$\begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 1 \end{bmatrix}$$

(U.P.P.C.S 2000; U.P.T.U. 2007)

Also find eigenvalues and eigenvectors of this matrix.

Solution :

The characteristic equation of the matrix is

$|A - \lambda I| X = 0$

$$\begin{bmatrix} 1-\lambda & 2 & 2 \\ 0 & 2-\lambda & 1 \\ -1 & 2 & 2-\lambda \end{bmatrix} = 0$$

$\Rightarrow \lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$

$\Rightarrow (\lambda - 1)(\lambda^2 - 4\lambda + 4) = 0$

or $\lambda = 1, 2, 2$

The eigenvectors corresponding to $\lambda_1 = 1$ is

$[A - 1I] X = 0$

$$\begin{bmatrix} 1-1 & 2 & 2 \\ 0 & 2-1 & 1 \\ -1 & 2 & 2-1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 2 & 2 \\ 0 & 1 & 1 \\ -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$2x_2 + 2x_3 = 0$

$x_2 + x_3 = 0 \Rightarrow x_2 = -x_3 = k_1$ (say), $K_1 \neq 0$

$-x_1 + 2x_2 + x_3 = 0$

$x_1 = 2x_2 + x_3 = 2k_1 - k_1 = k_1$

Hence, the required eigenvector is

$$X_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k_1 \\ k_1 \\ -k_1 \end{bmatrix} = k_1 \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

Now, for the eigenvector corresponding to $\lambda_2 = 2$

$[A - 2I] X = 0$

$$\begin{bmatrix} 1-2 & 2 & 2 \\ 0 & 2-2 & 1 \\ -1 & 2 & 2-2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 2 & 2 \\ 0 & 0 & 1 \\ -1 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

i.e. $-x_1 + 2x_2 + 2x_3 = 0$

$x_3 = 0$

$-x_1 + 2x_2 = 0$

$x_1 = 2x_2 = k_2$ (say), $k_2 \neq 0$

$x_1 = k_2,\ x_2 = \frac{1}{2}k_2,\ x_3 = 0$

Hence, the corresponding vector is

$$X_2 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k_2 \\ \frac{1}{2}k_2 \\ 0 \end{bmatrix} = \frac{1}{2}k_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

Diagonalization of Matrices :

We have referred to the elements a_{ii} of a square matrix as its main diagonal elements. All other elements are called off- diagonal elements.

Diagonal Matrix : - Definition

A square matrix having all off - diagonal elements equal to zero is called a diagonal matrix.

We often write a diagonal matrix having main diagonal elements d_1, d_2................d_n as

$$\begin{bmatrix} d_1 & & O \\ & d_2 & \\ O & \cdot & d_n \end{bmatrix}$$

with O in the upper right and lower left corners to indicate that all off - diagonal elements are zero.

Diagonalizable Matrix : An $n \times n$ matrix A is diagonalizable if there exists an $n \times n$ matrix P such that $P^{-1}AP$ is a diagonal matrix.

When such P exists, we say that P is diagonalizes A.

Example 8 : Diagonalize the following matrix

$$A = \begin{bmatrix} -1 & 4 \\ 0 & 3 \end{bmatrix}$$

Solution : The eigenvalues of the given matrix are -1 and 3, and corresponding eigenvectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ respectively

From

$$P = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Because the eigenvectors are linearly independent, this matrix is non singular (as $|P| \neq 0$), we find that

$$P^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

Now compute

$$P^{-1}AP = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 4 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}$$

which has the eigenvalues down the main diagonal, corresponding to the order in which the eigentvectors were written as column of P.

If we use the other order in writing the eigenvectors as columns and define

$$Q = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

then we get

$$Q^{-1}AQ = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$$

Example 19 : Diagonalize the matrix

$$\begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

(U.P.T.U. 2006)

Solution : The characteristic equation of the given matrix is $|A - \lambda I| = 0$ i.e.

$$\begin{bmatrix} 1-\lambda & 6 & 1 \\ 1 & 2-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{bmatrix} = 0$$

$\lambda = 3, 4, -1$

Now, eigenvectors corresponding to

$\Rightarrow \lambda = -1$ is

$[A - \lambda_1 I]\, X_1 = 0$

$$\text{i.e.} \begin{bmatrix} 1+1 & 6 & 1 \\ 1 & 2+1 & 0 \\ 0 & 0 & 3+1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 & 6 & 1 \\ 1 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

i.e. $2x_1 + 6x_2 + x_3 = 0$

$x_1 + 3x_2 = 0$

$4x_3 = 0 \Rightarrow x_3 = 0$

$x_1 = -3x_2$

suppose $x_2 = k$, then $x_1 = -3k$, $k \neq 0$

The eigenvector is

$$X_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3k \\ k \\ 0 \end{bmatrix} = k\begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$$

Eigenvector corresponding to $\lambda_2 = 3$ is

$[A - \lambda_2 I]X_2 = 0$

$$\text{i.e.} \begin{bmatrix} 1-3 & 6 & 1 \\ 1 & 2-3 & 0 \\ 0 & 0 & 3-3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -2 & 6 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

i.e. $-2x_1 + 6x_2 + x_3 = 0$

$x_1 - x_2 = 0$ $x_1 = x_2 = k$ (say), $k \neq 0$

$x_3 = 2x_1 - 6x_2 = 2k - 6k = -4k$

The eigenvector is

$$X_2 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k \\ k \\ -4k \end{bmatrix} = k\begin{bmatrix} 1 \\ 1 \\ -4 \end{bmatrix}$$

Eigenvector corresponding to $\lambda_3 = 4$ is

$[A - \lambda_3 I] X_3 = 0$

$$\begin{bmatrix} 1-4 & 6 & 1 \\ 1 & 2-4 & 0 \\ 0 & 0 & 3-4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -3 & 6 & 1 \\ 1 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore -3x_1 + 6x_2 + x_3 = 0$

$x_1 - 2x_2 = 0 \Rightarrow x_1 = 2x_2$

$-x_3 = 0 \Rightarrow x_3 = 0$

Let $x_2 = k$ then $x_1 = 2k$, Thus, the eigenvector is

$$X_3 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2k \\ k \\ k \end{bmatrix} = k\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

Thus, modal matrix P is

$$P = \begin{bmatrix} -3 & 1 & 2 \\ 1 & 1 & 1 \\ 0 & -4 & 0 \end{bmatrix}$$

Now $P^{-1} = \dfrac{1}{20}\begin{bmatrix} 4 & -8 & -1 \\ 0 & 0 & 5 \\ -4 & -12 & -4 \end{bmatrix}$

For diagonalization

$D = P^{-1}AP$

$$= -\frac{1}{20}\begin{bmatrix} 4 & -8 & -1 \\ 0 & 0 & 5 \\ -4 & -12 & -4 \end{bmatrix}\begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} -3 & 1 & 2 \\ 1 & 1 & 1 \\ 0 & -4 & 0 \end{bmatrix}$$

$$= -\frac{1}{20}\begin{bmatrix} 4-8+0 & 24-16+0 & 4+0-3 \\ 0+0+0 & 0+0+0 & 0+0+15 \\ -4-12+0 & -24-24+0 & -4+0-12 \end{bmatrix}\begin{bmatrix} -3 & 1 & 2 \\ 1 & 1 & 1 \\ 0 & -4 & 0 \end{bmatrix}$$

$$= -\frac{1}{20}\begin{bmatrix} -4 & 8 & 1 \\ 0 & 0 & 15 \\ -16 & -48 & -16 \end{bmatrix}\begin{bmatrix} -3 & 1 & 2 \\ 1 & 1 & 1 \\ 0 & -4 & 0 \end{bmatrix}$$

$$= -\frac{1}{20}\begin{bmatrix} 12+8+0 & -4+8-4 & -8+8+0 \\ 0+0+0 & 0+0-60 & 0+0+\ 0 \\ 48-48 & -16-48+64 & -32-48+0 \end{bmatrix}$$

$$= -\frac{1}{20}\begin{bmatrix} 20 & 0 & 0 \\ 0 & -60 & 0 \\ 0 & 0 & -80 \end{bmatrix}\begin{bmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

D = dia (-1, 3, 4) Answer.

Complex Matrices :

A matrix is said to be complex if its elements are complex number. For example

$$A = \begin{bmatrix} 2+3i & 4i \\ 2 & -i \end{bmatrix}$$

is a complex matrix.

Unitary Matrix :

A square matrix A is said to be unitary if

$A^{\theta}A = I$

Where $A^{\theta} = (\overline{A})'$ i.e transpose of the complex conjugate matrix.

Example 20 : Show that the matrix

$\frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$ is unitary

(U.P.T.U. 2002)

Solution :

$$A = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix} \text{and } A' = \begin{bmatrix} 1 & 1-i \\ 1+i & -1 \end{bmatrix}$$

$$(\overline{A})' = A^{\theta} = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$$

$$A^{\theta}A = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix} \times \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$$

$$= \frac{1}{\sqrt{3}}\begin{bmatrix} 1.1+(1+i).(1-i) & 1(1+i)+(1+i).1 \\ (1-i).1+(-1)(1-i) & (1-i)(1+i)+(-1)(-1) \end{bmatrix}$$

$$= \frac{1}{\sqrt{3}}\begin{bmatrix} 1+1-i^2 & 0 \\ 0 & 1-i^2+1 \end{bmatrix}$$

$$= \frac{1}{\sqrt{3}}\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$\Rightarrow$ A is unitary matrix. Proved

Hermitian matrix :

A square matrix A is said to be a Hermitian matrix if the transpose of the conjugate matrix is equal to the matrix itself i.e

$A^{\theta} = A \Rightarrow \overline{a}_{ij} = a_{ji}$

where $A = [a_{ij}]_{n \times n}$; $a_{ij} \in C$

For example

$$\begin{bmatrix} a & b+ic \\ b-ic & d \end{bmatrix}, \begin{bmatrix} 1 & 2-3i & 3+4i \\ 2+3i & 0 & 4-5i \\ 3-4i & 4+5i & 2 \end{bmatrix}$$

are Hermitian matrices.
if A is a Hermitian matrix, then
$\bar{a}_{ii} = a_{ii} \Rightarrow \alpha - i\beta = \alpha + i\beta$
$\Rightarrow 2i\beta = 0$
$\Rightarrow \beta = 0$
$\therefore a_{ii} = \alpha + i(0)$
$a_{ii} = \alpha$
= which is purely real
Thus every diagonal element of Hermitian Matrix must be real.

(U.P.P.C.S 2005)

Skew - Hermitian Matrix :

A square matrix A is said to be skew - Hermitian if $A^{\theta} = -A \Rightarrow \bar{a}_{ij} = -a_{ij}$
For principal diagonal
$j = i$
$\Rightarrow \bar{a}_{ii} = -a_{ii}$
$\Rightarrow \bar{a}_{ii} + a_{ii} = 0$
$\Rightarrow$ realpart of $a_{ii} = 0$
$\Rightarrow$ diagonal elements are parely imaginary
Thus the diagonal elements of a skew - Hermitian matrix must be pure imaginary numbers or zero.
For example

$$\begin{bmatrix} 0 & -2-i \\ 2-i & 0 \end{bmatrix}, \begin{bmatrix} -i & 3+4i \\ -3+4i & 0 \end{bmatrix}$$

are Skew - Hermitian matrices.

Problem set

Exercise

1. Using elementary transformations find the inverse of matrix A where

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$$

[U.P.T.U. (C.O.) 2007]

$$\text{Ans}.\ A^{-1} = \begin{bmatrix} -1/4 & 3/4 & 0 \\ 3/4 & -1/4 & 0 \\ -1/4 & -1/4 & 1 \end{bmatrix}$$

2. Using elementary transformations, find the inverse of the matrix A where

$$A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$$

Ans. $A^{-1} = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

3. Using elementary transformations, find the inverse of the matrix A where

$$A = \begin{bmatrix} i & -1 & 2i \\ 2 & 0 & 2 \\ -1 & 0 & 1 \end{bmatrix}$$

Ans. $\begin{bmatrix} 0 & 1/4 & -i/2 \\ -1 & (3/4)i & i/2 \\ 0 & 1/4 & 1/2 \end{bmatrix}$

4. Using elementary transformations, find the inverse of matrix A where

$$A = \begin{bmatrix} 2 & 5 & 3 & 3 \\ 2 & 3 & 3 & 4 \\ 3 & 6 & 3 & 2 \\ 4 & 12 & 0 & 8 \end{bmatrix}$$

Ans. $A^{-1} = \frac{1}{48}\begin{bmatrix} -144 & 36 & 60 & 21 \\ 48 & -20 & -12 & -5 \\ 48 & -4 & -12 & -13 \\ 0 & 12 & -12 & 3 \end{bmatrix}$

5. Find the rank of the following matrix by reducing it to normal form.

(i) $A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 4 & 1 & 2 & 1 \\ 3 & -1 & 1 & 2 \\ 1 & 2 & 0 & 1 \end{bmatrix}$

(U.P.T.U. (C.O) 2002)

(ii) $A = \begin{bmatrix} 0 & 1 & 2 & -2 \\ 4 & 0 & 2 & 6 \\ 2 & 1 & 3 & 1 \end{bmatrix}$

(U.P.T.U. (C.O.) 2007)

(iii) $\begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$

(U.P.T.U. 2006)

(iv) Find the rank of the matrix, $A = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix}$

(U.P.T.U. 2000, 2003)

Ans. (i) 3 (ii) 2 (iii) 3 (iv) 2

6. If $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$, find two non - singular matrices P and Q such that PAQ = I Hence find A^{-1}.

(U.P.T.U. 2002)

Ans. $P = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ -2 & 3 & -3 \end{bmatrix}$ and $Q \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

and $A^{-1} = QP = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$

7. Find the non singular matrices P and Q such that PAQ in the normal form for the matrices below.

(i) $\begin{bmatrix} 1 & 2 & 3 & -2 \\ 2 & -2 & 1 & 3 \\ 3 & 0 & 4 & 1 \end{bmatrix}$

(ii) $\begin{bmatrix} 1 & -3 & 1 & 2 \\ 0 & 1 & 2 & 3 \\ 3 & 4 & 1 & -2 \end{bmatrix}$

Ans. (i) $P=\begin{bmatrix}1 & 0 & 0\\ -2 & 1 & 0\\ -1 & -1 & 1\end{bmatrix}$, $Q=\begin{bmatrix}1 & 1/3 & 4/3 & -1/3\\ 0 & -1/6 & -5/6 & 7/6\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$

(ii) $P=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 3/28 & 13/28 & -1/28\end{bmatrix}$

$Q=\begin{bmatrix}1 & 3 & -7 & 21/28\\ 0 & 1 & -2 & 10/28\\ 0 & 0 & 1 & -47/28\\ 0 & 0 & 0 & 1\end{bmatrix}$

8. Use elementary transformation to reduce the following matrix A to upper triangular form and hence find the rank A where

$A=\begin{bmatrix}2 & 3 & -1 & -1\\ 1 & -1 & -2 & -4\\ 3 & 1 & 3 & -2\\ 6 & 3 & 0 & -7\end{bmatrix}$ (U.P.T.U. 2005)

Ans. 3

9. Check the consistency of the following system of linear non-homogenous equations and find the solution, if it exists.

$7x_1 + 2x_2 + 3x_3 = 16$; $2x_1 + 11x_2 + 5x_3 = 25$, $x_1 + 3x_2 + 4x_3 = 13$

(U.P.T.U. 2008)

Ans. $x_1 = \frac{95}{91}$, $x_2 = \frac{100}{91}$, $x_3 = \frac{197}{91}$

10. For what values of λ and μ, the following system of equations

$2x + 3y + 5z = 9$, $7x + 3y - 2z = 8$, $2x + 3y + \lambda z = \mu$

will have (i) unique solution and (ii) no solution

Ans. (i) $\lambda \neq 5$ (ii) $\mu \neq 9, \lambda = 5$

11. Determine the values of λ and μ for which the following system of equations

$3x - 2y + z = \mu$, $5x - 8y + 9z = 3$, $2x + y + \lambda z = -1$ has

(i) Unique solution (ii) No solution and

(iii) Infinite solutions.

Ans. (i) $\lambda \neq -3$ (ii) $\lambda = -3$, $\mu \neq \frac{1}{3}$ (iii) $\lambda = -3$, $\mu = 1/3$

12. Verify the Cayley - Hamilton theorem for the following matrices and also find its inverse using this theorem

(i) $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$

(U.P.T.U. 2007)

(ii) $A = \begin{bmatrix} 2 & 2 & 1 \\ 0 & 1 & -1 \\ 3 & -1 & 1 \end{bmatrix}$

(U.P.T.U. (CO) 2007)

Ans. (i) $A^{-1} = \begin{bmatrix} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{bmatrix}$ (ii) $A^{-1} = \frac{1}{9}\begin{bmatrix} 0 & 3 & 3 \\ 3 & 1 & -2 \\ 3 & -8 & -2 \end{bmatrix}$

13. Find the characteristic equation of the symmetric matrix

$$\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -2 & 2 \end{bmatrix}$$

and hence also find A^{-1} by Cayley - Hamilton theorem. Find the value of $A^6 - 6A^5 + 9A^4 - 2A^3 - 12A^2 - 23A - 9I$

(U.P.T.U. 2003, 2004)

$$A^{-1} = \frac{1}{3}\begin{bmatrix} 2 & 0 & -1 \\ 0 & 3 & 1 \\ 0 & 3 & 3 \end{bmatrix}$$

and Value = $\begin{bmatrix} 84 & -102 & 80 \\ -80 & 106 & -80 \\ 102 & -138 & 106 \end{bmatrix}$

14. Find the characteristic equation of the matrix

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$$

verify Cayley-Hamilton theorem and hence evaluate the matrix equation
$A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$

(U.P.T.U. 2002)

Ans. $\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$

and $\begin{bmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{bmatrix}$

15. Find the eigenvalues and corresponding eigen vectors of the following

(i) $\begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix}$

(ii) $\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

(iii) $\begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}$

Ans. (i) 1, 1, 7; $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

(ii) 5, –3, –3; $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$

(iii) 2, 2, 3, $\begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} -5 \\ -2 \\ 5 \end{bmatrix}$ Dependent eigenvectors.

16. Diagonalize the given matrix

(i) $\begin{bmatrix} 0 & -1 \\ 4 & 3 \end{bmatrix}$

(ii) $\begin{bmatrix} 5 & 3 \\ 1 & 3 \end{bmatrix}$

(iii) $\begin{bmatrix} 5 & 0 & 0 \\ 1 & 0 & 3 \\ 0 & 0 & -2 \end{bmatrix}$

(iv) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & 1 & 0 \\ 0 & 0 & -3 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}$

Ans. (i) $D = P^{-1}AP = \begin{bmatrix} \frac{3+\sqrt{7}i}{2} & 0 \\ 0 & \frac{3-\sqrt{7}i}{2} \end{bmatrix}$

(ii) $P^{-1}AP = \begin{bmatrix} 2 & 0 \\ 0 & 6 \end{bmatrix}$

(iii) $P^{-1}AP = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

(iv) $P^{-1}AP = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & \frac{-5+\sqrt{5}}{2} & 0 \\ 0 & 0 & 0 & \frac{-5-\sqrt{5}}{2} \end{bmatrix}$

17. Prove that $\begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix}$ is not diagonalizable.

OBJECTIVE PROBLEMS

Each of the following questions has four alternative answers, one of them is correct. Tick mark the correct answer.

1. The rank of the matrix $A = \begin{bmatrix} 1 & -3 & 2 \\ 3 & -9 & 6 \\ -2 & 6 & -4 \end{bmatrix}$ is

(U.P.P.C.S. 1990)

(A) 0 (B) 1
(C) 2 (D) 3

Ans. (B)

2. The equations
$x - y + 2z = 4$
$3x + y + 4z = 6$
$x + y + z = 1$ have

(A) Unique solution (B) Infinite solution
(C) No solution (D) None of these

Ans. (B)

3. If $A = \begin{bmatrix} 5 & 0 & 2 \\ 0 & 1 & 0 \\ -4 & 0 & -1 \end{bmatrix}$ and I be 3×3 unit matrix. If M = I - A, then rank of I - A is

(I.A.S. 1994)

(A) 0 (B) 1
(C) 2 (D) 3

Ans. (B)

4. If $\rho(A)$ denotes rank of a matrix A, then $\rho(AB)$ is

(I.A.S. 1994)

(A) $\rho(A)$ (B) $\rho(B)$
(C) Is less than or equal to min $[\rho(A), \rho(B)]$
(D) $> \min [\rho(A), \rho(B)]$

Ans. (C)

5. If $3x + 2y + z = 0$
$x + 4y + z = 0$
$2x + y + 4z = 0$
be a system of equations then

(I.A.S. 1994)

(A) It is inconsistent
(B) It has only the trival solution $x = 0, y = 0, z = 0$
(C) It can be reduced to a single equation and so a solution does not exist.
(D) The determinant of the matrix of coefficient is zero.

Ans. (B)

6. Consider the Assertion (A) Reason (R) given below :
Assertion (A) the system of linear equations
$x - 4y + 5z = 8$
$3x + 7y - z = 3$
$x + 15y - 11z = -14$ is inconsistent.
Reason (R) Rank $\rho(A)$ of the coefficient matrix of the system is equal to 2, which is less than the number of variables of the system.

(I.A.S. 1993)

The correct answer is -
(A) Both A and R are true and R is the correct explanation of A.
(B) Both A and R are true but R is not a correct explanation of A.
(C) A is true but R is false.
(D) A is false but R is true.

Ans. (B)

7. Consider Assertion (A) and Reason (R) given below :

Assertion (A) : The inverse of $\begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix}$ dose not exist

Reason (R) : The matrix is non sigular.

(I.A.S. 1993)

The correct answer is ______

(A) Both A and R are true and R is the correct explanation of A.
(B) Both A and R are true but R is not a correct explanation of A.
(C) A is true but R is false.
(D) A is false but R is true.

Ans. (D)

8. If A = Diag ($\lambda_1, \lambda_2, \ldots\ldots\ldots\lambda_n$), then the roots of the equation det (A - xI) =0 are___

(I.A.S. 1993)

(A) All equal to 1
(B) All equal to zero
(C) $\lambda_i, 1 \le i \le n$
(D) $-\lambda_i, 1 \le i \le n$

Ans. (C)

9. The matrix $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 5 & 7 \\ 2 & 1 & -1 \end{bmatrix}$ then inverse matrix is given by

(A) $\begin{bmatrix} 3 & 1/2 & 1/2 \\ 4 & 3/4 & -5/4 \\ 2 & 1/4 & -3/4 \end{bmatrix}$

(B) $\begin{bmatrix} 3 & -1/2 & -1/2 \\ -4 & 3/4 & 5/4 \\ 2 & -1/4 & -3/4 \end{bmatrix}$

(C) $\begin{bmatrix} 3 & 1/2 & 1/2 \\ 4 & 3/4 & 5/4 \\ 2 & -1/4 & -3/4 \end{bmatrix}$

(D) $\begin{bmatrix} 3 & 1/2 & 1/2 \\ 4 & 3/4 & 5/4 \\ 2 & 1/4 & 3/4 \end{bmatrix}$

Ans. (B)

10. The rank of the matrix $A = \begin{bmatrix} 3 & 1 & 2 \\ 6 & 2 & 4 \\ 3 & 1 & 2 \end{bmatrix}$ given by

(A) 0 (B) 1
(C) 1 (D) 3

Ans. (B)

11. The value of λ for which the system of equation $x + 2y + 3z = \lambda x$, $3x + y + 2z = \lambda y$, $2x + 3y + z = \lambda x$, have non - trival solution is given by

(A) 1 (B) 2
(C) 4 (D) None of these

Ans. (D)

12. Let the matrix be $A = \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$, which one of the following is true.

(A) A^{-1} exists (B) B^{-1} exist
(C) $AB = BA$ (D) None of these

Ans. (D)

13. The equations

$x + y + z = 3$
$x + 2y + 3z = 4$
$2x + 3y + 4z = 7$

have the solution -

(A) $x = 2, y = 1, z = 1$
(B) $x = 1, y = 2, z = 1$
(C) $x = 3, y = 1, z = 1$
(D) $x = 1, y = 0, z = 3$

Ans. (C)

14. If A and B are square matrices of same order, then which one of the following is true.

(A) $(AB)' = A'B'$ (B) $(AB)^{-1} = A^{-1}B^{-1}$
(C) $(A^{-1})' = (A')^{-1}$ (D) $B'AB = BA'B$

Ans. (C)

15. The inverse of the matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ is

(A) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$ (B) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$

(C) $\begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$ (D) None of these

Ans. (A)

16. The rank of the matrix $\begin{bmatrix} 1 & -2 \\ -2 & 4 \\ -1 & 2 \end{bmatrix}$ is

(A) 0 (B) 1
(C) 2 (D) 3

Ans. (B)

17. Let A be an $n \times n$ matrix from the set of real numbers and $A^3 - 3A^2 + 4A - 6I = 0$
where I is $n \times n$ is unit matrix if A^{-1} exists, then

(I.A.S. 1994)

(A) $A^{-1} = A - I$
(B) $A^{-1} = A + 6I$
(C) $A^{-1} = 3A - 6I$
(D) $A^{-1} = \frac{1}{6}(A^2 - 3A + 4I)$

Ans. (D)

18. Consider the following statements. Assertion (A) : If a 2×2 matrix, commutes with every 2×2 matrix, than it is a scalar matrix.
Reason (R) : A 2×2 scalar matrix commutes with every 2×2 matrix of these statements -

(I.A.S. 1995, 2007)

(A) Both A and R are true and R is the correct explanation of A.
(B) Both A and R are true but R is not a correct explanation of A.
(C) A is true but R is false.
(D) A is false but R is true.

Ans. (B)

19. The inverse of the matrix $\begin{bmatrix} -0.5 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is

(A) $\begin{bmatrix} 0.5 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ (B) $\begin{bmatrix} 0.5 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(C) $\begin{bmatrix} -2 & 0 & 0 \\ 0 & 0.25 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (D) $\begin{bmatrix} 2 & 0 & 0 \\ 0 & -0.25 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

Ans. (D)

20. The system of equation
$x + 2y + z = 9$
$2x + y + 3z = 7$
can be expressed as

(I.A.S. 1995, 2004)

(A) $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 9 \\ 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

(B) $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} 9 \\ 7 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

(C) $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 7 \end{bmatrix}$

(D) None of these

Ans. (D)

21. The points (x_1, y_1), (x_2, y_2), (x_3, y_3) are collinear if the rank of the matrix.
$A = \begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix}$ is

(A) 1 or 2 (B) 2 or 3
(C) 1 or 3 (D) 2

Ans. (A)

22. To convert the Hermitian matrix into Skew Hermitian one, the Hermitian matrix, must be multiplied by

(A) –1 (B) i

(C) –i (D) None of these

Ans. (B)

23. $\begin{bmatrix} 1 & i+1 & 3 \\ -2 & 1 & 5+i \\ \sqrt{2} & 3-i & 0 \end{bmatrix}$ is a 3 × 3 matrix over the set of

(A) Natural numbers (B) Integers

(C) Real numbers (D) Complex numbers

Ans. (D)

24. The system of equations

$x + 2y + 3z = 4$

$2x + 3y + 8z = 7$

$x - y + 9z = 1$ have

(A) Unique solution (B) No solution

(C) Infinite solution (D) None of these

Ans. (C)

25. A system of equation is said to be consistent if there exist..............solutions for the system -

(A) No (B) One

(C) At least one (D) Infinite

Ans. (C)

26. If the determinant of coefficient of the system of homogeneous linear equation is zero, then the system have -

(A) Trivial solution

(B) Non - trivial solution

(C) Infinite solution

(D) None of these

Ans. (B)

27. Who among the following is associated with a technique of solving a system of linear squations ?

(A) Sarrus (B) Cayley

(C) Cramer (D) Hermite

Ans. (C)

28. Matrix $A = \begin{bmatrix} 3 & 2 & 0 \\ 2 & 0 & i \\ 0 & -i & 0 \end{bmatrix}$ is

(A) Unitary (B) Hermitian

(C) Skew-Hermitian (D) None of these

Ans. (B)

29. Given matrix $\begin{bmatrix} -1 & 0 & 3-i \\ 0 & 1 & 0 \\ 3+i & 0 & 0 \end{bmatrix}$ is

(A) Hermitian (B) Non - Hermitian
(C) Unitary (D) None of these

Ans. (A)

30. Matrix $\begin{bmatrix} i & 1 & 0 \\ -1 & 0 & 2i \\ 0 & 2i & 0 \end{bmatrix}$ is

(A) Hermitian (B) Skew-Hermitian
(C) Unitary (D) None of these

Ans. (B)

31. Characteristic roots of the matrix
$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ -\sin\theta & -\cos\theta \end{bmatrix}$ is

(A) $\pm i$ (B) ± 1
(C) ± 2 (D) None of these

Ans. (B)

32. The eigen values of the matrix
$A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$ is

(A) 5, -3, -3 (B) -5, 3, 3
(C) -5, 3, -3 (D) None of these

Ans. (A)

33. The matrix A is defined by
$A = \begin{bmatrix} -1 & 0 & 0 \\ 2 & -3 & 0 \\ 1 & 4 & 2 \end{bmatrix}$ the eigenvalues of A^2 is

(A) -1, -9, -4 (B) 1, 9, 4
(C) -1, -3, 2 (D) 1, 3, -2

Ans. (B)

34. If the matrix is $A = \begin{bmatrix} -1 & 2 & 3 \\ 0 & 3 & 5 \\ 0 & 0 & -2 \end{bmatrix}$ then the eigenvalues of $A^3 + 5A + 8I$ are

(A) -1, 27, -8 (B) - 1, 3, -2
(C) 2, 50, -10 (D) 2, 50, 10

Ans. (C)

35. The eigenvalue of a matrix A are 1, -2, 3 the eigenvalues of $3I - 2A + A^2$ are

(A) 2, 11, 6 (B) 3, 11, 18
(C) 2, 3, 6 (D) 6, 3, 11

Ans. (A)

36. The matrix $\begin{bmatrix} 3i & 0 & 0 \\ -1 & 0 & i \\ 0 & -i & 0 \end{bmatrix}$ is

(A) Unitary (B) Hermitian
(C) Skew - Hermitian (D) None of these

Ans. (D)