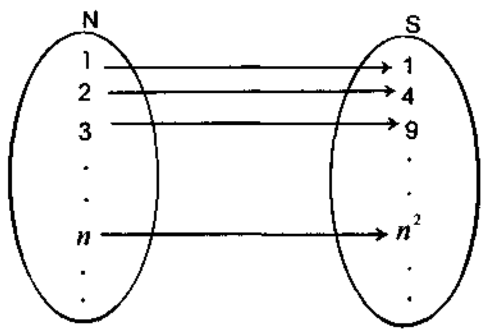
Sequences of Series

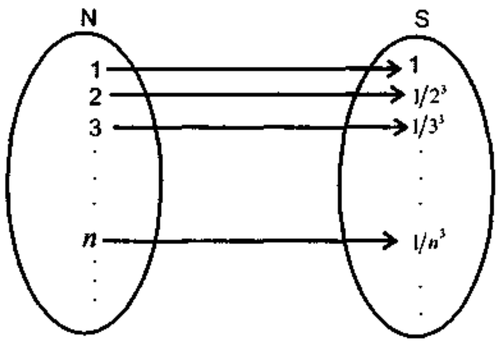
6.0 Sequence

A function f:N $\rightarrow$ S, where S is any nonempty set is called a *Sequence* i.e., for each n$\in$ N, $\exists$ a unique element f(n) $\in$ S. The sequence is written as f(1), f(2), f(3),......f(n)....,and is denoted by {f(n)}, or <f(n)>, or (f(n)). If f(n) $= a_n$,the sequence is written as $a_1, a_2.....a_n$ and denoted by , $\{a_n\}$ $or < a_n > or \left(a_n\right)$. Here f(n) or a_n are the n^{th} terms of the Sequence.

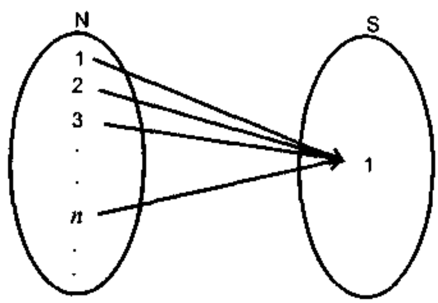
6.1.1 Example : 1 , 4 , 9 , 16 ,......... n^2 ,.....(or) $< n^2 >$



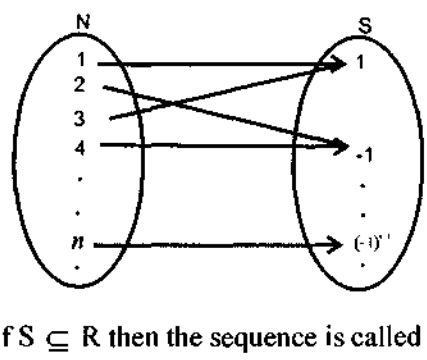
6.1.2 Example : $\frac{1}{1^3}, \frac{1}{2^3}, \frac{1}{3^3}, \ldots\ldots \frac{1}{n^3} \ldots (or) \left(\frac{1}{n^3}\right)$



6.1.3 Example : 1, 1, 1......1..... or <1>



6.1.4 Example : 1 , −1, 1, −1, ... or $\left\langle (-1)^{n-1} \right\rangle$



Note : 1. If S $\subseteq$ R then the sequence is called a *real sequence.*
2. The range of a sequence is almost a countable set.

6.1.5 Kinds of Sequences

1. *Finite Sequence* :A sequence $< a_n >$ in which $a_n = 0 \ \forall n > m \in N$ is said to be a finite Sequence. i.e., A finite Sequence has a finite number of terms.
2. *Infinite Sequence :* A sequence ,which is not finite is an infinite sequence.

6.1.6 Bounds of a Sequence and Bounded Sequence

1. If $\exists$ a number 'M' $\ni a_n \le$ M, $\forall n \in$ N, the Sequence $< a_n >$ is said to be bounded above or bounded on the right.

   Ex: $1, \frac{1}{2}, \frac{1}{3},$,....... here $a_n \le 1 \ \forall n \in \mathrm{N}$

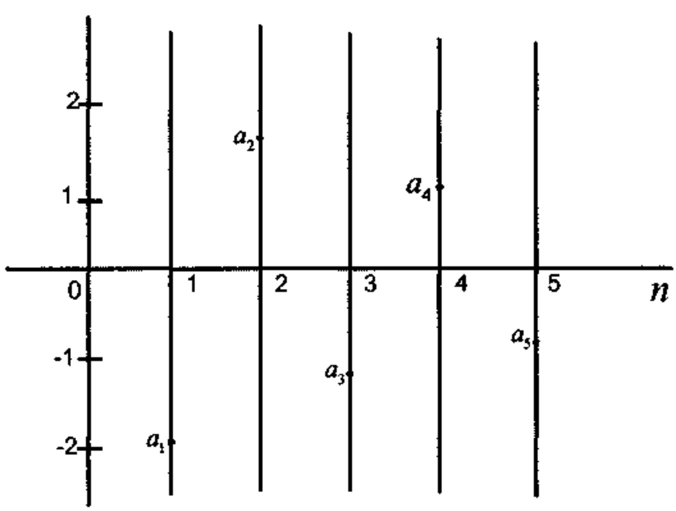
2. If $\exists$ a number 'm' $\ni a_n \ge m, \forall n \in \mathrm{N}$, the sequence $< a_n >$ is said to be bounded below or bounded on the left.

   Ex : 1 , 2 , 3 ,.....here $a_n \ge 1 \ \forall n \in \mathrm{N}$

3. A sequence which is bounded above and below is said to be bounded.

   Ex : Let $a_n = (-1)^n \left(1 + \frac{1}{n}\right)$

n	1	2	3	4	......
a_n	-2	3/2	-4/3	5/4	......



From the above figure (see also table) it can be seen that m = –2 and M = $\frac{3}{2}$.

$\therefore$ The sequence is bounded.

6.1.7 Limits of a Sequence

A Sequence $< a_n >$ is said to tend to limit 'l' when, given any + ve number '$\in$', however small, we can always find an integer 'm' such that $|a_n - l| < \in, \forall n \geq m$, and we write $\underset{n\to\infty}{Lt}\, a_n = l$ or $\langle a_n \to l \rangle$

Ex : If $a_n = \dfrac{n^2+1}{2n^2+3}$ then $< a_n > \to \dfrac{1}{2}$.

6.1.8 Convergent, Divergent and Oscillatory Sequences

1. *Convergent Sequence :* A sequence which tends to a finite limit, say 'l' is called a *Convergent Sequence.* We say that the sequence converges to 'l'
2. *Divergent Sequence :* A sequence which tends to $\pm\infty$ is said to be *Divergent* (or is said to diverge).
3. *Oscillatory Sequence :* A sequence which neither converges nor diverges ,is called an *Oscillatory Sequence.*

Examples

1. Consider the sequence $2, \dfrac{3}{2}, \dfrac{4}{3}, \dfrac{5}{4}, ...$ here $a_n = 1 + \dfrac{1}{n}$

   The sequence $< a_n >$ is convergent and has the limit 1

   $$a_n - 1 = 1 + \frac{1}{n} - 1 = \frac{1}{n} \text{ and } \frac{1}{n} < \in \text{ whenever } n > \frac{1}{\in}$$

   Suppose we choose $\in = .001$, we have $\dfrac{1}{n} < .001$ when n > 1000.

2. If $a_n = 3 + (-1)^n \dfrac{1}{'n'} < a_n >$ converges to 3.
3. If $a_n = n^2 + (-1)^n .n, < a_n >$ diverges.
4. If $a_n = \dfrac{1}{n} + 2(-1)^n, < a_n >$ oscillates between -2 and 2.

6.2 Infinite Series

6.2.1

If $< u_n >$ is a sequence, then the expression $u_1 + u_2 + u_3 + ... + u_n + ...$ is called an infinite series. It is denoted by $\sum_{n=1}^{\infty} u_n$ or simply $\sum u_n$

The sum of the first n terms of the series is denoted by s_n

i.e., $s_n = u_1 + u_2 + u_3 + ... + u_n; s_1, s_2, s_3, ... s_n$ are called *partial sums.*

6.2.2 Convergent, Divergent and Oscillatory Series

Let $\sum u_n$ be an infinite series. As $n \to \infty$, there are three possibilities.

(a) *Convergent series :* As $n \to \infty, s_n \to$ a finite limit, say 's' in which case the series is said to be convergent and 's' is called its sum to infinity. Thus $\underset{n\to\infty}{Lt}\, s_n = s$ (or) simply $Lt s_n = s$

This is also written as $u_1 + u_2 + u_3 + ... + u_n + ...to\infty = s$. (or) $\sum_{n=1}^{\infty} u_n = s$

(or) simply $\sum u_n = s$.

(b) *Divergent series :* If $s_n \to \infty$ or $-\infty$, the series said to be divergent.

(c) *Oscillatory Series:* If s_n does not tend to a unique limit either finite or infinite it is said to be an *Oscillatory Series.*

Note : Divergent or Oscillatory series are sometimes called non convergent series.

6.2.3 Geometric Series

The series, $1 + x + x^2 + ...x^{n-1} + ...$ is

(i) Convergent when $|x| < 1$, and its sum is $\frac{1}{1-x}$

(ii) Divergent when $x \geq 1$.

(iii) Oscillates finitely when $x = -1$ and oscillates infinitely when $x < -1$.

Proof:

The given series is a geometric series with common ratio 'x'

$\therefore \quad s_n = \frac{1-x^n}{1-x}$ when $x \neq 1$ [By actual division – verify]

(i) When $|x| < 1$:

$$\underset{n\to\infty}{Lt}\, s_n = \underset{n\to\infty}{Lt}\left(\frac{1}{1-x}\right) - \underset{n\to\infty}{Lt}\left(\frac{x^n}{1-x}\right) = \frac{1}{1-x}$$

$$\left[\text{since } x^n \to 0 \text{ as } n \to \infty\right]$$

$\therefore$ The series converges to $\frac{1}{1-x}$

(ii) When $x \geq 1 : s_n = \frac{x^n - 1}{x-1}$ and $s_n \to \infty$ as $n \to \infty$

$\therefore$ The series is divergent.

(iii) When $x = -1$: when n is even, $s_n \to 0$ and when n is odd, $s_n \to 1$

$\therefore$ The series oscillates finitely.

(iv) When $x < -1, s_n \to \infty$ or $-\infty$ according as n is odd or even.

$\therefore$ The series oscillates infinitely.

6.2.4 Some Elementary Properties of Infinite Series

1. The convergence or divergence of an infinites series is unaltered by an addition or deletion of a finite number of terms from it.
2. If some or all the terms of a convergent series of positive terms change their signs, the series will still be convergent.
3. Let $\sum u_n$ converge to 's'

   Let 'k' be a non – zero fixed number. Then $\sum ku_n$ converges to ks.

   Also, if $\sum u_n$ diverges or oscillates, so does $\sum ku_n$
4. Let $\sum u_n$ converge to 'l' and $\sum v_n$ converge to 'm'. Then

   (i) $\sum(u_n + v_n)$ converges to ($l + m$) and

   (ii) $\sum(u_n - v_n)$ converges to ($l - m$)

6.2.5 Series of Positive Terms

Consider the series in which all terms beginning from a particular term are +ve . Let the first term from which all terms are +ve be u_1.

Let $\sum u_n$ be such a convergent series of +ve terms. Then, we observe that the convergence is unaltered by any rearrangement of the terms of the series.

6.2.6 Theorem

If $\sum u_n$ is convergent, then $\underset{n\to\infty}{Lt}\, u_n = 0$.

Proof:

$$s_n = u_1 + u_2 + \ldots\ldots + u_n$$

$$s_{n-1} = u_1 + u_2 + \ldots\ldots + u_{n-1}, \text{ so that, } u_n = s_n - s_{n-1}$$

Suppose $\sum u_n = l$ then $\underset{n\to\infty}{Lt}\, s_n = l$ and $\underset{n\to\infty}{Lt}\, s_{n-1} = l$

$$\therefore \quad \underset{n\to\infty}{Lt}\, u_n = \underset{n\to\infty}{Lt}\,(s_n - s_{n-1}) \; ; \quad \underset{n\to\infty}{Lt}\, s_n - \underset{n\to\infty}{Lt}\, s_{n-1} = l - l = 0$$

Note : The converse of the above theorem need not be always true. This can be Observed from the following examples.

(i) Consider the series , $1+\frac{1}{2}+\frac{1}{3}+.......+\frac{1}{n}+....$; $u_n=\frac{1}{n}, \underset{n\to\infty}{Lt}\, u_n=0$

But from p-series test (2.7) it is clear that $\sum\frac{1}{n}$ is divergent.

(ii) Consider the series, $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+.....+\frac{1}{n^2}+......$

$u_n=\frac{1}{n^2}, \underset{n\to\infty}{Lt}\, u_n=0$, by p series test, clearly $\sum\frac{1}{n^2}$ converges,

Note : If $\underset{n\to\infty}{Lt}\, u_n \neq 0$ the series is divergent;

Ex: $u_n=\frac{2^n-1}{2^n}$, here $\underset{n\to\infty}{Lt}\, u_n=1$

$\therefore$ $\sum u_n$ is divergent.

Tests for the Convergence of an Infinite Series

In order to study the nature of any given infinite series of +ve terms regarding convergence or otherwise, a few tests are given below.

6.2.7 P-Series Test

The infinite series, $\sum_{n=1}^{\infty}\frac{1}{n^p}=\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+......$, is

(i) Convergent when $p>1$, and (ii) Divergent when $p\le 1$. [JNTU Dec 2002, A 2003]

Proof :

Case (i) Let $p>1$; $p>1, 3^p>2^p$; $\Rightarrow$ $\frac{1}{3^p}<\frac{1}{2^p}$

$$\therefore \quad \frac{1}{2^p}+\frac{1}{3^p}<\frac{1}{2^p}+\frac{1}{2^p}=\frac{2}{2^p}$$

Similarly, $$\frac{1}{4^p}+\frac{1}{5^p}+\frac{1}{6^p}+\frac{1}{7^p}<\frac{1}{4^p}+\frac{1}{4^p}+\frac{1}{4^p}+\frac{1}{4^p}=\frac{4}{4^p}$$

$$\frac{1}{8^p}+\frac{1}{9^p}+....+\frac{1}{16^p}<\frac{8}{8^p}$$, and so on.

Adding we get

$$\sum\frac{1}{n^p}<1+\frac{2}{2^p}+\frac{4}{4^p}+\frac{8}{8^p}+....$$

i.e., $$\sum\frac{1}{n^p}<1+\frac{1}{2^{(p-1)}}+\frac{1}{2^{2(p-1)}}+\frac{1}{2^{3(p-1)}}+......$$

The RHS of the above inequality is an infinite geometric series with common ratio $\frac{1}{2^{p-1}} < 1 \left(\text{since } p > 1\right)$ The sum of this geometric series is finite.

Hence $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is also finite.

$\therefore$ The given series is convergent.

Case **(ii)** : Let p =1; $\sum \frac{1}{n^p} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + ...$

We have,

$$\frac{1}{3} + \frac{1}{4} > \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

$$\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2}$$

$$\frac{1}{9} + \frac{1}{10} + ...\frac{1}{16} > \frac{1}{16} + \frac{1}{16} + ...\frac{1}{16} = \frac{1}{2} \text{ and so on}$$

$$\therefore \quad \sum \frac{1}{n^p} = 1 + \left(\frac{1}{2} + \frac{1}{3}\right) + \left(\frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7}\right) + ...$$

$$\geq 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + ...$$

The sum of RHS series is ∞ $\left(\text{since } s_n = 1 + \frac{n-1}{2} = \frac{n+1}{2} \text{ and } \underset{n\to\infty}{Lt}\, s_n = \infty\right)$

$\therefore$ The sum of the given series is also ∞; $\therefore$ $\sum_{n=1}^{\infty} \frac{1}{n^p}$ ($p = 1$) diverges.

Case (iii) : Let p<1, $\sum \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + ...$

Since $p < 1, \frac{1}{2^p} > \frac{1}{2}, \frac{1}{3^p} > \frac{1}{3}$...... and so on

$$\therefore \quad \sum \frac{1}{n^p} > 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + ...$$

From the Case (ii), it follows that the series on the RHS of above inequality is divergent.

$\therefore$ $\sum \frac{1}{n^p}$ is divergent, when P < 1

Note : *This theorem is often helpful in discussing the nature of a given infinite series.*

6.2.8 Comparison Tests

1. Let $\sum u_n$ and $\sum v_n$ be two series of +ve terms and let $\sum v_n$ be convergent. Then $\sum u_n$ converges,

   1. If $u_n \le v_n, \forall \; n \in N$

   2. or $\frac{u_n}{v_n} \le k \forall n \in N$ where k is > 0 and finite.

   3. or $\frac{u_n}{v_n} \to$ a finite limit > 0

Proof:

1. Let $\sum v_n = l$ (finite)

   Then, $u_1 + u_2 + ... + u_n + ... \le v_1 + v_2 + ...v_n + ... \le l > 0$

   Since l is finite it follows that $\sum u_n$ is convergent

2. $\frac{u_n}{v_n} \le k \Rightarrow u_n \le kv_n, \forall n \in N$, since $\sum v_n$ is convergent and k (>0) is finite, $\sum kv_n$ is convergent $\therefore$ $\sum u_n$ is convergent.

3. Since $\underset{n\to\infty}{Lt} \frac{u_n}{v_n}$ is finite, we can find a +ve constant $k, \ni \frac{u_n}{v_n} < k \forall n \in N$

   $\therefore$ from (2), it follows that $\sum u_n$ is convergent

2. Let $\sum u_n$ and $\sum v_n$ be two series of +ve terms and let $\sum v_n$ be divergent. Then $\sum u_n$ diverges,

   * 1. If $u_n \ge v_n, \forall n \in N$

   or * 2. If $\frac{u_n}{v_n} \ge k, \forall n \in N$ where k is finite and $\neq 0$

   or * 3. If $\underset{n\to\infty}{Lt} \frac{u_n}{v_n}$ is finite and non-zero.

Proof:

1. Let M be a +ve integer however large it may be . Science $\sum v_n$ is divergent, a number m can be found such that

$$v_1 + v_2 + ... + v_n > M, \forall n > m$$

$$\therefore \quad u_1 + u_2 + ... + u_n > M, \forall n > m\left(u_n \ge v_n\right)$$

$\therefore \quad \sum u_n$ is divergent

2 $\quad u_1 \geq kv_n \forall n$

$\sum v_n$ is divergent $\Rightarrow \sum kv_n$ is divergent

$\therefore \quad \sum u_n$ is divergent

3. Since $\underset{n\to\infty}{Lt} \frac{u_n}{v_n}$ is finite, a + ve constant k can be found such that $\frac{u_n}{v_n} > k, \forall n$

(probably except for a finite number of terms)

$\therefore$ From (2), it follows that $\sum u_n$ is divergent.

Note :

1. In (1) and (2), it is sufficient that the conditions with * hold $\forall n > m \in N$

Alternate form of comparison tests : The above two types of comparison tests 2.8.(1) and 2.8.(2) can be culbed together and stated as follows :

If $\sum u_n$ and $\sum v_n$ are two series of + ve terms such that $\underset{n\to\infty}{Lt} \frac{u_n}{v_n} = k$,

where k is non- zero and finite, then $\sum u_n$ and $\sum v_n$ both converge or both diverge.

Note :

1. The above form of comparison tests is mostly used in solving problems.
2. In order to apply the test in problems, we require a certain series $\sum v_n$ whose nature is already known i.e., we must know whether $\sum v_n$ is convergent are divergent. For this reason, we call $\sum v_n$ as an 'auxiliary series'.
3. In problems, the geometric series (2.3.) and the p-series (2.7) can be conveniently used as 'auxiliary series'.

Solved Examples

6.2.9 Example

Test the convergence of the following series:

(a) $\frac{3}{1}+\frac{4}{8}+\frac{5}{27}+\frac{6}{64}+.....$ (b) $\frac{4}{1}+\frac{5}{4}+\frac{6}{9}+\frac{7}{16}+.....$

(c) $\sum_{n=1}^{\infty}\left[\left(n^4+1\right)^{1/4}-n\right]$

Solution

(a) *Step 1:*

To find "u_n" the n^{th} term of the given series. The numerators 3, 4, 5, 6......of the terms, are in AP.

n^{th} term $t_n = 3+(n-1).1 = n+2$

Denominators are $1^3, 2^3, 3^3, 4^3.....n^{th}$ term = n^3 ; $\therefore\ u_n = \dfrac{n+2}{n^3}$

Step 2:

To choose the auxiliary series $\sum v_n$. In u_n, the highest degree of n in the numerator is 1 and that of denominator is 3.

$\therefore$ we take, $v_n = \dfrac{1}{n^{3-1}} = \dfrac{1}{n^2}$

Step 3 :

$$\underset{n\to\infty}{Lt}\frac{u_n}{v_n} = \underset{n\to\infty}{Lt}\frac{n+2}{n^3}\times n^2 = \underset{n\to\infty}{Lt}\frac{n+2}{n} = \underset{n\to\infty}{Lt}\left(1+\frac{2}{n}\right) = 1,$$

which is non- zero and finite.

Step 4 :

Conclusion:

$$\underset{n\to\infty}{Lt}\frac{u_n}{v_n} = 1$$

$\therefore\ \sum u_n$ and $\sum v_n$ both converge or diverge (by comparison test). But $\sum v_n = \sum \dfrac{1}{n^2}$ is convergent by p-series test (p = 2 > 1); $\therefore\ \sum u_n$ is convergent.

(b) $\dfrac{4}{1}+\dfrac{5}{4}+\dfrac{6}{9}+\dfrac{7}{16}+.....$

Step 1 : 4 , 5, 6, 7, ...in AP , $t_n = 4+(n-1)1 = n+3$ $\quad\therefore\ u_n = \dfrac{n+3}{n^2}$

Step 2 : Let $\sum v_n = \dfrac{1}{n}$ be the auxiliary series

Step 3 : $\underset{n\to\infty}{Lt}\dfrac{u_n}{v_n} = \underset{n\to\infty}{Lt}\left(\dfrac{n+3}{n^2}\right)\times n = \underset{n\to\infty}{Lt}\left(1+\dfrac{3}{n}\right) = 1$, which is non-zero and finite.

Step 4 : $\therefore$ By comparison test, both $\sum u_n$ and $\sum v_n$ converge are diverge together .

But $\sum v_n = \sum \frac{1}{n}$ is divergent, by p-series test (p = 1); $\therefore$ $\sum u_n$ is divergent.

(c) $\sum_{n=1}^{\infty}\left[\left(n^4+1\right)^{1/4}-n\right]$

$$=\left\{n^4\left(1+\frac{1}{n^4}\right)\right\}\frac{1}{4}-n=n\left[\left(1+\frac{1}{n^4}\right)\frac{1}{4}-1\right]$$

$$=n\left[1+\frac{1}{4n^4}+\frac{\frac{1}{4}\left(\frac{1}{4}-1\right)}{2!}.\frac{1}{n^8}+.....-1\right]=n\left[\frac{1}{4n^4}-\frac{3}{32n^8}+.....\right]$$

$$=\frac{1}{4n^3}-\frac{3}{32n^7}+....=\frac{1}{n^3}\left[\frac{1}{4}-\frac{3}{32n^4}+.....\right]$$

Here it will be convenient if we take $v_n = \frac{1}{n^3}$

$$\underset{n\to\infty}{Lt}\frac{u_n}{v_n}=\underset{n\to\infty}{Lt}\left(\frac{1}{4}-\frac{1}{32n^4}+.....\right)=\frac{1}{4},$$ which is non-zero and finite

$\therefore$ By comparison test, $\sum u_n$ and $\sum v_n$ both converge or both diverge. But by p-series test $\sum v_n = \frac{1}{n^3}$ is convergent. (p = 3 > 1); $\therefore$ $\sum u_n$ is convergent.

6.2.10 Example

If $u_n = \frac{\sqrt[3]{3n^2+1}}{\sqrt[4]{2n^3+3n+5}}$ show that $\sum u_n$ is divergent

Solution

As n increases, u_n approximates to

$$\frac{\sqrt[3]{3n^2}}{\sqrt[4]{2n^3}}=\frac{3^{1/3}}{2^{1/4}}\times\frac{n^{2/3}}{n^{3/4}}=\frac{3^{1/3}}{2^{1/4}}.\frac{1}{n^{1/12}}$$

$\therefore$ If we take $v_n = \dfrac{1}{n^{1/12}}$, $\underset{n\to\infty}{Lt}\dfrac{u_n}{v_n} = \dfrac{3^{1/3}}{2^{1/4}}$ which is finite.

[(or) *Hint:* Take $v_n = \dfrac{1}{n^{l_1 - l_2}}$, where l_1 and l_2 are indices of 'n' of the largest terms in denominator and nominator respectively of u_n. Here $v_n = \dfrac{1}{n^{\frac{3}{4}-\frac{2}{3}}} = \dfrac{1}{n^{\frac{1}{12}}}$]

By comparison test, $\sum v_n$ and $\sum u_n$ converge or diverge together. But

$$\sum v_n = \sum \frac{1}{n^{1/12}} \text{ is divergent by } p-\text{series test (since } p = \frac{1}{12} < 1)$$

$\therefore \sum u_n$ is divergent.

6.2.11 Example

Test for the convergence of the series. $\sqrt{\dfrac{1}{2}} + \sqrt{\dfrac{2}{3}} + \sqrt{\dfrac{3}{4}} + \sqrt{\dfrac{4}{5}} + \ldots\ldots$

Solution

Here, $\quad u_n = \sqrt{\dfrac{n}{n+1}}$;

Take $\quad v_n = \dfrac{1}{n^{\frac{1}{2}-\frac{1}{2}}} = \dfrac{1}{n^0} = 1, \quad \underset{n\to\infty}{Lt}\dfrac{u_n}{v_n} = \underset{n\to\infty}{Lt}\sqrt{\dfrac{1}{1+\dfrac{1}{n}}} = 1$ (finite)

$\sum v_n$ is divergent by p – series test. ($p = 0 < 1$)

$\therefore$ By comparison test, $\sum u_n$ is divergent, (Students are advised to follow the procedure given in ex. 1.2.9(a) and (b) to find "u_n" of the given series.)

6.2.12 Example

Show that $1 + \dfrac{1}{\lfloor 1} + \dfrac{1}{\lfloor 2} + \ldots\ldots + \dfrac{1}{\lfloor n} + \ldots..$ is convergent.

Solution

$$u_n = \frac{1}{\lfloor n} \text{ (neglecting } 1^{st} \text{ term)}$$

$$= \frac{1}{1.2.3\ldots\ldots n} < \frac{1}{1.2.2.2\ldots..n-1 times} = \frac{1}{(2^{n-1})}$$

$$\therefore \quad \sum u_n < 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots\ldots$$

which is an infinite geometric series with common ratio $\frac{1}{2} < 1$

$\therefore \sum \frac{1}{2^{n-1}}$ is convergent. (1.2.3(a)). Hence $\sum u_n$ is convergent.

6.2.13 Example

Test for the convergence of the series, $\frac{1}{1.2.3} + \frac{1}{2.3.4} + \frac{1}{3.4.5} + \ldots\ldots$

Solution

$$u_n = \frac{1}{n(n+1)(n+2)};$$

Take $v_n = \frac{1}{n^3}$

$$\underset{n\to\infty}{Lt} \frac{u_n}{v_n} = \underset{n\to\infty}{Lt} \frac{n^3}{n^3\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)} = 1 \text{ (finite)}$$

$\therefore$ By comparison test, $\sum u_n$, and $\sum v_n$ converge or diverge together. But by p-series test, $\sum v_n = \sum \frac{1}{n^3}$ is convergent ($p = 3 > 1$); $\therefore \sum u_n$ is convergent .

6.2.14 Example

If $u_n = \sqrt{n^4+1} - \sqrt{n^4-1}$, show that $\sum u_n$ is convergent. [JNTU, 200s]

Solution

$$u_n = n^2\left(1+\frac{1}{n^4}\right)^{\frac{1}{2}} - n^2\left(1-\frac{1}{n^4}\right)^{\frac{1}{2}}$$

$$= n^2\left[\left(1+\frac{1}{2n^4}-\frac{1}{8n^8}+\frac{1}{16n^{12}}-\ldots\right)-\left(1-\frac{1}{2n^4}-\frac{1}{8n^8}-\frac{1}{16n^{12}}-\ldots\right)\right]$$

$$= n^2\left[\frac{1}{n^4}+\frac{1}{8n^{12}}+\ldots\right] = \frac{1}{n^2}\left[1+\frac{1}{8n^{10}}+\ldots\right]$$

Take $\quad v_n = \frac{1}{n^2}$, hence $\underset{n\to\infty}{Lt} \frac{u_n}{v_n} = 1$

$\therefore$ By comparison test, $\sum u_n$ and $\sum v_n$ converge or diverge together. But $\sum v_n = \frac{1}{n^2}$ is convergent by p –series test ($p = 2 > 1$) $\therefore$ $\sum u_n$ is convergent.

6. 2.15 Example

Test the series $\frac{1}{1+x} + \frac{1}{2+x} + \frac{1}{3+x} + \ldots\ldots$ for convergence.

Solution

$$u_n = \frac{1}{n+x};$$

Take $$v_n = \frac{1}{n},$$

then $$\frac{u_n}{v_n} = \frac{n}{n+x} = \frac{1}{1+\frac{x}{n}}$$

$$\underset{n\to\infty}{Lt}\left(\frac{1}{1+\frac{x}{n}}\right) = 1; \sum v_n = \sum \frac{1}{n}$$ is divergent by p-series test ($p = 1$)

$\therefore$ By comparison test, $\sum u_n$ is divergent.

6.2.16 Example

Show that $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$ is divergent.

Solution

$$u_n = \sin\left(\frac{1}{n}\right); \qquad \text{Take} \qquad v_n = \frac{1}{n}$$

$$\underset{n\to\infty}{Lt}\frac{u_n}{v_n} = \underset{n\to\infty}{Lt}\frac{\sin\left(\frac{1}{n}\right)}{\left(\frac{1}{n}\right)} = \underset{t\to 0}{Lt}\frac{\sin t}{t} \text{ (where } t = 1/n) = 1$$

$\therefore$ $\sum u_n, \sum v_n$,both converge or diverge . But $\sum v_n = \sum \frac{1}{n}$ is divergent (p -series test, $p = 1$); $\therefore$ $\sum u_n$ is divergent.

6.2.17 Example

Test the series $\sum \sin^{-1}\left(\frac{1}{n}\right)$ for convergence.

Solution

$$u_n = \sin^{-1}\frac{1}{n};$$

Take $$v_n = \frac{1}{n}$$

$$\underset{n\to\infty}{Lt}\frac{u_n}{v_n} = \underset{n\to\infty}{Lt}\frac{\sin^{-1}\left(\frac{1}{n}\right)}{\left(\frac{1}{n}\right)}; = \underset{\theta\to 0}{Lt}\left(\frac{\theta}{\sin\theta}\right) = 1\left(Taking\sin^{-1}\frac{1}{n} = \theta\right)$$

But $\sum v_n$ is divergent .Hence $\sum u_n$ is divergent.

6.2.18 Example

Show that the series $1+\frac{1}{2^2}+\frac{2^2}{3^3}+\frac{3^3}{4^3}+.....$ is divergent.

Solution

Neglecting the first term, the series is $1+\frac{1}{2^2}+\frac{2^2}{3^3}+\frac{3^3}{4^4}+.....$ .Therefore

$$u_n = \frac{n^n}{(n+1)^{n+1}} = \frac{n^n}{(n+1)(n+1)}n = \frac{n^n}{n\left(1+\frac{1}{n}\right).n^n\left(1+\frac{1}{n}\right)^n}$$

$$= \frac{1}{n\left(1+\frac{1}{n}\right)\left(1+\frac{1}{n}\right)^n}; \quad \text{Take} \quad v_n = \frac{1}{n}$$

$$\therefore \qquad \underset{n\to\infty}{Lt}\frac{u_n}{v_n} = \underset{n\to\infty}{Lt}\frac{1}{\left(1+\frac{1}{n}\right)\left(1+\frac{1}{n}\right)^n} = \underset{n\to\infty}{Lt}\frac{1}{\left(1+\frac{1}{n}\right)^n.1} = \frac{1}{e}$$

which is finite and $\sum v_n = \sum\frac{1}{n}$ is divergent by p –series test ($p = 1$)

$\therefore \qquad \sum u_n$ is divergent.

6.2.19 Example

Show that the series $\frac{1}{1.2.3}+\frac{3}{2.3.4}+\frac{5}{3.4.5}+\ldots\ldots\infty$ is convergent.

Solution

$$\frac{1}{1.2.3}+\frac{3}{2.3.4}+\frac{5}{3.4.5}+\ldots\ldots\infty$$

$$n^{th} \text{ term} = u_n = \frac{2n-1}{n(n+1)(n+2)} = \frac{1}{n^2}.\frac{\left(2-\frac{1}{n}\right)}{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}$$

Take $\qquad v_n = \frac{1}{n^2}$

$$\underset{n\to\infty}{Lt}\frac{u_n}{v_n} = \underset{n\to\infty}{Lt}\frac{1}{n^2}\frac{\left(2-\frac{1}{n}\right)}{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)} \div \left(\frac{1}{n^2}\right)$$

$$\underset{n\to\infty}{Lt}\frac{u_n}{v_n} = \frac{2-0}{(1+0)(1+0)} = 2 \text{ which is finite and non-zero}$$

$\therefore$ By comparison test $\sum u_n$ and $\sum v_n$ converge or diverge together

But $\sum v_n = \sum \frac{1}{n^2}$ is convergent. $\therefore$ $\sum u_n$ is also convergent.

6.2.20 Example

Test whether the series $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}+\sqrt{n+1}}$ is convergent

Solution

The given series is $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}+\sqrt{n+1}}$

$$u_n = \frac{1}{\sqrt{n}+\sqrt{n+1}}$$

$$= \frac{\sqrt{n+1}-\sqrt{n}}{\left(\sqrt{n}+\sqrt{n+1}\right)\left(\sqrt{n+1}-\sqrt{n}\right)} = \sqrt{n+1}-\sqrt{n}$$

$$u_n = \sqrt{n}\left\{\left(1+\frac{1}{n}\right)^{1/2} - 1\right\} = \sqrt{n}\left\{\left(1+\frac{1}{2n}-\frac{1}{8n^2}+.....\right)-1\right\}$$

$$u_n = \sqrt{n}\left\{\frac{1}{2n}-\frac{1}{8n^2}+....\right\} = \sqrt{n}\left\{\frac{1}{2}-\frac{1}{8n}+....\right\}$$

Take $\quad v_n = \dfrac{1}{\sqrt{n}}$

$$\underset{n\to\infty}{Lt}\frac{u_n}{v_n} = \underset{n\to\infty}{Lt}\frac{1}{\sqrt{n}}\left\{\frac{1}{2}-\frac{2}{8n}+......\right\} \div \left(\frac{1}{\sqrt{n}}\right) = \frac{1}{2}$$

which is finite and non-zero

Using comparison test $\sum u_n$ and $\sum v_n$ converge or diverge together.

But $\sum v_n = \sum \dfrac{1}{\sqrt{n}}$ is divergent $\left(\text{since } p = 1/2\right)$

$\therefore \quad \sum u_n$ is also divergent.

6.2.21 Example

Test for convergence $\sum\limits_{n=1}^{\infty}\left[\sqrt[3]{n^3+1}-n\right]$ [JNTU 1996, 2003, 2003s]

n^{th} term $\quad u_n = n\left[\left(1+\dfrac{1}{n^3}\right)^{1/3}-1\right] = n\left[1+\dfrac{1}{3n^3}+\dfrac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{1.2}\cdot\dfrac{1}{n^6}+.....-1\right]$

$$= \frac{1}{3n^2}-\frac{1}{9n^5}+...... = \frac{1}{n^2}\left(\frac{1}{3}-\frac{1}{9n^3}+......\right) \text{; Let } v_n = \frac{1}{n^2}$$

Then $\quad \underset{n\to\infty}{Lt}\, u_n/v_n = \underset{n\to\infty}{Lt}\left(1/3 - 1/9n^3 + ...\right) = 1/3 \neq 0$

$\therefore$ By comparison test, $\sum u_n$ and $\sum v_n$ both converge or diverge.

But $\sum v_n$ is convergent by p -series test $\left(\text{since } p = 2 > 1\right)$ $\therefore$ $\sum u_n$ is convergent.

6.2.22 Example

Show that the series, $\dfrac{2}{1^p}+\dfrac{3}{2^p}+\dfrac{4}{3^p}+.......$ is convergent for $p > 2$ and divergent for $p \leq 2$

Solution

n^{th} term of the given series $= u_n = \dfrac{n+1}{n^p} = \dfrac{n\left(1+\frac{1}{n}\right)}{n^p} = \dfrac{\left(1+\frac{1}{n}\right)}{n^{p-1}}$

Let us take $v_n = \dfrac{1}{n^{p-1}}$; $\underset{n\to\infty}{Lt} \dfrac{u_n}{v_n} = 1 \neq 0$;

$\therefore \sum u_n$ and $\sum v_n$ both converge or diverge by comparison test.

But $\sum v_n = \sum \dfrac{1}{n^{p-1}}$ converges when $p-1>1$; i.e., $p>2$ and diverges when $p-1 \leq 1$ i.e $p \leq 2$; Hence the result.

6.2.23 Example

Test for convergence $\displaystyle\sum_{n=1}^{\infty}\left(\frac{2^n+3}{3^n+1}\right)^{1/2}$

Solution

$$u_n = \left[\frac{2^n\left(1+\frac{3}{2^n}\right)}{3^n\left(1+\frac{1}{3^n}\right)}\right]^{1/2};$$

Take $\qquad v_n = \sqrt{\dfrac{2^n}{3^n}}$; $\dfrac{u_n}{v_n} = \left(\dfrac{1+\frac{3}{2^n}}{1+\frac{1}{3^n}}\right)^{1/2}$

$\underset{n\to\infty}{Lt} \dfrac{u_n}{v_n} = 1 \neq 0$; $\therefore$ By comparison test, $\sum u_n$ and $\sum v_n$ behave the same way.

But $\sum v_n = \displaystyle\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^{n/2} = \sqrt{\frac{2}{3}} + \frac{2}{3} + \left(\frac{2}{3}\right)^{3/2} + \ldots\ldots$, which is a geometric series with common ratio $\sqrt{2/3}$ (<1) $\therefore \sum v_n$ is convergent. Hence $\sum u_n$ is convergent.

6.2.24 Example

Test for convergence of the series, $\dfrac{1}{4.7.10} + \dfrac{4}{7.10.13} + \dfrac{9}{10.13.16} + \ldots\ldots$

Solution

	4, 7, 10,..............is an A . P;	$t_n = 4+(n-1)3 = 3n+1$
	7, 10, 13,............is an A . P;	$t_n = 7+(n-1)3 = 3n+4$
and	10 , 13 , 16 ,.............is an A. P;	$t_n = 10+(n-1)3 = 3n+7$

$$\therefore \quad u_n = \frac{n^2}{(3n+1)(3n+4)(3n+7)} = \frac{n^2}{3n\left(1+\frac{1}{3n}\right).3n\left(1+\frac{4}{3n}\right).3n\left(1+\frac{7}{3n}\right)}$$

$$= \frac{1}{27n\left(1+\frac{1}{3n}\right)\left(1+\frac{4}{3n}\right)\left(1+\frac{7}{3n}\right)};$$

Taking $v_n = \frac{1}{n}$, we get

$\underset{n\to\infty}{Lt} \frac{u_n}{v_n} = \frac{1}{27} \neq 0$; $\therefore$ By comparison test, both $\sum u_n$ and $\sum v_n$ behave in the same manner. But by p –series test, $\sum v_n$ is divergent, since $p = 1$. $\therefore$ $\sum u_n$ is divergent .

6.2.25 Example

Test for convergence $\sum \frac{\sqrt{2n^2-5n+1}}{4n^3-7n^2+2}$

Solution

n^{th} term of the given series = $u_n = \frac{\sqrt{2n^2-5n+1}}{4n^3-7n^2+2}$

Let $\quad v_n = \frac{1}{n^2}$

$$\underset{n\to\infty}{Lt} \frac{u_n}{v_n} = \underset{n\to\infty}{Lt} . \left[\frac{n\sqrt{2-\frac{5}{n}+\frac{1}{n^2}}}{n^3\left(4-\frac{7}{n}+\frac{2}{n^3}\right)} \times \frac{n^2}{1}\right]$$

$$= \underset{n\to\infty}{Lt} \left[\frac{\sqrt{2-\frac{5}{n}+\frac{1}{n^2}}}{\left(4-\frac{7}{n}+\frac{2}{n^3}\right)}\right] = \frac{\sqrt{2}}{4} \neq 0$$

$\therefore$ By comparison test, $\sum u_n$ and $\sum v_n$ both converge or diverge. But $\sum v_n$ is convergent. [p series test $-p=2>1$] $\therefore$ $\sum u_n$ is convergent .

6.2.26 Example

Test the series $\sum u_n$, whose n^{th} term is $\frac{1}{\left(4n^2-i\right)}$

Solution

$$u_n = \frac{1}{\left(4n^2 - i\right)};$$

Let $$v_n = \frac{1}{n^2},\ \underset{n\to\infty}{Lt}\frac{u_n}{v_n} = \underset{n\to\infty}{Lt}\left[\frac{n^2}{n^2\left(4 - \frac{i}{n^2}\right)}\right] = \frac{1}{4} \neq 0$$

$\therefore\ \sum u_n$ and $\sum v_n$ both converge or diverge by comparison test. But $\sum v_n$ is convergent by p-series test $(p = 2 > 1)$; $\therefore\ \sum u_n$ is convergent.

6.2.27 Example

If $u_n = \left(\frac{1}{n}\right).\sin\left(\frac{1}{n}\right)$, show that $\sum u_n$ is convergent .

Solution

Let $v_n = \frac{1}{n^2}$,so that $\sum v_n$ is convergent by p-series test .

$$\underset{n\to\infty}{Lt}\left(\frac{u_n}{v_n}\right) = \underset{n\to\infty}{Lt}\frac{\sin\left(1/n\right)}{\left(1/n\right)} = \underset{n\to\infty}{Lt}\left(\frac{\sin t}{t}\right)$$

where $t = 1/n$. Thus $\underset{n\to\infty}{Lt}\left(\frac{u_n}{v_n}\right) = 1 \neq 0$

$\therefore$ By comparison test, $\sum u_n$ is convergent.

6.2.28 Example

Test for convergence $\sum \frac{1}{\sqrt{n}}\tan(1/n)$

Take $v_n = 1/n^{3/2}$; $\underset{n\to\infty}{Lt}\left[u_n/v_n\right] = 1 \neq 0$ (as in above example)

Hence by comparison test , $\sum u_n$ converges as $\sum v_n$ converges.

6.2.29 Example

Show that $\sum_{n=1}^{\infty}\sin^2\left(\frac{1}{n}\right)$ is convergent.

Solution

Let $u_n = \sin^2\left(\frac{1}{n}\right)$; Take $v_n = \frac{1}{n^2}$

$$\underset{n\to\infty}{Lt}\left(\frac{u_n}{v_n}\right) = \underset{n\to\infty}{Lt}\left[\frac{\sin\left(1/n\right)}{1/n}\right]^2 = \underset{t\to 0}{Lt}\left(\frac{\sin t}{t}\right)^2$$

where $t = 1/n$; $\underset{n\to\infty}{Lt}\left(u_n/v_n\right) = 1^2 = 1 \neq 0$

$\therefore$ By comparison test, $\sum u_n$ and $\sum v_n$ behave the same way.

But $\sum v_n$ is convergent by p- series test, since p = 2 > 1; $\therefore$ $\sum u_n$ is convergent.

6.2.30 Example :

Show that $\sum_{n=2}^{\infty} 1/\log(n^n)$ is divergent.

Solution

$$u_n = 1/n\log n \; ; \; \log 2 < 1 \Rightarrow 2\log 2 < 2 \Rightarrow 1/2\log 2 > 1/2 \; ;$$

Similarly $\quad 1/3\log 3 > 1/3, \ldots\ldots 1/n\log n > 1/n, n \in N$

$\therefore \quad \Sigma\, 1/n\log n > \Sigma\, 1/n$; But $\Sigma\, 1/n$ is divergent by p-series test.

By comparison test, given series is divergent. [If $\sum v_n$ is divergent and $u_n \geq v_n \forall n$ then $\sum u_n$ is divergent.]

(*Note :* This problem can also be done using Cauchy's integral Test. See ex1.6.2)

6.2.31 Example

Test the convergence of the series $\sum_{n=1}^{\infty}(c+n)^{-r}(d+n)^{-s}$, where c, d, r, s are all +ve.

Solution

The n^{th} term of the series $= u_n = \dfrac{1}{(c+n)^r(d+n)^s}$.

Let $\quad v_n = \dfrac{1}{n^{r+s}}$ Then $\dfrac{u_n}{v_n} = \dfrac{n^{r+s}}{n^r\left(1+\frac{c}{n}\right)^r . n^s\left(1+\frac{d}{n}\right)^s} = \dfrac{1}{\left(1+\frac{c}{n}\right)^r\left(1+\frac{d}{n}\right)^s}$

$\underset{n\to\infty}{Lt}\frac{u_n}{v_n}=1\neq 0$, $\therefore \sum u_n$ and $\sum v_n$ both converge are diverge, by comparison test.

But by p-series test, $\sum v_n$ converges if $(r+s)>1$ and diverges if $(r+s)\leq 1$

$\therefore \sum u_n$ converges if $(r+s)>1$ and diverges if $(r+s)\leq 1$.

6.2.32 Example

Show that $\sum_{1}^{\infty} n^{-\left(1+1/n\right)}$ is divergent.

Solution

$$u_n = n^{-\left(1+1/n\right)} = \frac{1}{n.n^{1/n}}$$

Take $\quad v_n=\frac{1}{n}$; $\underset{n\to\infty}{Lt}\frac{u_n}{v_n}=\underset{n\to\infty}{Lt}\frac{1}{n^{1/n}}=1\neq 0$

For let $\quad \underset{n\to\infty}{Lt}\frac{1}{n^{1/n}}=y$ say; $\log y = \underset{n\to\infty}{Lt}-\frac{1}{n}.\log n = -\underset{n\to\infty}{Lt}\frac{\frac{1}{n}}{1}=0$

$\therefore \quad y=e^0=1 \qquad$ ($\left(\frac{\infty}{\infty}\right)$ using L Hospitals rule)

By comparison test both $\sum u_n$ and $\sum v_n$ converge or diverge. But p-series test, $\sum v_n$ diverges (since $p=1$); Hence $\sum u_n$ diverges.

6.2.33 Example

Test for convergence the series $\sum_{n=1}^{\infty}\frac{(n+a)^r}{(n+b)^p(n+c)^q}$, a, b, c, p, q, r, being +ve.

Solution

$$u_n=\frac{(n+a)^r}{(n+b)^p(n+c)q}=\frac{n^r\left(1+a/n\right)^r}{n^p\left(1+b/n\right)^p.n^q\left(1+c/n\right)^q}$$

$$=\frac{1}{n^{p+q-r}}.\frac{\left(1+a/n\right)^r}{\left(1+b/n\right)^p\left(1+c/n\right)^q};$$

Take $\quad v_n = \dfrac{1}{n^{p+q-r}}$; $\underset{n\to\infty}{Lt} \dfrac{u_n}{v_n} = 1 \neq 0$;

Applying comparison test both $\sum u_n$ and $\sum v_n$ converge or diverge.

But by p-series test, $\sum v_n$ converges if $(p+q-r) > 1$ and diverges if $(p+q-r) \leq 1$.

Hence $\sum u_n$ converges if $(p+q-r) > 1$ and diverges if $(p+q-r) \leq 1$.

6.2.34 Example

Test the convergence of the following series whose n^{th} terms are :

(a) $\dfrac{(3n+4)}{(2n+1)(2n+3)(2n+5)}$; (b) $\tan\dfrac{1}{n}$;

(c) $\left(\dfrac{1}{n^2}\right)\left(\dfrac{n+1}{n+3}\right)^n$ (d) $\dfrac{1}{(3^n+5^n)}$;

(e) $\dfrac{1}{n.3^n}$

Solution

(a) *Hint :* Take $v_n = \dfrac{1}{n^2}$; $\sum v_n$ is convergent; $\underset{n\to\infty}{Lt}\left(\dfrac{u_n}{v_n}\right) = \dfrac{3}{8} \neq 0$ (Verify)

Apply comparison test:

$\sum u_n$ is convergent [the student is advised to work out this problem fully]

(b) Proceed as in **1. 2.16**; $\sum u_n$ is convergent.

(c) *Hint :* Take $v_n = \dfrac{1}{n^2}$; $\underset{n\to\infty}{Lt}\left(\dfrac{u_n}{v_n}\right) = \underset{n\to\infty}{Lt} \dfrac{(1+1/n)^n}{\left(1+3/n\right)^n} = \dfrac{e}{e^3} = \dfrac{1}{e^2} \neq 0$

$v_n = \dfrac{1}{n^2}$ is convergent (work out completely for yourself)

(d) $u_n = \dfrac{1}{3^n+5^n} = \dfrac{1}{5^n} \cdot \dfrac{1}{\left[1+\left(\dfrac{3}{5}\right)^n\right]}$; Take $v_n = \dfrac{1}{5^n}$; $\underset{n\to\infty}{Lt}\left(\dfrac{u_n}{v_n}\right) = 1 \neq 0$

$\sum u_n$ and $\sum v_n$ behave the same way. But $\sum v_n$ is convergent since it is a geometric series with common ratio $\frac{1}{5} < 1$

$\therefore \sum u_n$ is convergent by comparison test .

(e) $\frac{1}{n.3^n} \leq \frac{1}{3^n}, \forall n \in N$, since $n.3^n \geq 3^n$;

$$\therefore \qquad \sum \frac{1}{n.3^n} \leq \sum \frac{1}{3^n} \qquad .....(1)$$

The series on the R.H .S of (1) is convergent since it is geometric series with $r = \frac{1}{3} < 1$.

$\therefore$ By comparison test $\sum \frac{1}{n.3^n}$ is convergent.

6.2.35 Example

Test the convergence of the following series.

(a) $1 + \frac{1+2}{1^2+2^2} + \frac{1+2+3}{1^2+2^2+3^2} + \frac{1+2+3+4}{1^2+2^2+3^2+4^2} + \ldots\ldots\ldots\ldots$

(b) $1 + \frac{1^2+2^2}{1^3+2^3} + \frac{1^2+2^2+3^2}{1^3+2^3+3^3} + \frac{1^2+2^2+3^2+4^2}{1^3+2^3+3^3+4^3} + \ldots\ldots\ldots\ldots$

Solution

(a) $$u_n = \frac{1+2+3+....+n}{1^2+2^2+3^2+.....n^2} = \frac{n\frac{(n+1)}{2}}{n(n+1)\frac{(2n+1)}{6}} = \frac{3}{(2n+1)}$$

Take $v_n = \frac{1}{n}$; $\underset{n\to\infty}{Lt} \frac{u_n}{v_n} = \underset{n\to\infty}{Lt}\left(\frac{3n}{2n+1}\right) = \frac{3}{2} \neq 0$

$\sum u_n$ and $\sum v_n$ behave alike by comparison test.

But $\sum v_n$ is diverges by *p*-series test. Hence $\sum u_n$ is divergent.

(b) $$u_n = \frac{1^2+2^2+....+n^2}{1^3+2^3+....n^3} = \frac{n(n+1)\frac{(2n+1)}{6}}{n^2\frac{(n+1)^2}{4}} = \frac{2(2n+1)}{3n(n+1)}$$

Hint : Take $v_n = \frac{1}{n}$ and proceed as in (a) and show that $\sum u_n$ is divergent.

Exercise 6(a)

1. Test for convergence the infinite series whose n^{th} term are:

(a) $\dfrac{1}{n-\sqrt{n}}$ **[Ans :** divergent**]**

(b) $\dfrac{\sqrt{n+1}-\sqrt{n}}{n}$ **[Ans :** convergent**]**

(c) $\sqrt{n^2+1}-n$ **[Ans :** divergent**]**

(d) $\dfrac{\sqrt{n}}{n^2-1}$ **[Ans :** convergent**]**

(e) $\sqrt{n^3+1}-\sqrt{n^3}$ **[Ans :** divergent**]**

(f) $\dfrac{1}{\sqrt{n(n+1)}}$ **[Ans :** divergent**]**

(g) $\dfrac{\sqrt{n}}{n^2+1}$ **[Ans :** convergent**]**

(h) $\dfrac{2n^3+5}{4n^5+1}$ **[Ans :** convergent**]**

2. Determine whether the following series are convergent or divergent.

(a) $\dfrac{1}{1+3^{-1}}+\dfrac{2}{1+3^{-2}}+\dfrac{3}{1+3^{-3}}+\ldots\ldots\ldots\ldots$ **[Ans :** divergent**]**

(b) $\dfrac{12}{1^3}+\dfrac{22}{2^3}+\dfrac{32}{3^3}+\ldots\ldots\ldots+\dfrac{2+10n}{n^3}+\ldots\ldots\ \ldots\ldots$ **[Ans :** convergent**]**

(c) $\dfrac{1}{\sqrt{1}+\sqrt{2}}+\dfrac{1}{\sqrt{2}+\sqrt{3}}+\dfrac{1}{\sqrt{3}+\sqrt{4}}+\ldots\ldots\ \ldots\ldots$ **[Ans :** divergent**]**

(d) $\dfrac{2}{3^2}+\dfrac{3}{4^2}+\dfrac{4}{5^2}+\ldots\ldots\ \ldots\ldots$ **[Ans :** divergent**]**

(e) $\dfrac{1}{1^2}+\dfrac{1}{2^3}+\dfrac{1}{3^4}+\ldots\ldots\ldots$ **[Ans :** convergent**]**

(f) $\sum_{n=1}^{\infty} \frac{\sqrt[3]{n^2+1}}{\sqrt[4]{4n^2+2n+3}}$ .... [Ans : divergent]

(g) $\sum_{1}^{\infty}\left(8^{1/n}-1\right)$ .... [Ans : divergent]

(h) $\sum_{1}^{\infty}\frac{3n^3+8}{5n^5+9}$ .... [Ans : convergent]

(i) $\frac{1}{1.3}+\frac{2}{3.5}+\frac{3}{5.7}+$ .... [Ans : divergent]

6.3

6.3.1 D' Alembert's Ratio Test

Let (i) $\sum u_n$ be a series of +ve terms and (ii) $\underset{n\to\infty}{Lt}\frac{u_{n+1}}{u_n}=k(\geq 0)$

Then the series $\sum u_n$ is (i) convergent if $k<1$ and (ii) divergent if $k>1$.

Proof : *Case* (i) : $\underset{n\to\infty}{Lt}\frac{u_{n+1}}{u_n}=k(<1)$

From the definition of a limit, it follows that

$$\exists m>0 \text{ and } l(0<l<1) \ni \frac{u_{n+1}}{u_n}<l \forall n\geq m$$

i.e., $\frac{u_{m+1}}{u_m}<l$, $\frac{u_{m+2}}{u_{m+1}}<l$,..........

$$\therefore \quad u_m+u_{m+1}+u_{m+2}+\ldots\ldots\ldots\ldots = u_m\left[1+\frac{u_{m+1}}{u_m}+\frac{u_{m+2}}{u_m}+\ldots..\right]$$

$$= u_m=\left[1+\frac{u_{m+1}}{u_m}+\frac{u_{m+2}}{u_{m+1}}.\frac{u_{m+1}}{u_m}+\ldots..\right]$$

$$<u_m\left(1+l+l^2+\ldots\right)=u_m.\frac{1}{1-l}(l<1)$$

But $u_m.\frac{1}{1-l}$ is a finite quantity $\therefore \sum_{n=m}^{\infty}u_n$ is convergent

By adding a finite number of terms $u_1+u_2+\ldots\ldots+u_{m-1}$, the convergence of the series is unaltered. $\sum_{n=m}^{\infty}u_n$ is convergent .

Case **(ii)** $\underset{n\to\infty}{Lt}\frac{u_{n+1}}{u_n}=k>1$

There may be some finite number of terms in the beginning which do not satisfy the condition $\frac{u_{n+1}}{u_n}\geq 1$. In such a case we can find a number 'm' $\ni \frac{u_{n+1}}{u_n}\geq 1, \forall n\geq m$

Omitting the first 'm' terms, if we write the series as $u_1+u_2+u_3+$ ...., we have

$$\frac{u_2}{u_1}\geq 1, \frac{u_3}{u_2}\geq 1, \frac{u_4}{u_3}\geq 1 \text{ .... and so on}$$

$$\therefore \quad u_1+u_2+......+u_n=u_1\left(1+\frac{u_2}{u_1}+\frac{u_3}{u_2}.\frac{u_2}{u_1}+.....\right) \quad \text{(to } n \text{ terms)}$$

$$\geq u_n \text{ (1 + 1 + 1.1 +.......to } n \text{ terms)}$$

$$= nu_1$$

$$\underset{n\to\infty}{Lt}\sum_{n=1}^{n}u_n\geq \underset{n\to\infty}{Lt}\, n.u_1 \text{ which } \to\infty \text{; } \therefore \sum u_n \text{ is divergent}$$

6.3.2

Note **: 1**

The ratio test fails when $k=1$. As an example, consider the series, $\sum_{n=1}^{\infty}\frac{1}{n^p}$

Here $\underset{n\to\infty}{Lt}\frac{u_{n+1}}{u_n}=\underset{n\to\infty}{Lt}\left(\frac{n}{n+1}\right)^p=\underset{n\to\infty}{Lt}\left(\frac{1}{1+\frac{1}{n}}\right)^p=1$

i.e., $k=1$ for all values of p.

But the series is convergent if $p>1$ and divergent if $p\leq 1$, which shows that when $k=1$, the series may converge or diverge and hence the test fails .

Note : 2 Ratio test can also be stated as follows :

If $\sum u_n$ is series of +ve terms and if $\underset{n\to\infty}{Lt}\frac{u_n}{u_{n+1}}=k$, then $\sum u_n$ is convergent

If $k>1$ and divergent if $k<1$ (the test fails when $k=1$).

Solved Examples

Tests for convergence of Series

6.3.3 Example

(a) $$\frac{x}{1.2}+\frac{x^2}{2.3}+\frac{x^3}{3.4}+\ldots\ldots\ldots\ldots\ldots$$

Solution

$$u_n=\frac{x^n}{n(n+1)};u_{n+1}=\frac{x^{n+1}}{(n+1)(n+2)},$$

$$\frac{u_{n+1}}{u_n}=\frac{x^{n+1}}{(n+1)(n+2)}\cdot\frac{n(n+1)}{x^n}=\frac{1}{\left(1+\frac{2}{n}\right)}x.$$

Therefore $\underset{n\to\infty}{Lt}\frac{u_{n+1}}{u_n}=x$

$\therefore$ By ratio test $\sum u_n$ is convergent When $|x|<1$ and divergent when $|x|>1$;

When $x=1$, $u_n=\frac{1}{n^2(1+1/n)}$; Take $v_n=\frac{1}{n^2}$; $\underset{n\to\infty}{Lt}\frac{u_n}{v_n}=1$

$\therefore$ By comparison test $\sum u_n$ is convergent.

Hence $\sum u_n$ is convergent when $|x|\le 1$ and divergent when $|x|>1$

(b) $$1+3x+5x^2+7x^3+\ldots\ldots$$

Solution

$$u_n=(2n-1)x^{n-1};u_{n+1}=(2n+1)x^n$$

$$\underset{n\to\infty}{Lt}\frac{u_{n+1}}{u_n}=\underset{n\to\infty}{Lt}\left(\frac{2n+1}{2n-1}\right)x=x$$

$\therefore$ By ratio test $\sum u_n$ is convergent when $|x|<1$ and divergent when $|x|>1$

When $x=1:u_n=2n-1;\ \underset{n\to\infty}{Lt}u_n=\infty;\ \therefore\ \sum u_n$ is divergent.

Hence $\sum u_n$ is convergent when $|x|<1$ and divergent when $|x|\ge 1$

(c) $$\sum_{n=1}^{\infty}\frac{x^n}{n^2+1}\ldots\ldots\ldots\ldots$$

Solution

$$u_n = \frac{x^n}{n^2+1} \; ; \; u_{n+1} = \frac{x^{n+1}}{(n+1)^2+1}$$

Hence $$\frac{u_{n+1}}{u_n} = \left(\frac{n^2+1}{n^2+2n+2}\right)x$$

$$\underset{n\to\infty}{Lt} \frac{u_{n+1}}{u_n} = \underset{n\to\infty}{Lt}\left[\frac{n^2\left(1+\frac{1}{n^2}\right)}{n^2\left(1+\frac{2}{n}+\frac{2}{n^2}\right)}\right](x) = x$$

$\therefore$ By ratio test, $\sum u_n$ is convergent when $|x|<1$ and divergent when $|x|>1$

When $x=1: u_n = \dfrac{1}{n^2+1}$; Take $v_n = \dfrac{1}{n^2}$

$\therefore$ By comparison test, $\sum u_n$ is convergent when $|x| \le 1$ and divergent when $|x|>1$

6.3.4 Example

Test the series $\sum\limits_{n\to\infty}^{\infty}\left(\dfrac{n^2-1}{n^2+1}\right)x^n, x>0$ for convergence.

Solution

$$u_n = \left(\frac{n^2-1}{n^2+1}\right)x^n ; u_{n+1} = \left[\frac{(n+1)^2-1}{(n+1)^2+1}\right]x^{n+1}$$

$$\underset{n\to\infty}{Lt} \frac{u_{n+1}}{u_n} = \underset{n\to\infty}{Lt}\left[\left(\frac{n^2+2n}{n^2+2n+2}\right)\frac{(n^2+1)}{(n^2-1)}\right].x$$

$$= \underset{n\to\infty}{Lt}\left[\frac{n^4(1+2/n)(1+1/n)}{n^4\left(1+2/n+2/n^2\right)(1-1/n)}\right] = x$$

$\therefore$ By ratio test, $\sum u_n$ is convergent when $x<1$ and divergent when $x>1$ when $x=1$,

$$u_n = \frac{n^2-1}{n^2+1} \qquad \text{Take } v_n = \frac{1}{n^0}$$

Applying p-series and comparison test, it can be seen that $\sum u_n$ is divergent when $x = 1$.

$\therefore \sum u_n$ is convergent when $x < 1$ and divergent $x \geq 1$

6.3.5 Example

Show that the series $1 + \frac{2^p}{\lfloor 2} + \frac{3^p}{\lfloor 3} + \frac{4^p}{\lfloor 4} + \ldots\ldots$, is convergent for all values of p.

Solution

$$u_n = \frac{n^p}{\lfloor n}\ ;\ u_{n+1} = \frac{(n+1)^p}{\lfloor n+1}$$

$$\underset{n\to\infty}{Lt}\ \frac{u_{n+1}}{u_n} = \underset{n\to\infty}{Lt}\left[\frac{(n+1)^p}{\lfloor n+1} \times \frac{\lfloor n}{n^p}\right] = \underset{n\to\infty}{Lt}\left\{\frac{1}{(n+1)}\left(\frac{n+1}{n}\right)^p\right\}$$

$$= \underset{n\to\infty}{Lt}\ \frac{1}{(n+1)} \times \underset{n\to\infty}{Lt}\left(1+\frac{1}{n}\right)^p = 0 < 1;$$

$\sum u_n$ is convergent for all 'p'.

6.3.6 Example

Test the convergence of the following series

$$\frac{1}{1^p} + \frac{1}{3^p} + \frac{1}{5^p} + \frac{1}{7^p} + \ \ldots\ldots\ldots\ldots$$

Solution

$$u_n = \frac{1}{(2n-1)^p}\ ;u_{n+1} = \frac{1}{(2n+1)^p}$$

$$\frac{u_{n+1}}{u_n} = \frac{(2n-1)^p}{(2n+1)^p} = \frac{2^p.n^p(1-1/2n)^p}{2^p n^p(1+1/2n)^p}; \qquad \underset{n\to\infty}{Lt}\ \frac{u_{n+1}}{u_n} = 1$$

$\therefore$ Ratio test fails.

Take $v_n = \frac{1}{n^p}$; $\frac{u_n}{v_n} = \frac{n^p}{(2n-1)^p} = \frac{1}{2^p\left(1-\frac{1}{2n}\right)^p}$; $\underset{n\to\infty}{Lt}\ \frac{u_n}{v_n} = \frac{1}{2^p}$,

which is non – zero and finite 7

$\therefore$ By comparison test, $\sum u_n$ and $\sum v_n$ both converge or both diverge.

But by p – series test, $\sum v_n = \sum \frac{1}{n^p}$ converges when $p > 1$ and diverges when $p \le 1$

$\therefore \sum u_n$ is convergent if $p > 1$ and divergent if $p \le 1$.

6.3.7 Example

Test the convergence of the series $\sum_{n=1}^{\infty} \frac{(n+1)x^n}{n^3}, x > 0$

Solution

$$u_n = \frac{(n+1)x^n}{n^3}; u_{n+1} = \frac{(n+2)x^{n+1}}{(n+1)^3}$$

$$\frac{u_{n+1}}{u_n} = \frac{n+2}{(n+1)^3} . x^{n+1} . \frac{n^3}{(n+1)x^n} = \left(\frac{n+2}{n+1}\right)\left(\frac{n}{n+1}\right)^3 . x$$

$$\underset{n\to\infty}{Lt} \frac{u_{n+1}}{u_n} = \underset{n\to\infty}{Lt} \left(\frac{1+\frac{2}{n}}{1+\frac{1}{n}}\right) \frac{1}{\left(1+\frac{1}{n}\right)^3} . x = x$$

$\therefore$ By ratio test, $\sum u_n$ converges when $x < 1$ and diverges when $x > 1$.

When $x = 1$, $u_n = \frac{n+1}{n^3}$

Take $v_n = \frac{1}{n^2}$; By comparison test $\sum u_n$ is convergent (give proof)

$\therefore \sum u_n$ is convergent if $x \le 1$ and divergent if $x > 1$.

6.3.8 Example

Test the convergence of the series

(i) $\sum_{n=1}^{\infty} \left(\frac{n^2}{2^n} + \frac{1}{n^2}\right)$ (ii) $1 + \frac{2.5.8}{1.5.9} + \frac{2.5.8.11}{1.5.9.13} + \ldots$ (iii) $\frac{1}{3} + \frac{1.2}{3.5} + \frac{1.2.3}{3.5.7} + \ldots$

Solution

(i) $\sum_{n=1}^{\infty} \left(\frac{n^2}{2^n} + \frac{1}{n^2}\right) = \sum_{n=1}^{\infty} \frac{n^2}{2^n} + \sum_{n=1}^{\infty} \frac{1}{n^2}$ Let $u_n = \frac{n^2}{2^n}; v_n = \frac{1}{n^2}$

$$u_{n+1} = \frac{(n+1)^2}{2^{n+1}}; \frac{u_{n+1}}{u_n} = \frac{(n+1)^2}{2^{n+1}} \cdot \frac{2^n}{n^2}$$

$$\underset{n\to\infty}{Lt} \frac{u_{n+1}}{u_n} = \underset{n\to\infty}{Lt} \frac{1}{2} \cdot \left(1+\frac{1}{n}\right)^2 = \frac{1}{2} < 1$$

$\therefore$ By ratio test $\sum u_n$ is convergent .By p –series test, $\sum v_n$ is convergent.

$\therefore$ The given series $\left(\sum u_n + \sum v_n\right)$ is convergent.

(ii) Neglecting the first term, the series can be taken as,

$$\frac{2.5.8}{1.5.9} + \frac{2.5.8.11}{1.5.9.13} +$$

Here, 1[st] term has 3 fractions ,2[nd] term has 4 fractions and so on .

$\therefore$ n^{th} term contains $(n+2)$ fractions

2. 5. 8.......are in A. P.

$\therefore$ $(n+2)^{th}$ term $= 2 + (n+1)3 = 3n+5$;

$\therefore$ 1. 5. 9,.......are in A. P.

$\therefore$ $(n+2)^{th}$ term $= 1 + (n+1)4 = 4n+5$

$$\therefore \quad u_n = \frac{2.5.8.....(3n+5)}{1.5.9.....(4n+5)}$$

$$u_{n+1} = \frac{2.5.8.....(3n+5)(3n+8)}{1.5.9.....(4n+5)(4n+9)}$$

$$\frac{u_{n+1}}{u_n} = \frac{(3n+8)}{(4n+9)};$$

$$\underset{n\to\infty}{Lt} \frac{u_{n+1}}{u_n} = \underset{n\to\infty}{Lt} \frac{n\left(3+\frac{8}{n}\right)}{n\left(4+\frac{9}{n}\right)} = \frac{3}{4} < 1$$

$\therefore$ By ratio test, $\sum u_n$ is convergent.

(iii) 1 , 2 , 3 , ...are in A. P n^{th} term $= n$; 3. 5. 7..........are in A.P. n^{th} term $= 2n+1$

$$\therefore \qquad u_n = \left[\frac{1.2.3.....n}{3.5.7.....(2n+1)}\right]$$

$$u_{n+1} = \left[\frac{1.2.3.....n(n+1)}{3.5.7.....(2n+1)(2n+3)}\right]$$

$$\frac{u_{n+1}}{u_n} = \left(\frac{n+1}{2n+3}\right)$$

$$\underset{n\to\infty}{Lt} \frac{u_{n+1}}{u_n} = \underset{n\to\infty}{Lt} \frac{n.\left(1+\frac{1}{n}\right)}{n\left(2+\frac{3}{n}\right)} = \frac{1}{2} < 1$$

$\therefore$ By ratio test, $\sum u_n$ is convergent.

6.3.9 Exercise

Test for the convergence of the series $\sum \frac{1.2.3......n}{3.5.7......(2n+1)}$

Solution

$$u_n = \frac{1.2.3......n}{3.5.7......(2n+1)} \; ; \; u_{n+1} = \frac{1.2.3......(n+1)}{3.5.7......(2n+3)} \; ;$$

$$\frac{u_{n+1}}{u_n} = \frac{n+1}{2n+3} \; ; \quad \underset{n\to\infty}{Lt} \frac{u_{n+1}}{u_n} = \underset{n\to\infty}{Lt} \frac{n\left(1+1/n\right)}{n\left(2+3/n\right)} = \frac{1}{2} < 1 \; ;$$

$\therefore$ By ratio test, $\sum u_n$ is convergent.

6.3.10 Example

Test for convergence $\sum_{n=1}^{\infty} \frac{1.3.5....(2n-1)}{2.4.6....2n}.x^{n-1} \left(x > 0\right)$

Solution

The given series of +ve terms has $u_n = \frac{1.3.5....(2n-1)}{2.4.6....2n}.x^{n-1}$

and $\qquad u_{n+1} = \frac{1.3.5....(2n+1)}{2.4.6....(2n+2)} x^n$

$$\underset{n\to\infty}{Lt}\frac{u_{n+1}}{u_n}=\underset{n\to\infty}{Lt}\left(\frac{2n+1}{2n+2}\right)x=\underset{n\to\infty}{Lt}\frac{2n\left(1+\frac{1}{2n}\right)}{2n\left(1+\frac{2}{2n}\right)}.x=x$$

$\therefore$ By ratio test, $\sum u_n$ is converges when $x<1$ and diverges when $x>1$ when $x=1$, the test fails.

Then $\quad u_n=\dfrac{1.3.5....(2n-1)}{2.4.6.....2n}<1 \quad$ and $\underset{n\to\infty}{Lt}u_n\neq 0$

$\therefore$ $\sum u_n$ is divergent. Hence $\sum u_n$ is convergent when $x<1$, and divergent when $x\geq 1$

6.3.11 Example

Test for the convergence of $1+\frac{2}{5}x+\frac{6}{9}x^2+........+\left(\frac{2^n-2}{2^n+1}\right)x^{n-1}+.....(x>0)$

Solution

Omitting 1st term, $u_n=\left(\dfrac{2^n-2}{2^n+1}\right)x^{n-1},(n\geq 2)$ and 'u_n' are all +ve.

$$u_{n+1}=\frac{\left(2^{n+1}-2\right)}{\left(2^{n+1}+1\right)}x^n\ ;\ \underset{n\to\infty}{Lt}\left(\frac{u_{n+1}}{u_n}\right)=\underset{n\to\infty}{Lt}.\left(\frac{2^{n+1}-2}{2^{n+1}+1}\right)\times\left(\frac{2^n+1}{2^n-2}\right).x$$

$$=\underset{n\to\infty}{Lt}\left[\frac{2^{n+1}\left(1-\frac{1}{2^n}\right)}{2^{n+1}\left(1+\frac{1}{2^{n+1}}\right)}.\frac{2^n\left(1+\frac{1}{2^n}\right)}{2^n\left(1-\frac{2}{2^n}\right)}.x\right]=x\ ;$$

Hence, by ratio test, $\sum u_n$ converges if $x<1$ and diverges if $x>1$.

When $x=1$, the test fails. Then $u_n=\dfrac{2^n-2}{2^n+1}$; $\underset{n\to\infty}{Lt}u_n=1\neq 0$; $\therefore$ $\sum u_n$ diverges

Hence $\sum u_n$ is convergent when $x<1$ and divergent $x>1$

6.3.12 Example

Using ratio test show that the series $\displaystyle\sum_{n=0}^{\infty}\frac{(3-4i)^n}{n!}$ converges

Solution

$$u_n={(3-4i)^n}/{n!}\ ;\ u_{n+1}={(3-4i)^{n+1}}/{(n+1)!}\ ;$$

$$\underset{n\to\infty}{Lt}\left(\frac{u_{n+1}}{u_n}\right) = \underset{n\to\infty}{Lt}\left(\frac{3-4i}{n+1}\right) = 0 < 1$$

Hence, by ratio test, $\sum u_n$ converges.

6.3.13 Example

Discuss the nature of the series, $\frac{2}{3.4}x + \frac{3}{4.5}x^2 + \frac{4}{5.6}x^3 + ...\infty (x > 0)$

Solution

Since $x > 0$, the series is of +ve terms ;

$$u_n = \frac{(n+1)}{(n+2)(n+3)}x^n > u_{n+1} = \frac{(n+2)}{(n+3)(n+4)}x^{n+1}$$

$$\underset{n\to\infty}{Lt}\frac{u_{n+1}}{u_n} = \left[\frac{(n+2)^2 .x}{(n+1)(n+4)}\right]$$

$$= \underset{n\to\infty}{Lt}\left[\frac{n^2 (1+2/n)^2 .x}{n^2\left(1+5/n + 4/n^2\right)}\right] = x;$$

Therefore by ratio test, $\sum u_n$ converges if $x < 1$ and diverges if $x > 1$

When $x = 1$, the test fails; Then $u_n = \frac{(n+1)}{(n+2)(n+3)}$;

Taking $v_n = \frac{1}{n}$; $\underset{n\to\infty}{Lt} = \frac{u_n}{v_n} = 1 \neq 0$

$\therefore$ By comparison test $\sum u_n$ and $\sum v_n$ behave same way. But $\sum v_n$ is divergent by p-series test. ($p = 1$);

$\therefore$ $\sum u_n$ is diverges when $x = 1$

$\therefore$ $\sum u_n$ is convergent when $x < 1$ and divergent when $x \geq 1$

6.3.14 Example

Discuss the nature of the series $\sum \frac{3.6.9.....3n.5^n}{4.7.10.....(3n+1)(3n+2)}$

Solution

Here, $$u_n = \frac{3.6.9.....3n}{4.7.10.....(3n+1)} \frac{5^n}{(3n+2)};$$

$$u_{n+1} = \frac{3.6.9.....3n(3n+3)5^{n+1}}{4.7.10.....(3n+1)(3n+4)(3n+5)};$$

$$\underset{n\to\infty}{Lt} \frac{u_{n+1}}{u_n} = \underset{n\to\infty}{Lt} \frac{(3n+2)(3n+3).5}{(3n+4)(3n+5)}$$

$$= \underset{n\to\infty}{Lt} \left[\frac{5.9n^2\left(1+\frac{2}{3n}\right)\left(1+\frac{3}{3n}\right)}{9n^2\left(1+\frac{4}{3n}\right)\left(1+\frac{5}{3n}\right)} \right] = 5 > 1$$

$\therefore$ By ratio test, $\sum u_n$ is divergent.

6.3.15 Example

Test for convergence the series $\sum_{n=1}^{\infty} n^{1-n}$

Solution

$$u_n = n^{1-n};\ u_{n+1} = (n+1)^{-n};$$

$$\frac{u_{n+1}}{u_n} = \frac{(n+1)^{-n}}{n^{1-n}} = \frac{n}{n(n+1)^n} = \frac{1}{n}\left(\frac{n}{n+1}\right)^n$$

$$\underset{n\to\infty}{Lt} \frac{u_{n+1}}{u_n} = \underset{n\to\infty}{Lt} \frac{1}{n}.\left(\frac{1}{1+\frac{1}{n}}\right)^n = 0.\frac{1}{e} = 0 < 1$$

$\therefore$ By ratio test $\sum u_n$, is convergent

6.3.16 Example

Test the series $\sum_{n=1}^{\infty} \frac{2n^3}{\lfloor n}$, for convergence.

Solution

$$u_n = \frac{2n^3}{\lfloor n};\ u_{n+1} = \frac{2(n+1)^3}{\lfloor n+1}$$

$$\frac{u_{n+1}}{u_n} = \frac{2(n+1)^3}{\lfloor n+1} \times \frac{\lfloor n}{2n^3} = \frac{(n+1)^2}{n^3} = \frac{\left(1+\frac{1}{n}\right)^2}{n};$$

$$\underset{n\to\infty}{Lt} \frac{u_{n+1}}{u_n} = 0 < 1;$$

$\therefore$ By ratio test, $\sum u_n$ is convergent.

6.3.17 Example

Test convergence of the series $\sum \frac{2^n n!}{n^n}$

Solution

$$u_n = \frac{2^n n!}{n^n};\ u_{n+1} = \frac{2^{n+1}(n+1)!}{(n+1)^{n+1}};$$

$$\frac{u_{n+1}}{u_n} = \frac{2^{n+1}(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{2^n n!} = 2\left(\frac{n}{n+1}\right)^n$$

$$\underset{n\to\infty}{Lt} \frac{u_{n+1}}{u_n} = 2\underset{n\to\infty}{Lt} \frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{2}{e} < 1 \quad (\text{since } 2 < e < 3)$$

$\therefore$ By ratio test, $\sum u_n$ is convergent.

6.3.20 Example

Test the convergence of the series $\sum u_n$ where u_n is

(a) $\frac{n^2+1}{3^n+1}$ (b) $\frac{x^{n-1}}{(2n+1)^a}, (a>0)$

(c) $\left(\frac{1.2.3....n}{4.7.10.....3n+3}\right)^2$ (d) $\frac{\sqrt{1+2^n}}{\sqrt{1+3^n}}$

(e) $\left(\frac{3n^3+7n^2}{5n^9+11}\right)x^n$

Solution

(a) $$\underset{n\to\infty}{Lt}\left(\frac{u_{n+1}}{u_n}\right) = \underset{n\to\infty}{Lt}\left[\frac{(n+1)^2+1}{3^{n+1}+1} \times \frac{3^n+1}{n^2+1}\right]$$

$$= \underset{n\to\infty}{Lt}\left[\frac{n^2\left(1+2/n+2/n^2\right)}{n^2\left(1+1/n\right)}\cdot\frac{3^n\left(1+1/3^n\right)}{3^{n+1}\left(1++1/3^{n+1}\right)}\right]$$

$$= \frac{1}{3} < 1$$

$\therefore$ By ratio test, $\sum u_n$ is convergent.

(b)
$$\underset{n\to\infty}{Lt}\left(\frac{u_{n+1}}{u_n}\right) = \underset{n\to\infty}{Lt}\left[\frac{x^n}{(2n+3)^a}\times\frac{(2n+1)^a}{x^{n-1}}\right]$$

$$= \underset{n\to\infty}{Lt}\left[\frac{2^a n^a\left(1+1/2n\right)^a}{2^a n^a\left(1+3/2n\right)^a}.x\right] = x$$

By ratio test, $\sum u_n$ convergence if $x < 1$ and diverges if $x > 1$.

When $x = 1$, the test fails; Then, $u_n = \frac{1}{(2n+1)^a}$; Taking $v_n = \frac{1}{n^a}$ awe have,

$$\underset{n\to\infty}{Lt}\left(\frac{u_n}{v_n}\right) = \underset{n\to\infty}{Lt}\left(\frac{n}{2n+1}\right)^a = \underset{n\to\infty}{Lt}\frac{1}{\left(2+1/n\right)^a} = \frac{1}{2^a} \neq 0 \text{ and finite (since } a > 0\text{)}.$$

$\therefore$ By comparison test, $\sum u_n$ and $\sum v_n$ have same property

But p –series test, we have

(i) $\sum v_n$ convergent when $a > 1$

and (ii) divergent when $a \leq 1$

$\therefore$ To sum up, (i) $x < 1$, $\sum u_n$ is convergent $\forall a$.

(ii) $x > 1$, $\sum u_n$ is divergent $\forall a$.

(iii) $x = 1$, $a > 1$, $\sum u_n$ is convergent, and

(iv) $x = 1$, $a \leq 1$, $\sum u_n$ is divergent.

(c) $$\underset{n\to\infty}{Lt}\frac{u_{n+1}}{u_n}=\underset{n\to\infty}{Lt}\left[\frac{1.2.3....n(n+1)}{4.7.10....(3n+3)(3n+6)}\times\frac{4.7.10....(3n+3)}{1.2.3....n}\right]^2$$

$$=\underset{n\to\infty}{Lt}\left[\frac{(n+1)}{3(n+2)}\times\right]^2=\frac{1}{9}<1 \quad ;$$

$\therefore$ By ratio test, $\sum u_n$ is convergent

(d) $$\underset{n\to\infty}{Lt}\frac{u_{n+1}}{u_n}=\underset{n\to\infty}{Lt}\left[\frac{\left(1+2^{n+1}\right)}{\left(1+3^{n+1}\right)}\times\frac{\left(1+3^n\right)}{\left(1+2^n\right)}\right]^{1/2}$$

$$=\underset{n\to\infty}{Lt}\left[\frac{2^{n+1}\left(1+\frac{1}{2^{n+1}}\right)}{3^{n+1}\left(1+\frac{1}{3^{n+1}}\right)}\times\frac{3^n\left(1+\frac{1}{3^n}\right)}{2^n\left(1+\frac{1}{2^n}\right)}\right]^{1/2}=\left(\frac{2}{3}\right)^{1/2}<1$$

$\therefore$ By ratio test, $\sum u_n$ is convergent.

(e) $$\underset{n\to\infty}{Lt}\frac{u_{n+1}}{u_n}=\underset{n\to\infty}{Lt}\left[\frac{3(n+1)^3+7(n+1)^2}{5(n+1)^9+11}\times\frac{5n^9+11}{3n^3+7}\times x\right]$$

$$=\underset{n\to\infty}{Lt}\frac{u_{n+1}}{u_n}=\underset{n\to\infty}{Lt}\left[\frac{3n^3\left(1+1/n\right)^3+7n^2\left(1+1/n\right)^2}{5n^9\left(1+1/n\right)^9+11}\times\frac{5n^9\left(1+11/5n^9\right)}{3n^3\left(1+7/3n^3\right)}\times x\right]$$

$$=\underset{n\to\infty}{Lt}\left[\frac{3n^3\left\{\left(1+1/n\right)^3+\frac{7}{3n}\left(1+1/n\right)^2\right\}}{5n^9\left\{\left(1+1/n\right)^9+\frac{11}{5n^9}\right\}}\times\frac{5n^9\left(1+11/5n^9\right)}{3n^3\left(1+7/3n^3\right)}\times x\right]=x$$

$\therefore$ By ratio test, $\sum u_n$ converges when $x<1$ and diverges when $x>1$.
When $x=1$, the test fails,

Then $$u_n=\frac{3n^3\left(1+7/3n\right)}{5n^9\left(1+11/5n^9\right)}=\frac{3}{5n^6}\frac{\left(1+7/3n\right)}{\left(1+11/5n^9\right)}$$

Taking $v_n=\frac{1}{n^6}$, we observe that $\underset{n\to\infty}{Lt}\frac{u_n}{v_n}=\frac{3}{5}\neq 0$

$\therefore$ By comparison test and p series test, we conclude that $\sum u_n$ is convergent.

$\therefore$ $\sum u_n$ is convergent when $x \le 1$ and divergent when $x > 1$.

Exercise - 1(b)

1. Test the convergency or divergency of the series whose general term is :

(a) $\dfrac{x^n}{n}$ .... [Ans : $|x| < 1 cgt, |x \ge| 1 dgt$]

(b) nx^{n-1} .... [Ans : $|x| < 1 cgt, |x \ge| 1 dgt$]

(c) $\left(\dfrac{2^n - 2}{2^n + 1}\right)x^{n-1}$ .... [Ans : $|x| < 1 cgt, |x \ge| 1 dgt$]

(d) $\left(\dfrac{n^2 + 1}{n^2 - 1}\right)x^n$ .... [Ans : $|x| < 1 cgt, |x \ge| 1 dgt$]

(e) $\dfrac{\lfloor n}{n^n}$ .... [Ans: cgt.]

(f) $\dfrac{4^n \lfloor n}{n^n}$ .... [Ans: dgt.]

(g) $\dfrac{(n^3 + 1)^n}{(3^n + 1)}$ .... [Ans: cgt.]

2. Determine whether the following series are convergent or divergent :

(a) $\dfrac{x}{1.2} + \dfrac{x^2}{3.4} + \dfrac{x^3}{5.6} +$ .... [Ans : $|x| \le 1 cgt, |x| > 1 dgt$]

(b) $1 + \dfrac{x}{2^2} + \dfrac{x^2}{3^2} + \dfrac{x^3}{4^2} +$ .... [Ans : $|x| \le 1 cgt, |x| > 1 dgt$]

(c) $\dfrac{1}{1.2.3} + \dfrac{x}{4.5.6} + \dfrac{x^2}{7.8.9} +$ ...... [Ans : $|x| \le 1 cgt, |x| > 1 dgt$]

(d) $1 + \dfrac{x}{2} + \dfrac{x^2}{5} + \dfrac{x^3}{10} + \ldots . \dfrac{x^n}{n^2 + 1} + \ldots .$ [Ans : $|x| \le 1 cgt, |x| > 1 dgt$]

(e) $\dfrac{1.2}{x} + \dfrac{2.3}{x^2} + \dfrac{3.4}{x^3} +$ .... [Ans : $|x| > 1 cgt, |x| \le 1 dgt$]

6.4 Raabe's Test

Let $\sum u_n$ be series of +ve terms and let $\underset{n\to\infty}{Lt}\left\{n\left(\frac{u_n}{u_{n+1}}-1\right)\right\}=k$

Then

(i) If $k>1$, $\sum u_n$ is convergent.

(ii) If $k<1$, $\sum u_n$ is divergent. (The test fails if $k=1$)

Proof:

Consider the series $\sum v_n=\sum\frac{1}{n^p}$

$$n\left[\frac{v_n}{v_{n+1}}-1\right]=n\left[\left(\frac{n+1}{n}\right)^p-1\right]=n\left[\left(1+\frac{1}{n}\right)^p-1\right]$$

$$=n\left[\left(1+\frac{p}{n}+\frac{p(p-1)}{\underline{|2}}\cdot\frac{1}{n^2}+\ldots\right)-1\right]$$

$$=p+\frac{p(p-1)}{\underline{|2}}\cdot\frac{1}{n}+\ \ldots\ldots\ldots$$

$$\underset{n\to\infty}{Lt}\ n\left\{\frac{v_n}{v_{n+1}}-1\right\}=p$$

Case **(i) :**

In this case,

$$\underset{n\to\infty}{Lt}\ n\left\{\frac{u_n}{u_{n+1}}-1\right\}=k>1$$

We choose a number 'p' $\ni k>p>1$; Comparing the series $\sum u_n$ with $\sum v_n$ which is convergent, we get that $\sum u_n$ will converge if after some fixed number of terms

$$\frac{u_n}{u_{n+1}}>\frac{v_n}{v_{n+1}}=\left(\frac{n+1}{n}\right)^p$$

i.e. If., $$n\left(\frac{u_n}{u_{n+1}}-1\right)>p+\frac{p(p-1)}{\angle 2}\cdot\frac{1}{n}+\ \ldots\ldots\ldots\text{from (1)}$$

i.e., If $$\underset{n\to\infty}{Lt}\ n\left(\frac{u_n}{u_{n+1}}-1\right)>p$$

i.e., If $k > p$, which is true . Hence $\sum u_n$ is convergent .The second case also can be proved similarly.

Solved Examples

6.4.1 Example

Test for convergence the series

$$x+\frac{1}{2}.\frac{x^3}{3}+\frac{1.3}{2.4}.\frac{x^5}{5}+\frac{1.3.5}{2.4.6}.\frac{x^7}{7}+.....$$

Solution

Neglecting the first tem ,the series can be taken as ,

$$\frac{1}{2}.\frac{x^3}{3}+\frac{1.3}{2.4}.\frac{x^5}{5}+\frac{1.3.5}{2.4.6}.\frac{x^7}{7}+.....$$

1.3.5,...are in A.P. n^{th} term $= 1+(n-1)2 = 2n-1$

2.4.6...are in A.p. n^{th} term $= 2+(n-1)2 = 2n$

3.5.7.....are in A.P n^{th} term $= 3+(n-1)2 = 2n+1$

$\therefore$ u_n (n^{th} term of the series) =

$$=\frac{1.3.5....(2n-1)}{2.4.6....(2n)}.\frac{x^{2n+1}}{2n+1}$$

$$u_{n+1}=\frac{1.3.5....(2n-1)(2n+1)}{2.4.6....(2n)(2n+2)}.\frac{x^{2n+3}}{2n+3}$$

$$\frac{u_{n+1}}{u_n}=\frac{1.3.5....(2n+1)}{2.4.6....(2n+2)}.\frac{x^{2n+3}}{(2n+3)}.\frac{2.4.6....2n}{1.3.5....(2n-1)}.\frac{(2n+1)}{x^{2n+1}}$$

$$=\frac{(2n+1)^2x^2}{(2n+2)(2n+3)}$$

$$\therefore \quad \underset{n\to\infty}{Lt}\frac{u_{n+1}}{u_n}=\underset{n\to\infty}{Lt}\frac{4n^2\left(1+\frac{1}{2n}\right)^2}{4n^2\left(1+\frac{2}{2n}\right)\left(1+\frac{3}{2n}\right)}x^2=x^2$$

$\therefore$ By ratio test, $\sum u_n$ converges if $|x|<1$ and diverges if $|x|>1$

If $|x|=1$ the test fails.

Then $\quad x^2 = 1 \quad$ and $\quad \dfrac{u_n}{u_{n+1}} = \dfrac{(2n+2)(2n+3)}{(2n+1)^2}$

$$\frac{u_n}{u_{n+1}} - 1 = \frac{(2n+2)(2n+3)}{(2n+1)^2} - 1 = \frac{6n+5}{(2n+1)^2}$$

$$\underset{n\to\infty}{Lt}\left\{n\left(\frac{u_n}{u_{n+1}} - 1\right)\right\} = \underset{n\to\infty}{Lt}\left(\frac{6n^2+5n}{4n^2+4n+1}\right)$$

$$= \underset{n\to\infty}{Lt}\frac{n^2\left(6+\frac{5}{n}\right)}{n^2\left(4+\frac{4}{n}+\frac{1}{n^2}\right)} = \frac{3}{2} > 1$$

By Raabe's test, $\sum u_n$ converges. Hence the given series is convergent when $|x| \le 1$ an divergent when $|x| > 1$.

6.4.2 Example

Test for the convergence of the series

$$1 + \frac{3}{7}x + \frac{3.6}{7.10}x^2 + \frac{3.6.9}{7.10.13}x^3 + ...; x > 0$$

Solution

Neglecting the first term,

$$u_n = \frac{3.6.9....3n}{7.10.13....3n+4}.x^n$$

$$u_{n+1} = \frac{3.6.9....3n(3n+3)}{7.10.13....(3n+4)(3n+7)}.x^{n+1}$$

$$\frac{u_{n+1}}{u_n} = \frac{3n+3}{3n+7}.x \quad ; \quad \underset{n\to\infty}{Lt}\frac{u_{n+1}}{u_n} = x$$

$\therefore$ By ratio test, $\sum u_n$ is convergent when $x < 1$ and divergent when $x > 1$.

When $x = 1$ The ratio test fails. Then

$$\frac{u_n}{u_{n+1}} = \frac{3n+7}{3n+3}; \frac{u_n}{u_{n+1}} - 1 = \frac{4}{3n+3}$$

$$\underset{n\to\infty}{Lt}\left\{n\left(\frac{u_n}{u_{n+1}} - 1\right)\right\} = \underset{n\to\infty}{Lt}\left(\frac{4n}{3n+3}\right) = \frac{4}{3} > 1$$

$\therefore$ By Raabe's test, $\sum u_n$ is convergent .Hence the given series converges if $x \le 1$ and diverges if $x > 1$.

6.4.3 Example

Examine the convergence of the series

$$\sum_{n=1}^{\infty} \frac{1^2.5^2.9^2....(4n-3)^2}{4^2.8^2.12^2....(4n)^2}$$

Solution

$$u_n = \frac{1^2.5^2.9^2....(4n-3)^2}{4^2.8^2.12^2....(4n)^2}$$

$$u_{n+1} = \frac{1^2.5^2.9^2....(4n-3)^2(4n+1)^2}{4^2.8^2.12^2....(4n)^2(4n+4)^2}$$

$$\underset{n\to\infty}{Lt} \frac{u_{n+1}}{u_n} = \underset{n\to\infty}{Lt} \frac{(4n+1)^2}{(4n+4)^2} = 1 \quad \text{(verify)}$$

$\therefore$ The ratio test fails.

Hence by Raabe's test, $\sum u_n$ is convergent.

6.4.4 Example

Find the nature of the series $\sum \frac{(\underline{|n})^2}{\underline{|2n}} x^n, (x > 0)$

Solution

$$u_n = \frac{(\underline{|n})^2}{\underline{|2n}}.x^n \,;\; u_{n+1} = \frac{(\underline{|n+1})^2}{\underline{|2n+2}}.x^{n+1}$$

$$\frac{u_{n+1}}{u_n} = \frac{(n+1)^2}{(2n+1)(2n+2)} x;$$

$$\underset{n\to\infty}{Lt} \frac{u_{n+1}}{u_n} = \underset{n\to\infty}{Lt} \frac{n^2\left(1+\frac{1}{n}\right)^2}{4n^2\left(1+\frac{1}{2n}\right)\left(1+\frac{2}{2n}\right)}.x = \frac{x}{4}$$

$\therefore$ By ratio test, $\sum u_n$ converges when $\frac{x}{4} < 1$, i. e ; $x < 4$; and diverges when $x > 4$; When $x = 4$, the test fails.

When $x = 4, \dfrac{u_n}{u_{n+1}} = \dfrac{(2n+1)(2n+2)}{4(n+1)^2}$

$$\frac{u_n}{u_{n+1}} - 1 = \frac{-2n-2}{4(n+1)^2} = \frac{-1}{2(n+1)};$$

$$\underset{n\to\infty}{Lt}\left[n\left(\frac{u_n}{u_{n+1}}-1\right)\right] = \frac{-1}{2} < 1$$

$\therefore$ By ratio test, $\sum u_n$ is divergent

Hence $\sum u_n$ is convergent when $x < 4$ and divergent when $x > 4$

6.4.5 Example

Test for convergence of the series $\sum \dfrac{4.7....(3n+1)}{1.2.3....n} x^n$.

Solution

$$u_n = \frac{4.7....(3n+1)}{1.2.3....n} x^n \; ; \; u_{n+1} = \frac{4.7....(3n+1)(3n+4)}{1.2.3....n(n+1).} x^{n+1}$$

$$\underset{n\to\infty}{Lt} \frac{u_{n+1}}{u_n} = \underset{n\to\infty}{Lt}\left[\frac{(3n+4)}{(n+1)}.x\right] = 3x$$

$\therefore$ By ratio test $\sum u_n$ converges if $3x < 1$ i.e., $x < \dfrac{1}{3}$ and diverges if $x > \dfrac{1}{3}$;

If $x = \dfrac{1}{3}$, the test fails

When $x = \dfrac{1}{3}$, $n\left[\dfrac{u_n}{u_{n+1}} - 1\right] = n\left[\dfrac{(n+1)3}{3n+1} - 1\right] = n\left[\dfrac{-1}{3n+4}\right] = -\dfrac{1}{\left(3+\dfrac{4}{n}\right)}$

$$\underset{n\to\infty}{Lt}\, n\left[\frac{u_n}{u_{n+1}} - 1\right] = -\frac{1}{3} < 1$$

$\therefore$ By Raabe's test, $\sum u_n$ is divergent.

$\therefore$ $\sum u_n$ is convergent when $x < \dfrac{1}{3}$ and divergent when $x \geq \dfrac{1}{3}$

6.4.6 Example

Test for convergence $2+\frac{3x}{2}+\frac{4x^2}{3}+\frac{5x^3}{4}+..........(x>0)$

Solution

The n^{th} term $\quad u_n=\frac{(n+1)}{n}x^{n-1}$;

$$u_{n+1}=\frac{(n+2)}{(n+1)}x^n;\ \frac{u_{n+1}}{u_n}=\frac{n(n+2)}{(n+1)^2}.x$$

$$\underset{n\to\infty}{Lt}\frac{u_{n+1}}{u_n}=\underset{n\to\infty}{Lt}\frac{n^2\left(1+2/n\right)}{n^2\left(1+1/n\right)^2}.x=x$$

$\therefore$ By ratio test, $\sum u_n$ is convergent if $x<1$ and divergent if $x>1$

If $x=1$, the test fails.

Then $$\underset{n\to\infty}{Lt}\ n\left[\frac{u_n}{u_{n+1}}-1\right]=\underset{n\to\infty}{Lt}\ n\left[\frac{(n+1)^2}{n(n+2)}-1\right]$$

$$=\underset{n\to\infty}{Lt}\ n\left[\frac{1}{n(n+2)}\right]=0<1$$

$\therefore$ By Raabe's test $\sum u_n$ is divergent

$\therefore$ $\sum u_n$ is convergent when $x<1$ and divergent when $x\geq 1$

6.4.7 Example

Find the nature of the series $\frac{3}{4}+\frac{3.6}{4.7}+\frac{3.6.9}{4.7.10}+......\infty$

Solution

$$u_n=\frac{3.6.9.....3n}{4.7.10.....(3n+1)};\ u_{n+1}=\frac{3.6.9.....3n(3n+3)}{4.7.10.....(3n+1)(3n+4)}$$

$$\frac{u_{n+1}}{u_n}=\frac{3n+3}{3n+4};\ \underset{n\to\infty}{Lt}\frac{u_{n+1}}{u_n}=\underset{n\to\infty}{Lt}\frac{3n\left(1+3/3n\right)}{3n\left(1+4/3n\right)}=1$$

Ratio test fails.

$\therefore$ $$\underset{n\to\infty}{Lt}\left[n\left\{\frac{u_n}{u_{n+1}}-1\right\}\right]=\underset{n\to\infty}{Lt}\left[n\left(\frac{3n+4}{3n+3}-1\right)\right]$$

$$= \underset{n\to\infty}{Lt}\frac{n}{3(n+1)} = \underset{n\to\infty}{Lt}\frac{n}{3n\left(1+\frac{1}{n}\right)} = \frac{1}{3} < 1$$

$\therefore$ By Raabe's test $\sum u_n$ is divergent.

6.4.8 Example

If $p, q > 0$ and the series

$$1+\frac{1}{2}\frac{p}{q}+\frac{1.3.p(p+1)}{2.4.q(q+1)}+\frac{1.3.5}{2.4.6}\frac{p(p+1)(p+2)}{q(q+1)(q+2)}+....$$

is convergent , find the relation to be satisfied by p and q.

Solution

$$u_n = \frac{1.3.5.....(2n-1)}{2.4.6.....2n}\frac{p(p+1).....(p+n-1)}{q(q+1).....(q+n-1)} \text{ [neglecting } 1^{st} \text{ term]}$$

$$u_{n+1} = \frac{1.3.5.....(2n-1)(2n+1)}{2.4.6.....2n(2n+2)}\frac{p(p+1).....(p+n-1)(p+n)}{q(q+1).....(q+n-1)(q+n)}$$

$$\frac{u_{n+1}}{u_n} = \frac{(2n+1)}{(2n+2)}\frac{(p+n)}{(q+n)};$$

$$\underset{n\to\infty}{Lt}\frac{u_{n+1}}{u_n} = \underset{n\to\infty}{Lt}\left[\frac{2n\left(1+\frac{1}{2n}\right)}{2n\left(1+\frac{1}{2n}\right)}.\frac{n\left(1+\frac{p}{n}\right)}{n\left(1+\frac{q}{n}\right)}\right] = 1$$

$\therefore$ ratio test fails.

Let us apply Raabe's test

$$\underset{n\to\infty}{Lt}\left[n\left(\frac{u_n}{u_{n+1}}-1\right)\right] = \underset{n\to\infty}{Lt}\left[n\left\{\frac{(q+n)(2n+2)}{(p+n)(2n+1)}-1\right\}\right]$$

$$\underset{n\to\infty}{Lt}\left[n\left\{\frac{2q(n+1)-p(2n+1)+n}{n^2\left(1+\frac{p}{n}\right)\left(2+\frac{1}{n}\right)}\right\}\right]$$

$$\underset{n\to\infty}{Lt}\left[\frac{2q\left(1+\frac{1}{n}\right)-p\left(2+\frac{1}{n}\right)+1}{2}\right] = \frac{2q-2p+1}{2}$$

Since $\sum u_n$ is convergent, by Raabe's test, $\frac{2q-2p+1}{2}>1$

$\Rightarrow q-p>\frac{1}{2}$, is the required relation.

Exercise 1 (c)

1. Test whether the series $\sum_{1}^{\infty} u_n$ is convergent or divergent where

$$u_n=\frac{2^2.4^2.6^2.....(2n-2)^2}{3.4.5.....(2n-1)(2n)}.x^{2n}$$

[Ans : $|x|\le 1 cgt, |x|>1 dgt$ **]**

2. Test for the convergence the series

$$\sum_{1}^{\infty}\frac{4.7.10.....(3n+1)}{\underline{|n}}x^n$$

[Ans : $|x|<\frac{1}{3} cgt, |x|\ge\frac{1}{3} dgt$ **]**

3. Test for the convergence the series :

(i) $$\frac{2^2.4^2}{3^2.3^2}+\frac{2^2.4^2.5^2.7^2}{3^2.3^2.6^2.6^2}+\frac{2^2.4^2.5^2.7^2.8^2.10^2}{3^2.3^2.6^2.6^2.9^2.9^2}+...$$

[Ans : divergent]

(ii) $$\frac{3.4}{1.2}x+\frac{4.5}{2.3}x^2+\frac{5.6}{3.4}x^3+.....(x>0)$$

[**Ans :** cgt if $x\le 1$ dgt if, $x>1$]

(iii) $$\sum\frac{1.3.5.....(2n-1)}{2.4.6.....2n}.\frac{x^n}{(2n+2)}(x>0)$$

[**Ans :** cgt if $x\le 1$ dgt if, $x>1$]

(iv) $$1+\frac{(\underline{|1})^2}{\underline{|2}}x+\frac{(\underline{|2})^2x^2}{\underline{|4}}+\frac{(\underline{|3})^2x^3}{\underline{|6}}+......(x>0)$$

[**Ans :** cgt if $x<4$ and dgt if, $x\ge 4$]

6.5 Cauchy's Root Test

Let $\sum u_n$ be a series of +ve terms and let $\underset{n\to\infty}{Lt}\, u_n^{1/n}=l$. Then $\sum u_n$ is convergent when $l<1$ and divergent when $l>1$

Proof :

(i) $\underset{n\to\infty}{Lt}\, u_n^{1/n} = l < 1 \Rightarrow \exists a$ +ve number 'λ'$(l < \lambda < 1) \ni u_n^{1/n} < \lambda, \forall n > m$

(or) $u_n < \lambda^n, \forall n > m$

Since $\lambda < 1, \sum \lambda^n$ is a geometric series with common ratio < 1 and therefore convergent.

Hence $\sum u_n$ is convergent.

(ii) $\underset{n\to\infty}{Lt}\, u_n^{1/n} = l > 1$

$\therefore$ By the definition of a limit we can find a number $r \ni u_n^{1/n} > 1 \forall n > r$

i.e., $u_n > \forall n > r$

i.e., after the 1st 'r' terms , each term is > 1.

$\underset{n\to\infty}{Lt} \sum u_n = \infty \quad \therefore \sum u_n$ is divergent.

Note :

When $\underset{n\to\infty}{Lt}\left(u_n^{1/n}\right) = 1$, the root test can't decide the nature of $\sum u_n$. The fact of this statement can be observed by the following two examples.

1. Consider the series $\sum 1/n^3 :- \underset{n\to\infty}{Lt} u_n^{1/n} = \underset{n\to\infty}{Lt}\left(\frac{1}{n^3}\right)^{1/n} = \underset{n\to\infty}{Lt}\left(\frac{1}{n^{1/n}}\right)^3 = 1$

2. Consider the series $\sum 1/n$, in which $\underset{n\to\infty}{Lt} u_n^{1/n} = \underset{n\to\infty}{Lt} \frac{1}{n^{1/n}} = 1$

In both the examples given above, $\underset{n\to\infty}{Lt} u_n^{1/n} = 1$. But series (1) is convergent (p-series test)

And series (2) is divergent. Hence when the *limit*=1, the test fails.

Solved Examples

6.5.1 Example

Test for convergence the infinite series whose n^{th} terms are:

(i) $\frac{1}{n^{2n}}$ (ii) $\frac{1}{(\log n)^n}$ (iii) $\frac{1}{\left[1+\frac{1}{n}\right]^{n^2}}$

Solution

(i) $u_n = \dfrac{1}{n^{2n}}, u_n^{1/n} = \dfrac{1}{n^2}$; $\underset{n\to\infty}{Lt}\, u_n^{1/n} = \underset{n\to\infty}{Lt}\, \dfrac{1}{n^2} = 0 < 1;$

By root test $\sum u_n$ is convergent.

(ii) $u_n = \dfrac{1}{(\log n)^n}; u_n^{1/n} = \dfrac{1}{\log n}$; $\underset{n\to\infty}{Lt}\, u_n^{1/n} = \underset{n\to\infty}{Lt}\, \dfrac{1}{\log n} = 0 < 1;$

$\therefore$ By root test , $\sum u_n$ is convergent.

(iii) $u_n = \dfrac{1}{\left(1+\dfrac{1}{n}\right)^{n^2}}; u_n^{1/n} = \dfrac{1}{\left(1+\dfrac{1}{n}\right)^n}$

$$\underset{n\to\infty}{Lt}\, u_n^{1/n} = \underset{n\to\infty}{Lt}\, \frac{1}{\left(1+\dfrac{1}{n}\right)^n} = \frac{1}{e} < 1;$$

$\therefore$ By root test $\sum u_n$ is convergent.

6.5.2 Example

Find whether the following series are convergent or divergent.

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n - 1}$

Solution

$$u_n^{1/n} = \left(\frac{1}{3^n - 1}\right)^{1/n} = \left(\frac{1}{3^n\left(1 - \dfrac{1}{3^n}\right)}\right)^{1/n}$$

$$\underset{n\to\infty}{Lt}\, u_n^{1/n} = \underset{n\to\infty}{Lt}\left(\frac{1}{3^n\left(1 - \dfrac{1}{3^n}\right)}\right)^{1/n} = \frac{1}{3} < 1;$$

By root test, $\sum u_n$ is convergent.

(ii) $1 + \dfrac{1}{2^2} + \dfrac{1}{3^3} + \dfrac{1}{4^4} + \ldots..$

Solution

$$u_n = \frac{1}{n^n};\ \underset{n\to\infty}{Lt}\ u_n^{1/n} = \underset{n\to\infty}{Lt}\left(\frac{1}{n^n}\right)^{1/n} = 0 < 1$$

By root test, $\sum u_n$ is convergent.

(iii) $$\sum_{n=1}^{\infty} \frac{\left[(n+1)x\right]^n}{n^{n+1}}$$

Solution

$$u_n = \frac{\left[(n+1)x\right]^n}{n^{n+1}}$$

$$\underset{n\to\infty}{Lt}\ u_n^{1/n} = \underset{n\to\infty}{Lt}\left[\frac{\{(n+1)x\}^n}{n^{n+1}}\right]^{1/n}$$

$$\underset{n\to\infty}{Lt}\left[\left\{\frac{(n+1)x}{n}\right\}^n.\frac{1}{n}\right]^{1/n} = \underset{n\to\infty}{Lt}\left(\frac{n+1}{n}\right)x.\frac{1}{n^{1/n}}$$

$$\underset{n\to\infty}{Lt}\left(1+\frac{1}{n}\right)x.\frac{1}{n^{1/n}} = \underset{n\to\infty}{Lt}\ x.\frac{1}{n^{1/n}} = x \qquad \left(\text{since } \underset{n\to\infty}{Lt}\ x.\frac{1}{n^{1/n}} = 1\right)$$

$\therefore$ $\sum u_n$ is convergent if $|x| < 1$ and divergent if $|x| > 1$ and when $|x| = 1$ the test fails.

Then $u_n = \dfrac{(n+1)^n}{n^{n+1}}$; Take $v_n = \dfrac{1}{n}$

$$\frac{u_n}{v_n} = \frac{(n+1)^n}{n^{n+1}}.n = \frac{(n+1)^n}{n^n} = \left(1+\frac{1}{n}\right)^n; \qquad \underset{n\to\infty}{Lt}\frac{u_n}{v_n} = e > 1$$

$\therefore$ By comparison test, $\sum u_n$ is divergent.

($\sum v_n$ diverges by p –series test)

Hence $\sum u_n$ is convergent if $|x| < 1$ and divergent $|x| \geq 1$

6.5.3 Example

If $u_n = \dfrac{n^{n^2}}{(n+1)^{n^2}}$, show that $\sum u_n$ is convergent

$$\underset{n\to\infty}{Lt}\, u_n^{1/n} = \underset{n\to\infty}{Lt}\left[\frac{n^{n^2}}{(n+1)^{n^2}}\right]^{1/n} ;= \underset{n\to\infty}{Lt} = \frac{n^n}{(n+1)^n} = \underset{n\to\infty}{Lt}\left(\frac{n}{n+1}\right)^n$$

$$= \underset{n\to\infty}{Lt}\left(\frac{1}{1+\frac{1}{n}}\right)^n = \frac{1}{e} < 1 ;\ \therefore\ \sum u_n \text{ converges by root test .}$$

6.5.4 Example

Establish the convergence of the series $\frac{1}{3}+\left(\frac{2}{5}\right)^2+\left(\frac{3}{7}\right)^3+$ ....

Solution

$$u_n = \left(\frac{n}{2n+1}\right)^n \text{ ....(verify);}$$

$$\underset{n\to\infty}{Lt}\, u_n^{1/n} = \underset{n\to\infty}{Lt}\left(\frac{n}{2n+1}\right) = \frac{1}{2} < 1$$

By root test, $\sum u_n$ is convergent.

6.5.5 Example

Test for the convergence of $\sum_{n=1}^{\infty}\sqrt{\frac{n}{n+1}}.x^n$

Solution

$$u_n = \left(\frac{1}{1+\frac{1}{n}}\right)^{\frac{1}{2}} .x^n;\ \underset{n\to\infty}{Lt}\, u_n^{1/n} = \underset{n\to\infty}{Lt}\left(\frac{1}{1+\frac{1}{n}}\right)^{\frac{1}{2}} .x = x$$

$\therefore$ By root test, $\sum u_n$ is convergent if $|x| < 1$ and divergent if $|x| > 1$.

When $|x| = 1\ :\ u_n = \sqrt{\frac{n}{n+1}}$, taking $v_n = \frac{1}{n^0}$ and applying comparison test , it can

be seen that is divergent

$\sum u_n$ is convergent if $|x| < 1$ and divergent if $|x| \geq 1$.

6.5.6 Example

Show that $\sum_{n=1}^{\infty}\left(n^{1/n}-1\right)^{n}$ converges.

Solution

$$u_n=\left(n^{1/n}-1\right)^{n}$$

$$\underset{n\to\infty}{Lt}\, u_n^{1/n}=\underset{n\to\infty}{Lt}\left(n^{1/n}-1\right)=1-1=0<1\left(\text{since } \underset{n\to\infty}{Lt}\, n^{1/n}=1\right);$$

$\therefore \sum u_n$ is convergent by root test.

6.5.7 Example

Examine the convergence of the series whose n^{th} term is $\left(\dfrac{n+2}{n+3}\right)^{n}.x^{n}$

Solution

$$u_n=\left(\frac{n+2}{n+3}\right)^{n}.x^{n};\ \underset{n\to\infty}{Lt}\, u_n^{1/n}=\underset{n\to\infty}{Lt}\left(\frac{n+2}{n+3}\right)x=x$$

$\therefore$ By root test, $\sum u_n$ converges when $|x|<1$ and diverges when $|x|>1$

When $|x|=1$: $u_n=\left(\dfrac{n+2}{n+3}\right)^{n}$; $\underset{n\to\infty}{Lt}\, u_n=\underset{n\to\infty}{Lt}\dfrac{\left(1+\dfrac{2}{n}\right)^{n}}{\left(1+\dfrac{3}{n}\right)^{n}}$

$$=\frac{e^2}{e^3}=\frac{1}{e}\neq 0 \quad \text{and the terms are all +ve .}$$

$\therefore \sum u_n$ is divergent . Hence $\sum u_n$ is convergent if $|x|<1$ and divergent if $|x|\geq 1$.

6.5.8 Example

Show that the series,

$$\left[\frac{2^2}{1^2}-\frac{2}{1}\right]^{-1}+\left[\frac{3^3}{2^3}-\frac{3}{2}\right]^{-2}+\left[\frac{4^4}{3^4}-\frac{4}{3}\right]^{-3}+\ldots\ldots \text{ is convergent}$$

$$u_n=\left[\frac{(n+1)^{n+1}}{n^{n+1}}-\frac{n+1}{n}\right]^{-n};=\left(\frac{n+1}{n}\right)^{-n}\left[\left(\frac{n+1}{n}\right)^{n}-1\right]^{-n}$$

$$\left(1+\frac{1}{n}\right)^{-n}\left[\left(1+\frac{1}{n}\right)^{n}-1\right]^{-n};\ u_n^{1/n}=\left(1+\frac{1}{n}\right)^{-1}\left[\left(1+\frac{1}{n}\right)^{n}-1\right]^{-1}$$

$$=\frac{1}{\left(1+\frac{1}{n}\right)}\frac{1}{\left\{\left(1+\frac{1}{n}\right)^{n}-1\right\}}$$

$$\therefore \quad \underset{n\to\infty}{Lt}\, u_n^{1/n}=\frac{1}{1}\cdot\frac{1}{e-1}=\frac{1}{e-1}<1$$

$\therefore$ By root test, $\sum u_n$ is convergent.

6.5.9 Example

Test $\sum_{m=1}^{\infty} u_m$ for convergence when $u_m=\dfrac{e^{-m}}{\left(1+2/m\right)^{-m^2}}$

Solution

$$\underset{m\to\infty}{Lt}\left(u_m^{1/m}\right)=\underset{m\to\infty}{Lt}\left[\frac{\left(1+2/m\right)^{m^2}}{e^m}\right]^{1/m};\ \underset{m\to\infty}{Lt}\frac{1}{e}\left(1+\frac{2}{m}\right)^{m}=\frac{e^2}{e}=e>1$$

Hence Cauchy's root tells us that $\sum u_m$ is divergent.

6.5.11 Example :

Test the convergence of the series $\sum \dfrac{n}{e^{n^2}}$

Solution

$$\underset{n\to\infty}{Lt}\, u_n^{1/n}=\underset{n\to\infty}{Lt}\frac{n^{1/n}}{e^n}=0<1$$

$\therefore$ By root test, $\sum u_n$ is convergent.

6.5.12 Example

Test the convergence of the series, $\dfrac{2}{1^2}x+\dfrac{3^2}{2^3}x^2+\ldots\dfrac{(n+1)^n.x^n}{n^{n+1}}+\ldots\ldots,x>0$

Solution

$$\underset{n\to\infty}{Lt}\, u_n^{1/n} = \underset{n\to\infty}{Lt}\left[\frac{(n+1)^n . x^n}{n^{n+1}}\right]^{1/n} = \underset{n\to\infty}{Lt}\left[\left(\frac{n+1}{n}\right).\frac{1}{n^{1/n}}.x\right]$$

$$= \underset{n\to\infty}{Lt}\left[\left(1+\frac{1}{n}\right).\frac{1}{n^{1/n}}.x\right] = 1.1.x = x\left[\text{since } \underset{n\to\infty}{Lt}\, n^{1/n} = 1\right]$$

$\therefore$ By root test, $\sum u_n$ converges if $x < 1$ and diverges when $x > 1$.

When $x = 1$, the test fails.

Then $\quad u_n = \left(1+\frac{1}{n}\right)^n . \frac{1}{n}$; Take $v_n = \frac{1}{n}$

$$\underset{n\to\infty}{Lt}\frac{u_n}{v_n} = \underset{n\to\infty}{Lt}\left(1+\frac{1}{n}\right)^n = e \neq 0$$

$\therefore$ By comparison test and p-series test, $\sum u_n$ is divergent.

Hence $\sum u_n$ is convergent when $x < 1$ and divergent when $x \geq 1$.

Exercise 1 (d)

1. Test for convergence the infinite series whose n^{th} terms are :

(a) $\frac{1}{2^n - 1}$ .... [Ans : convergent]

(b) $\frac{1}{(\log)^{2n}}.(n \neq 1)$ .... [Ans : convergent]

(c) $\left(\frac{3n+1}{4n+3}.x\right)^n$ .... [Ans : $|x| < \frac{4}{3} cgt, |x| \geq \frac{4}{3} dgt$]

(d) $\frac{x^n}{\lfloor n}$ .... [Ans : cgt for all $x \geq 0$]

(e) $\frac{\lfloor n}{n^n}$ .... [Ans : convergent]

(f) $\frac{3^n.\angle n}{n^3}$ .... [Ans : convergent]

(g) $\dfrac{(2n^2-1)^n}{(2n)^{2n}}$ .... [**Ans :** convergent]

(h) $\left(n^{1/n}-1\right)^{2n}$ .... [**Ans :** convergent]

(i) $\left(\dfrac{n-1}{n}\right)^{-n^2}$ .... [**Ans :** divergent]

(j) $\left(\dfrac{nx}{n+1}\right)^n, (x>0)$ .... [**Ans :** $x<1$ cgt , $x \geq 1$ dgt]

2. Examine the following series for convergence :

(a) $1+\dfrac{x}{2}+\dfrac{x^2}{3^2}+\dfrac{x^3}{4^3}+\ldots\ldots, x>0$ .... [**Ans :** $x \leq 1 cgt, x>1 dgt$]

(b) $\dfrac{1}{4}+\left(\dfrac{2}{7}\right)^2+\left(\dfrac{3}{10}\right)^3+\ldots\ldots$ .... [**Ans :** convergent]

6.6

6.6.1 Integral Test

+ve term series,

$$\phi(1)+\phi(2)+\ldots\ldots+\phi(n)+\ldots\ldots$$

where $\phi(n)$ decreases as n increases is convergent or divergent according as the integral $\int_1^\infty \phi(x)\,dx$ is finite or infinite.

Proof :

Let $S_n=\phi(1)+\phi(2)+\ldots\ldots+\phi(n)$

From the above figure, it can be seen that the area under the curve $y = \phi(x)$ between any two ordinates lies between the set of exterior and interior rectangles formed by the ordinates at

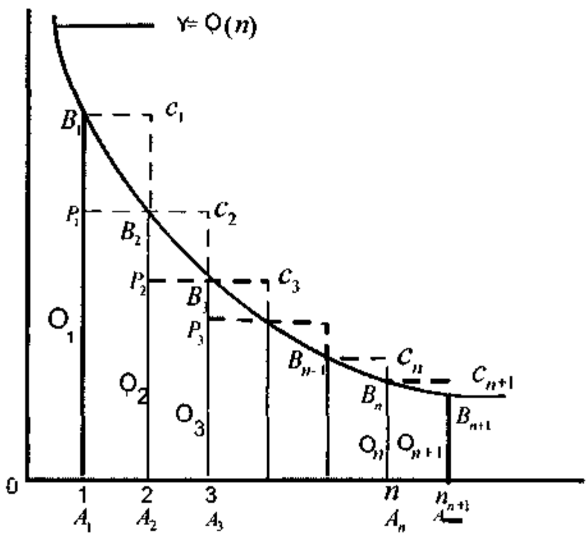
$n = 1, 2, 3, \ldots\ldots n, n+1, \ldots\ldots$

Hence the total area under the curve lies between the sum of areas of all interior rectangles and sum of the areas of all the exterior rectangles.

Hence

$$\{\phi(1)+\phi(2)+\ldots\ldots+\phi(n)\} \geq \int_1^{n+1} \phi(x)\,dx \geq \{\phi(2)+\phi(3)+\ldots\ldots+\phi(n+1)\}$$

$$\therefore\ S_n \geq \int_1^\infty \phi(x)\,dx \geq S_{n+1}-\phi(1)$$



As $n \to \infty$, Lt S_n is finite or infinite according as $\int^{\infty} \phi(x)dx$ is finite or infinite.

Hence the theorem.

Solved Examples

6.6.2 Example

Test for convergence the series $\sum_{n=2}^{\infty} \frac{1}{n \log n}$

Solution

$$\int_2^{\infty} \frac{1}{x \log x} dx = \underset{n\to\infty}{Lt}\left[\int_2^{n} \frac{1}{x \log x} dx\right] = \underset{n\to\infty}{Lt}\left[\log\log x\right]_2^n = \infty$$

$\therefore$ By integral test, the given series is divergent.

6.6.3 Example

Test for convergence of the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$

Solution

$$\int_1^{\infty} \frac{1}{x^p} dx = \underset{n\to\infty}{Lt}\left[\int_1^{n} \frac{1}{x^p} dx\right] = \underset{n\to\infty}{Lt}\left[\frac{x^{-p+1}}{-p+1}\right]_1^n;$$

$$= \frac{1}{1-p} \underset{n\to\infty}{Lt}\left[n^{1-p} - 1\right]$$

Case (i)

If $p > 1$, this limit is finite; $\therefore \sum \frac{1}{n^p}$ is convergent.

Case (ii)

If $p < 1$, the limit is in finite; $\therefore \sum \frac{1}{n^p}$ is divergent.

Case (iii)

If $p = 1$, the limit $\underset{n\to\infty}{Lt} \log x \Big|_1^n = \underset{n\to\infty}{Lt} (\log n) = \infty$; $\therefore \sum \frac{1}{n^p}$ is divergent.

Hence $\sum \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \le 1$

6.6.4 Example

Test the series $\sum_1^\infty \frac{n}{e^{n^2}}$ for convergence.

Solution

$$u_n = \frac{n}{e^{n^2}} = \phi(n)(say);$$

$\phi(n)$ is +ve and decreases as n increases. So let us apply the integral test.

$$\int_1^\infty \phi(x)\,dx = \int_1^\infty x e^{-x^2}\,dx = \frac{1}{2}\int_1^\infty e^{-t}\,dt \left\{t = x^2, dt = 2x\,dx\right\}$$

$$= -\frac{1}{2} e^{-t} \Big|_1^\infty = -\frac{1}{2}\left(0 - \frac{1}{e}\right) = \frac{1}{2e}, \text{ which is finite.}$$

By integral test, $\sum u_n$ is convergent.

6.6.5 Example

Apply integral test to test the convergence of the series $\sum_2^\infty \frac{1}{n^2} \sin\left(\frac{\pi}{n}\right)$

Solution

Let $\phi(n) = \frac{1}{n^2}\sin\left(\frac{\pi}{n}\right)$; $\phi(n)$ decreases as n increases and is +ve .

$$\int_2^\infty \phi(x)\,dx = \int_2^\infty \frac{1}{x^2}\sin\left(\frac{\pi}{x}\right)dx$$

Let $\pi/x = t$

$$-\frac{1}{\pi}\int_{\pi/2}^{0}\sin t\,dt = \frac{1}{\pi}\cos t\Big|_{\pi/2}^{0} = \frac{1}{\pi} \text{ finite, } -\pi/x^2\,dx = dt$$

$$1/x^2\,dx = -1/\pi\,dt$$

$\therefore$ By integral test, $\sum u_n$ converges $x = 2 \Rightarrow t = \pi/2 \quad x = \infty \Rightarrow t = 0$

6.6.6 Example

Apply integral test and determine the convergence of the following series.

(a) $\sum_{1}^{\infty}\frac{3n}{4n^2+1}$ (b) $\sum_{1}^{\infty}\frac{2n^3}{3n^4+2}$ (c) $\sum_{1}^{\infty}\frac{1}{3n+1}$

Solution

(a) $\phi(n) = \frac{3n}{4n^2+1}$ is +ve and decreases as n increases

$$\int_1^{\infty}\phi(x)dx = \int_1^{\infty}\frac{3x}{4x^2+1}dx \qquad \begin{pmatrix} 4x^2+1 = t \Rightarrow xdx = 1/8\,dt \\ x = 1 \Rightarrow t = 5, x = \infty \Rightarrow t = \infty \end{pmatrix}$$

$$\int_1^{\infty}\phi(x)\,dx = \underset{n\to\infty}{Lt}\left[\frac{3}{8}\int_5^{t}\frac{dt}{t}\right] = \underset{n\to\infty}{Lt}\left[\frac{3}{8}\log t - \log 5\right] = \infty$$

$\therefore$ By integral test, $\sum u_n$ diverges.

(b) $\phi(n) = \frac{2n^3}{3n^4+2}$ decreases as n increases and is +ve

$$\int_1^{\infty}\phi(x)\,dx = \int_1^{\infty}\frac{2x^3}{3x^4+2}dx$$

$$= \frac{1}{6}\int_5^{\infty}\frac{dt}{t} = \frac{1}{6}\left[\log t\right]_5^{\infty} = \infty \quad [\text{where } t = 3x^4+2]$$

By integral test, $\sum u_n$ is divergent.

(c) $\phi(n) = \frac{1}{3n+1}$ is +ve, and decreases as n increases.

$$\int_1^{\infty}\phi(x)\,dx = \int_1^{\infty}\frac{1}{3x+1}dx = \int_4^{\infty}\frac{1}{3}\frac{dt}{t}\left[t = 3x+1\right] = \frac{1}{3}\log t\Big|_t^{\infty} = \infty$$

$\therefore$ By integral test, $\sum u_n$ is divergent.

6.7

6.7.1 Alternating Series

A series, $u_1 - u_2 + u_3 - u_4 + - + ... + (-1)^{n-1} u_n + ...$, where u_n are all +ve, is an alternating series .

6.7.2 Leibneitz Test

If in an alternating series $\sum_{n=1}^{\infty} (-1)^{n-1} .u_n$, where u_n are all +ve,

(i) $u_n > u_{n+1}, \forall n,$ and

(ii) $\underset{n\to\infty}{Lt}\, u_n = 0$, then the series is convergent.

Proof :

Let $u_1 - u_2 + u_3 - u_4 + ...$ be an alternating series (' u_n 'are all +ve)

Let $u_1 > u_2 > u_3 > u_4 ......$, Then the series may be written in each of the following two forms :

$$(u_1 - u_2) + (u_3 - u_4) + (u_5 - u_6) + ... \qquad ....(A)$$

$$u_1 - (u_2 - u_3) - (u_4 - u_5) - ..... \qquad ....(B)$$

(A) Shows that the sum of any number of terms is +ve and

(B) Shows that the sum of any number of terms is $< u_1$.

Hence the sum of the series is finite. $\therefore$ The series is convergent.

Note :

If $\underset{n\to\infty}{Lt}\, u_n \neq 0$, then the series is oscillatory.

Solved Examples

6.7.3 Example

Consider the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} .....$

In this series, each term is numerically less than its preceding term and n^{th} term $\to 0$ as $n \to \infty$.

$\therefore$ By Leibneitz's test, the series is convergent.

(Note the sum of the above series is $Log_e 2$)

6.7.4 Example

Test for convergence $\sum \frac{(-1)^{n-1}}{2n-1}$

Solution

The given series is an alternating series $\sum(-1)^{n-1}u_n$, where $u_n=\dfrac{1}{2n-1}$

We observe that (i) $u_n>0,\forall n$ (ii) $u_n>u_{n+1},\forall n$ (iii) $\underset{n\to\infty}{Lt}u_n=0$

$\therefore$ By Leibneitz's test, the given series is convergent.

6.7.5 Example

Show that the series $S=1-\dfrac{1}{3}+\dfrac{1}{9}-\dfrac{1}{27}+......$ converges.

Solution

The given series is $\sum_{1}^{\infty}\dfrac{(-1)^{n-1}}{3^{n-1}}=\sum(-1)^{n-1}u_n$, where $u_n=\dfrac{1}{3^{n-1}}$ is an alternating series in which 1. $u_n>0,\forall n$ 2. $u_n>u_{n+1},\forall n$ and 3. $\underset{n\to\infty}{Lt}u_n=0$;

Hence by Leibneitz's test, it is convergent.

6.7.6 Example

Test for convergence of the series, $\dfrac{x}{1+x}-\dfrac{x^2}{1+x^2}+\dfrac{x^3}{1+x^3}-+.....,0<x<1$

Solution

The given series is of the form $\sum\dfrac{(-1)^{n-1}x^n}{1+x^n}=\sum(-1)^{n-1}u_n$,

where $u_n=\dfrac{x^n}{1+x^n}$ Since $0<x<1$, $u_n>0,\forall n$;

Further, $u_n-u_{n+1}=\dfrac{x^n}{1+x^n}-\dfrac{x^{n+1}}{1+x^{n+1}}$

$$=\frac{x^n-x^{n+1}}{(1+x^n)(1+x^{n+1})}=\frac{x^n(1-x)}{(1+x^n)(1+x^{n+1})}$$

$0<x<1\Rightarrow$ all terms in numerator and denominator of the above expression are +ve.

$\therefore$ $u_n>u_{n+1},\forall n.$

Again, $x^n\to 0$ as $x^n\to\infty$ since $0<x<1$; $\therefore$ $\underset{n\to\infty}{Lt}u_n=\dfrac{0}{1+0}=0$

$\therefore$ By Leibneitz's test, the given series is convergent.

6.7.7 Example

Test for convergence $\sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n(n+1)(n+2)}}$

Solution

The given series is an alternating series $\sum (-1)^{n-1} u_n$

where $u_n = \frac{1}{\sqrt{n(n+1)(n+2)}}$; $u_n > 0, \forall n$;

Again, $\sqrt{(n+1)(n+2)(n+3)} > \sqrt{n(n+1)(n+2)}$

$\therefore$ $\frac{1}{\sqrt{(n+1)(n+2)(n+3)}} < \frac{1}{\sqrt{n(n+1)(n+2)}}, \forall n$

i.e., $u_{n+1} < u_n, \forall n$

Further, $\underset{n\to\infty}{Lt}\, u_n = \underset{n\to\infty}{Lt} \frac{1}{\sqrt{n(n+1)(n+2)}} = 0$

$\therefore$ By Leibnitz's test, $\sum_{2}^{\infty} (-1)^{n-1} u_n$ is convergent

6.7.8 Example

Test for the convergence of the following series,

$$\frac{1}{6} - \frac{2}{11} + \frac{3}{16} - \frac{4}{21} + \frac{5}{26} - + \dots$$

Solution

Given series, $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{5n+1} = \sum (-1)^{n-1} u_n$ is an alternating series

$$u_n = \frac{n}{5n+1} > 0 \forall n \ ;$$

$$\frac{n}{5n+1} - \frac{n+1}{5n+6} = \frac{-1}{(5n+1)(5n+6)} \Rightarrow u_n < u_{n+1}, \forall n$$

Again, $\underset{n\to\infty}{Lt}\, u_n = \underset{n\to\infty}{Lt} \frac{n}{5n+1} = \frac{1}{5} \neq 0$

Thus conditions (ii) or (iii) of Leibnitz's test are not satisfied. The given series is not convergent. It is oscillatory.

6.7.9 Example

Test the nature of the following series.

(a) $\sum_{1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}+\sqrt{n+1}}$ (b) $\sum \frac{(-1)^{n-1}}{n^2+1}$ (c) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\lfloor n+1}$

Solution

(a) $u_n = \frac{1}{\sqrt{n}+\sqrt{n+1}} > 0 \forall n$;

$$u_n - u_{n+1} = \frac{1}{\sqrt{n}+\sqrt{n+1}} - \frac{1}{\sqrt{n+1}+\sqrt{n+2}}$$

$$= \frac{\sqrt{n+2}-\sqrt{n}}{\left(\sqrt{n}+\sqrt{n+1}\right)\left(\sqrt{n+1}+\sqrt{n+2}\right)} = \frac{2}{\left(\sqrt{n+2}+\sqrt{n}\right)\left(\sqrt{n}+\sqrt{n+1}\right)\left(\sqrt{n+1}+\sqrt{n+2}\right)} > 0$$

$\therefore$ By Leibnitz's test the series converges.

(b) $u_n = \frac{1}{n^2+1} > 0, \forall n;\ \frac{1}{n^2+1} > \frac{1}{(n+1)^2+1} \Rightarrow u_n > u_{n+1}, \forall n;$

$\underset{n\to\infty}{Lt}\, u_n = 0$ $\therefore$ By Leibnitz's test, given series converges.

(c) $u_n = \frac{1}{\lfloor n+1} > 0, \forall n;$

$$\lfloor n+2 > \lfloor n+1 \Rightarrow \frac{1}{\lfloor n+2} < \frac{1}{\lfloor n+1} \Rightarrow u_n > u_{n+1}, \forall n$$

By Leibnitz's test, given series converges.

6.7.10 Example

Test the convergence of the series $\frac{1}{5\sqrt{2}} - \frac{1}{5\sqrt{3}} + \frac{1}{5\sqrt{4}} - + \ldots\ldots$

Solution

The series can be written as $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{5\sqrt{n+1}}$; $u_n = \frac{1}{5\sqrt{n+1}}$

(i) $u_n > 0 \forall n$

(ii) $5\sqrt{n+2} > 5\sqrt{n+1} \Rightarrow \frac{1}{5\sqrt{n+2}} < \frac{1}{5\sqrt{n+1}} \Rightarrow u_n > u_{n+1} \forall n$

(iii) $\underset{n\to\infty}{Lt}\, u_n = 0$;

By Leibnitz's test, the given series converges.

6.7.11 Example

Test for convergence the series, $1-\frac{1}{2}+\frac{1}{4}-\frac{1}{6}+.....$

Solution

The given series can be written as $\sum\frac{(-1)^n}{2n}$ (omitting 1st term)

$$\frac{1}{2n}>0\forall n;\ \frac{1}{2n}>\frac{1}{2n+2}\Rightarrow u_n>u_{n+1},\forall n;\ \underset{n\to\infty}{Lt}\frac{1}{2n}=0$$

$\therefore$ By Leibnitz's test, $\sum\frac{(-1)^n}{2n}$ is convergent.

6.7.12 Example

Test for convergence the series, $1-\frac{1}{3!}+\frac{1}{5!}-\frac{1}{7!}+......$

Solution

General term of the series is $\frac{(-1)^{n-1}}{(2n-1)!}$

The series is an alternating series; $\frac{1}{(2n-1)!}>0\forall n$

$$\frac{1}{(2n-1)!}>\frac{1}{(2n-1)!}\Rightarrow u_n>u_{n+1},\forall n\in N;\ \underset{n\to\infty}{Lt}\frac{1}{(2n-1)!}=0$$

By Leibnitz's test, given series is convergent.

6.8 Absolute convergence

A series $\sum u_n$ is said to be absolutely convergent if the series $\sum|u_n|$ is convergent

6.8.1

Consider the series

$$\sum u_n=1-\frac{1}{2^3}+\frac{1}{3^3}-\frac{1}{4^3}+-+......$$

$$\sum|u_n|=1+\frac{1}{2^3}+\frac{1}{3^3}+\frac{1}{4^3}+......=\sum_1^\infty\frac{1}{n^3}$$

By $p-$ series test, $\sum|u_n|$ is convergent ($p=3>1$)

Hence $\sum u_n$ is absolutely convergent.

Note :

1. If $\sum u_n$ is a series of +ve terms, then $\sum u_n = \sum |u_n|$.

   For such a series, there is no difference between convergence and absolute convergence. Thus a series of +ve terms is convergent as well as absolutely convergent.

2. An absolutely convergent series is convergent. But the converse need not be true.

   Consider $$\sum_{1}^{\infty}(-1)^{n-1}.\frac{1}{n}=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+......$$

   This series is convergent (1.7.3)

   But $\sum\left|(-1)^{n-1}.\frac{1}{n}\right|=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+.......$ is divergent (p-series test).

   Thus $\sum u_n$ is convergent need not imply that $\sum |u_n|$ is convergent (i.e., $\sum u_n$ is not absolutely convergent).

6.9

6.9.1 Conditional Convergence

If the series $\sum|u_n|$ is divergent and $\sum u_n$ is convergent, then $\sum u_n$ is said to be conditionally convergent.

6.9.2 Consider the Series

$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}.......$ $\sum u_n$ is convergent by Leibnitz's test. (Ex.1.7.3)

But $\sum|u_n|=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}.....$ is divergent by $p-$ series test.

$\therefore$ $\sum u_n$ is conditionally convergent.

6.10

6.10.1 Power Series and Interval of Convergence

A series, $a_0+a_1x+a_2x^2+.....+a_nx^n+.....$ where 'a_n' are all constants is a power series in x.

It may converge for some values of x.

$$\underset{n\to\infty}{Lt}\frac{u_{n+1}}{u_n}=\underset{n\to\infty}{Lt}\frac{a_{n+1}}{a_n}.x \quad (1^{st} \text{ term is omitted.})$$

$$=kx \text{ (say) where } \quad \underset{n\to\infty}{Lt}\frac{a_{n+1}}{a_n}=k$$

Then, by ratio test, the series converges when $|kx|<1$.

i.e., it converges $\forall x\in\left(\frac{-1}{k},\frac{1}{k}\right)(k\neq 0)$

The interval $\left(\frac{-1}{k},\frac{1}{k}\right)$ is known as the interval of convergence of the given power series.

Solved Examples

6.10.2 Example

Find the interval of convergence of the series $\sum_{n=1}^{\infty}\frac{x^n}{n^3}$

Solution

$$u_n=\frac{x^n}{n^3};u_{n+1}=\frac{x^{n+1}}{(n+1)^3}$$

$$\underset{n\to\infty}{Lt}\left(\frac{u_{n+1}}{u_n}\right)=\underset{n\to\infty}{Lt}\left(\frac{n}{n+1}\right)^3.x=\underset{n\to\infty}{Lt}\left(\frac{1}{1+\frac{1}{n}}\right)^3.x=x$$

By ratio test, the given series converges when $|x|<1$, i.e., $x\in(-1,1)$

When $x=1,\sum u_n=\sum 1/n^3$, which, is convergent by p series test.

$\therefore \sum u_n$ is convergent when $x=1$

Hence, the interval of convergence of the given series is (-1, 1]

6.10.3 Example

Test for the convergence of the following series.

(a) $1-1/\sqrt{2}+1/\sqrt{3}-1/\sqrt{4}+\dots\dots\dots$

(b) $1+1/2^2-1/3^2-1/4^2+1/5^2+1/6^2-1/7^2-1/8^2+\dots\dots$

(c) $1 - x^2/\underline{|2} + x^4/\underline{|4} - x^6/\underline{|6} + \dots\dots\dots$

(d) $\sum_0^{\infty}(-1)^n(n+1)x^n$, with $x < \dfrac{1}{2}$

Solution

(a) The series is of the form $\sum(-1)^{n-1}u_n$ where $u_n = 1/\sqrt{n}$

It is an alternating series where (i) $u_n > 0\forall_n$ (ii) $u_n > u_{n+1}\forall n$ and (iii) $\underset{n\to\infty}{Lt}\, u_n = 0; \therefore$ By Leibnitz test, the series is convergent.

Again the series $1 + 1/\sqrt{2} + 1/\sqrt{3} + 1/\sqrt{4} + \dots\dots$ is divergent, by $p-$ series test.

Hence the given series is conditionally convergent.

(b) $\sum|u_n| = \sum_{p=1}^{\infty} 1/n^2$ which is convergent by $p-$ series test.

$\therefore$ The given series is absolutely convergent.

$\therefore$ It is convergent.

(c) The given series is

$$\sum(-1)^{n-1}.\frac{x^{2n-2}}{(2n-2)!}. = \sum(-1)^{n-1}u_n; \quad \therefore |u_n| = \frac{x^{2n-2}}{(2n-2)!}$$

$$u_{n+1} = \frac{x^{2n}}{2n!}; \quad \left|\frac{u_{n+1}}{u_n}\right| = \frac{1}{(2n-1)(2n)}.\left|x^2\right|; \quad \underset{n\to\infty}{Lt}\left|\frac{u_{n+1}}{u_n}\right| = 0 < 1$$

By ratio test, the series $\sum|u_n|$ converges $\forall x$; i.e., $\sum u_n$ is absolutely convergent $\forall x$;

(d) Here, $|u_n| = (n+1)x^n; |u_{n+1}| = (n+2)x^{n+1}$ (neglect 1[st] term)

$$\underset{n\to\infty}{Lt}\frac{|u_{n+1}|}{|u_n|} = \underset{n\to\infty}{Lt}\frac{(n+2)}{(n+1)}|x| = \underset{n\to\infty}{Lt}\frac{(1+2/n)}{(1+1/n)}|x| = |x| < 1 \quad (\because x < 1/2)$$

$\therefore \sum|u_n|$ is convergent $\forall x$, i.e., given series is absolutely convergent and hence convergent.

6.10.4

Show that the series $1 + x + x^2/\lfloor 2 + x^2/\lfloor 3 + \ldots\ldots$ converges absolutely $\forall x$

Solution

$$\underset{n\to\infty}{Lt} \frac{|u_{n+1}|}{|u_n|} = \frac{|x|}{n} = 0 < 1 \text{ when } x \neq 0 \text{ [since } |u_n| = \frac{|x^{n-1}|}{(n-1)!}; |u_{n+1}| = \frac{|x^n|}{n!} \text{]}$$

$\therefore$ By ratio test, $\sum |u_n|$ is convergent $\forall x \neq 0$.

When $x = 0$, the series is (1 + 0 + 0 + ...) and is convergent

$\therefore$ $\sum |u_n|$ converges $\Rightarrow \sum u_n$ is absolutely convergent $\forall x$.

6.10.5 Example

Show that the series, $1 - \frac{1}{3} + \frac{1}{3^2} - \frac{1}{3^4} + \ldots\ldots..$ is absolutely convergent.

Solution

$\sum |u_n| = \sum_{n=1}^{\infty} 1/3^{n-1}$, which is a geometric series with common ratio $\frac{1}{3} < 1$

$\therefore$ It is convergent. Hence given series is absolutely convergent.

6.10.6 Example

Test for convergence, absolute convergence and conditional convergence of the series,

$$1 - \frac{1}{5} + \frac{1}{9} - \frac{1}{13} + \ldots\ldots\ldots$$

Solution

The given alternating series is of the form $\sum (-1)^{n-1} u_n$, where, $u_n = \frac{1}{4n-3}$.

Hence, $u_n > 0 \forall n \in N$; $u_{n+1} = \frac{1}{4(n+1)-3} = \frac{1}{4n+1}$

$$u_n - u_{n+1} = \frac{1}{4n-3} - \frac{1}{4n+1}$$

$$= \frac{4n+1-4n+3}{(4n-3)(4n+1)} = \frac{4}{(4n-3)(4n+1)} > 0, \forall n \in N$$

i.e., $u_n > u_{n+1}, \forall n \in N$ $\underset{n\to\infty}{Lt} u_n = \underset{n\to\infty}{Lt} \frac{1}{4n-3} = 0$;

All conditions of Leibnitz's test are satisfied.

Hence $\sum (-1)^{n-1} u_n$ is convergent.

$|u_n| = \frac{1}{4n-3}$; Take $v_n = \frac{1}{n}$; $\underset{n\to\infty}{Lt} \frac{|u_n|}{v_n} = \underset{n\to\infty}{Lt} \frac{n}{n(4-3/n)} = \frac{1}{4} \neq 0$ and finite.

$\therefore$ By comparison test, $\sum |u_n|$ and $\sum v_n$ behave alike.

But by $p-$ series test, $\sum v_n$ is divergent (since $p = 1$) .

$\sum |u_n|$ is divergent and $\therefore$ The given series is conditionally convergent.

6.10.7 Example

Test the series $\sum_{n=1}^{\infty} (-1)^{n-1} . \frac{1}{3\sqrt{n}}$, for absolute / conditional convergence .

Solution

The given series is an alternating series of the form $\sum (-1)^{n-1} u_n$.

Here

(i) $u_n = \frac{1}{3\sqrt{n}}, \forall n \in N$

(ii) $3(n+1) > 3n \Rightarrow 3\sqrt{n+1} > 3\sqrt{n}, \forall n$.

$\therefore \frac{1}{3\sqrt{n+1}} < \frac{1}{3\sqrt{n}}$, i.e., $u_{n+1} < u_n, \forall n \in N$

And $\underset{n\to\infty}{Lt}\, u_n = \underset{n\to\infty}{Lt} \frac{1}{3\sqrt{n}} = 0$

$\therefore$ By Leibnitz's test, the given series is convergent.

But $\sum \left| (-1)^{n-1} . \frac{1}{3\sqrt{n}} \right| = \sum \frac{1}{3\sqrt{n}}$ is divergent by $p-$ series test (since $p = \frac{1}{2} < 1$)

$\therefore$ The given series is conditionally convergent.

6.10.8 Example

Test the following series for absolute / conditional convergence.

(a) $\sum_{n=1}^{\infty} (-1)^{n-1} , \frac{\sin(n\alpha)}{n^2}$ (b) $\sum_{n=1}^{\infty} (-1)^{n-1} . \frac{n^2}{n^3+1}$

(c) $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}$ (d) $\sum (-1)^{n-1} . \frac{n\pi^n}{e^{3n+1}}$

Solution

(a) $|u_n| = \dfrac{|\sin n\alpha|}{n^2} < \dfrac{1}{n^2}$ $[\text{since} |\sin n\alpha| < 1]$ considering $v_n = 1/_{n^2}$ and using comparison and p - series tests, we get that $\sum |u_n|$ is convergent $\sum u_n$ is absolutely convergent .

(b) By Leibnitz's test, the series converges.

Taking $v_n = \dfrac{1}{n}$, by comparison and p - series tests , $\sum \dfrac{n^2}{n^3+1}$, is seen to be divergent.

Hence given series is conditionally convergent.

(c) Take $|u_n| = \dfrac{1}{2n!}$; $\underset{n\to\infty}{Lt} \dfrac{|u_{n+1}|}{|u_n|} = 0 < 1$; By ratio test, $\sum |u_n|$ is convergent;

Hence given series is absolutely convergent.

(d) $|u_n| = \dfrac{n\pi^n}{e^{3n+1}}$; By root test, is convergent, $\therefore$ given series is absolutely convergent.

[In problems (a) to (d) above, hints only are given. Students are advised to do the complete problem themselves]

6.10.9 Example

Find the interval of convergence of the following series.

(a) $\sum\limits_{n=1}^{\infty} (-1)^{n-1} x^n/_{n^3}$ (b) $\sum\limits_{n=1}^{\infty} (-1)^{n-1} \dfrac{n(x+2)^n}{3^n}$

(c) $\log(1+x)$

Solution

(a) Let the given series be $\sum u_n$; Then $|u_n| = \dfrac{|x^n|}{n^3}$; $|u_{n+1}| = \dfrac{|x^{n+1}|}{(n+1)^3}$

$$\underset{n\to\infty}{Lt} \frac{|u_{n+1}|}{|u_n|} = \underset{n\to\infty}{Lt} \left(\frac{n}{n+1}\right)^3 .|x| = \underset{n\to\infty}{Lt} \left(\frac{1}{1+\frac{1}{n}}\right)^3 .|x| = |x|$$

$\therefore$ By ratio test, $\sum |u_n|$ is convergent if $|x| < 1$

i.e., $\sum u_n$ is absolutely convergent if $|x| < 1$;

$\therefore$ $\sum u_n$ is convergent if $|x| < 1$

If $x = 1$, the given series becomes $1-\frac{1}{2^3}+\frac{1}{3^3}-\frac{1}{4^3}+........$

which is convergent, since $\sum\frac{1}{n^3}$ is convergent.

Similarly, if $x = -1$, the series becomes $\sum-\frac{1}{n^3}=-\sum\frac{1}{n^3}$ which is also convergent.

Hence the interval of convergence of $\sum u_n$ is $(-1\le x\le 1)$

(b) Proceeding as in (a),

$$\underset{n\to\infty}{Lt}\frac{|u_{n+1}|}{|u_n|}=\frac{|x+2|}{3}$$

$\therefore \sum u_n$ is convergent if $|x+2|<3$, i.e., if $-3<x+2<3$, i.e., if $-5<x<1$.

If $x=-5$, $\sum u_n=\sum(-1)^{2n-1}.n$, and is divergent (in both these cases

If $x=1$, $\sum u_n=\sum(-1)^{n-1}.n$, and is divergent $\underset{n\to\infty}{Lt}\, u_n\ne 0$)

Hence the interval of convergence of the series is $(-5<x<1)$

(c) $\log(1+x)= x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+.......$

$$=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}=\sum u_n \quad \text{(say)}$$

$$|u_n|=\frac{|x^n|}{n};\ |u_{n+1}|=\frac{|x^{n+1}|}{n+1}$$

$$\underset{n\to\infty}{Lt}\frac{|u_{n+1}|}{|u_n|}=\underset{n\to\infty}{Lt}\frac{1}{\left(1+\frac{1}{n}\right)}|x|=|x|$$

By ratio test, $\sum|u_n|$ is convergent when $|x|<1$

i.e., $\sum u_n$ is absolutely convergent and hence convergent when $-1<x<1$.

When $x=-1$, $\sum u_n=\sum(-1)^{n-1}.\frac{1}{n}=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+.......$,

which is convergent by Leibnitz's test. (give the proof)

When $x = 1$, $\sum u_n = -\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+......\right)$ which is divergent,

since $\sum \frac{1}{n}$ is divergent by p –series test (prove).

Hence $\sum u_n$ is convergent when $-1 < x \le 1$

Interval of convergence is ($-1 < x \le 1$).

Exercise -1 (e)

1. Use integral test and determine the convergence or divergence of the following series:

1 $\sum \frac{1}{n^2}$ .... [**Ans** : convergent]

2. $\sum_2^\infty \frac{1}{n(\log n)^2}$ .... [**Ans** : convergent]

2. Test for convergence of the following series:

1 $1-\frac{1}{\lfloor 2}+\frac{1}{\lfloor 4}-\frac{1}{\lfloor 6}+....$ .... [**Ans** : convergent]

2. $\sum_1^\infty \frac{(-1)^{n-1}}{(2n-1)(2n)}$ .... [**Ans** : convergent]

3. $\sum_1^\infty (-1)^{n-1} n^{-5/2}$ .... [**Ans** : convergent]

3. Classify the following series into absolutely convergent and conditionally convergent series :

1. $\sum \frac{(-1)^n}{n^3}$ .... [**Ans** : abs.cgt]

2. $\sum \frac{\sin\sqrt{n}}{n^{3/2}}$ .... [**Ans** : abs.cgt]

3. $\sum \frac{(-1)^n}{n(\log n)^2}$ .... [**Ans** : abs.cgt]

4. Find the interval of convergence of the following series :

1. $\sum \frac{2^n x^n}{\lfloor n}$ .... [**Ans** : $-\infty < x < \infty$]

2. $\sum \frac{x^n}{n^2}$ .... **[Ans : $-1 \le x \le 1$]**

3. $x - \frac{x^2}{\sqrt{2}} + \frac{x^3}{\sqrt{3}} - \frac{x^4}{\sqrt{4}} + \ldots\ldots$ .... **[Ans : $-1 < x \le 1$]**

5. (a) Show that $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots.$ is absolutely convergent.

(b) Show that $1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \ldots.$ is conditionally convergent.

Tests of Convergence – A Summary

1. The geometric series $\sum_{n=1}^{\infty} x^{n-1}$ converges if $|x| < 1$, diverges if $x \ge 1$, and oscilates when $x \le -1$

2. If $\sum u_n$ is convergent, $\underset{n\to\infty}{Lt}\, u_n = 0$ [The convergent need not be necessary]

3. p – series test :- $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \le 1$

4. *Comparison test :-* The series $\sum u_n$ and $\sum v_n$ are both convergent or both divergent if $\underset{n\to\infty}{Lt} \frac{u_n}{v_n}$ is finite and non – zero.

5. *D'Alembert'sRatio test :-* $\sum u_n$ converges or diverges according as

$$\underset{n\to\infty}{Lt} \frac{u_{n+1}}{u_n} < 1 \quad \text{or} \quad > 1$$

$\left(\text{Alternately, if } \underset{n\to\infty}{Lt} \frac{u_n}{u_{n+1}} > 1 \quad \text{or} \quad < 1 \right)$. If the limit = 1, the test fails

6. *Raabe's test :* $\sum u_n$ converges or diverges according as

$$\underset{n\to\infty}{Lt} \left[n \left\{ \frac{u_n}{u_{n+1}} - 1 \right\} \right] > 1 \text{ or } < 1 \ .$$

7. *Cauchy's root test :* $\sum u_n$ converges or diverges according as $\underset{n\to\infty}{Lt} \left(u_n^{\frac{1}{n}} \right) < 1 \text{ or } > 1$

(If limit = 1, the test fails.)

8. *Integral test :* A series $\sum\phi(n)$ of +ve terms where $\phi(n)$ decreases as n increases is convergent or divergent according as the integral $\int_1^\infty \phi(x)dx$ is finite or infinite.

9. *Alternating series – Leibnitz's test:* An alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}u_n$ convergent if (i) $u_n = u_{n+1}, \forall n$ and (ii) $\underset{n\to\infty}{Lt}\, u_n = 0$

10. Absolute / conditional convergence :
    (a) $\sum u_n$ is absolutely convergent if $\sum|u_n|$ is convergent.
    (b) $\sum u_n$ is conditionally convergent if $\sum u_n$ is convergent and $\sum|u_n|$ is divergent.
    (c) An absolutely convergent series is convergent, but converse need not be true . i.e., a convergent series need not be convergent.

Miscellaneous Exercise - 1 (e)

1. Examine the convergence of the following series:

1. $\frac{1}{1.2}+\frac{1}{3.4}+\frac{1}{5.6}+......$ .... [cgt.]
2. $\frac{1^2}{1^3+1}+\frac{2^2}{2^3+1}+\frac{3^2}{3^3+1}+......$ .... [dgt.]
3. $\frac{2}{1}+\frac{2^2}{2}+\frac{2^3}{3}+.......$ .... [dgt.]
4. $\frac{1}{1+2}+\frac{2}{1+2^2}+\frac{3}{1+2^3}+.....$ .... [cgt.]
5. $\frac{x}{1+x}+\frac{x^2}{1+x^2}+\frac{x^3}{1+x^3}+.....(x>0)$ .... [cgt. if $x\le 1$ dgt. if $x>1$]
6. $2x+\frac{3x^2}{8}+\frac{4x^3}{27}+.....(x>0)$ .... [cgt. if $x\le 1$ dgt. if $x>1$]
7. $1+\frac{1}{2}+\frac{1.3}{2.4}+\frac{1.3.5}{2.4.6}+....$ .... [dgt.]
8. $\frac{3^2}{6^2}+\frac{3^2.5^2}{6^2.8^2}+\frac{3^2.5^2.7^2}{6^2.8^2.10^2}+......$ .... [cgt]
9. $\frac{3.4}{1.2}+\frac{4.5}{2.3}+\frac{5.6}{3.4}+....$ .... [dig.]

10. $\frac{(\lfloor 1)^2}{\lfloor 2}.x+\frac{(\lfloor 2)^2}{\lfloor 4}x^2+\frac{(\lfloor 3)^3}{\lfloor 6}x^3+....(x>0)$ .... [cgt. if $x<4$, dgt. if $x \geq 4$]

11. $1+\frac{x}{2^2}+\frac{x^2}{3^2}+\frac{x^3}{4^2}+.....(x>0)$ .... [cgt.if $x \leq 1$, dgt. if $x>1$]

12. $\frac{3x}{4}+\left(\frac{4}{5}\right)^2 x^2+\left(\frac{5}{6}\right)^3+x^3+.....(x>0)$ .... [cgt if $x<1$, dgt. if $x \geq 1$]

13. $\sum\left(1+\frac{1}{n}\right)^{n^2}$ .... [dgt.]

14. $\sum\frac{2^{3n}}{3^{2n}}$ .... [cgt.]

15. $\sum\frac{a^n}{1+n^2}, a<1$ .... [cgt.]

16. $1-\frac{1}{2.2}+\frac{1}{3.3}-\frac{1}{4.4}+-+-.....$ .... [Abs. cgt.]

2. Examine for absolute and conditional convergence of the following series :

1. $\sum(-1)^n\frac{3^{3n}}{3^{2n}}$ .... [Abs. cgt.]

2. $\sum\frac{(-1)^n.n}{2^n}$ .... [Abs. cgt.]

3. $1-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{4}}+\frac{1}{\sqrt{5}}$ .... [Cond. cgt]

4. $\sum(-1)^n\frac{(n^2+1)}{n^3}$ .... [Cond.cgt.]

3. Determine the interval of convergence of the following series :

1 $x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+.....$ .... [$-1 \leq x<1$]

2. $\sum\frac{(x+1)^n}{n.2^n}$ .... [$-3<x<1$]

Exercise - 1 (g) (Objective type questions)

1. The infinite series $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ is

(i)	convergent	(ii)	divergent	
(iii)	oscillatory	(iv)	none of these	**[Ans :(i)]**

2. The series $\frac{1+n}{1+n^2}$ is

(i)	convergent	(ii)	divergent	
(iii)	oscillatory	(iv)	none of these	**[Ans :(ii)]**

3. The series $\frac{1}{1\sqrt{1}} - \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} - \frac{1}{4\sqrt{4}} + \ldots..$

(i)	oscillatory	(ii)	absolutely convergent	
(iii)	conditionally convergent	(iv)	none of these	**[Ans :(ii)]**

4. The series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \ldots\ldots$ is

(i)	oscillatory	(ii)	divergent	
(iii)	convergent	(iv)	none of these	**[Ans :(iii)]**

5. The interval of convergence of the series $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots$,is

(i)	$-\infty < x < \infty$	(ii)	$-1 < x < 2$	
(iii)	$-1 < x \leq 1$	(iv)	none of these	**[Ans :(iii)]**

6. The series $\frac{1}{1.2} + \frac{2}{3.4} + \frac{3}{5.6} + \ldots\ldots\ldots\infty$ is

(i)	convergent	(ii)	divergent	
(iii)	oscillatory	(iv)	none of these	**[Ans :(ii)]**

7. The series $\frac{1}{1.3} + \frac{2}{3.5} + \frac{3}{5.7} + \ldots\ldots\ldots\infty$, is

(i)	conditionally convergent	(ii)	convergent	
(iii)	divergent	(iv)	none of these	**[Ans :(iii)]**

8. The series $\frac{2}{1^p} + \frac{3}{2^p} + \frac{4}{3^p} + \frac{5}{4^p} + \ldots\ldots\infty$ is convergent if

(i)	$p < 2$	(ii)	$p = 2$	
(iii)	$p > 2$	(iv)	none of these	**[Ans :(iii)]**

9. The series 6 – 10 + 4 + 6 – 10 + 4 + 6 – 10 + 4 + 6............ ∞ is

(i)	convergent	(ii)	oscillatory	
(iii)	divergent	(iv)	none of these	**[Ans :(ii)]**

10. The series $\frac{1}{2.4}+\frac{1}{4.6}+\frac{1}{6.8}+........$ is

(i) convergent (ii) divergent
(iii) oscillatory (iv) none of these **[Ans :(i)]**

2. Indicate whether the following statements are true or false :

1. The series $\sum\frac{1}{1+2^{-n}}$ is convergent . ... [False]
2. The series $\sum\frac{n^2+5}{2n^2+7}$ is not convergent . ... [True]
3. The series $\frac{1}{\underline{|1}}+\frac{1}{\underline{|2}}+\frac{1}{\underline{|3}}+........$ is divergent................... [False]
4. The series $x-\frac{x^3}{3}+\frac{x^5}{5}-+-.......,$ converges when $-1\le x\le 1$ .. [True]
5. The series $\sum\frac{(-1)^{n-1}}{n.5^n}$ is absolutely convergent. .... [True]
6. The series $x+2x^2+3x^3+4x^4+....\infty$ is convergent if $x>1$........ [False]
7. The series $x+\frac{x^2}{2}+\frac{x^3}{3}+........\infty$ is divergent if $x\ge 1$ .... [True]
8. The series $1+\frac{x}{2}+\frac{2!}{3^2}x^3+\frac{3!}{4^3}x^3+\frac{4!}{5^4}x^4+......+\infty$ is convergent if $x>e$ .... [True]
9. The series $\sum\left(1+\frac{1}{\sqrt{n}}\right)^{-n^{3/2}}$ is divergent .... [False]
10. The series $\frac{1}{2}+\frac{2}{3}x+\left(\frac{3}{4}\right)^2x^2+\left(\frac{4}{5}\right)^3x^3+.......$ is convergent if $x<1$ . ... [True]
11. The series $1-2x+3x^2-4x^3+.....\infty(x<1)$ is divergent........... [False]
12. The series $\frac{x}{1+x}-\frac{x^2}{1+x^2}+\frac{x^3}{1+x^3}-\frac{x^4}{1+x^4}+......\infty$ is convergent ...... [True]
13. The series $\frac{\sin x}{1^3}-\frac{\sin 2x}{2^3}+\frac{\sin 3x}{3^3}+......\infty$ converges absolutely...... [True]

14. The series $\sum \frac{(-1)^{n-1}}{\sqrt{n}}$ is conditionally convergent. .... [True]

15. The series whose n^{th} term is $\frac{3n^2+5}{(n+2)^a}$ is convergent. [False]

3. Fill in the Blanks :

1. The geometric series $\sum_{n=1}^{\infty} ar^{n-1}$ converges if__________. [**Ans:** $|r|<1$]
2. If a series of +ve terms $\sum u_n$ is convergent, $\underset{n\to\infty}{Lt}\, u_n =$ ______. . [**Ans:** 0]
3. $\sum_{n=1}^{\infty}\left\{\sqrt[3]{n^3+1}-n\right\}$ is __________. [**Ans:** convergent.]
4. If $\sum_{n=1}^{\infty}\frac{3n^3-4}{(n+5)^p}$ is divergent ,value of p is ________. [**Ans:** ≤ 4]
5. The interval of convergence of $\sum u_n$ where $u_n = \left(\frac{n^2-2}{n^2+2}\right)^{2n} x$, is ______.

   [**Ans:** $-1<x<1$]
6. $\sum u_n$ is convergent series of +ve series. Then $\underset{n\to\infty}{Lt}\left(u_n^{1/n}\right)$ is ________.

   [**Ans:** <1]
7. The series 8 – 12 + 4 + 8 – 12 – 4 + ...is _________. [**Ans** : Oscillatory]
8. If $u_n>0, \forall n$ and $\sum u_n$ is convergent, then $\underset{n\to\infty}{Lt}\left[n\left\{\frac{u_n}{u_{n+1}}-1\right\}\right]$ is ______.

   [**Ans:** >1]
9. If the series $\sum_{n=1}^{\infty}(-1)^n a_n, (a_n>0\forall n)$ is convergent, then for all values of n, $\frac{a_n}{a_{n+1}}$

   is______. [**Ans:** >1]
10. If $u_n = \left(1+\frac{1}{n}\right)^{-n^2}$, $\underset{n\to\infty}{Lt}\, u_n^{1/n} =$ ____________. [**Ans:** $1/e$]

Tests of Convergence – A Summary

1. The geometric series $\sum_{n=1}^{\infty} x^{n-1}$ converges if $|x| < 1$, diverges if $x \geq 1$ and oscilates when $x \leq 1$.
2. If $\sum u_n$ is convergent, $\underset{n\to\infty}{Lt}\, u_n = 0$ [The converse need not be necessary]
3. p – series test:- $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if p > 1 and divergent if $p \leq 1$
4. *Comparison test :* The series $\sum u_n$ and $\sum v_n$ are both convergent or both divergent if $\underset{n\to\infty}{Lt} \frac{u_n}{v_n}$ is finite and non – zero.
5. *D'ALEMBERT'S Ratio test :* $\sum u_n$ converges or diverges according as $\underset{n\to\infty}{Lt} \frac{u_{n+1}}{v_n} < 1$ or > 1 (Alternately, $\underset{n\to\infty}{Lt} \frac{u_{n+1}}{v_n} < 1$ or > 1). If the limit = 1, the test fails.
6. *Raabe's test:* $\sum u_n$ converges or diverges according as $\underset{n\to\infty}{Lt}\left[n\left\{\frac{u_n}{u_{n+1}} - 1\right\}\right] > 1$ or < 1
7. *Cauchy's root test:* $\sum u_n$ converges or diverges according as $\underset{n\to\infty}{Lt}\left(u_n^{\frac{1}{n}}\right) < 1$ or > 1; (If the Limit = 1, The test fails).
8. *Integral test :* A series $\sum \phi(n)$ of +ve terms where $\phi(n)$ decreases as n increases is convergent or divergent according as integral $\int_1^{\infty} \phi(x)\,dx$ is finite or infinite.
9. *Alternating series – Leibnitz's test :* An alternating series $\sum_{n=1}^{\infty} (-1)^{n-1} u_n$ (where $u_n > 0 \forall n$) is convergent if (i) $u_n > u_{n+1}$, $\forall n$ (ii) $\underset{n\to\infty}{Lt}\, u_n = 0$
10. Absolute/ conditional convergence :
    (a) $\sum u_n$ is absolutely convergent if $\sum |u_n|$ is convergent .
    (b) $\sum u_n$ is conditionally convergent if $\sum u_n$ is convergent and $\sum |u_n|$ is divergent.
    (c) An absolutely convergent series is convergent but converse need not be true. i.e., a convergent series need not be convergent.

Solved University Questions

1. Test the convergence of the series:

$$\frac{1}{1.2.3}+\frac{2}{2.3.4}+\frac{3}{3.4.5}+\dots\dots\dots$$

Solution

Let u_n be the n^{th} term of the series;

Then, $$u_n=\frac{n}{n(n+1)(n+2)}=\frac{1}{(n+1)(n+2)}$$

Let $$v_n=\frac{1}{n^2}\text{; then, } \underset{n\to\infty}{Lt}\frac{u_n}{v_n}=\underset{n\to\infty}{Lt}\frac{n^2}{(n+1)(n+2)}$$

$$=\underset{n\to\infty}{Lt}\frac{1}{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}=1,$$

Which is non-zero and finite.

$\therefore$ By comparison test, both $\sum u_n$ and $\sum v_n$ converge or diverge together.

But $\sum v_n$ is convergent by p-series test ($p > 1$) $\therefore$ $\sum u_n$ is convergent.

2. Show the every convergent sequence is bounded

Solution

Let $\langle a_n \rangle$ be a sequence which converges to a limit 'l 'say.

$\underset{n\to\infty}{Lt}\, a_n = l \Rightarrow$ given any +ve number $\in$, however small ,

we can always find an integer ' m ' , $\ni$, $|a_n - l| < \in$, $\forall n \ge m$

Taking $\in = 1$, we have, $|a_n - l| < 1$;

i.e., $$(l-1) < a_n < (l+1),\ \forall n \ge m$$

Let λ = min $\{a_1, a_2, \dots\dots\dots a_{m-1}, (l-1)\}$, and μ = max$\{a_1, a_2, \dots\dots\dots a_{m-1}, (l+1)\}$

Then obviously, $\lambda \le a_n \le \mu$, $\forall n \in N$;

Hence $\langle a_n \rangle$ is bounded.

3. Show that the series,

$S = 1-\frac{1}{3!}+\frac{1}{5!}-\frac{1}{7!}+\dots\dots\dots$ converges.

Solution

The given series is an alternating series $\sum(-1)^{n-1}u_n$,

where $u_n = \dfrac{1}{(2n-1)!}$ We observe that,

(i) $u_n > 0$ and $u_n > u_{n+1}, \forall n$ and

(ii) $\underset{n\to\infty}{Lt}\, u_n = \underset{n\to\infty}{Lt}\, \dfrac{1}{(2n-1)!} = 0$

$\therefore$ By Leibneitz's test [7 .2] the given series converges .

4. Show that the geometric series $\sum_{m=0}^{\infty} q^m = 1 + q + q^2 + \ldots\ldots$ converges to the sum

$\dfrac{1}{1-q}$ when $|q| < 1$ and diverges when $|q| \geq 1$

Solution

See theorm 2.3 (replace 'x' by 'q') .

5. Define the convergence of a series. Explain the absolute convergence and conditional convergence of a series. Test the convergence of series

$$\sum\left[1+\frac{1}{\sqrt{n}}\right]^{-n^2}$$

Solution

For theory part , refer 2 .1 , 2.2 , 8.1 , 9.1, and 9.2

Problem : Let $u_n = \left(1+\dfrac{1}{\sqrt{n}}\right)^{-n^2}$; $\underset{n\to\infty}{Lt}\left(u_n^{1/n}\right) = \underset{n\to\infty}{Lt}\left(1+\dfrac{1}{\sqrt{n}}\right)^{-n}$

$$= \underset{n\to\infty}{Lt}\frac{1}{\left[1+\frac{1}{\sqrt{n}}\right]^n} = \frac{1}{e^2} < 1$$

By Cauchy's root test, $\sum u_n$ is convergent.

6. Test the convergence of the series, $1 + \dfrac{1}{2}x + \dfrac{1.3}{2.4}x^2 + \dfrac{1.3.5}{2.4.6}x^3 + \ldots..$

Given that $x > 0$.

Solution

Omitting the first term of the series, we have, $\dfrac{1.3.5\ldots..(2n-1)}{2.4.6\ldots..2n}x^n$ and

$$u_n = \frac{1.3.5.(2n-1)}{2.4.6....2n}x^n \;;\; u_{n+1} = \frac{1.3.5.....(2n+1)}{2.4.6.....(2n+2)}.x^{n+1};$$

$$\underset{n\to\infty}{Lt}\frac{u_{n+1}}{u_n} = \underset{n\to\infty}{Lt}\left(\frac{2n+1}{2n+2}\right).x = x$$

By ratio test, $\sum u_n$ is convergent when $x < 1$, and divergent when $x > 1$

The ratio test fails when $x = 1$

When $x = 1$, $\quad \frac{u_n}{u_{n+1}} - 1 = \frac{2n+2}{2n+1} - 1 = \frac{1}{2n+1}$

$$\underset{n\to\infty}{Lt}\left[n\left(\frac{u_n}{u_{n+1}} - 1\right)\right] = \underset{n\to\infty}{Lt}\left(\frac{n}{2n+1}\right) = \frac{1}{2} < 1\;;$$

$\therefore$ By Raabe's test, $\sum u_n$ diverges.

$\therefore$ The given series converges when $x < 1$ and diverges when $x \geq 1$.

7. Test the convergence of the series, $\frac{1}{2} + \left(\frac{2}{3}\right)x + \left(\frac{3}{4}\right)^2 x^2 + \left(\frac{4}{5}\right)^3 x^3 + ...x > 0$

Solution

Neglecting the 1st term,

$$u_n = \left[\left(\frac{n+1}{n+2}\right)x\right]^n;$$

$$u_n^{1/n} = \left(\frac{n+1}{n+2}\right)x = \left(\frac{1+1/n}{1+2/n}\right)x$$

$\underset{n\to\infty}{Lt}\, u_n^{1/n} = x$; By Cauchy's root test, $\sum u_n$ is cgt. when $x < 1$ and dgt. when $x > 1$; when $x = 1$, the test fails.

When $\quad x = 1,\ u_n = \frac{\left(1+1/n\right)^n}{\left(1+2/n\right)^n};\ \underset{n\to\infty}{Lt}\, u_n = \frac{e}{e^2} = \frac{1}{e} \neq 0$

$\therefore$ $\sum u_n$ is divergent.

$\therefore$ is cgt. when $x < 1$ and dgt. when $x \geq 1$.

8. Test the series whose n^{th} term is $(3n-1)/2^n$ for convergence.

Solution

$$u_n = \frac{(3n-1)}{2^n} \quad ; \quad u_{n+1} = \frac{\{3(n+1)-1\}}{2^{n+1}} \quad ;$$

$$\frac{u_{n+1}}{u_n} = \frac{(3n+2)}{2(3n-1)} \qquad \underset{n\to\infty}{Lt} \frac{u_{n+1}}{u_n} = \frac{1}{2} < 1 \quad ;$$

$\therefore$ By ratio test, $\sum u_n$ is convergent.

9. Show by Cauchy's integral test that the series $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$ converges if $p > 1$ and diverges if $0 < p \le 1$

Solution

Let $\phi(x) = \frac{1}{x(\log x)^p}$; $x \ge 2$; Then $\phi(x)$ decreases as x increases in $[2, \infty]$

$$\int_2^{\infty} \phi(x)dx = \int_2^{\infty} \frac{dx}{x(\log x)^p} = \int_{\log 2}^{\infty} \frac{du}{u^p} = \frac{u^{1-p}}{1-p}\Big|_{\log 2}^{\infty} \quad ;$$

[Taking $\log x = u$, $\frac{1}{x}dx = du$ $\quad x = 2 \Rightarrow u = \log 2$ and $x = \infty \Rightarrow u = \infty$]

***Case* (i) :** $p > 1 \Rightarrow 1 - p < 0 \Rightarrow$ Integral is finite , and

***Case* (ii) :** $0 < p \le 1 \Rightarrow$ Integral is infinite.

Hence, by integral test, the given series converges if $p > 1$ and diverges when $0 < p \le 1$.

10. Test the convergence of the series $\sum \left(1 + \frac{1}{\sqrt{n}}\right)^{-n^{3/2}}$

Solution

$$u_n^{1/n} = \left\{\left(1 + \frac{1}{\sqrt{n}}\right)^{-n^{3/2}}\right\}^{1/n} = \frac{1}{\left(1 + \frac{1}{\sqrt{n}}\right)^{\sqrt{n}}} \quad ;$$

$$\underset{n\to\infty}{Lt}\, u_n^{\frac{1}{n}} = \frac{1}{e} < 1 \quad [\,2 < e < 3\,].$$

By Cauchy's root test, $\sum u_n$ is convergent.

11. Test the convergence of the series, $\sum_{n=2}^{\infty} \frac{(-1)^n . x^n}{n(n-1)}$, $0 < x < 1$

Solution

The given series is of the form $\sum (-1)^n u_n$, where $u_n = \frac{x^n}{n(n-1)}$.

This is an alternating series in which (i) $u_n > 0$ and $u_n > u_{n+1} \forall n \in N$.

Further $\underset{n\to\infty}{Lt}\, u_n = 0$. Hence, by Leibnitz test, the series is convergent.

12. Discuss the convergence of the series, $\frac{1}{2\sqrt{1}} + \frac{x^2}{3\sqrt{2}} + \frac{x^4}{4\sqrt{3}} + \frac{x^6}{5\sqrt{4}} + \ldots\ldots\ldots\ldots$

Solution

n^{th} term of the series $= u_n = \frac{x^{2n}}{(n+2)\sqrt{n+1}}$ (omitting 1^{st} term)

$$u_{n+1} = \frac{x^{2n+2}}{(n+3)\sqrt{n+2}}\,;\ \frac{u_{n+1}}{u_n} = \frac{\sqrt{n+2}\sqrt{n+1}}{(n+3)} . x^2$$

$$\underset{n\to\infty}{Lt} \frac{u_{n+1}}{u_n} = \underset{n\to\infty}{Lt} \left[\frac{\sqrt{1+2/n}\sqrt{1+1/n}}{\left(1+3/n\right)} . x^2 \right] = x^2\,;$$

$\therefore$ By ratio test, $\sum u_n$ converges if $x^2 < 1$, i.e., if $|x| < 1$, and diverges if $x^2 > 1$, i.e., if $|x| > 1$;

When $x^2 = 1$, $u_n = \frac{1}{(n+2)\sqrt{n+1}}$; taking $v_n = \frac{1}{n^{3/2}}$,

$$\underset{n\to\infty}{Lt} \frac{u_n}{v_n} = \underset{n\to\infty}{Lt} \frac{n^{3/2}}{n^{3/2}\left(1+2/n\right)\sqrt{1+1/n}} = 1$$

$\therefore$ By comparison test, $\sum u_n$ and $\sum v_n$ both converge or diverge together;

But $\sum v_n$ is convergent by p-series test.

$\therefore$ $\sum u_n$ is convergent if $|x| \leq 1$ and divergent if $|x| > 1$.

13. Test the convergence of the series $\sum_{n=1}^{\infty} \frac{x^{2n}}{(n+1)\sqrt{n}}$

Solution

$$u_n = \frac{x^{2n}}{(n+1)\sqrt{n}}; \quad u_{n+1} = \frac{x^{2n+2}}{(n+2)\sqrt{n+1}}$$

$$\frac{u_{n+1}}{u_n} = \frac{\sqrt{n}\sqrt{n+1}}{n+2}.x^2 = \frac{\sqrt{1+\frac{1}{n}}}{\left(1+\frac{2}{n}\right)}.x^2; \ \underset{n\to\infty}{Lt} \frac{u_{n+1}}{u_n} = x^2;$$

$\therefore$ By ratio test, $\sum u_n$ converges when $|x| < 1$ and diverges for $|x| > 1$.

When $|x| = 1$, $u_n = \frac{1}{n^{3/2}\left(1+\frac{1}{n}\right)}$ taking $v_n = \frac{1}{n^{3/2}}$ and applying the comparision test, we observe that $\sum u_n$ is convergent .

Hence $\sum u_n$ converges when $|x| \leq 1$ and diverges when $|x| > 1$.

14. Find the interval of convergence of the series, $\frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + ...\infty$

Solution

For the given series, $u_n = \frac{x^{n+1}}{n+1}$; $u_{n+1} = \frac{x^{n+2}}{n+2}$

$$\underset{n\to\infty}{Lt} \frac{u_{n+1}}{u_n} = \underset{n\to\infty}{Lt} \left(\frac{1+\frac{1}{n}}{1+\frac{2}{n}}\right) x = x$$

By ratio test, $\sum u_n$ converges when $|x| < 1$ i.e., $-1 < x < 1$

When $x = 1$, $u_n = \frac{1}{n+1}$

Taking $\qquad u_n = \frac{1}{n}$; $\frac{u_n}{v_n} = \frac{1}{1+\frac{1}{n}}$

and, $\underset{n\to\infty}{Lt}\frac{u_n}{v_n}=1\neq 0$ and finite.

$\therefore$ Both $\sum u_n$ and $\sum v_n$ converge or diverge together.

But $\sum v_n$ diverges $\therefore$ $\sum u_n$ also diverges when $x = 1$.

When $x = -1$, the given series is

$\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\frac{1}{5}+$ which is alternating series with

$u_n > u_{n+1}\forall n$ and $u_n \to 0$ as $n\to\infty$

$\therefore$ By Leibneitz's test $\sum u_n$ converges when $x=-1$

$\therefore$ Interval of convergence is [(–1, 1) i.e., $-1\leq x<1$

15. Test the convergence of the series $\sum_{n=1}^{\infty}\frac{1.3.5....(2n+1)}{2.5.8....(3n+2)}$

Solution

$$u_n=\frac{1.3.5....(2n+1)}{2.5.8....(3n+2)};\quad u_{n+1}=\frac{1.3.5....(2n+3)}{2.5.8....(3n+5)}$$

$$\frac{u_{n+1}}{u_n}=\frac{2n+3}{3n+5};\qquad \underset{n\to\infty}{Lt}\frac{u_{n+1}}{u_n}=\underset{n\to\infty}{Lt}\left[\frac{2+\left(3/n\right)}{3+\left(5/n\right)}\right]=\frac{2}{3}<1$$

$\therefore$ By ratio test, $\sum u_n$ is convergent.

16. Prove that the series $\sum\frac{(-1)^n}{n(\log n)^3}$ converges absolutely.

Solution

$$|u_n|=\frac{1}{n(\log n)^3}\quad;\quad \int_2^{\infty}\frac{dx}{x(\log x)^3}=\int_{\log 2}^{\infty}\frac{dt}{t^2}$$

(where $t=\log x$) $=\left.\frac{-1}{t}\right|_{\log 2}^{\infty}=\frac{1}{\log 2}$, which is finite.

$\therefore$ By integral test $\sum|u_n|$ is convergent.

$\therefore$ $\sum u_n$ converges absolutely.

17. Test the convergence of the series $\sum \frac{(2n+1)}{n^3+1}x^n, x>0$

Solution

n^{th} term of the given series, $u_n = \frac{2n+1}{n^3+1}x^n$;

$$u_{n+1} = \left[\frac{2(n+1)+1}{(n+1)^3+1}\right]x^{n+1} = \frac{2n+3}{(n+1)^3+1}x^{n+1}$$

$$\underset{n\to\infty}{Lt}\frac{u_{n+1}}{u_n} = \underset{n\to\infty}{Lt}\frac{(2n+3).x^{n+1}}{\left\{(n+1)^3+1\right\}}\times\frac{(n^3+1)}{x^n(2n+1)}$$

$$\underset{n\to\infty}{Lt}\left[\frac{2n\left(1+\frac{3}{2n}\right).n^3\left(1+\frac{1}{n^3}\right)}{n^3\left\{\left(1+\frac{1}{n}\right)^3+\frac{1}{n^3}\right\}.2n\left(1+\frac{1}{2n}\right)}\right]x = x$$

By ratio test, $\sum u_n$ converges if $x < 1$ and diverges if $x > 1$. If $x = 1$ the test fails.

When $x = 1$, $u_n = \frac{2n+1}{n^3+1}$; Taking $v_n = \frac{1}{n^2}$;

$$\underset{n\to\infty}{Lt}\frac{u_n}{v_n} = \underset{n\to\infty}{Lt}\frac{2n+1}{n^3+1}\times n^2 = 2 \neq 0 \quad \text{and finite}$$

$\therefore$ $\sum u_n$ and $\sum v_n$ converge or diverge together.

But $\sum v_n$ converges $\therefore$ $\sum u_n$ also converges.

Thus, $\sum u_n$ converges when $x \le 1$ and diverges when $x > 1$.

18. Test the series $\sum_{n=1}^{\infty}\frac{(-1)^n(\log n)}{n^2}$, for absolute/conditional convergence

Solution

$$u_n = \frac{(-1)^n(\log n)}{n^2}\ ;\ |u_n| = \frac{(\log n)}{n^2}\ ;$$

$$\int_2^{\infty}\frac{\log x}{x^2}dx = \int_{\log 2}^{\infty} te^{-t}dt \text{ [taking } \log x = t,\ x = e^t,\ \tfrac{1}{x}dx = \log t \text{]}$$

$$= -te^{-t} + e^{-t}\Big|_{\log 2}^{\infty} = 0-[1-\log 2].e^{-\log 2} = \frac{1}{2}(\log 2-1),$$

which is finite.

$\therefore$ By integral test $\sum|u_n|$ is convergent $\Rightarrow \sum u_n$ converges absolutely.

(Note that $\sum u_n$ is cgt. by Leibnitz's test).

19. Test the convergence of the series $\sum\frac{1}{(\log\log n)^n}$

Solution

Given that $u_n = \dfrac{1}{(\log\log n)^n}$;

$$\underset{n\to\infty}{Lt}\, u_n^{1/n} = \underset{n\to\infty}{Lt}\left[\frac{1}{\log\log n}\right] = 0 < 1$$

By Cauchy's root test, $\sum u_n$ is convergent.

20. Find the interval of convergence of the series,

$$x+\frac{1}{2}.\frac{x^3}{3}+\frac{1}{2}.\frac{3}{4}.\frac{x^5}{5}+\frac{1.3.5}{2.4.6}.\frac{x^7}{7}+\ldots\ldots$$

Solution

Term of the series, $u_n = \dfrac{1.3.5\ldots..(2n-1)}{2.4.6\ldots..2n}.\dfrac{x^{2n+1}}{(2n+1)}$ (neglecting 1st term)

$$u_{n+1} = \frac{1.3.5\ldots..(2n-1)(2n+1)}{2.4.6\ldots..2n(2n+2)}.\frac{x^{2n+3}}{(2n+3)} \quad ;$$

$$\underset{n\to\infty}{Lt}\frac{u_{n+1}}{u_n} = \underset{n\to\infty}{Lt}\left[\frac{(2n+1)^2}{(2n+2)(2n+3)}.x^2\right] \quad \underset{n\to\infty}{Lt}\left[\frac{n^2\left(4+\frac{4}{n}+\frac{1}{n^2}\right)}{n^2\left(4+\frac{10}{n}+\frac{6}{n^2}\right)}.x^2\right] = x^2$$

By ratio test, $\sum u_n$ converges when $x^2 < 1$, i.e., $|x| < 1 \Rightarrow -1 < x < 1$

When $x^2 = 1$, the test fails;

Then $\frac{u_n}{u_{n+1}}-1=\left(\frac{4n^2+10n+6}{4n^2+2n+1}-1\right)=\frac{8n+5}{4n^2+2n+1}$

$$\underset{n\to\infty}{Lt}\left[n\left(\frac{u_n}{u_{n+1}}-1\right)\right]=\underset{n\to\infty}{Lt}\frac{n^2\left(8+\frac{5}{n}\right)}{n^2\left(4+\frac{2}{n}+\frac{1}{n^2}\right)}=2>1$$

$\therefore$ By Raabe's test, $\sum u_n$ converges when $x^2=1$, i.e., $x=\pm 1$.

$\therefore$ Interval of convergence of $\sum u_n$ is $(-1\le x\le 1)$