# 7

# Vector Differentiation

### 7.1.1 Vector Point function and vector field

Let P be any point in a region 'D' of space. Let r be the position vector of P. If there exists a vector function F corresponding to each P, then such a function F is called a vector point function and the region D is called a vector field.

*Note :* In what follows $i, j, k$ are unit vectors along X, Y, Z axes respectively

For example, consider the vector function

$$F = (x - y)\, i + xyj + yzk \qquad .....(1)$$

Let P be a point whose position vector is

$r = 2i + j + 3k$ in the region D of space.

At P, the value of F is obtained by putting $x = 2, y = 1, z = 3$ in F.

i.e. At P, $F = i + 2j + 3k$

Thus, to each point P of the region D, there corresponds a vector F given by the vector function (1).

Hence F is a vecotr point function (of scalar variables $x, y, z$) and the region D *is a vector field.*

***Scalar point function and scalar field.***

If there exists a scalar 'f' given by a scalar function 'f' corresponding to each point P (with position vector r) in a region D of space, 'f' is called a scalar point function and D is called a scalar field.

As an example, let P be a point whose position vector is $r = 2i + j + 3k$.

Consider $f = xyz + xy + z$

Then the value of f at P is obtained by putting $x = 2, y = 1, z = 3$

i.e., At P, $f = 2.1.3 + 2.1 + 3 = 11$

Hence the scalar '11' is attached to the point P.

The function 'f' is a scalar point function (of scalar variables $x, y, z$), and D is a scalar field.

*Note :* There can be vector and scalar function of one or more scalar variables.

### 7.1.2 Differentiation of a vector

If $r(u) = r_1(u)i + r_2(u)j + r_3(u)k$, (where $r_1, r_2, r_3$, are scalar functions of 'u') be a vector function of 'u', then,

$$\frac{dr}{du} = \underset{\delta u \to 0}{\mathrm{Lt}} \frac{\delta r}{\delta u} = \underset{\delta u \to 0}{\mathrm{Lt}} \frac{r(u+\delta u) - r(u)}{\delta u}$$

$$= \underset{\delta u \to 0}{\mathrm{Lt}} \Sigma\left[\frac{r_1(u+\delta u) - r_1(u)}{\delta u} i\right]$$

$$= \Sigma \frac{dr_1}{du} i = \frac{dr_1}{du} i + \frac{dr_2}{du} j + \frac{dr_3}{du} k$$

**Example**

If $r(u) = (3u^2 + 5u + 6)\,i + 3u^2 j - 4uk$, Find $\frac{dr}{du}$, when u = 1

$$\frac{dr}{du} = \left\{\frac{d}{du}(3u^2 + 5u + 6)\right\}i + \left\{\frac{d}{du}(3u^2)\right\}j + \left\{\frac{d}{du}(-4u)\right\}k$$

*Note :* We can apply the above rule of derivative to the case of partial derivatives also

ex : If $A = (x^2yz)i + (xy^2z)j - (3x^3y^2z^2)k$

find $\frac{\partial^2 A}{\partial x \partial v}$ at the point $(1, -1, 2)$

$$\frac{\partial A}{\partial y} = \left\{\frac{\partial}{\partial y}\left(x^2yz\right)\right\}i + \frac{\partial}{\partial y}\left\{xy^2z\right\}j - \frac{\partial}{\partial y}\left\{3x^3y^2z^2\right\}k$$

$$= (x^2z)i + (2xyz)j - (6x^3yz^2)\,k$$

$$\frac{\partial^2 A}{\partial x\partial y} = \left\{\frac{\partial}{\partial x}\left(x^2z\right)\right\}i + \frac{\partial}{\partial x}\left\{2xyz\right\}j - \frac{\partial}{\partial x}\left\{\left(6x^3yz^2\right)\right\}k$$

$$= 2xz\,i + 2yz\,j - 18x^2yz^2\,k$$

At the point (1, −1, 2)

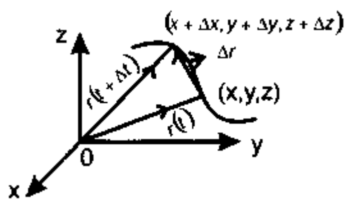
$$\frac{\partial^2 A}{\partial x\partial y} = 4i - 4j + 72k$$

### 7.1.3 Application to space curves

Let $r(t) = x(t)i + y(t)j + z(t)\,k$ represent the position vector of a point $(x, y, z)$ on a space curve whose equations are given by $x = x(t)$, $y = y(t)$, $z = z(t)$, where 't' is time.

Then $\frac{dr}{dt} = \frac{dx}{dt}i + \frac{dy}{dt}j + \frac{dz}{dt}k$

and $\frac{d^2r}{dt^2} = \frac{d^2x}{dt^2}i + \frac{d^2y}{dt^2}j + \frac{d^2z}{dt^2}k$




(i) $\frac{dr}{dt}$ represents the velocity vector $v$ (or tangent vector) of the point $(x, y, z)$

(ii) $\frac{d^2r}{dt^2}$ represents the acceleration vector **a** at the point $(x, y, z)$

Ex : If a particle moves along a curve $x = e^{-t}$, $y = 2\cos 2t$, $z = 2\sin 2t$, where 't' is time.

1) find velocity and acceleration at time t = 0, and

2) find also their magnitudes

**Sol :** $r = xi + yj + zk$

$$= (e^{-t})i + (2\cos 2t)\,j + (2\sin 2t)k$$

$$v = \frac{dr}{dt} = \frac{d}{dt}(e^{-t})i + \frac{d}{dt}(2\cos 2t)j + \frac{d}{dt}(2\sin 2t)k$$

$$= (-e^{-t})i - (4\sin 2t)j + (4\cos 2t)k$$

$$a = \frac{d^2r}{dt^2} = \frac{dv}{dt} = \frac{d}{dt}(-e^{-t})i - \frac{d}{dt}(4\sin 2t)j + \frac{d}{dt}(4\cos 2t)k$$

$$= (e^{-t})\, i - (8\cos 2t)\, j - (8\sin 2t)\, k \qquad \text{.... (2)}$$

Putting t = 0 in (1), velocity as t = 0 is $v = -i + 4k$

Magnitude = $\sqrt{17}$

putting t = 0 in (2) acceleration at t = 0 is a = i–8j

Magnitude = $\sqrt{65}$

## 7.2 Gradient of Scalar Function

### 7.2.1 The Vector differential operator 'DEL' or 'NABLA', denoted as '$\nabla$' is defined by

$$\nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z} \qquad (i, j, k \text{ are unit vectors in } x, y, z \text{ directions})$$

This operator '$\nabla$' is used in defining the gradient, divergence and curl.

Properties of '$\nabla$' are similar to those of vectors. The operator is appled to both vector and scalar functions.

### 7.2.2 Gradient

If $\phi(x, y, z)$ is a scalar function, defined at each point $(x, y, z)$ in a certain region of space and is differentiable, the gradent of $\phi$ (shortly written as grad $\phi$) is defined as,

$$\text{grad } \phi = \nabla \phi = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\phi$$

$$= \left(\frac{\partial \phi}{\partial x}\right)i + \left(\frac{\partial \phi}{\partial y}\right)j + \left(\frac{\partial \phi}{\partial z}\right)k,$$

(which is a vector function)

$\therefore$ If $\phi$ defines a scalar field, 'grad' $\phi$ or $\nabla\phi$ defines a vector field.

### 7.2.3 Physical significance of 'grad $\phi$' :

If $\phi(x, y, z) = c$ (c being a constant) represents a surface, then 'grad $\phi$' represents the normal vector to the surface at the point $(x, y, z)$

For, if $r = xi + yj + zk$, is the position vector of the point $(x, y, z)$ on the surface, we have, $dr = (dx)\,i + (dy)\,j + (dz)\,k$ which is in the tangent plane to the surface of $(x, y, z)$

$$\text{Again, } \nabla\phi, \mathrm{dr} = \left[\frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial z}k\right].\,[\,dxi + dyj + dzk\,]$$

$$= \left(\frac{\partial\phi}{\partial x}\right)dx + \left(\frac{\partial\phi}{\partial y}\right)dy + \left(\frac{\partial\phi}{\partial z}\right)dz = d\phi = 0 \qquad (\because \phi = c)$$

$\therefore$ The vector '$\nabla\phi$' which is $\perp r$ to the tangent plane is the normal vector to $\phi = c$ at $(x, y, z)$

### 7.2.4 Directional, Derivative

If **a** be any vector, $\frac{\nabla\phi.\mathbf{a}}{|\mathbf{a}|}$ which represents the component of $\nabla\phi$ in the direction of

**a** is known as the directional derivative of '$\phi$' in the direction of **a**,

(1) Physically the directional derivative is the rate of change of '$\phi$' in the direction of **a**.

(2) The directional derivative will be maximum in the direction of $\nabla\phi$ (i.e., $a = \nabla\phi$) and the maximum value of the directional derivate $= \frac{\nabla\phi.\nabla\phi}{|\nabla\phi|} = |\nabla\phi|$.

### 7.2.5 Some basic properties of the gradient

If $\phi$ and $\psi$ are two scalar functions,

(1) grad $(\phi + \psi)$ = grad $\phi$ + grad $\psi$ (or) $\nabla(\phi + \psi) = \nabla\phi + \nabla\psi$

$$\text{Proof : grad } (\phi + \psi) = \nabla(\phi + \psi) = \left\{\frac{\partial}{\partial x}(\phi + \psi)\right\}i + \left\{\frac{\partial}{\partial y}(\phi + \psi)\right\}j + \left\{\frac{\partial}{\partial z}(\phi + \psi)\right\}k$$

$$= \left(\frac{\partial \phi}{\partial x} i + \frac{\partial \phi}{\partial y} j + \frac{\partial \phi}{\partial z} k\right) + \left(\frac{\partial \psi}{\partial x} i + \frac{\partial \psi}{\partial y} j + \frac{\partial \psi}{\partial z} k\right)$$

$$= \nabla\phi + \nabla\psi$$

(2) grad $(\phi\psi) = \phi\,(grad\,\psi) + \psi(grad\phi)$ (or) $\nabla(\phi\psi) = \phi\,(\nabla\psi) + (\nabla\phi)\psi$

Proof: grad $(\phi\psi) = \nabla(\phi\psi)$

$$= \left\{\frac{\partial}{\partial x}(\phi\psi)\right\} i + \left\{\frac{\partial}{\partial y}(\phi\psi)\right\} j + \left\{\frac{\partial}{\partial z}(\phi\psi)\right\} k$$

$$= \left(\phi\frac{\partial \psi}{\partial x} + \psi\frac{\partial \phi}{\partial x}\right) i + \left(\phi\frac{\partial \psi}{\partial y} + \frac{\partial \phi}{\partial y}\right) j + \left(\phi\frac{\partial \psi}{\partial z} + \psi\frac{\partial \phi}{\partial z}\right) k$$

$$= \phi\left(\frac{\partial \psi}{\partial x} i + \frac{\partial \psi}{\partial y} j + \frac{\partial \psi}{\partial z} k\right) + \psi\left(\frac{\partial \phi}{\partial x} i + \frac{\partial \phi}{\partial y} j + \frac{\partial \phi}{\partial z} k\right) = \phi(\nabla\psi) + \psi(\nabla\phi)$$

$$= \phi(grad\,\psi) + \psi(grad\,\phi)$$

(3) If $\psi \neq 0$, grad $\left(\frac{\phi}{\psi}\right) = \frac{\psi(grad\phi) - \phi(grad\psi)}{(\psi)^2}$

(Proof is left to the reader.)

### Solved Examples

**Ex. 7.2.6** If $f = x^2yz$, find grad $f$ at the point (1, –2, 1).

**Sol :** $f = x^2yz$ ; $\because$ grad $\phi = \nabla\phi = \frac{\partial \phi}{\partial x} i + \frac{\partial \phi}{\partial y} j + \frac{\partial \phi}{\partial z} k$,

$$\text{grad } f = \frac{\partial}{\partial x}(x^2yz)\, i + \frac{\partial}{\partial y}(x^2yz) j + \frac{\partial}{\partial z}(x^2yz) k$$

$$= (2xyz) i + (x^2z) j + (x^2y) k$$

$\therefore$ At the point (1, –2, 1), grad $\phi = -4i + j - 2k$

**Ex. 7.2.7** Find the unit normal to the surface $xy + yz + zx = 3$ at the point (1, 1, 1).

**Sol:** If $\phi = c$ is a surface, $\nabla\phi$ is the normal to it.

Here f = xy +yz + zx

$$\therefore \text{ normal to } \phi = \nabla\phi = \frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial z}k.$$

$$= \left\{\frac{\partial}{\partial x}(xy + yz + zx)\right\}i + \left\{\frac{\partial}{\partial y}(xy + yz + zx)\right\}j + \left\{\frac{\partial}{\partial z}(xy + yz + zx)\right\}k$$

= (y + z)i + (x + z)j + (y + x)k

$\therefore$ normal at (1,1,1) = 2i + 2j + 2k

$$\therefore \text{ Unit normal} = \frac{2i + 2j + 2k}{\sqrt{2^2 + 2^2 + 2^2}} = \frac{i + j + k}{\sqrt{3}}$$

**Ex. 7.2.8** (a) Find the directional derivative of $f = 2e^{2x-y+z}$ at (1, 3, 1) in a direction towards the point (2, 1, 3).

**Sol :** $f = 2e^{2x-y+z}$ ; $\nabla f = 2e^{2x-y+z}(2i - j + k)$

$$\nabla f\big|_{(1,3,1)} = 2.e^{2-3+1}(2i - j + k) = 4i - 2j + 2k$$

Let A = (1, 3, 1) and B = (2, 1, 3),

AB = (2–1)i + (1–3)j + (3–1)k = i – 2j + 2k = a (say)

$$\text{Directional derivative in the direction of a} = \frac{\nabla f . a}{|a|}$$

$$= \frac{(4i - 2j + 2k).(i - 2j + 2k)}{\sqrt{1 + 4 + 4}} = \frac{(4 + 4 + 4)}{3} = 4$$

(b) In the 'Problem (a)' find the maximum value of the directional derivative

Ans : Maximum value of directional derivative = $|\nabla f|$

$$= |\,4i - 2j + 2k\,| = \sqrt{16 + 4 + 4} = 2\sqrt{6}$$

**Ex. 7.2.9** Find the acute angle between the surface $xy^2z = 2$ and $x^2 + y^2 + z^3 = 6$ at the point (2, 1, 1).

**Sol :** Let $f = xy^2z = 4$ be the surface (1)

Normal vector to (1) at (2, 1, 1) = $\nabla f\big|_{(2,1,1)} = |(y^2z)i + (2xyz)j + (xy^2)k|_{(2,1,1)}$

$= i + 4j + 2k$ = a (say).

Let $g = (x^2 y^2 + z^2) = 6$ be the surface (2)

Normal vector to (2) at (2,1,1) = $\nabla g|_{(2,1,1)} = (2xi + 2yj + 2zk)|_{(2,1,1)}$

$= 4i + 2j + 2k$ = b (say)

$\therefore$ Angle between the surfaces

= Angle between the normals to them

= Angle between a and b

$$= \left|\cos^{-1}\left(\frac{a,b}{ab}\right)\right| = \cos^{-1}\left|\frac{4+8+4}{\sqrt{1+16+4}\sqrt{16+4+4}}\right|$$

$$= \cos^{-1}\left(\frac{16}{\sqrt{21}\sqrt{24}}\right) = \cos^{-1}\left(\frac{16}{6\sqrt{14}}\right) = \cos^{-1}\left(\frac{8}{3\sqrt{14}}\right)$$

**7.2.10** Find the constants $p$ and $q$ such that the surfaces $px^2 - qyz = (p + 2)x$ and $4x^2y + z^3 = 4$ are orthogonal at the point $(1, -1, 2)$

**Sol :** Let $f = px^2 - qyz - (p + 2)x = 0$ be surface (1), and

Let $g = 4x^2y + z^3 = 4$ be surface (2)

Normal to (1) at (1, –1, 2) = $\nabla f\Big|_{(1,-1,2)} = \left(\frac{\partial f}{\partial x}i + \frac{\partial f}{\partial y}j + \frac{\partial f}{\partial z}k\right)\Big|_{(1,-1,2)}$

$= [(2px - p - 2)i - (qz)j - (qy)k]|_{(1,-1,2)} = (p-2)i - (2q)j - (q)k$ = a (say)

Normal to (2) at (1, – 1, 2) = $\nabla g|_{(1,-1,2)}$

$= [(8xy)i + (4x^2)j + (3z^2)k]\Big|_{(1,-1,2)} = -8i + 4j + 12k$ = b (say)

Since the surfaces (1) and (2) are orthogonal, a.b = 0

$\therefore -8(p-2) + 4(-2q) + 12(q) = 0$

$\Rightarrow -8p + 16 - 8q + 12q = 0$

$\Rightarrow -8p + 4q + 16 = 0$

$\Rightarrow 2p - q = 4$ ..... (i)

Since the point $(1, -1, 2)$ lies on (1), we have,

$p + 2q - p - 2 = 0 \Rightarrow q = 1$

from (i) we get, $p = 5/2$ $\quad \therefore p = 5/2, q = 1$

**7.2.11** If $\mathbf{r} = xi + yj + zk$ and $r = |\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}$ show that grad $(r^3) = 3r\,\mathbf{r}$

**Sol :** Let $\phi = r^3 = (x^2 + y^2 + z^3)^{3/2}$

Then, $\dfrac{\partial\phi}{\partial x} = \dfrac{3}{2}(x^2 + y^2 + z^2)^{3/2-1} \cdot 2x = 3x\,r$

similarly $\dfrac{\partial\phi}{\partial y} = 3yr$; and $\dfrac{\partial\phi}{\partial z}\,3zr$

grad $\phi = \dfrac{\partial\phi}{\partial x}i + \dfrac{\partial\phi}{\partial y}j + \dfrac{\partial\phi}{\partial z}k$

$= (3xr)\,i + (3yr)\,j + (3zr)\,k$

$= 3r\,(xi + yj + zk) = 3r\,\mathbf{r}$

***Aliter :*** If $r^2 = x^2 + y^2 + z^2$, we have, $2r\dfrac{\partial r}{\partial x} = 2x \Rightarrow \dfrac{\partial r}{\partial x} = \dfrac{x}{r}$

Similarly $\dfrac{\partial r}{\partial y} = \dfrac{y}{r}$ ; $\dfrac{\partial r}{\partial z} = \dfrac{z}{r}$

grad $\phi = \nabla r^3$

$= \dfrac{\partial}{\partial x}(r^3)i + \dfrac{\partial}{\partial y}(r^3)j + \dfrac{\partial}{\partial z}(r^3)k = 3r^2\dfrac{\partial r}{\partial x}i + 3r^2\dfrac{\partial r}{\partial y}j + 3r^2\dfrac{\partial r}{\partial z}k$

$= 3r^2\left[\dfrac{x}{r}i + \dfrac{y}{r}j + \dfrac{z}{r}k\right] = 3r\,\mathbf{r}$

**7.2.12** Evaluate grad $r^n$

**Sol :** Let $\phi = r^n = (x^2 + y^2 + z^2)^{n/2}$

$\dfrac{\partial\phi}{\partial x} = \dfrac{n}{2}\left(x^2 + y^2 + z^2\right)^{n/2-1}.2x = nx\left(x^2 + y^2z^2\right)^{n/2-1}$

similarly $\dfrac{\partial\phi}{\partial x} = ny\left(x^2+y^2+z^2\right)^{n/2-1}$ and $\dfrac{\partial\phi}{\partial z} = nz\left(x^2+y^2+z^2\right)^{n/2-1}$

$$\text{grad } r^n = \text{grad } \phi = \frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial z}k$$

$$= n, \left(x^2+y^2+z^2\right)^{n/2-1} . (xi + yj + zk) = n\, r^{n-2}.r$$

Aliter: $\text{grad } \phi \left(\dfrac{\partial\phi}{\partial x}\right)i + \left(\dfrac{\partial\phi}{\partial y}\right)j + \left(\dfrac{\partial\phi}{\partial z}\right)k = \dfrac{\partial\phi}{\partial r}.\dfrac{\partial r}{\partial x}i + \dfrac{\partial\phi}{\partial r}.\dfrac{\partial r}{\partial y}j + \dfrac{\partial\phi}{\partial r}.\dfrac{\partial r}{\partial z}k$

$$= nr^{n-1} . \frac{x}{r}i + nx^{n-1} . \frac{y}{r}j + nx^{n-1} . \frac{z}{r}k = nr^{n-2}.r$$

**Ex.7.2.13** If A is a constant vector prove that grad (r . A) = A

**Sol :** Let $A = A_1 i + A_2 j + A_3 k$ ($A_1, A_2, A_3$ being constant functions)

$r = xi + yj + zk$

$r . A = A_1 x + A_2 y + A_3 z = f$ (say)

$$\frac{\partial f}{\partial x} = A_1, \frac{\partial f}{\partial y} = A_2, \frac{\partial f}{\partial z} = A_3$$

$$\therefore \text{grad } (r.A) = \text{grad } f = \frac{\partial f}{\partial x}i + \frac{\partial f}{\partial y}j + \frac{\partial f}{\partial z}k = A_1 i + A_2 j + A_3 k = A$$

**Ex. 7.2.14** Prove $\nabla\ (\phi(r)) = \dfrac{\phi'(r).r}{r}$

**Sol :** Let $f = \phi\ (r)$

$\dfrac{\partial f}{\partial x} = \phi'\ (r)\ \dfrac{\partial r}{\partial x} = \phi'(r).\dfrac{x}{r}$ $\qquad \because \dfrac{\partial r}{\partial x} = \dfrac{x}{r}$

$\dfrac{\partial f}{\partial y} = \phi'(r)\dfrac{\partial r}{\partial y} = \phi'(r).\dfrac{y}{r}$ $\qquad \dfrac{\partial r}{\partial y} = \dfrac{y}{r}$

$\dfrac{\partial f}{\partial z} = \phi'(r)\dfrac{\partial r}{\partial z} = \phi'(r).\dfrac{z}{r}$ $\qquad \dfrac{\partial r}{\partial z} = \dfrac{z}{r}$ (see Aliter of ex. 7.2.11)

$$\therefore \nabla\ (\phi\ (r)) = \nabla f = \frac{\partial f}{\partial x}i + \frac{\partial f}{\partial y}j + \frac{\partial f}{\partial z}k$$

$$= \frac{\phi'(r)}{(r)}\ (xi + yj + zk) = \frac{\phi'(r)r}{(r)}$$

**Ex. 7.2.15** Find the equations for the tangent plane and normal line to the surface $z = x^2 + y^2$ at the point (2, 1, 5).

**Sol :** Let $r = xi + yj + zk$ be the position vector of any point $P(x, y, z)$ on the surface.

Let $r_1 = x_1 i + y_1 j + z_1 k$ be the position vector of fixed point $A(x_1, y_1, z_1)$ on the surface.

Then $AP = (x - x_1)\, i + (y - y_1)\, j + (z - z_1)\, k = r - r_1$

Let n be the normal to the surface at A.

Then, since AP is perpendicular to n, we have,

$$(r - r_1)\, n = 0 \qquad \text{..... (1)}$$

which is the equation to the tangent plane at A.

Here, in the given problem

$r - r_1 = (x - 2)\, i + (y - 1)\, j + (z - 5)k$

and $n = \nabla (x^2 + y^2 - z)$ at A (2, 1, 5)

$$= (2xi + 2yj - k)\Big|_{(2,1,5)} = 4i + 2j - k$$

$\therefore$ The tangent plane at (2, 1, 5) is, (from (1)),

$$4(x - 2) + 2(y - 1) - 1(z - 5) = 0 \Rightarrow 4x + 2y - z = 5 \qquad \text{.... (2)}$$

From (2), the direction ratios of the normal line at A are 4, 2, –1

$\therefore$ Equation to the normal line at A are,

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} \text{ where}$$

$(x_1, y_1, z_1) = (2, 1, 5)$ and (a, b, c) = (4, 2, –1)

$\therefore$ The equations of normal line are $\dfrac{x-2}{4} = \dfrac{y-1}{2} = \dfrac{z-5}{-1}$

*From the above example, we have to remember the following :*

Let $\phi = c$ be any given surface and $(x_1, y_1, z_1)$ be a point on it; then

(1) Equation to the tangent plane to $\phi = c$ at $(x_1, y_1, z_1)$ is

$$(x - x_1)\frac{\partial \phi}{\partial x} + (y - y_1)\frac{\partial \phi}{\partial y} + (z - z_1)\frac{\partial \phi}{\partial z} = 0$$

(2) Equations to Normal line at $(x_1, y_1, z_1)$ are

$$\frac{x-x_1}{\partial\phi/\partial x}=\frac{y-y_1}{\partial\phi/\partial y}=\frac{z-z_1}{\partial\phi/\partial z}$$

## Exercise 7(a)

1. If $\phi = 2xz^3 - 3x^2yz$, find $\nabla\phi$ and $|\nabla\phi|$ at the point $(2, 2, -1)$

   [Ans : (i) $22i + 12j - 12k$ (ii) $2\sqrt{193}$ ]

2. If $V = 2x\,i - 3y^2\,j + z^3k$, and $\phi = 2xyz - 3z^2$, find $V.\nabla\phi$ and $V\times\nabla\phi$ at the point $(1, 2, 3)$

   [Ans . (1) $-426$ (2) $6i + 352j + 156k$]

3. If $f = 2xyz$ and $g = x^2y + z$, fing $\nabla\ (f+g)$ and $\nabla\ (fg)$ at the point $(1, -1, 0)$

   [Ans : $-2i + j - k$; $2k$ ]

4. Evaluate $\nabla\ (3r^2 - 4\sqrt{r} + \frac{6}{\sqrt[3]{r}})$ [Ans : $(6 - 2r^{3/2} - 2r^{-\frac{7}{3}})\,r$ ]

5. If $\phi = r^2 e^{-r}$, show that grad $\phi = (2-r)\,e^{-r}\,r$

6. Find a unit normal vector to the surface $z = x^2 + y^2$ at the point $(1, -2, 5)$

   [Ans : $\frac{1}{\sqrt{21}}(2i - 4j - k)$

7. Find the equations to the tangent plane and the normal line to the surface $xz^2 + x^2y = z - 1$ at the point $(1, -3, 2)$

   [Ans: (i) $2x - y - 3z + 1 = 0$ (ii) $\frac{x-1}{-2}=\frac{y+3}{1}=\frac{z-2}{3}$

8. Find the equations to the Tangent plane and normal line to the surface $y = x^2 + z^2$ at the point $(1, 5, -2)$

   [ Ans : (i) $2x - y - 4z = 5$ (ii) $\frac{x-1}{2}=\frac{y-5}{-1}=\frac{z+2}{-4}$

9. Find the directional derivative of $U = 4xz^3 - 3x^2y^2z$ at $(2, 1, -2)$ in the direction of $(3i - 2j + 6k)$.

   [Ans : $\frac{384}{7}$ ]

10. Find the directional derivative of $\phi = 4e^{2x-y+z}$ at the point $(1, 1, -1)$ in a direction towards the point $(2, 3, 1)$

    [ Ans : 8/3]

11. Find the values of the constants a, b, c so that the directional derivative of f = $axy^2 + byz + cz^2x^3$ at (1, 2, –1) has a maximum magnitude 64 in a direction parallel to $z$-axis.

[Hint : $\Delta f$ at (1, 2, –1) is $||^{lel}$ to $z$ - axis.

$\therefore$ Equate coefficients of i and j to zero and $|\nabla f| = 64$. Thus get 3 equations in a, b, c and solve them]

[Ans: a = 6, b = 24, c = – 8]

12. Find the acute angle between the surfaces $xy^2z = 3x + z^2$ and $3x^2 - y^2 + 2z = 1$ at the point (1, –2, 1)

[Ans : $\cos^{-1}\dfrac{\sqrt{3}}{7\sqrt{2}}$]

13. Find grad $\psi$ if r = $xi + yj + zk$, r = $|r|$ and

(i) $\psi$ = Log r (ii) $\psi = \dfrac{1}{r}$ (iii) $\psi$ = r

[Ans : (i) r/r$^2$ (ii) –r/r$^3$ (iii) r/r]

14. Find the directional derivative of g = $x^2y^2 + y^2z^2 + z^2x^2$ at the point (1, 1, –2) in the direction of the tangent to the curve $x = e^{-t}$, $y = 2\sin t + 1$, $z = t - \cos t$ at t = 0.

[Hint : Tangent vector to the given curve is

$$\frac{dx}{dt}i + \frac{dy}{dt}j + \frac{dz}{dt}k$$

[Ans : $\dfrac{2}{\sqrt{6}}$]

15. Find the acute angle between the normals to the surface $xy = z^2$ at the points (1, 9, 3) and (3, 3, –3)

[Ans : $\cos^{-1}\left(\dfrac{1}{\sqrt{177}}\right)$]

16. If r = $xi + yj + zk$ and $\phi = x^3 + y^3 + z^3 - 3xyz$, show that r, grad $\phi = 3\phi$.

## 7.3 The Divergence of a Vector Function

**7.3.1** If $A = A_1 i + A_2 k$ is a vector function, defined and differentiable at each point $(x, y, z)$ in a certain region of space [i.e., A defines a vector field], then the divergence of A (abbreviated as 'Div A') is defined as,

$$\text{Div A} = \nabla . A$$

$$= \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right) . (A_1 i + A_2 j + A_3 k)$$

$$= \left(\frac{\partial A_1}{\partial x} i + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right)$$

(since $i.i = j.j = k.k = 1$)

*Note* : (1) Div A is a scalar field

(2) $\nabla . A \neq A . \nabla$

### 7.3.2 Physical significance of the divergence

If A represents the velocity of fluid in a fluid flow, Div A represents the rate of fluid flow through unit volume. (or) Div A gives the rate at which fluid is originating at a point per unit volume.

Similarly if A represents the Electric flux or heat flux, Div A represents the amount of electric flux or heat flux that diverges per unit volume in unit time.

### 7.3.3 Some properties of Divergence

If A, B , are vector functions and '$f$' is a scalar function, then, prove that

(1) Div (A + B) = Div A + Div B (i.e) $\nabla . (A + B) = \nabla . A + \nabla . B$

*Proof:* Let $A = A_1 i + A_2 j + A_3 k$

$B = B_1 i + B_2 j + B_3 k$

$A + B = (A_1 + B_1)\, i + (A_2 + B_2)\, j + (A_3 + B_3)\, k$

$$\text{Div A} = \frac{\partial}{\partial x}(A_1 + B_1) + \frac{\partial}{\partial y}(A_2 + B_2) + \frac{\partial}{\partial z}(A_3 + B_3)$$

$$= \left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right) + \left(\frac{\partial B_1}{\partial x} + \frac{\partial B_2}{\partial y} + \frac{\partial B_3}{\partial z}\right)$$

= Div A + Div B.

(2) Prove that, Div $(fA) = (\text{grad } f) .A + f(\text{Div } A)$ i.e. $\nabla .(fA) = (\nabla f).A + f(\nabla A)$

Proof : Let $A = A_1 i + A_2 j + A_3 k$ then $fA = fA_1 i + fA_2 j + fA_3 k$

$$\nabla .(fA) = \frac{\partial}{\partial x}(fA_1) + \frac{\partial}{\partial y}(fA_2) + \frac{\partial}{\partial z}(fA_3)$$

$$= f\frac{\partial A_1}{\partial x} + A_1\frac{\partial f}{\partial x} + f\frac{\partial A_2}{\partial y} + A_2\frac{\partial f}{\partial y} + f\frac{\partial A_3}{\partial z} + A_3\frac{\partial f}{\partial z}$$

$$= f\left[\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right] + \left[\left(\frac{\partial f}{\partial x}\right)A_1 + \left(\frac{\partial f}{\partial y}\right)A_2 + \left(\frac{\partial f}{\partial z}\right)A_3\right] \quad \text{..... (1)}$$

$$(\nabla f).A = \left(\frac{\partial f}{\partial x}i + \frac{\partial f}{\partial y}j + \frac{\partial f}{\partial z}k\right) . (A_1 i + A_2 j + A_3 k)$$

$$= \left(\frac{\partial f}{\partial x}\right)A_1 + \left(\frac{\partial f}{\partial y}\right)A_2 + \left(\frac{\partial f}{\partial z}\right)A_3 \quad \text{..... (2)}$$

$$f(\nabla .A) = f\left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right) \quad \text{..... (3)}$$

(1), (2), (3) $\Rightarrow \nabla .(fA) = (\nabla f).A + f(\nabla .A)$

**7.3.4** Solenoidal vectors : A vector A is said to be solenoidal if Div A = 0

## Solved Examples

**Ex. 7.3.5** If $A = (x^2y)\, i + (xy^2z)\, j + (xyz)k$, find div A at the point $(1, -1, 2)$.

Sol : $A = (x^2y)\, i + (xy^2z)j + (xyz)k$

$$\text{Div } A = \frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}$$

$$= \frac{\partial}{\partial x}(x^2y) + \frac{\partial}{\partial y}(xy^2z) + \frac{\partial}{\partial z}(xyz)$$

$$= 2xy + 2xyz + xy$$

$$= xy\,(2z + 3)$$

$\therefore$ At $(1, -1, 2)$, Div $A = (1)(-1)[4 + 3] = -7$

**Ex. 7.3.6** If $V = 2xyi + 3x^2yj - 3pyz\,k$ is solenoidal at (1, 1, 1), find '$p$'.

Sol : Div V $= \frac{\partial}{\partial x}(2xy) + \frac{\partial}{\partial y}(3x^2 y) + \frac{\partial}{\partial z}(-3pyz)$

$= 2y + 3x^2 - 3py$

At (1, 1, 1), Div V $= 5 - 3p$

Since V is solenoidal, Div V $= 0$ $\therefore p = \frac{5}{3}$

**Ex. 7.3.7** If $r = xi + yz + zk$, and $r = |r|$, show that Div $(r^3 r) = 6\,r^3$

Sol : $r = \sqrt{x^2 + y^2 + z^2}$ $\therefore r^3 r = (x^2 + y^2 + z^2)^{3/2}(xi + yj + zk)$

$= A_1 i + A_2 j + A_3 k$ (say)

then, $A_1 = x(x^2 + y^2 + z^2)^{3/2}$

$A_2 = y(x^2 + y^2 + z^2)^{3/2}$

$A_3 = z(x^2 + y^2 + z^2)^{3/2}$

$$\frac{\partial A_1}{\partial x} = x\frac{3}{2}\left(x^2 + y^2 + z^2\right)^{\frac{3}{2}-1} 2x + \left(x^2 + y^2 + z^2\right)^{3/2} .\, 1$$

$= 3x^2 r + r^3$

similarly $\frac{\partial A_2}{\partial y} = 3y^2 r + r^3$; $\quad \frac{\partial A_3}{\partial z} = 3z^2 r + r^3$

$$\text{Div A} = \frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}$$

$= 3r\,(x^2 + y^2 + z^2) + 3r^3 = 3r^3 + 3r^3 = 6r^3$

Aliter : $r^3\, r = r^3\, xi + r^3\, yi + r^3 zk.$

$$\frac{\partial}{\partial x}(r^3 x) = r^3.1 + x \,.\, 3r^2\,\frac{\partial r}{\partial x} = r^3 + x \,.\, 3r^2 \,.\, \frac{x}{r}$$

$= r^3 + 3x^2 r$ $\quad \left(\because \frac{\partial r}{\partial x} = \frac{x}{r}\right)$

Similarly $\frac{\partial}{\partial y}$ $(r^3 y) = r^3 + 3y^2 r$ and $\frac{\partial}{\partial z}(r^3 z) = r^3 + 3z^2 r$

Div A = $6r^3$

**Ex. 7.3.8** Evaluate Div [ r grad $(r^{-3})$ ] or $\nabla \cdot \left\{ r\nabla\left(\frac{1}{r^3}\right)\right\}$

Sol : grad $(r^3) = \left\{\frac{\partial}{\partial x}(r^3)\right\}i + \left\{\frac{\partial}{\partial y}(r^3)\right\}j + \left\{\frac{\partial}{\partial z}(r^3)\right\}k$

$$= -3r^4 \frac{\partial r}{\partial x} i - 3r^4 \frac{\partial r}{\partial y} j - 3r^{-4} \frac{\partial r}{\partial z} k = -3r^{-4}\left(\frac{x}{r}i + \frac{y}{r}j + \frac{z}{r}k\right)$$

$$= -3r^{-5}(xi + yj + zk)$$

r grad $(r^3) = -3r^{-4}(xi + yj + zk)$

$= A_1 i + A_2 j + A_3 k$ (say)

where $A_1 = -3r^{-4}x$, $A_2 = -3r^{-4}y$, $A_3 = -3r^{-4}z$

$$\frac{\partial A_1}{\partial x} = \frac{\partial}{\partial x}[-3r^{-4}x]$$

$$= -3r^{-4} \cdot 1 + x \cdot -3 \cdot -4 r^{-5} \cdot \frac{\partial r}{\partial x} = -3r^{-4} + 12\, x\, r^{-5} \cdot \frac{x}{r}$$

$$= -3r^{-4} + 12\, x^2 r^{-6}$$

Similarly, $\frac{\partial A_2}{\partial y} = -3r^{-4} + 12y^2 r^{-6}$

$$\frac{\partial A_3}{\partial z} = -3r^{-4} + 12z^2 r^{-6}$$

$\therefore$ Div (r grad $r^{-3}$) = $\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}$

$= -9r^{-4} + 12\, r^{-6}(x^2 + y^2 + z^2) = -9r^{-4} + 12\, r^{-6} \cdot r^2 = 3r^{-4}$

(Alternate method is left to the reader).

**Ex. 7.3.9** Show that $\nabla.(x^n r) = (n + 3) r^n$. Hence show that $r/r^3$ is solenoidal.

**Sol :** $r^n r = r^n (xi + yj + zk)$

$$\nabla.(r^n r) = \frac{\partial}{\partial x}(xr^n) + \frac{\partial}{\partial y}(yr^n) + \frac{\partial}{\partial z}(zr^n)$$

$$= \left[r^n + xnr^{n-1}.\frac{\partial r}{\partial x}\right] + \left[r^n + y.nr^{n-1}\frac{\partial r}{\partial y}\right] + \left[r^n + z.n\, r^{n-1}\frac{\partial r}{\partial z}\right]$$

$$= 3r^n + n\, r^{n-1}\left(x.\frac{x}{r} + y.\frac{y}{r} + z.\frac{z}{r}\right)$$

$$= 3r^n + n\,\frac{r^{n-1}}{r}(x^2 + y^2 + z^2)$$

$$= 3r^n + n\, r^{n-2}.\, r^2 = (n+3)\, r^n$$

If $n = -3$, $(r^{-3} r) = (-3 + 3)\, r^{-3} = 0$

$\therefore \frac{r}{r^3}$ is solenoidal

**Ex. 7.3.10** Prove that Div $(C_1A + C_2B) = C_1$ Div A + $C_2$ Div B, where $C_1$, $C_2$ are constants.

**Sol :** Let $A = A_1 i + A_2 j + A_3 k$

$B = B_1 i + B_2 j + B_3 k$

$C_1A + C_2B = (C_1A_1 + C_2B_1)\, i + (C_1A_2 + C_2B_2) j + (C_1A_3 + C_2B_3) k$

$$\text{Div}\,(C_1A + C_2B) = \frac{\partial}{\partial x}(C_1A_1 + C_2B_1) + \frac{\partial}{\partial y}(C_1A_2 + C_2B_2) + \frac{\partial}{\partial z}(C_1A_3 + C_2B_3)$$

$$= C_1\left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right) + C_2\left(\frac{\partial B_1}{\partial x} + \frac{\partial B_2}{\partial y} + \frac{\partial B_3}{\partial z}\right)$$

$= C_1$ div A + $C_2$ div B

**Ex. 7.3.11** If $A = 2xi + 3yj + 5zk$ and $f = 2xyz$, find div $(fA)$ at $(1, 2, 3)$.

**Sol :** $fA = 2xyz\,(2xi + 3yj + 5zk)$

$= (4x^2yz)i + (6xy^2z)j + (10xyz^2)k$

$$\text{Div}\,(fA) = \frac{\partial}{\partial x}(4x^2yz) + \frac{\partial}{\partial y}(6xy^2z) + \frac{\partial}{\partial z}\,(10xyz^2)$$

$= 40\,xyz$

$\therefore$ At $(1, 2, 3)$, div $(fA) = 240$

Aliter: div $(fA) = D.(fA) = (\nabla f).\,A + f(\nabla A)$

$\nabla f = 2yzi + 2xzj + 2xyk$

$\nabla f.A = 4xyz + 6xyz + 10xyz = 20xyz$

$f(\nabla .A) = 2xyz\,(2 + 3 + 5) = 20xyz$

Hence $\nabla .(fA) = 40xyz$ and at $(1,2,3)$ $\nabla .(fA) = 240$

**Ex.7.3.12** If f, g are scalar fields show that $\nabla f \times \nabla g$ is solenoidal

**Sol :**
$$\nabla f = \left(\frac{\partial f}{\partial x}\right)i + \left(\frac{\partial f}{\partial y}\right)j + \left(\frac{\partial f}{\partial z}\right)k$$

$$\nabla g = \left(\frac{\partial g}{\partial x}\right)i + \left(\frac{\partial g}{\partial y}\right)j + \left(\frac{\partial g}{\partial z}\right)k$$

$$\nabla f \times \nabla g = \begin{vmatrix} i & j & k \\ \partial f/\partial x & \partial f/\partial y & \partial f/\partial z \\ \partial g/\partial x & \partial g/\partial y & \partial g/\partial z \end{vmatrix}$$

$= \Sigma\,(f_y g_z - f_z g_y)i$ (Suffixes denote partial derivatives)

$$\text{Div}\,(\nabla f \times \nabla g) = \Sigma\,\frac{\partial}{\partial x}\left(f_y g_z - f_z g_y\right)$$

$= \Sigma\,(f_y\,g_{xz} + g_z f_{xy} - f_z\,g_{xy} - g_y f_{xz})$

$= 0$

$\therefore$ $\nabla f \times \nabla g$ is solenoidal

## Exercise 6(b)

1. If $V = (x^2z)i - (2y^3z^2)j + (xy^2z)k$, find div A at the point (1, –1, 1)

[Ans. – 3]

2. If $r = xi + yj + zk$, find div r [Ans. 3]

3. If $F = (3xyz^2)i + (2xy^3)j - (x^2yz)k$, and $\phi = 3x^2 - yz$,

find (i) Div F (ii) Div ($\phi$F) and (iii) Div (grad$\phi$); at the point (1, –1, 1)

[ Ans. (i) 4 (ii) 1 (iii) 6 ]

4. If $V = (3x^2y - z)i + (xz^3 + y^4)j - 2x^3z^2k$, find grad (Div V) at the point (2, –1, 0)

[Ans. $-6i + 24j - 32k$ ]

5. Evaluate : (1) Div ($r^2$r) (2) Div (r r) (3) grad Div (r/r) (4) div (r/$r^3$)

[ Ans. (1) $5r^2$ (2) 4r (3) $-2r/r^3$ (4) 0 ]

6. Show that $V = 3y^4z^2i + 4x^3z^2j + 6x^2y^3k$ is solenoidal

7. Show that the vector $F = (2x^2 + 8xy^2z)i + (3x^3y - 3xy)j - (4y^2z^2 + 2x^3z)k$ is not solenoidal, but $G = xyz^2$ F is solenoidal.

8. If a is a constant vector and V = a × r, where $r = xi + yj + zk$, show that V is a solenoidal vector.

9. Determine the constant 'b' such that the vector, $V = (2x + 3y)i + (by - 3z)j + (6x - 12z)k$ is solenoidal

10. If $r_1$ and $r_2$ are vectors joining fixed points A $(x_1, y_1, z_1)$ and B$(x_2, y_2, z_2)$ to a variable point P$(x, y, z)$ prove that $r_1 \times r_2$ is solenoidal.

11. If $r = xi + yj + zk$ and $r = |r|$ show that, Div (grad $r^n$) = $n(n+1)r^{n-2}$.

12. If $g = r^{-2n}$, find div (grad g) amd find 'n' such that 'g' is solenoidal.

[Ans. $\dfrac{2n(2n-1)}{r^{2n+2}}; n = \dfrac{1}{2}$ ]

### 7.4 Curl of a vector function

**7.4.1** If A is a differential vector function, then curl A is defined as, curl A = $\nabla \times A$

$$\text{If } A = A_1 i + A_2 j + A_3 k, \text{ then Curl } A = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_1 & A_2 & A_3 \end{vmatrix}$$

$$= \left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right)i + \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right)j + \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)k$$

$$= \Sigma\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right)i$$

Note : The curl of a vector is also a vector

### 7.4.2 Physical significance of curl

Let $r = xi + yj + zk$ be the position vector of a point $P(x, y, z)$ of a rigid body rotating about a fixed axis about the origin O with an angular velocity $\omega = \omega_1 i + \omega_2 j + \omega_3 k$. Then the velocity V of the particle P is given by,

$$V = \omega \times r = \begin{vmatrix} i & j & k \\ \omega_1 & \omega_2 & \omega_3 \\ x & y & z \end{vmatrix}$$

$$= (\omega_2 z - \omega_3 y)i + (\omega_3 x - \omega_1 z)j + (\omega_1 y - \omega_2 x)k$$

$$\text{Curl } V = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \omega_2 z - \omega_3 y & \omega_3 x - \omega_1 z & \omega_1 y + \omega_2 x \end{vmatrix}$$

$$= i\left[\frac{\partial}{\partial y}(\omega_1 y - \omega_2 x) - \frac{\partial}{\partial z}(\omega_3 x - \omega_1 z)\right] + j\left[\frac{\partial}{\partial z}(\omega_2 z - \omega_3 y) - \frac{\partial}{\partial x}(\omega_1 y - \omega_2 x)\right]$$

$$+ k\left[\frac{\partial}{\partial x}(\omega_3 x - \omega_1 z) - \frac{\partial}{\partial y}(\omega_2 z - \omega_3 y)\right]$$

$$= i(\omega_1 + \omega_1) + j(\omega_2 + \omega_2) + k(\omega_3 + \omega_3) = 2\omega$$

Thus the curl of velocity vector is twice the angular velocity of rotation.

**7.4.3 Irrotational Vector :** A vector V whose curl is zero is said to be an irrotational vector.

**7.4.4 Properties :**(1) $\nabla \times (A + B) = \nabla \times A + \nabla \times B$ (or) curl (A + B) = curl A + curl B

Proof: Let $A = A_1 i + A_2 j + A_3 k$ and

$B = B_1 i + B_2 j + B_3 k$ so that

$A + B = (A_1 + B_1)i + (A_2 + B_2)j + (A_3 + B_3)k$

$$\text{Curl}\,(A+B) = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_1 + B_1 & A_2 + B_2 & A_3 + B_3 \end{vmatrix}$$

$$= \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_1 & A_2 & A_3 \end{vmatrix} + \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ B_1 & B_2 & B_3 \end{vmatrix}$$

$= \text{curl } A + B$

(2) If $\phi$ is a scalar function and A is a vector function

Curl $(\phi A) = \phi(\text{curl } A) + (\text{grad}\phi) \times A$

(or)

$\nabla \times (\phi A) = \phi(\nabla \times A) + (\nabla \phi) \times A$

*Proof :* If $A = A_1 i + A_2 j + A_3 k$, then $\phi A = \phi A_1 i + \phi A_2 j + \phi A_3 k$

$$\Delta \times (\phi A) = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \phi A_1 & \phi A_2 & \phi A_3 \end{vmatrix}$$

$$= i\left[\frac{\partial}{\partial y}(\phi A_3) - \frac{\partial}{\partial z}(\phi A_2)\right] + j\left[\frac{\partial}{\partial z}(\phi A_1) - \frac{\partial}{\partial x}(\phi A_3)\right] + k\left[\frac{\partial}{\partial x}(\phi A_2) - \frac{\partial}{\partial y}(\phi A_1)\right]$$

$$= \sum i\left[\phi\frac{\partial A_3}{\partial y} + A_3\frac{\partial \phi}{\partial y} - \phi\frac{\partial A_2}{\partial z} - A_2\frac{\partial \phi}{\partial z}\right]$$

$$= \phi\Sigma\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right)i + \Sigma\left(A_3\frac{\partial \phi}{\partial y} - A_2\frac{\partial \phi}{\partial z}\right)i$$

$$= \phi\begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_1 & A_2 & A_3 \end{vmatrix} + \begin{vmatrix} i & j & k \\ \partial\phi/\partial x & \partial\phi/\partial y & \partial\phi/\partial z \\ A_1 & A_2 & A_3 \end{vmatrix}$$

$= \phi(\nabla \times A) + (\nabla \phi) \times A$

### 7.4.5 Conservative vector field :

A vector field **F**, which can be derived from a scalar field $\phi$ such that $\mathbf{F} = \nabla\phi$, is called a conservative vector field and $\phi$ is called the scalar potential of **F**.

## Solved Examples

**Ex.7.4.6** If A = $(xy)i + (yz)j + (zx)k$; find (a) curl A and (b) curl curl A at (1, 2, –3)

**Sol :** A = $xyi + yzj + zxk$

(a) $$\text{curl A} = \nabla \times A = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xy & yz & zx \end{vmatrix}$$

$$= i\left[\frac{\partial}{\partial y}(zx) - \frac{\partial}{\partial z}(yz)\right] + j\left[\frac{\partial}{\partial z}(xy) - \frac{\partial}{\partial x}(zx)\right] + k\left[\frac{\partial}{\partial x}(yz) - \frac{\partial}{\partial y}(xy)\right]$$

$= i(0 - y) + j(0 - z) + \text{k}\ (0 - x)$

$= -yi - zj - xk$

$\therefore$ curl A at (1, 2, –3) = $-2i + 3j - k$

(b) curl A = $\nabla \times (\nabla \times A)$

$= \nabla \times (-yi - zj - xk)$

$$= \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -y & -z & -x \end{vmatrix}$$

$$= i\left[\frac{\partial}{\partial y}(-x) - \frac{\partial}{\partial z}(-z)\right] + j\left[\frac{\partial}{\partial z}(-y) - \frac{\partial}{\partial x}(-x)\right] + k\left[\frac{\partial}{\partial x}(-z) - \frac{\partial}{\partial y}(-y)\right]$$

$= i(0 - (-1) + j(0 - (-1)) + k(0 - (-1))$

$= \text{i} + \text{j} + \text{k}$

$\therefore$ curl A at (1, 2, –3) = $i + j + k$

**Ex. 7.4.7** Show that V = $xi + y^2j + z^3k$ is irrotational

**Sol :** Curl V = $\nabla \times$ V

$$= i\left[\frac{\partial}{\partial y}(z^3) - \frac{\partial}{\partial z}(y^2)\right] + j\left[\frac{\partial}{\partial z}(x) - \frac{\partial}{\partial x}(z^3)\right] + k\left[\frac{\partial}{\partial x}(y^2) - \frac{\partial}{\partial y}(x)\right]$$

$= 0 \quad \therefore$ V is irrotational

**Ex. 7.4.8** If F = $(4x + 3y + az)i + (bx - y + z)j + (2x + cy + z)k$ is irrotational, find the constants a, b, c

**Sol :** Curl F = $\begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ (4x+3y+az) & (bx-y+z) & (2x+cy+z) \end{vmatrix}$

$$= i\left[\frac{\partial}{\partial y}(2x+cy+z) - \frac{\partial}{\partial z}(bx-y+z)\right] + j\left[\frac{\partial}{\partial z}(4x+3y+az) - \frac{\partial}{\partial x}(2x+cy+z)\right]$$

$$+ k\left[\frac{\partial}{\partial x}(bx-y+z) - \frac{\partial}{\partial y}(4x+3y+az)\right]$$

$= (c - 1)i + (a - 2)j + (b - 3)k$

Since F is irrotational, curl F = 0

$\therefore c - 1 = 0, a - 2 = 0, b - 3 = 0$

i.e., $a = 2, b = 3, c = 1$

**Ex. 7.4.9** If r = $xi + yj + zk$, and r = |r|, find curl ($r^n$ r)

**Sol :** r = $xi + yj + zk$, r = |r| = $\sqrt{x^2 + y^2 + z^2}$

$r^n$ r = $(x^2 + y^2 + z^2)^{n/2}(xi + yj + zk)$

curl ($r^n$ r) = $\begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x(x^2+y^2+z^2)^{n/2} & y(x^2+y^2+z^2)^{n/2} & z(x^2+y^2+z^2)^{n/2} \end{vmatrix}$

$$= i\left[z.\frac{n}{2}(x^2+y^2+z^2)2y - y.\frac{n}{2}(x^2+y^2+z^2)2z\right]$$

$$+ j\left[x.\frac{n}{2}(x^2+y^2+z^2).2z - z.\frac{n}{2}(x^2+y^2+z^2).2x\right]$$

$$+ k\left[y.\frac{n}{2}(x^2+y^2+z^2).2x - x.\frac{n}{2}(x^2+y^2+z^2).2y\right]$$

$= i(0) + j(0) + k(0)$

$= 0$

Aliter: $r^n \mathbf{r} = r^n (xi + yj + zk)$

$$\text{curl}(r^n \mathbf{r}) = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xr^n & yr^n & zr^n \end{vmatrix}$$

$$= \Sigma i\left[\frac{\partial}{\partial y}(zr^n) - \frac{\partial}{\partial z}(yr^n)\right] = \Sigma i\left[z.nr^{n-1}\frac{\partial r}{\partial y} - y.nr^{n-1}\frac{\partial r}{\partial z}\right]$$

$$= \Sigma i\left[nzr^{n-1}\frac{y}{r} - y.n.r^{n-1}\frac{z}{r}\right]$$

$$= i(o) + j(o) + k(o) = 0$$

**Ex. 7.4.10** Prove that, if $F = (x + y + 1)i + j - (x + y)k$, F. curl F = 0

**Sol :** $$\text{Curl } F = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x+y+1 & 1 & -x-y \end{vmatrix}$$

$$= i[-1-0] + j[0+1] + k[0-1]$$

$$= -i + j - k$$

$$\therefore F \text{ curl } F = -1(x + y + 1) + 1.1 + 1(x + y) = 0$$

**Ex. 7.4.11** $P(x, y, z)$ is a variable point and $Q(x_1, y_1, z_1)$, $R(x_2, y_2, z_2)$ are fixed points. If U = QP and V = RP; Prove that curl (U × V) is equal to 2(U – V).

**Sol :** $P(x, y, z)$, $Q = (x_1, y_1, z_1)$, $R = (x_2, y_2, z_2)$

$$\therefore QP = (x - x_1)i + (y - y_1)j + (z - z_1)k = U$$

$$RP = (x - x_2)i + (y - y_2)j + (z - z_2)k = V$$

$$U \times V = \begin{vmatrix} i & j & k \\ (x-x_1) & (y-y_1) & (z-z_1) \\ (x-x_2) & (y-y_2) & (z-z_2) \end{vmatrix}$$

$$= \Sigma\{(y - y_1)(z - z_2) - (z - z_1)(y - y_2)\}\, i$$

$$= \Sigma\, \{y(z_1 - z_2) - z(y_1 - y_2) + (y_1 z_2 - y_2 z_1)\}\, i$$

$$\text{Curl } (U \times V) = \sum\left[\frac{\partial}{\partial y}\{x(y_1 - y_2) - y(x_1 - x_2) + (x_1y_2 - x_2y_1)\}\right.$$

$$\left. - \frac{\partial}{\partial z}\{z(x_1 - x_2) - x(z_1 - z_2) + (z_1x_2 - z_2x_1)\}\right]i$$

$$= \Sigma(-(x_1 - x_2) - (x_1 - x_2)i$$

$$= \Sigma -2(x_1 - x_2)i = 2\Sigma(x_2 - x_1)i$$

$$= 2(U - V)$$

**Ex. 7.4.12** If F is a conservative vector field show that curl F = 0

**Sol.** F is a conservative vecor field.

$\therefore$ There exists a scalar field '$\phi$' such that F = $\nabla\phi = \frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial z}k$

$$\therefore \text{Curl } F = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \partial\phi/\partial x & \partial\phi/\partial y & \partial\phi/\partial z \end{vmatrix}$$

$$= \Sigma i\left(\frac{\partial^2\phi}{\partial y\partial z} - \frac{\partial^2\phi}{\partial z\partial y}\right) = 0$$

**Ex. 7.4.13** Show that $F = (6xy + z^3)i + (3x^2 - z)j + (3xz^2 - y)k$ is irrotational. Find $\phi$ such that $F = \nabla\phi$

**Sol :**

$$\text{Curl } F = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 6xy + z^3 & 3x^2 - z & 3xz^2 - y \end{vmatrix}$$

$$= i(-1 + 1) + j(3z^2 - 3z^2) + k(6x - 6x)$$

$$= 0$$

$\therefore$ F is irrotational

Let $F = \nabla\phi = \frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial z}k$

$$\therefore \frac{\partial\phi}{\partial x} = 6xy + z^3 \qquad \text{..... (1)}$$

$$\frac{\partial \phi}{\partial y} = 3x^2 - z \qquad ..... (2)$$

$$\frac{\partial \phi}{\partial z} = 3xz^2 - y \qquad ..... (3)$$

Integrating equation (1) with respect to $x$ we get

$$\phi = 3x^2y + xz^3 + f(y, z) \qquad ..... (4)$$

Differentiating (4) partially w.r.t '$y$'

$$\frac{\partial \phi}{\partial y} = 3x^2 + \frac{\partial f(y,z)}{\partial y} \qquad ..... (5)$$

From (2) and (5) we have

$$\frac{\partial f(y,z)}{\partial y} = -z \qquad ..... (6)$$

Integrating (6) w.r.t 'y' we get

$$f(y, z) = -yz + h(z)$$

Hence $\phi = 3x^2 + zx^3 - yz + h(z)$ ..... (7)

differentiating (7) w.r.t '$z$' we get

$$\frac{\partial \phi}{\partial z} = 3xz^2 - y + h^1(z) \qquad ..... (8)$$

Comparing (3) and (8) we have

$$h^1(z) = 0, \therefore h(z) = \text{constant, 'c' say}$$

Hence $\phi = 3x^2y + xz^3 - yz + c$ (from 7)

Aliter : $\therefore$

$$\frac{\partial \phi}{\partial x} = 6xy + z^3 \qquad .... (1)$$

$$\frac{\partial \phi}{\partial y} = 3x^2 - z \qquad .... (2)$$

$$\frac{\partial \phi}{\partial z} = 3xz^2 - y \qquad ..... (3)$$

Integrating the above equations respectively w.r.t $x$, $y$, and $z$, we get

$$\phi = 3x^2y + xz^3 + f(y, z) \quad \text{.... (4)}$$

$$\phi = 3x^2y - yz + g(z, x) \quad \text{.... (5)}$$

$$\phi = xz^3 - yz + h(x, y) \quad \text{..... (6)}$$

$\phi$ should satisfy all the above three equations simultaneously

(i.e.) (4), (5) & (6)

$\therefore \phi = 3x^2y + xz^3 - yz + c$, where c is a numerical constant.

*Note*: [Here $f = -yz$, $g = xz^3$, $h = 3x^2y$ will satisfy (4), (5) & (6)

**Ex.6.4.14** If f(r) is differentiable, show that f(r)r is irrotational

**Sol :** $f(r)\, r = f(r)\,(xi + yj + zk)$

$$\text{Curl } (f(r)\, r) = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x f(r) & y f(r) & z f(r) \end{vmatrix}$$

$$= \sum i\left(\frac{\partial}{\partial y}\{z f(r)\} - \frac{\partial}{\partial z}\{y f(r)\}\right)$$

$$= \sum i\left(z.f'(r)\frac{y}{r} - y.f'(r)\frac{\partial r}{\partial z}\right)$$

$$= \sum i\left(z.f'(r)\frac{y}{r} - y.f'(r)\frac{z}{r}\right) = 0$$

$\therefore$ $f(r)r$ is irrotational

**Ex. 7.4.15** If U and V are irrotational, prove that U × V is solenoidal

**Sol :** Let $U = U_1 i + U_2 j + U_3 k$

$V = V_1 i + V_2 j + V_3 k$

U is irrotational $\therefore$ curl U = 0

$$\Rightarrow \Sigma i\left[\frac{\partial U_3}{\partial y} - \frac{\partial U_2}{\partial z}\right] = 0 \quad \text{..... (1)}$$

$$\text{similarly, } \Sigma i\left[\frac{\partial V}{\partial y}3 - \frac{\partial V}{\partial z}2\right] = 0 \quad \text{..... (2)}$$

$$U \times V = \begin{vmatrix} i & j & k \\ U_1 & U_2 & U_3 \\ V_1 & V_2 & V_3 \end{vmatrix} = \Sigma i(U_2V_3 - U_3\,V_2)$$

$$\text{Div}\,(U \times V) = \Sigma \frac{\partial}{\partial x}(U_2\,V_3 - U_3V_2)$$

$$= \sum\left(U_2\frac{\partial V_3}{\partial x} + V_3\frac{\partial U_2}{\partial x} - U_3\frac{\partial V_2}{\partial x} - V_2\frac{\partial U_3}{\partial x}\right) \qquad \text{..... (3)}$$

From (1), we have, $\frac{\partial U_3}{\partial y} = \frac{\partial U_2}{\partial z}; \frac{\partial U_1}{\partial z} = \frac{\partial U_3}{\partial x}; \frac{\partial U_2}{\partial x} = \frac{\partial U_1}{\partial y}$ ..... (4)

and from (2), we have, $\frac{\partial V_3}{\partial y} = \frac{\partial V_2}{\partial z}; \frac{\partial V_1}{\partial z} = \frac{\partial V_3}{\partial x}; \frac{\partial V_2}{\partial x} = \frac{\partial V_1}{\partial y}$ ..... (5)

Substituting the six equations of (4) & (5) in (3), we observe that all the 12 terms of (3) will get cancelled. Hence Div $(U \times V) = 0$

$\Rightarrow U \times V$ is solenoidal

## Exercise - 7(c)

1. If $V = (2xz^2)i - (yz)j + (3xz^3)k$, and $f = x^2yz$, find the following at the point (1, 1, 1) (a) curl V (b) curl $(fV)$ (c) curl (curl V)

   [Ans. (a) $i + j$ (b) $5i - 3j - 4k$ (c) $5i + 3k$]

2. If 'g' is a scalar function, show that curl (g grad g) = 0

3. Find the value of the constant 'p' for which the vector $V = (pxy - z^3) + (p - 2)\,x^2j + (1 - p)\,xz^2k$ is irrotational.

   [Ans : 4]

4. Find the constants a, b, c so that the curl of the vector $A = (x + 2y + az)i + (bx - 3y - z)j + (4x + cy + 2z)k$ is identically equal to zero

   [Ans. 4, 2, –1]

5. If $r = xi + yj + zk$, $r = |r|$, show that curl $\left(\frac{r}{r^2}\right) = 0$ and find a scalar function 'f' such that $\frac{r}{r^2} = -\nabla f, f(a) = 0$ where $a > 0$.

   [Ans : $f = \log\left(\frac{a}{r}\right)$]

6. If $r = xi + yj + zk$, and $p$, $q$ are constant vectors, show that (1) curl $[(r \times p) \times q] = (p \times q)$ and (2) curl $[(p.q)r)] = 0$.

## 7.5 Laplacian Operator : $\nabla^2$

**7.5.1** $\nabla^2 = \nabla.\nabla = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right).\left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)$

$= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)$ is called the Laplacian Operator

'$\nabla^2$' can be applied to both scalar and vector functions as shown below.

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$$

where '$\phi$' is scalar function

If $A = A_1 i + A_2 j + A_3 k$, is a vector function, then

$$\nabla^2 A = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)(A_1 i + A_2 j + A_3 k)$$

$$= \left(\nabla^2 A_1\right)i + \left(\nabla^2 A_2\right)j + \left(\nabla^2 A_3\right)k$$

**7.5.2 Vector Identities :** we shall give below some vector identities with proofs.

1. If '$\phi$' is a scalar function curl grad $\phi = 0$, (or) $\nabla \times \nabla\phi = 0$

***Proof :*** Curl grad $\phi = \nabla \times \left(\frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial z}k\right)$

$$= \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \partial\phi/\partial x & \partial\phi/\partial y & \partial\phi/\partial z \end{vmatrix}$$

$$= \sum i\left[\frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{\partial\phi}{\partial y}\right)\right]$$

$$= \sum i\left(\frac{\partial^2\phi}{\partial y\partial z} - \frac{\partial^2\phi}{\partial z\partial y}\right)$$

$= 0$, assuming that '$\phi$' possesses continuous second order partial derivatives.

(2) If V is a vector function, Div (Curl A) = 0 (or) $\nabla.(\nabla \times A) = 0$

Proof: Let A = $A_1 i + A_2 j + A_3 k$

$$\text{Curl A} = \sum i\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right)$$

$$\text{Div (curl A)} = \sum \frac{\partial}{\partial x}\left\{\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right\}$$

$$= \frac{\partial^2 A_3}{\partial x \partial y} - \frac{\partial^2 A_2}{\partial x \partial z} + \frac{\partial^2 A_1}{\partial y \partial z} - \frac{\partial^2 A_3}{\partial y \partial x} + \frac{\partial^2 A_2}{\partial z \partial x} - \frac{\partial^2 A_1}{\partial z \partial y}$$

$$= 0$$

(3) If A is a vector function, curl (curl A) = grad (Div A) − $\nabla^2 A$ (or) $\nabla \times (\nabla \times A) = \nabla(\nabla.A) - \nabla^2 A$

***Proof:*** $\text{curl A} = \sum i\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right)$

$$\text{Curl (curl A)} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right) & \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right) & \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right) \end{vmatrix}$$

$$= \sum \left[\frac{\partial}{\partial y}\left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right)\right] i$$

$$= \left[\left(-\frac{\partial^2 A_1}{\partial x^2} - \frac{\partial^2 A_1}{\partial y^2} - \frac{\partial^2 A_1}{\partial z^2}\right) i + \left(-\frac{\partial^2 A_2}{\partial x^2} - \frac{\partial A_2}{\partial y^2} - \frac{\partial^2 A_2}{\partial z^2}\right) j\right.$$

$$\left. + \left(-\frac{\partial^2 A_3}{\partial x^2} - \frac{\partial^2 A_3}{\partial y^2} - \frac{\partial^2 A_3}{\partial z^2}\right) k\right] + \left[\left(\frac{\partial^2 A_1}{\partial x^2} + \frac{\partial^2 A_2}{\partial y \partial x} + \frac{\partial^2 A_3}{\partial z \partial x}\right) i\right.$$

$$\left. + \left(\frac{\partial^2 A_1}{\partial x \partial y} + \frac{\partial^2 A_2}{\partial y^2} + \frac{\partial^2 A_3}{\partial z \partial y}\right) j + \left(\frac{\partial^2 A_1}{\partial x \partial z} + \frac{\partial^2 A_2}{\partial y \partial z} + \frac{\partial^2 A_3}{\partial z^2}\right) k\right]$$

(adding and subtracting $\frac{\partial^2 A_1}{\partial x^2} i$, $\frac{\partial^2 A_2}{\partial y^2} j$ and $\frac{\partial^2 A_3}{\partial z^2} k$).

$$= -\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)(A_1 i + A_2 j + A_3 k)$$

$$+ i\frac{\partial}{\partial x}\left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right) + j\frac{\partial}{\partial y}\left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right) + k\frac{\partial}{\partial z}\left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right)$$

$$= -\nabla^2 A + \nabla\left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right)$$

$$= -\nabla^2 A + \nabla(\nabla . A)$$

(4) $\nabla.(A \times B) = B.(\nabla \times A) - A.(\nabla \times B)$, (A, B are vector functions) (or) Div(A×B)=B. curl A – A. curl B.

***Proof:*** Let $A = A_1 i + A_2 j + A_3 k$ and $B = B_1 i + B_2 j + B_3 k$

$$A \times B = \begin{vmatrix} i & j & k \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}$$

$$= \sum i(A_2 B_3 - A_3 B_2)$$

$$\text{Div}(A \times B) = \sum \frac{\partial}{\partial x}(A_2 B_3 - A_3 B_2)$$

$$= \sum \left(A_2 \frac{\partial B_3}{\partial x} + B_3 \frac{\partial A_2}{\partial x} + A_3 \frac{\partial B_2}{\partial x} + B_2 \frac{\partial A_3}{\partial x}\right) \quad .....(1)$$

$$\nabla \times A = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_1 & A_2 & A_3 \end{vmatrix} = \sum\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right) i$$

$$B.(\nabla \times A) = \sum B_1\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right) \quad ..... (2)$$

Similarly,

$$A.(\nabla \times B) = \sum A_1\left(\frac{\partial B_3}{\partial y} - \frac{\partial B_2}{\partial z}\right) \quad \text{..... (3)}$$

Expanding the summations of (1), (2) & (3), we observe that,

Div (A × B) = B. (∇ × A) – A. (∇ × B)

### 7.5.3 Operation of ∇ on product of two functions

Suppose $\phi$ and $\Psi$, are two scalar or vector point functions. When $\nabla$ is operated on the product of $\phi$ and $\psi$, the following rule is useful.

$\nabla(\phi\Psi) = \nabla(\phi_0\Psi) + \nabla(\phi\Psi_0)$ wherein the suffix '0' indicates that the function is not to be varied, that is, $\nabla$ is not to be operated on that function. After the completion of operation of $\nabla$ the suffixes are dropped.

While proving the identities the following are useful.

(a) a.b = b.a

(b) a × b = –b × a

(c) a × a = 0

(d) a. (b × c) = (a × b) .c

(e) a × (b × c) = (a. c)b – (a. b) c

(f) (a. c) b = a × (b × c) + (a. b) c

(g) a. a = $a^2$

(h) × operation is always between vectors only

vector identities using $\nabla$ operation:

(i) $\nabla.(FG) = \nabla.(FG_0) + \nabla.(F_0G)$

$= \nabla F.G_0 + F_0\, \nabla.G$

$\therefore$ div (fG) = G. grad F + F div G

(ii) $\nabla \times (FG) = \nabla \times (FG_0) + \nabla \times (F_0G)$

$\nabla \times (FG_0) = \nabla F \times G_0$ due to (h)

$\nabla \times (F_0G) = (\nabla \times G)F_0 = F_0(\nabla \times G)$

$\therefore\ \nabla \times (FG) = \nabla F \times G + F(\nabla \times G)$

(i.e.) curl (FG) = (grad F) × G + F curl G

(iii) $\nabla . (F \times G) = \nabla .(F \times G_0) + \nabla . (F_0 \times G)$

$\nabla . (F \times G_0) = (\nabla \times F) .G_0$ due to (d)

$= G_0 .(\nabla \times F)$ due to (a)

$\nabla . (F_0 \times G) = - \nabla . (G \times F_0)$ due to (b)

$= - \nabla \times (G. F_0)$ due to (d)

$= - (\nabla \times G). F_0$

$= - F_0 (\nabla \times G)$ due to (a)

$\therefore \nabla . (F \times G) = G. (\nabla \times F) - F. (\nabla \times G)$

Thus div $(F \times G) = G.$ curl $F - F.$ curl $G$

(iv) $\nabla \times (F \times G) = \nabla \times (F \times G_0) + \nabla \times (F_0 \times G)$

$\nabla \times (F \times G_0) = (\nabla . G_0) F - (\nabla . F) G_0$ due to (e)

$= (G_0 . \nabla) F - G_0 (\nabla . F)$ due to (a)

$\nabla \times (F_0 \times G) = (\nabla .G) F_0 - (\nabla . F_0) G$ due to (e)

$= F_0 (\nabla . G) - (F_0 . \nabla) G$ due to (a)

$\therefore \nabla \times (F \times G) = (G . \nabla) F - G(\nabla .F) + F (\nabla .G) - (F. \nabla) G$

Thus curl $(F \times G) = (G. \nabla) F - (F. \nabla) G + F (\nabla .G) - G (\nabla .F)$

(v) $\nabla (F . G) = \nabla (F_0 . G) + \nabla (F .G_0)$

$\nabla (F_0 . G) = F_0 \times (\nabla \times G) + (F_0 . \nabla) G$ due to (f)

$\nabla (F . G_0) = \nabla (G_0 . F) = G_0 \times (\nabla \times F) + (G_0 . \nabla) F$ due to (f)

$\therefore \nabla (F . G) = F \times (\nabla \times G) + G \times (\nabla \times F) + (F . \nabla) G + (G . \nabla) F$

Thus grad $(F . G) = F \times$ curl $G + G \times$ curl $F + (F . \nabla) G + (G . \nabla) F$

(vi) curl (grad F) $= \nabla \times (\nabla F) = (\nabla \times \nabla) F = 0$ due to (c)

(vii) div (curl F) $= \nabla .(\nabla \times F) = \nabla \times (\nabla . F)$ due to (d)

$= (\nabla \times \nabla). F = 0$ due to (c)

(viii) curl (curl F) $= \nabla \times (\nabla \times F)$

$= (\nabla . F)\nabla - (\nabla . \nabla) F$ due to (e)

$= \nabla (\nabla . F) - \nabla^2 F$ (since $\nabla .$ must not appear at the end)

$=$ grad (div F) $- \nabla^2 F$

## Solved examples

**Ex. 7.5.4** If $f = x^2y^3z^2$, find $\nabla^2 f$ at (1, 2, 1)

**Sol:** $$\nabla^2 f = \frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2}$$

$$= 2y^3z^2 + 6x^2yz^2 + 2x^2y^3$$

$\therefore$ at (1, 2, 1), $\nabla^2 f = 16 + 12 + 16 = 44$.

**Ex. 7.5.5** Show that, if $r = xi + yj + zk$, $r = |r|$, then $\nabla^2 r^n = n(n+1)\, r^{n-2}$

**Sol:** $r^n = (x^2 + y^2 + z^2)^{n/2}$, $(\because r = \sqrt{x^2 + y^2 + z^2}\,)$

$$\frac{\partial}{\partial x}(r^n) = \frac{n}{2}(x^2+y^2+z^2)^{\frac{n-2}{2}-1}.2x = nx(x^2+y^2+z^2)^{\frac{n-2}{2}}$$

$$\frac{\partial^2}{\partial x^2}(r^n) = n\left[x.\frac{n-2}{2}(x^2+y^2+z^2)^{\frac{n-2}{2}-1}.2x + (x^2+y^2+z^2)^{\frac{n-2}{2}}\right]$$

$$= n\left[(n-2)x^2.(x^2+y^2+z^2)^{\frac{n-4}{2}} + (x^2+y^2+z^2)^{\frac{n-2}{2}}\right]$$

$$= n(n-2)\ x^2.r^{n-4} + nr^{n-2} \qquad \ldots(1)$$

Similarly, $$\frac{\partial^2}{\partial y^2}(r^n) = n(n-2)y^2r^{n-4} + nr^{n-2} \qquad \ldots(2)$$

and $$\frac{\partial^2}{\partial z^2}(r^n) = n(n-2)z^2r^{n-4} + nr^{n-2} \qquad \ldots(3)$$

$$\nabla^2(r^n) = \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)r^n$$

$= 3nr^{n-2} + n(n-2)\, r^{n-4}\,(x^2 + y^2 + z^2)$ adding (1), (2) & (3)

$= 3nr^{n-2} + n(n-2)\, r^{n-2}$ $(\because x^2 + y^2 + z^2 = r^2)$

$= n(n+1)\, r^{n-2}$

Aliter: $$\frac{\partial}{\partial x}\left(r^n\right) = nr^{n-1}.\frac{\partial r}{\partial x} = nr^{n-1}.\frac{x}{r}$$

$$= nr^{n-2}.x \qquad \left(\because \frac{\partial r}{\partial x} = \frac{x}{r}\right)$$

$$\frac{\partial^2}{\partial z^2}\left(r^n\right) = n\left(r^{n-2}.1 + x.(n-2)r^{n-3}.\frac{\partial r}{\partial x}\right)$$

$$= n\left(r^{n-2} + (n-2)r^{n-3}.\frac{x^2}{r}\right)$$

$$= nr^{n-2} + n(n-2)\, r^{n-4}.\, x^2 \qquad \ldots(1)$$

Similarly, $$\frac{\partial^2}{\partial y^2}\left(r^n\right) = nr^{n-2} + n(n-2)r^{n-4}.y^2 \qquad \ldots(2)$$

and $$\frac{\partial^2}{\partial z^2}\left(r^n\right) = nr^{n-2} + n(n-2)r^{n-4}.z^2 \qquad \ldots(3)$$

Adding (1), (2) & (3) we get,

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)r^n = 3nr^{n-2} + n(n-2)r^{n-4}\left(x^2 + y^2 + z^2\right)$$

$$\Rightarrow \nabla^2 r^n = 3nr^{n-2} + n(n-2)r^{n-4}.r^2$$
$$= r^{n-2}[3n + n^2 - 2n]$$
$$= n(n+1)\, r^{n-2}$$

*Note:* If n = –1, we have $\nabla^2\left(\frac{1}{r}\right) = 0$, which means that $\left(\frac{1}{r}\right)$ satisfies the Laplace's equation

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0 \text{ or } \nabla^2\phi = 0$$

**Ex.7.5.6** If $\nabla.U = 0$, $\nabla.V = 0$, $\nabla \times U = -\frac{\partial V}{\partial t}$, $\nabla \times V = \frac{\partial U}{\partial t}$, show that U and V satisfy the wave equation $\nabla^2 U = \frac{\partial^2 U}{\partial t^2}$

**Sol:** $$\nabla \times (\nabla \times U) = \nabla \times \left(-\frac{\partial V}{\partial t}\right) = \frac{-\partial}{\partial t}(\nabla \times V)$$

$$= \frac{-\partial}{\partial t}\left(\frac{\partial U}{\partial t}\right) = \frac{-\partial^2 U}{\partial t^2} \qquad \ldots(1)$$

But $\nabla \times (\nabla \times U) = \nabla(\nabla . U) - \nabla^2 U$

$= -\nabla^2 U \quad (\because \nabla U = 0)$ ...(2)

from (1) & (2), $\frac{\partial^2 U}{\partial t^2} = \nabla^2 U$, which shows that U satisfies the wave equation.

Similarly,

$$\nabla \times (\nabla \times V) = \nabla \times \frac{\partial U}{\partial t} = \frac{\partial}{\partial t}(\nabla \times U) = \frac{-\partial^2 V}{\partial t^2} \quad ...(3)$$

and $\nabla \times (\nabla \times V) = \nabla(\nabla . V) - \nabla^2 V = -\nabla^2 V (\because \nabla . V = 0)$

...(4)

(3) and (4) $\Rightarrow$ V satisfies the wave equation.

**Ex. 7.5.7** If $\nabla . V = 0$, show that $\nabla \times [\nabla \times \{\nabla \times (\nabla \times V)\}] = \nabla^4 V$ (or)

curl [curl{curl (curl V)}] = $\nabla^4 V$

**Sol:** We know that

$\nabla \times (\nabla \times V) = \nabla(\nabla . V) - \nabla^2 V$ [from 7.5.2 (3)]

$= -\nabla^2 V \; (\because \nabla . V = 0)$

$= -U$ (say)

Then the given expression

$= -\nabla \times (\nabla \times U) = \nabla^2 U - \nabla(\nabla . U)$ [from 7.5.2 (3)]

$= \nabla^2(\nabla^2 V) - \nabla(\nabla . U) \qquad (\because \nabla^2 V = U)$

$= \nabla^4 V - \nabla(\nabla . U)$

$$\nabla . U = \nabla . (\nabla^2 V)$$

$$= \left\{\sum i \frac{\partial}{\partial x}\right\} . \{\nabla^2 V\}$$

$$= \sum \left\{i \frac{\partial}{\partial x}\right\} . \left(\nabla^2 (V_1 i + V_2 j + V_3 k)\right)$$

$$= \nabla^2 \left(\frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}\right)$$

$= \nabla^2 (Div V) = \nabla^2(0) = 0 \qquad (\because \text{Div } V = \nabla . V = 0)$

Hence the problem

## Exercise 7(d)

(1) Show that $\nabla^2(\log r) = \frac{1}{r^2}$

(2) Prove that $\nabla^2(fg) = f(\nabla^2 g) + 2(\nabla f).(\nabla g) + g(\nabla^2 f)$

(3) Show that $\nabla^2\left(\nabla.\frac{r}{r^2}\right) = \frac{2}{r^4}$

(4) Show that $\nabla^2\phi(r) = \frac{d^2\phi}{dr^2} + \frac{2}{r}\frac{df}{dr}$

show also that, if $\nabla^2\phi = 0$, then $\phi = C_1 + \frac{C_2}{r}$ where $C_1$, $C_2$ are constants

(5) If $r = xi + yj + zk$, prove that, curl ($k \times$ grad $\frac{1}{r}$) + grad ($k$ . grad $\frac{1}{r}$) = 0

## 7.6 VECTOR INTEGRATION

### 7.6.1 Ordinary integration of vectors

(1) If $A(u) = A_1(u)i + A_2(u)j + A_3(u)k$ be a vector function of a scalar variable 'u' ($A_1(u), A_2(u), A_3(u)$ assumed to be continuous in any given interval), the indefinite integral of A(u) is given by.

$$\int A(u)du = i\int A_1(u)du + j\int A_2(u)du + k\int A_3(u)du$$

(2) If there exists a vector B(u) such that A(u) = $\frac{d}{du}$ (B(u)), we can write

$$\int A(u)du = \int \frac{d}{du}(B(u))du = B(u) + c$$

where c is a constant of integration independent of u.

(3) The definite integral of A(u) between the limits 'a' and 'b' in the above, is written as,

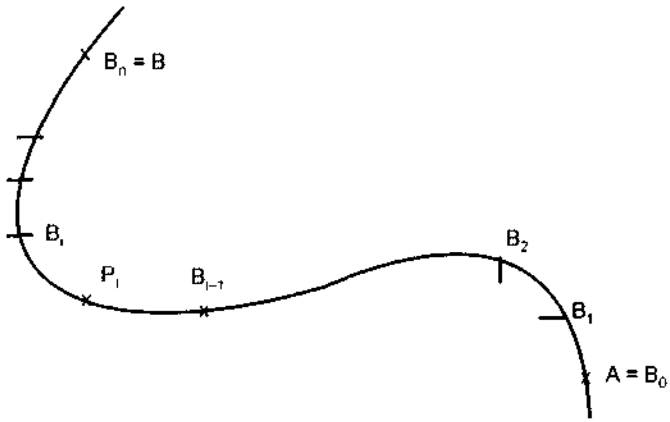
$$\int_a^b A(u)du = \int_a^b \frac{d}{du}(B(u))du = B(u) + c\Big|_a^b = B(b) - B(a)$$

### 7.6.2 Line Integrals:

Let $A(x,y,z)$ be a continuous vector function defined in the entire region of space. Let c be any curve in the region. Divide c into n intervals by taking points

$A = B_0, B_1, B_2 \ldots B_n (= B)$.

Let $P_i$ be any point in the interval $B_{i-1} B_i$




Let $r_0, r_1......r_n$ be the position vectors of points $B_0, B_1, B_2 ......B_n$ respectively. Let us consider the sum,

$$\sum A(P_i)\delta r_i$$

The limit of this sum as $n \to \infty$ and $|\delta r_i| \to 0$ is defined as the line integral of A along the curve c and is denoted symbolically by

$$\int_c A.dr \text{ or } \int_c A.\frac{dr}{dt}dt; \qquad \text{which is a scalar.}$$

If c is a closed curve, the integral is written as $\oint A.dr$

**Cartesian form of line integral:**

If $\quad A = A_1 i + A_2 j + A_3 k$

$\quad dr = (dx)i + (dy)j + (dz)k,$

$$\int_c A.dr = \int_c A_1 dx + A_2 dy + A_3 dz$$

*Note:* $\int_c \phi dr$, and $\int_c A \times dr$ are also examples of line integrals.

### 7.6.3 Physical appliations:

**(1) Work done by a force**

(1) If A represents a force and dr is an element of the path of the particle along a curve c, then the line integral

$$\int_P^Q A.dr \qquad \text{(P, Q are 2 points on c)}$$

represents the work done by force A in moving the particle from P to Q.

**(2) flow or circulation:**

(2) If A is the electric field strength, the line integral given above i.e., $\int_c A.dr$, is called the flow of A along c. If c is a closed curve it is often referred to as circulation of A around c.

In general, the line integral $\int_P^Q A.dr$, will depend on the path from P to Q

**7.6.4** ***Theorem:*** Prove that the necessary and sufficient condition for the integral $\int_c A.dr$, to be independent of the path c joining any two points is that A is a conservative vector field, (or) there exists a scalar field $\phi$ such that $A = \nabla\phi$ (or) curl $A = 0$, [i.e., the work done by the force A in moving the particle from one point to another is independent of the path if $A = \nabla\phi$ ]

*Proof:* Let $P = (x_1, y_1, z_1)$, $Q = (x_2, y_2, z_2)$ be any two given points on the curve c. Let $A = \nabla\phi$ where $\phi$ is single–valued and has continuous derivatives.

(1) Work done $= \int_P^Q A.dr$

$$= \int_P^Q \nabla\phi.dr = \int_P^Q \left( i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z} \right) \quad (dxi + dyj + dzk)$$

$$= \int_P^Q \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz = \int_P^Q d\phi = \phi(Q) - \phi(P)$$

$$= \phi(x_2, y_2, z_2) - \phi(x_1, y_1, z_1)$$

i.e., the integral depends upon the two points only but not on the path joining them. (This is true only if $\phi$ is single–valued at all points P and Q).

(2) The integral is independent of the path. Then,

$$\phi(x,y,z) = \int_{(x_1,y_1,z_1)}^{(x,y,z)} A.dr = \int_{(x_1,y_1,z_1)}^{(x,y,z)} A.\frac{dr}{ds}ds$$

Differentiation with respect to 's' gives

$$\frac{d\phi}{ds} = A.\frac{dr}{ds} \qquad \text{...... (1)}$$

But $\frac{d\phi}{ds} = \nabla\phi \frac{dr}{ds}$ ...... (2)

$(1) - (2) \Rightarrow (A - \nabla\phi).\ \frac{dr}{ds} = 0$, which is true irrespective of $\frac{dr}{ds}$

$\therefore A = \nabla\phi$

## Solved Examples

**Ex. 7.6.5:** $F(t) = (3t^2 - t)i + (2-6t)j - 4tk$, find (a) $\int F(t)dt$ (b) $\int_2^4 F(t)dt$

**Sol:** (a) $\int F(t)dt = i\int(3t^2 - t)dt + j\int(2-6t)dt - k\int 4t\,dt$

$$= \left(t^3 - \frac{t^2}{2}\right)i + \left(2t - 3t^2\right)j - 2t^2k + c$$

(b) $\int_2^4 F(t)dt = \left(t^3 - \frac{t^2}{2}\right)i + \left(2t - 3t^2\right)j - 2t^2k + c\Big|_2^4 = 50i - 32j - 24k$

**Ex. 7.6.6** Evaluate: $\int_0^{\pi/2}[(3\sin\theta)i + (2\cos\theta)j]d\theta$

**Sol:** Given integral

$$= i\int_0^{\pi/2}(3\sin\theta)d\theta + j\int_0^{\pi/2}(2\cos\theta)d\theta = -3\cos\theta\Big|_0^{\pi/2} i + 2\sin\theta\Big|_0^{\pi/2} j = 3i + 2j$$

**Ex.7.6.7** The acceleration a of a particle at any time 't' $\geq 0$ is given by,

$a = e^{-t}i - 6(t+1)j + (3\sin t)k$

Find the velocity V and displacement r at any time 't' given that V = 0 when t = 0 and r = 0 when t = 0.

**Sol:** $a = e^{-t}i - 6(t+1)j + (3\sin t)k = \frac{d^2r}{dt^2}$

$$\therefore V = \frac{dr}{dt} = \int a\,dt = -e^{-t}i - 6\left(\frac{t^2}{2} + t\right)j - (3\cos t)k + C_1 \quad .....(1)$$

($C_1$ is constant of integration)

But V = 0 when t = 0

$\therefore -i - 3k + C_1 = 0 \Rightarrow C_1 = i + 3k$

Substituting in (1), we get

Velocity $V = (1 - e^{-t})i - (3t^2 + 6t)j + (3 - 3\cos t)k$

Integrating,

$r = (1 + e^{-t})i - (t^3 + 3t^2)j + (3t - 3\sin t)k + C_2$ ..... (2)

($C_2$ being constant of integration)

But r = 0 when t = 0.

$\therefore$ (2) $\Rightarrow + i + C_2 = 0 \therefore C_2 = - i$

From (2), $r = (t - 1 + e^{-t})i - (t^3 + 3t^2)j + (3t - 3\sin t)k$

**Ex. 7.6.8** Show that

$$\int F \times \frac{d^2F}{dt^2} = F \times \frac{dF}{dt} + c$$

**Sol:** We know that

$$\frac{d}{dt}\left(F \times \frac{dF}{dt}\right) = F \times \frac{d}{dt}\left(\frac{dF}{dt}\right) + \frac{dF}{dt} \times \frac{dF}{dt}$$

$$= F \times \frac{d^2F}{dt^2} \qquad \left(\because \frac{dF}{dt} \times \frac{dF}{dt} = 0\right)$$

Hence the result.

**Ex. 7.6.9** If $A = 2ti + 3t^2j - (4t + 1)k$, and $B = ti + 2j + t^2k$, find

(i) $\int_0^2 (A.B)dt$ (ii) $\int_0^2 (A \times B)dt$.

**Sol:** (i) $A.B = (2t)t + (3t^2)(2) - t^2(4t + 1) = 2t^2 + 6t^2 - 4t^3 - t^2 = 7t^2 - 4t^3$

$$\int_0^2 (A.B)dt = \frac{7t^3}{3} - t^4\Big|_0^2 = \frac{8}{3}$$

(ii) $A \times B = \begin{vmatrix} i & j & k \\ 2t & 3t^2 & -(4t+1) \\ t & 2 & t^2 \end{vmatrix}$

$= (3t^4 + 8t + 2)i + (-4t^2 - t - 2t^3)j + (4t - 3t^3)k$

$$\int_0^2 (A \times B)dt = \left[\left(\frac{3t^5}{5} + 4t^2 + 2t\right)i + \left(\frac{-4t^3}{3} - \frac{t^2}{2} - \frac{2t^4}{4}\right)j + \left(2t^2 - \frac{3t^4}{4}\right)k\right]_0^2$$

$$= \frac{196}{5}i - \frac{62}{3}j - 4k$$

**Ex.7.6.10** If F(2) = $i + 2j - 2k$ and F(3) = $6i - 2j + 3k$,

evaluate $\int_2^3 F.\frac{dF}{du}du$

**Sol:** From vector differentiation,

We get, $\frac{d}{du}(F.F) = F.\frac{dF}{du} + \frac{dF}{du}.F = 2\left(F.\frac{dF}{du}\right)$, so that

$$F.\frac{dF}{du} = \frac{1}{2}\frac{d}{du}(F.F) = \frac{1}{2}\frac{d}{du}|F|^2$$

$$\therefore \int_2^3 F.\frac{dF}{du}du = \frac{1}{2}\int_2^3\left\{\frac{d}{du}|F|^2\right\}du = \left[\frac{1}{2}|F|^2\right]_2^3$$

But F(2) = $i + 2j - 2k \Rightarrow |F|^2 = 1 + 4 + 4 = 9$, when u = 2

and F(3) = $6i + 2j - 3k \Rightarrow |F|^2 = 36 + 4 + 9 = 49$, when u = 3

$$\therefore \int_2^3 F.\frac{dF}{du}du = \frac{1}{2}[49 - 9] = 20$$

**Ex. 7.6.11** If F = $(x^2 - 2y)i - 6yzj + 8xz^2k$, evaluate $\int_c F.dr$ from the point (0,0,0) to the point (1,1,1) along the following paths

(1) $x = t$, $y = t^2$, $z = t^3$.

(2) the straight line from (0,0,0) to (1,0,0), then to (1,1,0) and then to (1,1,1) and

(3) the straight line joining (0,0,0) to (1,1,1).

**Sol.** (1) $x = t$; $y = t^2$; $z = t^3$;

$dx = dt$; $dy = 2t\,dt$; $dz = 3t^2\,dt$;

when t = 0, the point is (0,0,0), when t = 1, the point is (1,1,1)

$$\int_c F.dr = \int_c (x^2 - 2y)dx - 6yz\,dy + 8xz^2\,dz$$

$$= \int_{t=0}^{t=1}(t^2 - 2t^2)dt - 6.t^2.t^3(2t)dt + 8t(t^3)^2(3t^2)dt = \int_0^1 -t^2dt - 12t^6dt + 24t^9dt$$

$$= \int_0^1 \left(24t^9 - 12t^6 - t^2\right)dt = \frac{24t^{10}}{10} - \frac{12t^7}{7} - \frac{t^3}{3}\Big|_0^1$$

$$= \frac{12}{5} - \frac{12}{7} - \frac{1}{3} = \frac{252-180-35}{105} = \frac{37}{105}$$

Aliter: Along C, F = $(t^2 - 2t^2)i - 6.\,t^2\,t^3 j + 8.t.t^6\,k = -t^2 i - 6t^5 j + 8t^7 k$

dr = $(dx)i + (dy)j + (dz)k = dt\;i + (2t\;dt)j + (3t^2\;dt)k$

$$\int_c F.dr = \int_{t=0}^1 \left(-t^2 - 12t^6 + 24t^9\right)dt \qquad \text{(Taking dot product of F and dr)}$$

$$= \frac{37}{105}$$

(2) Let O = (0,0,0), P = (1,0,0), Q = (1,1,0), R = (1,1,1).

Then along OP, $y = 0$, $z = 0$, $dy = 0$, $dz = 0$ and $x$ varies from 0 to 1.

$$\therefore \int_{OP} F.dr = \int_{x=0}^1 \left(x^2 - 2.0\right)dx - 6(0)(0)(0) + 8x(0)^2(0) = \int_0^1 x^2 dx = \frac{1}{3} \qquad \text{.... (1)}$$

Along PQ, $x = 1$, $z = 0$, $dx = 0$, $dz = 0$ and $y$ varies from 0 to 1.

$$\therefore \int_{PQ} F.dr = \int_{y=0}^1 \left(1^2 - 2y\right)0 - 6y(0)dy + 8.1.(0)^2(0) = \int 0 = 0 \qquad \text{.... (2)}$$

Along QR, $x = 1$, $y = 1$, $dx = 0$, $dy = 0$ and $z$ varies from 0 to 1.

$$\therefore \int_{QR} F.dr = \int_{z=0}^1 \left(1^2 - 2.1\right)(0) - 6.1.z(0) + 8.1.z^2.dz = \int_0^1 8z^2 dz = \frac{8}{3} \qquad \text{.... (3)}$$

Adding (1), (2), (3), $\int F.dr = \frac{1}{3} + 0 + \frac{8}{3} = 3$

(3) The equation of the straight line joining (0,0,0) to (1,1,1) is $\frac{x}{1} = \frac{y}{1} = \frac{z}{1} = t$ (say), so that $x = t$, $y = t$, $z = t$, $dx = dy = dz = dt$ 't' takes values from 0 to 1.

$$\therefore \int_c F.dr = \int_0^1 \left[\left(t^2 2t\right) - 6t.t + 8t.t^2\right]dt = \int_0^1 \left(8t^3 - 5t^2 - 2t\right)dt$$

$$= \left[2t^4 - \frac{5t^3}{3} - t^2\right]_0^1 = 2 - \frac{5}{3} - 1 = \frac{-2}{3}$$

**Ex.7.6.12** Find the total work done by a force F = $2xyi - 4zj + 5xk$ along the curve $x = t^2$, $y = 2t + 1$, $z = t^3$, from the points t = 1 to t = 2.

**Sol:** $x = t^2$, $dx = 2t\,dt$; $y = 2t + 1$, $dy = 2dt$; $z = t^3$, $dz = 3t^2\,dt$

Total work done = $\int_c F.dr$

$$= \int_c (2xyi - 4zj + 5xk).\{(dx)i + (dy)j + (dz)k\} = \int_c 2xy\,dx - 4z\,dy + 5x\,dz$$

$$= \int_{t=1}^{2} [2t^2(2t+1)]2t\,dt - (4t^3)2dt + (5t^2)3t^2\,dt = \int_1^2 (8t^4 + 4t^3 - 8t^3 + 15t^4)dt$$

$$= \int_1^2 (23t^4 - 4t^3)dt = \left[23\frac{t^5}{5} - t^4\right]_1^2 = \frac{23}{5}(32-1) - (16-1) = \frac{713}{5} - 15 = \frac{638}{5}$$

**Ex.7.6.13** If c is the curve $y = 3x^2$ in the $xy$ – plane and F = $(x + 2y)i - xyj$, evaluate $\int F.dr$, from the point (0,0) to (1,3).

**Sol:** Since c is a curve in $xy$ – plane, we take $r = xi + yj$, so that

F. dr = $\{(x + 2y)i - xyj\} . \{(dx)i + (dy)j\} = (x + 2y)\,dx - xy\,dy$.

***1st Method***: By taking the curve in parametric coordinates as,

$x = t$, $y = 3t^2$, $dx = dt$, $dy = 6t\,dt$, so that t varies from 0 to 1 to get the points (0,0) & (1,3): We have

$$\int_{(0,0)}^{(1,3)} F.dr = \int_{t=0}^{1} (t + 6t^2)\,dt - t(3t^2)(6t\,dt) = \int_0^1 (t + 6t^2 - 18t^4)\,dt$$

$$= \frac{t^2}{2} + \frac{6t^3}{3} - \frac{18t^5}{5}\Big|_0^1 = \frac{1}{2} + 2 - \frac{18}{5} = \frac{5}{2} - \frac{18}{5} = -\frac{11}{10}$$

***2nd Method:*** $y = 3x^2$, $dy = 6x\,dx$ and $x$ varies from 0 to 1.

$$\therefore \int_c F.dr = \int_0^1 (x + 6x^2)dx - x.3x^2.6x.dx$$

$$= \int_0^1 (x + 6x^2 - 18x^4)dx = -\frac{11}{10}$$

**Ex.7.6.14** Find the work done by the force F in moving a particle once around the circle 'C' in $xy$ – plane, if the centre of the circle is origin and radius is '2' and

$F = (x + y + z)i + (2x + y)j + (2x - y + z)k.$

**Sol:** $xy$ – plane is $z = 0$,

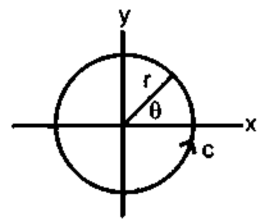
$\therefore F = (x + y)i + (2x + y)j + (2x - y)k$ and $r = xi + yj \Rightarrow dr = (dr)i + (dy)j$

$F.\,dr = (x + y)dx + (2x + y)dy.$

The equation of the circle is $x = 2\cos\theta$, $y = 2\sin\theta$

$\therefore dx = -2\sin\theta\, d\theta$, $dy = 2\cos\theta\, d\theta$

$\theta$ varies from 0 to $2\pi$




Work done = $\int F.dr$

$$= \int_0^{2\pi} (2\cos\theta + 2\sin\theta)(-2\sin\theta)d\theta + (2.2\cos\theta + 2\sin\theta)(2\cos\theta)d\theta$$

$$= \int_0^{2\pi} \left(-2\sin 2\theta - 4\sin^2\theta + 8\cos^2\theta + 2\sin 2\theta\right)d\theta = \int_0^{2\pi} \left(8\cos^2\theta - 4\sin^2\theta\right)d\theta$$

$$= \int_0^{2\pi} [4(1 + \cos 2\theta) - 2(1 - \cos 2\theta)]d\theta = \int_0^{2\pi} (2 + \cos 2\theta)d\theta = [2\theta + 3\sin 2\theta]_0^{2\pi}$$

$= 4\pi$

**Ex.7.6.15** Show that the necessary and sufficient condition for a vector field V to be conservative is curl V = 0

**Sol:** a) Necessary condition: If V is conservative, $\exists$ a '$\phi$' $\ni$ $V = \nabla \phi$.

curl V = curl ($\nabla \phi$) = 0 (see 7.5.2 (1))

b) Sufficient condition: Let $V = V_1 i + V_2 j + V_3 k$

$$\text{Curl } V = \nabla \times V = 0 \Rightarrow \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ V_1 & V_2 & V_3 \end{vmatrix} = 0$$

$$\Rightarrow \sum \left( \frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z} \right) i = 0$$

$$\Rightarrow \frac{\partial V_3}{\partial y} = \frac{\partial V_2}{\partial z}; \quad \frac{\partial V_1}{\partial z} = \frac{\partial V_3}{\partial x}; \quad \frac{\partial V_2}{\partial x} = \frac{\partial V_1}{\partial y} \qquad \text{.... (1)}$$

The work done by the force field V in moving a particle from $(x_1,y_1,z_1)$ to $(x,y,z)$

is $\int_c V.dr$

$= \int_c V_1(x,y,z)dx + V_2(x,y,z)dy + V_3(x,y,z)dz$

where V is a path joining $(x_1,y_1,z_1)$ to $(x,y,z)$

Let us choose a particular path consisting straight line segments from $(x_1,y_1,z_1)$ to $(x,y_1,z_1)$ to $(x,y,z_1)$ to $(x,y,z)$ and denote the work done along this path by a scalar function $\phi(x,y,z)$;

$$\therefore\ \phi(x,y,z) = \int_{x_1}^{x} V_1(x,y_1,z_1)dx + \int_{y_1}^{y} V_2(x,y,z_1)dy + \int_{z_1}^{z} V_3(x,y,z)dz \qquad \text{..... (2)}$$

From (2), it can be seen that,

$$\frac{\partial\phi}{\partial z} = V_3(x,y,z) \qquad \text{.... (3)}$$

$$\frac{\partial\phi}{\partial y} = V_2(x,y,z_1) + \int_{z_1}^{z} \frac{\partial V_3}{\partial y}(x,y,z)dz = V_2(x,y,z_1) + \int_{z_1}^{z} \frac{\partial V_2}{\partial z}(x,y,z)dz \text{ [from(1)]}$$

$$= V_2(x,y,z_1) + V_2(x,y,z)\Big|_{z_1}^{z} = V_2(x,y,z_1) - V_2(x,y,z) - V_2(x,y,z_1)$$

$$= V_2(x,y,z) \qquad \text{.... (4)}$$

$$\frac{\partial\phi}{\partial x} = V_1(x,y_1,z_1) + \int_{y_1}^{y} \frac{\partial V_2}{\partial x}(x,y,z_1)dy + \int_{z_1}^{z} \frac{\partial V_3}{\partial x}(x,y,z_1)dz$$

$$= V_1(x,y_1,z_1) + \int_{y_1}^{y} \frac{\partial V_1}{\partial y}(x,y,z_1)dy + \int_{z_1}^{z} \frac{\partial V_1}{\partial z}(x,y,z) \qquad \text{from} \qquad (1)$$

$$= V_1(x,y_1,z_1) + V_1(x,y,z_1)\Big|_{y_1}^{y} + V_1(x,y,z)\Big|_{z_1}^{z}$$

$$= V_1(x,y_1,z_1) + V_1(x,y,z_1) - V_1(x,y_1,z_1) + V_1(x,y,z) - V_1(x,y_1,z_1)$$

$$= V_1(x,y,z) \qquad \text{.... (5)}$$

$$(3), (4), (5) \Rightarrow \frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial z}k = V_1(x,y,z)i + V_2(x,y,z)j + V_3(x,y,z)k = V$$

$\Rightarrow V = \nabla\phi$

Hence the proof.

**Ex. 7.6.16** a) Show that $F = y^2 i + (2xy + z^2) j + 2yzk$ is a conservative force field.

b) Find its scalar potential.

c) Find the work done in moving an object in this field from (1,2,1) to (3,1,4)

**Sol:** a) $\nabla \times F = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y^2 & 2xy + z^2 & 2yz \end{vmatrix}$

$= i(2z - 2z) + j(0 - 0) + k(2y - 2y) = 0$

$\therefore$ F is a conservative force field.

b) Let '$\phi$' be the scalar potential of F.

***1st method:***

$$F = \nabla\phi = \frac{\partial\phi}{\partial x} i + \frac{\partial\phi}{\partial y} j + \frac{\partial\phi}{\partial z} k$$

$$\therefore \frac{\partial\phi}{\partial x} = y^2 \quad .... (1) \quad \frac{\partial\phi}{\partial y} = 2xy + z^2 \quad .... (2) \quad \frac{\partial\phi}{\partial z} = 2yz \quad .... (3)$$

Integrating (1) w.r.t x, (2) w.r.t. y, & (3) w.r.t. z, respectively, we get,

$\phi = xy^2 + f(y,z)$, $\quad \phi = xy^2 + yz^2 + g(z,x)$, and $\quad \phi = yz^2 + h(x,y)$.

These equations will be consistent if $f$, $g$, $h$ are taken as

$f(y,z) = yz^2$, $\quad g(z,x) = 0$, $h(x,y) = xy^2$.

Hence $\phi = xy^2 + yz^2$ + constant

***2nd Method:*** Since F is conservative, $\int_c F.dr$ is independent of path joining $(x_1,y_1,z_1)$ to $(x,y,z)$

using method of problem 7.6.15(b),

$$\phi = \int_{x_1}^{x} (y_1^2) dx + \int_{y_1}^{y} (2xy + z_1^2) dy + \int_{z_1}^{z} 2yz dz = xy_1^2 \Big|_{x_1}^{x} + (xy^2 + z_1^2 y)\Big|_{y_1}^{y} + yz^2 \Big|_{z_1}^{z}$$

$= xy_1^2 - x_1 y_1^2 + xy^2 + z_1^2 y - x^2 y_1 - z_1^2 y_1 + yz^2 - yz_1^2$

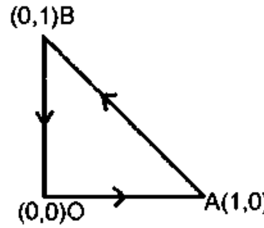
$= xy^2 + yz^2 - x_1 y_1^2 - z_1^2 y_1 = xy^2 + yz^2$ + constant.

***3rd Method:*** Since F. dr $= \nabla\phi.dr = \frac{\partial\phi}{\partial x} dx + \frac{\partial\phi}{\partial y} dy + \frac{\partial\phi}{\partial z} dz = d\phi$,

$d\phi = (y^2)dx + (2xy + z^2)dy + (2yz)dz = (y^2dx + 2xydy) + (z^2dy + 2yz)dz$

$= d(xy^2) + d(yz^2) = d(xy^2 + yz^2)$

$\Rightarrow \phi = xy^2 + yz^2$ +constant




c) work done = $\int_c F.dr = \int_{P_1=(1,2,1)}^{P_2=(3,1,4)} d\phi$

$= \phi(P_2) - \phi(P_1) = xy^2 + xz^2 \Big|_{(1,2,1)}^{(3,1,4)}$

$= (3.1^2 + 3.4^2) - (1.2^2 + 1.1^2) = 51 - 5 = 46.$

**Ex. 7.6.17** Evaluate $\int_c (x^3 dy + y^2 dx)$ where c is the boundary of the triangle whose vertices are (0,0), (1,0), (0,1).

**Sol:** Let I = $\int_c (x^3 dy + y^2 dx)$

$I = I_1 + I_2 + I_3$, where $I_1 = \int_{OA}$, $I_2 = \int_{AB}$, $I_3 = \int_{BO}$

(i) Along OA, $y = 0$, $\Rightarrow dy = 0$

$\therefore I_1 = \int x^3.0 + 0^2.dx = 0$

(ii) Along AB, $x + y = 1$, $y = 1 - x \Rightarrow dy = -dx, x$ varies from 1 to 0.

$\therefore I_2 = \int_1^0 x^3(-dx) + (1-x)^2 dx$

$= \int_0^1 (x^2 - x^3 + 1 - 2x) dx = -\frac{x^3}{3} - \frac{x^4}{4}\Big|_1^0 = 0 - \left(\frac{1}{3} - \frac{1}{4}\right) = -\frac{1}{12}$

(iii) Along BO, $x = 0 \Rightarrow dx = 0$ $\therefore$ $I_3 = \int 0.dy + y^2.0 = 0$

$\therefore$ $I = I_1 + I_2 + I_3 = -\frac{1}{12}$

**Ex.7.6.18:** If $f = xy^2z^2$, evaluate $\int_c f dr$ where the curve '$c$' is given by

$x = t$, $y = t^2$, $z = t^3$ from $t = 0$ to $1$ .

**Sol:** $f = xy^2z^2 = t\left(t^2\right)^2.\left(t^3\right)^2 = t^{11}$

$$dr = dx\ i + dy\ j + dz\ k$$

$$dr = \left(dt\right)i + \left(2tdt\right)j + \left(3t^2dt\right)k = \left(i + 2tj + 3t^2k\right)dt$$

$$\int_c f dr = \int_{t=0}^{1} t^{11}\left(i + 2tj + 3t^2k\right)dt = i\int_0^1 t^{11} + dt + j\int_0^1 2t^{12}dt + k\int_0^1 3t^{13}dt$$

$$= i\frac{t^{12}}{12}\Big|_0^1 + j\frac{2t^{13}}{13}\Big|_0^1 + k\frac{3t^{14}}{14}\Big|_0^1 = \frac{1}{12}i + \frac{2}{13}j + \frac{3}{14}k$$

**Ex.7.6,18:** If A = $3zi - 2xj + yk$, and $c$ is the curve given by $x = \cos t, y = \sin t$, $z = 2\cos t$,

evaluate $\int_c \mathbf{A} \times dr$ from $t = 0$ to $t = \frac{\pi}{2}$.

**Sol:**

$$\mathbf{A} \times dr = \begin{vmatrix} i & j & k \\ 3z & -2x & y \\ dx & dy & dz \end{vmatrix}$$

$$= \left(-2x\ dz - y\ dy\right)i + \left(y\ dx - 3z\ dz\right)j + \left(3z\ dy + 2x\ dx\right)k$$

$$x = \cos t, y = \sin t, z = 2\cos t$$

$$\Rightarrow dx = \left(-\sin t\right)dt, dy = \left(\cos t\right)dt, dz = \left(-2\sin t\right)dt$$

$\therefore$ (1) $\Rightarrow$ A $x$ d$r$ = $i$[($-2\cos t$)($-2\sin t$) $-$ $\sin t$ . $\cos t$]d$t$ +$j$[($\sin t$)($-\sin t$) $-$ 3($2\cos t$) ($-2\sin t$)]d$t$ + $k$[3($2\cos t$)($\cos t$) + ($2\cos t$)($-\sin t$)]d$t$.

= $i$($3\sin t \cos t$) d$t$ + $j$[(12 $\sin t \cos t$) $-$ $\sin^2 t$)] d$t$ + $k$($6\cos 2t - 2\sin t \cos t$) d$t$.

$$\therefore \int_c A x dr = i\int_0^{\pi/2} \frac{3}{2}\sin 2t dt + j\int_0^{\pi/2}\left[6\sin 2t - \frac{1}{2}\left(1 - \cos 2t\right)\right]dt$$

$$+ k\int_0^{\pi/2}\left[3\left(1 + \cos 2t\right) - \sin 2t\right]dt$$

$$= i\frac{3}{2}\left(\frac{-\cos 2t}{2}\right)_0^{\pi/2} + j\left[-3\cos 2t - \frac{t}{2} + \frac{\sin 2t}{4}\right]_0^{\pi/2} + k\left[3t + \frac{3}{2}\sin 2t + \frac{\cos 2t}{2}\right]_0^{\pi/2}$$

$$= i\left(\frac{-3}{4}\right)\left(-1-1\right) + j\left[-3\left(-1-1\right) - \frac{\pi}{4} + 0\right] + k\left[\frac{3\pi}{2} + 0 + \frac{1}{2}\left(-1-1\right)\right]$$

$$= \frac{3}{2}i + \left(6 - \frac{\pi}{4}\right)j + \left(\frac{3\pi}{2} - 1\right)k$$

## Exercise – 7(e)

1) If $U(t) = (2t^2 - 1)i + 3tj + (2 - t)k$. Find (a) $\int U(t)dt$, (b) $\int_2^4 U(t)dt$

[Ans: (a) $\left(\frac{2t^3}{3} - t\right)i + \frac{3t^2}{2}j + \left(2t - \frac{t^2}{2}\right)k$, (b) $\frac{106}{3}i + 18j - 2k$ ]

2) Evaluate: $\int_0^{\pi/2} (6\sin u)i - (3\cos u)j + uk$

[Ans: $6i - 3j + \frac{\pi^2}{8}k$ ]

3) If $A(s) = si - s^2j + (s + 1)k$, $B(s) = 2si + 6sk$ find (a) $\int_0^2 A.Bds$, (b) $\int_0^2 A \times Bds$.

[Ans: (a) $\frac{100}{3}$, (b) $-48i - 12j + 16k$]

4) If $A = ti - j + 2tk$, $B = t^2i - tj + 2k$, $C = i - 2j + 2k$, evaluate

(a) $\int_0^1 \{A.(B \times C)\}dt$, (b) $\int_0^1 \{A \times (B \times C)\}dt$

[Ans: (a) $-\frac{1}{3}$, (b) $-\frac{5}{2}i + \frac{37}{6}j + \frac{8}{3}k$ ]

5) The acceleration of a particle a at any time $t \geq 0$ is given by $a = e^t i + (2\cos 2t)j + (2\sin 2t)k$. If the velocity V and displacement r are both zero at t = 0, find V and r at any time t.

[Ans: $V = (e^t - 1)i + (\sin 2t)j + (1 - \cos 2t)k$,

$r = (e^t - t - 1)i + \frac{1}{2}(1 - \cos 2t)j + \left(t - \frac{1}{2}\sin 2t\right)k$ ]

6) Evaluate $\int_2^3 A.\frac{dA}{du}$ du, given that, $A(2) = 4i - 2j + 3k$ and $A(3) = 2i + j + 2k$

[Ans: $-10$]

## Exercise 7(f)

1. If $\phi = xyz$, valuate $\int_c \phi dr$, where c is the curve $x = t^3, y = t^2, z = t$, from t = 0 to 1

[Ans: $\frac{1}{3}i + \frac{1}{4}j + \frac{1}{7}k$]

2. If $F = xi - yzj + z^2k$, and c is the curve given by $x = t, y = t^3, z = t^2$, evaluate (i) $\int_c F.dr$ and (ii) $\int_c F.dr$, from t = 0 to t = 1.

[Ans: (i) $-\frac{5}{7}i - \frac{7}{15}j + \frac{11}{12}k$, (ii) $\frac{23}{24}$]

3. If $A = (2x + 3)i + xyj + (zx - y)k$, evaluate $\int_c F.dr$ along c where c is

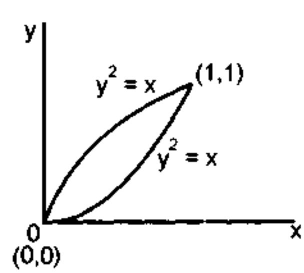
(a) the curve $x = t^3, y = 2t^2, z = t$ from t = 0 to 1.
(b) The straight lines from (0,0,0) to (1,0,0), then to (1,0,1) and then to (1,2,1)
(c) The straight line joining (0,0,0) and (1,2,1).

[Ans: (a) $\frac{491}{105}$, (b) $\frac{13}{2}$, (c) $\frac{14}{3}$]

4. If $A = (3xy - 2y^2)i + (x - y)j$, evaluate $\int_c A.dr$ along the curve c in $xy$ – plane given by $y = x^3$ from the point (0,0) to (2,8).

[Ans: $\frac{1308}{35}$]

5. If $F = (2x - y)i + (x - 2y)j$, evaluate $\int_c A.dr$ where c is the closed curve shown in the figure below.




[Ans: ($\frac{1}{3}$)]

6. If A = $(3x + 2y)i + (x + y)j$, and c is the boundary of the tringle whose vertices are (0,0), (1,0), (0,1), evaluate $\int_c A.dr$.

[Ans: $\frac{3}{2}$]

7. If A = $(2x + y)i + (3x - 2y)j$, compute the circulation of A about the circle C: $x^2 + y^2 = 4$, traversed in the positive direction.

[Ans: $8\pi$]

8. Find the work done in moving a particle in the force field.
F = $2x^2i + (2yz - x)j + yk$, along (a) the straight line from (0,0,0) to (3,1,2)
(b) the space curve $x = 3t^2$, $y = t$, $z = 3t^2 - t$ from t = 0 to t = 1

[Ans: (a) $\frac{113}{6}$, (b) $\frac{58}{3}$]

9. (a) Prove that V = $(2x \sin y - 3)i + (x^2 \cos y + z^2)j + 2(yz + 1)k$ is a conservative force field. (b) Find the scalar potential of V. (c) Find the work done in moving an object in this field from (1,0,–1) to $(2, \frac{\pi}{2}, 1)$.

[Ans: (b) $x^2 \sin y + yz^2 - 3x + 2z$ + constant, (c) $\frac{\pi}{2} + 5$]

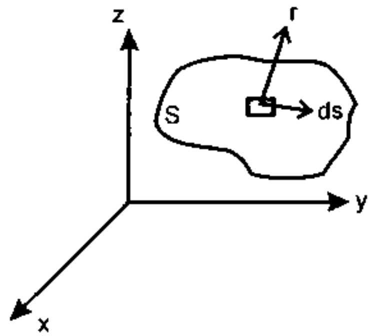
10. If A = $(9x^2y - 2xz^3)i + 3x^3j - 3x^2z^2k$, (a) prove that $\int_c A.dr$ is independent of the curve 'C' joining two given points. (b) show that there is a differentiable function $\phi$ such that A = $\nabla\phi$ and find it.

[Ans: $\phi = 3x^3y - x^2z^3 + c$]

## 7.7 SURFACE INTEGRALS

**7.7.1** Let S be a two–sided surface. Let one side be taken as the positive side. If S is a closed surface, the outer side is considered as the positive side. Let A be a vector function. Consider an element of area 'ds' in the surface. Let n be the unit normal vector to ds in the positive direction. It can be seen that A.n = A cos$\theta$. (where '$\theta$' is the angle between A and n and A = |A|) is the normal component of A. Let ds be a vector whose magnitude is ds and whose direction is that of n.

$\therefore$ ds = n ds.




Then the integral,

$$\iint_s A.ds = \iint_s A.n.ds \qquad \text{..... (1)}$$

is an example of a surface integral which is also called as flux of A over s.
If 'f' is a scalar function,

$$\iint_s \phi ds \qquad \text{..... (2)}$$

$$\iint_s \phi n ds \qquad \text{..... (3)}$$

$$\iint_s A \times ds = \iint_s A \times n ds \qquad \text{..... (4)}$$

are some other examples of surface integrals.

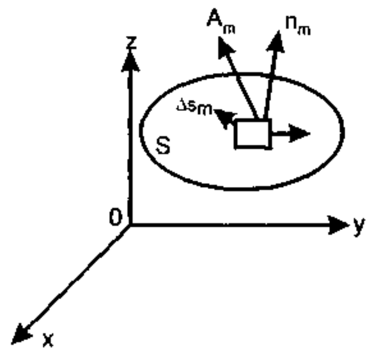
*Note* (1) The surface integrals can also be defined in terms of limits of sums. (see 7.7.2)

(2) The notation $\oiint_s$ is also used to denote a surface integral over the closed surface s.

(3) Sometimes the notation $\oint_s$ may also be used for surface integrals.

(4) Surface integrals can be conveniently evaluated by expressing them as double integrals over the projected area of s on one of the coordinate planes (see 7.7.3)

### 7.7.2 Definition of surface integral as the limit of a sum:




The area S is divided into 'L' elements of area $\Delta S_m$, m = 1,2, ....L,

let $P_m = (x_m, y_m, z_m)$ be any point in $\Delta S_m$. Let $A(x_m, y_m, z_m) = A_m$

Let $n_m$ be the positive unit normal to $\Delta S_m$ at $P_m$. Then $((A_m.n_m)$ is the normal component of $A_m$ at $P_m$. Consider the sum,

$$\sum_{m=1}^{L} A_m.n_m \Delta S_m \qquad ..... (1)$$

The limit of the sum (1) as L $\to \infty$ such that the largest dimension of each $\Delta S_m \to 0$ (if the limit exists) is known as the surface integral of the normal component of A over S and denoted by $\iint_s A.n.ds$

### 7.7.3 Evaluation of a surface integral

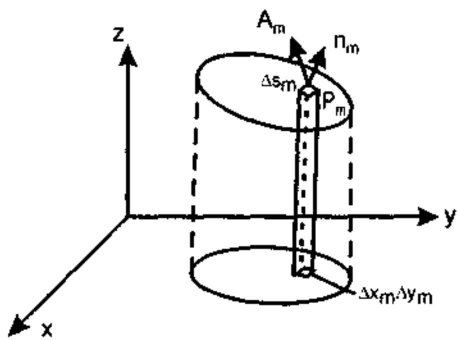
To evaluate surface integrals, it is convenient to express them as double integrals taken over the projected area of the surface S on one of the coordinate planes (*xy*,*yz*, or *zx* planes).

If R is the projection of S on the *xy* – plane, it can be shown that

$$\iint_s A.nds = \iint_R A.n \frac{dxdy}{|n.k|}$$

From 7.7.2, the surface integral is the limit of the sum

$$\sum_{m=1}^{L} A_m.n_m \Delta S_m \qquad ..... (1)$$




The projection of $\Delta S_m$ on the $xy-$plane is $|n_m \Delta S_m .k|$ (or) $|n_m .k| \Delta S_m$ which is equal to $\Delta x_m \Delta y_m$

$$\therefore \ \Delta S_m = \frac{\Delta x_m \Delta y_m}{|n_m .k|}$$

$\therefore$ The sum (1) becomes

$$\sum_{m=1}^{L} A_m .n_m \frac{\Delta x_m \Delta y_m}{|n_m .k|}$$

By the fundamental theorem of integral calculus the limit of this sum as $L \to \infty$ in such a manner that the largest $\Delta x_m$ and $\Delta y_m$ approach zero is

$$\iint_R A.n \frac{dxdy}{|n.k|}$$ which is the required result.

*Note:* Similarly if R is the projection of S on yz and zx planes respectively, it can be seen as

$$\iint_S A.nds = \iint_R A.n \frac{dydz}{|n.i|}$$

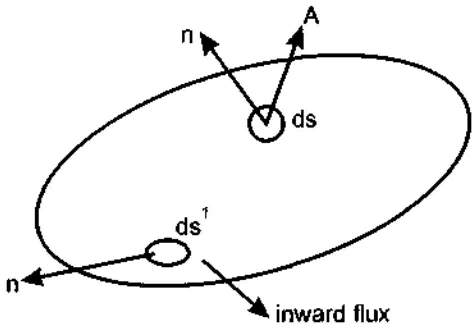
and
$$\iint_S A.nds = \iint_R A.n \frac{dzdx}{|n.j|}$$

### 7.7.4 Physical interprettion of surface integrals:

Let A denote the velocity of a moving fluid. Let S be a fixed surface in the fluid. Let ds be an element of surface. Then A.n ds = A.dS represents the amount of fluid that passes normally through dS in unit time at any point. If the direction of n is outward or positive, the amount of fluid flow is positive. Similarly if dS' is another element for which n is in the negative direction, the fluid flow is negative at that point.

to $\int_S A.nds$ and it is known as the total flux of A through the entire surface S.

A can be a vector denoting physical quantities such as electric force, magnetic




force, flux of heat or gravitational force etc. In all these cases, $\int_S A.nds$ denotes total flux of A through S.

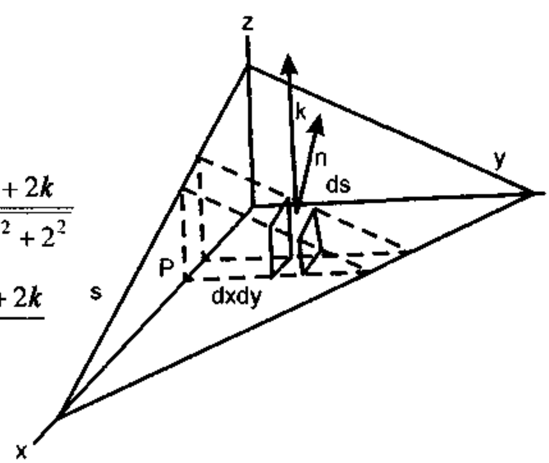
### Sloved Examples

**Ex. 7.7.5** Evaluate $\iint_S A.nds$ where $A = (x+y^2)i - 2xj + 2yzk$ and S is the surface of the plane $2x + y + 2z = 6$ in the first octant.

**Sol:** $A = (x + y^2)i - 2xj + 2yzk$

Let $\phi = 2x + y + 2z - 6$

$$\nabla\phi = \frac{\partial\phi i}{\partial x} + \frac{\partial\phi j}{\partial y} + \frac{\partial\phi k}{\partial z} = 2i + j + 2k$$




$$\text{Unit normal n to S} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2i + j + 2k}{\sqrt{2^2 + 1^2 + 2^2}}$$

$$= \frac{2i + j + 2k}{3}$$

$$A.n = \frac{1}{3}\left[2\left(x + y^2\right) - 2x + 4yz\right]$$

$$= \frac{1}{3}\left[2x + 2y^2 - 2x + 2y(6 - 2x - y)\right]$$

[Substituting $2z = 6 - 2x - y$ ($\because 2x + y + 2z = 6$)]

$$= \frac{1}{3}(12y - 4xy)$$

If R be the projection of S on the $xy$ – plane.

$$|n.k| = \frac{2}{3} \Rightarrow ds = \frac{dxdy}{|n.k|} = \frac{3}{2}dxdy$$

$$\therefore \iint_R A.nds = \iint_R (A.n)\frac{dxdy}{|n.k|}$$

$$= \iint_R \frac{1}{3}(12y - 4xy)\frac{3}{2}dxdy$$

$$= \iint_R (6y - 2xy)dxdy \qquad \text{..... (1)}$$

To evaluate this double integral over R,

(i) Keep $x$ fixed and integrate w.r.t. y from $y = 0$ to $(6 - 2x)$, ii) and then integrate w.r.t. $x$ from $x = 0$ to $x = 3$.

$\therefore$ Given integral

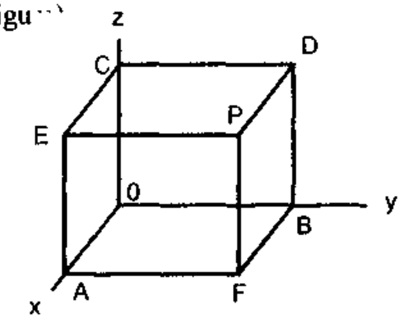
$$\int_{x=0}^{3}\int_{y=0}^{6-2x}(6y - 2xy)dydx$$

$$= \int_{x=0}^{3}\left[3y^2 - xy^2\right]_{y=0}^{6-2x}dx = \int_{x=0}^{3}(3 - x)(6 - 2x)^2 dx = \int_{x=0}^{3}\left(108 - 108x + 36x^2 - 4x^3\right)dx$$

$$= \left[108x - 54x^2 + 12x^3 - x^4\right]_0^3 = 324 - 486 + 324 - 81 = 81$$

**Ex. 7.7.6** If $F = 4xzi - y^2j + yzk$, evaluate $\iint_s F.nds$ where S is the surface of the cube bounded by $x = 0, x = 1, y = 0, y = 1, z = 0$ and $z = 1$.

**Sol.** The surface S can be divided into 6 faces (see figu...




(i) $S_1$: Face EPFA

(ii) $S_2$: Face OBDC

(iii) $S_3$: Face PFBD

(iv) $S_4$: Face OCEA

(v) $S_5$: Face PDCE

(vi) $S_6$: Face OBFA

$$\iint_S F.nds = \iint_{S_1} F.nds + \iint_{S_2} F.nds + \iint_{S_3} F.nds + \iint_{S_4} F.nds + \iint_{S_5} F.nds + \iint_{S_6} F.nds$$

On $S_1$: n = $i$ $x = 1$

$$\iint_{S_1} F.nds = \int_0^1 \int_0^1 \left(4zi - y^2 j + yzk\right) idydz = \int_0^1 \int_0^1 4zdydz = \int_0^1 2z^2\Big|_0^1 dy = \int_0^1 2dy = 2$$

On $S_2$: n = $-i$, $x = 0$

$$\iint_{S_2} F.n.ds = \int_0^1 \int_0^1 \left(-y^2 j + yzk\right)(-i)dydz = \int_0^1 \int_0^1 (0)dydz = 0$$

On $S_3$: n = $j$, $y = 1$

$$\iint_{S_3} F.n.ds = \int_0^1 \int_0^1 \left(4xzi - j + zk\right) jdxdz = \int_0^1 \int_0^1 (-1)dxdz = \int_0^1 -z\Big|_0^1 \, dx = \int_0^1 -1dx = -1$$

On $S_4$: n = $-j$, $y = 0$

$$\therefore \iint_{S_4} F.n.ds = \int_0^1 \int_0^1 (4xzi)(-j)dxdz = \int_0^1 \int_0^1 (0)dydz = 0$$

On $S_5$: n = $k$, $z = 1$

$$\therefore \iint_{S_5} F.n.ds = \int_0^1 \int_0^1 \left(4xi - y^2 j + yk\right) kdxdy = \int_0^1 \int_0^1 ydxdy = \int_0^1 \frac{y^2}{2}\Big|_0^1 \, dx = \int_0^1 \frac{1}{2} dx = \frac{1}{2}$$

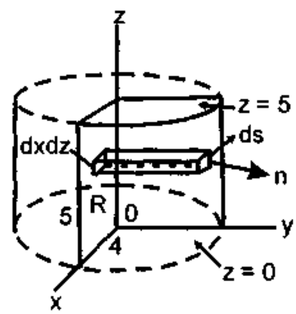
On $S_6$: n = – k, z = 0

$$\iint_{S_6} F.n.ds = \int_0^1 \int_0^1 \left(-y^2 j\right)(-k)dxdy = 0$$

$$\therefore \iint_S F.n.ds = 2 + 0 - 1 + 0 + \frac{1}{2} + 0 = \frac{3}{2}$$

**Ex. 7.7.7** If $A = z^2 i + x^2 j - y^2 zk$, and S is the surface of the cylinder $x^2 + y^2 = 16$ included in the first octant between $z = 0$ and $z = -5$, evaluate $\iint_S A.n.ds$

**Sol:**




Project s on $xz$ plane and let the projection be R. (See figure)

$$\iint_S A.n.ds = \iint_R A.n \frac{dxdy}{|n.j|} \qquad \text{..... (1)}$$

The normal to $x^2 + y^2 = 16$ is

$$\nabla(x^2 y^2) = \frac{\partial}{\partial x}(x^2 + y^2)i + \frac{\partial}{\partial y}(x^2 + y^2)j$$

$$= 2xi + 2yj$$

Unit normal $n = \dfrac{2xi + 2yj}{\sqrt{(2x)^2 + (2y)^2}} = \dfrac{2(xi + yj)}{2\sqrt{x^2 + y^2}} = \dfrac{xi + yj}{4} \quad (\because x^2 + y^2\ 16)$

$$A.n = \frac{z^2 x + x^2 y}{4}$$

$$n.j = \frac{y}{4}$$

$\therefore$ From (1) $\displaystyle\iint_S A.nds = \iint_R \frac{z^2 x + x^2 y}{4} \frac{dxdz}{y/4}$

$$= \iint_R \frac{(z^2 x + x^2 y)}{y} dxdz = \int_{z=0}^{5} \left[ \int_{x=0}^{4} \left( \frac{xz^2}{\sqrt{16 - x^2}} + x^2 \right) dx \right] dz$$

$$= \int_{z=0}^{5} \left( 4z^2 \frac{64}{3} \right) dz \qquad \left[ \because \int_0^4 \frac{x}{\sqrt{16 - x^2}} dx = 4, \quad \int_0^4 x^2 dx = \frac{64}{3} \right]$$

$$= \frac{4z^3}{3} + \frac{64z}{3}\Big|_0^5 = \frac{820}{3}$$

**Ex. 7.7.8** Evaluate $\iint_S \phi n ds$ where S is the surface of problem 7.7.7 above and $\phi = \frac{xyz}{8}$

**Sol:** We have $\iint_S \phi n ds = \iint_R \phi n \frac{dxdz}{|n.j|}$

using $n = \frac{xi+yj}{4}$, $n.j = \frac{y}{4}$, the integral on R.H.S. becomes,

$$\iint_R \frac{1}{8} xyz \frac{(xi+yj)}{4} \cdot \frac{dxdz}{y/4}$$

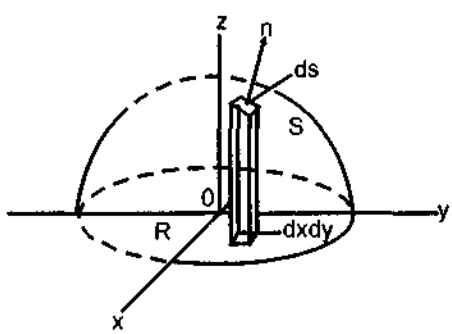
$$\frac{1}{8}\int_{z=0}^{5}\int_{x=0}^{4} xz(xi+yj)dxdz = \frac{1}{8}\int_{z=0}^{5}\left[\int_0^4 \left(x^2zi + xz\sqrt{16-x^2}\right)jdx\right]dz$$

$$\frac{1}{8}\int_0^5\left(\frac{64z}{3}i + \frac{64z}{3}j\right)dz \qquad [\because \int_0^4 x\sqrt{16-x^2}dx = \frac{64}{3}\int_0^4 x^2dx = \frac{64}{3}]$$

$$\frac{1}{8}\cdot\frac{64}{3}\left(\frac{z^2}{2}i + \frac{z^2}{2}j\right)\Big|_0^5 = \frac{8}{3}\cdot\frac{25}{2}(i+j) = \frac{100}{3}(i+j)$$

**Ex. 7.7.9** If $A = yi + (x-2xz)j - xyk$, evaluate $\iint_S (curl A).nds$ where S is the surface of the sphere $x^2+y^2+z^2 = 4$ above the xy plane

**Sol:**




$$\text{Curl A} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & x-2xz & -xy \end{vmatrix}$$

$= i\{\frac{\partial}{\partial y}(-xy) - \frac{\partial}{\partial z}(x-2xz)\} + j\{\frac{\partial}{\partial z}(y) - \frac{\partial}{\partial x}(-xy)\} + k\{\frac{\partial}{\partial x}(x-2xz) - \frac{\partial}{\partial y}(y)\}$

$= xi + yj - 2zk$

The normal to the surface is $\nabla(x^2 + y^2 + z^2) = 2xi + 2yj + 2zk$

Unit normal n = $\frac{2xi + 2yj + 2zk}{\sqrt{(2x)^2 + (2y)^2 (2z)^2}}$

$= \frac{xi + yj + zk}{2}$ $\quad (\because x^2 + y^2 + z^2 = 4)$

(Curl A). n = $\frac{x^2 y^2 - 2z^2}{2}$

The projection of S on $xy$ plane is the circle $x^2 + y^2 = 4$, $z = 0$ (see figure)

$\therefore \iint_S (curl\ A.n)ds = \iint_R (curl\ A.n)\frac{dxdy}{|n.k|}$

$= \iint_R \frac{(x^2 + y^2 - 2z^2)}{2}\frac{dxdy}{z/2}$

$= \int_{x=-2}^{2} \int_{y=-\sqrt{4-x^2}}^{4-x^2} \frac{x^2 + y^2 - 2(4 - x^2 - y^2)}{\sqrt{4 - x^2 - y^2}} dxdy \quad (\because z^2 = 4 - x^2 - y^2)$

$= \int_{x=-2}^{2} \int_{y=-\sqrt{4-x^2}}^{4x^2} \frac{3(x^2 + y^2) - 8}{\sqrt{4 - x^2 y^2}} dxdy$

Changing into polar coordinates by taking $x$ = r cosθ, $y$ = r sinθ, d$x$ d$y$ =r d$r$ dθ, the integral becomes

$\int_{\theta=0}^{2\pi} \int_{r=0}^{2} \left(\frac{3(r^2 - 4) + 4}{\sqrt{4 - r^2}}\right) r\ drd\theta$

$= \int_{\theta=0}^{2\pi} [\int_{r=0}^{2} \left(-3r\sqrt{4 - r^2} + \frac{4r}{\sqrt{4 - r^2}}\right) dr] d\theta$

$= = \int_{\theta=0}^{2\pi} (4 - r^2)^{3/2} - 4\sqrt{4 - r^2} \Big|_{r=0}^{2} d\theta = \int_{\theta=0}^{2} (8 - 8) d\theta = 0$

**Ex.7.7.10** Evaluate $\iint_S A.nds$ where A = $yzi + zxj + xyk$ and S is the part of the sphere $x^2 + y^2 + z^2 = 9$ which lies in the first octant.

Sol: Unit normal n to S = $\dfrac{\nabla(x^2+y^2+z^2)}{|\nabla(x^2+y^2+z^2)|}$

$$= \frac{2xi + 2yj + 2zk}{\sqrt{4x^2+4y^2+4z^2}} = \frac{xi + yj + zk}{3} \qquad (\because x^2+y^2+z^2 = 9)$$

A.n = $3xyz$; n.$k$ = $\dfrac{z}{3}$

If R is the projection of S on $xy$ – plane, we have,

$$\iint_S A.nds = \iint_R A.n\frac{dxdy}{|n.k|}$$

$$= \iint_R 3xyz\frac{dxdy}{z/3} = 9\iint_R xy\ dxdy$$

The region R is bounded by $x$–axis, $y$–axis and the circle $x^2 + y^2 = 9$; z = 0 changing to polar coordinates, the last integral becomes

$$9\int_{\theta=0}^{\pi/2}\int_{r=0}^{3}(r\cos\theta)(r\sin\theta)rdrd\theta = 9\int_0^{\pi/2}[\int_0^3 r^3\cos\theta\sin\theta dr]d\theta$$

$$= 9\int_0^{\pi/2}\left(\frac{r^4}{4}\right)\Bigg|_0^3 \cos\theta\sin\theta\ d\theta = 9.\frac{81}{4}\int_0^{\pi/2}\cos\theta\sin\theta d\theta$$

$$= \frac{729}{4}.\frac{1}{2} = \frac{729}{8}. \qquad \left(\because \int_0^{\pi/2}\cos\theta\sin\theta d\theta = \frac{1}{2}\right)$$

## Exercise – 7(g)

1. If F = $18zi - 12xj + 3yk$, evaluate $\iint_S F.nds$ where S is that part of the plane $2x + 3y + 6z = 12$ which is located in the first octant

   (Hint: Take projection of S on the xy plane and ds = $\dfrac{dxdy}{|n.k|}$)

   [Ans: 24]

2. Evaluate $\iint_S V.nds$, where $V = yi + 2xj - zk$ and s is the surface of the plane $2x + y = 6$ in the first octant cut off by the plane $z = 4$

[Ans: 108]

3. If $r = xi + yj + zk$ find the value of the integral $\iint_S r.nds$ over

   a) the surface S of the unit cube bounded by the coordinate planes and the planes $x = y = z = 1$ and
   b) the surface of the sphere of radius 'a' with centre at the origin

[Ans: a) 3, b) $4\pi a^3$]

4. Evaluate $\iint_S F.ds$ over the entire surface of the region above the $xy$ plane bounded by the cone $z^2 = x^2 + y^2$ and the plane $z = 4$, if $F = 4xzi + xy^2zj + 3zk$

[Ans: $320\pi$]

5. If S is the surface of the parabolic cylinder $y^2 = 8x$ in the first octant bounded by the planes $y = 4$ and $z = 6$, evaluate $\iint_S F.nds$, where $F = 2yi - zj + x^2k$

   [Hint: $\iint_S F.nds = \iint_R F.n\frac{dydz}{|n.i|}$, where R is the projection of S on the yz plane]

[Ans: 132]

6. Evaluate $\iint_S V.nds$ over the entire surface S of region bounded by the cylinder $x^2 + z^2 = 9$, $x = 0$, $y = 0$, $z = 0$ and $y = 8$ if $V = 6zi + (2x + y)j - xk$

[Ans: $18\pi$]

7. Evaluate $\iint_S A.nds$ where $A = zi + xj - 3y^2zk$ and S is the surface of the cylinder $x^2 + y^2 = 16$ included in the first octant between $z = 0$ and $z = 5$

[Ans: 90]

8. If $V = yzi + zxj + xyk$ and S is that part of the sphere $x^2 + y^2 + z^2 = 1$ which lies in the first octant, find the value of $\iint_S V.nds$.

[Ans: $\frac{3}{8}$]

9. Evaluate $\iint_S \{(x^3 - yz^2)i - (2x^2y)j + 2k\}.ds$ over a cube with edges of length 'r' parallel to the coordinate axes

[Ans: $\frac{1}{2}$]

10. If V = $xi + yj + zk$, and S is the triangle with vertices at (1,0,0), (0,1,0) and (0,0,1), find the value of $\iint_S V.ds$.

[Ans: $\frac{1}{2}$]

11. If A = $xi - yj + (z^2 - 1)k$, find the value of $\iint_S A.nds$, where S is the closed surface bounded by the planes z = 0, z = 1 and the cylinder $x^2 + y^2 = 1$

[Ans: $\pi$]

## 7.8 VOLUME INTEGRALS

**7.8.1** Consider a closed surface in space enclosing a volume V. Then, integrals of the form $\iiint_V Adv$ and $\iiint_V \phi dv$, [A is a vector function, f is a scalar function] are examples of volume integrals.

### 7.8.2 Expression of volume integral as the limit of a sum:

Let A be a continuous vector function. Let S be a surface enclosing the region D. Divide this region D into a finite number of subregions $D_i$ ...,= 1,2, .....n.

Let $\Delta v_i$ be the volume of the subregion $D_i$ enclosing any point whose position vector is $r_i$.

Consider the sum

$$V = \sum_{i=1}^{n} A(r_i)\Delta v_i$$

The limit of this sum as n $\to \infty$ such that $\Delta v_i \to 0$, is called the volume integral of A over D and denoted by $\iiint_D Adv$

If A = $A_1(x,y,z)i + A_2(x,y,z)j + A_3(x,y,z)k$, so that $dv = dx\,dy\,dz$

$$\iiint_D A\,dv = i\iiint_D A_1(x,y,z)dx\,dy\,dz + j\iiint_D A_2(x,y,z)dx\,dy\,dz +$$

$$k\iiint_D A_3(x,y,z)dx\,dy\,dz$$

## Solved Examples

**Ex. 7.8.3** If $F = (2x^2 - 3z)i - 2xyj - 4xk$, evaluate $\iiint_V \nabla.F dv$ where V is the closed region bounded by the planes $x = 0$, $y = 0$, $z = 0$ and $2x + 2y + z = 4$.

**Sol:** $\nabla.F = \frac{\partial}{\partial x}(2x^2 - 3z) - \frac{\partial}{\partial y}(2xy) - \frac{\partial}{\partial z}(4x) = 4x - 2x = 2x$

$$\therefore \iiint_V \nabla.F dv = \int_{x=0}^{2}\int_{y=0}^{2-x}\int_{z=0}^{4-2x-2y} 2x\,dz\,dy\,dx = \int_{x=0}^{2}\int_{y=0}^{2-x}[\int_{z=0}^{4-2x-2y} 2x\,dz]dy\,dx$$

$$= \int_{x=0}^{2}\int_{y=0}^{2-x} 2xz\Big|_{z=0}^{4-2x-2y} dy\,dx = \int_{x=0}^{2}\int_{y=0}^{2-x} 2x(4-2x-2y)dy\,dx = \int_{x=0}^{2}[\int_{y=0}^{2-x}(8x - 4x^2 - 4xy)dy]dx$$

$$= \int_{x=0}^{2} 8xy - 4x^2y - 2xy^2\Big|_{y=0}^{2-x} dx \quad = \int_{x=0}^{2}[8x(2-x) - 4x^2(2-x) - 2x(2-x)^2]dx$$

$$= \int_{x=0}^{2}(8x - 8x^2 + 2x^3)dx = (4x^2 - \frac{8x^3}{3} + \frac{2x^4}{4})\Big|_0^2 \quad = 16 - \frac{64}{3} + 8 = \frac{8}{3}$$

**Ex. 7.8.4** Evaluate $\iiint_V (\nabla.A)dv$ over the region bounded by $x^2 + y^2 = 4$, $z = 0$ and $z = 3$, where $A = 4xi - 2y^2j + z^2k$.

**Sol:** $\nabla.A = \frac{\partial}{\partial x}(4x) - \frac{\partial}{\partial y}(2y^2) + \frac{\partial}{\partial z}(z^2) = 4 - 4y + 2z$

$$\therefore \iiint_V (\nabla.A)dv = \iiint_V (4 - 4y + 2z)dv = \int_{x=-2}^{x=2}\int_{y=-\sqrt{4-x^2}}^{y=\sqrt{4-x^2}}[\int_{z=0}^{3}(4 - 4y + 2z)dz]dy\,dx$$

$$= \int_{x=-2}^{2}\int_{y=-\sqrt{4-x^2}}^{y=\sqrt{4-x^2}} (4z-4yz+z^2)\Big|_0^3 \; dydx = \int_{x=-2}^{2}[\int_{y=-\sqrt{4-x^2}}^{y=\sqrt{4-x^2}}(21-12y)dy]dx$$

$$= \int_{x=-2}^{2}(21y-6y^2)\Big|_{y=-\sqrt{4-x^2}}^{y=\sqrt{4-x^2}} dx = \int_{x=-2}^{2} 42\sqrt{4-x^2}\,dx = 84\int_0^2\sqrt{4-x^2}\,dx$$

$$= 84[\frac{x}{2}\sqrt{4-x^2} + 2\sin^{-1}(\frac{x}{2})]_0^2 = 84[0+2(\frac{\pi}{2})-0] = 84\pi$$

**Ex. 7.8.5** Evaluate $\iiint_V \phi dv$ taken over the rectangular parallelopiped $0 \le x < a$, $0 \le y < b$, $0 \le z < c$ and $\phi = 2(x+y+z)$

**Sol:** $$\iiint_V \phi dv = \iiint_V 2(x+y+z)dv = \int_{x=0}^{a}\int_{y=0}^{b}\left[\int_{z=0}^{c}2(x+y+z)\right]dydx$$

$$= \int_{x=0}^{a}\int_{y=0}^{b} 2xz+2yz+z^2\Big|_{z=0}^{c} \; dydx = \int_{x=0}^{a}[\int_{y=0}^{b}(2cx+2cy+c^2)dy]dx$$

$$= \int_{x=0}^{a} 2cxy+cy^2+c^2y\Big|_0^b \; dx = \int_{x=0}^{a}(2bcx+cb^2+c^2b)dx = bcx^2+(b^2c+bc^2)x\Big|_0^a$$

$$= a^2bc+a(b^2c+bc^2) = abc(a+b+c)$$

**Ex. 7.8.6** If $\phi = 4y + 2xz$, evaluate $\iiint_V \phi dv$ over the region in the first octant bounded by $x^2+y^2=9$, $z=0$, $z=2$.

**Sol:** $$\iiint_V \phi dv = \iiint_V (4y+2xz)dv$$

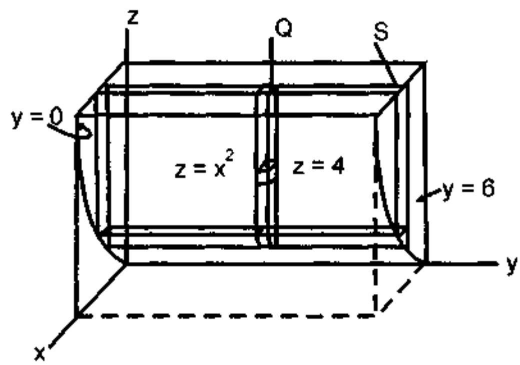
$$= \int_{x=0}^{x=3}\int_{y=0}^{y=\sqrt{9-x^2}}[\int_{z=0}^{z=2}(2xz+4y)dz]dydx = \int_0^3\int_0^{\sqrt{9-x^2}}(4yz+xz^2)\Big|_0^2 \; dydx$$

$$= \int_0^3[\int_0^{\sqrt{9-x^2}}(8y+4x)dy]dx = \int_0^3(4y^2+4xy)\Big|_0^{\sqrt{9-x^2}} dx$$

$$= \int_0^3[4(9-x^2)+4x\sqrt{9-x^2}]dx = 108$$

**Ex. 7.8.7** Evaluate $\iiint_V F dv$ where $F = xzi - 2xj + 2y^2k$ and V is the region bounded by the surfaces $x = 0$, $y = 0$, $y = 6$, $z = x^2$, and $z = 4$

**Sol:**




The region V is covered by (see the figure) (a) keeping $x$ and $y$ fixed and integrating from $z = x^2$ to $z = 4$ (base top of column PQ) (b) then by keeping $x$ fixed and integrating from $y = 0$ to $y = 6$ (R to S in the slab) and (c) finally integrating from $x = 0$ to $x = 2$ (where $z = x^2$ meets $z = 4$).

$\therefore$ The required integral is

$$\int_{x=0}^{2}\int_{y=0}^{6}\left[\int_{z=x^2}^{4}(xzi - 2xj + 2y^2k)dz\right]dydx$$

$$= i\int_{x=0}^{2}\int_{y=0}^{6}\left[\int_{z=x^2}^{4}xzdz\right]dydx + j\int_{x=0}^{2}\int_{y=0}^{6}\left[\int_{z=x^2}^{4}-2xdz\right]dydx + k\int_{x=0}^{2}\int_{y=0}^{6}\left[\int_{z=x^2}^{4}2y^2dz\right]dydx$$

$$= i\int_{x=0}^{2}\int_{y=0}^{6}\frac{xz^2}{2}\Bigg|_{z=x^2}^{4}dydx + j\int_{x=0}^{2}\int_{y=0}^{6}-2xz\Bigg|_{z=x^2}^{4}dydx + k\int_{x=0}^{2}\int_{y=0}^{6}2y^2z\Bigg|_{z=x^2}^{4}dydx$$

$$= i\int_{x=0}^{2}[\int_{y=0}^{6}(8x-\frac{x^5}{2})dy]dx + j\int_{x=0}^{2}[\int_{y=0}^{6}(2x^3-8x)dy]dx + k\int_{x=0}^{2}[\int_{y=0}^{6}(8y^2-2x^2y)dy]dx$$

$$= i\int_{x=0}^{2} 8xy-\frac{x^5y}{2}\Big|_{y=0}^{6} dx + j\int 2x^3y-8xy\Big|_{y=0}^{6} dx + k\int_{x=0}^{2}\frac{8y^3}{3}-\frac{2x^2y^3}{3}\Big|_{y=0}^{6} dx$$

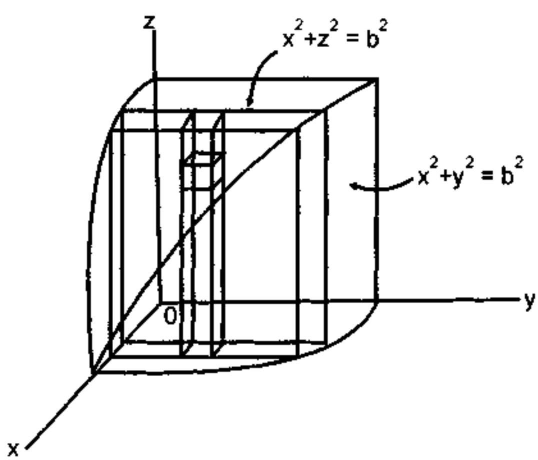
$$= i\int_{0}^{2}(48x-3x^5)dx + j\int_{0}^{2}(12x^3-48x)dx + k\int_{0}^{2}(576-144x^2)dx$$

$$= i(24x^2-\frac{1}{2}x^6)\Big|_{x=0}^{2} + j(3x^4-24x^2)\Big|_{x=0}^{2} + k(576x-48x^3)\Big|_{x=0}^{2}$$

$$= 64i - 48j + 768k$$

**Ex. 7.8.8** Find the volume of the region common to the intersecting cylinders $x^2 + y^2 = b^2$ and $x^2 + z^2 = b^2$.

**Sol:**




Required volume is equal to 8 times the volume of the region shown in the figure (as the axes cut the volume into 8 equal parts one in each octant)

$$\text{Volume} = 8\int_{x=0}^{b}\int_{y=0}^{\sqrt{b^2-x^2}}\left(\int_{z=0}^{\sqrt{b^2-x^2}} dz\right)dydx = 8\int_{x=0}^{b}\left(\int_{y=0}^{\sqrt{b^2-x^2}}\sqrt{b^2-x^2}\,dy\right)dx$$

$$= 8\int_{x=0}^{b}\left(b^2-x^2\right)dx \quad = 8\left[\left(b^2x-\frac{x^3}{3}\right)_0^3\right] = 16\frac{b^2}{3}.$$

## Exercise – 7(h)

1. Evaluate $\iiint_V (2x+y)\,dv$ where V is the closed region bounded by the cylinder $z = 4 - x^2$ and the planes $x = 0, y = 2$ and $z = 0$. [ Ans: $\frac{80}{3}$ ]

2. If $\mathbf{A} = \left(2x^2 - 3z\right)i - 2xyj - 4xk$, and V is the closed region bounded by the planes $x$=0,$y$=0,$z$=0and $2x + 2y + z = 4$,findthevalueof $\iiint_V (\text{curl}\mathbf{A})\,dv$

[Ans: $\frac{8}{3}(j-k)$ ]

3. Evaluate $\iiint_V f\,dv$ where $f = 45x^2y$ and v is the closed region bounded by the planes $4x + 2y + z = 8, x = 0, y = 0$ and $z = 0$ [Ans: 128 ]

4. If $\mathbf{A} = 2xzi - xj + y^2k$ and v is the region bounded by the surfaces $x = 0, y = 0, y = 6,$ $z = x^2$ and z = 4, find the value of $\iiint_V \mathbf{A}\,dv$ Ans: $128i - 24j + 384k$ ]

5. Evaluate $\iiint_V (\text{Div } \mathbf{A})dv$ taken over the rectangular parallelepiped, $0 \le x \le 1,\ 0 \le y \le 2,$ $0 \le z \le 3$, where $\mathbf{A} = \left(x^2 - yz\right)i + \left(y^2 - zx\right)j + \left(z^2 - xy\right)k$ [Ans:36]

6. Evaluate $\iiint_V (\text{Div } \mathbf{F})dv$ for the volume of a cube with edges of length unity parallel to the coordinate axes where $\text{F} = \left(x^3 - yz^2\right)i - \left(2x^2y\right)j + 2k$

[Ans: 1/3 ]

[**Ans:** (i) 0 (ii) $4r\sin\theta+\frac{1}{r^2}\frac{\cos 2\theta}{\sin\theta}$ (iii) $3\cos\theta$ ]

**9.** Show that the following vector fields are solenoidal.

(i) $\mathbf{A}=(z\cos\theta)\mathbf{e}_\rho-(z-\sin\theta)\mathbf{e}_\theta$ (ii) $\mathbf{F}=\left(\frac{1}{r^3}\cot\theta\right)\mathbf{e}_r-\frac{1}{r^3}\mathbf{e}_\theta+(r\cos\theta)\mathbf{e}_\phi$

**10.** Find the curl of the following vector fields :

(i) $\mathbf{A}=(z\sin\theta)\mathbf{e}_\rho+(z\cos\theta)\mathbf{e}_\theta-(\rho\cos\theta)\mathbf{e}_z$

(ii) $\mathbf{F}=(r\sin\theta)\mathbf{e}_r+\left(\frac{1}{r}\cos\theta\right)\mathbf{e}_r+\left(\frac{1}{r}\right)\mathbf{e}_\phi$

(iii) $\mathbf{V}=\frac{1}{r}\tan\frac{\theta}{2}\mathbf{e}_\phi$

[**Ans:**(i) $(\sin\theta\text{-}\cos\theta)\mathbf{e}_\rho+(\sin\phi+\cos\phi)\mathbf{e}_\theta$ (ii) $\left(\frac{1}{r^2}\cot\theta\right)\mathbf{e}_r-(\cos\theta)\mathbf{e}_\phi$ (iii) $\frac{1}{r^2}\mathbf{e}_\phi$ ]

**11.** If '$f$' is a scalar function in orthogonal curvilinear coordinates $v_1,v_2,v_3$ prove that '$\nabla f$' is irrotational.

## 7.10 GREEN'S THEOREM IN THE PLANE

### 7.10.1 Green's Theorem:

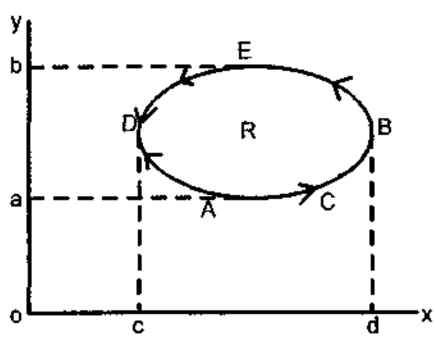
Let

(i) R be a closed region of the xy plane bounded by a simple closed curve C

(ii) P(x,y) and Q(x,y) be continuous functions of x and y having continous derivatives in R. Then $\int_C Pdx+Qdy=\iint_R\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dxdy$

where C is in the positive direction.

**Proof :**



Let the equations of the curves DAB and DEB (see the figure) be respectively $y = f_1(x)$ and $y = f_2(x)$. Let R be the region bounded by the curve C. Then,

$$\iint_R \frac{\partial P}{\partial y} dxdy = \int_{x=c}^{d} \left[ \int_{y=f_1(x)}^{f_2(x)} \frac{\partial P}{\partial y} dy \right] dx$$

$$= \int_{x=c}^{d} P(x,y) \Big|_{y=f_1(x)}^{y=f_2(x)} dx = \int_{x=c}^{d} [P(x,f_2) - P(x,f_1)] dx = -\int_{c}^{d} P(x,f_1) dx - \int_{d}^{c} P(x,f_2) dx$$

$$= -\oint_c P(x,y) dx$$

i.e., $\oint_c P(x,y)dx = -\iint_R \frac{\partial P}{\partial y} dxdy$ ..... (1)

|||$^{ly}$ if the equations of the curves ADE be and ABE be taken respectively as $x = g_1(y)$ and $x = g_2(y)$, we have,

$$\iint_R \frac{\partial Q}{\partial x} dxdy = \int_{y=a}^{b} \left[ \int_{x=g_1(y)}^{g_2(y)} \frac{\partial Q}{\partial x} dx \right] dy$$

$$= \int_a^b [Q(g_2,y) - Q(g_1,y)] dy = \int_b^a Q(g_1,y) dy + \int_a^b Q(g_2,y) dy$$

$$= \oint_c Q(x,y) dy$$

Thus $\oint_c Q(x,y)dy = \iint_R \frac{\partial Q}{\partial x} dxdy$ ..... (2)

Adding (1) and (2),

$$\oint_c Pdx + Qdy = \iint_R \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy$$

### 7.10.2 Vector notation of Green's theorem

Green's Theorem in the plane can be put in vector notation in the following way.

Let $F = P(x, y)i + Q(x, y)j$

and $r = xi + yj$, so that

$$dr = (dx)i + (dy)j$$

$$\therefore \quad F.\,dr = P\,dx + Q\,dy$$

$$\text{Again, } \nabla \times F = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & 0 \end{vmatrix}$$

$$= \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)k \text{, so that}$$

$$(\nabla \times F).k = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

Taking dR = dxdy, Green's theorem in the plane can be stated in the vector form as,

$$\oint_C F.dr = \iint_R (\nabla \times F).k\,dR \qquad \text{..... (A)}$$

### 7.10.3 Physical interpretation of Green's theorem

1. Let F denote the force field acting on a particle.

   Then $\oint_C F.dr$ represents the work done in moving the particle around a closed curve C.

   $\therefore$ From (A) it follows that the work done is determined by curl $F = \nabla \times F$.

2. In particular, if $\nabla \times F = 0$ i.e., if F is conservative (or $F = \nabla f$)

   Then $\oint_C F.dr = 0$. i.e., the work done is independent of the path.

3. Conversely, if the integral is independent of the path, i.e., if

   $\oint_C F.dr = 0$, then $\nabla \times F = 0$

In the plane, $\nabla \times F = 0$ is equivalent to saying that $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ where $F = Pi + Qj$.

### 7.10.4 Application of Green's theorem to the evaluation of area of a simple closed curve.

The area bounded by a simple closed curve C = $\frac{1}{2}\oint_C xdy - ydx$

*Proof:* By Green's theorem, we have,

$$\oint_C Pdx + Qdy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dxdy$$

if we put P = – $y$, and Q = $x$,

i.e., $\frac{\partial P}{\partial y} = -1$, $\qquad \frac{\partial Q}{\partial x} = 1$, $\qquad \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 2$,

∴ We get,

$\oint_C xdy - ydx = \iint_R 2dxdy = 2A$, where A is the required area.

i.e., A = $\frac{1}{2}\oint_C xdy - ydx$.

### Solved Examples

**Ex. 7.10.5** Find the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

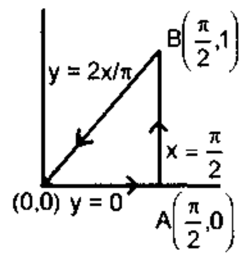
Sol: Parametric equations of the ellipse are,
$x$ = a cosθ, $\qquad y$ = b sinθ
d$x$ = – a sinθ dθ, $\quad$ d$y$ = b cosθ dθ.
∴ By Green's theorem,

Area of the ellipse = $\frac{1}{2}\oint_C xdy - ydx$

$$= \frac{1}{2}\int_0^{2\pi}(a\cos\theta b\cos\theta + b\sin\theta a\sin\theta)d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} ab\,d\theta = \pi ab$$

**Ex. 7.10.6** Evaluate $\oint_c (y-\sin x)dx + (\cos x)dy$, a) directly and b) using Green's theorem, where c is the boundary of the triangle in $xy$–plane whose vertices are (0,0), $\left(\frac{\pi}{2},0\right)$ and $\left(\frac{\pi}{2},1\right)$ traversed in the positive direction.




**Sol:** Let $I = \oint_c (y-\sin x)dx + (\cos x)dy$

a) along OA: $y = 0$, $dy = 0$

$$\therefore \int_{OA} (y-\sin x)dx + (\cos x)dy = \int_0^{\pi/2} -\sin x dx = \cos x\Big|_0^{\pi/2} = -1$$

along AB: $x = \frac{\pi}{2}$, $dx = 0$

$$\therefore \int_{AB} (y-\sin x)dx + (\cos x)dy = \int 0dy = 0$$

along BO: Equation of OB is $y = \frac{2x}{\pi}$, $dy = \frac{2}{\pi}dx$,

$$\therefore \int_{BO} (y-\sin x)dx + \cos x dy = \int_{\pi/2}^{0}\left[\left(\frac{2x}{\pi} - \sin x\right) + \frac{2}{\pi}\cos x\right]dx$$

$$= \frac{x^2}{\pi} + \cos x + \frac{2}{\pi}\sin x\Big|_{\pi/2}^{0} = (0+1+0) - \left(\frac{\pi}{4} + 0 + \frac{2}{\pi}\right) = 1 - \frac{\pi}{4} - \frac{2}{\pi}.$$

$\therefore$ I = Sum of the integrals along OA, AB and BO

$$= -1 + 0 + 1 - \frac{\pi}{4} - \frac{2}{\pi} = -\frac{\pi}{4} - \frac{2}{\pi}$$

(b) By Green's theorem,

$$\oint_c Pdx + Qdy = \iint_R \left( \frac{\partial Q}{dx} - \frac{\partial P}{dy} \right) dx\, dy, \quad \text{Here } P = y - \sin x, \quad Q = \cos x$$

$$\frac{\partial Q}{\partial x} = -\sin x, \quad \frac{\partial P}{\partial y} = 1$$

$$\therefore \quad I = \iint_R (-\sin x - 1) dx\, dy = \int_{x=0}^{\pi/2} \left[ \int_{y=0}^{2x/\pi} (-\sin x - 1) dy \right] dx = \int_{x=0}^{\pi/2} -y \sin x - y \int_0^{2x/\pi} dx$$

$$= \int_0^{\pi/2} \left( -\frac{2x}{\pi} \sin x - \frac{2x}{\pi} \right) dx = -\frac{2}{\pi} \int_{x=0}^{\pi/2} x \sin x \, dx - \frac{2}{\pi} \int_0^{\pi/2} x dx$$

$$= -\frac{2}{\pi} \left[ -x \cos x + \sin x + \frac{x^2}{2} \right]_0^{\pi/2} - \frac{2}{\pi} \left[ 0 + 1 + \frac{\pi^2}{8} \right] = -\frac{2}{\pi} - \frac{\pi}{4}$$

**Ex.6.10.7** Evaluate $\oint_c (3x + 4y) dx + (2x - 3y) dy$ where C is the circle in $xy$ plane with centre at origin and radius 2 units.

**Sol:** By Green's theorem ,

$$\oint Pdx + Qdy = \iint_R \left( \frac{\partial Q}{dx} - \frac{\partial P}{dy} \right) dx\, dy,$$

Here $P = 3x + 4y, \; Q = 2x - 3y$ ; $\frac{\partial P}{\partial y} = 4, \; \frac{\partial Q}{\partial x} = 2$

$\therefore$ The given integral = $\iint_R (2-4) dxdy$

$$= -2 \iint_R dxdy = -2\text{A},$$

where A is the area of the circle C.

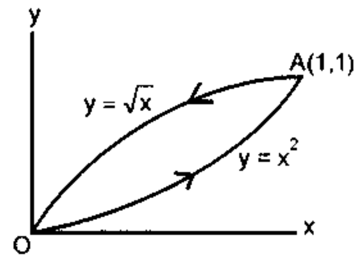
$$= -2 \times \pi \times 2^2 = -8\pi$$

**7.10.8** Verify Green's theorem in the plane for the integral $\oint_c (3x^2 - 8y^2)dx + 4(4y - 6xy)dy$

where C is the boundary of the region given by

(1) $y = \sqrt{x}$, $y = x^2$ (2) $x = 0, y = 0$, $x + y = 1$

(1) The given region is shown in the figure below.




let I = $\oint_c (3x^2 - 8y^2)dx + (4y - 6xy)dy$.

The points of intersection of $y = x^2$ and $y = \sqrt{x}$ are (0, 0) and (1, 1).
We have to integrate I along
(1) $y = x^2$ from 0 to A.
(2) along $y = \sqrt{x}$ from A to O, and add the two values.

Along $y = x^2$, $dy = 2xdx$

$$\therefore I = \int_{AO} (3x^2 - 8x^4)dx + (4x^2 - 6x^3)2xdx = \int_{x=0}^{x=1} (3x^2 - 20x^4 + 8x^3)dx = x^3 - 4x^5 + 2x^4 \Big|_0^1 = -1$$

Along $y = \sqrt{x}$ ; $x = y^2$, $dx = 2ydy$

$$\therefore I = \int_{AO} (3y^4 - 8y^2)dy + (4y - 6x^3)dy = \int_{y=1}^{y=0} (6y^5 - 22y^3 + 4y)(2y)dy = y^6 - \frac{11}{2}y^4 + 2y^2 \Big|_1^0 = \frac{5}{2}$$

$$\therefore \quad I = -1 + \frac{5}{2} = \frac{3}{2}$$

**By Green's Theorem,**

$$\oint_c P\,dx + Qdy = \iint_R \left(\frac{\partial Q}{\partial x} \cdot \frac{\partial P}{\partial y}\right) dxdy$$

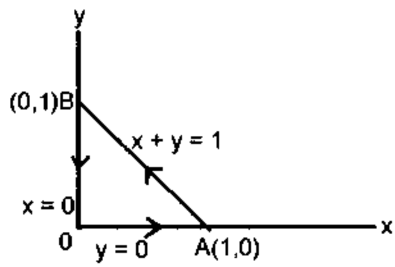
Here, $P = 3x^2 - 8y^2$ ; $Q = 4y - 6xy$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = -6y - (-16y) = 10y$$

$$\therefore I = \int\int_R 10\,y\,dx\,dy = \int_{x=0}^{1}\left[\int_{y=x^2}^{\sqrt{x}} 10\,y\,dy\right]dx = \int_{x=0}^{1} 5y^2 \Big|_{x^2}^{\sqrt{x}} dx = \int_{x=0}^{1}(5x - 5x^4)dx = \frac{3}{2}$$

Green's theorem is verified.

2. The given region is shown in the figure below.




Along OA, $y = 0 \Rightarrow dy = 0$

$$\therefore \text{Given integral} = \int_{x=0}^{1} 3x^2 dx = x^3 \Big|_0^1 = 1$$

Along AB, $y = 1 - x \Rightarrow dy = -\,dx$

$\therefore$ Given integral

$$= \int_{x=1}^{0}\{3x^2 - 8(1-x^2)\}dx + \{4(1-x) - 6x(1-x)(-dx)\} = \int_{x=1}^{0}(-11x^2 + 26x - 12)\,dx$$

$$= -\frac{11}{3}x^3 + 13x^2 - 12x\Big|_1^0 = \frac{11}{3} - 13 + 12 = \frac{8}{3}$$

Along BO, $x = 0 \Rightarrow dx = 0$

$$\text{Given integral} = \int_{y=1}^{y=0} 4y\,dy = 2y^2\Big|_1^0 = -2$$

$$\therefore \text{The given integral} = 1 + \frac{8}{3} - 2 = \frac{5}{3}$$

**By green's theorem,**

$$\oint_c P dx + Q dy = \int\int_R \left(\frac{\partial Q}{\partial x} . \frac{\partial P}{\partial y}\right) dx dy$$

Here, $P = 3x^2 - 8y^2$, $Q = 4y - 6xy$

$$\frac{\partial Q}{\partial x} \cdot \frac{\partial P}{\partial y} = -6y - (-16y) = 10y$$

$$\therefore \text{The given integral} = \iint_R 10y\,dx\,dy = \int_{x=0}^{1}\left[\int_{y=0}^{y=1-x} 10y\,dy\right]dx = \int_{x=0}^{1} 5y^2 \Big|_0^{1-x} dx = \int_0^1 5(1-x)^2 dx$$

$$= \frac{5(1-x)^3}{3x-1}\Big|_0^1 = -\frac{5}{3}(0-1) = \frac{5}{3}$$

Hence the theorem is verified.

**Ex. 7.10.9** Evaluate $\int_{(0,0)}^{(2,1)} (10x^4 - 2xy^3\ dx - 3x^2y^2\,dy$ along the path $x^4 - 6xy^3 - 4y^2 = 0$

**Sol:** $P = 10x^4 - 2xy^3$, $Q = 3x^2y^2$

$$\frac{\partial P}{\partial y} = -6xy^2 = \frac{\partial Q}{\partial x}$$

The integral is independent of the path. Hence we can use any path. For example, if we use the path from points (0, 0) to (2, 0) and then from (2, 0) to (2, 1); we can evaluate the integral.

(i) From (0, 0) to (2, 0) ; $y = 0$, $dy = 0$

$$\therefore \text{The integral} = \int_0^2 10x^4\,dx - 2x^5\Big|_0^2 = 64$$

(ii) From (2, 0) to (2, 1) ; $x = 2$, $dx = 0$

$$\therefore \text{The integral} = \int_0^2 10x^4\,dx = 2x^5 \int_0^2 = 64 \quad \int_0^2 12y^2\,dy = -4y^3\Big|_0^2 = -4$$

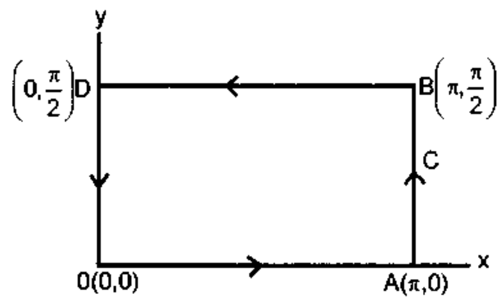
$\therefore$ The value of the integral = 64 – 4 = 60

Aliter : Since $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, we know that $\{10x^4 - 2xy^3)dx - 3x^2y^2\,dy\}$ is an exact differential of $(2x^5 - x^2y^3)$.

$$\text{The given integral} = \int_{(0,0)}^{(2,1)} d\left(2x^5 - x^2y^3\right) = 2x^5 - x^2y^3 \int_{(0,0)}^{(2,1)} = 2.2^5 - 2^2.1^3 = 60$$

**Ex. 7.10.10** Verify Green's theorem for $\int_c (e^{-x}\sin y)\,dx + (e^{-x}\cos y)\,dy$ where C is the boundary of the rectangle whose vertices are (0, 0) (π, o) (π, $\frac{\pi}{2}$ ) and (0, $\frac{\pi}{2}$ ) traversed in the +ve direction.

**Sol:**




By Green's theorm,

$$\oint_c Pdx + Qdy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$$

Here, $P = e^{-x}\sin y$, $Q = e^{-x}\cos y$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = -e^{-x}\cos y - e^{-x}\cos y = -2e^{-x}\cos y$$

$$\therefore \quad \oint_c Pdx + Qdy = \iint_R -2e^{-x}\cos y\, dxdy$$

$$= -2\int_{x=0}^{\pi} [\int_{y=0}^{\pi/2} e - x\cos dy]\, dx = -2\int_{x=0}^{\pi} e^{-x}\sin y \int_0^{\pi/2} dx$$

$$= -2\int_c^{\pi} e^{-x} dx = 2e^{-x}\Big|_0^{\pi} = 2(e^{-\pi} - 1) \qquad .....(i)$$

Again, $\oint_c P\,dx + Q\,dy = \int_c \int_{OA} + \int_{AB} + \int_{BD} + \int_{DO}$

Along OA ; $\quad y = 0, \qquad dy = 0$

$$\therefore \quad \int_{OA} Pdx + Qdy = 0$$

Along AB : $\quad x = \pi, \qquad dx = 0$

$$\int_{AB} Pdx + Qdy = \int_0^{\pi/2} e^{-\pi}\cos y\,dy = e^{-\pi}\sin y\Big|_0^{\pi/2} = e^{-\pi}$$

Along BD ; $\quad y = \dfrac{\pi}{2}, \quad dy = 0$

$$\int_{BD} P\,dx + Qdy = \int_{\pi}^{0} e^{-x}dx = \frac{e^{-x}}{-1}\Bigg|_{\pi}^{0} = -1 + e^{-\pi}$$

Along DO ; $\quad x = 0, \quad dx = 0$

$$\therefore \int_{DO} P\,dx + Qdy = \int_{\pi/2}^{0} \cos y\,dy = \sin y\Big|_{\pi/2}^{0} = -1$$

$$\therefore \oint_{c} P\,dx + Qdy = 0 + e^{-\pi} - 1 + e^{-\pi} - 1 = 2(e^{-\pi} - 1) \qquad \text{.....(ii)}$$

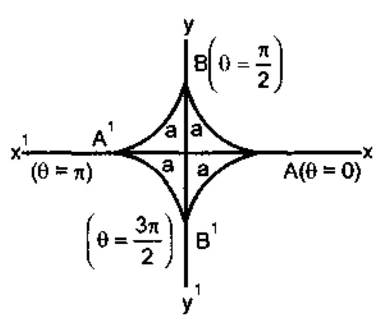
$\therefore$ From (i) and (ii) it is proved that

$$\oint_{c} P\,dx + Qdy = \iint_{R} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$$

Hence the theorem is verified.

**Ex. 7.10.11** Apply Green's theorem to obtain the area bounded by the curve
$x^{2/3} + y^{2/3} = a^{2/3}, \quad a > 0$.

**Sol:** The parametric equations of the curve are $x = a\cos^3\theta$, $y = \sin^3\theta$. A rough sketch of the curve is given below :




$x = a\cos^3\theta, \qquad y = a\sin^3\theta$

$dx = -3\,a\cos^2\theta\,\sin\theta\,d\theta,\ dy = 3a\,\sin^2\theta\cos\theta\,d\theta$

By Green's theorem, the area bounded by a simple closed curve C is given by

$$\frac{1}{2}\oint_{C}(xdy - ydx)$$

$\therefore$ The area bounded by the given curve C

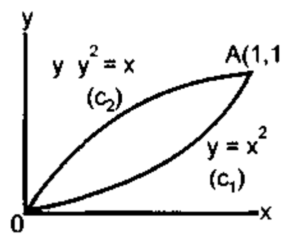
$$= \frac{1}{2}\int_{\theta=0}^{2\pi}\{a\cos^3\theta\,.\,3a\sin^2\theta\cos\theta + a\sin^3\theta . 3a\cos^2\theta\sin\theta\}d\theta$$

$$= \frac{3a^2}{2}\int_0^{2\pi}\cos^2\theta\sin^2\theta(\cos^2\theta+\sin^2\theta)d\theta$$

$$= \frac{3a^2}{2}\int_0^{2\pi}\cos^2\theta\sin^2\theta\,d\theta = \frac{3a^2}{8}\int_0^{2\pi}\sin^2 2\theta\,d\theta = \frac{3a^2}{16}\int_0^{2\pi}(1-\cos 4\theta)\,d\theta = \frac{3\pi a^2}{8}$$

**Ex. 7.10.12** Verify Green's theorem for $\int_C (2xy - x^2)dx + (x+y^2)dy$, where C is the closed curve in $xy$-plane bounded by the curves $y = x^2$ and $y^2 = x$.

**Sol:** By Green's theorem,




$$\int_C Pdx + Qdy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$$

Here, $P = 2xy - x^2$, $Q = x + y^2$, $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1 - 2x$

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy = \int_{x=0}^{1}\left[\int_{y=x^2}^{y=\sqrt{x}}(1-2x)dy\right]dx = \int_0^1 y - 2xy\Big|_{y=x^2}^{y=\sqrt{x}} dx$$

$$= \int_0^1 (\sqrt{x} - 2x\sqrt{x} - x^2 + 2x^3)dx = \frac{2}{3}x^{\frac{3}{2}} - 2.\frac{2}{5}x^{\frac{5}{2}} - \frac{x^3}{3} + 2.\frac{x^4}{4}\Big|_0^1$$

$$= \frac{1}{2} - \frac{4}{5} - \frac{1}{3} + \frac{1}{2} = \frac{1}{30} \qquad \text{..... (i)}$$

Again $\int_C Pdx + Qdy = \int_{C_1}(Pdx + Qdy) + \int_{C_2}(Pdx + Qdy)$

Along $C_1$, $y = x^2$, $dy = 2xdx$, "$x$" varies from 0 to 1

$$\therefore \int_{C_1} Pdx + Qdy = \int_{x=0}^{1}(2x^3 - x^2)dx + (x + x^4)2xdx = \int_0^1 (2x^3 + x^2 + 2x^5)dx$$

$$= \frac{x^4}{2} + \frac{x^3}{3} + \frac{x^6}{3}\Big|_0^1 = \frac{7}{6}$$

Along $C_2$, $x = y^2$, $dx = 2ydy$, '$y$' varies from 1 to 0

$$\therefore \int_{C_2} Pdx + Qdy = \int_{y=1}^{0} (2y^3 - y^4)2ydy + (y^2 + y^2)dy = \int_{1}^{0} (4y^4 - 2y^5 + 2y^2)dy$$

$$= \frac{4}{5}y^5 - \frac{2}{6}y^6 + \frac{2}{3}y^3\Big|_1^0 = -\frac{4}{5} + \frac{1}{3} - \frac{2}{3} = -\frac{17}{15}$$

$$\therefore \int_c \text{P}dx + \text{Q}dy = \frac{7}{6} - \frac{17}{15} = \frac{1}{30} \qquad \text{.....(ii)}$$

From (i) and (ii), the theorem is verified.

## Exercise – 7(j)

1. Evaluate $\oint (x^2 + y^2)dx + 3xy^2dy$, (a) directly (b) by Green's theorem, where c is the circle $x^2 + y^2 = 4$, traversed in the +ve direction.

   (Ans : $12\pi$)

2. Evaluate $\oint (x^2 + 2xy)dx + (x^2y + 3)dy$ around the boundary C of the region given by $y^2 = 8x$ and $x = 2$, (a) directly and (b) by Green's theorem.

   (Ans : 128)

3. Verify Green's theorem for the integral $\oint (3x^2 + 2y)dx - (x + 3\cos y)dy$ where C is the boundary of the parallelogram with vertices at (0, 0), (2, 0), (3, 1) and (1, 1)

   (Ans : –6)

4. Evaluate $\int_{(0,0)}^{(\pi,2)} (6xy - y^2)dx + (3x^2 - 2xy)dy$ along the cycloid $x = \theta - \sin\theta$, $y = 1 - \cos\theta$. Verify the result by using Green's theorem.

   (Ans : $6\pi^2 - 4\pi$)

5. Using Green's theorem find the area bounded by one arch of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1-\cos\theta), a > 0$, and x-axis.

**[Ans:** $3\pi a^2$ **]**

6. Evaluate $\oint_c (2x^2 - y^2)dx + (x^2 + y^2)$ by Green' theorem where C is the boundary of the surface in the $xy$ plane enclosed by $x$ axis and the semi-circle $y = \sqrt{1-x^2}$

**[Ans:** $4/3$ **]**

7. Evaluate $\oint_c (\cos x \sin y - xy)dx + \sin x \cos y$ ,using Green's theorem where c is the circle $x^2 + y^2 = 1$

**[Ans: 0]**

8. Verify Green's Theorem in the plane for $\oint_c (x^2 - xy^3)dx + (y^2 - 2xy)$where C is the square with vertices at ( 0,0) , ( 2,0) and (0 ,2)

**[Ans: 8]**

9. Evaluate $\int_{(0,0)}^{(2,1)} (12x^3 - 2xy^3)dx - 3x^2y^2dy$ along the path $x^3 - y^3 + y - 4xy = 0$

[ **Hint** : Proceed as in aliter of 7.10.9]

**[Ans:** $3x^4 - x^2y^3\Big|_{(0,0)}^{(2,1)} = 44$ **]**

**Sol:** By the divergence theorem ,

$$\iint_S (\mathbf{F}\,.n)\,ds = \iiint_v (\nabla.\mathbf{F})dv \qquad ....(1)$$

$$\nabla.\mathbf{F} = \frac{\partial}{\partial x}(2x)+\frac{\partial}{\partial y}(3y)+\frac{\partial}{\partial y}\left(\frac{z^3}{3}\right)=5+z^2$$

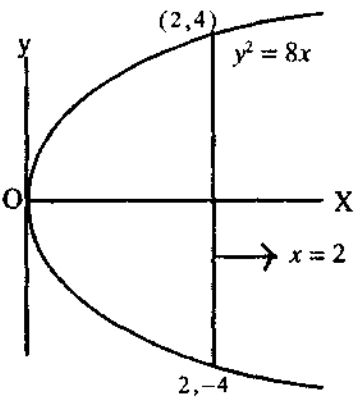
From (1), $\iint_S (\mathbf{F}\,.n)\,ds = \iiint_v (5+z^2)dv$ ,

$$= \int_{x=0}^{2}\int_{y=2\sqrt{2x}}^{2\sqrt{2x}}\left[\int_{z=0}^{3}\left(5+z^2\right)dz\right]dydx$$

$$= \int_{x=0}^{2}\int_{y=2\sqrt{2x}}^{2\sqrt{2x}} 5z+\frac{z^3}{3}\Big|_0^3\, dydx$$

$$= \int_{x=0}^{2}\left[\int_{y=2\sqrt{2x}}^{2\sqrt{2x}} 24dy\right] \int_{x=0}^{2} 24\,y\Big|_{-2\sqrt{2x}}^{2\sqrt{2x}} dx$$

$$= \int_{x=0}^{2} 96\sqrt{2}xdx = 96\sqrt{2}\,\frac{x^2}{2}\Big|_0^2 = 192\sqrt{2}$$




**Ex.7.11.11** Evaluate $\iint_S \left(2xi-3y^2j+z^2k\right).n\ ds$ over the surface bounded by $x^2+y^2=1,\ z=0,\ z=2$ , using Gauss' theorem.

**Sol:** By the divergence theorem ,

$$\iint_S (\mathbf{F}\,.n)\,ds = \iiint_v (\nabla.\mathbf{F})dv \ ,$$

Here , $\mathbf{F} = 2xi-3y^2j+z^2k$

$$\text{Div } \mathbf{F} = \frac{\partial}{\partial x}(2x)-\frac{\partial}{\partial y}\left(3y^2\right)+\frac{\partial}{\partial y}\left(z^2\right)$$

$$= 2-6y+2z$$

$$\therefore\ \iint (\mathbf{F}\,.n)ds = \int_{x=1}^{+1}\int_{y=\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_0^2 \left(2-6y+2z\right)dzdydx$$

$$= \int_{x=-1}^{+1} \int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \left[ \int_{z=0}^{2} (2-6y+2z)\,dz \right] dy\,dx = \int_{x=-1}^{+1} \int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 2z-6yz+z^2 \Big|_{-1}^{1} dy\,dx$$

$$= \int_{x=-1}^{+1} \left[ \int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (8-12y)\,dy \right] dx = \int_{x=-1}^{1} 8y-6y^2 \Big|_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dx = \int_{-1}^{1} 16\sqrt{1-x^2}\,dx$$

$$= 16\left[ \frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1} x \right]_{-1}^{1} = 8\pi$$

**Ex.7.11.12 :** Evaluate using the divergence theorem $\iint_S (F.n)\,dS$ where S is the surface of the sphere $x^2 + y^2 + z^2 = b^2$ in the first octant and $F = yi + zj + xk$

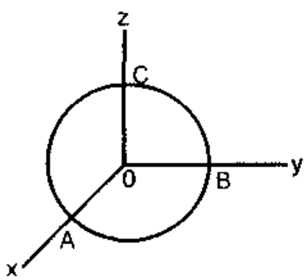
**Sol :** By divergence theorem,

$$\therefore \iint_S F.n\,ds = \iiint_V \nabla.F\,dv \qquad \text{..... (i)}$$

$F = yi + zj + xk$

$\nabla.F = 0$

$$\therefore \iint_V (\nabla.F)\,dv = 0 \qquad \text{..... (ii)}$$




Let us evaluate the surface integrals over the faces OAB, OBC and OCA.

$$\int\int_{OAB} F.n\,ds = -\int_{x=0}^{b} \int_{y=0}^{\sqrt{b^2-x^2}} x\,dx\,dy \qquad (\because n = -k)$$

$$= -\int_0^b xy\Big|_0^{\sqrt{b^2-x^2}} dx = -\int_0^b x\sqrt{b^2-x^2}\,dx = -\pi\frac{b^3}{3}$$

Similarly $\iint\limits_{OAB} \mathbf{F}.\mathbf{n}ds = \iint\limits_{OAB} \mathbf{F}.\mathbf{n}ds + \iint\limits_{ABC} \mathbf{F}.\mathbf{n}ds + \iint\limits_{OCA} \mathbf{F}.\mathbf{n}ds + \iint\limits_{ABC} \mathbf{F}.\mathbf{n}ds$

$$= -\pi b^3 + \iint\limits_{ABC} \mathbf{F}.\mathbf{n}ds \qquad \text{....(iii)}$$

From (i), (ii), and (iii), we get

$$0 = -\pi b^3 + \iint\limits_{ABC} \mathbf{F}.\mathbf{n}ds$$

$$\iint\limits_{ABC} \mathbf{F}.\mathbf{n}ds = \pi b^3$$

**Ex. 7.11.13:** Verify divergence theorem for $\mathbf{F} = 4xi - 2y^2 j + z^2 k$ taken over the region bounded by

$x^2 + y^2 = 4,\ z = 0$ and $z = 3$.

**Sol:** By the divergence theorem , we have

$$\iiint\limits_{v} Div\vec{F}dv = \iint\limits_{S} \vec{F}.\vec{n}\, ds \qquad \text{....( 1 )}$$

(1) $Div\vec{F} = \dfrac{\partial}{\partial x}(4x) - \dfrac{\partial}{\partial y}(2y^2) + \dfrac{\partial}{\partial z}(z^2) = 4 - 4y + 2z$

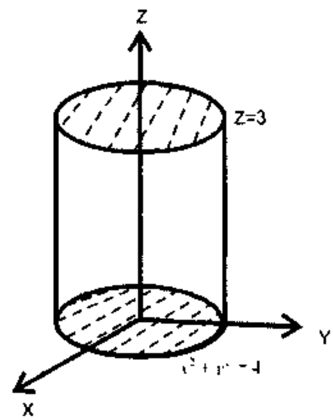
$$\therefore \quad \text{L.H.S. of (1)} = \int\limits_{x=2}^{2} \int\limits_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \left[ \int\limits_{z=0}^{3} (4 - 4y + 2z) dz \right] dydx$$

$$= \int\limits_{x=-z}^{2} \int\limits_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \left[ 4z - 4yz + z^2 \right]_{z=0}^{3} dydx = \int\limits_{x=-2}^{2} \left[ \int\limits_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (21 - 12y) dy \right] dx$$

$$= 42 \int\limits_{x=-2}^{2} \sqrt{4 - x^2} dx = 84\pi$$

[Do the integration w.r.t.$x$ yourself, taking $x = 2\sin\theta$ ]

(2) Evaluate of surface integral $\iint\limits_{S} \vec{F}.\bar{n}\, ds$




The given surface of the cylinder can be divided into 3parts, namely

(a) $S_1$ : the circular surface z = 0

(b) $S_2$ : the surface z = 3 (circular ) and

(c) $S_3$ : the cylindrical portion of $S : x^2 + y^2 = 4,\ z = 0,\ z = 3$

we now find $\iint \vec{F}.\vec{n}\, ds$ over $S_1, S_2, S_3$ .If we add them, we get R.H.S of (1).

(a) on $S_1 : z = 0$ ; $\vec{n} = -k$ ; $\vec{F}.\vec{n} = -\left(4xi - 2y^2 j\right).k = 0$ ; $\therefore \iint\limits_{S_1} \vec{F}.\vec{n}\, ds = 0.$

(b) on $S_2 : z = 3$ ; $\vec{n} = k$ ; $\vec{F}.\vec{n} = \left(4xi - 2y^2 j + 9k\right).k = 9$ ; $ds = \dfrac{dxdy}{\left|\bar{n}.k\right|} = dxdy$

$\therefore \iint\limits_{S_2} \vec{\mathbf{F}}.\vec{\mathbf{n}}\, ds = \iint\limits_{S_2} 9dxdy = 9\mathbf{A}$ , where A is the area of the circle

$x^2 + y^2 = 4,\ = 9\pi\left(2^2\right) = 36\pi$

(c) on $S_3$ : Let $\phi = x^2 + y^2 - 4 = 0$ ;

$$\vec{\mathbf{n}} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2(xi + yi)}{2\sqrt{x^2 + y^2}} = \frac{xi + yi}{2}\left(\text{since } x^2 + y^2 = 4\right)$$

$$\vec{\mathbf{F}}.\vec{\mathbf{n}} = \frac{4x^2 - 2y^3}{2} = 2x^2 - y^3 ;$$

To evaluate $\iint\limits_{S_3} \vec{\mathbf{F}}.\vec{\mathbf{n}}\, ds$ , take $x = 2\cos\theta, y = 2\sin\theta,$

and $ds = 2d\theta\ dz$ ; limits of z are 0 to 3 and those of $\theta$ are 0 to $2\pi$ .

Hence $\iint\limits_{S_3} \bar{\mathbf{F}}.\bar{\mathbf{n}}\ ds = \int\limits_{\theta=0}^{2\pi}\int\limits_{z=0}^{3}\left(8\cos^2\theta - 8\sin^3\theta\right)2d\ \theta\ dz$

$$= 16\int\limits_{\theta=0}^{2\pi}\left[\left(\cos^2\theta-\sin^3\theta\right)z\right]_{z=0}^{3} d\theta = 48\int\limits_{\theta=0}^{2\pi}\left(\cos^2\theta-\sin^3\theta\right)d\theta = 48\pi$$

[ Do the integration w.r.t. $\theta$ yourself ]

$\therefore$ R.H.S of(1) = $0+36\pi+48\pi = 84\pi$ ; $\therefore$ L.H.S = R.H.S

Hence the theorem is verified.

## Exercise – 7K

1. Verify Gauss's divergence theorem for $\mathbf{A} = (x^2 - yz)i + (y^2 - zx)j + (z^2 - xy)k$ taken over the rectangular parallalopiped $0 \le x \le 2,\ 0 \le y \le 3,\ 0 \le z \le 1$.[Ans:36]

2. Use the divergence theorem to find $\iint\limits_S \mathbf{F}.ndS$, where $f = (3x+2z^2)i - (z^2 - 2y)j + (y^3 - 2z)k$ and S is the surface of the sphere with centre at (2, –1, 3) and radius 2 units. [Ans:32 $\pi$ ]

3. Verify the divergence theorem for the vector function, $\mathbf{A} = (4xz)i - (y^2)j + (yz)k$, taken over the unit cube bounded by $x=0,\ x=1,\ y=0,\ y=1,\ z=0$ and $z=1$.[Ans:3/2]

4. If $\mathbf{r} = xi + yj + zk$, and S is the surface of the rectangular parallelepiped bounded by planes $x=0, y=0, z=0, x=a, y=b$ and $z=c$, find the value of $\iint\limits_S \mathbf{r}.\mathbf{n}dS$ using Gauss's theorem. Verify your answer by direct evaluation of the integral. [Ans:3abc]

5. Use the divergence theorem to evaluate $\iint\limits_S \mathbf{A}.\mathbf{n}ds$ for $A = (2x)i - (2y)j + (3z)k$ where $s$ is the sphere given by $(x-1)^2 + y^2 + z^2 = 1$ [Ans:4 $\pi$ ]

6. If $\mathbf{V} = (lx)i + (my)j + (nz)k$ and $l,\ m,\ n$ being constants show that $\iint\limits_S \mathbf{V}.\mathbf{ds} = \frac{32\pi}{3}(l+m+n)$. where S is the surface the sphere $(x-3)^2 + (y-2)^2 + (z-1)^2 = 4$.

## 7.12 Stoke's Theorem

### 7.12.1 Stoke's Theorem

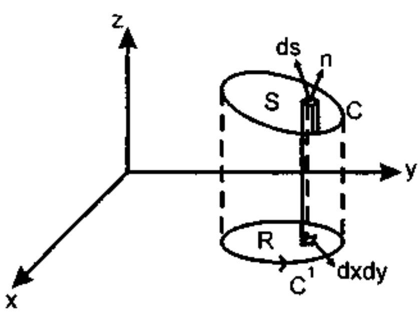
Let (1) S be an open, two-sided surface bounded by a simple closed curve C.

(2) A be a vector function having continuous derivatives

Then, $\oint A.dr = \iint_s (\nabla \times A).nds = \iint_s (\nabla \times A).ds$

where C travels in the +ve direction and n is the unit +ve (outward drawn) normal to S.

*Proof :*




Let S be the surface. Let the projections of S on the coordinate planes be regions bounded by simple closed curves.

Let 'R' the projection of S on xy plane be bounded by $C^1$. (see the figure above).

Let the equation of S be $z = \phi_1(x, y)$ where $\phi_1$ is a single valued, continuous and differentiable function.

Let $\quad A = A_1 i + A_2 j + A_3 k$

Then $\quad \nabla \times (A_1 i) = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_1 & 0 & 0 \end{vmatrix} = \dfrac{\partial A_1}{\partial z} j - \dfrac{\partial A_1}{\partial y} k$

$$\{\nabla\times(A_1 i)\}.n\,ds=\left(\frac{\partial A_1}{\partial z}-\frac{\partial A_1}{\partial y}\right).nds=\left\{\frac{\partial A_1}{\partial z}(n.j)-\frac{\partial A_3}{\partial y}(n.k)\right\}ds \qquad \text{..... (i)}$$

The position vector r of any point on S can be taken as

$$r = xi + yj + zk$$

$$= xi + yj + \phi_1(x, y)k$$

and $\dfrac{\partial r}{\partial y}=j+\dfrac{\partial \phi_1}{\partial y}k$

But $\dfrac{\partial r}{\partial y}$ being the vector tangent to S, it is $\perp_r$ to n.

$\therefore \dfrac{\partial r}{\partial y}.\,n = 0 \Rightarrow n.j = -\dfrac{\partial \phi_1}{\partial y}\,(n.k)$

$\therefore (1) \Rightarrow \{\nabla \times A_1 i)\}\,.\,n\,ds = \left(-\dfrac{\partial A_1}{\partial z}\dfrac{\partial \phi_1}{\partial y}n.k-\dfrac{\partial A_1}{\partial y}n.k\right)ds$

$$=-\left(\frac{\partial A_1}{\partial z}\frac{\partial z}{\partial y}+\frac{\partial A_1}{\partial y}\right)(n.k)ds \qquad (\because z=\phi_1) \qquad \text{..... (ii)}$$

on S, $\quad A_1(x\,y,\,z) = A_1(x,\,y,\,\phi_1(y)) = G(x,\,y)$ (say)

$$\frac{\partial A_1}{\partial y}+\frac{\partial A_1}{\partial z}.\frac{\partial z}{\partial y}=\frac{\partial G}{\partial y}$$

$\therefore$ (ii) becomes,

$$\{\nabla \times (A_1 i)\}.n\,ds = -\frac{\partial G}{\partial y}\,(n.k)\,ds = -\frac{\partial G}{\partial y}\,dxdy \qquad [\because n.k.\ ds = dxdy]$$

$\iint\limits_S (\nabla\times A_1 i)\}.n\,ds=\iint\limits_R -\dfrac{\partial G}{\partial y}dxdy=\oint\limits_C G\,dx,$ by Green's theorem in the plane. Now at each point $(x, y)$ of C, the value of G is the same as the value of $A_1$ at each point $(x, y, z)$ of C, and since $dx$ is same for both curves, we have

$$\oint\limits_C G dx=\oint\limits_C A_1\,dx$$

Hence, $$\iint_S \{\nabla \times (A_1 i)\}.n\, ds = \oint_C A_1 dx \qquad \text{.....(iii)}$$

$|||^{ly}$ by projecting S on $yz$ and $zx$ planes, it can be shown that,

$$\iint_S \{\nabla \times (A_2 i)\}.n\, ds = \oint_C A_2 dy \qquad \text{..... (iv)}$$

$$\iint_S \{\nabla \times (A_3 k)\}.n\, ds = \oint_C A_3 dz \qquad \text{..... (v)}$$

Adding (iii), (iv) and (v), $\iint_S \nabla \times A.n\, ds = \oint_C A.\, dr$

(since $A.\, dr = A_1\, dx + A_2 dy + A_3 dz$)

Hence the theorem is proved.

**Ex.7.12.1 :(a)** Express Stoke's theorem in words and (b) obtain its cartesian form.

**Sol :(a)** The line integral of the tangential component of a vector A taken around a simple closed curve C is equal to the surface integral of the normal component of curl A taken over a surface S having C as its boundary.

(b) As in 7.11.2(b)

$A = A_1 i + A_2 j + A_3 k$
$n = (\cos\alpha) i + (\cos\beta) j + (\cos\gamma) k$

Then, $$\nabla \times A = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_1 & A_2 & A_3 \end{vmatrix}$$

$$= \left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right) i + \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right) j + \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right) k$$

$$(\nabla \times A).n = \left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right)\cos\alpha + \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right)\cos\beta + \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)\cos y$$

$A.\, dr = (A_1 i + A_2 j + A_3 k).\, (dx\, i + dy\, j + dz\, k) = A_1 dx + A_2 dy + A_3\, dz$

Hence the cartesian form of the Stoke's theorem can be stated as,

$$\iint_s [(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z})\cos\alpha + (\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x})\cos\beta + (\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y})\cos y]ds$$

$$= \oint_C A_1 dx + A_2 dy + A_3 dz$$

## Solved Examples

**Ex.7.12.3 :** Verify Stoke's Theorem for $A = (x-2y)i + yz^2 j + y^2 zk$, where S is the upper half of the sphere $x^2 + y^2 + z^2 = 1$ and C is its boundary.

**Sol :** The boundary of the projection of S in the xy-plane is a circle with centre at origin and unit radius. Its parametric equations are $x = \cos\theta, y = \sin\theta, z = 0, 0 \le \theta < 2\pi$

$\therefore \quad dx = (-\sin\theta)d\theta,\ dy = (\cos\theta)d\theta.$

$$\therefore \quad \oint_C A.dr = \oint_C (x-2y)dx + yz^2 dy + y^2 z dz$$

$$= \int_{\theta=0}^{2\pi} [\cos\theta - 2\sin\theta]\,(-\sin\theta)d\theta \qquad (\because z = 0)$$

$$= \int_0^{2\pi} [\frac{-\sin 2\theta}{2} + 1 - \cos 2\theta]d\theta = \frac{\cos 2\theta}{4} + \theta - \frac{\sin 2\theta}{2}\Bigg|_0^{2\pi} = 2\pi$$

$$\nabla \times A = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ (x-2y) & yz^2 & y^2 z \end{vmatrix} = 2k$$

$$\iint_s (\nabla \times A).nds = \iint_s 2(n.k)ds = 2\iint_R dxdy$$

($n.k\ ds = dxdy$ and R is the projection of S on the $xy$ plane)

$$\iint_s (\nabla \times A).nds = 2\int_{x=-1}^{+1} \int_{y=-\sqrt{1-x^2}}^{y=\sqrt{1-x^2}} dy\ \}\,dx = 4\int_{-1}^{+1}\sqrt{1-x^2}\ dx = 8\int_0^1 \sqrt{1-x^2}\,dx = 2\pi$$

Hence Stoke's theorem is verified.

**Ex.7.12.4 :** Prove that a necessary and sufficient condition that $\oint_C F.dr$ for every closed curve C is that $\nabla \times \mathbf{F} = 0$.

***Proof:*** (a) The condition is necessary : Let $\nabla \mathbf{F} = 0$;

Then by Stoke's theorem , $\int_C \mathbf{F}.dr = \iint_S (\nabla \times \mathbf{F}).\mathbf{n}\ ds = 0$

(b) The condition is sufficient:

Suppose that $\oint_C F.dr = 0$ around every closed path c.

Assume that $\nabla \times \mathbf{F} \neq 0$ at some point P. then , assuming that $\nabla \mathbf{F}$ is continuous , there exists a region with P as its interior point where $\nabla \mathbf{F} = 0$.Let S be surface contained in this region and let the normal n to S at each point has the same direction as $\nabla \mathbf{F}$ .

Then $\nabla \mathbf{F} = a\ n$, (a being a +ve constant); Let C be the boundary of S.

Then by Stokes theorem , $\int_C \mathbf{F}.dr = \iint_S (\nabla \mathbf{F}).\mathbf{n}\ ds = a \iint_S n.n\ ds > 0$

Which is a contradiction to the hypothesis that $\int \mathbf{F}.dr = 0$ ; $\therefore \nabla \mathbf{F} = 0$

*Note: It* follows that $\nabla \mathbf{F} = 0$ is also a necessary and sufficient condition for the line integral $\int_{P_1}^{P_2} \mathbf{F}.dr$ to be independent of path joining the points $P_1$ and $P_2$.(see 7.6.4)

**Ex.7.12.5** If $r = xi + yj + zk$, show that $\int_C r.dr = 0$

By Stoke's theorem , $\int_C r.dr = \iint_C (Curl\ r).\mathbf{n}\ ds$ .....(1)

But curl $\mathbf{r} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/dz \\ x & y & z \end{vmatrix} = 0$

$\therefore (1) \Rightarrow \int_C r.dr = 0$

**Ex.7.12.6 :** If '$f$' and '$g$' are scalar functions, show that

$$\int_C f(grad\ g).\ dr = -\int_C g(grad\ f).dr$$

**Sol :** By stoke's theorem,

$\int_C \{grad\ (fg).dr = \iint_S [curl\{grad(fg)\}]n\ ds = 0$, since curl grad $(fg) = 0$

$$\therefore \int_C \{grad\ (fg)\}.dr = 0 \qquad \text{..... (1)}$$

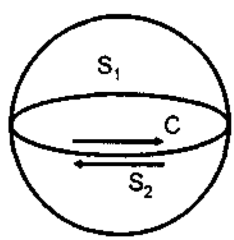
But grad$(fg) = f(grad\ g) + g(grad\ f)$ ..... (2)

Hence the result [from (1) and (2)]

**Ex.7.12.7 :** If A is any vector function, prove by stoke's theorem that div curl A = 0.

**Sol :** Let V be any volume enclosed by a closed surface S. Then by Gauss' divergence theorem. We get,

$$\iiint_V \nabla.(curl\ \mathrm{A})dv = \iint_S (curl\ \mathrm{A}).nds \qquad \text{..... (1)}$$




Divide the surface S into two portions $S_1$ and $S_2$ by a closed curve C.

Then $\iint_S (curl\ \mathrm{A}).n\ ds$

$$= \iint_{S_1} (curl\ \mathrm{A}).n\ ds_1 + \iint_{S_2} (curl\ \mathrm{A}).\mathrm{n}\ \mathrm{ds}_2$$

$$= \int_C \mathrm{A}.dr - \int_C \mathrm{A}.dr$$

= 0, by Soke's theorem, since the +ve directions along the boundaries of $S_1$ and $S_2$ are opposite.

$\therefore$ (1) $\Rightarrow \iiint \nabla.(curl\, A) dv = 0$

Since this is true for all volume elements V, we have,

$\nabla$ . curl A = 0 $\Rightarrow$ div(curl A) = 0

**Ex.7.12.8 :** Use Stoke's theorem and prove that curl grad $f$ = 0, where 'f' is a scalar function.

**Sol :** If S is a surface enclosed by a simple closed curve C, we have, by Stoke's theorem,

$$\iint_S \{curl(grad\ f)\}.n\, ds = \int_C (grad\ f).dr \qquad \text{.... (1)}$$

Now, grad $f$. dr

$$= (\frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k).(dx\, i + dy\, j + dz\, k)$$

$$= \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz = df$$

$$\therefore \int_C (grad\ f).dr = \int_A^A df$$

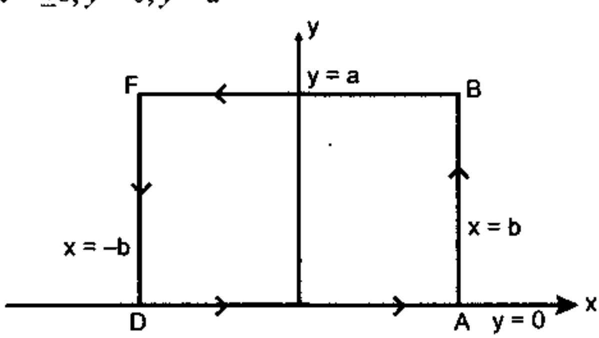
= f(A) – f(A) = 0, where A is any point on C

$$\therefore (1) \Rightarrow \iint_S \{curl(grad\ f)\}.nds = 0$$

Since this equation is true for all surface elements S,

we have, curl (grad $f$) = 0

**Ex.7.12.9 :** Verify stoke's theorem for A = $y^2 j - 2xyj$ taken round the rectangle bounded by $x = \pm$b, $y$ = 0, $y$ = a



**Sol :** curl A = $\begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y^2 & -2xy & 0 \end{vmatrix} = -4yk$

For the given surface S, n = k

$\therefore$ (curl A). n = $-4y$

Hence $\iint_s (curlA).n\, ds$

$$= \iint_s -4y\,dx\,dy = \int_{y=0}^{a}\left[\int_{x=-b}^{b} -4y dx\right] dy = \int_0^a -4xy\Big|_{-b}^{b} dx = \int_0^a -8by\,dy$$

$$= -4by^2\Big|_0^a = -4a^2b \qquad \text{..... (1)}$$

$$\int_C A.dr = \int_{DA} + \int_{AB} + \int_{BF} + \int_{FD}$$

$$\int A.dr = y^2dx - 2xydy$$

Along DA, $y = 0, dy = 0, \Rightarrow \int_{DA} A.dr = 0$ $\qquad (\because A.dr = 0)$

Along AB, $x = b, dx = 0$

$$\therefore \int_{AB} A.dr = \int_{y=0}^{a} -2by\,dy = -by^2\Big|_0^a = -a^2b$$

Along BF, $y = a, dy = 0$

$$\therefore \int_{BF} A.dr = \int_b^{-b} a^2\ dx = -2a^2b$$

Along FD, $x = -b, dx = 0$

$$\therefore \int_{FD} A.dr = \int_a^0 2by\,dy = -by^2\Big|_a^0 = -a^2b$$

$$\therefore \int_C A.dr = 0 - a^2b - 2a^2b - a^2b = -4a^2b \qquad \text{..... (2)}$$

From (1) and (2), $\int_C A.dr = \iint_S curl(A).nds$

Hence the theorem is verified

**Ex.7.12.10:** Use stoke's theorem to evaluate the integral $\int_C A.dr$ where $A = 2y^2 i + 3x^2 j - (2x + z)k$, and C is the boundary of the triangle whose vertices are (0,0,0), (2,0,0), (2,2,0).

**Sol:** $\text{Curl } A = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2y^2 & 3x^2 & -2x - z \end{vmatrix}$

$= 2j + (6x - 4y)k$

Since the z–coordinate of each vertex of the triangle is zero, the triangle lies in the xy–plane.

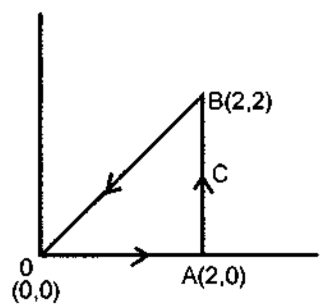
$\therefore n = k.$

$\therefore$ (curl A). n $= 6x - 4y$

consider the triangle in xy-plane.

Equation of the straight line OB is $y = x$

By Stoke's theorem,




$$\int_C A.dr = \iint_S (Curl A).n ds$$

$$= \int_{x=0}^{2} \int_{y=0}^{y=x} (6x - 4y) dx dy = \int_{x=0}^{2} \left[ \int_{y=0}^{x} (6x - 4y) dy \right] dx$$

$$= \int_{x=0}^{2} 6xy - 2y^2 \Big|_0^x dx = \int_0^2 (6x^2 - 2x^2) dx = 4 \frac{x^3}{3} \Big|_0^2 = \frac{32}{3}.$$

**Ex. 7.12.11** Use Stoke's theorem to evaluate $\iint (curl A).n ds$, where $A = 2yi + (x - 2zx)j + xyk$, and S is the surface of the sphere $x^2 + y^2 + z^2 = b^2$ above the xy-plane.

**Sol:** The boundary C of the surface S is the circle $x^2 + y^2 + z^2 = b^2$, $z = 0$.

The parametric equations of C are $x = b\cos\theta$, $y = b\sin\theta$, $z = 0$, $0 \le \theta < 2\pi$

By Stoke's theorem, we have,

$$\iint_S (Curl A).n ds = \int_C A.dr$$

$$= \int_C 2ydx + (x-2zx)dy + xydz = \int_C 2ydx + xdy \qquad (\because z = 0, dz = 0 \text{ on C})$$

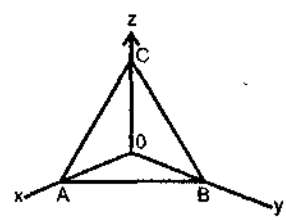
$$= \int_0^{2\pi} (2b\sin\theta)(-b\sin\theta)d\theta + b\cos\theta . b\cos\theta . d\theta \qquad [\because x = b\cos\theta \Rightarrow dx = -b\sin\theta\, d\theta$$

and $y = b\sin\theta \Rightarrow dy = b\cos\theta\, d\theta]$

$$= b^2 \int_0^{2\pi} (\cos^2\theta - 2\sin^2\theta)d\theta = b^2 \int_0^{2\pi} \left[\frac{1+\cos 2\theta}{2} - (1-\cos 2\theta)\right] d\theta$$

$$= \frac{b^2}{2}\int_0^{2\pi} (-1 + 3\cos 2\theta) = +\frac{b^2}{2}\left[-\theta + \frac{3\sin 2\theta}{2}\right]_0^{2\pi} = -\frac{b^2}{2}.2\pi = -\pi b^2$$

**Ex. 7.12.12** Apply Stoke's theorem to evaluate $\int A.dr$ .where A = $(x-y)i + (2y+z)j + (y-z)k$ and C is the boundary of the triangle whose vertices are $\left(\frac{1}{6},0,0\right)$ $\left(0,\frac{1}{3},0\right)$ and $\left(0,0,\frac{1}{2}\right)$.




**Sol:** Let A = $\left(\frac{1}{6},0,0\right)$ B = $\left(0,\frac{1}{3},0\right)$ C = $\left(0,0,\frac{1}{2}\right)$.

The equation of the plane ABC is (by intercept form),

$$\frac{x}{1/6} + \frac{y}{1/3} + \frac{z}{1/2} = 1 \Rightarrow 6x + 3y + 2z = 1 \qquad ..... (1)$$

The direction ratios of the normal to (1) are 6, 3, 2

$\therefore$ Direction cosines are $\frac{6}{7},\frac{3}{7},\frac{2}{7}$ $\qquad \left(\because \sqrt{6^2+3^2+2^2} = 7\right)$

If n is the unit normal to the plane, n = $\frac{6}{7}i + \frac{3}{7}j + \frac{2}{7}k$

A = $(x-y)i + (2y+z)j + (y-z)k$

$$\nabla \times A = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x-y & 2y+z & y-z \end{vmatrix} = k$$

$$\therefore (\nabla \times A).n = \frac{2}{7}$$

$\therefore$ By Stoke's theorem,

$$\int_C \mathbf{A}.dr = \iint_S (\nabla \mathbf{F}).\mathbf{n}\ ds = \frac{2}{7}\iint_S ds = \frac{2}{7} \quad \text{(Area of triangle ABC)} \qquad .....(2)$$

To find the area of triangle $ABC$:

$$AB = \sqrt{\left(\frac{1}{6}\right)^2 + \left(\frac{1}{3}\right)^2} = \frac{\sqrt{5}}{6},$$

$$AC = \sqrt{\left(\frac{1}{6}\right)^2 + \left(\frac{1}{2}\right)^2} = \frac{\sqrt{10}}{6}$$

Direction ratios of AB are $\frac{-1}{6}, \frac{1}{3}, 0$ .

Direction ratios of AC are $\frac{-1}{\sqrt{5}}, \frac{2}{\sqrt{5}}, 0$

Direction ratios of AC are $\frac{-1}{\sqrt{10}}, 0, \frac{3}{\sqrt{10}}$

$$\cos C\hat{A}B = \left(\frac{-1}{\sqrt{5}}\right)\left(\frac{-1}{\sqrt{10}}\right) + 0 + 0 = \frac{1}{\sqrt{50}}$$

$$\sin C\hat{A}B = \frac{7}{\sqrt{50}}$$

$$\therefore \text{ Area of triangle } ABC = \frac{1}{2} AB.AC \sin C\hat{A}B = \frac{1}{2}.\frac{\sqrt{5}}{6}.\frac{\sqrt{10}}{6}.\frac{7}{\sqrt{50}} = \frac{7}{72}$$

$$\int_C \mathbf{A}.dr = \frac{2}{7} \times \frac{7}{72} = \frac{1}{36} \text{ [from(2)]}$$

**Ex.7.12.13** Evaluate $\iint_S (curl\ \mathbf{A}).n\ ds$ taken over the portion $s$ of the surface $x^2 + y^2 + z^2 - 2fx + fz = 0$ above the $xy$ *plane* $z = 0$, if $A = \sum (x^2 + y^2 - z^2)i$ and verify Stoke's theorem.

**Solution:** Let 'S' denote the portion of the surface, $x^2 + y^2 + z^2 - 2fx + fz = 0$ above the *xy-plane z= 0.*

The surface S meets the $xy$-plane in the circle 'C', whose equations are $x^2 + y^2 - 2fx = 0, z = 0$.

$\Rightarrow (x-f)^2 + y^2 = f^2$, z = 0

$\therefore$ The parametric equations of 'C' can be taken as

$x = f + f\cos\theta, \qquad y = f\sin\theta, \qquad z = 0 \qquad\qquad (0 \le \theta < 2\pi)$

Let $S_1$ denote the plane region bounded by C. If $S^1$ is the surface consisting of S and $S_1$, $S^1$ is a closed surface.

$\therefore$ From example 7.11.8 on divergence theorem, we have

$$= 0$$

i.e., $\int\int_S (Curl\ A).2\,ds + \int\int_{S_1} (Curl\ A).n\,ds = 0$ $\qquad$ [$\because$ $S^1$ consists of S and $S_1$]

i.e., $\int\int_S (Curl\ A).n\,ds - \int\int_{S_1} (curlA).kds = 0$ $\qquad$ [$\because$ n = – k on $S_1$]

$\therefore \int\int_S (Curl\ A).n\,ds = \int\int_{S_1} (curlA).kds$ $\qquad$ ..... (1)

Now, curl A = $\begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2+y^2-z^2 & y^2+z^2-x^2 & z^2+x^2-y^2 \end{vmatrix}$

$$= i(-2y-2z) + j(-2z-2x) + k(-2x-2y)$$

$\therefore$ (curl A). $k = -2(x+y)$

$\therefore$ From (1), $\int\int_S (curlA).nds = -2\int\int_{S^1} (x+y)ds$

Polar equation of $S_1$ is $r = 2f\cos\theta$.

$\therefore$ changing to polar coordinates,

$$\int\int_{S_1} (curlA).nds = -2\int_{\theta=0}^{\pi}\int_{r=0}^{2f\cos\theta} (r\cos\theta + r\sin\theta) r d\theta dr$$

$$= -2\int_{\theta=0}^{\pi}\left[\int_{r=0}^{2f\cos\theta} (\cos\theta+\sin\theta) r^2.dr\right] d\theta = -2\int_{\theta=0}^{\pi} (\cos\theta+\sin\theta)\frac{r^3}{3}\Big|_0^{2f\cos\theta} d\theta$$

$$= -2\times\frac{8f^3}{3}\int_0^{\pi}(\cos\theta+\sin\theta)\cos^3\theta d\theta = -\frac{16f^3}{3}\int_0^{\pi}\cos^4\theta d\theta \quad [\because \int_0^{\pi}\cos^3\theta\sin\theta d\theta = 0]$$

$$= \frac{16f^3}{3}\times 2\int_0^{\pi/2}\cos^4\theta d\theta = -\frac{32f^3}{3}.\frac{3}{4}\times\frac{1}{2}\times\frac{\pi}{2} = -2\pi f^3 \qquad \text{..... (2)}$$

Again, $\int_C A.dr$

$$= \int_C (x^2+y^2-z^2)dx + (y^2+z^2-x^2)dy + (z^2+x^2-y^2)dz$$

$$= \int_C (x^2+y^2)dx + (y^2-x^2)dy \qquad [\because \text{ on C} \quad z=0, \quad dz=0]$$

$$= \int_0^{2\pi}\left[(f+f\cos\theta)^2+f^2\sin^2\theta\right](-f\sin\theta)d\theta$$

$$+ \int_0^{2\pi}\left[f^2\sin^2\theta-(f+f\cos\theta)^2\right](f\cos\theta)d\theta$$

$$f^3\left[\int_0^{2\pi}\left[\left\{(1+\cos\theta)^2+\sin^2\theta\right\}(-\sin\theta)+\left\{\sin^2\theta-\left(1+\cos\theta^2\right)\right\}\cos\theta\right]d\theta\right.$$

$$= f^3\left[\int_0^{2\pi}-2\sin\theta d\theta+\int_0^{2\pi}-2\sin 2\theta d\theta+\int_0^{2\pi}-2\cos\theta d\theta+\int_0^{2\pi}-\cos^3\theta d\theta\right.$$

$$\left.-\int_0^{2\pi}(1+\cos 2\theta)d\theta+\int_0^{2\pi}\sin^2\theta\cos\theta d\theta\right] \qquad \text{(on simplification)}$$

$$= f^3(-2\pi) = -2\pi f^3 \qquad \text{(all other integrals vanish)} \qquad \text{.....(3)}$$

from (2) and (3), we have

$$\iint_S (Curl A).n ds = \int_C A.dr$$

which verifies Stoke's theorem.

## Exercise-7(*I*)

1. *f* $F=(xe^{x})i+(3y^{2})j-(z)dz$, and C is the $x^{2}+y^{2}=9, z=2$, evaluate $\oint_C F.dr$ using Stoke's theorem.

[Ans: 0]

2. Apply Stoke's theorem to obtain the value of the integral $\oint_C V.dr$, where $V=(3y^{2})i+(2x^{2})j-(x+2z)k$ and C is the boundary of the triangle whose vertices are (0,0,0),(1,0,0) and (1,1,0)

[Ans: 1]

3. Verify Stoke's theorem for $F=(2x-y)i-(yz^{2})j-(y^{2}z)k$ if S is the upper half surface of the sphere $x^{2}+y^{2}+z^{2}=1$ and C is the boundary .

[Ans: $\pi$ ]

4. If $F=(y-z+2)i+(yz+4)j-(xz)k$ and S represents the surface of the cube $x=0,\ y=0,\ z=0,\ x=2,\ y=2,\ z=2$ above the *xy plane* , verify that $\iint_S (Curl\ F).ds=\oint_C F.dr$, C being the boundary of S traversed in the +ve direction.

[Ans: Each integral =-4]

5. Find the value of the integral $\oint_C (yz)dx+(zx)dy+(xy)dz$, using Stoke's theorem where C is the Curve $x^{2}+y^{2}=4, z=y^{2}$

[Ans:0]

6. Verify Stoke's theorem for the function $V=(3x^{2})i+(2xy)j$ , integrated along the square x = 0, y = 0, x = 1,y =1 in the *xy-plane*.. [Ans: 0]

7. Evaluate by Stoke's theorem the integral $\oint_C A.dr$ where $A=(2\sin z)i-(3\cos x)j+(\sin y)k$; where C is the boundary of the rectangle $0\le x\le\pi,\ 0\le y\le 1,\ z=0$
[Ans:6]

## Exercise – 7(m)

1. If $f(x,y,z) = x^l y^m z^n - 1$ , find the directional derivative of $f$ at (1,1,1) in the direction of (i+2j+2k)

[Ans: $\frac{1}{3}(l+2m+2n)$]

2. Find the acute angle between the surfaces

$x^2+y^2+z^2=6$ and $3xyz+y^2z-xy+3=0$ at (1, -1, 2)

[Ans: $\cos^{-1}\left(\frac{\sqrt{5}}{3}\right)$]

3. If $r = xi + yj + zk$, and $p,q$ are constant vectors, show that $Div\{(r\times p)\times q\} = -2(p,q)$

4. If $F=(x^2y)i-(y^2z)j+(z^2x)k$, find curl F at (1,-2,3)

5. If $f = xyz(x+y+z)$, prove that $curl\ grad\, f = 0$

6. A fluid motion is given by $V=(z^3)i-(y^2)j+(3xz^2)k$. Show that it is irrotational. Find its velocity potential $\phi$ such that $V=\nabla\phi$

[Ans : $\phi = xz^3 - \frac{y^3}{3} + c$]

7. Evaluate $\int_{(0,0)}^{(2,1)}(4x^3-12x^2y^2)dx-(8x^3y)dy$ along the path $x^3-3xy^2=2y^3$

[Ans : 16]

8. If $V=(xy)i-(yz)j+(zx^2)k$ and S is the surface of the cube bounded by $x=0, x=2, y=0, y=2, z=0, z=2$, evaluate $\int_S V.n\ ds$

[Ans : 32/3]

9. If $f = 4x + yz$, evaluate $\iiint_V f dv$ over the region in the first octant bounded by

$x^2+y^2=1,\ \ z=0,\ \ z=3$ [Ans : 11/2]

10. Express $F = xi - y^2 j + zk$ in (a) cylindrical polar coordinates (b) spherical polar coordinates.

(Ans: (a) $\rho(\cos^2\theta - \rho\sin^3\theta).e_\rho - \rho\sin\theta\cos\theta(1+\rho\sin\theta).e_\theta + ze_z$

(b) $(r\sin^2\theta\cos^2\phi - r^2\sin^3\theta\sin^3\phi + r\cos^2\theta)e_r$

$+(r\sin\theta\cos\theta\cos^2\phi - r^2\sin^2\theta\cos\theta\sin^3\phi - r\sin\theta\cos\theta)e_\theta$

$+ (-r\sin\theta\sin\phi\cos\phi - r^2\sin^2\theta\sin^2\phi\cos\phi)e_\phi$]

11. If $f = \rho z\sin 2\theta$, find $grad\, f$ in cylindrical coordinates

[$(z\sin 2\theta)e_\rho + (2z\cos 2\theta)e_\theta + (\rho\sin 2\theta)e_z$

12. If $\mathbf{A} = (r\cos\theta)e_r - (\frac{1}{r}\sin\theta)e_\theta + re_\phi$, find the $curl\mathbf{A}$ in spherical coordinates

[Ans: $(\cot\theta)e_r + 2e_\theta - (2\sin\theta)e_\phi$]

13. Verify Green's theorem in the plane for $\int_C (x^3 - y^2)dx + (x^2 - 2xy)dy$, where C is the boundary of the square bounded by $0 \le x \le 1,\ \ 0 \le y \le 1$ [Ans:1]

14. Using Gauss divergence theorem, prove that $\iint_S F.n\, ds = \pi r^2 l^2$, where $f = (y^2 z)i + (xz)j + (z^2)k$, and S is the surface bounded by $x^2 + y^2 = r^2,\ \ z = 0,$ and $z = l$

15. Show that the Stoke's theorem, when restricted to the *xy-plane*, is Green's theorem in the plane.

(Hint : In Stoke's theorem, take $\mathbf{A} = Pi + Qj;\ \ n = k;$ and $ds = dxdy$)

## Exercise-7(n)

**I. Choose the correct answer in the following questions**

1. The tangent vector at the point t=1 on the curve $x = t^2 + 1, y = 4t - 3, z = t^3$ is

(a) 2i-4j+3k (b) 2i+4j+3k (c) 2i-4j-3k (d) 2i+4j-3k [b]

2. The magnitude of acceleration at $\theta = 0$ on the curve $x = 2\cos 3\theta, y = 2\sin 3\theta, z = 3\theta$ is

(a) 6 (b) 9 (c) 18 (d) 3 [c]

3. if $f = xyz$, the value of $|grad\ f|$ at the point (1, 2, -1) is

(a) 0 (b) 1 (c) 2 (d) 3 [d]

4. The maximum rate of change of $f = xy^2 + yz + zx^2$ at the point (1, 1, 1) is

(a) $\sqrt{11}$ (b) 0 (c) 3 (d) none [d]

5. The angle between the normals to the sphere $x^2 + y^2 + z^2 = 9$ at the points (1, 2, 2) and (2, 1, 2) is

(a) $\cos^{-1}(8/9)$ (b) $\frac{\pi}{2}$ (c) $\cos^{-1}(3/4)$ (d) $\frac{\pi}{4}$ [a]

6. If a is a constant vector and $r = xi + yj + zk$, then $\nabla(a.r)$ is

(a) 0 (b) a (c) **r** (d) r [b]

7. If $(x+3y)i + (2-3y)j + (x+az)k$ is a solinoidal vector, the value of a is

(a) 0 (b) 1 (c) 2 (d) 3 [c]

8. If $r = xi + yj + zk$, and **a** $= \frac{1}{3}r$, $div\ \vec{a} =$

(a) 0 (b) 1 (c) -1 (d) 2 [b]

## Exercise - 7(O)

1. If $f(x,yx) = x^l y^m z^n - 1$, find the directional derivative of f at (1,1,1) in the direction of (i+2j+2k) [Ans: $\frac{1}{3}(l+2m+2n)$]

2. Find the acute angle between the surfaces $x^2 + y^2 + z^2 = 6$ and $3xyz + y^2z - xy + 3 = 0$ at $(1,-1,2)$ [Ans: $\cos^{-1}\left(\frac{\sqrt{5}}{3}\right)$]

3. If $r = xi + yj + zk$, and p, q, are constant vectors show that
Div $\{r \times p\} \times q\} = -2(p.q)$

4. If $F = (x^2y)i - (y^2z)j + (z^2x)k$, find curl curl F at $(1,-2,3)$
[Ans: $6i - 2j + 4k$]

5. If $f = xyz(x + y + z)$, prove that curl grad $f = 0$

6. A fluid motion is given by $V = (z^3)i - (y^2)j + (3xz^2)k$. Show that it is irrotational. Find its velocity potential $\phi$ such that $V = \nabla\phi$
[Ans: $\phi = xz^3 - \frac{y^3}{3} + c$]

7. Evaluate $\int_{(0,0)}^{(2,1)} (4x^3 - 12x^2y^2)dx - (8x^3y)dy$ along the path $x^3 - 3xy^2 = 2y^3$
[Ans: 16]

8. If $V = (xy)i - (yz)j + (zx^2)k$ and S is the surface of the cube bounded by $x = 0$, $x = 2$, $y = 0$, $y = 2$, $z = 0$ and $z = 2$, evaluate $\int_S V.nds$
[Ans: 32/3]

9. If f = 4x + yz, evaluate $\iiint_V f dv$ over the region in the first octant bounded by $x^2 + y^2 = 1$, z = 0, z = 3
[Ans: 11/2]

10. Express $F = xi - y^2j + zk$ in

    a) cylindrical polar coordinates

    b) spherical polar coordinates.

[Ans: a) $\rho(\cos^2\theta - \rho\sin^3\theta)e_\rho - \rho\sin\theta\cos\theta(1 + \rho\sin\theta).e_\theta + ze_z$

b) $(r\sin^2\theta\cos^2\phi - r^2\sin^3\theta\sin^3\phi + r\cos^2\theta)e_r + (r\sin\theta\cos\theta\cos^2\phi - r^2\sin^2\theta\cos\theta\sin^3\phi - r\sin\theta\cos\theta)e_\theta + (-r\sin\theta\sin\phi\cos\phi - r^2\sin^2\theta\sin^2\phi\cos\phi)e_\phi$]

11. If $f = \rho z\sin 2\theta$, find grad $f$ in cylindrical coordinates

[Ans: $(z\sin 2\theta)e_\rho + (2x\cos 2\theta)e_\theta + (\rho\sin 2\theta)e_z$]

12. If $A = (r\cos\theta)e_r - \left(\frac{1}{r}\sin\theta\right)e_\theta + re_\phi$, find curl A in spherical coordinates

[Ans: $(\cot\theta)e_r + 2e_\theta - (\sin\theta)e_\phi$]

13. Verify Green's theorem in the plane for $\int_C (x^3 - y^2)dx + (x^2 - 2xy)dy$, where C is the boundary of the square bounded by $0 \le x \le 1$, $0 \le y \le 1$

[Ans: 1]

14. Using Gauss' divergence theorem, prove that $\iint_S F.nds = \pi r^2 l^2$, where $F = (y^2z)i + (xz)j + (z^2)k$, and S is the surface bounded by $x^2 + y^2 = r^2$, z = 0, and z = 1

15. Show that the Stoke's theorem, when restricted to the xy plane, is Green's theorem in the plane

    [*Hint:* In Stoke's theorem, take A = P$i$ + Q$j$; $n = k$; and d$s$ = d$x$ d$y$]