Unit - V

Higher Order Differential Equations.

* Higher Order linear Differential Equations with Constant Coefficients :-

The general form of a linear differential equation (L.D.E) with constant co-efficient of order n is

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots + a_{n-1} \frac{dy}{dx} + a_n y$$

$$= Q(x).$$

Let $D = \frac{d}{dx}$, then above equation can be written as

$$a_0 D^n y + a_1 D^{n-1} y + a_2 D^{n-2} y + \cdots + a_{n-1} D y + a_n y = Q(x)$$

i.e. $[a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + \cdots + a_{n-1} D + a_n] y = Q(x)$

i.e. $f(D) y = Q(x)$ ——— (1).

• where ~~f(D) y = Q~~

$f(D)$ is a polynomial in D of degree n. The general solution of equation (1) is given by

$$\boxed{y = C.F. + P.I}$$

where C.F i.e. Complementory function is the solution of the equation $f(D) y = 0$ which is called homogeneous equation (i.e $Q(x) = 0$) and P.I = Particular Integral which is defined by

$$P.I = \frac{1}{f(D)} Q(x)$$

* • Method to find Complementary Function :- (C.F)

1) To find Complementary function (C.F), we consider

$f(D)y = 0$, where.

$f(D) = a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + \dots + a_{n-1} D + a_n$

2) Put $D = m$ if $f(D) = 0$, we get

$a_0 m^n + a_1 m^{n-1} + a_2 m^{n-2} + \dots + a_{n-1} m + a_n = 0.$

which is called as auxillary equation (A.E)

In general, The (A.E) has 'n' roots say $m_1, m_2 \dots m_n$

3) Depending on the natures of the roots there arises four cases for C.F. as below.

Case I :-

If all the roots are real and different, then

$C.F = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \dots + c_n e^{m_n x}$

Case II :-

If two of the roots are real and equal i.e. $m_1 = m_2 = m$ and $m_3, m_4, \dots m_n$ are as usual real and different then

$C.F = (c_1 + c_2 x) e^{mx} + c_3 e^{m_3 x} + c_4 e^{m_4 x} + \dots + c_n e^{m_n x}$

Similarly, when $m_1 = m_2 = m_3 = m$ and remaining $m_4, m_5, \dots m_n$ are real and different then

$C.F = (c_1 + c_2 x + c_3 x^2) e^{mx} + c_4 e^{m_4 x} + \dots + c_n e^{m_n x}$ and so on

Case - III :-

If one pair of the roots be imaginary i.e. $m_1 = \alpha + i\beta$ & $m_2 = \alpha - i\beta$ where α and β being real and all other roots are as usual, then

$$C.F = [c_1 \cos\beta x + c_2 \sin\beta x] e^{\alpha x} + c_3 e^{m_3 x} + \cdots + c_n e^{m_n x}$$

Case - IV :-

If two pairs of imaginary roots be equal i.e. $m_1 = m_2 = \alpha + i\beta$ & $m_3 = m_4 = \alpha - i\beta$ & all all other roots i.e. $m_5, m_6, \ldots, m_n$ are real and different then

$$C.F = [(c_1 + c_2 x)\cos\beta x + (c_3 + c_4 x)\sin\beta x] e^{\alpha x} + c_5 e^{m_5 x} + c_6 e^{m_6 x} + \cdots + c_n e^{m_n x}$$

Que.1 Solve $(D^3 + D^2 + 4D + 4)y = 0$.

Soln:- The given D.E is

$(D^3 + D^2 + 4D + 4)y = 0$.

The A.E is given by

$$m^3 + m^2 + 4m + 4 = 0$$
$$m(m^2+4) + m^2 + 4 = 0$$
$$(m+1)(m^2+4) = 0.$$

Its roots are $m = -1$ & $\pm 2i$

$\therefore$ Let $m_1 = -1$, $m_2 = 2i$ & $m_3 = -2i$

$$C.F = c_1 e^{m_1 x} + e^{\alpha x}[c_2 \cos\beta x + c_3 \sin\beta x]$$

$$= c_1 e^{-x} + e^{0x}[c_2 \cos 2x + c_3 \sin 2x]$$

$[\because \alpha = 0 \text{ \& } \beta = 2]$

$\therefore$ The required solution is,

$$\boxed{C.F = c_1 e^{-x} + (c_2 \cos 2x + c_3 \sin 2x)}$$

* Methods to Find Particular Integral (P.I) :-

Consider $F(D)y = Q(x)$, where.

$$F(D) = (a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + \dots + a_{n-1} D + a_n)$$

Then $P.I = \frac{1}{F(D)} Q(x)$.

Now, Depending on the form of the function $Q(x)$, there arises six different cases for P.I as below.

Case - I :-

Let $Q(x) = e^{ax}$, where a is any constant.

Then $P.I = \frac{1}{F(D)} e^{ax}$.

$= \frac{1}{F(a)} e^{ax}$, Provided $F(a) \neq 0$.

i.e. replace 'D' by 'a' in $F(D)$.

Note :- If $F(a) = 0$ then.

$P.I = x \cdot \frac{1}{F'(a)} e^{ax}$ Provided $F'(a) \neq 0$.

If $F'(a) = 0$ then.

$P.I. = x^2 \frac{1}{F''(a)} e^{ax}$, Provided $F''(a) \neq 0$.

If $F''(a) = 0$.

$P.I = x^3 \frac{1}{F'''(a)} e^{ax}$, Provided $F'''(a) \neq 0$ & so on

Remark :-

If $Q(x) = K$, any constant, then.

$$P.I = \frac{1}{f(D)} K = K \frac{1}{f(D)} e^{0x}$$

$$= K \frac{1}{f(0)} e^{0x}$$, provided $f(0) \neq 0$ & proceding as case - I.

Examples ON Case - I.

Que.2 Solve :- $(D^3+8)y = 4 + e^{-2x}$.

Sol^n:- The given D.E is $(D^3+8)y = 4 + e^{-2x}$

Its A.E is $m^3+8=0$

$\Rightarrow m = -2, 1 \pm \sqrt{3}i$

$\therefore C.F = C_1 e^{-2x} + e^{x}(C_2 \cos\sqrt{3}x + C_3 \sin\sqrt{3}\cdot x)$ ——— (1)

Now

$$P.I = \frac{1}{D^3+8}(4+e^{-2x})$$

$$= \frac{1}{D^3+8}.4 + \frac{1}{D^3+8} e^{-2x}$$

$$= 4.\frac{1}{D^3+8} e^{0x} + \frac{1}{(-2)^3+8} e^{-2x}.$$

$$= 4.\frac{1}{0+8} + \frac{1}{0} e^{-2x}$$ .... (case fail's for 2nd term)

$$= \frac{1}{2} + x.\frac{1}{3D^2} e^{-2x}.$$

$$= \frac{1}{2} + x \frac{1}{3D^2} e^{-2x}$$

$$= \frac{1}{2} + x . \frac{1}{3(-2)^2} e^{-2x}$$

$$= \frac{1}{2} + \frac{x}{12} e^{-2x} \quad \text{------ (2)}$$

$\therefore$ The general solution of given D.E is given by

$$y = C.F + P.I$$

$$= C_1 e^{-2x} + e^{x}(C_2 \cos\sqrt{3}x + C_3 \sin\sqrt{3}x) + \frac{1}{2} + \frac{x}{12} e^{-2x}$$

[$\because$ By (1) and (2)]

$\therefore$ The required solution is

$$\boxed{y = C_1 e^{-2x} + e^{x}(C_2 \cos\sqrt{3}x + C_3 \sin\sqrt{3}x) + \frac{1}{2} + \frac{x}{12} e^{-2x}}$$

Que 3 Solve $(D+2)(D-1)^2 y = e^{-2x} + 2\sinh x$.

Soln:- The given D.E. is $(D+2)(D-1)^2 y = e^{-2x} + 2\sinh x$

$\therefore$ Its A.E is

$$(m+1)(m-1)^2 = 0.$$

$$\Rightarrow m = 1, 1, -2$$

$$\therefore C.F = (C_1 + C_2 x) e^{x} + C_3 e^{-2x} \quad \text{------ (1)}$$

Now $P.I = \frac{1}{f(D)} [e^{-2x} + 2\sinh x]$

$$= \frac{1}{(D+2)(D-1)^2} \left[e^{-2x} + 2 . \frac{e^{x} - e^{-x}}{2} \right].$$

$$\left[\because \sinh x = \frac{e^{x} - e^{-x}}{2} \right.$$

$$= \frac{1}{(D+2)(D-1)^2} e^{-2x} + \frac{1}{(D+2)(D-1)^2} e^{x} - \frac{1}{(D+2)(D-1)^2} e^{-x}$$

$$= \frac{1}{(-2+2)(2-1)^2} e^{-2x} + \frac{1}{(1+2)(1-1)^2} e^{x} - \frac{1}{(-1+2)(-1-1)^2} e^{-x}.$$

$$= \frac{1}{0} e^{-2x} + \frac{1}{0} e^{x} - \frac{1}{4} e^{x}.$$

$$= \text{Case fails} + \text{case fails} - \frac{1}{4} e^{-x}.$$

$$\therefore P.I = x . \frac{1}{(D-1)^2 + 2(D+2)(D-1)} e^{-2x} + x . \frac{1}{(D-1)^2 + 2(D+2)(D-1)} e^{x} - \frac{1}{4} e^{-x}.$$

$$= x . \frac{1}{(-2-1)^2 + 0} e^{-2x} + x . \frac{1}{0 + 2(1+2)(1-1)} e^{x} - \frac{1}{4} e^{-x}$$

$$= \frac{x}{9} e^{-2x} + \text{Case fails} - \frac{1}{4} e^{-x}.$$

$$\therefore P.I = \frac{x}{9} e^{-2x} - \frac{1}{4} e^{-x} + x^2 \frac{1}{2(D-1) + 2(D-1) + 2(D+2)} e^{x}$$

$$= \frac{x}{9} e^{-2x} - \frac{1}{4} e^{-x} + x^2 \frac{1}{0+0+2(1+2)} e^{x}.$$

$$= \frac{x}{9} e^{-2x} - \frac{1}{4} e^{-x} + x^2 \frac{1}{6} e^{x}$$

$$\therefore P.I = \frac{x^2}{6} e^{x} + \frac{x}{9} e^{-2x} - \frac{1}{4} e^{-x} \quad \text{——— (2)}$$

$\therefore$ The general solution of given D.E is

$$y = C.F + P.I$$
$$= (c_1 + c_2 x)e^{x} + c_3 e^{-2x} + \frac{x^2}{6}e^{x} + \frac{x}{9}e^{-2x} - \frac{1}{4}e^{-x}.$$

$\therefore$ The required solution is
$$\boxed{y = (C_1 + C_2 x)e^{x} + C_3 e^{-2x} + \frac{x^2}{6}e^{x} + \frac{x}{9}e^{-2x} - \frac{1}{4}e^{-x}}$$

* **Case II :-**

Let $Q(x)$ = $\sin ax$ or $\cos ax$, where a is any constant.

Then

$$P.I = \frac{1}{f(D)} \sin ax \text{ or } \cos ax.$$
$$= \frac{1}{\phi(D^2)} \sin ax \text{ or } \cos ax.$$
$$= \frac{1}{\phi(-a^2)} \sin ax \text{ or } \cos ax, \text{ provided } \phi(-a^2) \neq 0$$

i.e. First convert the given $F(D)$ as a function of D^2.
i.e. $\phi(D^2)$ if it is not, then replace D^2 by $-(a)^2$.

Note :- If $(\phi(-a)^2 = 0$ then

$$P.I = x.\frac{1}{\phi'(D^2)} \sin ax \text{ or } \cos ax$$

$$\therefore P.I = x.\frac{1}{\phi'(-a)^2} \sin ax \text{ or } \cos ax, \text{ Provided } \phi'(-a)^2 \neq 0.$$

If $\phi'(-a)^2 = 0$ then

$$\therefore P.I = x^2 \frac{1}{\phi''(-a)^2} \sin ax \text{ or } \cos ax, \text{ provided } \phi''(-a)^2 \neq 0.$$

Remark :- If after substituting $D^2 = -a^2$ in the first step, denominator turns into an expression of the type. $(pD+q)$ where p and q are constants. then multiply the numerator and denominator by $(pD-q)$ factor and then put $D^2 = -a^2$ in denominator and finally operate $(pD-q)$ on $\sin ax$ or $\cos ax$.

* Examples on CASE - II :-

Que 4 Solve $(D^2+4)y = \cos 2x$, where $D = \frac{d}{dx}$.

Soln :- Given D.E. is $(D^2+4)y = \cos 2x$.

Its A.E. is

$m^2+4 = 0$

$m = \pm 2i$

$\therefore$ C.F $= c_1 \cos 2x + c_2 \sin 2x$. ——— (1)

Now P.I $= \frac{1}{D^2+4} (\cos 2x)$

$= \frac{1}{-4+4} \cos 2x$ = case fails $[\because D^2 = -(2)^2]$

P.I $= x \cdot \frac{1}{2D} \cos 2x = \frac{x}{2} \frac{1}{D} \cos 2x$

$= \frac{x}{2} \int \cos 2x \, dx = \frac{x}{2} \frac{\sin 2x}{2}$ $\left[\because \frac{1}{D} = \int\right]$

$\therefore$ P.I $= \frac{x \sin 2x}{4}$ — (2)

$\therefore$ General solution is given by

$y =$ C.F + P.I

$= c_1 \cos 2x + c_2 \sin 2x + \frac{x}{4} \sin 2x$. (from (1) & (2))

$\therefore$ The required soln is,

$y = c_1 \cos 2x + c_2 \sin 2x + \frac{x}{4} \sin 2x$.

Que. 5 Solve $(D^4 - 3D^2 - 4)y = 5\sin 2x - e^{-2x}$.

Sol^n:- Given $(D^4 - 3D^2 - 4)y = 5\sin 2x - e^{-2x}$

The A.E. is

$m^4 - 3m^2 - 4 = 0$.

$(m^2-4)(m^2+1) = 0$.

$m^2 = 4 \quad m^2 = -1$

$m = \pm 2, \pm i$

$\therefore$ C.F. $= c_1 e^{2x} + c_2 e^{-2x} + c_3 \cos x + c_4 \sin x$ ——— (1).

Now

$$P.I. = \frac{1}{D^4 - 3D^2 - 4}[5\sin 2x - e^{-2x}]$$

$$= 5 \cdot \frac{1}{D^4 - 3D^2 - 4}\sin 2x - \frac{1}{D^4 - 3D^2 - 4}e^{-2x}.$$

$$= 5\frac{1}{(-4)^2 - 3(-4) - 4}\sin 2x - \frac{1}{(-2)^4 - 3(-2)^2 - 4}e^{-2x}.$$

(using case I & II)

$$= \frac{5}{24}\sin 2x + \text{case fails.}$$

$$\therefore P.I. = \frac{5}{24}\sin 2x - x\frac{1}{4D^3 - 6D}e^{-2x}.$$

$$= \frac{5}{24}\sin 2x - x\frac{1}{4(-2)^3 - 6(-2)}e^{-2x} \quad (\because D = -2)$$

$$= \frac{5}{24}\sin 2x - x\frac{1}{-32+12}e^{-2x}$$

$$P.I. = \frac{5}{24}\sin 2x + \frac{x}{20}e^{-2x} \quad \text{——— (2)}.$$

$\therefore$ The general solution of give D.E is.

$y = C.F. + P.I.$

$= c_1 e^{2x} + c_2 e^{-2x} + c_3 \cos x + c_4 \sin x + \frac{5}{24} \sin 2x$

$+ \frac{x}{20} e^{-2x}$ [$\because$ by (1) & (2)]

$\therefore$ The required solution is

$y = c_1 e^{2x} + c_2 e^{-2x} + c_3 \cos x + c_4 \sin x$
$+ \frac{5}{24} \sin 2x + \frac{x}{20} e^{-2x}.$

Que. 6 Solve $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = \sin hx + \sin\sqrt{2}\,x$.

Sol^n:- The given D.E can be written as.

$(D^2 - 2D + 2)y = \sin hx + \sin\sqrt{2}\,x$

Its A.E is

$m^2 - 2m + 2 = 0.$

$\Rightarrow m = \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm 2i}{2} = 1 \pm i$

$\therefore C.F = e^{x}(c_1 \cos x + c_2 \sin x)$ ——(1).

Now

$P.I = \frac{1}{D^2 - 2D + 2}[\sin hx + \sin\sqrt{2}\,x]$

$= \frac{1}{D^2 - 2D + 2} \sin hx + \frac{1}{D^2 - 2D + 2} \sin\sqrt{2}\,x.$

$$= \frac{1}{D^2-2D+2}\left[\frac{e^x - e^{-x}}{2}\right] + \frac{1}{(-2)-2D+2}\sin\sqrt{2}x.$$

$$\left[\because \sinh x = \frac{e^x - e^{-x}}{2} \text{ and } D^2 = -(\sqrt{2})^2.\right]$$

$$= \frac{1}{2}\frac{1}{D^2-2D+2}e^x - \frac{1}{2}\frac{1}{D^2-2D+2}e^{-x} - \frac{1}{2}\frac{1}{D}\sin\sqrt{2}x.$$

$$= \frac{1}{2}\frac{1}{1-2+2}e^x - \frac{1}{2}\frac{1}{1+2+2}e^{-x} - \frac{1}{2}\int \sin\sqrt{2}x\,dx$$

$$= \frac{1}{2}e^x - \frac{1}{10}e^{-x} - \frac{1}{2}\frac{(-\cos\sqrt{2}x)}{\sqrt{2}}$$

$$\therefore P.I = \frac{1}{2}e^x - \frac{1}{10}e^{-x} + \frac{1}{2\sqrt{2}}\cos\sqrt{2}x. \quad \text{——— (2).}$$

$\therefore$ The general solution of given D.E is

$$y = C.F. + P.I$$

$$= e^x(c_1\cos x + c_2\sin x) + \frac{1}{2}e^x - \frac{1}{10}e^{-x} + \frac{1}{2\sqrt{2}}\cos\sqrt{2}x \quad [\because \text{By (1) and (2)}]$$

$\therefore$ The required solution is

$$y = e^x(c_1\cos x + c_2\sin x) + \frac{1}{2}e^x - \frac{1}{10}e^{-x} + \frac{1}{2\sqrt{2}}\cos\sqrt{2}x.$$

Que. 7 Solve. $(D^3+1)y = \cos^2(x/2) + e^{-x}$.

Sol^n:- Given D.E is

$$(D^3+1)y = \cos^2(x/2) + e^{-x}$$

Its A.E is

$$m^3+1=0$$

$$(m+1)(m^2-m+1)=0$$

$$m = -1,\ \frac{1}{2} \pm \frac{\sqrt{3}}{2}i$$

$$\therefore C.F = c_1 e^{-x} + e^{x/2}\left[c_2 \cos\frac{\sqrt{3}}{2}x + c_3 \sin\frac{\sqrt{3}}{2}x\right] \quad \text{—(1)}$$

Now.

$$P.I = \frac{1}{D^3+1}\left[\cos^2(x/2) + e^{-x}\right]$$

$$= \frac{1}{D^3+1}\cos^2(x/2) + \frac{1}{D^3+1}e^{-x}$$

$$= \frac{1}{D^3+1}\left(\frac{\cos x + 1}{2}\right) + \frac{1}{(-1)+1}e^{-x}$$

$$\left[\because 1 + \cos x = 2\cos^2(x/2)\right]$$

$$= \frac{1}{D^3+1}\left(\frac{1}{2}\right) + \frac{1}{2}\frac{1}{D^3+1}\cos x + \text{case fails}$$

$$= \frac{1}{2}\frac{1}{D^3+1}e^{0x} + \frac{1}{2}\frac{1}{D\,D^2+1}\cos x + x \cdot \frac{1}{3D^2}e^{-x}$$

$$= \frac{1}{2}e^{0x} + \frac{1}{2}\frac{1}{D(-1)+1}\cos x + \frac{x}{3}e^{-x}$$

$$\left[\frac{1}{D^2} = \iint\right]$$

$$= \frac{1}{2} + \frac{x}{3}e^{-x} + \frac{1}{2}\frac{1}{1-D}\cos x.$$

$$= \frac{1}{2} + \frac{x}{3}e^{-x} + \frac{1}{2}\frac{(1+D)}{(1-D)(1+D)}\cos x.$$

$$= \frac{1}{2} + \frac{x}{3}e^{-x} + \frac{1}{2}\frac{1+D}{1-D^2}\cos x.$$

$$= \frac{1}{2} + \frac{x}{3}e^{-x} + \frac{1}{2}\frac{(1+D)}{1-(-1)}\cos x$$

$$= \frac{1}{2} + \frac{x}{3}e^{-x} + \frac{1}{4}(1+D)\cos x.$$

$$P.I = \frac{1}{2} + \frac{x}{3}e^{-x} + \frac{1}{4}\cos x - \frac{1}{4}\sin x. \quad \text{———— (2).}$$

General solution of given D.E is,

$$y = C.F + P.I$$

$$y = c_1 e^{-x} + e^{x/2}\left[c_2\cos\frac{\sqrt{3}}{2}x + c_3\sin\frac{\sqrt{3}}{2}x\right]$$

$$+ \frac{1}{2} + \frac{x}{3}e^{-x} + \frac{1}{4}(\cos x - \sin x)$$

$\because$ By (1) & (2)

Que. 2 Solve D.E $(D^4 - 3D^2 \ldots) y = 24 \sin 2x - 40e^{-2x}$

Que. 8 Solve D.E. $(D^4 - 3D^2 - 4)y = 24\sin 2x - 40e^{-2x}$

Sol^n :- Given D.E is

$(D^4 - 3D^2 - 4)y = 24\sin 2x - 40e^{-2x}$.

Its A.E is

$m^4 - 3m^2 - 4 = 0$

$\Rightarrow m^2 = 4 \quad , \quad m^2 = -1$

$m = \pm 2 \quad , \quad \pm i$

$\therefore$ C.F. $= c_1 e^{2x} + c_2 e^{-2x} + c_3\cos x + c_4 \sin x$ —— (1)

Now, P.I $= \dfrac{1}{D^4 - 3D^2 - 4}[24\sin 2x - 40e^{-2x}]$

$= 24\dfrac{1}{D^4 - 3D^2 - 4}\sin 2x - 40\dfrac{1}{D^4 - 3D^2 - 4}e^{-2x}$

$= 24\dfrac{1}{(-4)^2 - 3(-4) + 4}\sin 2x - 40\dfrac{1}{(-2)^4 - 3(-2)^2 - 4}e^{-2x}$

[$\because D^2 = -4$ in first term and $D = -2$ in 2^{nd} term]

$= 24\dfrac{1}{24}\sin 2x - \dfrac{40}{0}e^{-2x}$.

$= \sin 2x +$ case fails

$\therefore$ P.I $= \sin 2x - 40x\dfrac{1}{4D^3 - 6D}e^{-2x}$

$= \sin 2x - 40x\dfrac{1}{4(-2)^3 - 6(-2)}e^{-2x}$ [$D = -2$]

$= \sin 2x - 40x\dfrac{e^{-2x}}{-20}$

$$P.I = \sin 2x + 2x e^{-2x} \quad \text{------} (2).$$

General solution of given D.E is

$$y = C.F + P.I$$
$$= C_1 e^{2x} + C_2 e^{-2x} + C_3 \cos x + C_4 \sin x + \sin 2x + 2x e^{-2x}$$

[By (1) and (2)]

$\therefore$ The required solution is,

$$y = C_1 e^{2x} + C_2 e^{-2x} + C_3 \cos x + C_4 \sin x + \sin 2x + 2x e^{-2x}.$$

Que: Solve D.E

$$\frac{d^3y}{dt^3} + 3\frac{d^2y}{dt^2} + \frac{dy}{dt} - 5y = \cosh t + \sin t$$

Solⁿ:- The given D.E. can be written as

$$(D^3 + 3D^2 + D - 5)y = \cosh t + \sin t.$$

Its A.E. is given by

$$m^3 + 3m^2 + m - 5 = 0$$
$$\Rightarrow (m-1)(m^2 + 4m + 5) = 0$$
$$\Rightarrow (m-1)(m+2+i)(m+2-i) = 0.$$
$$\Rightarrow m = 1, -2 \pm i$$

$$C.F = c_1 e^{x} + e^{-2x}(c_2 \cos x + c_3 \sin x) \quad \text{------} (1).$$

Now,

$$P.I = \frac{1}{D^3 + 3D^2 + D - 5} \cosh t + \sin t.$$

$$= \frac{1}{D^3 + 3D^2 + D - 5} \cosh t + \frac{1}{D^3 + 3D^2 + D - 5} \sin t.$$

$$\frac{1}{D^3+3D^2+D-5}\cosh t + \frac{1}{D^3+3D^2+D-5}\sin t.$$

$$= \frac{1}{D^3+3D^2+D-5}\left(\frac{e^t+e^{-t}}{2}\right) + \frac{1}{D^3+3D^2+D-5}\sin t.$$

$$= \frac{1}{2}\frac{1}{D^3+3D^2+D-5}e^t + \frac{1}{2}\frac{1}{D^3+3D^2+D-5}e^{-t} + \frac{1}{D^3+3D^2+D-5}\sin t$$

$$= \frac{1}{2}\frac{1}{(1)^3+3(1)^2+(1)-5}e^t + \frac{1}{2}\frac{1}{(-1)^3+3(-1)^2+(-1)-5} + \frac{1}{D^3+3D^2+D-5}\sin t$$

[∴ By case I]

$$= \frac{1}{2}\frac{1}{0}e^t + \frac{1}{2}\left(\frac{-1}{4}\right)e^t + \frac{1}{D(-1)+3(-1)+D-5}\sin t$$

[By case II]

$= \text{Case fails} - \frac{1}{8}e^t - \frac{1}{8}\sin t$

$$= \frac{1}{2}x.\frac{1}{3D^2+6D+1}e^t - \frac{1}{8}e^t - \frac{1}{8}\sin t$$

$$= \frac{1}{2}x\frac{1}{3(1)^2+6(1)+1}e^t - \frac{1}{8}e^t - \frac{1}{8}\sin t$$

(By case I)

$$P.I = \frac{x}{20}e^t - \frac{1}{8}e^t - \frac{1}{8}\sin t \quad \text{———} (2).$$

∴ The solution of given D.E is given by

$y = C.F. + P.I.$

$$y = c_1e^x + e^{-2x}(c_2\cos x + c_3\sin x) + \frac{1}{20}xe^t - \frac{1}{8}e^{-t} - \frac{1}{8}\sin t$$

[∴ ...]

Que 10 Solve $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 10y + 37\sin 3x = 0$ & find the value of y when $x = \frac{\pi}{2}$, if it is given that $y = 3$, $\frac{dy}{dx} = 0$ when $x = 0$.

Sol^n:- Given equation can be written as.

$(D^2 + 2D + 10)y = -37\sin x$.

Its auxillary equation is,

$m^2 + 2m + 10 = 0$.

$$m = \frac{-2 \pm \sqrt{4 - 40}}{2} = \frac{-2 \pm 6i}{2}.$$

$m = -1 \pm 3i$

$\therefore$ C.F $= e^{-x}(C_1 \cos 3x + C_2 \sin 3x)$ ——— (1)

Now,

$$P.I = \frac{1}{D^2 + 2D + 10}(-37\sin 3x)$$

$$= -37 \frac{1}{D^2 + 2D + 10}\sin 3x.$$

$$= -37 \frac{1}{-9 + 2D + 10}\sin 3x. \quad [\because D^2 = -(3)^2]$$

$$= -37 \frac{1}{2D + 1}\sin 3x.$$

$$= -37 \frac{2D - 1}{4D^2 - 1}\sin 3x.$$

$$= \frac{-37}{4(-9)-1}(2D-1)\sin 3x$$

$$= \frac{-37}{-37}(2D-1)\sin 3x$$

$= 2D\sin 3x - \sin 3x$

$= 2\times 3\cos 3x - \sin 3x$

P.I. $= 6\cos 3x - \sin 3x$ ——— (2)

$\therefore$ General solution is given by

$y = C.F + P.I.$

$= e^{-x}(c_1\cos 3x + c_2\sin 3x) + 6\cos 3x - \sin 3x$

[$\because$ By (1) and (2)]

$\therefore$ The required solution is

$y = e^{-x}(c_1\cos 3x + c_2\sin 3x) + 6\cos 3x - \sin 3x$ ——— (3)

Now given condition is $y=3$, when $x=0$

$\therefore$ Equation (3) $\Rightarrow$

$3 = e^0(c_1\cos 0 + c_2\sin 0) + 6\cos 0 - \sin 0$

$= c_1 + 6$

$\Rightarrow c_1 = -3$

$\therefore$ Equation (3) becomes

$y = e^{-x}(-3\cos 3x + c_2\sin 3x) + 6\cos 3x - \sin 3x$

$\Rightarrow \frac{dy}{dx} = e^{-x}[9\sin 3x + 3c_2\cos 3x]$
$\quad - e^{-x}[-3\cos 3x + c_2\sin 3x]$

Now given that $\frac{dy}{dx} = 0$, when $x=0$

$\therefore$ Equation (4) $\Rightarrow$

$0 = e^0[9(0) + 3c_2] - e^0[-3(1) + c_2(0)] - 18(0) - 3(1)$

$0 = 3c_2 + 3 - 3$

$c_2 = 0$.

Putting $c_1 = -3$ & $c_2 = 0$ in eqn (3) we get

$y = 3e^{-x} \cos 3x + 6\cos 3x - \sin 3x$

$\therefore$ The required solution is

$y = -3e^{-x} \cos 3x + 6\cos 3x - \sin 3x$

CASE III :-

Let $Q(x) = x^m$ i.e. a polynomial of m^{th} degree

Then $P.I = \frac{1}{F(D)} x^m$

Taking least degree term (L.D.T), common from $F(D)$ and then write $F(D)$ in the form of $[1+\phi(D)]$.

Then $P.I = \frac{1}{L.D.T} [1+\phi(D)]^{-1} (x^m)$. Using binomial theorem

expand $[1+\phi(D)]^{-1}$ $\cancel{(x^m)}$ as far as the term in D^m & operate on x^m term by term, since $(m+1)^{th}$ derivative & higher derivative of x^m are zero.

Note :- Useful Binomial Expansions :-

(1) $(1-x)^{-1} = 1 + x + x^2 + \ldots\ldots + x^n + \ldots\ldots$

(2) $(1+x)^{-1} = 1 - x + x^2 - x^3 + \ldots\ldots + (-1)^n x^n + \ldots\ldots$

(3) $(1-x)^{-2} = 1 + 2x + 3x^2 + \ldots\ldots + (n+1)x^n + \ldots\ldots$

(4) $(1+x)^{-2} = 1 - 2x + 3x^2 + \ldots\ldots + (-1)^n (n+1) x^n + \ldots\ldots$