Swapnil Rao's MAC11 2022 Question Bank solved/solutions compilation

CSE(CY); 3rd semester

Preface: over 100-125 hours put on the book, would appreciate if credits are given during this PDF's circulation / distribution
apologies for the watermarking

General instructions:

>The main goal behind creating this was to make sure that the final answer of almost every question (most of the questions from the attached 2022 MAC11 QB have been solved) is written and it is done so with a "double underline"
so, if at all you come across a "double underline" for a question, then it means that its the final answer for that question

>The question number has been written on the left side of the margin, these correspond to the respective question number from the attached QB. if at all you come across a decimal based question number (like 6.5, 0.5 etc) ignore these questions as these are the cw questions that had been solved for my class

>If at all the steps are blurry/not clear/not understandable, i apologize. its mostly cuz i had to click a lot of photos and didnt have the patience to click every picture properly but made sure that the final "double underlined" answer is visible. also, i hadnt solved the questions back then with an intent of circulating it, it was mostly for my personal use and that should explain why its so shabby and dirty.

>This doesn't have ALL the Qs solved of course, but 95-97% of them have been solved

Disclaimer:

>I had verified almost every question back then and they seemed to have matched with the online solutions too. however, i am not responsible for any sort of marks lost in the tests; when youve tried to replicate what ive written

>Also, ive heard that MAC11 2023 QB has a couple new questions (added)
To obtain solutions for these (and for those questions that are missing from the QB), just search the question on youtube by typing 25-30% of the question and it most likely will appear in the search results. if not, then always try to search for an online calculator which does the same job. do so by typing:
[Topic name] online calculator
and then enter the question

>If both of them failed to give you your answer, then search the question by typing 25-30% of the question on google and it could potentially exist in numerade / Chegg both of them could later on be unblurred with online tools (i do not promote piracy of content however)

Unit I (MAC11)

Differential Calculus-I: Polar curves, angle between the radius vector and the tangent, angle between the curves, length of perpendicular from pole to the tangent, Pedal equations. Derivative of arc length & radius of curvature in cartesian, polar & parametric forms (All without proof).

- Pedagogy/Course delivery tools: Chalk and talk
- Links: https://nptel.ac.in/courses/111/105/111105121/
- Impartus recording: https://a.impartus.com/ilc/#/course/107625/1030

Unit II

Partial Differentiation: Partial derivatives, total differential coefficient, differentiation of composite and implicit functions. Jacobians and properties.

Vector differentiation: Scalar and vector fields, gradient of a scalar field, directional derivative, divergence of a vector field, solenoidal vector, curl of a vector field, irrotational vector, Laplacian operator, physical interpretation of gradient, divergence and curl.

- Pedagogy/Course delivery tools: Chalk and talk
- Links: https://nptel.ac.in/courses/111/105/111105134/
- Impartus recording: https://a.impartus.com/ilc/#/course/107625/1030

Unit III

Multiple Integrals: Evaluation of double and triple integrals, change of order of integration, change of variables. Areas and volumes using double and triple integrals. Beta and Gamma functions - properties (without proof).

- Pedagogy/Course delivery tools: Chalk and talk, power point presentation, videos
- Links: https://nptel.ac.in/courses/111/105/111105121/
- Impartus recording: https://a.impartus.com/ilc/#/course/107625/1030

Unit IV

Vector Integration: Line integrals, surface integrals and volume integrals. Green's theorem (with proof) and its applications, Stokes' theorem and Gauss divergence theorem (without proof) and its applications.

- Pedagogy / Course delivery tools: Chalk and talk, Power Point Presentation, Videos
- Links: https://nptel.ac.in/courses/111/105/111105134/
- Impartus recording: https://a.impartus.com/ilc/#/course/107625/1030

Unit V

Modular Arithmetic: Introduction to congruences, linear congruences, The Chinese remainder theorem, solving polynomials, linear diophantine equation, system of linear congruences, Euler's theorem, Wilson theorem and Fermat's little theorem. Applications of congruences-RSA algorithm.

- Pedagogy / Course delivery tools: Chalk and talk
- Links: https://nptel.ac.in/courses/111101137

Unit-I
Differential Calculus

Polar Curves

I Two and Four marks questions

1. Write the relation between cartesian and polar coordinates.
2. Write the expression for angle between the radius vector and the tangent. Also define the terms involved.
3. Write the expression for length of the perpendicular from pole to the tangent.
4. Find the angle between radius vector and the tangent for:

   a) $r = a(1+\cos\theta)$ b) $r^2\cos 2\theta = a^2$ c) $\frac{2a}{r} = 1-\cos\theta$ d) $r^m = a^m(\cos m\theta + \sin m\theta)$

5. For the following curves find the slope of the tangent at the indicated points:

   a) $\frac{2a}{r} = 1-\cos\theta$ at $\theta = \frac{2\pi}{3}$ b) $2r = a\sin 2\theta$ at $\theta = \frac{\pi}{4}$ c) $r^2\cos 2\theta = a^2$ at $\theta = \frac{\pi}{12}$

   d) $r\sec^2\left(\frac{\theta}{2}\right) = 4$ at $\theta = \frac{\pi}{2}$.

6. Show that at any point (r,θ), the tangent to the curve $r^n = a^n\sin(n\theta)$ makes an angle $(n+1)\theta$ with the initial line.
8. Find the pedal equation of the curve $r = a\theta$.

II Seven marks questions

9. Prove with usual notation $\tan\phi = r\frac{d\theta}{dr}$.
10. With usual notation prove that $p = r\sin\phi$ and hence prove that $\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$.
11. Show that the following pairs of curves intersect each other orthogonally:

    a) $r = ae^\theta$ and $re^\theta = b$ b) $r^n = a^n\cos n\theta$ and $r^n = a^n\sin n\theta$

    c) $r = a(1+\cos\theta)$ and $r = b(1-\cos\theta)$

12. Find the angle of intersection of the following pairs of curves:

    a) $r = \frac{a\theta}{1+\theta}$ and $r = \frac{a}{1+\theta^2}$ b) $r = a\log\theta$ and $r = \frac{a}{\log\theta}$

    c) $r^2\sin 2\theta = 4$ and $r^2 = 16\sin 2\theta$.

13. Find the length of perpendicular from pole to the tangent to the following curves:

    a) $r = a(1-\cos\theta)$ at $\left(a, \frac{\pi}{2}\right)$ b) $r^2\cos 2\theta = a^2$ at $\theta = \frac{\pi}{6}$.

14. Find the pedal equation to the following curves:

    a) $r^n = a^n\cos n\theta$ b) $r^2 = a^2\sin 2\theta$ c) $r = a\,\mathrm{sec}\,hn\theta$ d) $r = ae^{m\theta}$

Derivative of arc length and radius of curvature

I Two and Four marks questions

15. Write the formula to find the derivative of arc length of a curve in cartesian and parametric forms.
16. Write the formula to find the derivative of arc length of a curve in polar form.
17. Define (i) Curvature (ii) Radius of curvature.
18. Write the expression for radius of curvature for curves in cartesian and parametric forms.
19. Write the expression for radius of curvature for curves in polar and pedal form.
20. Prove that the curvature of a circle is constant.
21. With the usual notation, prove that $\sin\phi = r\dfrac{d\theta}{ds}$.
22. With the usual notation, prove that $\cos\phi = \dfrac{dr}{ds}$.
23. Find $\dfrac{ds}{dy}$ for the following curves:
    i) $a^2y^2 = a^3 - x^3$ at $(a,0)$ ii) $ax^2 = y^3$ iii) $y = a\log\sec(x/a)$.
24. Find $\dfrac{ds}{dx}$ for the following curves:
    $3ay^2 = x(x-a)^2$ ii) $x^{2/3} + y^{2/3} = a^{2/3}$ iii) $y = a\cosh(x/a)$.
25. Find $\dfrac{ds}{dt}$ for the curves: i) $x = e^t\sin t,\ y = \mathrm{e}^t\cos t$ ii) $x = a\cos t,\ y = b\sin t$.
26. Find $\dfrac{ds}{dr}, \dfrac{ds}{d\theta}$ for the curve $r\theta = a$.
27. Show that $\dfrac{ds}{d\theta} = \dfrac{a^2}{r}$ for the curve $r^2 = a^2\cos 2\theta$.
28. Find the radius of curvature for the following curves:
    a) $xy^3 = a^4$ at (a,a) b) $pa^2 = r^3$ c) $x = a\cos\theta,\ y = a\sin\theta$ at $\left(\dfrac{a}{\sqrt{2}}, \dfrac{a}{\sqrt{2}}\right)$.

II Seven marks questions

29. Write the expression for derivative of arc length in cartesian form and hence show that $\dfrac{ds}{dx} = \sec\psi$ and $\dfrac{ds}{dy} = \operatorname{cosec}\psi$.
30. Find $\psi, \dfrac{ds}{dt}, \dfrac{ds}{dx}$ and $\dfrac{ds}{dy}$ for the curve $x = a(\cos t + t\sin t),\ y = a(\sin t - t\cos t)$.
31. Find the radius of curvature for the following curves:
    a) $x^3 + y^3 = 3axy$ at $\left(\dfrac{3a}{2}, \dfrac{3a}{2}\right)$ b) $r^n = a^n\sin n\theta$ c) $x = a\cos^3 t,\ y = a\sin^3 t$.

32. Prove that the radius of curvature of the curve $x^4 + y^4 = 2$ at the point (1,1) is $\frac{\sqrt{2}}{3}$.

33. Show that for the curve $r == ae^{\theta\cot\alpha}$ where a and α are constants, $\frac{\rho}{r}$ is a constant.

34. Show that the radius of curvature of the curve $x^2 y = a(x^2 + y^2)$ at $(-2a, 2a)$ is $2a$.

35. Show that the radius of curvature of the curve $r^n = a^n \cos n\theta$ varies inversely as r^{n-1} .

Unit-II
Partial Differentiation and Vector differentiation

Partial derivatives

I Two & Four marks questions

1. Find $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ in each of the following cases:

   a) $u = \sin^{-1}\left(\frac{y}{x}\right)$ b) $u = x^y + \frac{y}{2x}$.

2. If $u = x\,f\left(\frac{y}{x}\right) + g\left(\frac{y}{x}\right)$ then show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = x\,f\left(\frac{y}{x}\right)$.

3. If $u = \frac{y}{z} + \frac{z}{x}$ the show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$.

4. If $z = e^{ax+by} f(ax - by)$ then show that $b\frac{\partial z}{\partial x} + a\frac{\partial z}{\partial y} = 2abz$.

5. If $u = e^{a\theta}\cos(a\log r)$ then prove that $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$.

6. The altitude of a right circular cone is 15cm and is increasing at 0.2 cm/s. The radius of the base is 10cm and is decreasing at 0.3cm/s. How fast is the volume changing?

7. Find the rate at which the area of a rectangle is increasing at a given instant when the sides of the rectangle are 4ft and 3ft and are increasing at the rate of 1.5ft/s and 0.5ft/s respectively.

II Seven marks questions

8. If $u = \log\left(x^3 + y^3 + z^3 - 3xyz\right)$ then prove that $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)u = \frac{3}{x+y+z}$ and hence show that $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \frac{-9}{(x+y+z)^2}$.

9. Find the value of n so that the equation $v = r^n\left(3\cos^2\theta - 1\right)$ satisfies the relation $\frac{\partial}{\partial r}\left(r^2\frac{\partial v}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial v}{\partial \theta}\right) = 0$.

Total differentiation

I Two & Four marks questions

10. Find $\frac{dy}{dx}$ in each of the following cases using partial derivatives:

    a) $e^x + e^y = 2xy$ b) $y^x = x$ c) $x^3 + y^3 - 3axy = 1$.

11. Find $\frac{du}{dx}$ in each of the following cases:

a) $u=\cos\left(x^2-y^3\right),\ 2x^2+3y^2=a^2$ b) $u=\tan^{-1}\left(\frac{y}{x}\right),\ x^2+y^2=a^2$.

12. If $u=x\log(xy)$ where $x^3+y^3+3xy=1$ then find $\frac{du}{dx}$.

13. If $u=\tan^{-1}\left(\frac{y}{x}\right)$ where $x=e^t-e^{-t}$, $y=e^t+e^{-t}$ then find $\frac{du}{dt}$.

14. Find $\frac{du}{dt}$ for the following functions:

a) $u=xy+yz+zx$ where $x=\frac{1}{t}$, $y=e^t$ and $z=e^{-t}$.

b) $u=x^3ye^z$ where $x=t$, $y=t^2$ and $z=\log t$ at $t=2$.

II Seven marks questions

15. If $u=e^{xy}\sin(yz)$ where $x=t^2$, $y=t-1$ and $z=\frac{1}{t}$ then find $\frac{du}{dt}$ at $t=1$ by partial derivatives.

16. Find the total derivative of the following functions and also verify the result by direct substitution

$u=xy^2+x^2y$ where $x=at^2$, $y=2at$ b) $u=\sin\left(\frac{x}{y}\right)$ where $x=e^t$, $y=t^2$.

17. If $z=f(x,y)$ and $x=u-v$, $y=uv$ then show that

a) $(u+v)\frac{\partial z}{\partial x}=u\frac{\partial z}{\partial u}-v\frac{\partial z}{\partial v}$ b) $(u+v)\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}$.

18. If $z=f(x,y)$ and $x=e^u+e^{-v}$, $y=e^{-u}-e^v$ then show that $\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}=x\frac{\partial z}{\partial x}-y\frac{\partial z}{\partial y}$.

19. If $u=f(y-z,z-x,x-y)$ then prove that $u_x+u_y+u_z=0$.

20. If $u=f(r,s,t)$ where $r=\frac{x}{y}$, $s=\frac{y}{z}$, $t=\frac{z}{x}$ then show that $xu_x+yu_y+zu_z=0$.

Jacobians

I Two & Four marks questions

21. Define Jacobian of u,v,w with respect to x,y,z.
22. State any two properties of Jacobians.
23. If $u=x+3y^2-z^2, v=4x^2yz, w=2z^2-xy$ then evaluate $\frac{\partial(u,v,w)}{\partial(x,y,z)}$ at (1,-1,0).
24. Find $\frac{\partial(u,v,w)}{\partial(x,y,z)}$ in each of the following cases:

a) $u = x(-y), v = xy.$ b) $u = 3x + 5y, v = 4x - 5y.$

25. Find $\frac{\partial(x,y)}{\partial(u,v)}$. in each of the following cases:

a) $x = \frac{u^2}{v}, y = \frac{v^2}{u}$ b) $x = u(1-v), y = uv$.

26. Find $\frac{\partial(u,v,w)}{\partial(x,y,z)}$ in each of the following cases:

$(a) u = x^2 - 2y, v = x + y + z, w = x - 2y + 3z$ $(b)\ u = \frac{yz}{x}, v = \frac{zx}{y}, w = \frac{xy}{z}$

II Seven marks questions

27. If $x = r\cos\theta, y = r\sin\theta$ then show that $JJ' = 1.$
28. If $u = \sqrt{yz}\ v = \sqrt{zx}, w = \sqrt{xy}$ and $x = r\cos\varphi\sin\theta, y = r\sin\varphi\sin\theta, z = r\cos\theta$ then find $\frac{\partial(u,v,w)}{\partial(r,\varphi,\theta)}$.
29. If $x = r\cos\varphi\sin\theta, y = r\sin\varphi\sin\theta, z = r\cos\theta$ then show that $\frac{\partial(x,y,z)}{\partial(r,\theta,\varphi)} = r^2\sin\theta.$
30. If $x = e^v\sec u$ and $y = e^v\tan u$ then find the Jacobian $\frac{\partial(x,y)}{\partial(u,v)}$ and $\frac{\partial(u,v)}{\partial(x,y)}$.
31. If $u = x^2 - 2y^2$ and $v = 2x^2 - y^2$ where $x = r\cos\theta, y = r\sin\theta$ then show that $\frac{\partial(u,v)}{\partial(r,\theta)} = 6r^3\sin 2\theta.$
32. If $u = x + y + z, v = y + z, w = z + x$ then find the inverse Jacobian by first confirming that u, v, w are functionally independent.
33. Prove that the functions $u = \tan^{-1}x + \tan^{-1}y$ and $v = \frac{x+y}{1-xy}$ are functionally dependent using the concept of Jacobians. Also express u in terms of v.
34. Show that the functions $u = x + y + z, v = x - y + z, w = x^2 + y^2 + z^2 - 2yz$ are not independent of one another. Also find the relation between them.
35. Find $\frac{\partial(u,v,w)}{\partial(x,y,z)}$ where $u = x^2 + y^2 + z^2, v = xy + yz + zx, w = x + y + z$ and also find the functional relation between u, v and w.

Scalar and vector field, gradient of scalar field, directional derivatives, divergence of a vector field and Solenoidal vector

I Two & Four-marks questions

36. Define scalar and vector field with an example.
37. Define gradient of a scalar field.
38. Show that $\nabla\phi$ is a vector perpendicular to the surface $\phi(x,y,z)=c$
39. If $\phi = f(r)$ where $r^2 = x^2+y^2+z^2$, then prove that $\nabla\phi = \frac{f'(r)}{r}\vec{r}$.
40. Show that $\nabla r^n = nr^{n-2}\vec{r}$.
41. Find $\nabla\phi$ if a) $\phi = \log|\vec{r}|$ b) $\phi = \frac{1}{r}$.
42. In a temperature field, heat flows in the direction of maximum decrease of temperature T. Find this direction at $p(2,1)$ when $T = x^3 - 3xy^2$.
43. Define directional derivative.
44. Explain the geometrical meaning of gradient of a scalar field.
45. Define the divergence of a vector field.
46. Define solenoidal vector.
47. Show that the maximum directional derivative takes place in the direction $\nabla\phi$.
48. Interpret the symbol $(\vec{A}\cdot\nabla)$. Is it same as $(\nabla\cdot\vec{A})$?
49. Verify $\nabla^2 u = 0$ if $u = x^2 - y^2 + 4z$.
50. If $\vec{A} = 2yz\,\hat{i} - x^2y\,\hat{j} + xz^2\hat{k}$ and $\vec{B} = x^2\,\hat{i} + yz\,\hat{j} - xy\,\hat{k}$ then find $(\vec{B}\cdot\nabla)\vec{A}$.
51. If $\phi = 2x^3y^2z^4$ then find $\nabla\cdot\nabla\phi$
52. Show that $\vec{F} = \frac{x\hat{i} + y\hat{j}}{x^2+y^2}$ is a solenoidal vector field.
53. Find $div\vec{F}$ at the point $(1,2,3)$ where $\vec{F} = (y^2+z^2-x^2)\hat{i} + (z^2+x^2-y^2)\hat{j} + (x^2+y^2-z^2)\hat{k}$
54. Show that the vector $\vec{F} = 3y^2z^2\hat{i} + 4x^3z^2\hat{j} + 3x^2y^2\hat{k}$ solenoidal.
55. If $\vec{F} = (x^2+y^2+z^2)^{-n}$ then find $div(grad\vec{F})$.

II Seven marks questions

56. Find the values of a and b so that the surfaces $ax^2 - byz = (a+2)x$ and $4x^2y + z^3 = 4$ may intersect orthogonally at the point $(1,-1,2)$.
57. Find the angle between the surfaces $x^2y + z = 3$ and $x\log z - y^2 = 4$ at the point $(1,-1,2)$.
58. Find the angle of intersection of the spheres $x^2+y^2+z^2+4x-6y-8z-47=0$ and $x^2+y^2+z^2=29$ at the point of intersection being $(4,-3,2)$.

Directional derivatives, divergence of a vector field and Solenoidal vector

II Seven marks questions

59. If $\phi = x^2y^3z^4$ then find the rate of change of ϕ at $(2,3,-1)$ in the direction making equal angles with the positive x, y and z axes.
60. Find the directional derivative of the function $\phi = xyz$ along the direction of the normal to the surface $xy^2 + yz^2 + zx^2 = 3$ at the point $(1,1,1)$.
61. Find the directional derivative of $\phi = x^2 - 2xy + z^3$ at the point $(1,-2,-1)$ along the vector $2\hat{i} - 4\hat{j} + 4\hat{k}$.
62. Find the directional derivative of $\phi = y^2x + yz^3$ at the point $(2,-1,1)$ in the direction of the normal to the surface $x\log z - y^2 = -4$ taken at the point $(-1,2,1)$.
63. If the directional derivative of $\phi = axy^2 + byz + cz^3x^3$ at $(-1,1,2)$ has a maximum magnitude of 32 units in the direction parallel to y-axis then find a, b, c.
64. In what direction from the point $(2,1,-1)$ is the directional derivative of $\phi = x^2yz^3$ a maximum? What is its magnitude?
65. What is the greatest rate of increase of $u = xyz^2$ at the point $(1,0,3)$.

Curl of a vector field, irrotational vector, laplacian operator and vector identities

I Two & Four marks questions

66. Define the curl of a vector field.
67. Give the physical meaning of curl of a vector field.
68. Define irrotational vectors.
69. Define Laplacian operator.
70. Write any two vector identities.
71. If $\vec{a}$ is the constant vector and $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ then prove that $\nabla \times (\vec{a} \times \vec{r}) = 2\vec{a}$.
72. Find $Curl(Curl\vec{A})$, where $\vec{A} = x^2y\hat{i} - 2xz\hat{j} + 2yz\hat{k}$ at the point $(1,0,2)$.
73. If $\vec{A} = 2yz\hat{i} - x^2y\hat{j} + xz^2\hat{k}$ and $\phi = 2x^2yz^3$ then find $(\vec{A} \times \nabla\phi)$.
74. If $\vec{F} = \frac{x}{r}\hat{i} + \frac{y}{r}\hat{j} + \frac{z}{r}\hat{k}$ where $r = \sqrt{x^2 + y^2 + z^2}$ then find $div(\vec{F})$ and $Curl(\vec{F})$.

II Seven marks questions

75. Find the value of the constant 'a' such that $\vec{A} = (axy - z^3)\hat{i} + (a-2)x^2\hat{j} + (1-a)xz^2\hat{k}$ is irrotational and hence find a scalar function ϕ such that $\vec{A} = \nabla\phi$.
76. Find the values of the constants a, b, c such that $\vec{F} = (x + 2y + az)\hat{i} + (bx - 3y - z)\hat{j} + (4x + cy + 2z)\hat{k}$ is conservative. Also find its scalar potential.
77. Show that $\vec{F} = (x^2 - y^2 + x)\hat{i} - (2xy + y)\hat{j}$ is irrotational. Also find a scalar function ϕ

such that $\vec{F}=\nabla\phi$.

78. Find the constants *a* and *b* so that $\vec{A}=(axy+z^3)\hat{i}+(3x^2-z)\hat{j}+(bxz^2-y)\hat{k}$ is irrotational and find ϕ such that $\vec{F}=\nabla\phi$.

79. If $\overline{F}=x^2y\hat{i}+2x^2yz\hat{j}-3y^2z\hat{k}$, find $div(\vec{F}), curl(\vec{F}), div(curl\vec{F})$ at $(2,1,1)$.

80. If $\vec{F}=(x+y+1)\hat{i}+\hat{j}-(x+y)\hat{k}$, show that $\vec{F}\cdot Curl(\vec{F})=0$.

81. If $\vec{F}=(3x^2y-z)\hat{i}+(xz^3+y)\hat{j}-2x^3z^2\hat{k}$, find $curl(curl\vec{F})$.

82. If $\vec{A}=xz^3\hat{i}-2x^2yz\hat{j}+2yz^4\hat{k}$ find (a) $\nabla\cdot\vec{A}$ (b) $\nabla\times\vec{A}$ (c) $\nabla\cdot(\nabla\times A)=0$ at $(1,-1,1)$.

83. Show that $\phi=\dfrac{1}{r}$ is a solution of Laplace equation $\nabla^2\phi=0$.

84. Find the value of the constant 'a' such that $\vec{F}=y(ax^2+z)\hat{i}+x(y^2-z^2)\hat{j}+2xy(z-xy)\hat{k}$ is solenoidal. For this value of 'a' show that $Curl(\vec{A})$ is also solenoidal.

85. Find the value of a if the vector $(ax^2y+yz)\hat{i}+(xy^2-xz^2)\hat{j}+(2xyz-2x^2y^2)\hat{k}$ has zero divergence. Find the curl of the above vector which has zero divergence.

86. Maxwell's equations of electromagnetic theory are given by $\nabla\cdot\vec{E}=0, \nabla\cdot\vec{H}=0, \nabla\times\vec{E}=-\dfrac{\partial\vec{H}}{\partial t}$ and $\nabla\times\vec{H}=\dfrac{\partial\vec{E}}{\partial t}$. Show that $\vec{E}$ and $\vec{H}$ satisfy the wave equation $\nabla^2u=\dfrac{\partial^2u}{\partial t^2}$.

87. If $\vec{\omega}=\vec{\omega}_1+\vec{\omega}_2+\vec{\omega}_3$ is the constant angular velocity and $\vec{v}$ is the velocity of a particle at a point $p(x,y,z)$ of the moving body having the position vector $\vec{r}$ such that $\vec{v}=\vec{\omega}\times\vec{r}$ then prove that $\vec{\omega}=\dfrac{1}{2}(\nabla\times\vec{v})$. Interpret the result when $\nabla\times\vec{v}=0$.

88. If $\vec{r}$ is the position vector of the point (x,y,z) and $r=\sqrt{x^2+y^2+z^2}$ then show that $\nabla\cdot\left(\dfrac{\vec{r}}{r^3}\right)=0$

Unit-III
Multiple Integrals

Double & Triple Integrals

I Two & Four marks questions

1. Define double integral of $f(x,y)$.
2. Define triple integral of $f(x,y,z)$.
3. With the help of a neat diagram mark the region of integration in the following double integrals:

   a) $\int_0^1\int_x^{\sqrt{x}} f(x,y)\,dy\,dx$ b) $\int_0^{\pi}\int_0^{\sin x} f(x,y)\,dy\,dx$ c) $\int_0^2\int_{2-x}^{\sqrt{4-x^2}} f(x,y)\,dy\,dx$ d) $\int_0^2\int_{y/4}^{3-y} f(x,y)\,dy\,dx$

4. With the help of a neat diagram mark the region of integration in the following double integrals:

   a) $\int_0^{\frac{\pi}{2}}\int_0^{2\cos\theta} f(r,\theta)dr\,d\theta$ b) $\int_0^{2\pi}\int_a^{b} f(r,\theta)dr\,d\theta$ c) $\int_0^{\frac{\pi}{2}}\int_0^{\infty} f(r,\theta)dr\,d\theta$.

5. Evaluate the following double integrals.

   a) $\int_1^2\int_2^3\left(x-\frac{1}{y}\right)^2 dx\,dy$ b) $\int_0^1\int_x^{\sqrt{x}} xy\,dy\,dx$ c) $\int_0^1\int_0^{x^2} e^{\frac{y}{x}}\,dy\,dx$ d) $\int_0^4\int_{x^2/4}^{2\sqrt{x}} y\,dy\,dx$.

6. Evaluate the following double integrals:

   a) $\int_0^{\pi}\int_0^{a\cos\theta} r\sin\theta\,dr\,d\theta$ b) $\int_0^{\pi/2}\int_0^{\infty} re^{-r^2}\,dr\,d\theta$ c) $\int_0^{\pi/2}\int_{a\cos\theta}^{a} \sin\theta\cos\theta\,dr\,d\theta$.

7. Evaluate $\iint_R \frac{\sin x}{x}\,dx\,dy$ where R is the triangle in the xy - plane bounded by the x – axis, the line $y=x$ and the line $x=1$.
8. Evaluate the following triple integrals:

   a) $\int_0^1\int_0^2\int_1^2 xyz^2\ dxdydz$ b) $\int_0^a\int_0^a\int_0^a (yz+zx+xy)\,dxdydz$ c) $\int_0^1\int_0^1\int_0^y xyz\ dxdydz$

II Seven marks questions

9. Evaluate the following double integrals:

   a) $\int_0^a\int_0^{\sqrt{a-x^2}} \sqrt{a^2-x^2-y^2}\,dy\,dx$ b) $\int_0^{a/\sqrt{3}}\int_0^{\sqrt{x^2+a^2}} \frac{x}{x^2+y^2+a^2}\,dy\,dx$

c) $\int_{-1}^{1}\int_{0}^{1-x} x^{1/3}y^{-1/2}(1-x-y)^{1/2}\,dy\,dx$

d) $\int_{0}^{1}\int_{0}^{\sqrt{1+x^2}} \frac{1}{1+x^2+y^2}\,dy\,dx$

10. Evaluate $\iint_R xy\,dx\,dy$ where R is the region bounded by $\frac{x}{a}+\frac{y}{b}=1;\ x=0$ and $y=0$.

11. Evaluate $\iint_R xy\,dx\,dy$ where R is the region bounded by the x- axis, ordinate at $x=2a$ and the curve $x^2=4ay$.

12. Evaluate $\iint_R y\,dx\,dy$ where R is the region bounded by y – axis, the curve $x^2=y$ and the line $x+y=2$ in the first quadrant.

13. Evaluate $\iint_R x^2\,dx\,dy$ where R is the region bounded by the curves $xy=16,\ x=y,\ y=0$ and $x=8$.

14. Evaluate $\iint_R r\sin\theta\,dr\,d\theta$, where R the region is bounded by the cardioid $r=a(1-\cos\theta)$ above the initial line.

15. Evaluate $\iint_R r^3 dr\,d\theta$ where R the region is included between the circles $r=2\sin\theta$ and $r=4\sin\theta$.

16. Evaluate the following triple integrals:

a) $\int_{-c}^{c}\int_{-b}^{b}\int_{-a}^{a}\left(x^2+y^2+z^2\right)dx\,dy\,dz$

b) $\int_{1}^{e}\int_{1}^{\log y}\int_{1}^{e^x}\log z\,dz\,dx\,dy$

c) $\int_{0}^{a}\int_{0}^{x}\int_{0}^{x+y} e^{x+y+z}\,dz\,dy\,dx$

d) $\int_{0}^{1}\int_{0}^{\sqrt{1-x^2}}\int_{0}^{\sqrt{1-x^2-y^2}} xyz\,dx\,dy\,dz$

17. Evaluate $\iiint x\,dx\,dy\,dz$ taken through the positive octant of the sphere $x^2+y^2+z^2=a^2$.

Change the order of integration

I Two & Four marks questions

18. Write the limits of integration after changing the order of integration with a neat diagram for the following double integrals:

a) $\int_{0}^{1}\int_{x}^{\sqrt{x}} f(x,y)\,dy\,dx$ b) $\int_{0}^{1}\int_{0}^{x^2} f(x,y)\,dy\,dx$ c) $\int_{0}^{\infty}\int_{0}^{x} f(x,y)\,dy\,dx$ d) $\int_{0}^{a}\int_{0}^{\sqrt{a^2-y^2}} f(x,y)\,dx\,dy$

II Seven marks questions

19. By changing the order of integration, evaluate the following double integrals:

a) $\int_{0}^{1}\int_{x^2}^{2-x} xy\,dy\,dx$ b) $\int_{0}^{1}\int_{\sqrt{y}}^{2-y} xy\,dx\,dy$ c) $\int_{0}^{a}\int_{\sqrt{ax}}^{a} \frac{y^2}{\sqrt{y^4-a^2x^2}}\,dy\,dx$

d) $\int_0^{\infty}\int_0^{x} x e^{-x^2/y}\, dy\, dx$ e) $\int_{-2}^{2}\int_{0}^{\sqrt{4-x^2}} (2-x)\, dy\, dx$ f) $\int_0^{4a}\int_{x^2/4a}^{2\sqrt{ax}} dy\, dx$

20. Show that $\int_0^6\int_{y/2}^{3} \frac{1}{x} e^{y/x}\, dy\, dx = 3(e^2-1).$

21. Show that $\int_0^{\pi/2}\int_x^{\pi/2} \frac{\sin y}{y}\, dydx = 1.$

Change of variables

I Two & Four marks questions

22. Write the procedure of evaluating double integral by changing into polar coordinates.
23. Write the procedure of evaluating triple integral by changing into cylindrical coordinates.
24. Write the procedure of evaluating triple integral by changing into spherical polar coordinates.
25. Write the limits of integration with respect to *r, θ* while evaluating the following integrals:

$\int_0^{4a}\int_{y^2/4a}^{y} f(x,y)\, dy\, dx$ b) $\int_0^{a}\int_{\sqrt{ax-x^2}}^{\sqrt{a^2-x^2}} f(x,y)\, dy\, dx$ c) $\int_0^a\int_x^a f(x,y)\, dy\, dx$

II Seven marks questions

26. Using the transformation $x+y=u; y=uv$, show that $\int_0^1\int_x^{1-x} e^{y/(y+x)} dy\, dx = \frac{1}{2}(e-1).$

27. If R is the region bounded by $x=0, y=0$ and $x+y=1$ then by using the transformation $x+y=u;\ x-y=v$, hence show that $\iint_R \sin\left(\frac{x-y}{x-y}\right) dxdy = 0.$

28. Evaluate the following integrals by changing to polar coordinates:

a) $\int_0^{\infty}\int_x^{\infty} \frac{e^{-y}}{y}\, dy\, dx$ b) $\int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)} dy\, dx$ c) $\int_0^{2a}\int_0^{\sqrt{2ax-x^2}} x^2 dy\, dx$

d) $\int_0^{2}\int_0^{\sqrt{2x-x^2}} \left(x^2+y^2\right) dy\, dx$ e) $\int_0^{a}\int_{\sqrt{ax-x^2}}^{\sqrt{a^2-x^2}} \frac{xy}{x^2+y^2} e^{-(x^2+y^2)} dy\, dx$

29. Transform the integral $\int_0^{\pi}\int_0^{a} r^3 \sin\theta \cos\theta\, dr\, d\theta$ to Cartesian form and hence evaluate.

30. By changing into polar coordinates, evaluate $\iint_R \frac{x^2y^2}{x^2+y^2}\, dx\, dy$ over the annular region

between circles $x^2+y^2=a^2$ and $x^2+y^2=b^2,(b>a)$.

31. Using spherical polar coordinates, evaluate $\iiint \frac{dx\,dy\,dz}{x^2+y^2+z^2}$, taken over the volume bounded by the sphere $x^2+y^2+z^2=a^2$.

32. Evaluate $\int_0^\infty\int_0^\infty\int_0^\infty \frac{dxdydz}{(1+x^2+y^2+z^2)^2}$, using spherical polar coordinates.

33. Evaluate $\iiint (x^2+y^2+z^2)dx\,dy\,dz$ taken over the volume enclosed by the sphere $x^2+y^2+z^2=1$, by transforming into spherical polar coordinates.

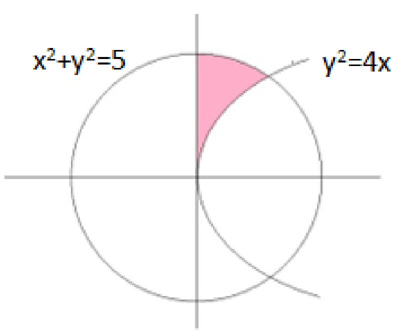
34. By transforming into cylindrical polar coordinates, evaluate $\iiint (x^2+y^2+z^2)dx\,dy\,dz$ taken over the region $0\le z\le x^2+y^2\le 1$.

Applications of double & triple integrals

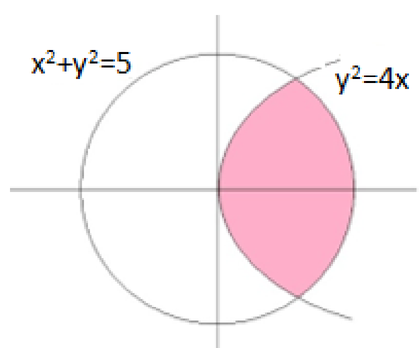
II Seven marks questions

35. Find the area bounded by ellipse using double integration.
36. Find the area bounded by the curves $y^2=4x$ and $x^2=4y$ using double integration.
37. Find the area lying inside the circle $r=a\sin\theta$ and outside the cardioid $r=a(1+\cos\theta)$ using double integration.
38. Find the area lying inside the cardioid $r=a(1+\cos\theta)$ and outside the circle $r=a$ using double integration.
39. Find the volume of the sphere $x^2+y^2+z^2=a^2$ using triple integration.
40. Find the volume of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ using triple integration.
41. Find the area of the following shaded regions using double integration:

a)



b)



Beta & Gamma functions

I Two & Four marks questions

42. Define beta function and hence write the trigonometric form of beta function.
43. Write the relation between beta and gamma function.
44. Define Gamma function. Find the value of $\Gamma(-3.5)$.
45. Evaluate a) $\Gamma\left(\frac{-1}{2}\right)$ b) $\beta\left(\frac{1}{3},\frac{2}{3}\right)$.
46. Prove that $\int_0^1 x^m \left[\log\left(\frac{1}{x}\right)\right]^n dx = \frac{\Gamma(n+1)}{(m+1)^{n+1}}$.
47. Evaluate the following:

a) $\int_0^\infty x^3 e^{-x^2}\, dx$ b) $\int_0^\infty e^{-ax^{1/a}}\, dx$.

II Seven marks questions

48. Show that $\int_0^{\pi/2} \sqrt{\sin\theta}\, d\theta \times \int_0^{\pi/2} \frac{1}{\sqrt{\sin\theta}} d\theta = \pi$.
49. Show that $\int_0^\infty \frac{e^{-x^2}}{\sqrt{x}} dx \times \int_0^\infty e^{-x^4} x^2\, dx = \frac{\pi}{4\sqrt{2}}$.

Unit-IV
Vector Integration

Line integrals, surface integrals and volume integrals

I Two/Four/Seven marks questions

1. Define a simple closed curve with an example.
2. Define line integral of a vector function.
3. Give the physical interpretation of $\int_c \vec{F}.d\vec{r}$ if $\vec{F}$ is a force on a particle moving along c.
4. If $\vec{F} = xy\ \hat{i} + yz\ \hat{j} + zx\ \hat{k}$ evaluate $\int_c \vec{F}.d\vec{r}$ where c is the curve represented by $x = t, y = t^2, z = t^3, -1 \le t \le 1$.
5. Evaluate $\int_c (xy\ \hat{i} + (x^2 + y^2)\ \hat{j}).d\vec{r}$ along the straight line joining origin and (1, 2).
6. Explain the method of evaluating the surface integral $\int_s \vec{F}\,.\,\hat{n}\,ds$.
7. Give the physical interpretation of $\int_s \vec{F}\,.\,\hat{n}\,ds$ when $\vec{F}$ represents the velocity of the fluid particle.
8. Evaluate $\vec{F} = 3xy\ \hat{i} - 5z\ \hat{j} + 10x\ \hat{k}$ where $\vec{F} = \cos y\ \hat{i} - x\sin y\ \hat{j}$ and c is the curve $y = \sqrt{1 - x^2}$ in xy-plane from (1, 0) to (0, 1).
9. Find the circulation of $\vec{F}$ round the curve c where $\vec{F} = y\ \hat{i} + z\ \hat{j} + x\ \hat{k}$ and c is the circle $x^2 + y^2 = 1,\ z = 0$.
10. Find the circulation of $\vec{F}$ round the curve c, where $\vec{F} = (x - y)\ \hat{i} + (x + y)\ \hat{j}$ and c is the circle $x^2 + y^2 = 4,\ z = 0$.
11. Find the total work done in moving particle in a force field $\vec{F} = 3xy\ \hat{i} - 5z\ \hat{j} + 10x\ \hat{k}$ along the curve $x = t^2 + 1,\ y = 2t^2$ and $z = t^3$ from $t = 1$ to $t = 2$.
12. If $\vec{F} = 3xy\ \hat{i} - 5y^2\ \hat{j}$ evaluate $\int_c \vec{F}.d\vec{r}$ where c is the curve $y = 2x^2$ in xy – plane from (0, 0) to (1, 2).
13. If $\vec{F} = (3x^2 + 6y)\,\hat{i} - 14yz\ \hat{j} + 20xz^2\ \hat{k}$ then evaluate $\int_c \vec{F}.d\vec{r}$ from (0, 0, 0) to (1, 1, 1) along the curve c given by $x = t,\ y = t^2, z = t^3$.
14. If $\vec{F} = (5xy - 6x^2)\,\hat{i} + (2y - 4x)\ \hat{j}$ then evaluate $\int_c \vec{F}.d\vec{r}$ where c is the curve $y = x^3$ from the point (1, 1) to the point (2, 8).
15. Find the total work done by a force $\vec{F} = 2xy\ \hat{i} - 4z\ \hat{j} + 5x\ \hat{k}$ along the curve $x = t^2,\ y = 2t\ + 1,\ z = t^3$ from $t = 0$ to $t = 1$.

16. If c is a simple closed curve in the $xy-$ plane not enclosing the origin then evaluate $\int_c \vec{F}.d\vec{r}$ where $\vec{F} = \dfrac{-y\hat{i} + x\hat{j}}{x^2 + y^2}$.

17. Prove that if $\int_{p_1}^{p_2} \vec{F}.d\vec{r}$ is independent of the path joining any two points p_1 and p_2 in a given region, then $\int_c \vec{F}.d\vec{r} = 0$ for all closed paths in the region.

18. If $\vec{F} = \left(x^2 - 2y\right)\hat{i} - 6yz\ \hat{j} + 8xz^2\ \hat{k}$ then evaluate $\int_c \vec{F}.d\vec{r}$ from the point (0, 0, 0) to (1, 1, 1) along the straight lines from (0, 0, 0) to (1, 0, 0), (1, 0, 0) to (1, 1, 0) and (1, 1, 0) to (1, 1, 1).

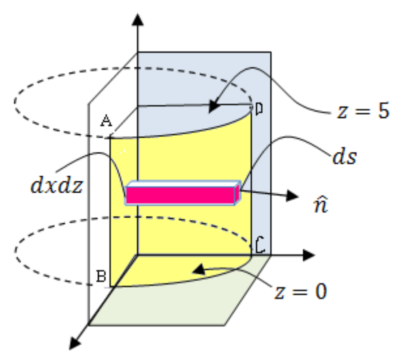
19. If $\vec{F} = y\ \hat{i} + (x - 2xz)\ \hat{j} - xy\ \hat{k}$ then evaluate $\int_s (\nabla\ X\ \vec{F}).\hat{n}\, ds$ where s is the surface of the sphere $x^2 + y^2 + z^2 = a$ above the xy-plane.

20. If $\vec{F} = 4xz\ \hat{i} - y^2\ \hat{j} + yz\ \ \hat{k}$ then evaluate $\int_s \vec{F}.\hat{n}\, ds$ directly where s is the surface of the cube bounded by $x = 0, x = 1, y = 0, y = 1,\ z = 0\ \&\ z = 1$.

21. Find the work done when a force $\vec{F} = \left(x^2 - y^2 + x\right)\hat{i} - (2xy + y)\ \hat{j}$ moves a particle in the $xy-$ plane from (0, 0) to (1, 1) along the parabola $y^2 = x$.

22. Evaluate $\int_s \vec{F}.\hat{n}ds$ where $\vec{F} = z\ \hat{i} + x\ \hat{j} - 3y^2 z\ \hat{k}$ and s is the surface of the cylinder $x^2 + y^2 = 16$ included in the positive octant between $z = 0$ and $z = 5$.

23. If $\vec{F} = z\ \hat{i} + x\ \hat{j} - 3y^2\ \hat{k}$ then evaluate $\int_s \vec{F}.\hat{n}\, ds$ where s is the curved surface ABCD as shown in the figure which is a part of the cylinder $x^2 + y^2 = 16$.



24. If $\vec{F} = 2xz\ \hat{i} - x\ \hat{j} - y^2\ \hat{k}$, then evaluate $\int_s \vec{F}.dv$ where v is the region bounded by the surface $x = 0$, $x = 2$, $y = 0$, $y = 6$, $z = x^2$ and $z = 4$.

Green's theorem

I Two/Four/Seven marks questions

30. State Green's theorem in a plane.
31. Using Green's theorem, evaluate $\int_c (y-\sin x)dx+\cos y\,dy$ where c is the plane triangle enclosed by the line $y=0,\ x=\frac{\pi}{2},\ y=\frac{2x}{\pi}$.
32. By using Green's theorem, evaluate $\int_c (x^2+xy)dx+(x^2+y^2)dy$ where c is the square formed by the lines $x=\pm 1;\ y=\pm 1$.
33. If c is a simple closed curve in the $xy-$plane by using Green's theorem that the integral $\int_c \frac{(xdy-ydx)}{2}$ represents the area A enclosed by c.
34. Evaluate $\int_c \left(x^2-\cosh y\right)dx+\left(y+\sin x\right)dy$ by Green's theorem where c is the rectangle with vertices (0, 0), (π, 0), (π, 1) and (0, 1).
35. State and prove Green's theorem in a plane.
36. Verify Green's theorem for $\int_c (xy+y^2)dx+x^2dy$ where c is bounded by $y=x$ and $y=x^2$
37. Verify Green's theorem for $\int_c (x^2+y^2)dx-2xydy$ where c is the rectangle bounded by $y=0, x=0, y=b$ and $x=a$.
38. Find the area of the ellipse using Green's theorem.
39. Find the area between the parabolas $y^2=4ax$ and $x^2=4ay$ using Green's theorem.
40. Find the area of the astroid using Green's theorem.

Stoke's theorem & Gauss divergence theorem

I Two/Four/Seven marks questions

41. State Stoke's theorem.
42. If $\vec{F}$ is irrotational, then show that $\int_c \vec{F}\,.\,d\vec{r}=0$ for any closed curve c.
43. Evaluate by Stoke's theorem $\int_c \left(e^x\,dx+2ydy-dz\right)$ where c is the curve $x^2+y^2=4$, $z=2$.
44. Evaluate $\int_s (yz\hat{i}+zx\hat{j}+xy\hat{k}).\hat{n}\,ds$ where s is the surface of the cube formed by $x=0$, $x=1$, $y=0$, $y=1$, $z=0$ and $z=1$ using Gauss divergence theorem.
45. Evaluate $\int_s \vec{r}.\hat{n}\,ds$ where s is a closed surface, using Gauss divergence theorem.
46. Show that $\int_s \nabla r^2.\hat{n}\,ds=6V$ where s is a closed surface enclosing a volume V using Gauss divergence theorem.

47. Evaluate $\int_c \vec{F}.d\vec{r}$ taken round the rectangle bounded by the lines $x = \pm a, y = 0 \; and \; y = b$ using Stoke's theorem where $\vec{F} = (x^2 + y^2)\,\hat{i} - 2xy\,\hat{j}$.

48. Using Stoke's theorem, evaluate $\int_c (x+y)dx + (2x-z)dy + (y+z)dz$ where c is the boundary of the triangle with vertices at (2, 0, 0), (0, 3, 0) and (0, 0, 6).

49. Evaluate $\int_c \vec{F}.d\vec{r}$ where $\vec{F} = y\,\hat{i} + z\,\hat{j} + x\,\hat{k}$ where c is the boundary of the upper half of the sphere $x^2 + y^2 + z^2 = 1$ using Stoke's theorem.

50. Evaluate $\int_c \vec{F}.d\vec{r}$ where $\vec{F} = y\,\hat{i} + z\,\hat{j} + x\,\hat{k}$ where c is the boundary of the upper half of the sphere $x^2 + y^2 + z^2 = 1$ using Stoke's theorem.

51. Evaluate $\int_s \vec{F}.\hat{n}\,ds$ where $\vec{F} = 4x\,\hat{i} - 2y^2\,\hat{j} + z^2\,\hat{k}$ and s is the surface bounding the region $x^2 + y^2 = 4$, $z = 0$ using Gauss divergence theorem.

52. Evaluate $\int_s (x^3\,\hat{i} + y^3\,\hat{j} + z^3\,\hat{k}).\hat{n}\,ds$ where s is the surface of the sphere $x^2 + y^2 + z^2 = a^2$ using Gauss divergence theorem.

53. Evaluate Gauss divergence theorem for $\vec{F} = 2xy\,\hat{i} + yz^2\,\hat{j} + xz\,\hat{k}$ and s is the rectangular parallelepiped bounded by $x = 0, y = 0, z = 0, x = 2, y = 1, z = 3$.

54. If $\vec{F} = 4xz\,\hat{i} - y^2\,\hat{j} + yz\,\hat{k}$ then evaluate $\int_s \vec{F}.\hat{n}ds$ using divergence theorem where s is the surface of the cube bounded by $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$ & $z = 1$.

Unit-V
Modular Arithmetic

GCD and Division Algorithm

I Two & Four marks questions

1. Define greatest common divisor.
2. Define proper divisor.
3. State division algorithm.
4. Prove that if $a|b$ and $b|c$ then $a|c$.
5. Prove that if $a|b$ and $a|c$ then $a|bm+nc$ where m,n are integers.
6. Prove that if $a|bc$ and $(a,b)=1$ then $a|c$.
7. Prove that if $a|c$, $b|c$ and $(a,b)=1$ then $ab|c$.
8. Prove that if $(a,b)=1=(a,c)$ then $(a,bc)=1$.
9. If $(a,b)=1$ and if $c\mid a$ and $d\mid b$, then $(c,d)=1$.
10. If $(a,b)=(a,c)=1$, then $(a,bc)=1$.
11. If $(a,b)=1$, then $(a^n,b^k)=1$ for all $n\geq 1,\ k\geq 1$.
12. If $(a,b)=1$, then $(a+b,a-b)$ is either 1 or 2.
13. Every integer $n>1$ is either a prime number or a product of prime numbers.
14. Prove that there are infinitely many prime numbers.
15. If a prime p does not divide a, then $(p,a)=1$
16. If a prime p divides ab, then p/a or p/b. More generally, if a prime p divides a product $a_1a_2....a_n$ then p divides at least one of the factors.

II Seven marks questions

17. Find gcd of the following using division algorithm
    (i) 27, 87 (ii) 165, 418 (iii) 30, 42, 70 (iv) 50, 90, 240 (v) 18, 24, 30, 38.
18. Let $d=(826,1890)$.Use the Euclidean algorithm to compute d, then express d as a linear combination of 826 and 1890.
19. If n,m, and $g>0$ are integers, then $g=(n,m)$ if and only if $(n/g,m/g)=1$
20. Using Division algorithm, express gcd (a,b) as a linear combination of a and b by finding gcd of a and b for the following:
    (i) 27, 87 (ii) 165, 418 (iii) 252, 595 (iv) 367, 671.

Diophantine equation

I Two & Four marks questions

21. Define Linear Diophantine Equation.
22. Write the condition for the existence of solution of LDE.

23. Determine whether the following LDE's are solvable or not:

(i) $12x+18y=30$ (ii) $2x+3y=4$ (iii) $6x+8y=25$ (iv) $12x+16y=18$.

II Seven marks questions

24. Find the general solution of the following LDE's:
(i) $28x+91y=119$ (ii) $63x-23y=-7$ (iii) $15x+21y=39$.
25. Find the general solution of the Diophantine equation
i) $1485x+1745y=15$
ii) $1492x+1066y=-4$

Congruences

I Two & Four marks questions

26. Define linear congruence modulo m.
27. Write the relation between linear congruences and LDE's.
28. Define Euler's Totient function.
29. State two properties of Euler's totient function.
30. State and prove Euler's theorem.
31. State and prove Fermat's little theorem.
32. State and prove Wilson's theorem.
33. State Chinese remainder theorem.
34. If the public key is the pair $n=10147$ and $j=119$, then what is the decoding transformation?

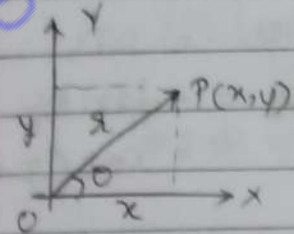
II Seven marks questions

35. Solve the following linear congruences:
(i) $27x \equiv 11 \pmod 7$ (ii) $37x \equiv 5 \pmod{11}$ (iii) $15x \equiv 12 \pmod{36}$ (iv) $36x \equiv 96 \pmod{156}$.
36. Express the linear congruence the following into linear Diophantine equations:
(i) $3x \equiv 4 \pmod 5$ (ii) $27x \equiv 11 \pmod 7$ (iii) $8x \equiv 5 \pmod{11}$.
37. Determine following if the congruences are solvable. Find the number of incongruent solutions when a congruence is solvable.
(i) $8x \equiv 10 \pmod 6$ (ii) $2x \equiv 3 \pmod 4$ (iii) $4x \equiv 7 \pmod 5$
38. Solve the following system of linear congruences by Chinese remainder theorem:
(i) $x \equiv 1 \pmod 3$ (ii) $x \equiv 2 \pmod 5$ (iii) $x \equiv 1 \pmod 3$ iv) $x \equiv 2 \pmod 7$
39. Using Fermat's theorem, find the remainder
(i) When 16^{53} is divided by 7.
(ii) When 3^{247} is divided by 17.
(iii) When 3^{181} is divided by 17.
(iv) When 3^{289} is divided by 23.
40. Using Fermat's theorem, Solve the following linear congruences:
(i) $12x \equiv 6 \pmod 7$ (ii) $37x \equiv 5 \pmod{11}$ (iii) $15x \equiv 12 \pmod{36}$ (iv) $3x \equiv 4 \pmod 5$.
41. Prove that the positive integer a is self invertible modulo p iff $a \equiv \pm 1 \pmod p$.
42. Let p be a prime and a be any integer such that $(a,p)=1$ then show that a^{p-2} is an inverse of a modulo p.
43. Let p be a prime and a be any integer such that $(a,p)=1$ then show that the solution

UNIT 1

Unit I

1. ⊛



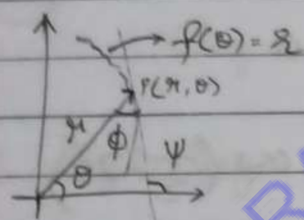
(x,y) P then $\cos\theta = \frac{x}{r}$ & $x = r\cos\theta$ ① CC

$\sin\theta = \frac{y}{r}$ & $y = r\sin\theta$ ② By DF $\sqrt{x^2+y^2} = |OP| = r$ ③

and $\tan\theta = \frac{y}{x}$, $\theta = \tan^{-1}(y/x)$ — ④

$r \rightarrow$ Radius vector & $\theta \rightarrow$ (polar angle) angle b/w RV & fixed line

2. ⊛ → 9.



$\tan\psi = \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{r\cos\theta + \sin\theta\,(dr/d\theta)}{-r\sin\theta + \cos\theta\,(dr/d\theta)}$

$y = r\sin\theta$
$x = r\cos\theta$ $\quad [\psi = \theta + \phi]$ $\left(\div \cos\theta \left(\frac{dr}{d\theta}\right)\right)$

$= \frac{r(d\theta/dr) + \tan\theta}{-r\tan\theta\,(d\theta/dr) + 1} = \tan\psi = \tan(\theta+\phi)$

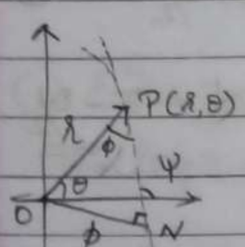
$= \frac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi}$

So $\tan\phi = r\left(\frac{d\theta}{dr}\right)$ or $\cot\phi = \frac{1}{r}\left(\frac{dr}{d\theta}\right)$ — ①

$r \rightarrow$ RV, $\theta \rightarrow$ angle b/w RV & init line

$\phi \rightarrow$ angle b/w RV & tangent to curve & ψ angle by tangent line with init

3. ⊛ → 10.

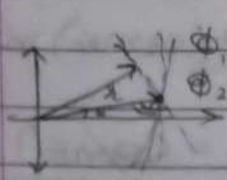


$\sin\phi = \frac{p}{r}$ & $p = r\sin\phi$ — ①

$\frac{1}{p^2} = \frac{1}{r^2}(1+\cot^2\phi) \Rightarrow \frac{1}{p^2} = \frac{1}{r^2}\left(1 + \frac{1}{r^2}\left(\frac{dr}{d\theta}\right)^2\right)$

$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$ — ②

⊛



$|\phi_2 - \phi_1|$ = angle b/w curves, so

$\tan(\phi_2 - \phi_1) = \frac{\tan\phi_2 - \tan\phi_1}{1 + \tan\phi_2\tan\phi_1}$

(orthogonal $\phi_2 - \phi_1 = \pi/2$ & $\tan\phi_1 \cdot \tan\phi_2 = -1$)

Depending on explicit avail. of ϕ, use LHS / RHS

4 (a) $r = a(1+\cos\theta)$; $\cos 2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$

$\frac{dr}{d\theta} = a(-\sin\theta)$ $\{ a = \frac{r}{1+\cos\theta}$

$\frac{1}{r}\frac{dr}{d\theta} = \frac{-\sin\theta}{1+\cos\theta} = -\frac{\cancel{2}\sin\theta/2 \cancel{\cos\theta/2}}{\cancel{2}\cos^2\theta/2} = -\tan\theta/2$
$= \cot\left(\frac{\pi}{2}+\frac{\theta}{2}\right)$

$\cot\phi$ so $\phi = \frac{\pi}{2}+\frac{\theta}{2}$

(b) $r^2\cos 2\theta = a^2$

$\frac{dr}{d\theta}\cancel{2}r\cos 2\theta - r^{\cancel{2}}\sin 2\theta . \cancel{2} = 0$

$\frac{1}{r}\frac{dr}{d\theta}\cos 2\theta = \sin 2\theta$

$\cot\phi = \tan 2\theta = \cot\left(\frac{\pi}{2}-2\theta\right)$

So $\phi = \frac{\pi}{2} - 2\theta$

(c) $\frac{2a}{r} = 1-\cos\theta$

$\log 2a - \log r = \log(1-\cos\theta) = 0 - \frac{1}{r}\frac{dr}{d\theta} = \frac{1}{1-\cos\theta}\sin\theta$

$-\frac{1}{r}\frac{dr}{d\theta} = \frac{\cancel{2}\sin\cancel{\theta/2}\cos\theta/2}{\cancel{2}\sin^{\cancel{2}}\theta/2} = \cot\theta/2 = -\cot\theta/2$

$= \cot\left(\pi - \theta/2\right) = \cot\phi$ So $\phi = \pi - \theta/2$

(d) $r^m = a^m(\cos m\theta + \sin m\theta)$

$m\log r = m\log a + \log(\cos m\theta + \sin m\theta)$

$\cancel{m}\frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{1}{\cos m\theta + \sin m\theta}(-\sin m\theta + \cos m\theta)\cancel{m}$

$\cot\phi = \frac{\cos m\theta - \sin m\theta}{\cos m\theta + \sin m\theta} = \frac{\cancel{\cos m\theta}(1-\tan m\theta)}{\cancel{\cos m\theta}(1+\tan m\theta)}$

$= \tan(\pi/4 - m\theta) = \cot(\pi/2 - \pi/4 + m\theta)$. $\phi = \frac{\pi}{4}+m\theta$

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5. (a) $\frac{2a}{r} = 1 - \cos\theta$ at $\theta = \frac{2\pi}{3}$:

$\log 2a - \log r = \log(1 - \cos\theta) = 0 - \frac{1}{r}\frac{dr}{d\theta} = \frac{1}{1-\cos\theta}\sin\theta$

$\cot\phi = -\frac{\cancel{2}\sin\theta/2\cos\theta/2}{\cancel{2}\sin^2\theta/2} = -\cot\theta/2 = \cot(\pi - \theta/2)$

$\psi = \theta + \phi = \pi - \frac{\theta}{2} + \theta = \pi + \frac{\theta}{2} = \pi + \frac{2\pi}{6} = \frac{8\pi}{6}$

$\tan(4\pi/3) = \sqrt{3}$

(b) $2r = a\sin 2\theta$ at $\theta = \pi/4$:

$2\log = \cancel{2}\frac{1}{r}\frac{dr}{d\theta} = a\cos 2\theta . \cancel{2} = \cot\phi = a\cos 2\theta$

OR

$\log 2 + \log r = \log a + \log\sin 2\theta = 0 + \frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{1}{\sin 2\theta}\cos 2\theta \, 2$

$\cot\phi = 2\cot 2\theta$ OR $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{a/2\, 2\sin 2\theta\cos 2\theta . 2}{\frac{a}{2}(2\cos\theta\cos 2\theta - \sin 2\theta\sin\theta)}$

$x = r\cos\theta = \frac{a\sin 2\theta\cos\theta}{2}$

$y = r\sin\theta = \frac{a\sin^2 2\theta}{2}$ $= \frac{4}{2}\frac{\sin 4\theta}{2\cos\cos 2\theta - \sin 2\theta\sin\theta}$

$= 2\left(\frac{\sin\pi}{B\ldots}\right) = 2\left(\frac{0}{\ldots}\right) = -1 = \tan\psi$ (?)

(c) $r^2\cos 2\theta = a^2$ at $\theta = \pi/12$:

$2\log r + \log\cos 2\theta = 2\log a \Rightarrow \cancel{2}\frac{1}{r}\frac{dr}{d\theta} + \frac{1 - \sin 2\theta . \cancel{2}}{\cos 2\theta} = 0$ $\tan 2\theta$

$\cot\phi = \tan 2\theta = \cot(\pi/2 - 2\theta)$ so $\phi = \frac{\pi}{2} - 2\theta$

so $\psi = \theta + \phi$ or $\frac{\pi}{2} - 2\theta + \theta = \frac{\pi}{2} - \theta = \frac{\pi}{2} - \frac{\pi}{12} = \frac{5\pi}{12}$

or $\tan\psi = \tan\frac{5\pi}{12} = 2 + \sqrt{3}$

(d) $r\sec^2(\theta/2) = 4$ at $\theta = \pi/2$

$\log r + 2\log(\sec(\theta/2)) = \log 4$ diff

$$\frac{1}{r}\frac{dr}{d\theta} + \cancel{2}\,\frac{1}{\sec(\theta/2)}\sec\theta/2\,\tan\theta/2\,\frac{1}{\cancel{2}} = 0$$

$$\cot\phi = -\tan\theta/2 = \cot\left(\frac{\pi}{2}+\frac{\theta}{2}\right)$$

$$\phi = \frac{\pi}{2}+\frac{\theta}{2} \quad \{ \theta+\phi = \psi$$

$$= \frac{\pi}{2}+\frac{\theta}{2}+\theta = \frac{\pi}{2}+\frac{3\theta}{2}$$

So $\tan\psi = \tan\left(\pi/2 + \frac{3\pi}{4}\right) = \tan\left(\frac{5\pi}{4}\right) = 1$

6. "tangent makes $(n+1)\theta$ with initial line" $= \psi = \theta+\phi$

So $n\log r = n\log a + \log(\sin n\theta) \Rightarrow \cancel{n}\frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{1\cos n\theta}{\sin n\theta}$

$\cot\phi = \cot n\theta \Rightarrow \phi = n\theta \quad \{ \phi+\theta = (n+1)\theta$ shown

14.5. $r = a(1+\cos\theta)$ $\pi/3 =$ (i) $\{ \perp^{(ii)} 2\pi/3$ (ii) θ

$$\frac{1}{r}\frac{dr}{d\theta} = a(0-\sin\theta) \text{ so } \frac{1}{r}\frac{dr}{d\theta} = -\frac{\sin\theta}{1+\cos\theta} = \frac{-2\sin\theta/2\cos\theta/2}{2\cos^2\theta/2}$$

$$\cot\phi = -\tan\theta/2 = \cot(\theta/2+\pi/2)$$

(i) $\phi = \frac{\theta}{2}+\frac{\pi}{2}$ and $\frac{\pi}{3}+\frac{\theta}{2} = \left(\frac{2\pi}{3}\right)\tan = 0$

$\psi = \theta+\phi = \theta+\frac{\theta}{2}+\pi/2 = 0$ Shown

(ii) $\phi+\theta = \psi = \frac{2\pi}{3}+\frac{\theta}{2}+\frac{\pi}{2}$ $\{ \tan(\psi) = \infty$ Shown

11. (a) $r = ae^{\theta}$ $\{ re^{\theta} = b$:

"int^s e/o $\perp^{lly}$ then $|\phi_2-\phi_1| = \pi/2$"

$\log r = \log a + \theta\log e$, $\log r + \theta\log e = \log b$

$$\frac{1}{r}\frac{dr}{d\theta} = 0+\theta \quad \{ \frac{1}{r}\frac{dr}{d\theta} = -\theta \Rightarrow \cot\phi_1 = \theta$$

$$\cot\phi_2 = -\theta$$

$$\Rightarrow \frac{1}{r}\frac{dr}{d\theta} = 0+1 \quad \{ \frac{1}{r}\frac{dr}{d\theta} = -1 \Rightarrow \frac{\pi}{2}+\frac{\pi}{4} \quad \phi_2-\phi_1$$

$\phi_1 = \pi/4$ $\quad \phi_2 = \frac{3\pi}{4}$ $\quad = \pi/2$ Shown

(b) $r^n = a^n \cos n\theta$, $r^n = a^n \sin n\theta$

$n \log r = n \log a + \log \cos n\theta$ $\quad$ $n \log r = n \log a + \log \sin n\theta$

$\cancel{n} \frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{1}{\cos n\theta} - \sin n\theta \cdot \cancel{n}$ $\quad$ $\cancel{n}\frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{1}{\sin n\theta} \cos n\theta \cancel{n}$

$$\cot\phi_1 = -\tan n\theta = \cot\left(\frac{\pi}{2}+n\theta\right) \quad \cot\phi_2 = \cot n\theta$$

$$\phi_1 = \frac{\pi}{2} + n\theta \qquad \phi_2 = n\theta$$

$|\phi_2 - \phi_1| = \pi/2$ so showed

(c) $r = a(1+\cos\theta)$, $r = b(1-\cos\theta)$:

$\log r = \log a + \log(1+\cos\theta)$ $\quad$ $\log r = \log b + \log(1-\cos\theta)$

$\frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{1}{1+\cos\theta} - \sin\theta$ $\quad$ $\frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{1}{1-\cos\theta}\sin\theta$

$\cot\phi_1 = \frac{-\sin\theta/_2 \cancel{\cos\theta/_2} \cancel{2}}{\cancel{2}\cos^{\cancel{2}}\theta/_2} = -\tan\theta/_2$ $\quad$ $\cot\phi_2 = \cot\theta/_2$

$= \cot\left(\frac{\pi}{2}+\frac{\theta}{2}\right)$ $\quad$ or $\frac{\cancel{2}\cancel{\sin\theta/_2}\cos\theta/_2}{\cancel{2}\sin^{\cancel{2}}\theta/_2}$

$\phi_1 = \pi/2 + \theta/2$ $\quad$ $|\phi_2 - \phi_1| = \theta/_2 - \theta/_2$ $\quad$ $\phi_2 = \theta/_2$ $\quad$ $= \cot\theta/_2$

so $|\phi_2 - \phi_1| = \cancel{\theta/_2} - \cancel{\theta/_2} + \pi/_2 = \pi/_2$ showed

12. (a) $r = \frac{a\theta}{1+\theta}$, $r = \frac{a}{1+\theta^2}$:

$\log r = \log a\theta - \log(1+\theta) = \log a + \log\theta - \log(1+\theta)$

$\frac{1}{r}\frac{dr}{d\theta} = \cancel{\frac{a}{\theta}}$ $\quad 0 + \frac{1}{\theta} - \frac{1}{1+\theta} = \frac{1}{\theta(1+\theta)} = \cot\phi_1$

$\log r = \log a + (-\log(1+\theta^2)) \Rightarrow \frac{1}{r}\frac{dr}{d\theta} = 0 - \frac{1}{1+\theta^2} 2\theta$

$\cot\phi_2 = -\frac{2\theta}{1+\theta^2}$; $\frac{a\theta}{1+\theta} = \frac{a}{1+\theta^2}$ $\quad$ ($f(\theta) = r_1 = r_2 = f(\theta)$)

$\cancel{\theta} + \theta^3 = 1 + \cancel{\theta}$; $\theta = 1$ $\quad$ $\phi_1 = \cot^{-1}\left(\frac{1}{2}\right)$ $\phi_2 = \cot^{-1}(-1)$

$\tan\phi_1 = 2$ $\quad$ $\tan^{-1}\phi_2 = -1$ $\quad$ $|\phi_2 - \phi_1| x$ $\quad$ so $\tan|\phi_2 - \phi_1|$

$= \left|\frac{2+1}{1+(-2)}\right| = 3$ so $\tan(\phi_2 - \phi_1) = 3$ or $\tan^{-1}(3)$

(b) $r = a\log\theta$, $r = \frac{a}{\log\theta}$

$\log r \Rightarrow \frac{dr}{dr} = a\frac{1}{\theta} = \frac{r}{\theta(\log\theta)}$ So $\frac{dr}{d\theta}\frac{1}{r} = \frac{1}{\theta(\log\theta)}$

$\cot\phi_1 = \frac{1}{\theta(\log\theta)}$ & $(\log\theta)^{-1}a = r \Rightarrow \frac{dr}{d\theta} = \times$ So

$r\log\theta = a$ & $\frac{dr}{d\theta}\log\theta + \frac{r}{\theta} = 0$ So, $\frac{dr}{d\theta}\frac{1}{r} = \frac{-1}{\theta(\log\theta)}$

$\cot\phi_2 = \frac{-1}{\theta(\log\theta)}$ & $\cancel{a}\log\theta = \frac{\cancel{a}}{\log\theta}$ & $\log_e^2\theta = 1 \Rightarrow \log_e\theta = 1$
$\Rightarrow e = \theta$

$\cot\phi_1 = \frac{1}{e(\log_e e)} \Rightarrow \tan\phi_1 = e$ & $\tan\phi_2 = -e$

So, $\tan|\phi_2 - \phi_1| = \left|\frac{-e-e}{1+(-e^2)}\right| = \frac{2e}{1-e^2} = \tan|\phi_2 - \phi_1|$

So, $\phi_2 - \phi_1 = \tan^{-1}\left(\frac{2e}{1-e^2}\right)$ is the ans

(c) $r^2\sin 2\theta = 4$ & $r^2 = 16\sin 2\theta$

$2\log r + \log\sin 2\theta = \log 4$

$\cancel{2}\frac{1}{r}\frac{dr}{d\theta} + \frac{1}{\sin 2\theta}\cos 2\theta\cdot\cancel{2} = 0$ So $\cot\phi_1 = \cancel{\tan^{-1}}\cot(\pi - 2\theta)$

$\cot 2\theta$ $\qquad \phi_1 = \pi - 2\theta$

$2\log r = \log 16 + \log\sin 2\theta \Rightarrow \cancel{2}\frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{1}{\sin 2\theta}\cos 2\theta\cdot\cancel{2}$

$\cot\phi_2 = \cot 2\theta$ & $\phi_2 = 2\theta$ So $\phi_2 - \phi_1 = \pi - 2\theta + 2\theta = \pi$ $\times$

Or $\pi - 4\theta$ & $r^2\sin 2\theta = 4$ equate

$\frac{16}{\sin 2\theta} = 4 \cdot 16\sin 2\theta$ or $\frac{1}{2} = \sin^2 2\theta \Rightarrow \frac{1}{2} = \sin 2\theta$

$\sin\left(\frac{\pi}{6}\right) = \sin 2\theta$ or $\theta = \frac{\pi}{12}$ So $\pi - \frac{\pi}{3} = \frac{2\pi}{3}$

13 (a) $\phi = r\sin\phi \rightarrow \frac{dr}{d\theta} = a(0 + \sin\theta) = \frac{r}{1-\cos\theta}\sin\theta$ or $\cot\phi = \frac{\sin\theta}{1-\cos\theta}$

$\frac{2\sin\theta/2\cos\theta/2}{2\sin^2\theta/2} = \cot\theta/2$ so $\phi = \theta/2$ and $\theta = \frac{\pi}{2}$ thus $p = r \times \sin\phi$

$p = r\sin\theta/2$ or $p = a\sin\frac{\pi}{4} = \frac{a}{\sqrt{2}} = p$

(b) $r^2\cos 2\theta = a^2$, $\theta = \pi/6$, $p = r\sin\phi$ so,

$2\log r + \log\cos 2\theta = 2\log a \Rightarrow \frac{2}{r}\frac{dr}{d\theta} + \frac{(-1)2\sin 2\theta}{\cos 2\theta} = 0$

$\cot\phi = \tan 2\theta = \cot(\pi/2 - 2\theta)$ so $\phi = \pi/2 - 2\theta = \frac{\pi}{2} - \frac{\pi}{3}$
$= \pi/6$ and $p = r\sin(\pi/6)$ and for r, $r^2 = 2a^2$

$r = a\sqrt{2}$ thus $p = \frac{a\sqrt{2}}{2}$ or $p = a/\sqrt{2}$ //

14. (a) $r^n = a^n\cos n\theta$ so $n\log r = n\log a + \log\cos n\theta$

$\frac{n}{r}\frac{dr}{d\theta} = 0 + \frac{1}{\cos n\theta}(-\sin n\theta \cdot n)$

$\cot\phi = -\tan n\theta = \cot\left(\frac{\pi}{2} + n\theta\right)$

so $\phi = \frac{\pi}{2} + n\theta$

and $p = r\sin\left(\frac{\pi}{2} + n\theta\right) = r\cos n\theta$, $\left(\frac{r}{a}\right)^n$

so $p = \frac{r^{n+1}}{a^n}$

(b) $r^2 = a^2\sin 2\theta$

$2\log r = 2\log a + \log 2\theta \rightarrow \frac{2}{r}\frac{dr}{d\theta} = 0 + \frac{1}{2\theta}\cdot 2$ or $\cot\phi = \frac{1}{2\theta}$

so, $\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$ or $\frac{1}{p^2} = \frac{1}{r^2}(1 + \cot^2\phi)$ $\quad \phi = 2\theta$

$2\log r = 2\log a + \log\sin 2\theta \Rightarrow \frac{2}{r}\frac{dr}{d\theta} = 0 + \frac{1}{\sin 2\theta}\cos 2\theta \cdot 2$ $\quad \cot\phi$, $\cot 2\theta$

$p = r\sin 2\theta$ or $p = r\left(\frac{r}{a}\right)^2 = r^3/a^2 = p$ $\quad$ Req. p

(c) $r^n = a^n\operatorname{sech} n\theta$

$\frac{dr}{d\theta} = a\operatorname{sech} n\theta\tanh n\theta \cdot n = \frac{r}{\operatorname{sech} n\theta}\tanh n\theta\, n\operatorname{sech} n\theta$

$\cot\phi = n\tanh n\theta$ and $\frac{1}{p^2} = \frac{1}{r^2}(1+n^2\tan^2 h n\theta)$ ✗ ϕ

$r^n = a^n \operatorname{sech} n\theta$

$n\log r = n\log a + \log \operatorname{sech} n\theta \Rightarrow n\frac{1}{r}\frac{dr}{d\theta} = 0 - \operatorname{sech} n\theta \tanh n\theta \, n$

$= n\frac{1}{r}\frac{dr}{d\theta} = 0 - \frac{1}{\cancel{\operatorname{sech} n\theta}}\cancel{\operatorname{sech} n\theta}\tanh h n\theta \, n \Rightarrow \cot\phi = -\tanh n\theta$

$\cot\phi = -\tanh n\theta$

$\frac{1}{p^2} = \frac{1}{r^2}(1+\tan^2 h n\theta) \Rightarrow \frac{1}{p^2} = \frac{1}{r^2}\underbrace{\sec^2 h n\theta}$, $\frac{r^n}{a^n}$

$\frac{1}{p^2} = \frac{1}{r^2}\left(\frac{r^n}{a^n}\right)^2 \Rightarrow \frac{1}{p^2} \Rightarrow$ ✗ So $\frac{1}{p^2} = \frac{1}{r^2}(1+1\cancel{r}\sec^2 h n\theta)$

$\frac{1}{p^2} = \frac{1}{r^2}\left(2-\left(\frac{r^n}{a^n}\right)^2\right)$

(c) $r = a\operatorname{sech} n\theta$

$\frac{dr}{d\theta} = -a\operatorname{sech} n\theta \tanh n\theta \, n = -\frac{r}{\cancel{\operatorname{sech} n\theta}}\tanh n\theta \cancel{\operatorname{sech} n\theta}\, n$

$\frac{dr}{d\theta}\frac{1}{r} = -n\tanh n\theta = \cot\phi$ So $\frac{1}{p^2} = \frac{1}{r^2}(1+\cot^2\phi)$

$\frac{1}{p^2} = \frac{1}{r^2}(1+n^2(1-\sec h n\theta)^2) \Rightarrow \frac{1}{p^2} = \frac{1}{r^2}\left(1+n^2\left(1-\frac{r}{a}\right)^2\right)$

$\frac{1}{p^2} = \frac{1}{r^2}\left(1+n^2\left(\frac{a-r}{a}\right)^2\right) \Rightarrow \frac{1}{p^2} = \frac{1}{r^2}($ ✗ as proper

$\frac{1}{p^2} = \frac{1}{r^2}(1+n^2\tan^2 h n\theta) = \frac{1}{r^2}\left(1+n^2(1-\sec^2 h n\theta)\right)$

$= \frac{1}{r^2}\left(1+n^2\left(1-\frac{r^2}{a^2}\right)\right)$ or $\frac{1}{p^2} = \frac{1}{r^2} + \frac{n^2}{r^2}\left(\frac{a^2-r^2}{a^2}\right)$

(d) $r = ae^{m\theta}$

$\log r = \log a + m\theta \Rightarrow \frac{1}{r}\frac{dr}{d\theta} = 0 + m$ or $\cot\phi = m$

So, $\frac{1}{p^2} = \frac{1}{r^2}(1+m^2)$

(e) $r^m = a^m(\cos m\theta + \sin m\theta)$

$m\log r = m\log a + \log(\cos m\theta + \sin m\theta)$

$\frac{1}{r}\frac{dr}{d\theta} m = 0 + \frac{1}{\cos m\theta + \sin m\theta}(-\sin m\theta + \cos m\theta) m = \cos m\theta(1-\tan$

$\cot\phi = \tan\left(\frac{\pi}{4} - m\theta\right) = \cot\left(\frac{\pi}{2} - \frac{\pi}{4} + m\theta\right)$ so $\phi = \frac{\pi}{4} + m\theta$

$\frac{1}{p} = \frac{1}{r} \times$ or $p = r\sin\phi = r\sin\left(\frac{\pi}{4} + m\theta\right)$

$p = r\left(\frac{1}{\sqrt{2}}\cos m\theta + \frac{1}{\sqrt{2}}\sin m\theta\right) \Rightarrow p = \frac{r}{\sqrt{2}}\left(\frac{r^m}{a^m}\right) = \frac{r^{m+1}}{(\sqrt{2})a^m}$

(f) $\frac{l}{r} = 1 + e\cos\theta$

$\log l - \log r = \log 1 + \log_e e^{\cos\theta}$ $\Rightarrow -\frac{1}{r}\frac{dr}{d\theta} = -\sin\theta$

$\cot\phi = \sin\theta$ & $\frac{1}{p^2} = \frac{1}{r^2}(1 + \sin^2\theta) = \frac{1}{r^2}\cos^2\theta$ & for θ.

$\frac{1}{p^2} = \frac{1}{r^2}(2 - \cos^2\theta)$? $\cos\theta = \left(\frac{l}{r} - 1\right)\frac{1}{e} = \frac{1}{e^2}\left(\frac{l}{r} - 1\right)^2$

~~So $\frac{1}{p^2} = \frac{1}{r^2}\left(2 - \frac{1}{e^2}\left(\frac{l}{r} - 1\right)^2\right) = \frac{1}{r^2}\left(2 - \frac{1}{e^2}\left(\frac{l^2}{r^2} + 1 - \frac{2l}{r}\right)\right)$~~

$\log l - \log r = \log(1 + e\cos\theta) \Rightarrow 0 - \frac{1}{r}\frac{dr}{d\theta} = \frac{1}{1 + e\cos\theta}(-e\sin\theta)$

$\cot\phi = \frac{e\sin\theta}{1 + e\cos\theta}$ & $\frac{1}{p^2} = \frac{1}{r^2}\left(1 + \frac{e^2\sin^2\theta}{(1 + e\cos\theta)^2}\right)$

So $\frac{1}{p^2} = \frac{1}{r^2}\left(\frac{1 + e^2\cos^2\theta + 2e\cos\theta + e^2\sin^2\theta}{1 + e^2\cos^2\theta + 2e\cos\theta}\right) = \left(\frac{1 + e^2 + 2e\cos\theta}{(1 + e\cos\theta)^2}\right)\frac{1}{r^2}$

$= \left(\frac{1 + e^2 + 2\left(\frac{l}{r} - 1\right)}{(l/r)^2}\right)\frac{1}{r^2}$ So $\frac{1}{p^2} = \frac{1}{l^2}\left(e^2 - 1 + \frac{2l}{r}\right)$

(g) $r = ae^{\theta\cot\alpha}$

$\frac{dr}{d\theta} = ae^{\theta\cot\alpha}\cdot\cot\alpha$ So $\cot\phi = \cot\alpha$, $\phi = \alpha$ or $p = r\sin\alpha$

8. $r = a\theta$

$\frac{dr}{d\theta} = a \Rightarrow \frac{dr}{d\theta} = \frac{r}{\theta}$ or $\cot\phi = \frac{1}{r}\frac{r}{\theta}$ or $= \frac{1}{\frac{r}{a}} = \frac{a}{r}$

So $\frac{1}{p^2} = \frac{1}{r^2}\left(1 + \left(\frac{a}{r}\right)^2\right) \Rightarrow \frac{1}{p^2} = \frac{1}{r^2} + \frac{a^2}{r^4}$

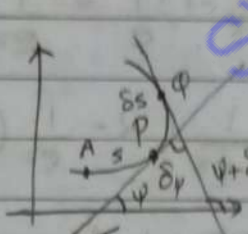
(★) Pedal eqn: eqn involving p & r w/o θ, ϕ or etc are p-r eqn where $r \to$ RV, $\phi \to \angle$OPFPTT & 'pedal' → tangent on which p dropped.

15. (★) $\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$ & $\frac{ds}{dy} = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}$ $s \to$ length of arc C
Cartesian

$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$ & $x = f(t)$, $y = f(t)$ & t is parameter, $t \leftrightarrow \theta$
$[t \leftrightarrow \theta]$ parametric

16. (★) $\frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}$ & $\frac{ds}{dr} = \sqrt{1 + r^2\left(\frac{d\theta}{dr}\right)^2}$ for $r = f(\theta)$
curve polar

17. (★) (i)



The curvature is as ROC of $\delta\psi$ (angle b/w Tangents) w.r.t δs as the point Q approaches P along curve.

So $C = \frac{d\psi}{ds} = \lim_{\delta s \to 0} \frac{\delta\psi}{\delta s}$

(ii) ROC: Reciprocal of curvature (if $C \neq 0$) of a curve at any point is ROC $\rho = \frac{ds}{d\psi}$

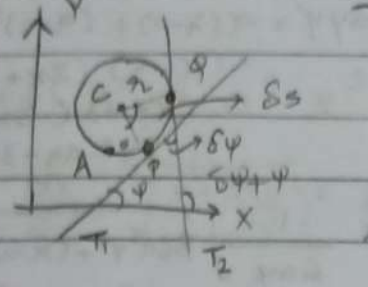
18. (★) Cartesian $\rho = \frac{(1 + (y_1)^2)^{3/2}}{y''}$ & $\frac{dy}{dx} = y'$, $\frac{d^2y}{dx^2} = y''$, $y = f(x)$

Parametric: $\rho = \frac{(\dot{x}^2 + \dot{y}^2)^{3/2}}{\dot{x}\ddot{y} - \dot{y}\ddot{x}}$ & $\dot{x} = \frac{dx}{dt}$, $\dot{y} = \frac{dy}{dt}$, $x = f(t)$, $y = f(t)$

↳

19. Polar $p = \dfrac{(r^2+(r')^2)^{3/2}}{r^2+2(r')^2-rr''}$ & $r' = \dfrac{dr}{d\theta}$, $r'' = \dfrac{d^2r}{d\theta^2}$, $r = f(\theta)$

Pedal $\rho = r\dfrac{dr}{dp}$ & $p = r\sin\phi$ (pedal / p-r eqn)

20. 

Let ψ & $\psi+\delta\psi$ be angle made by T_1 & T_2 resp wrt I.L & AQ be $s+\delta s$ as AP = s & PQ = δs and then $\angle PCQ = \delta\psi$ & by sector prop.

$\dfrac{\delta s}{r} = \delta\psi$ or $\delta s = r\,\delta\psi$ — ① and

By defn, $\lim\limits_{\delta s\to 0}\dfrac{\delta\psi}{\delta s} = \dfrac{d\psi}{ds} = \lim\limits_{\delta s\to 0}\dfrac{\delta\psi}{\delta\psi\, r} = \dfrac{1}{r}$ ∴ Constant @ every point $= 1/r$

(SL's C = 0 at every point)

21. $\sin\phi = r\dfrac{d\theta}{ds}$ proof:

Wkt $\psi = \theta+\phi$ and $p = r\sin\phi$ diff wrt s & r

$\dfrac{d\psi}{ds} = \dfrac{d\theta}{ds} + \dfrac{d\phi}{ds}$ $\quad \dfrac{dp}{dr} = \sin\phi + \cos\phi\dfrac{d\phi}{dr}r$ X

$\dfrac{ds}{d\theta} = \sqrt{r^2+\left(\dfrac{dr}{d\theta}\right)^2}$ & wkt $\cot\phi = \dfrac{1}{r}\dfrac{dr}{d\theta}$ & $\dfrac{dr}{d\theta} = r\cot\phi$ — ①

put ① $= \sqrt{r^2+r^2\cot^2\phi} = r\sqrt{\operatorname{cosec}^2\phi} = r\operatorname{cosec}\phi = \dfrac{ds}{d\theta}$

or, $\dfrac{d\theta}{ds}r = \sin\phi$ proved

22. $\cos\phi = \dfrac{dr}{ds}$ proof:

$\dfrac{ds}{dr} = \sqrt{1+r^2\left(\dfrac{d\theta}{dr}\right)^2}$ & wkt $\cot\phi = \dfrac{1}{r}\dfrac{dr}{d\theta}$ and $\dfrac{dr}{d\theta} = r\cot\phi$

$\dfrac{ds}{dr} = \sqrt{1+\cancel{r^2}\dfrac{\tan^2\phi}{\cancel{r^2}}} = \sec\phi$ or $\dfrac{dr}{ds} = \cos\phi$ $\quad \dfrac{d\theta}{dr} = \dfrac{1}{r}\tan\phi$

proved

24. (i) $3ay^2 = x(x-a)^2$.

$\frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}$ & $3a \cdot 2y = 3x^2 \frac{dx}{dy}$ & so w.r.t x

$x(x^2+a^2-2ax)$

$x^3+a^2x-2ax^2$

$6ay\frac{dy}{dx} = 3x^2+a^2-4ax$

$\frac{dy}{dx} = \left(\frac{3x^2+a^2-4ax}{6ay}\right)$

$\frac{ds}{dx} = \sqrt{1+\left(\quad\right)^2}$

OR $6ayy' = x(x-a)+(x-a)^2$

$(x-a)(2x+x-a)$

$(x-a)(3x-a)$

$3x^2-ax-3xa+a^2$

$= \sqrt{1+\frac{(x-a)^2(3x-a)^2}{36a^2y^2}}$

$= \sqrt{\frac{36a^2y^2+(x-a)^2(3x-a)^2}{36a^2y^2}} = \frac{1}{6ay}\sqrt{36a^2y^2+(x-a)^2(3x-a)^2}$ (2)

(ii) $x^{2/3}+y^{2/3}=a^{2/3}$

$\frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}$ so $\frac{2}{3}x^{-1/3}+\frac{2}{3}y^{-1/3}\frac{dy}{dx} = 0$

$\frac{dy}{dx}\left(\frac{2}{3}y^{-1/3}\right) = -\frac{2}{3}x^{-1/3} \Rightarrow \frac{dy}{dx} = -\frac{2}{3}x^{-1/3} \times \frac{3}{2}\frac{1}{y^{-1/3}} = \left(\frac{x}{y}\right)^{-1/3}$

$= \left(\frac{y}{x}\right)^{1/3}$

$\frac{ds}{dx} = \sqrt{1+\left(\frac{y}{x}\right)^{2/3}} = \sqrt{\frac{x^{2/3}+y^{2/3}}{x^{2/3}}} = \sqrt{\frac{a^{2/3}}{x^{2/3}}} = \left(\frac{a}{x}\right)^{1/3}$

(iii) $y = a\cosh(x/a)$

$\frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}$ & $\frac{dy}{dx} = a\sinh\left(\frac{x}{a}\right)\frac{1}{a} = \sinh\left(\frac{x}{a}\right)$

$\frac{ds}{dx} = \sqrt{1+\sinh^2\left(\frac{x}{a}\right)} = \sqrt{\cosh^2(x/a)} = \cosh\left(\frac{x}{a}\right) = \frac{y}{a}$

(iv) $y^2 = 4ax$

$2y\frac{dy}{dx} = 4a$ or $\frac{dy}{dx} = \frac{2a}{y}$ so $\frac{ds}{dx} = \sqrt{1+\left(\frac{2a}{y}\right)^2} = \sqrt{\frac{y^2+4a^2}{y^2}}$

$= \frac{1}{y}\sqrt{4a^2+y^2} = \sqrt{\frac{16a^2x^2+4a^2}{4ax}} = \sqrt{\frac{4a^2(4x^2+1)}{4ax}} = a$

$2yy' = 4a$

$y' = \frac{2a}{y} \rightarrow \frac{ds}{dx} = \sqrt{1+\frac{4a^2}{y^2}} = \sqrt{1+\frac{4a^2}{4ax}} = \sqrt{\frac{x+a}{x}}$

(v) $y = a\log\sec\left(\frac{x}{a}\right)$

$\frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}$ & $\frac{dy}{dx} = \cancel{a}\left[\frac{1}{\sec(x/a)}\sec x/a \tan x/a \cdot \frac{1}{\cancel{a}}\right] = \tan\left(\frac{x}{a}\right)$

so $\frac{ds}{dx} = \sqrt{1+\tan^2(x/a)} = \sec(x/a)$ //

23. (i) $a^2y^2 = a^3 - x^3$ at $(a,0)$:

$\frac{ds}{dy} = \sqrt{1+\left(\frac{dx}{dy}\right)^2}$ so $2a^2y = 0 - 3x^2\frac{dx}{dy} \Rightarrow 2a^2y = -3x^2\frac{dx}{dy}$

or $x' = \frac{2a^2y}{-3x^2}$ so $\sqrt{1+\left(\frac{2a^2y}{-3x^2}\right)^2} = \sqrt{1+\frac{2a^2\cancel{y}}{3x}} = 1$ //(?)

(ii) $ax^2 = y^3$:

$\frac{ds}{dy} = \sqrt{1+\left(\frac{dx}{dy}\right)^2}$ so $2axx' = 3y^2$ so $x' = \frac{3y^2}{2ax}$

$\frac{ds}{dy} = \sqrt{1+\frac{9y^4}{4a^2x^2}} = \sqrt{\frac{4a^2x^2+9y^4}{4a^2x^2}} = \frac{1}{2ax}\sqrt{4a^2x^2+9y^4}$ // (?)

(iii) $y = a\log\sec(x/a)$:

$\frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}$ so $\frac{dy}{dx} = \cancel{a}\left(\frac{1}{\sec(x/a)}\sec x/a \tan x/a\right)\frac{1}{\cancel{a}} = \tan x/a$

so $\frac{dx}{dy} = \cot x/a$ so $\sqrt{1+\cot^2 x/a} = \sqrt{\operatorname{cosec}^2 x/a} = \operatorname{cosec} x/a$ //

25. (i) $x = e^t\sin t$, $y = e^t\cos t$ $\left(\frac{ds}{dt}\right)$

$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}$ so $\frac{dx}{dt} = e^t\cos t$ $\frac{dy}{dt} = -e^t\sin t$

so $\sqrt{e^{2t}(1)} = e^t$ ✗ PR?st. $\frac{dx}{dt} = e^t\cos t + e^t\sin t$

$\frac{dy}{dt} = -e^t\sin t + e^t\cos t$ so $\frac{ds}{dt} = \sqrt{(e^t\cos t + e^t\sin t)^2 + (-e^t\sin t + e^t\cos t)^2}$

$= \sqrt{e^{2t}\cos^2 t + e^{2t}\sin^2 t + 2e^t\cos t\sin t + e^{2t}\sin^2 t + e^{2t}\cos^2 t - 2e^t\sin t\cos t}$

$= \sqrt{e^{2t}(1) + e^{2t}(1)} = \sqrt{2e^{2t}} = e^t\sqrt{2}$ //

(ii) $x = a\cos t$, $y = b\sin t$:

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \Rightarrow \frac{dx}{dt} = -a\sin t, \frac{dy}{dt} = b\cos t \text{ so}$$

$$\sqrt{a^2\sin^2 t + b^2\cos^2 t} = \sqrt{a^2\sin^2 t + b^2(1-\sin^2 t)}$$

$$= \sqrt{a^2\sin^2 t + b^2 - b^2\sin^2 t} = \sqrt{\sin^2 t(a^2-b^2)+b^2} //$$

26. $r\theta = a$, $\frac{ds}{dr}$ $\frac{ds}{d\theta}$:

$$\frac{ds}{dr} = \sqrt{1+r^2\left(\frac{d\theta}{dr}\right)^2} \quad : \theta + r\frac{d\theta}{dr} = 0 \text{ so } -\frac{\theta}{r} = \frac{d\theta}{dr} = -\frac{a}{r}\frac{\theta}{\cancel{a}}$$

$$= \sqrt{1+\cancel{r^2}\frac{\theta^2}{\cancel{r^2}}} = \sqrt{1+\theta^2} = \sqrt{1+(a/r)^2} = \frac{\sqrt{r^2+a^2}}{r} // (i)$$

$$\frac{ds}{d\theta} = \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2} \text{ so, } \frac{dr}{d\theta} = -\frac{r}{\theta} = \sqrt{r^2+\frac{r^2}{\theta^2}} = \sqrt{r^2\left(1+\frac{1}{\theta^2}\right)} = r\sqrt{\frac{\theta^2+1}{\theta^2}}$$

$$= \frac{r}{\theta}\sqrt{\theta^2+1} // (ii)$$

27. $\frac{ds}{d\theta} = \frac{a^2}{r}$ for $r^2 = a^2\cos 2\theta$ st.

$$\frac{ds}{d\theta} = \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2} \text{ so } \cancel{2}r\frac{dr}{d\theta} = -a^2\sin 2\theta\cdot\cancel{2} \quad \frac{dr}{d\theta} = \frac{-a^2\sin 2\theta}{r}$$

$$= \sqrt{r^2+\frac{a^4\sin^2 2\theta}{r^2}} = \sqrt{\frac{r^4+a^4\sin^2 2\theta}{r^2}} = \sqrt{\frac{a^4\cos^2 2\theta + a^4\sin^2 2\theta}{r^2}}$$

$$= \sqrt{\frac{a^4(1)}{r^2}} = \frac{a^2}{r} \text{ showed} //$$

14.95 $r^n = a^n\sin n\theta + b^n\cos n\theta$ st is $p^2(a^{2n}+b^{2n}) = r^{2n+2}$ pe

$p = r\sin\phi$ so $n\log r = \log(a^n\sin n\theta + b^n\cos n\theta)$

$$\cancel{n}\frac{1}{r}\frac{dr}{d\theta} = \frac{1}{a^n\sin n\theta + b^n\cos n\theta}\cdot a^n\cos n\theta - b^n\sin n\theta\cdot\cancel{n} = \cot\phi$$

$\frac{1}{p^2} = \frac{1}{r^2}(1+\cot^2\phi) \Rightarrow \frac{1}{p^2} = \frac{1}{r^2}\left(1+\left(\frac{a^n\cos n\theta - b^n\sin n\theta}{a^n\sin n\theta + b^n\cos n\theta}\right)^2\right)$

$\frac{1}{p^2} = \frac{1}{r^2}\left(\frac{a^{2n}\sin^2 n\theta + b^{2n}\cos^2 n\theta + b^{2n}\sin^2 n\theta + a^{2n}\cos^2 n\theta - \cancel{x} + \cancel{x}}{(a^n\sin n\theta + b^n\cos n\theta)^2}\right)$

$\frac{1}{p^2} = \frac{1}{r^2}\left(\frac{a^{2n}+b^{2n}}{r^{2n}}\right)$ so $p^2(a^{2n}+b^{2n}) = r^{2n+2}$ showed

28.2 $x = a(t+\sin t)$, $y = a(1-\cos t)$

(i) $\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}$ so $\frac{dx}{dt} = a(1+\cos t)$ $\frac{dy}{dt} = a(0+\sin t)$

$= \sqrt{a^2(1+\cos t)^2 + a^2\sin^2 t} = \sqrt{a^2 + a^2\cos^2 t + 2a^2\cos t + a^2\sin^2 t}$

$= \sqrt{a^2+a^2+2a^2\cos t} = \sqrt{2a^2(1+\cos t)} = a\sqrt{2}\sqrt{1+\cos t}$

$a\sqrt{2}\sqrt{2\cos^2 t/2} = 2a\cos t/2$ proved

(ii) $\frac{ds}{dy} = \sqrt{1+\left(\frac{dx}{dy}\right)^2}$ so $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\cancel{a}\sin t}{\cancel{a}(1+\cos t)} = \frac{2\sin t/2\cos t/2}{2\cos^2 t/2}$

$\sqrt{\frac{2a}{y}}$ $= \sqrt{1+\cot^2 t/2} = \sqrt{\operatorname{cosec}^2 t/2}$ $\frac{dy}{dx} = \tan t/2$

$= \operatorname{cosec} t/2$ $dx/dy = \cot t/2$

$\left(1-\frac{y}{a}\right) = \cos t$ or $-2\sin^2 t/2 + \cancel{1} = \cancel{1} - \frac{y}{a}$ or $\sin t/2 = \sqrt{y/a}$ ×

$\frac{ds}{dy} = \sqrt{1+\left(\frac{dx}{dy}\right)^2}$ so $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\cancel{a}\sin t}{\cancel{a}(1+\cos t)} = \frac{2\sin t/2\cos t/2}{2\cos^2 t/2}$

and $dx/dy = \cot t/2$ and $\sqrt{1+\cot t/2}$ $\frac{dy}{dx} = \tan t/2$

$(c^2 - c^2 = 1)$ $= \operatorname{cosec} t/2$ and $\left(\cancel{1}-\frac{y}{a}\right) = \cos t = \cancel{1} - 2\sin^2 t/2$

so $\sin t/2 = \sqrt{\frac{y}{2a}}$ and $\operatorname{cosec} t/2 = (2a/y)^{1/2} = \sqrt{\frac{2a}{y}}$ // proved

22.4 $x = a(\theta - \sin\theta)$, $y = a(1-\cos\theta)$

$\tan\psi = \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\cancel{a}(\sin\theta)}{\cancel{a}(1-\cos\theta)} = \frac{2\sin\theta/2\cos\theta/2}{2\sin^2\theta/2} = \cot\theta/2$ as $\theta + \phi$

$\tan\psi = \tan\left(\frac{\pi}{2}-\theta/2\right)$ so $\psi = \pi/2 - \theta/2$ //

$\frac{ds}{d\theta} \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}$ & $y = r\sin\theta$ $\quad a(\theta + \sin\theta) = r\cos\theta + \ldots$

$a(1-\cos\theta) = r\sin\theta \quad a\sin\theta = r\cos\theta + r'\ldots$

$r' \quad \frac{a\sin\theta - r\cos\theta}{\sin\theta} = a - r\cot\phi \quad r'$

$= \sqrt{r^2 + a^2 + r^2\cot^2\phi - 2ar\cot\phi}$

$\frac{ds}{d\theta} = \sqrt{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2} = \sqrt{a^2+a^2\cos^2\theta - 2a^2\cos\theta + a^2\sin^2\theta}$

$\frac{ds}{d\theta} = \sqrt{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2}$ & $\frac{dx}{d\theta} = a(1-\cos\theta)$ $\frac{dy}{d\theta} = a(\sin\theta)$

$\sqrt{a^2+a^2\cos^2\theta - 2a^2\cos\theta + a^2\sin^2\theta} = \sqrt{2a^2-2a^2\cos\theta}$

$= a\sqrt{2}\,\sqrt{2}\sin\theta/2 = 2a\sin\theta/2$

$\frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+\cot^2\phi/2} = \operatorname{cosec}\theta/2 \quad \text{|||y} \quad \frac{ds}{dy} = \sec\theta/2$

28.6 $y = a\sin t,\ x = a(\cos t + \log(\tan t/2))$ & ds/dt @ $t = \pi/4$

$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2} \rightarrow \frac{dx}{dt} = a\left(-\sin t + \frac{1}{\tan t/2}\sec^2 t/2 \cdot \frac{1}{2}\right) \quad \frac{dy}{dt} = a\cos t$

$a\left(-\sin t + \frac{1}{2\sin t/2\cos t/2}\right) = a\left(\frac{1-\sin^2 t}{\sin t}\right) = a\left(\frac{\cos^2 t}{\sin t}\right) = a\cot t\cos t$

So $\sqrt{a^2\cos^2 t\cot^2 t + a^2\cos^2 t} = \sqrt{a^2\cos^2 t(1+\cot^2 t)} = a\cos t\operatorname{cosec} t$

$= \frac{a\cos t}{\sin t} = a\cot t = a\cot(\pi/4) = a$ //

29. (a) $xy^3 = a^4$ at (a,a)

$\frac{(1+(y')^2)^{3/2}}{y''}$ So $3xy^2y' + y^3 = 0$

$y' = \frac{-y^3}{3xy^2} = \frac{-y}{3x} = \frac{-a}{3a} = -\frac{1}{3}$

So $\frac{\left(1+\frac{1}{9}\right)^{3/2}}{4/9a}$ $\quad y'' = -\left[\frac{3x(-1)y' - 3(y)}{9x^2}\right] = -\left[\frac{3a\left(-\frac{1}{3}\right) - 3a}{9a^2}\right]$

$\left(\frac{10}{9}\right)^{3/2}\frac{9a}{4} = \frac{10^{3/2}}{3^3}\cdot\frac{9a}{4} = \frac{10\sqrt{10}\,a}{12} = \frac{5\sqrt{10}\,a}{6}$ // $\quad = \frac{4}{9a}$

PAGE NO.
DATE / /

(b) $pa^2 = r^3$

$\rho = r\frac{dr}{dp}$ so $a^2 = 3r^2\frac{dr}{dp} \Rightarrow \frac{dr}{dp} = \frac{a^2}{3r^2} \Rightarrow \rho = \cancel{r}\frac{a^2}{3r^2} = \frac{a^2}{3r}$ //

(c) $x = a\cos\theta$ $y = a\sin\theta$ at $\left(\frac{a}{\sqrt2}, \frac{a}{\sqrt2}\right)$:

$\rho = \frac{(\dot{x}^2+\dot{y}^2)^{3/2}}{\dot{x}\ddot{y}-\ddot{x}\dot{y}}$ & $\dot{x} = -a\sin\theta$, $\dot{y} = a\cos\theta$ $\quad \frac{(a^2\sin^2\theta + \cos^2\theta a^2)^{3/2}}{a^2\sin^2\theta + a^2\cos^2\theta} = \frac{a^3}{a^2} = a$ //

$\ddot{x} = -a\cos\theta$ $\quad \ddot{y} = -a\sin\theta$

29. $\left.\begin{array}{l}\frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2} \\ \frac{ds}{dy} = \sqrt{1+\left(\frac{dx}{dy}\right)^2}\end{array}\right\}$ PF $\quad$ and, $y = r\sin\theta$, $x = r\cos\theta$ or $x^2+y^2 = r^2$

$\frac{y}{x} = \tan\theta \Rightarrow y = x\tan\theta$

X $\quad \frac{dy}{dx} = $ X

WKt $\frac{dy}{dx} = \tan\psi$ so $\sqrt{1+\tan^2\psi} = \sqrt{\sec^2\psi} \Rightarrow \frac{ds}{dx} = \sec\psi$ proved.

WKt $\frac{dy}{dx} = \tan\psi$ and $\sqrt{1+\cot^2\psi} = \sqrt{\operatorname{cosec}^2\psi} = \operatorname{cosec}\psi$ proved.

$dx/dy = \cot\psi$

31. (a) $x^3+y^3 = 3axy$ at $\left(\frac{3a}{2}, \frac{3a}{2}\right)$:

$\cancel{3}x^2 + \cancel{3}y^2\frac{dy}{dx} = \cancel{3}a(xy'+y) \Rightarrow x^2+y^2y' = axy'+ay$

$x^2 - ay = y'(ax - y^2)$ so $y' = \frac{x^2-ay}{ax-y^2}$ so $\frac{\frac{9a^2}{4} - \frac{\cancel{3}^{6}a^2}{4 \cancel{2}}}{\frac{3a^2}{2} - \frac{9a^2}{4}} = \frac{-\frac{3a^2}{4}}{\frac{3a^2}{4}} = -1$

$y' = -1$

$y'' = \frac{-16}{3a}$ $\quad y'' = \frac{(ax-y^2)(2x-ay') - (a-2yy')(x^2-ay)}{(ax-y^2)^2}$

$= \frac{\left(\frac{a\,3a}{2} - \frac{9a^2}{4}\right)\left(\frac{6a}{2}+a\right) - \left(a+\frac{6a}{2}\right)\left(\frac{9a^2}{4} - \frac{3a^2}{2}\right)}{\left(\frac{3a^2}{2} - \frac{9a^2}{4}\right)^2} = \frac{-\frac{6a^2}{4}\times 4a + 4a\times\frac{6a^2}{4}}{\frac{36a^4}{16}}$

$-3a^2/4 \quad 4a \quad 4a \quad 6a^2/4 \quad 3\,6a^2/4$

$= \frac{-12a^3}{\frac{36a^4}{16}} = -\frac{16\times 3}{3a}$

So $\rho = \frac{(1+(y')^2)^{3/2}}{y''} = \frac{(1+1)^{3/2}}{\frac{-16}{3a}} = \frac{2\sqrt2\times 3a}{-16} = -\frac{3\sqrt2 a}{8}$ But only +

So $3\sqrt2 a/8$ //

(b) $r^n = a^n \sin n\theta$

$$\rho = \frac{(r^2+(r_1)^2)^{3/2}}{r^2+2r_1^2-rr_2}$$

$n\log r = n\log a + \log \sin n\theta$

$$n\frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{1}{\sin n\theta}\cos n\theta \times n$$

$r\cot\phi = r\cot n\theta$

so $\frac{dr}{d\theta} = r\cot n\theta = r_1$

$$\frac{dr}{d\theta} = r\cot n\theta$$

$$\frac{d^2r}{d\theta^2} = -r\operatorname{cosec}^2 n\theta + \cot n\theta\, r_1$$

$$= \frac{(r^2 + r^2\cot^2 n\theta)^{3/2}}{r^2 + 2r^2\cot^2 n\theta + r^2\operatorname{cosec}^2 n\theta + r\cot n\theta(-1)}$$

$\rightarrow r^2 + 2r^2\cot^2 n\theta + r^2\operatorname{cosec}^2 n\theta - r^2\cot^2 n\theta$

$= r^2(1+\cot^2 n\theta) + r^2\operatorname{cosec}^2 n\theta = r^2(1+\operatorname{cosec}^2 n\theta)$

$= 2r^2\operatorname{cosec}^2 n\theta$ so $\frac{r^3}{r^2}\frac{(1+\cot^2 n\theta)^{3/2}}{2(\operatorname{cosec}^2 n\theta)}$ $\frac{r}{2}\operatorname{cosec} n\theta$

as n outside ×

$$\frac{(r^2+(r_1)^2)^{3/2}}{r^2+2(r_1)^2-rr_2}$$

$n\log r = n\log a + \log\sin n\theta$

$$n\frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{1}{\sin n\theta}\cos n\theta\, n$$

$r_2 = -nr\operatorname{cosec}^2 n\theta + \cot n\theta\, r_1$

$r_2 = -nr\operatorname{cosec}^2 n\theta + r\cot^2 n\theta$

$r_1 = r\cot n\theta$

$r_2 = -r\operatorname{cosec}^2 n\theta + \cot n\theta\, r_1$

$1/n$

$$\frac{(r^2 + r^2\cot^2 n\theta)^{3/2}}{r^2 + 2r^2\cot^2 n\theta + nr^2\operatorname{cosec}^2 n\theta - r^2\cot^2 n\theta} = \frac{r^3\operatorname{cosec}^3 n\theta}{r^2\operatorname{cosec}^2 n\theta(1+n)}$$

$\rightarrow = r^2 + r^2\cot^2 n\theta + nr^2\operatorname{cosec}^2 n\theta = r^2\operatorname{cosec}^2 n\theta + nr^2\operatorname{cosec}^2 n\theta$

$\rightarrow -r^2\operatorname{cosec}^2 n\theta(1+n)$

$$= \frac{r}{(1+n)}\operatorname{cosec} n\theta$$

(c) $x = a\cos^3 t$, $y = a\sin^3 t$

$$\rho = \frac{(\dot{x}^2+\dot{y}^2)^{3/2}}{\dot{x}\ddot{y}-\ddot{x}\dot{y}} = \frac{((-3a\cos^2 t\sin t)^2 + (3a\sin^2 t\cos t)^2)^{3/2}}{(-3a\cos^2 t\sin t)(-3a\sin^3 t + 6\sin t\cos^2 t) - (3a\sin^2 t\cos t)(-3a\cos^3 t + 6a\cos t\sin^2 t)}$$

$\ddot{y} = 3a(2\sin t\cos t\cos t - \sin^3 t)$

$= 3a(2\sin t\cos^2 t - \sin^3 t)$, $\ddot{x} = -3a(-2\cos t\sin^2 t + \cos^3 t)$

PAGE NO.
DATE / /

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3a\sin^2 t\cos t}{-3a\cos^2 t\sin t} = -\tan t = y'$ $\quad y'' = \frac{d^2y}{dx^2}$

$y'' = -\left(\sec^2 t\,\frac{dt}{dx}\right)$ $\{ \frac{dx}{dt} = -3a\cos^2 t\sin t$ $\left(-\frac{1}{3a}\sec^2 t\,\text{cosec}\,t\right)$

$= -\sec^2 t\left(-\frac{1}{3a}\sec^2 t\,\text{cosec}\,t\right) \Rightarrow \frac{1}{3a}\sec^4 t\,\text{cosec}\,t = y''$

$\frac{(1+(y')^2)^{3/2}}{y''}$

So $\frac{(1+\tan^2 t)^{3/2}}{\left(\frac{1}{3a}\sec^4 t\,\text{cosec}\,t\right)} = 3a\left[\frac{\sec^3 t\,\sin t}{\sec^4 t}\right] = 3a\cos t\sin t \Rightarrow \left(\times\frac{1}{2}\times 2\right)$

$= \left(\frac{3}{2}a\right)\sin 2t$

(d) $y = ax^2 + bx + c$ at $x = \frac{1}{2a}(\sqrt{a^2-1} - b)$

$y' = 2ax + b$ $\quad \frac{\left(1+\left(2a\frac{1}{2a}(\sqrt{a^2-1}-b+b)\right)^2\right)^{3/2}}{2a} = \frac{(1+a^2-1)^{3/2}}{2a} = \frac{a^3}{2a}$

$y'' = 2a$

$= a^2/2$ //

32. $x^4 + y^4 = 2$ at $(1,1)$, $\rho = \frac{\sqrt{2}}{3}$

$4x^3 + 4y^3 y' = 0$ $\quad \frac{(1+(y')^2)^{3/2}}{y''}$

$y' = -\frac{x^3}{y^3} = -1$ $\quad 3y^2y'(x^3) = 3y^2\cdot -\frac{x^3}{y^3}x^3 = \left(-\frac{3x^6}{y}\right)$

$y'' = -\left[\frac{3y^3x^2 - 3y^2y'x^3}{y^6}\right] = -\left[\frac{3y^3x^2 + \frac{3x^6}{y}}{y^6}\right] = -\left[\frac{3+3}{1}\right] = -6$

$= \frac{(1+(-1)^2)^{3/2}}{-6} = \frac{2\sqrt{2}}{-6_3} = -\frac{\sqrt{2}}{3}$ as ρ +ve, $-\frac{\sqrt{2}}{3}$ proved //

33. $r = ae^{\theta\cot\alpha}$, $a, \alpha \to$ const, ρ/r constant

$b = a\cot\alpha$

$\frac{dr}{d\theta} = a(\theta'\cot\alpha)e^{\theta\cot\alpha} = a\cot\alpha\, e^{\theta\cot\alpha} = (be^{\theta\cot\alpha} = r')$

$r'' = (b\cot\alpha)e^{\theta\cot\alpha} = (ce^{\theta\cot\alpha} = r'')$ $c = a\cot^2\alpha$

$\frac{(r^2+(r')^2)^{3/2}}{r^2+2(r')^2 - rr''} = \frac{(r^2 + a^2\cot^2\alpha\, e^{2\theta\cot\alpha})^{3/2}}{(r^2 + 2a^2\cot^2\alpha\, e^{2\theta\cot\alpha} - ra\cot^2\alpha\, e^{\theta\cot\alpha})}$

$$= \frac{\left(a^2 e^{2\theta\cot\alpha} + a^2\cot^2\alpha\, e^{2\theta\cot\alpha}\right)^{3/2}}{\left(a^2 e^{2\theta\cot\alpha} + 2a^2\cot^2\alpha\, e^{2\theta\cot\alpha} - a^2 e^{2\theta\cot\alpha}\cot^2\alpha\right)}$$

$$= \frac{\left(a^2 e^{2\theta\cot\alpha}(1+\cot^2\alpha)\right)^{3/2}}{\left(a^2 e^{2\theta\cot\alpha}(1+\cot^2\alpha)\right)} \to \frac{(a e^{\theta\cot\alpha}\operatorname{cosec}\alpha)^3}{(\operatorname{cosec}^2\alpha)^2} = a e^{\theta\cot\alpha}\operatorname{cosec}\alpha$$

$$= \frac{r\operatorname{cosec}\alpha}{r} = \frac{\rho}{r}$$

$\frac{\rho}{r} = \operatorname{cosec}\alpha$ = Constant proved

34. $x^2y = a(x^2+y^2)$ at $(-2a, 2a)$ is $2a$ (ρ).

$$\frac{(1+(y')^2)^{3/2}}{y''} \quad \text{so } y' \quad 2xy + x^2y' = a(2x+2yy')$$

$$2xy + x^2y' = 2xa + 2yay'$$

$$y'(2ay - x^2) = 2xy - 2xa$$

$$y' = \frac{-2xa+2xy}{2ay-x^2} = \frac{2xy-2xa}{2ay-x^2} = \frac{-8a^2+4a^2}{4a^2-4a^2} \quad \infty$$

$$\text{so } \frac{dx}{dy} = 0 = x' \quad \& \quad \frac{dx}{dy} = \frac{2ay-x^2}{2xy-2xa}$$

$$\frac{d^2x}{dy^2} = \frac{(2xy-2xa)(2a-2x\cancel{x'}) - (2(x+\cancel{x'y}) - \cancel{2x'a})(2ay-x^2)}{(2xy-2xa)^2}$$

$$= \frac{(2xy-2xa)(2a) - (2(x))(2ay-x^2)}{(2xy-2xa)^2}$$

$$x'' = \frac{(\overset{-4a^2}{-8a^2+4a^2})(2a) - (2(-2a)\cancel{(4a^2-4a^2)})}{(-8a^2+4a^2)^2} = \frac{-8a^3}{16a^4} = \frac{-1}{2a}$$

$$\text{so } \frac{(1+(x')^2)^{3/2}}{x''} = \frac{\left(1+\cancel{\frac{1}{4a^2}}\right)^{3/2}}{} = \frac{(1+0)^{3/2}}{-1/2a} = -2a$$

as +ve, $\rho = 2a$

35. RoC of $r^n = a^n\cos n\theta$ varies inversely as r^{n-1}.

$$\frac{(r^2+(r')^2)^{3/2}}{r^2+2(r')^2-rr''} = \rho \Rightarrow n\log r = n\log a + \log\cos n\theta$$

$$n\frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{1}{\cos n\theta}(-\sin n\theta\,\cancel{n}) \to -\tan n\theta$$

$$r' = -r\tan n\theta$$

$r' = (r\sec^2 n\theta \cdot n + \cancel{r}\tan n\theta)$ so $\dfrac{(r^2 + r^2\tan^2 n\theta)^{3/2}}{r^2 + 2r^2\tan^2 n\theta + r^2\sec^2 n\theta\, n - r^2\tan^2 n\theta}$

$= \dfrac{(r^2(1+\tan^2 n\theta))^{3/2}}{}$ $\quad r\sec n\theta$

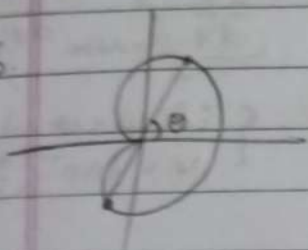
$r^2 + r^2\tan^2 n\theta + r^2\sec^2 n\theta \cdot n$

$r^2\sec^2 n\theta + \dots$

$r^2\sec^2 n\theta(1+n)$

$\rho = \dfrac{r\sec n\theta}{(1+n)}$ so $\rho = \dfrac{r}{1+n}\dfrac{1}{\cos n\theta}$, $\dfrac{r^n}{a^n} = \dfrac{r^{n+1}}{a^n(1+n)}$ so $\rho \propto \dfrac{1}{r^{n-1}}$ proved

36



ρ_1, ρ_2 { $r = a(1+\cos\theta)$; $r' = a(0-\sin\theta)$
$= a + a\cos\theta$ $\qquad r'' = a(0-\cos\theta)$

$\dfrac{(r^2 + a^2\sin^2\theta)^{3/2}}{r^2 + 2a^2\sin^2\theta + ra\cos\theta} = \dfrac{\left(a^2 + a^2\cos^2\theta + 2a^2\cos\theta + a^2\sin^2\theta\right)^{3/2}}{a^2 + a^2\cos^2\theta + 2a^2\cos\theta + 2a^2\sin^2\theta + a^2 + a^2\cos^2\theta}$

$\dfrac{(r^2 + (r')^2)^{3/2}}{r^2 + 2(r')^2 - rr''}$

$= \dfrac{(2a^2 + 2a^2\cos\theta)^{3/2}}{\underbrace{2a^2 + 2a^2}_{4a^2} + 2a^2\cos^2\theta} = \dfrac{(2a^2)^{3/2}\,(\cancel{1+\cos\theta})^{3/2}}{2a^2(\cancel{2+\cos^2\theta})}$ $\quad \frac{3}{2}-1 = \frac{1}{2}$

$= \sqrt{2a^2(1+\cos}$

$(a^2 + a^2\cos^2\theta + 2a^2\cos\theta + a^2\sin^2\theta)^{3/2}$ } $a^2 + a^2\cos^2\theta + 2a^2\cos\theta +$
$(2a^2)^{3/2}(1+\cos\theta)^{3/2}$ $\qquad 2a^2\sin^2\theta + a^2\cos\theta + a^2\cos^2\theta$

$= \dfrac{2\sqrt{2}\,a^{\cancel{3}}\,\cancel{(1+\cos\theta)}^{3/2}}{3\cancel{a^2}\cancel{(1+\cos\theta)}}$ $\quad (1+\cos\theta)^{1/2}$ $\qquad = \dfrac{a^2 + 2a^2 + 3a^2\cos\theta}{3a^2 + 3a^2\cos\theta}$
$\qquad 3a^2(1+\cos\theta)$

$\rho_1 = \dfrac{2\sqrt{2}}{3}a(1+\cos\theta)^{1/2}$ { $\rho_2, \theta_2 \Rightarrow \theta + \pi$ so $\theta+\pi$

$\rho_1^2 + \rho_2^2 = \left(\left(\dfrac{2\sqrt{2}a}{3}\right)^2 \times 2\right)(\cancel{1+\cos\theta+1+\cos(\theta+\pi)})$

$\rho_1^2 + \rho_2^2 = \dfrac{8}{3^2}a^2\,2 = \dfrac{16a^2}{9}$ proved

37 $\dfrac{1}{p^2} = \dfrac{1}{a^2} + \dfrac{1}{b^2} - \dfrac{r^2}{a^2b^2}$ at (p, r) ROC $= \dfrac{a^2b^2}{p^3}$ so $\rho = r\dfrac{dr}{dp}$ $\quad \rho = \dfrac{a^2b^2}{p^3}$ show

$= \dfrac{-2}{p^3} = 0 + 0 - \dfrac{2r}{a^2b^2}r'$ so $r' = \dfrac{a^2b^2}{p^3 r}$ so $r\dfrac{dr}{dp} = \dfrac{a^2b^2 r}{p^3 r}$

PAGE NO.
DATE

UNIT 2

Unit II

Q.5. (a) p.d of $ax^2+by^2+2hxy=z$.

$f_x = z_x = 2ax+2hy$, $z_y = 2by+2hx$

$z_{xx} = 2a+0 = 2a$ $\qquad z_{yy} = 2b$

$z_{xy} = 2h$ or $z_{yx} = 2b(0)+2h = 2h \Rightarrow z_{xy} = z_{yx}$

1\. (a) $u = \sin^{-1}(y/x)$

$u_x = \dfrac{1}{\sqrt{1-y^2/x^2}}\left(\dfrac{-y}{x^2}\right)$ and $u_y = \dfrac{1}{\sqrt{1-y^2/x^2}}\left(\dfrac{1}{x}\right)$

$= \dfrac{-yx^{-1}}{\sqrt{x^2-y^2}}$ and $= \dfrac{x}{\sqrt{x^2-y^2}}\dfrac{1}{x} = \dfrac{1}{\sqrt{x^2-y^2}}$

$u_x = \dfrac{-y}{x}\dfrac{1}{\sqrt{x^2-y^2}}$ $\qquad u_y = \dfrac{1}{\sqrt{x^2-y^2}}$

(b) $u = x^y + \dfrac{y}{2x}$

$u_x = yx^{y-1} + \dfrac{-y}{2x^2}$; $u_y = x^y \ln x + \dfrac{1}{2x}$

2\. $u = xf(y/x) + g(y/x)$ st $xu_x + yu_y = xf(y/x)$

$u_x = f(y/x) + g'(y/x)\left(\dfrac{-1}{x^2}\right)$ $\quad u_y = xf'(y/x)\left(\dfrac{1}{x}\right) + g'(y/x)\left(\dfrac{1}{x}\right)$

$xf(y/x) + xg'(y/x)\left(-\dfrac{1}{x^2}\right) + yxf'(y/x)\left(\dfrac{1}{x}\right) + yg'(y/x)(1/x)$

$= xf(y/x) - \dfrac{g'(y/x)}{x} + yf'(y/x) + \dfrac{y^2 g'(y/x)}{x^2}$

X

$xf'(y/x)(-y/x^2) + f(y/x) + g'(y/x)(-y/x^2) = u_x$

$xf'(y/x)(1/x) + g'(y/x)(1/x) = u_y$

$xu_x = -\cancel{yf'(y/x)} + xf(y/x) - \cancel{\dfrac{yg'(y/x)}{x}} = xf(y/x)$ Shown

$yu_y \quad + \cancel{yf'(y/x)} + \cancel{(y/x)(g'(y/x))}$

3. $u = \frac{y}{z} + \frac{z}{x}$ St $x u_x + y u_y + z u_z = 0$:

$u_x = \frac{-z}{x^2}$ $u_y = \frac{1}{z}$ $u_z = \frac{-y}{z^2} + \frac{1}{x}$ $\therefore \cancel{\frac{-z}{x}} + \cancel{\frac{y}{z}} - \cancel{\frac{y}{z}} + \cancel{\frac{z}{x}} = 0$ Shown

4. $z = e^{ax+by} f(ax-by)$ St $bz_x + az_y = 2abz$

$z_x = e^{ax+by} f'(ax-by).a + e^{ax+by} f(ax-by).a$

$z_y = e^{ax+by} f'(ax-by).-b + b\, e^{ax+by}(f(ax-by))$

$bz_x + az_y = ab e^{ax+by} (\cancel{f'(ax-by)} + f(ax-by))$

$+ ab e^{ax+by} (f(ax-by) - \cancel{f'(ax-by)})$

$= 2ab(e^{ax+by} f(ax-by)) = \underline{\underline{2abz}}$ Shown

5. $u = e^{a\theta}\cos(a\log r)$, pt $u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$

$u_r = -e^{a\theta}\sin(a\log r)\frac{a}{r}$; $u_{rr} = -a\left(-\frac{1}{r^2}e^{a\theta}\sin(a\log r) + \frac{1}{r}e^{a\theta}\cos(a\log r)\frac{a}{r}\right)$

$u_\theta = e^{a\theta}.a.\cos(a\log r) + e^{a\theta}\cancel{\cos}\sin(a\log r)$ $= \frac{a}{r^2}e^{a\theta}\sin(a\log r) - \frac{a^2}{r^2}e^{a\theta}\cos(a\log r)$

$- e^{a\theta}\sin(a\log r)\frac{a}{r}r_\theta$

$u = e^{a\theta}\cos(a\log r)$ so $u_\theta = ae^{a\theta}(a\log r) \Rightarrow a^2e^{a\theta}a\log r = u_{\theta\theta}$

$\cancel{\frac{a}{r^2}e^{a\theta}\sin(a\log r)} - \cancel{\frac{a^2}{r^2}e^{a\theta}\cos(a\log r)} - \cancel{\frac{a}{r^2}e^{a\theta}\sin(a\log r)} + \cancel{\frac{a^2}{r^2}e^{a\theta}\cos(a\log r)}$

$= 0$ // Proved

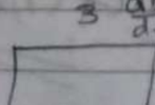
6. 

$h = 15cm$, $\frac{dh}{dt} = 0.2$, $r = 10$, $\frac{dr}{dt} = -0.3$

$\frac{dV}{dt} = ?$ So $\frac{1}{3}\pi r^2 h = V$

$\frac{1}{3}\pi\left(2r\frac{dr}{dt}h + \frac{dh}{dt}r^2\right) = \frac{dV}{dt}$

$= \frac{1}{3}\pi(2\times10\times-0.3\times15 + 0.2\times100) = -\frac{70}{3}\pi$ // decreasing

7. 

3 $\frac{db}{dt} = 0.5$, 4 $\frac{da}{dt} = 1.5$

$A = ab$ So $\frac{dA}{dt} = a\frac{db}{dt} + b\frac{da}{dt} = 4\times0.5 + 3\times1.5$

$= \underline{\underline{6.5 \text{ ft}^2/\text{s}}}$ (increase)

75. $u = \frac{1}{\sqrt{x^2+y^2+z^2}}$ ~~ST~~ ST $u_{xx}+u_{yy}+u_{zz}=0$

$u_x = \frac{1}{\cancel{2}\sqrt{x^2+y^2+z^2}} \cdot \cancel{2}x = \frac{x}{\sqrt{x^2+y^2+z^2}}$

$u_{xx} = \frac{x}{2\sqrt{x^2+y^2+z^2}}$ X $\frac{1}{\sqrt{u}} = u^{-1/2 - 1}$ $-\frac{1}{2}u^{3/2}$

$= \frac{1}{(x^2+y^2+z^2)^{3/2}} - \frac{3x}{(x^2+y^2+z^2)^{5/2}}$

$u_x = -\frac{1}{\cancel{2}} \frac{\cancel{2}x}{(x^2+y^2+z^2)^{3/2}}$

$= \frac{-x}{(x^2+y^2+z^2)^{3/2}}$

$u_{xx} = -\left[\frac{-3x}{\cancel{2}(x^2+y^2+z^2)^{5/2}} \cancel{2}x + \frac{1}{(x^2+y^2+z^2)^{3/2}}\right]$

$\equiv^{ly}$ $u_{yy} = \frac{1}{(\quad)} - \frac{3y}{(\quad)}$, $u_{zz} = \frac{1}{(\quad)} - \frac{3z}{(\quad)}$

So $u_{xx}+u_{yy}+u_{zz} = \left(\frac{1}{(\quad)} \times 3\right) - 3(\quad)^{\cancel{5/2}\ -3/2}$

$= 0$ Proved

(78.) $x^x y^y z^z = c$ ST at $x=y=z$, $z_{xy} = -(x\log(ex))^{-1}$

$x\log x + y\log y + z\log z = \log c$

$\cancel{x}\frac{1}{\cancel{x}} + \log x + 0 + \left(z\frac{1}{z} + \log z\right) z_x = 0$ So $z_x = -\frac{(1+\log x)}{(1+\log z)}$

$z_{xy} = -\left[\frac{\cancel{(1+\log z)(0+0)} - (0+\frac{1}{z}z_x)(1+\log x)}{(1+\log z)^2}\right]$ $z_x = -\frac{\cancel{(1+\log x)}}{\cancel{(1+\log x)}} = -1$

$= -\left[\frac{-(+\cancel{\frac{z_x}{z}})(-\frac{1}{z})(1+\log x)}{(1+\log z)^2}\right] = -(x(\log(ex))^{-1}$ Shown

$\left(\left(\because 1 = \log_e e\right) = \left(\log_e e + \log_e x = \log_e(ex)\right)\right)$

8. $u = \log(x^3+y^3+z^3-3xyz)$ $\nabla u = \frac{3}{x+y+z}$; $(\nabla)^2 u = \frac{-9}{(x+y+z)^2}$

$\nabla u = \left(\frac{1}{x^3+y^3+z^3-3xyz} \cdot 3x^2-3yz\right) + \left(\frac{3y^2-3xz}{x^3+y^3+z^3-3xyz}\right) + \left(\frac{3z^3-3xy}{x^3+y^3+z^3-3xyz}\right)$

$= \frac{3(x^2+y^2+z^2) - 3(xy+yz+zx)}{(x^3+y^3+z^3)-3xyz} = \frac{3(x^2+y^2+z^2-yx-zx-zy)}{(x+y+z)(x^2+y^2+z^2-yx-zx-zy)}$

$\nabla u = \frac{3}{x+y+z}$ Showed ; $\nabla^2 u$ So $\frac{(D)(6x)-(3x^2-3yz)(\quad)}{(D)^2}$

$N = 3x^2-3yz$

$$\left\{\frac{\begin{array}{l}(x^3+y^3+z^3-3xyz)(6x)-(3x^2-3yz)(3x^2-3yz)\\ +(x^3+y^3+z^3-3xyz)(6y)-(3y^2-3xz)(3y^2-3xz)\\ +(x^3+y^3+z^3-3xyz)(6z)-(3z^2-3xy)(3z^2-3xy)\end{array}}{(x^3+y^3+z^3-3xyz)^2}\right\} \times$$

& $\nabla\left(\frac{3}{(x+y+z)}\right) = \frac{-3}{(x+y+z)^2} - \frac{3}{(x+y+z)^2} - \frac{3}{(x+y+z)^2} = \frac{-9}{(x+y+z)^2}$ //

8.5 $u = r^m$, $r^2 = x^2+y^2+z^2$, st $u_{xx}+u_{yy}+u_{zz} = m(m+1)r^{m-2}$, $m \geq 2$

$u_x = m r^{m-1} \cdot r_x$; $r_x = \frac{1}{2\sqrt{x^2+y^2+z^2}}\,2x$ ≡ly $\frac{y}{\sqrt{\ }}$ & $\frac{z}{\sqrt{\ }}$

$u_{xx} = m[(m-1)r^{m-2} r_x + r^{m-1} r_{xx}]$

≡ly $u_{yy} = m[(m-1)r^{m-2} r_y + r^{m-1} r_{yy}]$ & $u_{zz} = m[(m-1)r^{m-2} r_z + r^{m-1} \cdot r_{zz}]$

$\Rightarrow m(m-1)r^{m-2}(r_x+r_y+r_z) + m \cdot r^{m-1}(r_{xx}+r_{yy}+r_{zz})$

$\Downarrow$

$\frac{x+y+z}{(x^2+y^2+z^2)^{1/2}}$, $\sqrt{x^2+y^2+z^2}$ $\rightarrow \frac{x+y+z}{\sqrt{x^2+y^2+z^2}}$

$r_{xx} = -\frac{x}{2}\frac{2x}{(x^2+y^2+z^2)^{3/2}}$ ≡ly $r_{xx}+r_{yy}+r_{zz} = \frac{-(x^2+y^2+z^2)'}{(x^2+y^2+z^2)^{3/2}}$

$= m(m-1)r^{m-2}\left(\frac{x+y+z}{\sqrt{x^2+y^2+z^2}}\right) + m r^{m-1}\left(-(x^2+y^2+z^2)^{-1/2}\right)$

$u = r^m$ & $2r r_x = 2x$ so $r_x = \frac{x}{r}$ ≡ly $r_y = \frac{y}{r}$ & $r_z = \frac{z}{r}$

$u_x = m r^{m-1} r_x$

$= m r^{m-2} x$

$u_{xx} = m((m-2)r^{m-3} x r_x + r^{m-2})$

$= m(m-2)r^{m-4}x^2 + m r^{m-2}$

≡ly $u_{yy} = m(m-2)r^{m-4}y^2 + m r^{m-2}$

$u_{zz} = m(m-2)r^{m-4}z^2 + m r^{m-2}$

Σ then

$= \cancel{3}m(m-2)r^{m-4}\underbrace{(x^2+y^2+z^2)}_{r^2} + 3m r^{m-2}$

$= m(m-2)r^{m-2} + 3m r^{m-2}$

$= (m^2-2m)r^{m-2} + 3m r^{m-2} = r^{m-2}(m^2-2m+3m+m) = m(m+1)r^{m-2}$

8.8 $u = x^2\tan^{-1}(y/x) - y^2\tan^{-1}(x/y)$, st $u_{xy} = \frac{x^2-y^2}{x^2+y^2}$

$u_x = \left(2x\tan^{-1}(y/x) - \frac{x^2 y}{x^2+y^2}\right) - \left(0 + \frac{y^3}{x^2+y^2}\right)$

$U_{xy} = 2\left(0 \cdot \tan^{-1}(y/x) + x\left(\frac{x^2}{x^2+y^2}\right)\right)$ $\quad$ $U_x = x^2\left(\frac{x^2}{x^2+y^2}\right)\left(\frac{-y}{x^2}\right) + 2x\left(\tan^{-1}(y/x)\right) - \left(0 + y^2\left(\frac{y^2}{x^2+y^2}\right)\left(\frac{1}{y}\right)\right)$

$= \left(\frac{-x^2y}{x^2+y^2}\right) + 2x\tan^{-1}(y/x) - \frac{y^3}{x^2+y^2}$

$U_{xy} = -\left[\frac{(x^2+y^2)(x^2)-(2y)(x^2y)}{(x^2+y^2)^2}\right] + 2\left[x\left(\frac{x^2}{x^2+y^2}\right)\left(\frac{1}{x}\right) + 0\right] - \left[\frac{(x^2+y^2)(3y^2)-2y^4}{(x^2+y^2)^2}\right]$

$= \left[\frac{-(x^2+y^2)(x^2)+(2y)(x^2y)}{(x^2+y^2)^2} + \frac{2x^2(x^2+y^2)}{(x^2+y^2)^2} - \frac{(x^2+y^2)(3y^2)+2y^4}{(x^2+y^2)^2}\right]$

$= \frac{-x^4 - x^2y^2 + 2x^2y^2 - 3x^2y^2 - 3y^4 + 2y^4 + 2x^4 + 2x^2y^2}{(x^2+y^2)^2}$

$a^2 - b^2 = (a+b)(a-b)$

$\Rightarrow \frac{x^4-y^4}{(x^2+y^2)^2} = \frac{(x^2+y^2)(x^2-y^2)}{(x^2+y^2)^2} = \frac{x^2-y^2}{x^2+y^2}$ Proved

8.9 $u = f(x+ay) + \phi(x-ay)$, p.t. $u_{yy} = a^2(u_{xx})$.

$u_y = f'(x+ay)\cdot a + \phi'(x-ay)\cdot -a$

$u_{yy} = f''(x+ay)a^2 + \phi''(x-ay)a^2 \Rightarrow a^2(f''(x+ay) + \phi''(x-ay))$

$u_x = f'(x+ay) + \phi'(x-ay)$ $\quad = a^2(u_{xx})$ Proved

$u_{xx} = f''(x+ay) + \phi''(x+ay)$

8.95. $u = \sin^{-1}(x/y) + \tan^{-1}(y/x)$, p.t. $xu_x + yu_y = 0$

$u_x = \frac{y\left(\frac{1}{y}\right)}{\sqrt{y^2+x^2}} + \frac{x^2\left(-y/x^2\right)}{x^2+y^2} = \left(\frac{1}{\sqrt{x^2+y^2}} - \frac{y}{x^2+y^2}\right)x \Rightarrow u_x x = \frac{x}{\sqrt{x^2+y^2}} - \frac{xy}{x^2+y^2}$

$u_y = \frac{y}{\sqrt{y^2+x^2}}\left(\frac{-x}{y^2}\right) + \frac{x^2}{x^2+y^2}\left(\frac{1}{x}\right) = \left(\frac{x}{x^2+y^2} - \frac{x}{y\sqrt{x^2+y^2}}\right)y \Rightarrow u_y y = \frac{xy}{x^2+y^2} - \frac{x}{\sqrt{x^2+y^2}}$

$ux + yu_y = \frac{xy}{x^2+y^2} - \frac{x}{\sqrt{x^2+y^2}} + \frac{x}{\sqrt{x^2+y^2}} - \frac{xy}{x^2+y^2} = 0$ Proved

8.96 $u = f(r)$, $r^2 = x^2+y^2$, $u_{xx} + u_{yy} = f''(r) + \frac{f'(r)}{r}$

$2r r_x = 2x$ $\quad$ $r_y = \frac{y}{r}$ $\quad$ so $f'(r) r_x = u_x$

$r_x = \frac{x}{r}$ $\quad$ $u_{xx} = r_{xx} f'(r) + f''(r)(r_x)^2$ $\quad + \frac{(r^2-x^2)}{r^3}f'(r) + f''(r)\frac{x^2}{r^2}$

$r_{xx} = \frac{r(1) - x r_x}{r^2} = \left(\frac{r - x r_x}{r^2}\right) = \frac{r^2-x^2}{r^3}$ $\quad = \left(\frac{r^2-x^2}{r^3}\right)f'(r) + f''(r)\left(\frac{x^2}{r^2}\right)$ $\quad \cdot f'(r)$

$u_x = f'(r)\, r_x = f'(r)\frac{x}{r}$ $\quad = 1/y \quad r/y = \frac{f'(y)y}{r} \;;\; r_y = \frac{y}{r}$

$u_{xx} = f'(r)\left(\frac{r(1) - (1) r_x \cdot x}{r^2}\right) + f''(r)\, r_x \cdot r_x$ $\qquad \frac{2r^2 - r^2}{r^3}$

$\qquad \frac{r^2 - x^2 + r^2 - y^2}{r^3} = \frac{1}{r}$

$u_{yy} = f'(r)\left(\frac{r(1) - (1) r_y \cdot y}{r^2}\right) + f''(r)\, r_y \cdot r_y$

$\Rightarrow u_{xx} + u_{yy} = f''(r)\left(\frac{x^2+y^2}{r^2}\right) + f'(r)\left(\frac{r - \frac{x^2}{r} + r - \frac{y^2}{r}}{r^2}\right)$

$1/r$

$= \frac{f'(r)}{r} + f''(r)$ proved

8.97 $\theta = t^n e^{-r^2/4t}$, $n = ?$ so $\frac{1}{r^2}\left(\frac{\partial}{\partial r}\left(r^2 \frac{\partial \theta}{\partial r}\right)\right) = \frac{\partial \theta}{\partial t}$

$\frac{\partial \theta}{\partial t} = \left(n t^{n-1} e^{-r^2/4t}\right) + \left(t^n \left(\frac{r^2}{4t^2}\right) e^{-r^2/4t}\right)$

$= n t^{n-1} e^{-r^2/4t} + t^n \frac{r^2}{4t^2} e^{-r^2/4t}$ $\qquad \frac{\partial \theta}{\partial r} = t^n \left(\frac{-2r}{4t}\right) e^{-r^2/4t}$

$\frac{\partial \theta}{\partial t} = e^{-r^2/4t}\, t^n \left(\frac{n}{t} + \frac{r^2}{4t^2}\right)$ — (i) $\qquad \frac{\partial \theta}{\partial r} = \left(-\frac{t^{n-1} \cdot r}{2}\right) e^{-r^2/4t}$

$\left(r^2\left(\frac{\partial \theta}{\partial r}\right)\right) = \left(\left(-\frac{r^3 t^{n-1}}{2}\right) e^{-r^2/4t}\right)\frac{\partial}{\partial r} \Rightarrow \left(-\frac{t^{n-1}}{2}\right)\left(3r^2 e^{-r^2/4t} + r^3\left(\frac{-2r}{4t}\right) e^{-r^2/4t}\right)$

$= \left(\left(-\frac{t^{n-1}}{2}\right)\left(3r^2 e^{-r^2/4t} - \frac{r^4}{2t} e^{-r^2/4t}\right)\right)\frac{1}{r^2}$

$= +t^n e^{-r^2/4t}\left(-\frac{3}{2t} - \frac{r^2}{4t}\right)$ — (ii) $\qquad$ (i) ≡ (ii) then $n = -\frac{3}{2}$

9. $v = r^n(3\cos^2\theta - 1)$, $n = ?$ if $\frac{\partial}{\partial r}\left(r^2 \frac{\partial v}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial v}{\partial \theta}\right) = 0$

$\frac{\partial v}{\partial r} \Rightarrow \left(3r^n \cos^2\theta - r^n\right)$ $\qquad 3n r^{n+1}\cos^2\theta - n r^{n+1}$

$= 3n r^{n-1}\cos^2\theta - n r^{n-1}$ $\qquad = 3n(n+1) r^n \cos^2\theta - n(n+1) r^n$

$r^2\left(n r^{n-1}(3\cos^2\theta - 1)\right)$ $\qquad = r^n (n^2+1)(3\cos^2\theta - 1)$ — (i)

$r^2\left(\frac{\partial v}{\partial r}\right) = \left(n r^{n+1}(3\cos^2\theta - 1)\right)\frac{\partial}{\partial r}$ $\qquad \frac{\partial}{\partial r}\left(r^2 \frac{\partial v}{\partial r}\right)$

$\frac{\partial v}{\partial \theta} = -6n r^n \cos\theta \sin\theta - 0$ $\qquad \rightarrow \left(-6n r^{n-1}(2\cos^2\theta \sin\theta - \sin^3\theta)\right) \times \frac{1}{\sin\theta}$

$\left(= -3n r^{n-1}(\sin 2\theta)\right)\sin\theta$

$\left(-6n r^{n-1}\cos\theta \sin^2\theta\right)\frac{\partial}{\partial \theta}$ $\qquad = -6n r^{n-1}(2\cos^2\theta - \sin^2\theta)$ — (ii)

PAGE NO.
DATE / /

(i) $\Rightarrow r^n(n^2+1)(3\cos^2\theta-1) - 6nr^{n-1}(2\cos^2\theta-\sin^2\theta)=0$

X $\quad n(n+1) - 3r^n \cdot n((n+1)\cos^2\theta - 1 - 4(\cos^2\theta + 2\sin^2\theta))=0$

$(\sin\theta\, v_\theta) = r^n(-6\cos\theta\sin^2\theta)$

$nr^{n+1}(3\cos^2\theta-1) = (v' r^2)$

$r^n(-6(2\cos^2\theta - \sin^2\theta))$

$n(n+1)r^n(3\cos^2\theta-1) = \dfrac{(v' r^2)\partial}{\partial r}$

$= \dfrac{\partial}{\partial\theta}\left(\sin\theta\dfrac{\partial v}{\partial\theta}\right)\dfrac{1}{\sin\theta}$

$n(n+1)r^n(3\cos^2\theta-1) + r^n(-12\cos^2\theta+6\sin^2\theta)=0$

$r^n \quad$ X $\quad n(n+1)v$

$nr^{n+1}(3\cos^2\theta-1) = n(n+1)r^n(3\cos^2\theta-1)$

$2\cos^2\theta - 1 + \cos^2\theta$

$r^n(-6\cos\theta\sin^2\theta) = -6r^n(2\cos^2\theta\sin\theta - \sin^3\theta) = -6r^n(2\cos^2\theta - \sin^2\theta)$

$= -6\underbrace{r^n(3\cos^2\theta-1)}_{v} = -6v$ so $n(n+1)\cancel{v} - 6\cancel{v} = 0$

$n^2+n-6=0, \quad n^2-2n+3n-6=0$

$n(n-2)+3(n-2)$ so $n=2, -3$

9.5. $u = e^{xyz}$, S.T. $\dfrac{\partial^3 u}{\partial x\partial y\partial z} = (1+3xyz+(xyz)^2)e^{xyz}$

$u_x = e^{xyz}\cdot yz$

$u_{xy} = z(y\cdot xz\, e^{xyz} + e^{xyz}) = z^2xy\, e^{xyz} + e^{xyz}$

$u_{xyz} = \Rightarrow xy[2z e^{xyz} + z^2\cdot xy\, e^{xyz}] + e^{xyz}\cdot xyz$

$= 2xyze^{xyz} + xyz^2e^{xyz} + \cancel{e^{xyz}\cdot xy}$ $\quad [xy(e^{xyz} + zxy e^{xyz})]$

$= 2xyze^{xyz} + xyz^2e^{xyz} + xye^{xyz} + x^2y^2z\, e^{xyz}$ X

$u = e^{xyz}$ so $u_x = yz\cdot e^{xyz}$

$u_{xy} = z(y\cdot xz e^{xyz} + e^{xyz}) = z^2xy\, e^{xyz} + ze^{xyz}$

$u_{xyz} = xy(2ze^{xyz} + z^2xye^{xyz}) + zyxe^{xyz} + e^{xyz}$

$= 2xyze^{xyz} + z^2x^2y^2e^{xyz} + zyxe^{xyz} + e^{xyz}$

$= e^{xyz}(1+3xyz+(xyz)^2)$ // Showed.

16. (a) ("total derivative": $\dfrac{du}{dt}$) $\dfrac{du}{dt} = \dfrac{dx}{dt}u_x + \dfrac{dy}{dt}u_y$

$u_x = y^2 + 2xy \quad \dfrac{dx}{dt} = 2at$

$= (y^2+2xy)(2at) + (2xy+x^2)(2a)$

$= 2ay^2t + 4xyat + 4xay + 2ax^2$

$u_y = 2xy + x^2 \quad \dfrac{dy}{dt} = 2a$

$= 2a(4a^2t^2 + t + 4(at^2)(2at)at \ldots$

$2ay^2t + 4xyat + 4xay + 2ax^2 = 2a(2at)^2t + 4(at^2)(2at)at$
$+ 4(at^2)(2at)a + 2a(at^2)^2$

$= 8a^3t^3 + 8a^3t^3 + 8a^3t^3 + 2a^3t^4 = 16a^3t^3 + 10a^3t^4 + 10a^3t^4$
$= 16a^3t^3 + 10a^3t^4$ — (i)

Direct sub: $(at^2)(2at)^2 + (at^2)^2(2at) = u$

$u = 4a^3t^4 + 2a^3t^5$ so $\frac{du}{dt} = 16a^3t^3 + 10a^3t^4$ — (ii)

(i) = (ii) verified.

(b) $u = \sin\left(\frac{x}{y}\right)$ $x = e^t,\ y = t^2$

So $\frac{du}{dt} = \frac{dx}{dt}u_x + u_y\frac{dy}{dt}$ ↗ u_x $\frac{\cancel{y}}{\sqrt{y^2-x^2}}\left(\frac{1}{y}\right) = \frac{1}{\sqrt{y^2-x^2}}$

u_y: $\frac{\cancel{y}}{\sqrt{y^2-x^2}}\left(\frac{-x}{y^2}\right) = \frac{-x}{y\sqrt{y^2-x^2}}$

$\frac{dx}{dt} = e^t$ $\frac{dy}{dt} = 2t$

So $\frac{du}{dt} = e^t\left(\frac{1}{\sqrt{y^2-x^2}}\right) + \left(\frac{-x}{y\sqrt{y^2-x^2}}\right)(2t)$

$\frac{du}{dt} = \frac{e^t \cdot t^2}{t^2\sqrt{t^4-e^{2t}}} - \frac{2te^t}{t^2\sqrt{t^4-e^{2t}}} = \frac{e^t t^2 - 2te^t}{t^2\sqrt{t^4-e^{2t}}}$

$= \frac{e^t t(t-2)}{t^2\sqrt{t^4-e^{2t}}} = \frac{e^t(t-2)}{t\sqrt{t^4-e^{2t}}}$ — (i)

$u = \sin\left(\frac{e^t}{t^2}\right) \Rightarrow \frac{du}{dt} = \frac{1}{\sqrt{t^4-e^{2t}}}\left(\frac{t^2e^t - 2te^t}{t^4}\right) \rightarrow \frac{te^t(t-2)}{t^4}$

$= \frac{e^t(t-2)}{t^3\sqrt{t^4-e^{2t}}}$

✗ $\sin^{-1}()$ so,

u_x: $\cos\left(\frac{x}{y}\right)\left(\frac{1}{y}\right)$

u_y $\cos\left(\frac{x}{y}\right)\left(-\frac{x}{y^2}\right)$

$\frac{dx}{dt} = e^t$ $\frac{dy}{dt} = 2t$

$\frac{du}{dt} = \frac{dx}{dt}u_x + \frac{dy}{dt}u_y = \frac{e^t}{y}\cos\left(\frac{x}{y}\right) - \frac{2tx}{y^2}\cos\left(\frac{x}{y}\right)$

$= \frac{e^t}{t^2}\cos\left(\frac{e^t}{t^2}\right) - \frac{2te^t}{t^4}\cos\left(\frac{e^t}{t^2}\right)$

By DS. $\sin\left(\frac{e^t}{t^2}\right)$ $\frac{e^t}{t^2}\cos\left(\frac{e^t}{t^2}\right)\left(1 - \frac{2\cancel{t}}{t\cancel{t^2}}\right)$ — (i)

So $\cos\left(\frac{e^t}{t^2}\right)\left(\frac{t^2e^t + 2te^t}{t^4}\right) \rightarrow \left(\frac{e^t}{t^2} + \frac{2e^t}{t^3}\right)$

So $\cos\left(\frac{e^t}{t^2}\right)\frac{e^t}{t^2}\left(1 + \frac{2}{t}\right)$ — (ii)

(i) = (ii) verified

16.5 $u = e^x \sin y$, $x = \log t$ & $y = t^2$, $\frac{du}{dt} = ?$

$\frac{du}{dt} = \frac{dx}{dt} u_x + \frac{dy}{dt} u_y$ so $u_x = e^x \sin y$, $\frac{dx}{dt} = \frac{1}{t}$

$u_y = e^x \cos y$, $\frac{dy}{dt} = 2t$

so $\frac{e^x}{t} \sin y + 2t e^x \cos y$

$= \frac{\cancel{e^{\log t}}}{t} \sin t^2 + 2t\, e^{\log t} \cos t^2$

$= \sin t^2 + 2t^2 \cos t^2$ //

10. (a) $\frac{dy}{dx} = -\frac{u_x}{u_y}$ so $e^x = 2y$ & $e^y = 2x$ so $-\frac{(e^x - 2y)}{(e^y - 2x)}$

$e^x - 2y = u_x$, $e^y - 2x = u_y$

(OR) $e^x + e^y \frac{dy}{dx} = 2\left(x\frac{dy}{dx} + y\right) = 2x\frac{dy}{dx} + 2y$

so $-\frac{(e^x - 2y)}{(e^y - 2x)} = \frac{dy}{dx}$ Same

(b) $y^x - x = u$ so $u_x = (y^x \ln y - 1)$ and $u_y = xy^{x-1}$

so $\frac{dy}{dx} = -\frac{(y^x \ln y - 1)}{xy^{x-1}}$ //

(c) $x^3 + y^3 - 3axy = 1$

$u_x = 3x^2 - 3ay$, $\frac{dy}{dx} = -\frac{(x^2 - ay)}{(y^2 - ax)}$ // $\left(= \frac{ay - x^2}{y^2 - ax}\right)$

$u_y = 3y^2 - 3ax$

(d) $x^y + y^x = c$

$u_x \Rightarrow x\log x + y\log y = \log c$ so $\left(\frac{1}{x} \cdot x + \log x\right) = 0 = (1 + \log x)$

change $u_x = yx^{y-1} + y^x \ln y = 0$

$u_y = x^y \ln x + xy^{x-1} = 0$ } $-\frac{(yx^{y-1} + y^x \ln y)}{(x^y \ln x + xy^{x-1})}$

(e) $(\cos x)^y = (\sin y)^x$

$u_x = (-y(\cos x)^{y-1} \sin x)$ $-$ $(\sin y)^x \ln(\sin y)$ } $u_y = (\cos x)^y \ln(\cos x) - (x(\sin y)^{x-1} \cos y)$

dy/dx so $\frac{+y \sin x (\cos x)^{y-1} + (\sin y)^x \ln \sin y}{(\cos x)^y \ln(\cos x) - x(\cos y)(\sin y)^{x-1}}$

11. (c) $u = x\log(xy)$, $x^3 + y^3 + 3xy = 1$, $du/dx = ?$

$\frac{du}{dx} = u_x + u_y \frac{dy}{dx}$; $\frac{dy}{dx} = -\frac{u_x}{u_y} = -\frac{(x\frac{1}{xy}y + \log(xy))}{(x\frac{1}{xy}x)}$

So $\frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{(3x^2+3y)}{(3y^2+3x)} = -\frac{(x^2+y)}{(y^2+x)}$

$\frac{du}{dx} = \left(x\frac{1}{xy}y + \log(xy)\right) + \left(x\frac{1}{xy}x\right)\left(-\frac{x^2+y}{y^2+x}\right)$

$= (1+\log(xy)) + -\frac{x}{y}\frac{(x^2+y)}{(y^2+x)}$

(b) $u = \cos(x^2 - y^3)$, $2x^2 + 3y^2 = a^2$

$\frac{du}{dx} = u_x \cancel{\frac{dx}{dx}} + u_y \frac{dy}{dx}$; $\frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{(4x)}{6y} = -\frac{2x}{3y}$

$\frac{du}{dx} = -\sin(x^2 - y^3)\cdot 2x + (-\sin(x^2-y^3)\cdot 3y^2)\left(-\frac{2x}{3y}\right)$

$= -2x\sin(x^2-y^3)(1+1) = -4x\sin(x^2-y^3)$ (?)

(c) $u = \tan^{-1}(y/x)$, $x^2 + y^2 = a^2$

$\frac{du}{dx} = u_x + \frac{dy}{dx}u_y$ so $\frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{(2x)}{(2y)} = -\frac{x}{y}$

$= \frac{x^2}{x^2+y^2}\left(\frac{1}{x}\right) + \frac{x^2}{y^2+x^2}\left(\frac{1}{x}\right)\left(-\frac{x}{y}\right)$

$\frac{x}{x^2+y^2} + \frac{x}{y^2+x^2} = \frac{2x}{x^2+y^2}$ // (?)

(d) $u = \sin(x^2+y^2)$, $ax^2 + by^2 = c^2$, $du/dx =$

$\frac{du}{dx} = u_x \cancel{\frac{dx}{dx}} + u_y \frac{du}{dx}$; $\frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{2ax}{2by} = \left(-\frac{ax}{by}\right)$

$= 2x\cos(x^2+y^2) + 2y\cos(x^2+y^2)\left(\frac{-ax}{by}\right)$

$= 2\cos(x^2+y^2)\left(x + y - \frac{ax}{by}\right)$ $\quad x + \frac{ax}{b}$, $x\left(\frac{b-a}{b}\right)$

$= 2x\cos(x^2+y^2)\left(\frac{b-a}{b}\right)$

13. $u = \tan^{-1}(y/x)$, $x = e^t - e^{-t}$, $y = e^t + e^{-t}$, $\frac{du}{dt} = ?$

(OR) $\frac{du}{dt} = \frac{dx}{dt} u_x + u_y \frac{dy}{dt}$ $\left(\frac{y}{x} = \frac{e^t + \frac{1}{e^t}}{e^t - \frac{1}{e^t}} = \frac{\frac{e^{2t}+1}{e^t}}{\frac{e^{2t}-1}{e^t}} = \frac{e^{2t}+1}{e^{2t}-1} \right)$ $u = \tan^{-1}\left(\frac{e^{2t}+1}{e^{2t}-1}\right)$

$\hookrightarrow \left((e^t + e^{-t}) \left(\frac{x^2}{x^2+y^2}\left(\frac{-y}{x^2}\right)\right)\right) + \left(\left(\frac{x^2}{x^2+y^2}\left(\frac{1}{x}\right)\right)(e^t - e^{-t})\right)$

$= \frac{-2e^t y}{x^2+y^2} + \frac{x(e^t - e^{-t})}{x^2+y^2}$ $xe^t - xe^{-t}$

$= \frac{e^t(-2y + xe^{-2t})}{x^2+y^2}$

$\Rightarrow \left(\frac{1}{(e^{2t}-1)^2 + (e^{2t}+1)^2} \cdot \cancel{(e^{2t}-1)^2} \right) \left(\frac{(e^{2t}-1)(2e^{2t}) - 2(e^{2t}-0)(e^{2t}+1)}{\cancel{(e^{2t}-1)^2}} \right)$ $-4e^{2t}$

$= \frac{2e^{2t}(\cancel{e^{2t}} - 1 - \cancel{e^{2t}} - 1)}{(\ )^2 + (\ )^2} = \frac{-4e^{2t}}{2(e^{4t}+1) - 2e^{2t} + e^{4t} + 1 + 2e^{2t}}$

$\frac{-2e^{2t} e^{-2t}}{e^{2t} + e^{-2t}} = \frac{-2e^{2t}}{e^{2t}+e^{-2t}} = \frac{-2}{e^{4t}+1} = \frac{-2}{e^{2t}+e^{-2t}}$ (?)

14. (a) $u = xy + yz + zx$, $x = 1/t$, $y = e^t$, $z = e^{-t}$ du/dt

$\frac{du}{dt} = \frac{dx}{dt} u_x + u_y \frac{dy}{dt} + u_z \frac{dz}{dt} = (y+z)\left(-\frac{1}{t^2}\right) + (x+z)(e^t) + (y+x)(-e^{-t})$

$= (e^t + e^{-t})\left(-\frac{1}{t^2}\right) + \left(\frac{1}{t} + e^{-t}\right)(e^t) + \left(e^t + \frac{1}{t}\right)(-e^{-t})$

$= -\frac{e^t}{t^2} - \frac{e^{-t}}{t^2} + \frac{e^t t}{t^2} + \cancel{\frac{t^2}{t^2}} - \cancel{1} - \frac{e^{-t} t}{t^2} = \frac{e^t(-1+t) + e^{-t}(-1-t)}{t^2}$ (?)

(b) $u = x^3 y e^z$ where $x = t$, $y = t^2$, $z = \log t$, at $t = 2$ du/dt

$\frac{du}{dt} = u_x \frac{dx}{dt} + u_y \frac{dy}{dt} + u_z \frac{dz}{dt} = 3x^2 y e^z (1) + x^3 e^z (2t) + x^3 y e^z \left(\frac{1}{t}\right)$

$= 3(4)(4)(e^{\log 2})(1) + (8)(e^{\log 2})(4) + (8)(4)(e^{\log 2})\left(\frac{1}{2}\right)$

$96 + 64 + 32 = 192$

15. $u = e^{xy} \sin(yz)$, $x = t^2$, $y = t-1$, $z = \frac{1}{t}$, $du/dt = ?$ at $t = 1$

$\frac{du}{dt} \Rightarrow e^{t^2(t-1)} \cdot \sin((t-1)(1/t))$; $= (e^{t^3 - t^2}) \sin\left(\frac{t-1}{t}\right)$

$= (e^{t^3 - t^2}) \cos\left(\frac{t-1}{t}\right)$ ...

$$\frac{dw}{dt} = \underbrace{e^{xy}\sin(yz)\cdot y}_{\times 2t} + \underbrace{e^{xy}\cos(yz)\cdot z}_{\times 1} + e^{xy}\sin(yz)\cdot x + \underbrace{e^{xy}\cos(yz)\cdot y}_{\times -1/t^2}$$

$$= \underbrace{e^{t^3-t^2}\cancel{\sin\left(\frac{t-1}{t}\right)}^{0}(t-1)(2t)} + e^{t^3-t^2}\cancel{\cos\left(\frac{t-1}{t}\right)}^{1}\left(\frac{1}{t}\right) + e^{t^3-t^2}\cancel{\sin\left(\frac{t-1}{t}\right)}$$

$$+ \underbrace{e^{t^3-t^2}\cos\left(\frac{t-1}{t}\right)\cancel{(t-1)}^{0}\left(-\frac{1}{t^2}\right)}, \quad t=1$$

$$= e^{t^3-t^2}(1) + 0 + 0 + 0 = e^{1-1}(1) = 1 \quad (3)$$

17. $z = f(x,y)$; $x = u - v$, $y = uv$, st

(a) $(u+v)\frac{\partial z}{\partial x} = u\frac{\partial z}{\partial u} - v\frac{\partial z}{\partial v}$

$$\left[\cancel{(u+v)} \quad \frac{\partial z}{\partial x} = \cancel{\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}} \quad \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}\right] \times$$

$$(u+v)\left(\frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}\right) = u\underbrace{\left(\frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial u}\frac{\partial u}{\partial y}\right)}_{(i)}$$

$$= \cancel{(u+v)}\left(\times \right) \text{ as LHS cannot be obtained, RHS;} \quad - v\underbrace{\left(\frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}\right)}_{(ii)}$$

(i): $\cancel{u\left(\frac{\partial z}{\partial u}\right.}$ ~~LHS~~ $\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} = \underbrace{z_x(1) + z_y(v)}_{(i)}$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} = \underbrace{z_x(-1) + z_y(u)}_{(ii)} \quad \text{so } u(i) - v(ii)$$

$$= uz_x + \cancel{uvz_y} + vz_x - \cancel{uvz_y}$$

$$= z_x(u+v) = \frac{\partial z}{\partial x}(u+v) \text{ showed = LHS}$$

(b) $(u+v)\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}$ } LHS so RHS: $\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} = \underbrace{z_x(-1) + z_y(u)}_{(ii)} \qquad = \underbrace{z_x(1) + z_y(v)}_{(i)}$$

$$(i) + (ii) = \cancel{z_x} - \cancel{z_x} + z_y(u+v) = \frac{\partial z}{\partial y}(u+v) \text{ showed = LHS}$$

18. $z = f(x,y)$; $x = e^u + e^{-v}$, $y = e^{u} - e^{v}$ st $z_u - z_v = xz_x - yz_y$

$$\frac{\partial z}{\partial x}\frac{\partial x}{\partial u}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial u}-\frac{\partial z}{\partial x}\frac{\partial x}{\partial v}-\frac{\partial z}{\partial y}\frac{\partial y}{\partial v} = z_x(e^u)+z_y(-e^{-u}) - z_x(-e^{-v})-z_y(-e^{v})$$

$$= z_x(e^u+e^{-v})+z_y(e^v-e^{-u}) = z_x \cdot x - z_y(y) = z_x x - z_y y \text{ shown}$$

19. $u=f(y-z, z-x, x-y)$, p.t. $u_x+u_y+u_z=0$

$p = y-z,\quad q = z-x,\quad r = x-y$

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial p}\frac{\partial p}{\partial x}+\frac{\partial u}{\partial q}\frac{\partial q}{\partial x}+\frac{\partial u}{\partial r}\frac{\partial r}{\partial x}$$

$\frac{\partial u}{\partial y} = u_p(1)+u_q(0)+u_r(-1) = u_p - u_r \quad \text{---(ii)}$

$= u_p(0)+u_q(-1)+u_r(1) = u_r - u_q \quad \text{---(i)}$

$\frac{\partial u}{\partial z} = u_p(-1)+u_q(1)+u_r(0) = u_q - u_p \quad \text{---(iii)}$

(i) + (ii) + (iii) $= \cancel{u_p} - \cancel{u_r} + \cancel{u_r} - \cancel{u_q} + \cancel{u_q} - \cancel{u_p}$

$= 0$ proved

20. $u = f(r,s,t)$, $r=\frac{x}{y}$, $s=\frac{y}{z}$, $t=\frac{z}{x}$, $xu_x+yu_y+zu_z=0$

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial r}\frac{\partial r}{\partial x}+\frac{\partial u}{\partial s}\frac{\partial s}{\partial x}+\frac{\partial u}{\partial t}\frac{\partial t}{\partial x} = u_r\left(\frac{1}{y}\right)+u_s(0)+u_t\left(-\frac{z}{x^2}\right)$$

$u_x = \frac{u_r}{y}-\frac{zu_t}{x^2}$; $u_y = u_r\left(-\frac{x}{y^2}\right)+\cancel{u_s(0)}+\cancel{u_t}\left(\cancel{-\frac{z}{x^2}}\right)+u_s\left(\frac{1}{z}\right)$

$xu_x = \underbrace{\frac{u_r x}{y}-\frac{zu_t}{x}}_{(i)}$; $yu_y = \frac{yu_s}{z}-\frac{xu_r}{y}$] (ii)

$u_z = u_r(0)+u_t\left(\frac{1}{x}\right)+u_s\left(-\frac{y}{z^2}\right)$

$zu_z = \frac{z}{x}u_t - \frac{yu_s}{z}$] (iii) So (i)+(ii)+(iii) $= \cancel{\frac{xu_r}{y}} - \cancel{\frac{zu_t}{x}} + \cancel{\frac{yu_s}{z}} + \cancel{\frac{z}{x}u_t} - \cancel{\frac{yu_s}{z}} - \cancel{\frac{xu_r}{y}}$

$= 0$ showed

21. Let u, v, w be ~~indep~~ fn of 3 indep variables x,y,z for which the 1st order PD exists, then $J(u,v,w) = \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z}\end{vmatrix}$

is the Jacobian of u,v,w wrt x,y,z.

22. Properties

1) $J \cdot J' = 1 \Rightarrow \frac{\partial(u,v)}{\partial(x,y)}\frac{\partial(x,y)}{\partial(u,v)}$: Jacobian of u,v wrt x,y is J & Jac of x,y wrt u,v is J'

then $J \cdot J' = 1$

2) If J_1 jacob of u,v wrt r,s & r,s wrt x,y jacob is J_2 then

$$J_1 J_2 = \frac{\partial(u,v)}{\partial(r,s)} \times \frac{\partial(r,s)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(x,y)}$$

23. $$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} 1 & 6y & -2z \\ 8xyz & 4xz & 4xy \\ -y & -x & 4z \end{vmatrix} = \begin{vmatrix} 1 & -6 & 0 \\ 0 & 0 & -4 \\ 1 & -1 & 0 \end{vmatrix}$$

@ $+1$ -1 0
x y z

$$= 0(0-0) - 0(0-0) - 4(-1+6) = -20$$

24. (a) $u = x(-y),\ v = xy$:

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} -y & -x & 0 \\ y & x & 0 \\ 0 & 0 & 0 \end{vmatrix} = -xy + xy = 0$$

(b) $u = 3x + 5y,\ v = 4x - 5y$:

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} 3 & 5 & 0 \\ 4 & -5 & 0 \\ 0 & 0 & 0 \end{vmatrix} = -15 - 20 = -35$$

25. (a) $x = \frac{u^2}{v},\ y = \frac{v^2}{u}$:

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{2u}{v} & -\left(\frac{u}{v}\right)^2 \\ -\left(\frac{v}{u}\right)^2 & \frac{2v}{u} \end{vmatrix} = 4 - 1 = 3$$

$\frac{2u}{v} \cdot \frac{2v}{u}$ ~~$\frac{u^2 v^2}{v^2 u^2}$~~

(b) $x = u(1-v),\ y = uv$:

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} -v & -u \\ v & u \end{vmatrix} = -uv + uv = 0$$

(c) $x = u^2 - v^2,\ y = 2uv$.

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \begin{vmatrix} 2u & -2v \\ 2v & 2u \end{vmatrix} = 4\begin{vmatrix} u & -v \\ v & u \end{vmatrix} = 4(u^2+v^2)$$

26. (a) $\frac{\partial(u,v,w)}{\partial(x,y,z)}$ for $u = x^2 - 2y$, $v = x+y+z$, $w = x-2y+3z$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} 2x & -2 & 0 \\ 1 & 1 & 1 \\ 1 & -2 & 3 \end{vmatrix}$$

$$= 1(-6+0) - 1(6x-0) + 1(-4x+2) = -6 - 6x - 4x + 2$$
$$= -(4+10x)$$

(b) $u = \frac{yz}{x}$, $v = \frac{zx}{y}$, $w = \frac{xy}{z}$

$$= \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} -\frac{yz}{x^2} & \frac{z}{x} & \frac{y}{x} \\ \frac{z}{y} & -\frac{zx}{y^2} & \frac{x}{y} \\ \frac{y}{z} & \frac{x}{z} & -\frac{xy}{z^2} \end{vmatrix}$$

$$= 1\begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix} = -1\cancel{(1-1)} - 1(-1-1) + 1(1+1)$$
$$= 2+2 = 4$$

27. $$J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix}$$
$$= r(\cos^2\theta + \sin^2\theta) = r \quad (i)$$

$$J' = \frac{\partial(r,\theta)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{1}{2\sqrt{x^2+y^2}} 2x & \frac{1}{2\sqrt{x^2+y^2}} 2y \\ \frac{x^2}{x^2+y^2}\left(\frac{-y}{x^2}\right) & \frac{x^2}{x^2+y^2}\left(\frac{1}{x}\right) \end{vmatrix}$$

$r = \sqrt{x^2+y^2}$, $\theta = \tan^{-1}(y/x)$

$$= \frac{x}{\sqrt{x^2+y^2}} \cdot \frac{x}{(x^2+y^2)} + \frac{y^2}{(\sqrt{x^2+y^2})(x^2+y^2)}$$

$1 - \frac{3}{2} = -\frac{1}{2}$ $\quad \frac{1}{2}+1 = 3/2$

$$= \frac{(x^2+y^2)^1}{(x^2+y^2)^{3/2}} = (x^2+y^2)^{-1/2} = (r^2)^{-1/2} = r^{-1}$$
$$= \frac{1}{r} \rightarrow (ii)$$

$$(i)\times(ii) = J \cdot J' = \frac{1}{r} r = 1 \text{ showed}$$

28. $$\frac{\partial(u,v,w)}{\partial(r,\phi,\theta)} = \begin{vmatrix} \frac{\partial u}{\partial r} & \frac{\partial u}{\partial \phi} & \frac{\partial u}{\partial \theta} \\ \frac{\partial v}{\partial r} & \frac{\partial v}{\partial \phi} & \frac{\partial v}{\partial \theta} \\ \frac{\partial w}{\partial r} & \frac{\partial w}{\partial \phi} & \frac{\partial w}{\partial \theta} \end{vmatrix} = \quad$$ So $\frac{\partial(u,v,w)}{\partial(x,y,z)} \cdot \frac{\partial(x,y,z)}{\partial(r,\phi,\theta)} = \frac{\partial(u,v,w)}{\partial(r,\phi,\theta)}$

(i) $$= \begin{vmatrix} 0 & \frac{1}{2\sqrt{yz}}\cdot z & \frac{1}{2\sqrt{yz}}\cdot y \\ \frac{1}{2\sqrt{xz}} z & 0 & \frac{1}{2\sqrt{xz}} x \\ \frac{1}{2\sqrt{xy}} y & \frac{1}{2\sqrt{xy}} x & 0 \end{vmatrix} = \frac{1}{8}\begin{vmatrix} 0 & z/\sqrt{yz} & y/\sqrt{yz} \\ z/\sqrt{xz} & 0 & x/\sqrt{xz} \\ y/\sqrt{xy} & x/\sqrt{xy} & 0 \end{vmatrix} = \frac{1}{8\sqrt{yz}\cdot\sqrt{xz}\cdot\sqrt{xy}}$$

$$\frac{1}{8xyz}\begin{vmatrix} 0 & z & y \\ z & 0 & x \\ y & x & 0 \end{vmatrix}$$

$$= \frac{1}{8xyz}\left[0(0-x^2) - z(0-xy) + y(zx-0)\right] = \frac{1}{8xyz}(xyz + xyz) = \frac{1}{8xyz} 2xyz = \frac{1}{4}$$

$$\begin{vmatrix} \cos\phi\sin\theta & -r\sin\phi\sin\theta & r\cos\phi\cos\theta \\ \sin\phi\sin\theta & r\cos\phi\sin\theta & r\sin\phi\cos\theta \\ \cos\theta & 0 & -r\sin\theta \end{vmatrix} = \cos\theta(-r^2\sin^2\phi\cos\theta\sin\theta - r^2\cos^2\phi\ldots) - 0(0) - r\sin\theta(r\cos^2\phi\sin^2\theta + r\sin^2\phi\sin^2\theta)$$

$= -r^2\cos^2\theta\sin^2\phi\sin\theta - r^2\cos^2\phi\ldots$

$- r^2\cos^2\phi\sin^3\theta - r^2\sin^2\phi\sin^3\theta \Rightarrow -r^2\sin\theta\cos^2\theta\underbrace{(\sin^2\phi+\cos^2\phi)}_{1}$

$- r^2\sin^3\theta\underbrace{(\cos^2\phi+\sin^2\phi)}_{1} = -r^2\sin\theta\underbrace{(\sin^2\theta+\cos^2\theta)}_{1}$

$= -r^2\sin\theta$ — (ii) So (i)×(ii) $= -\dfrac{r^2\sin\theta}{4}$

29. $\dfrac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = r^2\sin\theta$ (S.T.) $= \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial\theta} & \frac{\partial z}{\partial\phi} \end{vmatrix} = \begin{vmatrix} \cos\phi\sin\theta & r\cos\phi\cos\theta & -r\sin\phi\sin\theta \\ \sin\phi\cos\theta & r\sin\phi\cos\theta & r\cos\phi\sin\theta \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$

$= 0(0-0) + r\sin\theta(r\cos^2\phi\sin^2\theta + r\sin\theta\cos\theta\sin^2\phi) + \cos\theta(-r^2\sin^2\phi\cos\theta\sin\theta - r^2\cos^2\phi\sin\theta\cos\theta)$

$= (r^2\sin^3\theta\cos^2\phi + r^2\sin^2\theta\cos\theta\sin^2\phi - r^2\sin^2\phi\cos^2\theta\sin\theta - r^2\cos^2\phi\sin\theta\cos^2\theta)$

$= r^2\cancel{\sin^2\theta}(\quad -r^2\cos^2\theta\sin\theta\cancel{(\sin^2\phi+\cos^2\phi)} +$

$= r^2\sin^3\theta\cos^2\phi + r^2\sin^2_\theta\cos\theta\sin^2\phi - r^2\cos^2\theta\sin\theta$

$r^2\sin^2\theta(\sin\theta\cos^2\phi$

$= \begin{vmatrix} \cos\phi\sin\theta & r\cos\phi\cos\theta & -r\sin\phi\sin\theta \\ \sin\phi\sin\theta & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix} = \cos\theta[+r^2\cos\theta\sin^2\phi\sin\theta + r^2\sin\theta\cos\theta\cos^2\phi] + r\sin\theta[+r\sin^2\phi\sin^2\theta + r\sin^2\theta\cos^2\phi] + 0[\ ]$

$= -r^2\cos\theta\cos\theta\sin\theta[1]$
$\quad - r^2\sin\theta\sin^2\theta[1] = +r^2\sin\theta[1]$
$\quad = +r^2\sin\theta$ Proved.

30. $\dfrac{\partial(x,y)}{\partial(u,v)}$ $\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} e^v\sec u\tan u & e^v\sec u \\ e^v\sec^2 u & e^v\tan u \end{vmatrix} = e^{2v}\tan^2 u\sec u - e^{2v}\sec^3 u$

$= e^{2v}\sec u\,(\overbrace{\tan^2 u - \sec^2 u}^{-1}) = -e^{2v}\sec u$ and by $J\cdot J' = 1$,

PAGE NO.
DATE

$J' = \frac{1}{-e^{2v}\sec u} = -e^{-2v}\cos u$

31. $\frac{\partial(u,v)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial u}{\partial r} & \frac{\partial u}{\partial \theta} \\ \frac{\partial v}{\partial r} & \frac{\partial v}{\partial \theta} \end{vmatrix}$ so $\frac{\partial(u,v)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(r,\theta)} = \frac{\partial(u,v)}{\partial(r,\theta)}$

$\begin{vmatrix} 2x & -4y \\ 4x & -2y \end{vmatrix} \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = 2\cdot 2 \begin{vmatrix} x & -2y \\ 2x & -y \end{vmatrix} r\left[\cancel{\cos^2\theta + \sin^2\theta}\right]$

$= 4(-xy+4xy)\cdot r = 4(3xy)r = 12xyr = 12r^3\sin\theta\cos\theta$

$= 6r^3\sin 2\theta$ showed.

32. $J = \frac{\partial(u,v,w)}{\partial(x,y,z)} \neq 0 \Rightarrow$ fully indep $\Rightarrow \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix} = 0(\cancel{1-0}) - 1(\cancel{1-1}) + 1(0-1) = -1$

as $J \neq 0$, so fully indep

$J' = \frac{1}{J} = -1$ so $\frac{\partial(x,y,z)}{\partial(u,v,w)} = -1$ fully indep.

(OR) $\begin{cases} u = x+y+z \\ v = y+z \\ u-v = x \end{cases}$

$\cancel{v = y+z}$, $\cancel{u = x+y+z}$ $\quad v = y+z$, $u = x+y+z$ $\quad u-v = x$ (i), $z = w-u-v$ (ii), $y = v-w+u+v = 2v-w+u$

$= \begin{vmatrix} 1 & 1 & 0 \\ 1 & 2 & -1 \\ -1 & -1 & 1 \end{vmatrix} = 1(\cancel{2-1}) - 1(\cancel{1-1}) + 0(0) = 1$ inverse

33. fully dep if $J = 0$ so, $\frac{\partial(u,v)}{\partial(x,y)} = J = 0 \Rightarrow \begin{vmatrix} \frac{1}{1+x^2} & \frac{1}{1+y^2} \\ \frac{(1-xy)(1)-(0-y)(x+y)}{(1-xy)^2} & \frac{(1-xy)(1)-(0-x)(x+y)}{(1-xy)^2} \end{vmatrix} = \frac{1}{(1-xy)^2}\begin{vmatrix} \frac{1}{1+x^2} & \frac{1}{1+y^2} \\ 1+y^2 & 1+x^2 \end{vmatrix}$

$1-\cancel{xy}-(\cancel{-xy}-y^2) = 1+y^2 \quad 1-xy-(-x^2-xy)$

$= \frac{1}{(1-xy)^2}\left[\cancel{1-1}\right] = 0$ fully dep

$\tan^{-1}\left(\frac{x+y}{1-xy}\right) = \tan^{-1}x + \tan^{-1}y \rightarrow u = \tan^{-1}(v)$ so $\tan u = v$ soln

34. "not indep" "dep so $|J| = 0$

$\frac{\partial(u,v,w)}{\partial(x,y,w)} = 0$ (FD) $= \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2x & 2y-2z & 2z-2y \end{vmatrix}$ $\quad -1((2y-2z)-(2y-2z)) - 1((2z-2y)-2x) + 1((2y-2z)+2x)$

$= 2(x+y-z)+2(x+y-z)$

$= \frac{}{}(x+y-z)$ So $|J| \neq 0$ So fn's indep (?)

35. $\begin{vmatrix} 2x & 2y & 2z \\ y+z & x+z & y+x \\ 1 & 1 & 1 \end{vmatrix} = 2x(x+z-y-x) - 2y(y+z-y-x) + 2z(y+z-x-z)$

$= 2x(z-y) - 2y(z-x) + 2z(y-x)$

$= 2xz - 2xy - 2yz + 2xy + 2zy - 2zx = 0$

So fn's dep. Rel^n: $(x+y+z)^2 = x^2+y^2+z^2 + 2(xy+yz+zx)$

$w^2 = u + 2v$ Sol^n

36. Scalar field: If to each point $P(x,y,z)$ of Region in space, there corresponds a definite scalar $\phi(x,y,z)$ then we say ϕ is a SPF & Region in which scalar is specified = SF. $\forall P(x,y,z) \leftrightarrow \phi$ Fcon SPF = SF

Ex: Temp $T(x,y,z)$

Vector field: $\forall P(x,y,z)$ of Reg of space $\exists$ corresponds a definite vector $f(x,y,z)$ then VPF & Region in which spe. = VF

Ex: Velocity (flow of fluid) $v(x,y,z)$

37. If $\phi(x,y,z)$ is diff. scalar fn at each point x,y,z in certain Reg of space, then grad of ϕ denoted $\nabla\phi$ or grad(∇) defined by $\nabla\phi = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\phi$

$= \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$

38. Consider $\phi(x,y,z) = c$ of SF. $\vec{r}$ pos vec $= x\hat{i} + y\hat{j} + z\hat{k}$ of any pt on surface of $\phi(x,y,z) = c$, $d\phi = 0$ So $\nabla\phi \cdot d\vec{r} = \left(\hat{i}\frac{\partial\phi}{\partial x} \cdots\right) dx\hat{i}$

$= \frac{\partial\phi}{\partial x}dx + $

$= d\phi = 0$

Thus, $\nabla\phi$ & $d\vec{r}$ are $\perp$ to e/o & $d\vec{r}$ is a vector along tangent plane at P & $\nabla\phi$ is vector normal to surface $\phi(x,y,z) = c$

39. $\nabla\phi$ where $f(r) = \phi$ & $r^2 = x^2+y^2+z^2$ & $r_x = \frac{x}{r}$ ...

$\nabla\phi = f'(r) \cdot r_x\hat{i} + f'(r) \cdot r_y\hat{j} + f'(r) \cdot r_z\hat{k}$

$= \frac{f'(r)}{r}(\vec{r})$ as $\frac{f'(r)}{r}(x\hat{i} + y\hat{j} + z\hat{k})$

PAGE NO.
DATE / /

40 $nr^{n-1}\left(\frac{x}{r}\hat{i}+\frac{y}{r}\hat{j}+\frac{z}{r}\hat{k}\right) = nr^{n-2}\vec{r}$ ahead

41 (a) $\frac{1}{r}(r_x\hat{i}+r_y\hat{j}+r_z\hat{k}) = \frac{1}{r^2}(\vec{r})$

(b) $-\frac{1}{r^2}(r_x\hat{i}+r_y\hat{j}+r_z\hat{k})$
$= -\left(\frac{1}{r^3}(\vec{r})\right)$

(42) ~~max~~ decrease: $\nabla T = \cancel{3x^2\hat{i}} \; (3x^2-3y^2)\hat{i} + (-6xy)\hat{j}$
$-\nabla T$: $(3y^2-3x^2)\hat{i} + 6xy\hat{j}$ so $((3y^2-3x^2), 6xy)$
$\frac{2}{x}, \frac{1}{y}$ $= (3-12)\hat{i} + 12\hat{j} = -9\hat{i}+12\hat{j}$ dirn.

43 (A) The component of $\nabla\phi$ in the dir of a unit vector $\hat{a}$ in dir of $\vec{a}$ given by: $\nabla\phi\cdot\hat{a}$ is DD of ϕ at x,y,z in dir of $\vec{a}$.

44. (A) Geom meaning: $\phi(x,y,z)$ consider $= c$; $\nabla\phi$ is grad so $(\nabla\phi)\,d\vec{r} = d\phi = 0$
so $\nabla\phi \perp$ to surface, normal.
; physically: ROC of ϕ at x,y,z

45 (A) Let $\vec{F}(x,y,z) = F_1(x,y,z)\hat{i} + F_2(x,y,z)\hat{j} + F_3(x,y,z)\hat{k} = F_1\hat{i}+F_2\hat{j}+F_3\hat{k}$
be a VPF diff at each pt xyz in certain reg of space then div of $\vec{F}$
div F or $\nabla\cdot\vec{F}$
physically rate of outflow / unit volume.

46 (A) Defn of divergence first. Then If div of $F = 0$, solenoidal i.e. $\nabla\cdot\phi = 0$

47 (A) DD: $\nabla\phi\cdot\hat{a} = |\nabla\phi||\hat{a}|\cos\theta$; $|\hat{a}| = 1$ so $|\nabla\phi|\cos\theta$; so $-1 \le \cos\theta \le 1$ max 1 so $= \nabla\phi$; travel

48 (A) (3) $\nabla\cdot A$ is the divergence of a vector $A(x,y,z) = A_1\hat{i}+A_2\hat{j}+A_3\hat{k}$ (VPF diff at each pt x,y,z in a reg. of space). $\nabla\cdot A = \frac{\partial A_1}{\partial x}\hat{i} + \frac{\partial A_2}{\partial y}\hat{j} + \frac{\partial A_3}{\partial z}\hat{k}$

; $A\cdot\nabla = A_1\frac{\partial}{\partial x}\hat{i} + A_2\frac{\partial}{\partial y}\hat{j} + A_3\frac{\partial}{\partial z}\hat{k}$; operator ; Can be used on scalar $(A\cdot\nabla)\phi = A\cdot\nabla\phi$

49. $\nabla^2 u$ if $u = x^2 - y^2 + 2z^2$

$\nabla(\nabla u) = \nabla(2x\hat{i} - 2y\hat{j} + 4z\hat{k}) = 2 - 2 + 4$

$\nabla(\nabla u) = \nabla(2x\hat{i} - 2y\hat{j} + 4z\hat{k}) = 2 - 2 + 4$ (c) 4

50. $(\vec{B}\cdot\nabla)\vec{A} \Rightarrow \left(x^2\frac{\partial}{\partial x} + yz\frac{\partial}{\partial y} - xy\frac{\partial}{\partial z}\right)(2yz\hat{i} - x^2y\hat{j} + xz^2\hat{k})$

$= x^2 \frac{\partial(2yz)}{\partial x}$ (?)

555. $\hat{n} = \frac{\vec{n}}{|\vec{n}|} = \frac{\nabla\phi}{|\nabla\phi|}$ so $\nabla(x^2y + 2xz - 4) = \frac{(2xy + 2z)\hat{i} + (x^2)\hat{j} + (2x)\hat{k}}{|\nabla\phi|}$

so pt $(2,-2,3)$: $(-8+6)\hat{i} + 4\hat{j} + 4\hat{k} = \frac{-2\hat{i} + 4\hat{j} + 4\hat{k}}{\sqrt{4+16+16}}$

$= \frac{-2\hat{i} + 4\hat{j} + 4\hat{k}}{6} = -\frac{1}{3}\hat{i} + \frac{2}{3}\hat{j} + \frac{2}{3}\hat{k}$

51. $\nabla\cdot\nabla\phi = \nabla\cdot(6x^2y^2z^4\hat{i} + 4x^3yz^4\hat{j} + 8x^3y^2z^3\hat{k})$

$= 12xy^2z^4 + 4x^3z^4 + 24x^3y^2z^2 = 4xz^2(3y^2z^2 + x^2z^2 + 6x^2y^2)$

57. $(\nabla\phi_1)\cdot(\nabla\phi_2) = |\nabla\phi_1||\nabla\phi_2|\cos\theta$ so

so $\cos\theta = \frac{(2xy\hat{i} + x^2\hat{j} + \hat{k})(\log z\hat{i} - 2y\hat{j} + \frac{x}{z}\hat{k})}{|\nabla\phi_1||\nabla\phi_2|}$

$x \; y \; z$
$(1 \; -1 \; 2)$ so $2(1)(-1) + \log 2$
$(-2\hat{i} + 1\hat{j} + \hat{k})(\log 2\hat{i} + 2\hat{j} + \frac{1}{2}\hat{k})$

$= \frac{-2\log 2 + 2 + \frac{1}{2}}{\sqrt{6}\sqrt{\log^2 2 + \frac{17}{4}}}$ $\frac{5}{2}$ $\sqrt{4+1+1}$ $\sqrt{\log^2 2 + 4 + \frac{1}{4}}$ $\Rightarrow \theta = \cos^{-1}(0.209)$

$\approx 77.93^\circ$

56. If $\theta = 90^\circ$

so $|\nabla\phi_1||\nabla\phi_2|\cos 90^\circ = (\nabla\phi_1)\cdot(\nabla\phi_2) = 0$ $(2ax\hat{i} - bz\hat{j} - by\hat{k})\cdot(\,)$

$ax^2 - byz - (a+2)x = 0$

$x \; y \; z$
$1 \; -1 \; 2$ $[(2ax - (a+2))\hat{i} - bz\hat{j} - by\hat{k}][8xy\hat{i} + 4x^2\hat{j} + \ldots z^2\hat{k}]$

DATE / /

$[(2a-a-2)(-8)+(-2b+4)+(+b-12)]=0$

$-8a-16-2b+4+b-12=0$

$-8a-b-32=0 \Rightarrow 8a+b=-32$ (i)

$((2a-a-2)-8)+(-2b-4)+(+b-12)=0$

$(a-10)-2b-4+b-12=a-b$

$8xy(2ax-(a+2))+(-bz-4x^2)+(-by-3z^2)$

$=-8(2a-a-2)+(-2b-4)+(b-12)$

$=-16a+8a+16-2b-4+b-12$

first subs then multiply the two in D.P.

$-8a-b \Rightarrow ((a-2)\hat{i}-2b\hat{j}+b\hat{k})(-8\hat{i}+4\hat{j}+12\hat{k})$

$=-8(a-2)-8b+12b \Rightarrow -8a+16-8b+12b=0 \Rightarrow -8a+4b=-16$

$\Rightarrow 2a-b=4$ — (i) & subst $\Rightarrow$

$a+2b=a+2$ so $b=1$ & $a=\frac{4+1}{2}=\frac{5}{2}$ so $a=\frac{5}{2}$ & $b=1$

60. DD $\nabla\phi.\hat{a}$ (Comp. along UV) so $yz\hat{i}+xz\hat{j}+xy\hat{k}=\nabla\phi$ & $\hat{n}=\frac{\vec{n}}{|\vec{n}|}=\nabla\phi$

so $y^2\hat{i}+z^2\hat{j}+x^2\hat{k}=\nabla\phi=\vec{n}$

$x\ y\ z$ / $1\ 1\ 1$

so $=\frac{\hat{i}+\hat{j}+\hat{k}}{\sqrt{3}}=\hat{n}$ now,

$\nabla\phi=\hat{i}+\hat{j}+\hat{k}$

$(\nabla\phi).\hat{n}=\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{3}}=\frac{3}{\sqrt{3}}$

$=\sqrt{3}$ //

61. $\nabla\phi.\hat{n} \rightarrow \frac{2\hat{i}-4\hat{j}+4\hat{k}}{6}=\frac{1}{3}\hat{i}-\frac{2}{3}\hat{j}+\frac{2}{3}\hat{k}$ $\begin{array}{ccc}1 & -2 & -1\\ x & y & z\end{array}$

$\rightarrow (2x-2y)\hat{i}-2x\hat{j}+3z^2\hat{k}=2+4\hat{i}-2\hat{j}+3\hat{k}=6\hat{i}-2\hat{j}+3\hat{k}$

$=2+\frac{4}{3}+2=4+\frac{4}{3}=\frac{16}{3}$ //

62. $\phi=y^2x+4z^3$ at $\begin{array}{ccc}2 & -1 & 1\\ x & y & z\end{array}$ dir of $n=x\log z-y^2=-4$ $\begin{array}{ccc}-1 & 2 & 1\\ x & y & z\end{array}$

$+z^3$

$\nabla\phi=y^2\hat{i}+2yx\hat{j}+3yz^2\hat{k}$ & $\nabla\phi.\hat{a}$, $\hat{n}=\frac{\vec{n}}{|\vec{n}|}$ & $\vec{n}=\nabla n$

$\cancel{-4\hat{i}-4\hat{j}+}\ \hat{i}-\cancel{3}\hat{j}-3\hat{k}$

$\nabla\phi.\hat{n}=0$ $\quad \vec{n}=\log z\,\hat{i}-2y\hat{j}+\frac{x}{z}\hat{k}=-4\hat{j}-1\hat{k}$

$=0+\frac{12}{\sqrt{17}}+\frac{3}{\sqrt{17}}=\frac{15}{\sqrt{17}}$ // $\quad \hat{n}=\frac{\vec{n}}{|\vec{n}|}=\sqrt{\log^2 1+16+1}=\sqrt{17}$

$\hat{n}=-\frac{4}{\sqrt{17}}\hat{j}-\frac{1}{\sqrt{17}}\hat{k}$

59. $\phi = x^2y^3z^4$ RoC of ϕ at $(2,3,-1)$ in dir making equal angles w +ve x, y, z axes

$3\cos^2\alpha = 1$ so $\cos\alpha = \frac{1}{\sqrt{3}}$ Now, $\hat{n} = \frac{1}{\sqrt{3}}\hat{i} + \frac{1}{\sqrt{3}}\hat{j} + \frac{1}{\sqrt{3}}\hat{k}$ & RoC at x, y, z $\Downarrow$ $\nabla\phi$ grad

So $= 2xy^3z^4\hat{i} + 3x^2y^2z^4\hat{j} + 4x^2y^3z^3\hat{k}$

$\begin{matrix} 2 & 3 & -1 \\ x & y & z \end{matrix}$

$\nabla\phi = 108\hat{i} + 108\hat{j} - 432\hat{k}$ so $\nabla\phi \cdot \hat{n}$

$= \frac{108}{\sqrt{3}} + \frac{108}{\sqrt{3}} - \frac{432}{\sqrt{3}} = \frac{-216}{\sqrt{3}} = -72\sqrt{3}$

63. "dir ∥ to y axis" $\Rightarrow 0\hat{i} + \hat{j} + 0\hat{k} = \hat{n}$ & $\nabla\phi \cdot \hat{a}$ & $\nabla\phi$

$\begin{matrix} -1 & 1 & 2 \\ x & y & z \end{matrix}$

$\nabla\phi = \cancel{2axy\hat{i}}$ $(2axy + bz)\hat{i} + (ay^2 + 3cz^3x^2)\hat{j} + (by + 3cz^2x^3)\hat{k}$

So $\nabla\phi \cdot \hat{a} = |\nabla\phi|\,|\hat{a}|\cos\theta = |\nabla\phi| = 32 = ay^2 + 3cz^3x^2$

$\hat{a} \to 0\hat{i} + \hat{j} + 0\hat{k}$, $|\hat{a}| \to 1$, max, $\theta = 0$ as $\cos\theta = 1$

$a + 24c$

So $b - a = 16$ (i) $\quad$ $2b - 2a$ $\quad$ 0

$a + 24c = 0$, $b - 12c = 0 \Rightarrow$ $\quad$ $a + 24c = 0$, $2b - 24c = 0$ $\Rightarrow$ $a + 2b = 0$ $\quad$ $b - a = 16$, $2b + a = 0$ $\Rightarrow$ $b = 16/3$

$a = \frac{16}{3} - 16 = \frac{-32}{3}$ $\quad$ $c = 4/9$

64. "In dir of $\nabla\phi$" & mg: $|\nabla\phi|$ So $2xyz^3\hat{i} + x^2z^3\hat{j} + 3x^2yz^2\hat{k}$

$\begin{matrix} 2 & 1 & -1 \\ x & y & z \end{matrix}$

$\underbrace{\nabla\phi = -4\hat{i} - 4\hat{j} + 12\hat{k}}_{\text{dir}}$ & mg: $|\nabla\phi| = \sqrt{16 + 16 + 144}$

$= 4\sqrt{11}$

65. "greatest $\underbrace{\text{ROI}}_{\text{ROC at } xyz}$" $\Rightarrow \nabla\phi$ (gradient) & max $= |\nabla\phi|$ $\quad$ $yz^2\hat{i} + xz^2\hat{j} + 2xyz\hat{k}$ $= 0\hat{i} + 9\hat{j} + 0\hat{k}$

$\begin{matrix} 1 & 0 & 3 \\ x & y & z \end{matrix}$

$= \sqrt{0^2 + 0^2 + 81} = 9$

52. $\vec{F} = \frac{x\hat{i} + y\hat{j}}{x^2 + y^2}$ is solenoidal then divergence $= 0$

So $\nabla \cdot \left(\frac{x}{x^2+y^2}\hat{i} + \frac{y}{x^2+y^2}\hat{j}\right) = \left[\frac{(x^2+y^2)(1) - 2x(x)}{(x^2+y^2)^2}\right]$

$+ \left[\frac{(x^2+y^2)(1) - 2y(y)}{(x^2+y^2)^2}\right] = \frac{\cancel{x^2} + y^2 - 2x^2 + x^2 + \cancel{y^2} - 2y^2}{(x^2+y^2)^2} = 0$ Hence

(55) $\vec{F} = (x^2+y^2+z^2)^{-n}$ then div (grad $\vec{F}$)

$\nabla\cdot(\nabla\vec{F}) = \nabla\cdot\left[-n(x^2+y^2+z^2)^{-n-1}\cdot 2(x\hat{i}+y\hat{j}+z\hat{k})\right]$

$= -6n(x^2+y^2+z^2)^{-n-1}[x+y+z]$ X

So, $-6n[x(-n-1)(x^2+y^2+z^2)^{-n-2}\cdot 2x + (x^2+y^2+z^2)^{-n-1}(1)$
$+ y(-n-1)(x^2+y^2+z^2)^{-n-2}\, 2y + (x^2+y^2+z^2)^{-n-1}(1)$
$12n \quad + z(-n-1)(x^2+y^2+z^2)^{-n-2}\, 2z + (x^2+y^2+z^2)^{-n-1}(1)]$

$= +12n(n+1)(x^2+y^2+z^2)^{-n-2}(x^2+y^2+z^2)^1 + 3(x^2+y^2+z^2)^{-n-1}$

$(x^2+y^2+z^2)^{-n-1}[12n(n+1)+3] \rightarrow 12n^2+12+3 = 12n^2+15$

$\nabla\cdot(\nabla\vec{F}) = \nabla\cdot\left(-n(x^2+y^2+z^2)^{-n-1}\cdot 2x\hat{i} - n(x^2+y^2+z^2)^{-n-1}\cdot 2y\hat{j} - n(x^2+y^2+z^2)^{-n-1}\cdot 2z\hat{k}\right)$

$-2n(x(-n-1)(x^2+y^2+z^2)^{-n-2}\, 2x + (x^2+y^2+z^2)^{-n-1})$
$-2n(y(-n-1)(x^2+y^2+z^2)\cdot 2y + (x^2+y^2+z^2)^{-n-1})$
$-2n(z(-n-1)(x^2+y^2+z^2)\cdot 2z + (x^2+y^2+z^2)^{-n-1})$

$+4n(n+1)(x^2+y^2+z^2)^{-n-1} - 2n(x^2+y^2+z^2)^{-n-1}$

$= (x^2+y^2+z^2)^{n-1}(4n^2+4-6n)$ $\;-2n(x^2+y^2+z^2)^{-n-1}(1) = -6n(\quad)$

* (2) ==

66 (A) If $\vec{F}(x,y,z) = F_1(x,y,z)\hat{i} + F_2(x,y,z)\hat{j} + F_3(x,y,z)\hat{k} = F_1\hat{i}+F_2\hat{j}+F_3\hat{k}$ that is diff at each pt x, y, z of reg of space then curl $\vec{F}$

$= \nabla\times\vec{F} = \text{rot}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{vmatrix} = \hat{i}\left(\frac{\partial F_3}{\partial y} - \right.$

67 (A) Physical meaning: measure of rotation of a VF. / Circulates ...

68 (A) First define curl then, } irr vectors whose curl vanishes ...

69 (A) ∇^2 is the laplacian op, which is defined by $\nabla\cdot\nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

70 (A) (2) (a) curl (grad ϕ) = 0   CQD
(b) div (curl $\vec{F}$) = 0

\# If F is irrot then ... $\vec{F} = \nabla\phi$ & $\vec{F} \Rightarrow$ conservative.

71. $\nabla \times (\vec{a} \times \vec{r}) = 2\vec{a}$ so $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ & $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

So $\nabla \times ((a_2 z - a_3 y)\hat{i} - (a_1 z - x a_3)\hat{j} + (a_1 y - a_2 x)\hat{k})$

$$\begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix} \quad \hat{i}(a_2 z - y a_3) - \hat{j}(a_1 z - x a_3) + \hat{k}(a_1 y - a_2 x)$$

$$= \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ a_2 z - a_3 y & -a_1 z + a_3 x & a_1 y - a_2 x \end{vmatrix} = \hat{i}(a_1 + a_1) - \hat{j}(-a_2 - a_2) + \hat{k}(a_3 + a_3)$$

$$= 2(a_1\hat{i} + \hat{j}a_2 + a_3\hat{k})$$

$$= 2\vec{a} \text{ proved}$$

72. $\begin{matrix} 1 & 0 & 2 \\ x & y & z \end{matrix}$ $\quad \nabla \times (\nabla \times \vec{A})$

$$\begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2 y & -2xz & 2yz \end{vmatrix} = \hat{i}(2z + 2x) - \hat{j}(\cancel{0 - 0}) + \hat{k}(-2z - x^2)$$

$$= (2z + 2x)\hat{i} - \hat{k}(2z + x^2)$$

$$(\nabla \times \vec{A}) = 5\hat{i} - 5\hat{k}$$

$$= \begin{vmatrix} \hat{i} & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2z + 2x & 0 & -2z - x^2 \end{vmatrix}$$

$$= \hat{i}(\cancel{0 - 0}) - \hat{j}(-2x - 2) + \hat{k}(\cancel{0 - 0}) = (2x + 2)\hat{j} = 4\hat{j}$$

74. $\vec{F} = \frac{x}{r}\hat{i} + \frac{y}{r}\hat{j} + \frac{z}{r}\hat{k}$, $r = \sqrt{x^2 + y^2 + z^2}$, $div(\vec{F})$, $curl(\vec{F})$?

So, $r^2 = x^2 + y^2 + z^2$, $d\vec{r} = ?$ then, $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k} \Rightarrow d\vec{r} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$

$$div(\vec{F}) = \nabla \cdot \vec{F} = \frac{r - r_x \cdot x}{r^2} + \frac{r - r_y \cdot y}{r^2} + \frac{r - r_z \cdot z}{r^2}$$

$\cancel{d r r_x = 2x}$

$2 r r_x = 2x$

$r_x = x/r$ = $^{1y \& z}$

$$\frac{r - \frac{x \cdot x}{r}}{r^2} \quad \frac{r^2 - x^2 + r^2 - y^2 + r^2 - z^2}{r^3} = \frac{3r^2 - r^2}{r^3} = \frac{2r^2}{r^3}$$

$$= \frac{2}{r} \; div(\vec{F})$$

$\nabla \times \vec{F}$ = Curl F

$$\begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x/r & y/r & z/r \end{vmatrix} = \hat{i}(0 - 0) - \hat{j}(0 - 0) + \hat{k}(0 - 0) = 0\hat{i} + 0\hat{j} + 0\hat{k} \text{ so } \vec{F} \text{ is } \; curl(\vec{F})$$

73. $\vec{A} = 2yz\,\hat{i} - x^2 y\,\hat{j} + xz^2\,\hat{k}$. $\phi = 2x^2 y z^3$ $(\vec{A} \times \nabla\phi)$

$\nabla\phi = 4xyz^3\,\hat{i} + 2x^2 z^3\,\hat{j} + 6x^2 y z^2\,\hat{k}$ so $\vec{A} \times \nabla\phi$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2yz & -x^2 y & xz^2 \\ 4xyz^3 & 2x^2 z^3 & 6x^2 y z^2 \end{vmatrix} = \hat{i}(-6x^4 y^2 z^2 - 2x^3 z^5) - \hat{j}(12x^2 y^2 z^3 - 4x^2 y z^5) + \hat{k}(4x^2 y z^4 + 4x^3 y^2 z^3)$$

(?)

75. $\nabla \times \vec{A} = 0$ then $\vec{A} = \nabla\phi(x)$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ axy - z^3 & (a-2)x^2 & (1-a)xz^2 \end{vmatrix} = \hat{i}(-2x(0)+0) - \hat{j}((1-a)z^2 + 3z^2) + \hat{k}(2ax - 4x - ax)$$

$= 0\hat{i} - \hat{j}(4z^2 - az^2) + \hat{k}(2ax - ax - 4x) = 0$

$4z^2 = az^2$ so $a = 4$

$2ax - 4 = ax$, $a = 4$

$\vec{A} = (4xy - z^3)\hat{i} + 2x^2\hat{j} - 3xz^2\hat{k} = \frac{\partial}{\partial x}\phi_x \hat{i} + \frac{\partial}{\partial y}\phi_y \hat{j} + \frac{\partial}{\partial z}\phi_z \hat{k}$

$\frac{\partial}{\partial x}\phi_x = 4xy - z^3$

$\phi_x = 2x^2y - z^3x = 2x^2y - z^3x + f(y,z)$

$\phi_y = 2x^2y + f_2(x,z)$

$\phi_z = -\frac{3xz^3}{3} + f_3(x,y) = -xz^3 + f_3(x,y)$

so $\phi = 2x^2y - xz^3 + \sum f_i(\ldots) \Rightarrow \phi = 2x^2y - xz^3 + c$

76. "conservative" $\Rightarrow \vec{F} = \nabla\phi$ & $\nabla\times\vec{F} = 0$ so

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x+2y+az & bx-3y-z & 4x+cy+2z \end{vmatrix} \quad \hat{i}(c-1) - \hat{j}(4-a) + \hat{k}(b-2) = 0$$

$c - 1 = 0$, $a - 4 = 0$, $b - 2 = 0$

$c = 1$, $a = 4$, $b = 2$

so $(x+2y+4z)\hat{i} + (2x-3y-z)\hat{j} + (4x+y+2z)\hat{k} = \vec{F} = \nabla\phi$

so $\phi_x = \frac{x^2}{2} + 2yx + 4zx + f_1(y,z)$

$\phi_y = 2xy - \frac{3y^2}{2} - zy + f_2(x,z)$

$\phi_z = 4xz + yz + z^2 + f_3(y,x)$

$\phi = \frac{x^2}{2} + 2xy + 4xz - zy - \frac{3y^2}{2} + z^2 + C$

77. $\nabla\times\vec{F} = 0$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2-y^2+x & -2xy-y & 0 \end{vmatrix} \quad \hat{i}(0-0) - \hat{j}(0-0) + \hat{k}(-2y+2y) = 0\hat{i} + 0\hat{j} + 0\hat{k}$$ so irrotational

so $\vec{F} = \nabla\phi$, conservative

$\phi_x = \frac{x^3}{3} - xy^2 + \frac{x^2}{2} + f(y,z)$

$\vec{F} = (x^2 - y^2 + x)\hat{i} - (2xy + y)\hat{j}$

$\phi_y = -xy^2 - \frac{y^2}{2} + f_2(x,z)$

so $\phi = \frac{x^3}{3} - xy^2 + \frac{x^2}{2} - \frac{y^2}{2} + c$

$\phi_z = 0$

78.
$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ axy+z^3 & 3x^2-z & bxz^2-y \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \end{vmatrix} \quad -\hat{i}(-1+1) - \hat{j}(3z^2 - bz^2) + \hat{k}(ax - 6x) = 0$$

so $3z^2 = bz^2$, $b = 3$

$ax = 6x$, $a = 6$

$(6xy+z^3)\hat{i}+(3x^2-z)\hat{j}+(3xz^2-y)\hat{k}=\nabla\phi$ So cons $\vec{F}$

$\phi_x = 3x^2y+z^3x+f_1(y,z)$

$\phi_y = 3x^2y-zy+f_2(x,z)$

$\phi_z = xz^3-yz+f_3(x,y)$

$\Big\}\ \phi = 3x^2y+z^3x-zy+c$

79. $\nabla\cdot\vec{F} = 2xy+2x^2z-3y^2 = 2(2)(1)(1)+2(2\times2)(1)-3(1)(1)$

$\begin{pmatrix}2&1&1\\x&y&z\end{pmatrix} \qquad = 9$ div$(\vec{F})$

$\nabla\times\vec{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2y & 2x^2yz & -3y^2z \end{vmatrix} = \hat{i}(-6yz-2x^2y)-\hat{j}(0-0)+\hat{k}(4xyz-x^2)$

$= -14\hat{i}-0\hat{j}+4\hat{k}$ curl $\vec{F}$.

$\nabla\cdot(\nabla\times\vec{F})$ (By $V_1(B)=0$) But $\nabla\cdot(\nabla\times\vec{F}) = \cancel{9}\ 2xy+2x^2z-3y^2$

$-4xy+0+4xy = 0$ div(curl$\vec{F}$)

81. $\nabla\times(\nabla\times\vec{F}) \Rightarrow \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 3x^2y-z & xz^3+y & -2x^2z^2 \end{vmatrix} = \hat{i}(0-3xz^2)-\hat{j}(-6x^2z^2+1) + \hat{k}(z^3-3x^2)$

$= \hat{i}(-3xz^2)-\hat{j}(-6x^2z^2+1) + \hat{k}(z^3-3x^2)$

$= \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -3xz^2 & +6x^2z^2-1 & z^3-3x^2 \end{vmatrix} = \hat{i}(0-12x^2z)-\hat{j}(-6x+6xz) + \hat{k}(-12xz^2-0)$ $12xz^2-0$

$= -12x^2z\hat{i}-\hat{j}(-6x+6xz)+\hat{k}(+12xz^2)$

84. "Solenoidal" $\Rightarrow$ div $\vec{F}=0$ So $\nabla\cdot\vec{F}=0 \Rightarrow y(2ax)+x(2y)+2xy(1)$

$= 2axy+4xy = 2xy(a+2)=0$

So $a=-2$ value So, $\vec{F_n} = y(-2x^2+z)\hat{i}+x(y^2-z^2)\hat{j}+2xy(z-xy)\hat{k}$

$\nabla\times\vec{F_n} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y(-2x^2+z) & x(y^2-z^2) & 2xy(z-xy) \end{vmatrix} = \hat{i}(2x(-x)-x(-2z))-\hat{j}(2y(-y)-y(0)) + \hat{k}((y^2-z^2)-(-2x^2+z))$

$-2x^2+2xz+2y^2+2y+y^2-z^2+2x^2-z$

Solenoidal So $\nabla\cdot(\nabla\times\vec{F_n})=0$

But $-4x+2z+4y+2+0-2z-1$

PAGE NO. DATE / /

$\nabla\cdot(\nabla\times F_1) = 0 \Rightarrow \vec{F_1} = y(-2x^2+z)\hat{i} + x(y^2-z^2)\hat{j} + 2xy(z-xy)\hat{k}$

$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y(-2x^2+z) & x(y^2-z^2) & 2xy(z-xy) \end{vmatrix}$ $= \hat{i}(2x(-x) - (x(-2z))) + \hat{j}(2y(4y)+y(1)) + \hat{k}((0)-0)$

$= -4x + 2z + 4y + 1 - 2z - x$

$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -2x^2y+yz & xy^2-z^2x & 2xyz - 2x^2y^2 \end{vmatrix}$ $= \hat{i}(2xz - 4x^2y + 2zx) - \hat{j}(4xyz + 4xy^2) + (-x)) + \hat{k}(y^2 - z^2 + 2x^2 - z) \cdot \nabla$

$= 2z + 2z - 8xy - 2z + 8xy + x - 2z - x = 0$ Hence shown

88. $\nabla\cdot\left(\frac{x}{r^3}\hat{i} + \frac{y}{r^3}\hat{j} + \frac{z}{r^3}\hat{k}\right)$ st $= 0$ So $\frac{r^3(1) - 3r^2 \cdot r_x x}{r^6} + \frac{r^3(1) - 3r^2(r_y)y}{r^6} + \frac{r^3(1) - 3r^2 \cdot r_z \cdot z}{r^6}$

$r = f(x,y,z)$

$r_x = x/r$ ||ly r_y r_z

$= \frac{3r^3 - \frac{3r^2(r^2)}{r}}{r^6} = \frac{3r^4 - 3r^4}{r^7} = 0$ Proved.

83. $\phi = \frac{1}{r}$ P.T. $\nabla^2\phi = 0$ So $\nabla^2\left(\frac{1}{r}\right) \Rightarrow \frac{\partial^2}{\partial x^2}\left(\frac{1}{r}\right) + \frac{\partial^2}{\partial y^2}\left(\frac{1}{r}\right) + \frac{\partial^2}{\partial z^2}\left(\frac{1}{r}\right)$

$= \left(-\frac{1}{r^2} \cdot r_x\right) + \left(-\frac{1}{r^2}\cdot r_y\right) + \left(-\frac{1}{r^2} r_z\right)$

~~$r = \frac{x}{r}$~~

~~$r_{x} = 2(1)\, r_x$~~

$= -\left[\left(\frac{r^2(r_{xx}) - 2r(r_x)}{r^4}\right) + \left(\frac{r^2(r_{yy}) - 2r(r_y)}{r^4}\right) + \left(\frac{r^2(r_{zz}) - 2r(r_z)}{r^4}\right)\right]$

$r_x = \frac{x}{r}$ ||ly y, z

$r_{xx} = \frac{r(1) - r_x(x)}{r^2}$

$= -\left[\frac{r^2\left(\frac{r^2-x^2}{r^3}\right) - 2r\left(\frac{x}{r}\right)}{r^4}\right] = -\frac{1}{r^5}\left[r^2 - x^2 - 2xr + r^2 - y^2 - 2yr + r^2 - z^2 + 2zr\right]$

$\Rightarrow \frac{r - \frac{x^2}{r}}{r^2} = \frac{r^2 - x^2}{r^3}$ ||ly y, z

$= \frac{\frac{r^2 - x^2}{r} - 2x}{r^4}$

$= \frac{-r^2 - x^2 - 2xr}{r^5}$ $\quad -\frac{1}{r^5}\left[3r^2 - r^2 - 2r(x+y+z)\right]$

$r_x = \frac{x}{r}$

$= -\frac{r}{r^5}\left[2r^2 - 2r\right]$ $\quad$ simple — $\left[\frac{1}{r^3}\cdot\frac{x^3(1) - 3x^2(x)}{r^6}\right]$

$-\left[\frac{r^3 - 3r^2(x) + r^3 - 3r^2(y) + r^3 - 3r^2(z)}{r^6}\right] = -\frac{1}{r^6}\left[3r^3 - 3r^2(x+y+z)\right]$

$r\,dr = x(x+y+z)$ expand r as $\frac{1}{\sqrt{x^2+y^2+z^2}}$ and solve $= 0$ Shown

Use ~~the~~ r stuff, ~~Ignore~~ it ~~take~~ for answer $\to 0$ (Applicability)

UNIT 3

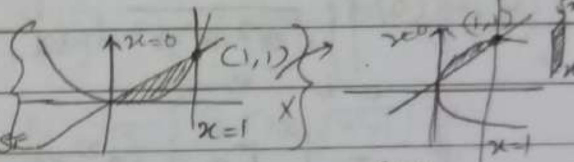
PAGE NO
DATE / /

Unit III

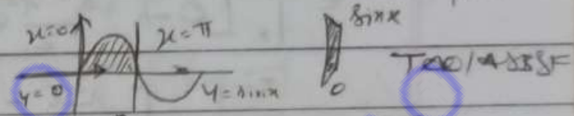
1. (A) The definite $\int$ of a bounded fn of 2 var in bounded domain of x-y plane ($\mathbb{R}^2$ space)
$\lim_{n\to\infty} \sum_{k=1}^{n} f(x_k, y_k)(\Delta x_k \times \Delta y_k)$ denoted $\iint_R f(x,y)\,dA$

2. (A) The definite $\int$ of a bounded fn of 3 variables in the bounded domain of $\mathbb{R}^3$ space / xyz plane
Denoted by $\iiint_V f(x,y,z)\,dV = \int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2} f(x,y,z)\,dx\,dy\,dz$

3. (a) $\int_0^1\int_x^{\sqrt{x}}$ $0 \le x \le 1$ $x=0$ $x=1$
$x \le y \le \sqrt{x}$ $y=x$ $y^2=x$

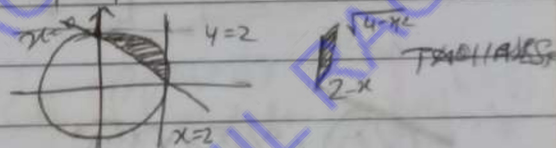


(b) $\int_0^{\pi}\int_0^{\sin x} f(x,y)\,dy\,dx$ $0 \le y \le \sin x$ $y=0$
$0 \le x \le \pi$ $y=\sin x$
$x=0$ $x=\pi$

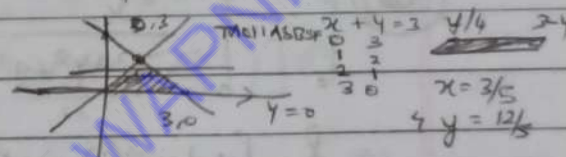


(c) $\int_0^2\int_{2-x}^{\sqrt{4-x^2}} f(x,y)\,dy\,dx$ $2-x \le y \le \sqrt{4-x^2}$
$0 \le x \le 2$
$y+x=2$ $y^2+x^2=2^2$

y	x
0	2
2	0



(d) $\int_0^2\int_{y/4}^{3-y} f(x,y)\,dy\,dx$ $y/4 \le x \le 3-y$ $y=2$
$0 \le y \le 2$ $y=0$
$4x=y$ $x+y=3$



5. (a) $\int_1^2\int_2^3 \left(x-\frac{1}{y}\right)^2 dx\,dy$: Constant limits so 1st w.r.t x so $\iint dx\,dy$ $\Big\}$ $\iint\left(x^2+\frac{1}{y^2}-\frac{2x}{y}\right)dx\,dy$

$= \int_1^2\int_2^3$ 1st w.r.t dx $\Rightarrow \int\left(\frac{1}{3}[x^3]_2^3 + \frac{1}{y^2}[x]_2^3 - \frac{\cancel{2}}{y\cancel{2}}[x^2]_2^3\right)dy = \int\left(\frac{1}{3}(27-8)+\frac{1}{y^2}(1) - \frac{1}{y}(9-4)\right)dy$

$= \int\left(\frac{19}{3} + \frac{1}{y^2} - \frac{5}{y}\right)dy = \frac{19}{3}[y]_1^2 - \left[\frac{1}{y}\right]_1^2 - 5\left[\ln|y|\right]_1^2$

$= \frac{19}{3} - \cancel{\left[\frac{1}{2}-\frac{1}{1}\right]}^{-1/2}_{+1/2} - 5[\ln 2 - \ln 1] = \frac{19}{3} + \frac{1}{2} - 5\ln 2 = \frac{41}{6} - (\ln 2)5$

(b) $\int_0^1\int_x^{\sqrt{x}} xy\,dy\,dx$ Nvar so w.r.t y 1st $\int_0^1 \frac{x}{2}[y^2]_x^{\sqrt{x}}\,dx = \int_0^1\frac{x}{2}[x-x^2]\,dx$

$= \int_0^1\left(\frac{x^2}{2}-\frac{x^3}{2}\right)dx = [x^3]_0^1\frac{1}{6} - \frac{1}{8}[x^4]_0^1 = \frac{1}{6}-\frac{1}{8} = \frac{2}{48} = \frac{1}{24}$

$\int uv = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \cdots$ # $\int \frac{1}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$

$e^{ax} \Rightarrow \frac{e^{ax}}{1/a}$ # $\int \sqrt{a^2-y^2} = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)$ — $\frac{1}{x^2+a^2+y^2}$ $\frac{1}{(\sqrt{x^2+a^2})^2+y^2}$

(c) $\int_0^1\int_0^{x^2} e^{y/x}\,dy\,dx = \int_0^1 \frac{1}{1/x}\left[e^{y/x}\right]_0^{x^2}dx = \int \frac{1}{1/x}(e^x-1)\,dx = \int (xe^x - x)\,dx$

$= (xe^x - e^x)_0^1 - \frac{1}{2}[x^2]_0^1 = (\cancel{e} - \cancel{e} - (-1)) - \frac{1}{2}(1-0) = e - \frac{3}{2} = \frac{1}{2}$

(d) $\int_0^4\int_{x^2/4}^{2\sqrt{x}} y\,dy\,dx = \int_0^4 \frac{1}{2}[y^2]_{x^2/4}^{2\sqrt{x}}\,dx = \frac{1}{2}\int_0^4 \left(4x - \frac{x^4}{16}\right)dx = \frac{1}{2}\cdot\frac{4}{2}[x^2]_0^4 - \frac{1}{16\times 2\times 5}[x^5]$

$= (16-0) - \frac{1}{32\times 5}(4^5) = 16 - \frac{4^5}{4^3\times 5} = 16 - \frac{64}{5} = \frac{16}{5} = 9.6$

9\. (a) $\int_0^a\int_0^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2-y^2}\,dy\,dx = \int_0^a \left[\frac{y}{2}\sqrt{a^2-x^2-y^2} + \frac{a^2-x^2}{2}\sin^{-1}\left(\frac{y}{\sqrt{a^2-x^2}}\right)\right]_0^{\sqrt{a^2-x^2}} dx$

$= \int_0^a \left[0 + \frac{a^2-x^2}{2}\sin^{-1}(1) - 0 - 0\right]dx = \int_0^a \left(\frac{a^2-x^2}{2}\right)\left(\frac{\pi}{2}\right)dx$

$= \int_0^a \left[\frac{a^2\pi}{4} - \frac{\pi}{4}x^2\right]dx = \frac{a^2\pi}{4}[x]_0^a - \frac{\pi}{4}\cdot\frac{1}{3}[x^3]_0^a = \frac{3a^3\pi}{12} - \frac{\pi}{12}a^3$

$= \frac{2a^3\pi}{12} = a^3\pi/6$ //

(b) $\int_0^{a/\sqrt{3}}\int_0^{\sqrt{x^2+a^2}} \frac{x}{x^2+y^2+a^2}\,dy\,dx = \int x\left[\frac{1}{\sqrt{x^2+a^2}}\tan^{-1}\left(\frac{y}{\sqrt{x^2+a^2}}\right)\right]_0^{\sqrt{x^2+a^2}} dx$

$= \int x\left[1\tan^{-1}(1) - \cancel{0\frac{1}{x^2+a^2}}\right]dy = \int_0^{a/\sqrt{3}} x\frac{\pi}{4}\,dx = \frac{\pi}{4\times 2}[x^2]_0^{a/\sqrt{3}} = \frac{\pi}{8}\left(\frac{a^2}{3}-0\right)$

$= \frac{a^2\pi}{24}$ ✗ $= \int \frac{x}{\sqrt{x^2+a^2}}\left(\frac{\pi}{4}\right)dx = \frac{\pi}{4}\int_0^{a/\sqrt{3}}\left(\frac{x}{\sqrt{x^2+a^2}}\right)dx$

$= \frac{\pi}{4}\int_0^{a/\sqrt{3}} \frac{x}{\sqrt{x^2+a^2}}\,dx$, $\sqrt{x^2+a^2} \to t$ So $x: 0 \to a/\sqrt{3}$, $t: a \to \frac{2a}{\sqrt{3}}$

($t^2 = x^2+a^2$, $\frac{a^2}{3}+a^2$, $2t\,dt = 2x\,dx$, $dx = \frac{t\,dt}{x}$)

So $\frac{\pi}{4}\int_0^{a/\sqrt{3}} \frac{x}{t}\frac{t\,dt}{x} = \frac{\pi}{4}[t]_a^{a/\sqrt{3}}$

$= \frac{\pi}{4}\left(\frac{a}{\sqrt{3}}\right) = \frac{a\pi}{4}$

We see, $\int_0^{a/\sqrt{3}}\int_0^{\sqrt{x^2+a^2}} \frac{x}{x^2+y^2+a^2}\,dy\,dx \Rightarrow \int_0^{a/\sqrt{3}}\int \frac{x}{(\sqrt{x^2+a^2})^2+y^2}\,dy\,dx = \int_0^{a/\sqrt{3}} x\left[\frac{1}{\sqrt{x^2+a^2}}\tan^{-1}\left(\frac{y}{\sqrt{x^2+a^2}}\right)\right]^{\sqrt{x^2+a^2}}$

$= \int_0^{a/\sqrt{3}} x\left[\frac{1}{\sqrt{x^2+a^2}}\tan^{-1}\left(\frac{\sqrt{x^2+a^2}}{\sqrt{x^2+a^2}}\right) - 0\right]dx = \int_0^{a/\sqrt{3}}\left(\frac{x}{\sqrt{x^2+a^2}}\right)\left(\frac{\pi}{4}\right)dx$

$= \frac{\pi}{4}\int_0^{a/\sqrt{3}}\left(\ \right)$ So $x^2+a^2 = t^2$, $2t\,dt = 2x\,dx$, $dx = \frac{t\,dt}{x}$

$x: 0 \to a/\sqrt{3}$, $t: a \to \frac{2a}{\sqrt{3}}$

$\frac{\pi}{4}\int \frac{x}{t}\frac{t\,dt}{x} = \frac{\pi}{4}[t]_a^{2a/\sqrt{3}}$

$= \frac{\pi}{4}\left(\frac{2a}{\sqrt{3}} - a\right)$

$\int_0^{\pi/2} \sin^n x = \frac{(n-1)(n-3)\cdots}{n(n-2)\cdots} \left(\times\frac{\pi}{2}\right)$ $\int_0^{\pi/2} \cos^n\theta$

$\int \frac{1}{\sqrt{x^2+a^2}} = \ln|x+\sqrt{x^2+a^2}|$

$\frac{x}{a}+\frac{y}{b}=1$

(c) $\int\int (x^{1/3}) y^{-1/2} (1-x-y)^{1/2} dy\,dx$: So $y = (1-x)\sin^2\theta$
$dy = (1-x)\, 2\sin\theta\cos\theta\, d\theta$

$= \int\int x^{1/3} (1-x)^{-1/2} \frac{1}{\sin\theta} (1-x-(1-x)\sin^2\theta)^{1/2} dy\,dx$ limits $y: 0 \to 1-x$, $\theta: 0 \to \pi/2$

$\int\int x^{1/3} (1-x)^{-1/2} \frac{1}{\cancel{\sin\theta}} (1-x-\sin^2\theta + x\sin^2\theta)^{1/2} (1-x) 2\cancel{\sin\theta}\cos\theta\, d\theta\, dx$

$1-x-\sin^2\theta(1+x)$
$= (1-x)(1-\sin^2\theta)$
$= (1-x)^{1/2}\cos\theta$

$\int 2x^{1/3}(1-x)\int_0^{\pi/2}\cos^2\theta\, d\theta\, dx$

$= \int\int 2x^{1/3}(1-x)(\cos^2\theta)\, d\theta\, dx$

$= 2\int x^{1/3} - x^{4/3} \left(\left[\frac{(2-1)}{\cancel{(2\cdot 0)}}\right]\frac{\pi}{2}\right) dx = \frac{\pi}{2}\int (x^{1/3}-x^{4/3})\, dx = \frac{\pi}{2}\frac{1}{3}x = -\frac{3\pi}{7}$

(d) $\int_0^1\int_0^{\sqrt{1+x^2}} \frac{1}{x^2+y^2+1} dy\,dx : \int_0^1\int \frac{1}{(\sqrt{x^2+1})^2+y^2} dy\,dx = \int_0^1 \left[\frac{1}{\sqrt{x^2+1}} \tan^{-1}\left(\frac{y}{\sqrt{x^2+1}}\right)\right]_0^{\sqrt{x^2+1}}$

$= \int_0^1 \frac{1}{\sqrt{x^2+1}}\left(\tan^{-1}(1)\right) dx = \frac{\pi}{4}\int_0^1 \frac{1}{\sqrt{x^2+1}} dx = \frac{\pi}{4}\left[\ln\left[x+\sqrt{x^2+1}\right]\right]_0^1$

$= \frac{\pi}{4}\left[\ln|1+\sqrt{2}| - \ln|0+1|\right] = \frac{\pi}{4}\ln|1+\sqrt{2}|$

10. $\iint_R xy\,dy\,dx$: $x=0$, $y=0$, b, a, $b(1-\frac{x}{a})$ So $\int_0^a\int_0^{b(1-\frac{x}{a})} xy\,dy\,dx = \int \frac{x}{2}[y^2]_0^{b(1-\frac{x}{a})} dx$

$= \int \frac{x}{2}\left[b^2\left(1-\frac{x}{a}\right)^2\right] dx = \int \frac{x}{2}\left[b^2\left(1+\frac{x^2}{a^2}-\frac{2x}{a}\right)\right] dx$

$= \int \left(\frac{b^2}{2}x + \frac{b^2}{2a^2}x^3 - \frac{2b^2}{2a}x^2\right) dx = \frac{b^2}{2\cdot 2}[x^2]_0^a + \frac{b^2}{2a^2\cdot 4}[x^4]_0^a - \frac{b^2}{a3}[x^3]_0^a$

$= \frac{2b^2a^2}{8} + \frac{b^2a^2}{8} - \frac{b^2a^2}{3} = \frac{3b^2a^2}{8} - \frac{b^2a^2}{3} = \frac{a^2b^2}{24}$

11. $\iint_R xy\,dx\,dy$: $y=0$, $x=2a$, $x^2=4ay$ $\int_0^{2a}\int_0^{a} x$ $x^2=4ay$, $(2a,a)$, $x=2a$, $x^2/4a$

$\int_0^{2a}\int_0^{x^2/4a} xy\,dx\,dy = \int_0^{2a} \frac{x}{2}[y^2]_0^{x^2/4a} dx = \int \frac{x}{2}\left[\frac{x^4}{16a^2}\right] dx = \frac{1}{32a^2}\frac{1}{6}[x^6]_0^{2a}$

$= \frac{1}{32a^2\times 6}[2^6a^6] = a^4/3$

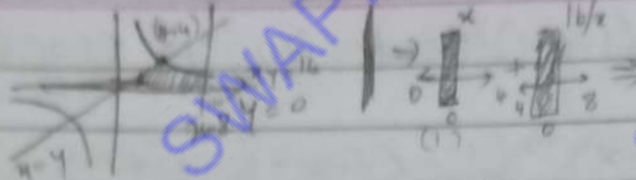
12. $y=x^2$, $(1,1)$, $(2,0)$, $(0,2)$, $2-x$, x^2

$\int_0^1\int_{x^2}^{2-x} y\,dx\,dy = \int \frac{1}{2}[y^2]_{x^2}^{2-x} dx = \frac{1}{2}\int (4+x^2-4x-x^4)\, dx$

$= \frac{1}{2}\left[4[x]_0^1 + \frac{1}{3}[x^3]_0^1 - \frac{4}{2}[x^2]_0^1 - \frac{1}{5}[x^5]_0^1\right]$

$= \frac{1}{2}\left[4+\frac{1}{3}-2-\frac{1}{5}\right] = \frac{16}{15}$

13. $\iint x^2 \, dx\, dy$



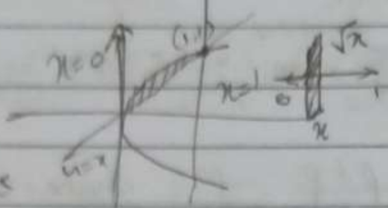
$\Rightarrow$ (1) $\int_0^4\int_0^x x^2 \, dy\, dx = \int_0^4 x^2[y]_0^x dx = \int_0^4 x^3 dx = \frac{1}{4}[x^4]_0^4 = 64$

(2) $\int_4^8\int_0^{16/x} x^2 \, dy\, dx = \int_4^8 x^2[y]_0^{16/x} dx = \int_4^8 x^2\left(\frac{16}{x}\right) dx = \frac{16}{2}[x^2]_4^8 = 384$

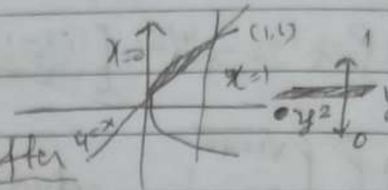
So (1)+(2) = 384+64 = 448

18. (a) $\int_0^1\int_x^{\sqrt{x}} f(x,y)\, dy\, dx$

$x \le y \le \sqrt{x}$, $0 \le x \le 1$; $x=0$, $x=1$, $y=x$, $y^2=x$



Before



After

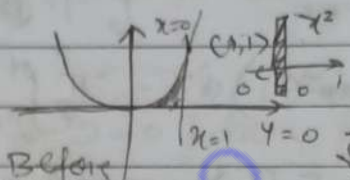
$x: y^2 \to y$

$y: 0 \to 1$

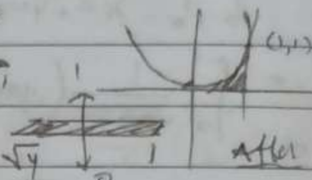
(ACCOT, ...)

(b) $\int_0^1\int_0^{x^2} f(x,y)\, dy\, dx$

$0 \le y \le x^2$, $0 \le x \le 1$; $y=0$, $y=x^2$, $x=0$, $x=1$



Before



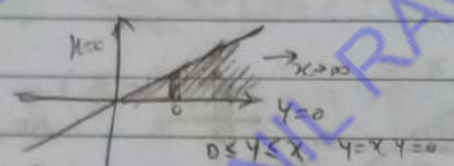
After

$x: \sqrt{y} \to 1$

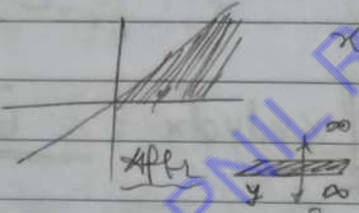
$y: 0 \to 1$

(c) $\int_0^\infty\int_0^x f(x,y)\, dy\, dx$

$0 \le y \le x$, $0 \le x \le \infty$; $y=x$, $y=0$, $x=0$, $x=\infty$



Before ① → ...



After

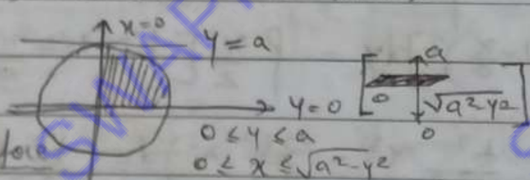
$x: y \to \infty$

$y: 0 \to \infty$

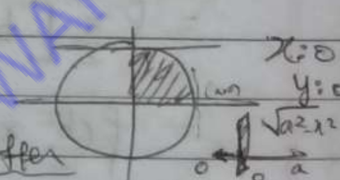
(d) $\int_0^a\int_0^{\sqrt{a^2-y^2}} f(x,y)\, dx\, dy$

$y=a$, $y=0$, $x=0$, $x^2+y^2=a^2$

$0 \le y \le a$, $0 \le x \le \sqrt{a^2-y^2}$



Before



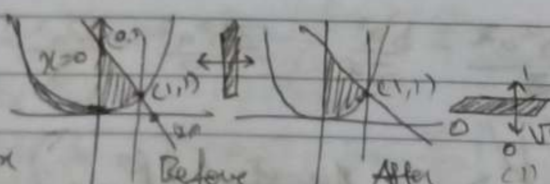
After

$x: 0 \to a$

$y: 0 \to \sqrt{a^2-x^2}$

19. (a) $\int_0^1\int_{x^2}^{2-x} xy\, dy\, dx$

$0 \le x \le 1$, $x^2 \le y \le 2-x$; $x=0$, $x=1$, $y=x^2$, $y+x=2$



Before / After

(1) + (2)

(1) $\int_0^1\int_0^{\sqrt{y}} xy\, dy\, dx = \int_0^1 \frac{y}{2}[x^2]_0^{\sqrt{y}} dy = \int_0^1 \frac{y}{2}(y)\, dy = \frac{1}{2}\cdot\frac{1}{3}[y^3]_0^1 = \frac{1}{6}$

(2) $\int_1^2\int_0^{2-y} xy\, dy\, dx = \int_1^2 \frac{y}{2}[x^2]_0^{2-y} dy$

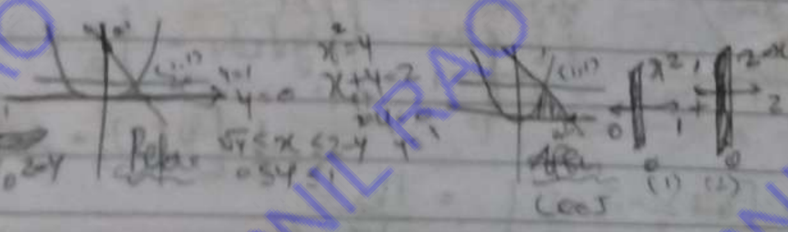
$= \frac{1}{2}\int_1^2 y(4+y^2-4y)\, dy = \frac{1}{2}\left[\frac{4}{2}[y^2]_1^2 + \frac{1}{4}[y^4]_1^2 - \frac{4}{3}[y^3]_1^2\right]$

$= \frac{1}{2}\left[2(4-1) + \frac{1}{4}(16-1) - \frac{4}{3}(8-1)\right] = \frac{1}{2}\left[6 + \frac{15}{4} - \frac{28}{3}\right] = \frac{5}{24}$ (2)

(1)+(2) = $\frac{1}{6} + \frac{5}{24} = 3/8$

(b) $\int_0^1\int_{\sqrt{y}}^{2-y} xy\, dx\, dy$

$x^2=4$, $x+y=2$



Before: $\sqrt{y} \le x \le 2-y$, $0 \le y \le 1$

After (COOJ)

New limits: $\int_0^1\int_0^{x^2} + \int_1^2\int_0^{2-x}$

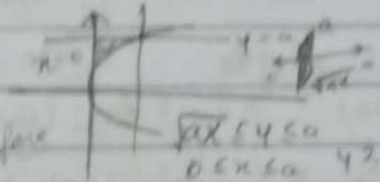
$\int \frac{1}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right)$ [illegible]

$\iint xy\,dy\,dx = \int_0^1 \frac{x}{2}\left[y^2\right]_0^{x^2} dx$ [illegible] $\int_1^2 \frac{x}{2}\left[y^2\right]_0^{2-x} dx$

$= \int_1^2 \frac{x}{2}\left[4+x^2-4x\right] dx = \frac{1}{2}\int_1^2 (4x + x^3 - 4x^2)\,dx = \frac{1}{2}\left[\frac{4}{2}[x^2]_1^2 + \frac{1}{4}[x^4]_1^2 - \frac{4}{3}[x^3]_1^2\right]$

$= \frac{1}{2}\left[2(4-1) + \frac{1}{4}(16-1) - \frac{4}{3}(8-1)\right] = \frac{5}{24}$; $\frac{2}{24} + \frac{5}{24} = \frac{7}{24}$ //

(c) $\int_0^a \int_{\sqrt{ax}}^a \frac{y^2}{\sqrt{y^4 - a^2x^2}}\,dy\,dx$



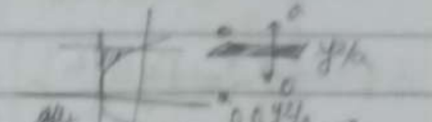
Before

$\sqrt{ax} \le y \le a$

$0 \le x \le a$

$y^2 = ax$

$y = a$



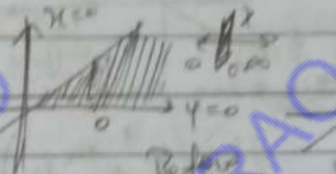
After

$\int_0^a \int_0^{y^2/a} \frac{y^2}{\sqrt{y^4 - a^2x^2}}$

$= \int_0^a y^2 \left[\sin^{-1}\left(\frac{ax}{y^2}\right)\right]_0^{y^2/a} dy = \int_0^a y^2\left[\sin^{-1}\left(\frac{ax}{yx}\right) - 0\right] dy$

$= \frac{\pi}{2 \cdot 3}\left[y^3\right]_0^a = \frac{a^3\pi}{6}$

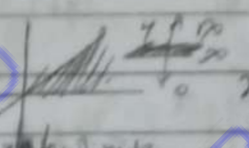
~~(d) $\int_0^\infty \int_0^x x e^{-x^2/y}\,dy\,dx$~~



$0 \le y \le x$

$0 \le x \le \infty$

Before



New limits

$x: y \to \infty$

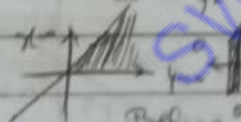
$y: 0 \to \infty$

$\int_0^\infty \int_y^\infty x e^{-x^2/y}\,dy\,dx$

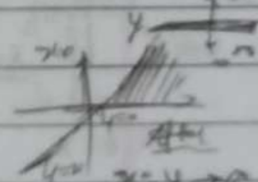
$x^2 = t$, $2x\,dx = dt$, $dx = \frac{dt}{2x}$; $x: y \to \infty$, $t: y^2 \to \infty$ $\Big\}$ $\int_0^\infty \int_{y^2}^\infty x e^{-t/y} \frac{dt}{2x}\,dy = \frac{1}{2}\left[\int_{y^2}\left[\frac{e^{-t/y}}{-1/y}\right] dy\right]$

$= -\frac{1}{2}\left[\int_{y^2}^\infty \left[y e^{-t/y}\right] dy\right] = -\frac{1}{2}\left[\int_{y^2}^\infty \right]$ ✗

$\int_0^\infty \int_0^x x e^{-x^2/y}\,dy\,dx$ $0 \le y \le x$, $0 \le x \le \infty$



Before



After

$x: y \to \infty$

$y: 0 \to \infty$

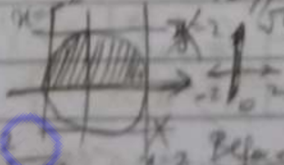
$\int_0^\infty \int_y^\infty x e^{-x^2/y}\,dy\,dx$ so $\left\{\begin{array}{l} x^2 = t \\ 2x\,dx = dt \\ dx = \frac{dt}{2x} \end{array}\right.$ $x: y \to \infty$, $t: y^2 \to \infty$ $\Big\}$ $\int_0^\infty \int_{y^2}^\infty x e^{-t/y}\,dy\,\frac{dt}{2x}$

$= \frac{1}{2}\left[\int_0^\infty \int_{y^2}^\infty e^{-t/y}\,dt\,dy\right] = \frac{1}{2}\left[\int\left[\frac{e^{-t/y}}{-1/y}\right]_{y^2}^\infty dy\right] = -\frac{1}{2}\left[\int\left[y e^{-t/y}\right]_{y^2}^\infty\right]$

$= -\frac{1}{2}\left[\int\left(y \cancel{\frac{1}{e^\infty}} - y e^{-y}\right) dy\right] = +\frac{1}{2}\left[-ye^{-y} + e^{-y}\right]_0^\infty = \frac{1}{2}\left[-\frac{\infty}{e^\infty} + \frac{1}{e^\infty} + 0 + e^0\right]$

$= 1/2$ //

(e) $\int_{-2}^2 \int_0^{\sqrt{4-x^2}} (2-x)\,dy\,dx$

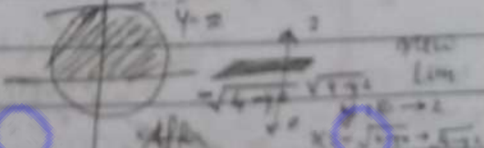


$x = -2, 2$

$-2 \le x \le 2$

$0 \le y \le \sqrt{4-x^2}$

Before



After

new lim

$y: 0 \to 2$

$x: -\sqrt{4-y^2} \to \sqrt{4-y^2}$

$y = 0$, $y^2 + x^2 = 4$

$\int_0^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} (2-x)\,dy\,dx = \int_0^2 \left[2[x] - \frac{1}{2}[x^2]\right] dy = \int_0^2 \left(2(2)(\sqrt{4-y^2}) - \frac{1}{2}[0]\right) dy$

$= 4\int_0^2 (\,)\,dy = 4\left[\frac{y}{2}\sqrt{4-y^2} + \frac{4}{2}\sin^{-1}\left(\frac{y}{2}\right)\right]_0^2 = 4\left[0 + 2\frac{\pi}{2} - 0 - 0\right] = 4\pi$

$$\int_0^{\pi/2} \sin^m\theta \cos^n\theta = \frac{(m-1)(m-3)\cdots(n-1)(n-3)\cdots}{(m+n)(m+n-2)\cdots} \times \pi/2 \quad m,n \in E$$

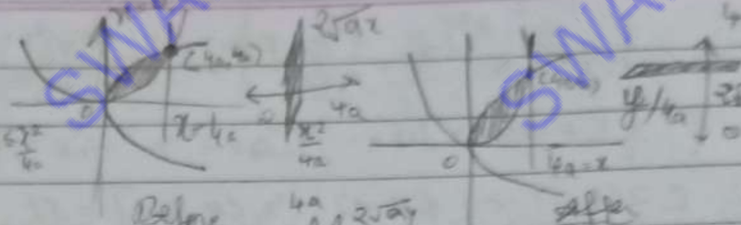
$\times 1$ otherwise

But $m=1$, $n=1$ then $=0$

(19) $\int_0^{4a}\int_{x^2/4a}^{2\sqrt{ax}} dy\,dx$

$0 \le x \le 4a$, $2\sqrt{ax} \le y \le \frac{x^2}{4a}$

$x=0, x=4a$, $y^2=4ax$, $4ay=x^2$, $y^2=4a(2\sqrt{ay})$



New lim: $x: \frac{y^2}{4a} \to 2\sqrt{ay}$, $y: 0 \to 4a$

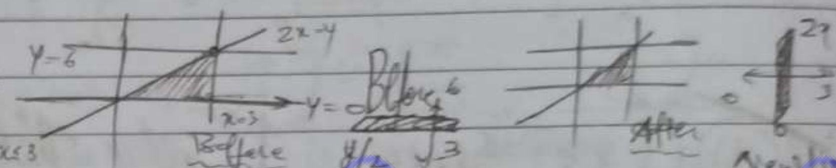
$$\int_0^{4a}\int_{y^2/4a}^{2\sqrt{ay}} dy\,dx = \int [x]_{y^2/4a}^{2\sqrt{ay}} dy = \int_0^{4a}\left(2\sqrt{ay} - \frac{y^2}{4a}\right)dy$$

$$\frac{2\cdot 2\sqrt{a}}{3}\left[y^{3/2}\right]_0^{4a} - \frac{1}{4a\,3}\left[y^3\right]_0^{4a} = \frac{4\sqrt{a}}{3}\left[(4a)^{3/2}\right] - \frac{1}{12a}\left[64a^3\right] = \frac{(16\sqrt{4})a^2}{3} - \frac{16a^2}{3}$$

$$= a^2\left(\frac{16\sqrt{4}}{3} - \frac{16}{3}\right) = a^2\left(\frac{32}{3} - \frac{16}{3}\right) = \frac{16a^2}{3}$$

20. $\int_0^6\int_{y/2}^3 \frac{1}{x}e^{y/x}\,dy\,dx$ (8t)

$y/2 \le x \le 3$, $x = y/2$ & $2x = y$, $0 \le y \le 6$



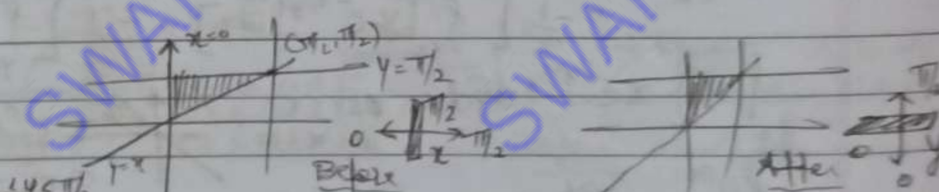
$x: 0 \to 3$, $y: 0 \to 2x$

$$\int_0^3\int_0^{2x} \frac{1}{x}e^{y/x}\,dy\,dx = \int_0^3 \frac{1}{x}\left(\int_0^{2x} e^{y/x}dy\right)dx = \int_0^3 \frac{1}{x}\left[\frac{e^{y/x}}{1/x}\right]_0^{2x}dx$$

$$= \int_0^3 \frac{x}{x}\left[e^{y/x}\right]_0^{2x}dx = \int_0^3 (e^2-1)\,dx = (e^2-1)[x]_0^3 = 3(e^2-1)$$

21. $\int_0^{\pi/2}\int_x^{\pi/2} \frac{\sin y}{y}\,dy\,dx$ (8t)

$x \le y \le \pi/2$, $0 \le x \le \pi/2$, $x=0, x=\pi/2, y=x, y=\pi/2$



New lim: $x: 0 \to y$, $y: 0 \to \pi/2$

$$\int_0^{\pi/2}\int_0^y \frac{\sin y}{y}\,dy\,dx = \int_0^{\pi/2}\frac{\sin y}{y}[x]_0^y\,dx = \int_0^{\pi/2}\frac{\sin y}{\cancel{y}}(\cancel{y}-0) = -\left[\cos y\right]_0^{\pi/2} = -(-1)$$

$= 1$ Showed

6 (a) $\int_0^{\pi}\int_0^{a\cos\theta} r\sin\theta\,dr\,d\theta = \int_0^{\pi}\frac{\sin\theta}{2}[r^2]_0^{a\cos\theta} = \frac{a^2}{2}\left[\int_0^{\pi}\sin\theta\cos^2\theta\right] = \frac{a^2}{2}2\left[\int_0^{\pi/2}\right]$

$$= a^2\left[\frac{(2-1)(1)}{(3-1)(3-2)}\right]\cancel{\left(\frac{\pi}{2}\times\frac{2}{\pi}\right)}$$

$$\int_0^{\pi}\frac{\sin\theta}{2}[r^2]_0^{a\cos\theta}d\theta = \int_0^{\pi}\frac{\sin\theta\, a^2\cos^2\theta}{2}$$

$$= \frac{a^2}{2}\int_0^{\pi}\sin\theta\cos^2\theta\,d\theta = \frac{a^2}{2}\times 2\left[\int_0^{\pi/2}\sin\theta\cos^2\theta\,d\theta\right] = a^2\left[\frac{(2-1)(1)}{(3-0)(3-2)}\right] = \frac{a^2}{3}$$

(b) $\int_0^{\pi/2}\int_0^{\infty} r e^{-r^2}\,dr\,d\theta = \int_0^{\pi/2}\left[\int_0^{\infty} r e^{-r^2}dr\right]d\theta$ $\left(\text{let } r^2 = t, \; 2r\,dr = dt, \; dr = \frac{dt}{2r}, \; r: 0 \to \infty, \; t: 0 \to \infty\right)$ $\int_0^{\pi/2}\int_0^{\infty} r e^{-t}\frac{dt}{2r}\,d\theta$

$= \frac{1}{2}\int_0^{\pi/2}\left[\frac{e^{-r^2}}{-1}\right]_0^\infty d\theta = \frac{-1}{2}\left[\int_0^{\pi/2}(\cancel{e^{-\infty}} - e^{0})\,d\theta\right] = \frac{1}{2}[\theta]_0^{\pi/2} = \pi/4$

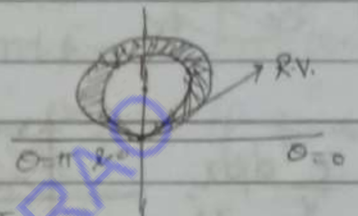
(c) $\int_0^{\pi/2}\int_{a\cos\theta}^{a} \sin\theta\cos\theta\, dr\, d\theta \quad \int_0^{\pi/2}\sin\theta\cos\theta\,[r]_{a\cos\theta}^{a}\, d\theta = \int \sin\theta\cos\theta\,(a-a\cos\theta)\,d\theta$

$= a\int \sin\theta\cos\theta\,(1-\cos\theta)\,d\theta = a\left[\int_0^{\pi/2}\sin\theta\cos\theta\,d\theta - \int_0^{\pi/2}\sin\theta\cos^2\theta\,d\theta\right]$

$= a\left[\left[\frac{(1)(1)}{(2)(1)}\right]\times 1 - \left[\frac{(1)(2-1)}{(3-0)(2-2)(1)}\right]\times 1\right] = a\left[\frac{1}{2}-\frac{1}{3}\right] = a/6$

(d) $\int_0^{\pi/2}\int_0^{a\cos\theta} r\,dr\,d\theta = \frac{1}{2}\int_0^{\pi/2}[r^2]_0^{a\cos\theta}\,d\theta = \frac{1}{2}\int_0^{\pi/2} a^2\cos^2\theta\,d\theta = \frac{a^2}{2}\left[\frac{(2-1)(1)}{(2-0)(1)}\right]\times\frac{\pi}{2} = \frac{a^2\pi}{8}$

15. $\iint_R r^3\,dr\,d\theta$ Where R b/w $r=2\sin\theta$ & $r=4\sin\theta$



$r = 2\sin\theta$
$r = 2\frac{y}{r}$
$r^2 = 2y$
$x^2+(y-1)^2=1$
$x^2+(y-2)^2=2^2$

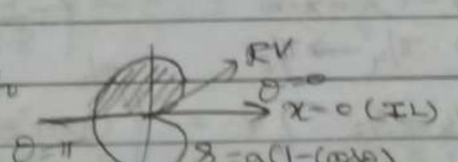
$r: 2\sin\theta \to 4\sin\theta, \quad \theta: 0 \to \pi$

$\int_0^{\pi}\int_{2\sin\theta}^{4\sin\theta} r^3\,dr\,d\theta = \int_0^{\pi}\frac{1}{4}[r^4]_{2\sin\theta}^{4\sin\theta}\,d\theta = \frac{1}{4}\int_0^{\pi}(4^4\sin^4\theta - 2^4\sin^4\theta)\,d\theta$

$= \frac{1}{4}\int_0^{\pi}\sin^4\theta\,(4^4-2^4)\,d\theta = \left[\left(\frac{4^4-2^4}{4}\right)\times 2\right]\int_0^{\pi/2}\sin^4\theta\,d\theta \quad \left[\frac{(4-1)(4-3)(1)}{(4-0)(4-2)(1)}\right]\times\pi/2$

$= \left[\left(\frac{4^4-2^4}{4}\times 2\right)\times\frac{\pi}{2}\right]\left[\frac{3\times 1}{4\times 2}\right] = \frac{\pi}{2}\times\frac{45\times 2}{2} = \frac{45\pi}{2}$

14. $\iint_R r\sin\theta\,dr\,d\theta$ R:



$\int_0^{\pi}\int_0^{a(1-\cos\theta)} r\sin\theta\,dr\,d\theta = \int\frac{1}{2}[r^2]_0^{a(1-\cos\theta)}\,d\theta$

$r: 0 \to a(1-\cos\theta)$
$\theta: 0 \to \pi$

$= \int\frac{1}{2}\left[a^2(1+\cos^2\theta-2\cos\theta)\right]d\theta = \left(\frac{a^2}{2}\right)\left[[\theta]_0^{\pi} + 2\left[\frac{2-1}{2-0}\frac{\pi}{2}\right] - 2[\sin\theta]_0^{\pi}\right]$

X $= \frac{a^2}{2}\left[(\pi-0) + \frac{\pi}{2} - 2(0-0)\right] = \frac{a^2}{2}\left[\pi+\frac{\pi}{2}\right] = \frac{a^2}{4}($

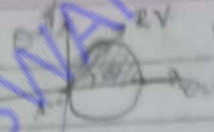
$\int_0^{\pi}\frac{\sin\theta}{2}[r^2]_0^{a(1-\cos\theta)}\,d\theta = \int_0^{\pi}\frac{\sin\theta}{2}\left[a^2(1-\cos\theta)^2-0\right]d\theta = \frac{a^2}{2}\int_0^{\pi}(\sin\theta+\sin\theta\cos^2\theta - 2\cos\theta\sin\theta)\,d\theta$

$= \frac{a^2}{2}\left[-[\cos\theta]_0^{\pi} + 2\left[\frac{(1)(2-1)}{(3-0)(2-2)}\right] - 2\times 2\left[\frac{(1)(1)}{(2-0)}\right]\right] = \frac{a^2}{2}\left[-[-1-1]+\frac{2}{3}-\frac{4}{2}\right]$

$= \frac{a^2}{2}\left[+2+\frac{2}{3}-\cancel{2}\right] = \frac{a^2}{2}[2/3] = \frac{a^2}{3}$ as $m=1$ $n=1$ $\int_0^{\pi}\sin x\cos x = 0$

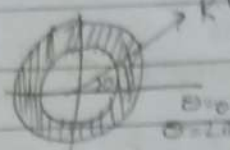
$\frac{1}{\sin^2\theta} = \csc^2\theta$

4. (a) $\int_0^{\pi/2}\int_0^{2\cos\theta} f(r,\theta)\,dr\,d\theta$



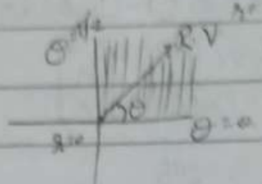
$0 \le r \le 2\cos\theta$, $x^2+y^2=r^2$, $r=2x$, $(x-1)^2+(y-0)^2=1^2$, $2x=r^2$

(b) $\int_0^{2\pi}\int_a^b f(r,\theta)\,dr\,d\theta$



$a \le r \le b$, $0 \le \theta \le 2\pi$, $r=a$, $r=b$, $x^2+y^2=r^2$

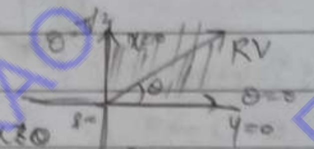
(c) $\int_0^{\pi/2}\int_0^{\infty} f(r,\theta)\,dr\,d\theta$



$0 \le r \le \infty$, $0 \le \theta \le \pi/2$ [or $\int\int dr\,d\theta$]

22. To change xy (CC) to $r\theta$ (PC) we take subs: $x=r\cos\theta$, $y=r\sin\theta$, $x^2+y^2=r^2$ and $|J| = \left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| = r$ so $\iint_R dx\,dy\, f(x,y) = \iint_R f(r,\theta)\, r\,dr\,d\theta$

28. (a) $\int_0^{\infty}\int_x^{\infty} \frac{e^{-y}}{y}\,dy\,dx$

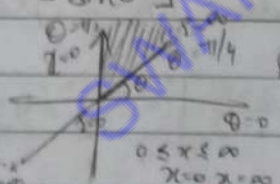


$\infty \ge y \ge 0$. So $r: 0 \to \infty$, $\theta: 0 \to \pi/2$

$\int_0^{\pi/2}\int_0^{\infty} \frac{e^{-r\sin\theta}}{r\sin\theta}\, r\,dr\,d\theta = \int_0^{\pi/2}\int_0^{\infty} \frac{e^{-r\sin\theta}}{\sin\theta}\,dr\,d\theta$

$= \int_0^{\pi/2} \frac{1}{\sin\theta}\left[\frac{e^{-r\sin\theta}}{-\sin\theta}\right] d\theta$ ✗

$\int_0^{\infty}\int_x^{\infty} \frac{e^{-y}}{y}\,dy\,dx$



$x \le y \le \infty$, $0 \le x \le \infty$

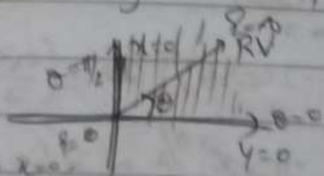
$r: 0 \to \infty$, $\theta: \pi/4 \to \pi/2$

So $\int_{\pi/4}^{\pi/2}\int_0^{\infty} \frac{e^{-r\sin\theta}}{r\sin\theta}\, r\,dr\,d\theta$, $\int_{\pi/4}^{\pi/2} -\frac{1}{\sin^2\theta}\left[e^{-r\sin\theta}\right]_0^{\infty} d\theta$

$= \int -\frac{1}{\sin^2\theta}\left[e^{-\infty} - e^0\right] = \int \frac{1}{\sin^2\theta}\,d\theta$

$\sin\theta = t$, $d\theta\cos\theta = dt$, $\theta: \pi/4 \to \pi/2$, $t: \frac{1}{\sqrt{2}} \to 1$ ✗ $= \int_{\pi/4}^{\pi/2} \csc^2\theta\,d\theta = -[\cot\theta]_{\pi/4}^{\pi/2} = -[0-1] = 1$

(b) $\int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)}\,dy\,dx$



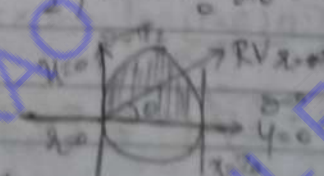
$0 \le x \le \infty$, $0 \le y \le \infty$, $r: 0 \to \infty$, $\theta: 0 \to \pi/2$

$\int_0^{\pi/2}\int_0^{\infty} e^{-r^2}\,dr\,d\theta\, r = \int_0^{\pi/2}$ So $r^2 = t$, $t: 0 \to \infty$, $2r\,dr = dt$, $dr = \frac{dt}{2r}$

$= \iint e^{-t}\,d\theta\, r\frac{dt}{2r}$

$= \frac{1}{2}\iint_0^{\infty} e^{-t}\,dt\,d\theta = \frac{1}{2}\int [e^{-t}]_0^{\infty}\,d\theta = -\frac{1}{2}\int [e^{-\infty} - 1]\,d\theta = \frac{1}{2}[\theta] = \frac{\pi}{4}$

(c) $\int_0^{2a}\int_0^{\sqrt{2ax-x^2}} x^2\,dy\,dx$



$0 \le y \le \sqrt{2ax-x^2}$, $0 \le x \le 2a$

$y=0$, $y^2+x^2=2ax$, $x=0$, $x=2a$, $y=\sqrt{2ax}$ or $r^2 = 2ar\cos\theta$

$r = 2a\cos\theta$, $r: 0 \to 2a\cos\theta$, $\theta: 0 \to \pi/2$

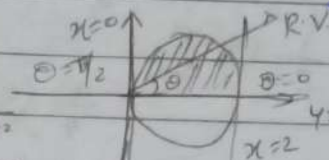
$\int_0^{\pi/2}\int_0^{2a\cos\theta} r^2\cos^2\theta\, r\,dr\,d\theta$

$= \int_0^{\pi/2}\left[\frac{\cos^2\theta\, r^4}{4}\right]_0^{2a\cos\theta} d\theta$

$= \int \frac{\cos^2\theta}{4}\left[16a^2\cos^2\theta\right]d\theta = a^2\int_0^{\pi/2}\cos^4\theta\, d\theta = a^2\left[\frac{(4-1)(4-3)}{(4-0)(4-2)}\right]\frac{\pi}{2} = \frac{3a^2\pi}{16}$ ✗

$= \int_0^{\pi/2}\frac{\cos^2\theta}{4}\left[2^4a^4\cos^4\theta\right]d\theta = \frac{2^4}{4}a^4\left[\frac{(6-1)(6-3)(6-5)}{(6-0)(6-2)(6-4)}\right]\frac{\pi}{2} = \frac{5a^4\pi}{8}$ //

(d) $\int_0^2\int_0^{\sqrt{2x-x^2}}(x^2+y^2)\,dy\,dx$



$0 \le x \le 2 \quad 0 \le y \le \sqrt{2x-x^2}$

$x=0 \;\; x=2 \;\; y=0 \;\; y^2+x^2=2x$

$(y-0)^2+(x-1)^2=1 \qquad \theta : 0 \to \pi/2$

$r^2 = 2r\cos\theta$

$r = 2\cos\theta \qquad r : 0 \to 2\cos\theta$

$\iint dx\,dy = \iint r\,dr\,d\theta$

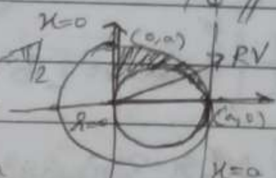
$\int_0^{\pi/2}\int_0^{2\cos\theta} r^2\, r\,dr\,d\theta = \int \frac{1}{4}\left[r^4\right]_0^{2\cos\theta}d\theta$

$= \frac{1}{4}\int\left[2^4\cos^4\theta\right]d\theta$

$= \frac{2^4}{4}\left[\frac{4-1}{4-0}\cdot\frac{4-3}{4-2}\cdot\frac{\pi}{2}\right]$

$= 3\pi/4$ //

(e) $\int_0^a\int_{\sqrt{ax-x^2}}^{\sqrt{a^2-x^2}}\frac{xy}{x^2+y^2}e^{-(x^2+y^2)}\,dy\,dx$



$\sqrt{ax-x^2} \le y \le \sqrt{a^2-x^2}$

$r : a\cos\theta \to a \qquad \theta : 0 \to \pi/2$

$y^2+x^2 = ax \qquad 0 \le x \le a \quad x=0,\ x=a$

$(y-0)^2+\left(x-\frac{a}{2}\right)^2 = \frac{a^2}{4} \qquad ar\cos\theta = r^2$

$y^2+x^2=a^2 \qquad r = a\cos\theta$

$a^2 = r^2 \quad r = a$

$\iint\frac{r^2\sin\theta\cos\theta}{r^2}e^{-r^2} = \iint \sin\theta\cos\theta\, e^{-r^2}\, r\,dr\,d\theta$

$r^2 = t \qquad r : a\cos\theta \to a \qquad t : a^2\cos^2\theta \to a^2$

$2r\,dr = dt$

$dr = \frac{dt}{2r}$

$\iint \sin\theta\cos\theta\, e^{-t}\, r\frac{dt}{2r}\,d\theta = \frac{1}{2}\iint \sin\theta\cos\theta\, e^{-t}\,dt\,d\theta$

$= \frac{1}{2}\int \sin\theta\cos\theta\left[e^{-t}\right]_{a^2\cos^2\theta}^{a^2}d\theta = -\frac{1}{2}\int sc\left[e^{-a^2}-e^{-a^2\cos^2\theta}\right]d\theta$

$u = a^2\cos^2\theta$

$du = 2a^2\cos\theta\sin\theta\, d\theta$

$\theta : 0 \to \pi/2$

$u : a^2 \to 0$

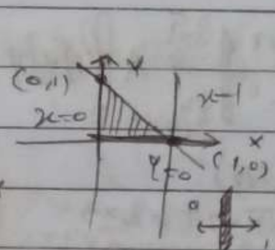
$\frac{-1}{2}\int_{a^2}^{0} \sin\theta\cos\theta\left[e^{-a^2}-e^{-u}\right]\frac{du}{\cos\theta\sin\theta\, 2a^2} = \frac{1}{4a^2}\int_{a^2}^{0}[\ ]du$

$= \frac{1}{4a^2}\left[e^{-a^2}[u]_{a^2}^{0} + \left[e^{-u}\right]_a^0\right] = \frac{1}{4a^2}\left[e^{-a^2}[0-a^2] + [1-e^{-a^2}]\right]$

$= \frac{1}{4a^2}\left[-a^2e^{-a^2}+1-e^{-a^2}\right] = \frac{1}{4a^2}\left(1-e^{-a^2}\right) - \frac{e^{-a^2}}{4}$ //

26. $x+y=u$
$y = uv$

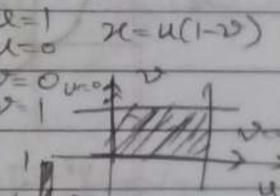
$\int_0^1\int_0^{1-x} e^{y/(y+x)}\,dy\,dx$



$0 \le y \le 1-x$
$0 \le x \le 1$

$x=0,\ x=1,\ y=0,\ x+y=1$

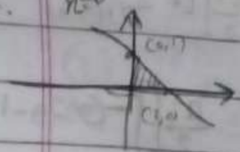
So $u=1$, $u=0$, $x = u(1-v)$
$v=0$, $v=1$



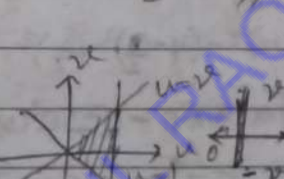
$= \int_0^1\int_0^1 e^{uv/u}\,|J|\,du\,dv \qquad |J| = \begin{vmatrix}1-v & -u\\ v & u\end{vmatrix} = u - uv + uv = u$

$= \frac{1}{2}\int_0^1 e^{v}\left[u^2\right]_0^1 = \frac{1}{2}\left[e^{v}\right]_0^1 = \frac{1}{2}(e-1)$ // Showed

27.



$x+y=u$
$x-y=v$
So $u=1$



$\int_{-v}^{v}\int_0^{1}\sin\left(\frac{v^2}{u}\right)|J|\,du\,dv \;\; -1/2$

$2y = u-v$ $\quad 2x = u+v$
$u=v$ $\quad 0 = u+v$
$v = -u$

$v : -u \to u$
$u : 0 \to 1$

$-\frac{1}{2}\int_0^1\int_{-u}^{u}\sin\left(\frac{v^2}{4}\right)du\,dv$

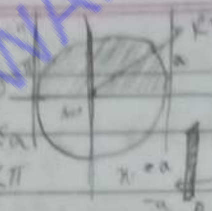
$= 0$ //

$x = \frac{u+v}{2}$
$y = \frac{u-v}{2}$

$\begin{vmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & -\frac{1}{2}\end{vmatrix} = -\frac{1}{4}-\frac{1}{4} = -\frac{1}{2}$

$= +\frac{1}{2}\int_0^1 u\left[\cos\left(\frac{v^2}{4}\right)\right]_{-u}^{u}du$

29. $\int\int r^3 \sin\theta\cos\theta\, dr\, d\theta$; $r=0, r=a$; $0\le r\le a$; $\theta=0, \theta=\pi$; $0\le\theta\le\pi$



$\int\int x y\, dx\, dy = \int \frac{x}{2}[y^2]_0^{\sqrt{a^2-x^2}} dx$

$\iint_R dx\,dy = \iint_{R'} r\,dr\,d\theta$ $= \int \frac{x}{2}[a^2-x^2-0]dx$ $= \int\left(\frac{a^2x}{2}-\frac{x^3}{2}\right)dx$

$= \frac{a^2}{2\cdot 2}[x^2]_{-a}^{a} - \frac{1}{2\cdot 4}[x^4]_{-a}^{a}$ $= \frac{a^2}{4}(a^2-a^2) - \frac{1}{8}(a^4-a^4) = 0$

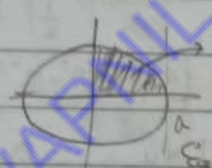
30. $\iint_R \frac{x^2y^2}{x^2+y^2}\,dx\,dy$

$\int_0^{2\pi}\int_a^b \frac{r^2}{r^2}\sin\theta\cos\theta\, r\,dr\,d\theta$ $= \frac{1}{2}\int_0^{2\pi}\sin\theta\cos\theta\,[r^2]_a^b\,d\theta$

$= \frac{1}{2}\int_0^{2\pi}\sin\theta\cos\theta\,[b^2-a^2]d\theta = \frac{b^2-a^2}{2}\times 4\times\left[\frac{(1)(1)}{(2-0)}\right]$ X

So. $\iint \frac{r^4\cdot r}{r^2}\sin^2\theta\cos^2\theta\, dr\,d\theta = \frac{1}{4}\int_0^{2\pi}\sin^2\theta\cos^2\theta\,[r^4]_a^b\,d\theta$

$= \frac{b^4-a^4}{4}\times 4\left[\frac{(2-1)(2-1)}{(4-0)(4-2)}\times\frac{\pi}{2}\right] = \frac{(b^4-a^4)\pi}{16}$

35. 

$4\times$ this area = area of ellipse

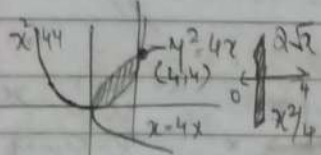
Eqn: $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$A = \iint_R dx\,dy$ so $= 4\times\int_0^a\int_0^{\frac{b}{a}\sqrt{a^2-x^2}} dx\,dy$

$y = \sqrt{\left(1-\frac{x^2}{a^2}\right)b^2} = \frac{b}{a}\sqrt{a^2-x^2}$

$= 4\times\int[y]_0^{(\sqrt{a^2-x^2})b/a}dx = 4\times\int\frac{b}{a}(\sqrt{a^2-x^2}-0)dx$

$= \frac{4\times b}{a}\left[\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]_0^a = \frac{4b}{a}\left[0+\frac{a^2}{2}\left(\frac{\pi}{2}\right)-0-0\right] = \boxed{\pi ab}$

36. 

$A = \iint_R dx\,dy = \int_0^4\int_{x^2/4}^{2\sqrt{x}} dx\,dy = \int_0^4[y]_{\frac{x^2}{4}}^{2\sqrt{x}}dx = \int_0^4\left(2\sqrt{x}-\frac{x^2}{4}\right)dx$

$y^2 = 4(2\sqrt{y})$

$y^2-8\sqrt{y}=0$; $y^2=8\sqrt{y}$; $y^4=64y$

$y\left(y-\frac{8}{y}\right)=0$ or $y^4+64y-2$

$= \frac{2\times 2}{3}[x^{3/2}]_0^4 - \frac{1}{4\cdot 3}[x^3]_0^4 = \frac{4\times 4\sqrt{4}}{3} - \frac{1}{12}$

$= \frac{8}{3}-\frac{64}{12} = \frac{16}{3}$

37. 

$r = \frac{ay}{r}$; $r^2 = ay$; $x^2+\left(y-\frac{a}{2}\right)^2 = \left(\frac{a}{2}\right)^2$; $r = a(1+\cos\theta)$

$\int_{\pi/2}^{\pi}\int_{a(1+\cos\theta)}^{a\sin\theta} dr\,d\theta\cdot r = \frac{1}{2}\int[r^2]_{a(1+\cos\theta)}^{a\sin\theta}d\theta$

$= \frac{1}{2}\int[a^2\sin^2\theta - a^2 - a^2\cos^2\theta - 2a^2\cos\theta]d\theta = \frac{a^2}{2}\int[-\cos2\theta-1-2\cos\theta]$

$= -\frac{a^2}{2}\left[\left[\frac{1}{2}\sin2\theta\right]_{\pi/2}^{\pi} + [\theta]_{\pi/2}^{\pi} + 2[\sin\theta]_{\pi/2}^{\pi}\right] = -\frac{a^2}{2}\left[\frac{1}{2}[0] + \pi - \frac{\pi}{2} + 2(0-1)\right]$

$= -\frac{a^2}{2}\left[\frac{\pi}{2}-2\right] = \frac{a^2}{2}\left[\frac{4-\pi}{2}\right] = \frac{a^2}{4}(4-\pi)$

$\int\sqrt{x^2+a^2} = \frac{x}{2}\sqrt{x^2+a^2}+\frac{a^2}{2}\sinh^{-1}\left(\frac{x}{a}\right)$

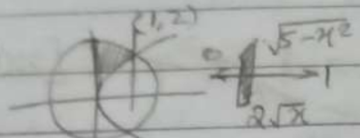
38. 

$r = a \to a(1+\cos\theta)$, $\theta: \frac{\pi}{2} \to -\frac{\pi}{2}$

$\int_{-\pi/2}^{\pi/2}\int_a^{a(1+\cos\theta)} r\,dr\,d\theta = \frac{1}{2}\int [r^2]_a^{a(1+\cos\theta)} d\theta = \frac{a^2}{2}\int (1+\cos^2\theta+2\cos\theta-1)\,d\theta$

$= \frac{a^2}{2}\left[\frac{1}{2}\left(\left[\frac{\sin 2\theta}{2}\right]_{-\pi/2}^{\pi/2}+[\theta]_{-\pi/2}^{\pi/2}\right)+2[\sin\theta]_{-\pi/2}^{\pi/2}\right]$

$= \frac{a^2}{2}\left[\frac{1}{2}\left(\underbrace{\frac{\pi}{2}+\frac{\pi}{2}}_{\pi}\right)+2(1+1)\right] = \frac{a^2}{2}\left[\frac{2\pi}{4}+4\right] = \frac{a^2}{4}\left[\frac{\pi+8}{1}\right]$

41. (i) 

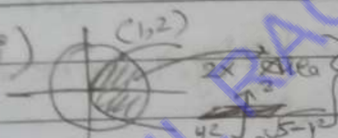
$x^2+4x-5=0$, $x=1$, $x'=5$, so $y=2$

$\int_0^1\int_{2\sqrt{x}}^{\sqrt{5-x^2}} dx\,dy = \int_0^1 [y]_{2\sqrt{x}}^{\sqrt{5-x^2}}dx = \int_0^1(\sqrt{5-x^2}-2\sqrt{x})\,dx$

$= \left[\frac{x}{2}\sqrt{5-x^2}+\frac{5}{2}\sin^{-1}\left(\frac{x}{\sqrt5}\right)\right]_0^1 - \frac{2\cdot2}{3}[x^{3/2}]_0^1$

$= \frac{1}{2}\cdot2+\frac{5}{2}\sin^{-1}\left(\frac{1}{\sqrt5}\right) - \cancel{0} - \cancel{0} - \frac{4}{3}(1-0) = 1+\frac{5}{2}\sin^{-1}\left(\frac{1}{\sqrt5}\right)-\frac{4}{3}$

$= 27.1757$ (or 0.8291) $= 0.825$

(ii) 

$2\times$ area $= 2\times\int_0^2\int_{y^2/4}^{\sqrt{5-y^2}}dx\,dy = 2\left[\int_0^2[x]_{y^2/4}^{\sqrt{5-y^2}}dy\right] = 2\left[\int\left(\sqrt{5-y^2}-\frac{y^2}{4}\right)dy\right]$

$A = \iint dx\,dy$

$= 2\left[\left[\frac{y}{2}\sqrt{5-y^2}+\frac{5}{2}\sin^{-1}\left(\frac{y}{\sqrt5}\right)\right]_0^2 - \frac{1}{4\cdot3}[y^3]_0^2\right]$

$= 2\left[\left[1\sqrt{1}+\frac{5}{2}\sin^{-1}\left(\frac{2}{\sqrt5}\right)-0-0\right]-\left[\frac{1}{12}\cdot2^3\right]\right] = 2\left(1+\frac{5}{2}\sin^{-1}\left(\frac{2}{\sqrt5}\right)-\frac{2}{3}\right)$

$= 2+5\sin^{-1}\left(\frac{2}{\sqrt5}\right)-\frac{4}{3} = 6.202$

8. (a) $\int_0^1\int_0^2\int_1^2 xyz^2\,dx\,dy\,dz = \iint \frac{yz^2}{2}[x^2]_1^2\,dy\,dz = \frac{3}{2}\iint yz^2\,dy\,dz = \frac{3}{2}\int\frac{z^2}{2}[y^2]_0^2\,dz$

$= \frac{3}{2}\int\frac{z^2}{2}\left[\frac{4}{2}\right] = \frac{3}{3}[z^3]_0^1 = \frac{3}{3}(1-0) = 1$

(b) $\int_0^a\int_0^a\int_0^a(yz+zx+xy)\,dx\,dy\,dz = \iint\left(yz[x]_0^a+\frac{z}{2}[x^2]_0^a+\frac{y}{2}[x^2]_0^a\right)dy\,dz$

$= \iint\left(\frac{a}{2}z[y^2]_0^a+\frac{za^2}{2}[y]_0^a+\frac{a^2}{4}[y^2]_0^a\right)dz = \frac{a^3}{4}[z^2]_0^a+\frac{a^3}{4}[z^2]_0^a+\frac{a^4}{4}[z]_0^a$

$= \frac{a^5}{4}+\frac{a^5}{4}+\frac{a^5}{4} = 3a^5/4$

(c) $\int_0^1\int_0^1\int_0^y xyz\,dx\,dy\,dz$ ... $\iint\frac{yz}{2}[x^2]_0^y\,dy\,dz$

$= \frac{1}{2}\iint y^3z\,dy\,dz = \frac{1}{2}\int\frac{y^3}{2}[z^2]_0^1\,dy = \frac{1}{4\cdot4}[y^4]_0^1 = \frac{1}{16}$

Usually $\int uv = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \dots$

But on failure, $\int uv = uv_1 - \int u'v_1$

16. (a) $\iiint (x^2+y^2+z^2)\,dx\,dy\,dz = \iint \frac{1}{3}[x^3]_{-a}^{a} + \iint y^2[x]_{-a}^{a} + \iint z^2[x]_{-a}^{a}$

$= \iint \frac{2}{3}a^3 + \iint y^2 \cdot 2a + \iint z^2 2a = \int \frac{2a^3}{3}[y]_{-b}^{b} + \int \frac{2a}{3}[y^3]_{-b}^{b} + \int z^2 2a[y]$

$= \int \frac{4a^3b}{3} + \int \frac{4ab^3}{3} + \int 4az^2b = \frac{4a^3b}{3}[z]_{-c}^{c} + \frac{4ab^3}{3}[z]_{-c}^{c} + \frac{4}{3}ab[z^3]_{-c}^{c}$

$= \frac{8a^3bc}{3} + \frac{8ab^3c}{3} + \frac{8abc^3}{3} = \frac{8}{3}(a^3bc + ab^3c + abc^3)$

ILATE $\frac{1}{1}.\log z$ (v, u)

(b) $\int_1^e \int_1^{\log y} \int_1^{e^x} \log z\,dz\,dx\,dy \quad \iint [\cancel{\log z \cdot z - \int \tfrac{1}{z} z}] \to [z(\log z - 1)]_{e^0=1}^{e^x} dx\,dy$

$= \iint [e^x(x-1) - 1(0-1)]\,dx\,dy = \iint_1^{\log y} [xe^x - e^x + 1]\,dx\,dy$ w.r.t x as z ✓ so,

$\left\{ \int [xe^x - e^x + e^x + x] \right\}^x = \int \left([xe^x - e^x]_1^{\ln y} - [e^x]_1^{\ln y} + [x]_1^{\ln y} \right) dy$ so

ILATE $y \log_e y$ (v, u)

$\int \left(\log_e y (e^{\log_e y}) - e^{\log_e y} - \cancel{e^1} + \cancel{e} - e^{\log_e y} + e^1 + \log_e y - 1 \right) dy$

$\int uv = uv_1 - \int u'v_1$

$= \int (y \log_e y - y - y + e + \log_e y - 1)\,dy = \int_{e^0=1}^{e} (y\log_e y - 2y + e + \log_e y - 1)\,dy$

$= \left[\frac{\ln y \cdot y^2}{2} - \int \frac{1}{y}\frac{y^2}{2} \to \frac{y^2}{4} \right]_1^e - \frac{2}{2}[y^2]_1^e + e[y]_1^e + [y(\ln y - 1)]_1^e - [y]_1^e$

$= \frac{\log_e e \cdot e^2}{2} - \frac{e^2}{4} - 0 + \frac{1}{4} - e^2 + 1 + e^2 - e + e(\log_e e - 1) - 1(\ln 1 - 1) - e + 1$

$= \frac{e^2}{2} - \frac{e^2}{4} + \frac{1}{4} - \cancel{e^2} + 1 + \cancel{e^2} - e + 1 - e + 1 = -\frac{e^2}{4} + \frac{e^2}{2} + 3 + \frac{1}{4} - 2e = \frac{13}{4} + \frac{e^2}{4} - \frac{8e}{8}$

$= \frac{e^2 + 13 - 8e}{4} = \frac{e(e-8)+13}{4}$ // (3)

(c) $\iiint e^{x+y+z}\,dz\,dy\,dx = \iiint e^{x+y} \cdot e^z\,dz\,dy\,dx = \iint e^{x+y}[e^z]_0^{x+y}\,dy\,dx = \iint e^{x+y}(e^{x+y} - 1)\,dy\,dx$

$= \iint (e^{2(x+y)} - e^{x+y})\,dy\,dx = \iint e^{2x} \cdot e^{2y} - \iint e^x \cdot e^y = \frac{1}{2}\int e^{2x}[e^{2y}]_0^x dx$

$= \frac{1}{2}\int e^{2x}(e^{2x} - 1)\,dx - \int e^{2x} - e^x\,dx = \frac{1}{2}\int e^{4x} - e^{2x}\,dx \quad \left\{ - \int e^x [e^y]_0^x dx \right.$

$= \frac{1}{2}\left[\frac{1}{4}[e^{4x}]_0^a - \frac{1}{2}[e^{2x}]_0^a - \frac{1}{2}[e^{2x}]_0^a - [e^x]_0^a \right] = \frac{1}{2}\left[\frac{1}{4}(e^{4a} - 1) - \frac{1}{2}(e^{2a} - 1) - \frac{1}{2}[e^{2a} - 1) - e^a + 1 \right]$

$= \frac{e^{4a}}{8} - \frac{e^{2a}}{4} - \frac{e^{2a}}{2} - e^a + \left(\frac{1}{4} + \frac{1}{2} + \frac{1}{2} + 1 \right)\frac{1}{2} = \frac{e^{4a}}{8} - \frac{e^{2a} \cdot 3}{4} + e^a \cdot \frac{2}{8}$ //

(d) $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} xyz\,dx\,dy\,dz \quad \iint \frac{xy}{2}[z^2]_0^{\sqrt{1-x^2-y^2}}\,dy\,dz = \iint \frac{xy}{2}[1 - x^2 - y^2]\,dy\,dz = \frac{1}{2}\iint xy - x^3y - xy^3$

$= \frac{1}{2}\int \left(\frac{x}{2}[y^2]_0^{\sqrt{1-x^2}} - \frac{x^3}{2}[y^2] - \frac{x}{4}[y^4] \right) dx = \frac{1}{2}\int \left(\frac{1}{2}(x - x^3) - \frac{1}{2}(x^3 - x^5) - \frac{1}{4}(x + x^5 - 2x^3) \right) dx$

$= \frac{1}{4}\left[\int_0^1 x - x^3 - (x^3 - x^5) - \frac{1}{2}(x + x^5 - 2x^3)\,dx \right] = \frac{1}{48}$ // (calc)

→ Cont after 2 pages

17. (Important & Ques) $\int_0^a\int_0^{\sqrt{a^2-x^2}}\int_0^{\sqrt{a^2-x^2-y^2}} xyz\,dxdydz = \iint \frac{xy}{2}[z^2]_0^{\sqrt{a^2-x^2-y^2}}\,dydz$

$= \iint \frac{xy}{2}[a^2-x^2-y^2]\,dydz = \iint\left(\frac{xya^2}{2} - \frac{x^3y}{2} - \frac{xy^3}{2}\right)dydz = \int \frac{x}{4}a^2[y^2]_0^{\sqrt{a^2-x^2}}$

$- \frac{x^3}{4}[y^2]_0^{\sqrt{a^2-x^2}} - \frac{x}{8}[y^4]_0^{\sqrt{a^2-x^2}})\,dz = \int \frac{x}{4} \}$ x & $\int x[z]_0^{\sqrt{a^2-x^2-y^2}}\,dydx =$

$\iint x\left(\sqrt{a^2-x^2-y^2}\right) = \int x\left[\frac{y}{2}\sqrt{a^2-x^2-y^2} + \frac{a^2-x^2}{2}\sin^{-1}\left(\frac{y}{\sqrt{a^2-x^2}}\right)\right]_0^{\sqrt{a^2-x^2}} dx =$

$(\sqrt{a^2-x^2})^2-(y)^2$

a^2-x^2

$\int x\left[0 + \frac{a^2-x^2}{2}\sin^{-1}(1) - 0 - 0\right]dx = \frac{\pi}{4}\int(a^2x - x^3)\,dx = \frac{\pi}{4}\left[\frac{a^2}{2}[x^2]_0^a - \frac{1}{4}[x^4]_0^a\right]$

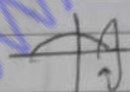
$= \frac{\pi}{4}\left[\frac{a^4}{2} - \frac{a^4}{4}\right] = \frac{\pi a^4}{16}$ //

24. ☆ for xyz (CC) to $r\theta\phi$ (SPC), we use foll subs: $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$, $|J| = r^2\sin\theta$, $\iiint dxdydz = \iiint$ ...

23. ☆ for xyz (CC) to $\rho\phi z$ (CPC), we use foll subs: $x = \rho\cos\phi$, $y = \rho\sin\phi$, $z = z$, $|J| = \rho$, & $x^2+y^2 = \rho^2$, $\iiint dxdydz = \iiint \rho\,d\rho\,d\phi\,dz$

31. $\iiint \frac{dxdydz}{x^2+y^2+z^2}$ where $x^2+y^2+z^2=a^2$. Subs: ~~$x = r\sin\theta\sin\phi$~~ ~~$y = r\theta$~~ $y = r\sin\theta\sin\phi$, $x = r\sin\theta\cos\phi$, $z = r\cos\theta$, $dxdydz = |J|\,drd\theta d\phi$, $|J| = r^2\sin\theta$

$= \iiint \frac{r^2\sin\theta\,dr\,d\theta\,d\phi}{r^2}$



$r: 0\to a$, $\theta: 0\to\pi$, $\phi: 0\to 2\pi$ $\}$ $\iiint \sin\theta\,dr\,d\theta\,d\phi = \iint \sin\theta[r]_0^a\,d\theta\,d\phi$

$= a\int[\cos\theta]_0^\pi\,d\phi = 2a\int d\phi = 4\pi a$

32. $\int_0^\infty\int_0^\infty\int_0^\infty \frac{dxdydz}{(1+x^2+y^2+z^2)^2}$ & $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$, $|J| = r^2\sin\theta$, $dxdydz = d\theta d\phi$ ..., for ~~new~~ lim: $\theta, \phi: 0\to\frac{\pi}{2}$, $t: 0\to\pi/2$; So $\iiint \frac{r^2\,dr\,d\theta\,d\phi\,\sin\theta}{(1+r^2)^2}$ $(1+\tan^2 t)^2$

So $r = \tan t$, $dr = \sec^2 t\,dt$; $\frac{\tan^2 t\,\sec^2 t\,dt\,d\theta\,d\phi\,\sin\theta}{\sec^4 t} = \frac{\sin^2 t}{\cos^2 t}\times\cos^2 t\,\sin\theta$ So

$\iint \sin\theta\,d\theta\,d\phi\left[\frac{2-1(1)}{2-0(1)}\frac{\pi}{2}\right] = -\frac{\pi}{4}\int[\cos\theta]_0^{\pi/2}\,d\phi = \frac{\pi}{4}[\phi]_0^{\pi/2} = \frac{\pi^2}{8}$ //

33. $\iiint(x^2+y^2+z^2)\,dxdydz$ So $y = r\sin\theta\sin\phi$, $x = r\sin\theta\cos\phi$, $z = r\cos\theta$, $|J| = r^2\sin\theta$; $\iiint r^4\sin\theta\,dr\,d\theta\,d\phi$

$dxdydz = r^2\sin\theta\,dr\,d\theta\,d\phi$; $= \iint \frac{\sin\theta}{5}[r^5]_0^1$

$r: 0\to 1$; $\theta: 0\to\pi$, $\phi: 0\to 2\pi$; $= \frac{4\pi}{5}$ //; $= -\frac{1}{5}\int[\cos\theta]_0^\pi\,d\phi = \frac{2}{5}[\phi]_0^{2\pi}$

39.  So $\iiint dxdydz$ or $\iiint r^2\sin\theta\,dr\,d\theta\,d\phi$; $r: 0\to a$, $\theta: 0\to\pi$, $\phi: 0\to 2\pi$ $\}$ $\iiint r^2\sin\theta\,dr\,d\theta\,d\phi = -\frac{a^3}{3}\int[\cos\theta]_0^\pi$

$= \frac{2a^3}{3}[\phi]_0^{2\pi} = \frac{4a^3\pi}{3} = \frac{4}{3}\pi a^3$ //

40. u … $x = au$, $y = bv$, $z = cw$, $u^2+v^2+w^2=1$ sphere

$\left\{ \begin{array}{l} x = au \\ y = bv \\ z = cw \end{array} \right.$ $u^2+v^2+w^2=1$ $\quad |J| = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc$

$dx\,dy\,dz = abc\,du\,dv\,dw$

$\iiint_V abc\,du\,dv\,dw = abc \frac{4\pi r^3}{3} = \frac{4}{3}\pi abc$

34. $\iiint (x^2+y^2+z^2)\,dx\,dy\,dz$ $\left\{ \begin{array}{l} x = \rho\cos\phi \\ y = \rho\sin\phi \\ z = z \end{array} \right.$ $dx\,dy\,dz = \rho\,d\rho\,d\phi\,dz$ $\quad |J| = \rho$ $\quad \rho: 0 \to 1$ so $z: 0 \to 1$, $\phi: 0 \to 2\pi$ $\quad \int_0^1\int_0^{2\pi}\int_0^1 (\rho^2+z^2)\,\rho\,d\rho\,d\phi\,dz$

$= \frac{1}{2}\iiint [\rho^2]_0^1 = \frac{1}{2}\int [\phi]_0^{2\pi} = \pi [z]_0^1 = \pi$, $\frac{z^2}{2}[\rho^2]_0^1$ $z^2[\phi]$ $= \frac{1}{4}[\rho^4]_0^1 = \frac{1}{4}[\phi]_0^{2\pi} = \frac{\pi}{2}[z]_0^1 = \frac{\pi}{2}$ (ii)

$\frac{z^3}{2}[\rho^2]_0^1 = \frac{z^2}{2}[\phi]_0^{2\pi} = z^2\pi$ so $\frac{\pi}{3}[z^3]_0^1 = \pi/3$ $\quad = \frac{\pi}{3}\text{(i)} + \frac{\pi}{2}\text{(ii)} = \frac{5\pi}{6}$

42. Beta fn denoted by $\beta(m,n)$ is defined by the DI $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$, $m,n>0$ aka Eulerian $\int^{al}$ of 1^{st} kind. Trig $2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\,d\theta$

$2m-1=p$, $2n-1=q$ so $\frac{\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)}{2} = \int_0^{\pi/2} \sin^p\theta \cos^q\theta\,d\theta$

Prop: (i) $\beta(m,n) = \beta(n,m)$ (ii) $\beta(m,n) = \frac{\Gamma m\,\Gamma n}{\Gamma(m+n)}$

43. Rel^n $\beta(m,n) = \frac{\Gamma m\,\Gamma n}{\Gamma(m+n)}$

44. Gamma fn denoted by Γn is defined by the improp $\int^{al}$: $\Gamma n = \int_0^\infty e^{-x} x^{n-1}dx$, $n>0$ aka Eulerian $\int^{al}$ of 2^{nd} kind. Other forms. $\Gamma n = 2\int_0^\infty e^{-x^2} x^{2n-1}dx$; $a^n\int_0^\infty e^{-ax}x^{n-1}dx$

$\frac{1}{n}\int e^{-x^{1/n}}dx$. Prop: (i) $\Gamma(n+1) = n\Gamma n$ (ii) $\Gamma 1 = 1$ (iii) $\Gamma\frac{1}{2} = \sqrt{\pi}$

(iv) $\Gamma n\,\Gamma(1-n) = \frac{\pi}{\sin n\pi}$ (v) if $n<0$, $\frac{\Gamma(n+1)}{n} = \Gamma n$

&, $\Gamma(-3.5)$ by (v), $n<0$ $\frac{\Gamma(n+1)}{n} = \Gamma(-3.5) = \frac{\Gamma(-2.5)}{-3.5} = \frac{\Gamma(-1.5)}{(3.5)(2.5)} = \frac{\Gamma(-0.5)}{(-3.5)(-2.5)(-1.5)} = \frac{\sqrt{\pi}}{(3.5)(2.5)(1.5)}$

45. (a) $\Gamma(-1/2)$ as $n<0$, $\frac{\Gamma(n+1)}{n} = \Gamma n$ so $\frac{\Gamma 1/2}{-1/2} = -2\sqrt{\pi}$

(b) $\beta(1/3, 2/3)$ so $\frac{\Gamma\frac{1}{3}\,\Gamma\frac{2}{3}}{\Gamma 1} = \Gamma\frac{2}{3}\,\Gamma\left(1-\frac{2}{3}\right) = \frac{\pi}{\sin\frac{2\pi}{3}} = \frac{2\pi}{\sqrt{3}}$ $\quad (= \Gamma n\,\Gamma(1-n))$

(c) $\Gamma\frac{9}{2} = \Gamma\left(1+\frac{7}{2}\right) = \frac{7}{2}\Gamma\frac{7}{2} = \frac{7}{2}\Gamma\left(1+\frac{5}{2}\right) = \frac{7}{2}\cdot\frac{5}{2}\Gamma\frac{5}{2} = (\;)\Gamma\left(1+\frac{3}{2}\right) = (\;)(\;)\left(\frac{3}{2}\right)\Gamma\left(1+\frac{1}{2}\right)$

$= (\;)(\;)(\;)\left(\frac{1}{2}\right)\sqrt{\pi} = \frac{105}{16}\sqrt{\pi}$

(46) $\int_0^1 x^m \left[\log\left(\frac{1}{x}\right)\right]^n dx = \frac{\Gamma(n+1)}{(m+1)^{n+1}}$ P.T.: $\log\left(\frac{1}{x}\right) = y \Rightarrow e^{-y} = x$ $x: 0 \to 1$, $y: \infty \to 0$, $dx = -e^{-y}dy$ (a lim)

$= \int_\infty^0 e^{-my} \cdot y^n (-e^{-y})\, dy = -\int_\infty^0 e^{-y(m+1)} y^n dy = \int_0^\infty e^{-y(m+1)} y^n dy = \int_0^\infty e^{-\alpha x} x^{n-1} dx = \frac{\Gamma n}{\alpha^n}$

$= \int_0^\infty e^{-y\underbrace{(m+1)}_{\alpha}} y^{\underbrace{(n+1)}_{n}-1}$

So, $\frac{\Gamma n}{\alpha^n} = \frac{\Gamma(n+1)}{(m+1)^{n+1}}$ Proved

47. (a) $\int_0^\infty x^3 e^{-x^2} dx \equiv \int_0^\infty e^{-x} x^{n-1} dx$ or $\equiv 2\int_0^\infty e^{-x^2} x^{2n-1} dx$ $2n-1=3$, $n=2$

So, $\frac{\Gamma n}{2} = \frac{\Gamma 2}{2} = \frac{1}{2}$ as $\left(\frac{\Gamma(1+1)}{2} = \frac{1\,\Gamma 1}{2} = \frac{1 \cdot 1}{2} = \frac{1}{2}\right)$

(b) $\int_0^\infty e^{-ax^{1/a}} dx$: $+ax^{1/a} = t$, $\left(\frac{t}{a}\right)^a = x$ so $dx = \left(\frac{a}{a^a}\right) t^{a-1} dt$ $x: 0 \to \infty$, $t: 0 \to \infty$ lim too $\int_0^\infty e^{-t} \cdot \left(\frac{a}{a^a}\right) t^{a-1} dt$

$\equiv \int_0^\infty e^{-x} x^{n-1} dx$

So $n = a$ $\frac{a}{a^a}\Gamma a = \frac{\Gamma(a+1)}{a^a}$

48. S.T. $\underbrace{\int_0^{\pi/2} \sqrt{\sin\theta}\, d\theta}_{(i)} \times \underbrace{\int_0^{\pi/2} \frac{1}{\sqrt{\sin\theta}} d\theta}_{(ii)} = \pi$ (i): $\int_0^{\pi/2} \sin^{1/2}\theta \cos^0\theta\, d\theta = \int_0^{\pi/2} \sin^p\theta \cos^q\theta = \frac{\beta(\,)}{2}$

$p = \frac{1}{2}$, $q = 0$

$\frac{\frac{1}{2}+1}{2} = \frac{3}{4}$, $\frac{0+1}{2} = \frac{1}{2}$ $= \frac{1}{2}\beta\left(\frac{3}{4}, \frac{1}{2}\right)$

(ii) $\int_0^{\pi/2} \sin^{-1/2}\theta \cos^0\theta\, d\theta$ $p = -1/2$ $q = 0$

So. $\frac{\beta\left(\frac{1}{4}, \frac{1}{2}\right)}{2}$ $\frac{p+1}{2}$ $\frac{-\frac{1}{2}+1}{2} = \frac{1}{4}$ $\frac{0+1}{2} = \frac{1}{2}$ $= \frac{1}{4} \frac{\cancel{\Gamma 3/4}\, \Gamma 1/2}{\Gamma 5/4} \frac{\Gamma 1/4\, \Gamma 1/2}{\cancel{\Gamma 3/4}}$

$= \frac{\pi}{4} \frac{\Gamma 1/4}{\Gamma 1+1/4} = \frac{\Gamma 1/4}{1/4\, \Gamma 1/4} = \frac{\pi}{4} \times 4 = \pi$ showed

(49) S.T. $\int_0^\infty \frac{e^{-x^2}}{\sqrt{x}} dx \times \int_0^\infty e^{-x^4} x^2 dx = \frac{\pi}{4\sqrt{2}}$: $\underbrace{\int_0^\infty e^{-x^2} x^{-1/2} dx}_{(i)} \times \underbrace{\int_0^\infty e^{-(x^2)^2} x^2 dx}_{(ii)}$

$\int_0^\infty e^{-t} t^{n-1} dx$ (i): $\equiv 2\int_0^\infty e^{-x^2} x^{2n-1} dx$ (ii): $x^2 = t$, $2x\,dx = dt$, $x: 0 \to \infty$, $t: 0 \to \infty$ $\int_0^\infty e^{-t^2} \cdot t \frac{dt}{2\sqrt{t}} = \frac{1}{2}\int_0^\infty e^{-t^2} t^{1/2} dt$

$2n-1 = -1/2$, $n = \frac{1}{4}$

So $\frac{\Gamma 1/4}{2}$ (i) $\equiv 2\int_0^\infty e^{-x^2} x^{2n-1} dx$, $2n-1 = \frac{1}{2}$, $n = 3/4$

$= \frac{1}{8} \Gamma 1/4\, \Gamma 1-1/4$ So $\frac{1}{4}\Gamma 3/4$ (ii)

$= \frac{1}{8} \frac{\pi}{\sin\frac{\pi}{4}} = \frac{1}{8} \frac{2\pi}{\sqrt{2}} = \frac{\pi}{4\sqrt{}} = \frac{\pi}{4\sqrt{2}}$ showed

UNIT 4

Unit IV

1. A ⊙ A scc is a curve that starts/ends at same point w/o cutting/intersecting itself. Ex: ○

2. Any $\int$ (al) to be eval along a curve i.e. $\int$ - If $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$ PV and $\vec{F}(x,y,z) = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ is VPF, s curves, s arc length given. $\frac{d\vec{r}}{ds}$ is unit tangent vector, comp along C, $F\left(\frac{d\vec{r}}{ds}\right)$, $\int F\cdot\left(\frac{d\vec{r}}{ds}\right)ds$
$= \int \vec{F}\cdot d\vec{r} = \int_C \vec{F}\cdot d\vec{r}$ is the line $\int$ along C but ...

3. If F, (VPF) acts on particle along C & its PV $= \vec{r}$ then UTV $\frac{d\vec{r}}{ds}$ so $\vec{F}$ compo along C $\vec{F}\cdot\frac{d\vec{r}}{ds}$ & small disp. WD $\vec{F}\cdot\frac{d\vec{r}}{ds}ds = \int_C \vec{F}\cdot d\vec{r}$ (WD)

4. $\vec{F} = xy\hat{i} + yz\hat{j} + zx\hat{k}$, $\int \vec{F}\cdot d\vec{r} = ?$ so $\int xy\,dx + yz\,dy + zx\,dz$
$\vec{r} \to d\vec{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}$ single var as x, y, z lin x, subs $x = t$, $dx = dt$
$\int_{-1}^{1} t^3 dt + 2t^6 dt + 3t^6 dt = \left[\frac{t^4}{4}\right]_{-1}^{1} + \frac{2}{7}[t^7]_{-1}^{1} + \frac{3}{7}[t^7]_{-1}^{1} = 0 + \frac{10}{7} = \frac{10}{7}$

9. $\oint_C \vec{F}\cdot d\vec{r} = \oint y\,dx + z\,dy + x\,dz$ $\oint_0^{2\pi} r\sin\theta$ and $x^2 + y^2 = 1 \to r = 1$
(θ: $0 \to 2\pi$) $y = r\sin\theta = r\cos\theta\,(dy)$
$x = r\cos\theta = -r\sin\theta\,(dx)$ $= \oint_0^{2\pi} -1^2\sin^2\theta + 0 + 0 = -\frac{1}{4}\left[\frac{2\cdot\pi}{2}\right]$
$z = 0,\ dz = 0\,d\theta$ $= -\pi$

11. WD $\int_C F\cdot d\vec{r} = \int_1^2 (3t^2+3)(2t^2)(2t)\,dt - 20t^4 + 30t^2 + 30)t^2$
$\int 3xy\,dx - 5z\,dy + 10x\,dz$ $x = t^2+1$, $y = 2t^2$, $z = t^3$ $\int_1^2 (3t^2+3)(2t^2)(2t)$ $= (6t^4 + 6t^2)2t$
$= \int_1^2 (12t^5 + 12t^3 - 20t^4 + 30t^4 + 30t^2)\,dt = \int_1^2 12t^5 + 10t^4 + 12t^3 + 30t^2\,dt$ $(12t^5 + 12t^3)$
$= 303$ (Calc)

8. $\int_C F\cdot d\vec{r} = \int_C \cos y\,dx - x\sin y\,dy = \int d(x\cos y)\,d\vec{r} = x\cos y - [x\cos y]_{(1,0)}^{(0,1)}$
$= 0\cos(1) - 1\cos(1)$ $\cos 1$ so $[x\cos y]$
$= 0\cos 1 - 1\cos 0 = -1$

13. $\int F\cdot d\vec{r}$ $\left(\int (3x^2+6y)\,dx - 14yz\,dy + 20xz^2\,dz \to \int (3t^2+6t^2)dt - 28t^6 dt\right.$
$(0, 0, 0 \to 1, 1, 1$ by subs$)$ $= \int 9t^2 - 28t^6 + 60t^9 dt$

$x=t$, $y=t^2$, $z=t^3$, $t: 0 \to 1$ at $t: 0 \to 1$, $x=t$

$\int_C \vec{F}\cdot d\vec{r} = \int_C F_1dx + F_2dy + F_3dz = \int_0^1 (3x^2+6y)dx - 14yz\,dy + 20xz^2\,dz$

$= \int_0^1 (3x^2+6x^2)dx - 28x^6dx + 60x^9dx = \int_0^1 9x^2 - 28x^6 + 60x^9$

$= \frac{9}{3}[x^3]_0^1 - \frac{28}{7}[x^7]_0^1 + \frac{60}{10}[x^{10}]_0^1 = 3 - 4 + 6 = 5$

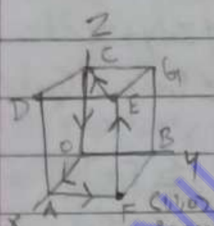
14. Either x or y i.e. $\int_C (5xy-6x^2)dx + (2y-4x)dy = \int_1^2 (5x^4-6x^2) + (2x^3-4x)3x^2dx$ $(x: 1\to 2)$, $dy = 3x^2dx$

$= \int_1^2 (5x^4 - 6x^2 + 6x^5 - 12x^3)dx = 35$ (at $\frac{5}{5}[x^5]_1^2 - \frac{6}{3}[x^3]_1^2 + \frac{6}{6}[x^6]_1^2 - \frac{12}{4}[x^4]_1^2$)

5. $(0,0) \to (1,2)$, $(y-0) = \left(\frac{2-0}{1-0}\right)(x-0)$, $y = 2x$, x on y $(0 \to 1)$

$\int_0^1 (2x^2dx + (x^2+4x^2)dy) = \int_0^1 2x^2dx + 10x^2dx = \int_0^1 12x^2dx$, $dy = 2dx$, $\frac{12}{3}[x^3]_0^1 = 4$

(Tutorial 4.)



$\int_C \vec{F}\cdot d\vec{r}$ AFECGDA $= \int_C = \int_{C_1} + \int_{C_2} + \dots + \int_{C_5}$ (Unit cube)

$\vec{F} = x^2\hat{i} - xz\hat{j} + y^2\hat{k}$. Now, $\int_{C_1}$: $z=0$, $dz=0$, $x=1$, $dx=0$, $y: 0\to 1$

$\int_{C_2}$: $x=1$, $dx=0$, $y=1$, $dy=0$, $z: 0 \to 1$; $\int 0 - 0 + y^2dz = \int_0^1 dz = [z]_0^1 = 1$ (ii); $\int_{C_1} 0 - 0 + y^2 dz = 0$ (i)

$x = y$, $dx = dy$, $x: 1 \to 0$, $y: 1 \to 0$, $z = 1$, $dz = 0$; $\int x^2dx - dy + 0$

$\int_{C_3} x^2dx - xz\,dy = \int_1^0 x^2dx - x\,dx = \frac{1}{3}[-1] - \frac{1}{2}[-1] = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$ (iii)

$\int_{C_4}$: $z: 1 \to 0$, $x=0$, $y=0$, $dx=0$, $dy=0$; $= 0$ (iv)

$\int_{C_5}$: $x: 0 \to 1$, $y=0$, $z=0$, $dy=0$, $dz=0$; $\int x^2dx - 0 + 0 = \frac{1}{3}[x^3]_0^1 = \frac{1}{3}$ (v)

$= 0 + 0 + \frac{1}{6} + \frac{1}{6} + 1 = \frac{3}{6} + 1 = \frac{3}{2}$

10. $\oint \vec{F}\cdot d\vec{r} = \oint (x-y)dx + (x+y)dy$

$z = 0$, $dz = 0$, $x = r\cos\theta$, $dx = -r\sin\theta\,d\theta$

and $r = 2$ $\{\theta: 0 \to 2\pi\}$, $y = r\sin\theta$, $dy = +r\cos\theta\,d\theta$

$\oint_0^{2\pi} (2\cos\theta - 2\sin\theta)(-2\sin\theta)d\theta$

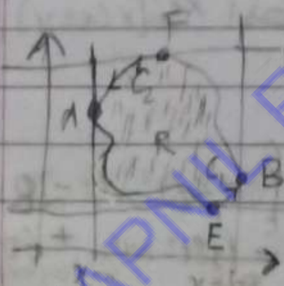
$= (-4\cos\theta\sin\theta + 4\sin^2\theta)d\theta + (2\cos\theta + 2\sin\theta)2\cos\theta\,d\theta$

$= -4\cos\theta\sin\theta + 4\sin^2\theta + 4\cos^2\theta + 4\cos\theta\sin\theta$

$4(1)d\theta = 4(\theta)_0^{2\pi} = 8\pi$ (3)

30. (a) Green's theorem: Let R be a closed bounded region in XY plane whose boundary is a scc C. Let $M(x,y)$ & $N(x,y)$ be cont. fn of x & y having cont. P.D. $\frac{\partial M}{\partial y}, \frac{\partial N}{\partial x}$ then $\oint M(x,y)dx + N(x,y)dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dxdy$.

36. (A)



Statement: Let R be a closed bounded region in XY plane whose boundary is a scc C and let $M(x,y)$ & $N(x,y)$ be cont. fn of x & y whose cont. P.D. $\frac{\partial M}{\partial y}$ & $\frac{\partial N}{\partial x}$ exist, then $\oint M(x,y)dx + N(x,y)dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dxdy$

) Proof: Let R be a closed bounded region in XY plane whose boundary is SCC C. Let M(x,y), N(x,y) be cont fns of x & y and whose cont PD $\frac{\partial M}{\partial y}$ & $\frac{\partial N}{\partial x}$ exist, then $\oint_C M(x,y)dx + N(x,y)dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$]

Let eqns of curves AEB (C_1) & BFA (C_2) be $y = f_1(x)$ & $y = f_2(x)$ resp^c. If R is bounded reg by C, then $\iint_R \frac{\partial M}{\partial y} dx\,dy = \int_a^b \int_{f_1(x)}^{f_2(x)} \frac{\partial M}{\partial y} dy\,dx$

$= \int_a^b [M(x,y)]_{f_1(x)}^{f_2(x)} dx = \int_a^b [M(x, f_1(x)) - M(x, f_2(x))] dy$

$= -\int_b^a M(x, f_2(x))\,dy - \int_a^b M(x, f_1(x))\,dy = -\int_{C_2} M(x,y)dx - \int_{C_1} M(x,y)dy$

$= -\left[\int_{C_1} M(x,y)dx + \int_{C_2} M(x,y)dx\right] = -\int_C M(x,y)dx$ thus

$-\iint_R \frac{\partial M}{\partial y} dx\,dy = \int_C M(x,y)\,dx$ — (i) $\equiv^{ly}$, $\iint_R \frac{\partial N}{\partial x} dx\,dy = \int_C N(x,y)dy$ — (ii)

Adding (i) & (ii), $\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \int_C M(x,y)dx + N(x,y)dy$ proved

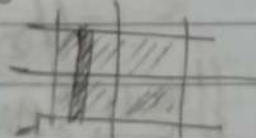
31. $\int_C (y - \sin x)dx + \cos y\,dy$ → $M(x,y) = y - \sin x$, $N(x,y) = \cos y$ $\quad \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = -1$

$\int_C M(x,y)dx + N(x,y)dy$ $\quad \frac{\partial M(x,y)}{\partial y} = 1$ $\quad \frac{\partial N}{\partial x} = 0$ $\quad -\iint_R dx\,dy$

$y = \left(\frac{2}{\pi}\right)x$ $\quad = -\int_0^{\pi/2}\int_0^{(2/\pi)x} dx\,dy = -\int\left(\frac{2}{\pi}x - 0\right)dx = -\frac{2}{2\pi}[x^2]_0^{\pi/2}$

$= -\frac{1}{\pi}\left[\frac{\pi^2}{4} - 0\right] = -\frac{\pi}{4}$ //

32. $\int_C (x^2 + xy)dx + (x^2 + y^2)dy$

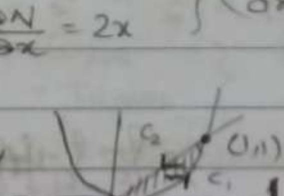


$M(x,y) = x^2 + xy$ $\quad N(x,y) = x^2 + y^2$

$\frac{\partial M}{\partial y} = x$ $\quad \frac{\partial N}{\partial x} = 2x$ $\Bigg\}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = x$ → $\iint_R x\,dx\,dy$ lim $-1 \to 1 : x$, $-1 \to 1 : y$

$= \frac{1}{2}\int_{-1}^{1} [x^2]_{-1}^{1} = 0$

36. $\int_C (xy + y^2)dx + x^2 dy$



$\int_C = \int_{C_1} + \int_{C_2}$ $\quad \oint_C M(x,y)dx + N(x,y)dy = \iint \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

Let's take LHS: $\int_{C_1}$: $y = x^2$, $dy = 2x\,dx$, lim $x : 0 \to 1$ $\Big\}$ $\int_0^1 (x^3 + x^4)dx + 2x^3 dx$

$= \frac{1}{4}[x^4]_0^1 + \frac{1}{5}[x^5]_0^1 + \frac{2}{4}[x^4]_0^1 = \frac{1}{4} + \frac{1}{5} + \frac{2}{4} = \frac{3}{4} + \frac{1}{5} = \frac{19}{20}$ — (i) $\int_{C_2}$ $y = x$, $dy = dx$, $x : 1 \to 0$

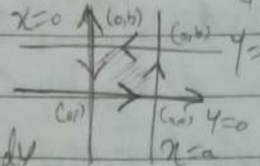
$= \int_1^0 x^2 + x^2 + x^2\,dx = \frac{3}{3}[x^3]_1^0 = -1$ (ii) $\frac{19}{20} - 1 = -\frac{1}{20}$ → LHS for RHS lim $y : x^2 \to x$, $x : 0 \to 1$

$\int_0^1\int_{x^2}^{x}$ $\frac{\partial M}{\partial y} = x + 2y$ $\frac{\partial N}{\partial x} = 2x$ $= (x - 2y)dx\,dy = x(x - x^2) - \frac{2}{2}(x^2 - x^4)$

$= \frac{1}{5}[x^5]_0^1 - \frac{1}{4}[x^4]_0^1 = \frac{-1}{20} \rightarrow RHS$ So LHS = RHS proved/verified

37. $\int_C (x^2+y^2)dx - 2xy\,dy$ So, LHS $\oint_C = \int_{C_1}+\int_{C_2}+\int_{C_3}+\int_{C_4} \Rightarrow \int_{C_1}: \begin{matrix} x: 0\to a \\ y=0 \\ dx \to 0 \\ dy = 0 \end{matrix} = \int_0^a x^2 dx - 0 = \frac{1}{3}[x^3]_0^a = \frac{a^3}{3}$ (i)

$\oint_C M(x,y)dx + N(x,y)dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$



$\int_{C_2}: \begin{matrix} y: 0\to b \\ x: a \\ dx = 0 \end{matrix} = \int_0^b \cancel{(a^2+y^2)0} - 2ay\,dy = -\frac{2a}{2}[y^2]_0^b = -ab^2$ (ii)

$\int_{C_3}: \begin{matrix} x: a\to 0 \\ y = b \\ dy = 0 \end{matrix} = \int_a^0 (x^2+b^2)dx = \frac{1}{3}[x^3]_a^0 + b^2[x]_a^0 = -\frac{a^3}{3} - b^2 a$ (iii)

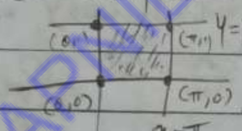
$\int_{C_4}: \begin{matrix} y: b\to 0 \\ x: 0 \quad dx = 0 \end{matrix} = \int_b^0 0 = 0$ (iv) So $0 + \cancel{ab^2} - \cancel{\frac{a^3}{3}} - ab^2 + \cancel{\frac{a^3}{3}} = -2ab^2 \rightarrow$ LHS

RHS: $\frac{\partial M}{\partial y} = 2y \quad \frac{\partial N}{\partial x} = -2y$

$\int_0^a \int_0^b \cancel{2x^3} - 4y = \left\{ \cancel{-\frac{4}{2}[y^2]_0^b} = -2a^2[x]_0^b \right\} = -4\iint y\,dx\,dy \rightarrow -4\int_0^a [y\ldots] y[x]_0^a dy = -\frac{4a}{2}[y^2]_0^b = -2ab^2 \rightarrow$ RHS

LHS = RHS, proved/verified

34. $\int_C (x^2 - \cosh y)dx + (y + \sin x)dy : \oint_C M(x,y)dx + N(x,y)dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$



$\frac{\partial M}{\partial y} = -\sinh y \quad \frac{\partial N}{\partial x} = +\cos x \} \quad +\cos x + \sinh y = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$

LHS, $\int_C = \int_{C_1} + \}$ avoid $\quad \int_0^1\int_0^\pi (\cos x + \sinh y)dx\,dy$

$= \int_0^1 ([\sin x]_0^\pi + \sinh y[x]_0^\pi)dy = \pi\int_0^1 \sinh y\,dy = \pi[\cosh y]_0^1 = \pi\cosh 1 - \pi$

33. Let's take $\int_C \left(\frac{x\,dy - y\,dx}{2}\right) = \int_C \frac{x\,dy}{2} - \frac{y\,dx}{2}$ By G.T. $\oint_C M(x,y)dx + N(x,y)dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dx\,dy$

Let's take RHS: $\frac{\partial N}{\partial x} = \frac{1}{2}$ } $\frac{\partial M}{\partial y} = -\frac{1}{2}$ So $\frac{1}{2}+\frac{1}{2} = 1$ So $\underbrace{\iint_R dx\,dy}$ → Area, Shown

38.

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ $\quad \int_C = \int_{C_1}+\int_{C_2}+\int_{C_3}$ } $\int_{C_1}: \begin{matrix} x: 0\to a \\ y = 0 \\ dy = 0 \end{matrix} = \int_0^a \frac{0-0}{2} = 0$ (i)

$4 \times$ Area → $\int_C \frac{x\,dy - y\,dx}{2}$

$\int_{C_2}: \begin{matrix} x: a\to 0 \\ y: 0\to b \\ \theta: 0 \to \pi/2 \end{matrix}$ { $x = a\cos\theta \; dx = -a\sin\theta$, $y = b\sin\theta \; dy = b\cos\theta$ } $\int_0^{\pi/2} \frac{(a\cos\theta)(b\cos\theta) - b\sin\theta(\ldots)}{2}$

$= \frac{1}{2}\int_0^{\pi/2} \underbrace{ab\cos^2\theta + ab\sin^2\theta}_{1} = \frac{ab}{2}[\theta]_0^{\pi/2} = \frac{\pi ab}{4}$ (ii)

$\int_{C_3}: \begin{matrix} x = 0 \; dx = 0 \\ y: b\to 0 \end{matrix}$ So $\int \frac{0-0}{2} = 0$ (iii) $\int_C$ = (i)+(ii)+(iii) $= 4\left(0+0+\frac{\pi ab}{4}\right) = \pi ab$

PAGE NO.
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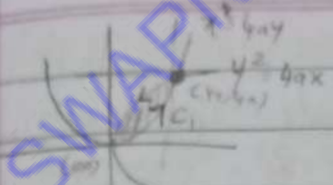
39. 

$\oint_C M(x,y)dx + N(x,y)dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$: Area

So $\oint_C \frac{x dy - y dx}{2}$ & $\oint_C = \int_{C_1} + \int_{C_2}$ So $\int_{C_1}$: $x: 0 \to 4a$, $y: 0 \to 4a$

$\int_{C_1}$: $2y\,dy = 4a\,dx$ $= 2\sqrt{ay}\left(\frac{4a\,dx}{4y}\right)$ ✗

$x = \frac{y^2}{4a} \Rightarrow \int_0^{4a} x$; $dx = \frac{2y\,dy}{4a} = \frac{y\,dy}{2a}$; $dy = (2a\,dx)/y$ ✗

$\int_C \frac{x dy - y dx}{2} = \int_0^{4a} \frac{2\sqrt{ay}\,dy}{2} - \frac{y}{2}\left(\frac{y\,dy}{2a}\right) = \int_0 \sqrt{ay}\,dy - \frac{y^2}{4a}dy$ $dx = \frac{2y\,dy}{4a} = \frac{y\,dy}{2a}$

$= \frac{2\sqrt{a}}{3}[y^{3/2}]_0^{4a} - \frac{1}{12a}[y^3]_0^{4a} = \frac{2\sqrt{a}}{3}(8a^{3/2}) - \frac{1}{12a}[64a^3] = \frac{2a^2}{3}\times 8 - \frac{16}{3}a^2$

& $\int_{C_1}$ $x : y : 0 \to 4a$, $x^2 = 4ay$, $2x\,dx = 4a\,dy$, $x\,dx = 2a\,dy$; $\int_0 \frac{x^2}{4a}\left(\frac{x\,dx}{2a}\right)$ ✗

By GT, wkt Area $= \int_C \frac{x dy - y dx}{2}$

$\int_C = \int_{C_1} + \int_{C_2}$ So for $\int_{C_1}$ x, y $0 \to 4a$

for $x^2 = 4ay$, ~~$2x\,dx = 4a\,dy$~~, $x\,dx = 2a\,dy$

$\int_0^{4a} \frac{x\left(\frac{x\,dx}{2a}\right) - \frac{x^2}{4a}(dx)}{2} = \frac{1}{2}\int_0^{4a} \frac{x^2 dx}{2a} - \frac{x^2 dx}{4a} = \frac{1}{4a}\int_0^{4a} x^2 dx - \frac{x^2}{2}$

$= \frac{1}{4a}\left[\frac{1}{3}[x^3]_0^{4a} - \frac{1}{6}[x^3]_0^{4a}\right] = \frac{1}{4a}\left[\frac{4^3a^3}{3} - \frac{4^3a^3}{6}\right] = \frac{a^2}{12}\left[4^3 - \frac{4^3}{2}\right]$

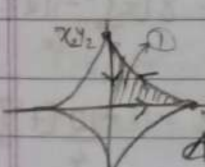
$= \frac{8}{3}a^2$ (i)

$\int_{C_2}$: $4a \to 0$ (x, y) ; $y^2 = 4ax$, $2y\,dy = 4a\,dx$, $y\,dy = 2a\,dx$

So $\int_{4a}^{0} \left(\frac{y^2}{4a}\right)dy - y\left(\frac{y\,dy}{2a}\right) = \int \left(\frac{y^2}{4a} - \frac{y^2}{2a}\right)dy = \frac{1}{2a}\int_{4a}^{0} \frac{y^2}{2} - y^2$

$= \frac{1}{2a}\left[\frac{1}{6}[y^3]_{4a}^{0} - \frac{1}{3}[y^3]_{4a}^{0}\right] = \frac{1}{2a}\left[\frac{-4^3a^3}{6} + \frac{4^3a^3}{3}\right] = \frac{a^2}{12}\left[-\frac{4^3}{2} + 4^3\right]$

$= \frac{a^2 \times 32}{12} = \frac{8a^2}{3}$ (ii) So $\int = \frac{8a^2}{3} + \frac{8a^2}{3} = \frac{16a^2}{3}$ //

40. 

Parametric $x = a\cos^3\theta$, $y = a\sin^3\theta$; $\int_C \frac{x dy - y dx}{2} \to$ Area by GT So,

$dx = -3a\cos^2\theta\sin\theta\,d\theta$
$dy = 3a\sin^2\theta\cos\theta\,d\theta$
$\theta : 0 \to \pi/2$

Asteroid ; $4 \times$ ① Area $=$ Tl. area ; $\int_{C_1}$ $x : 0 \to x_1$, $y : 0$ & $y = 0$

$\int_0^{\pi/2} \frac{a\cos^3\theta(a\sin^2\theta\cos\theta)d\theta + a\sin^3\theta(a\cos^2\theta\sin\theta)d\theta}{2}$ $= \frac{0 - 0}{2} = 0$ (i) ; $\int_{C_3}$ $y : y_2 \to 0$, $x = 0$, $dx = 0$; $\int_{C_3} \frac{0}{2} = 0$ (ii)

$= \frac{1}{2}\int_0^{\pi/2} a^2\cos^4\theta\sin^2\theta\,d\theta + a^2\sin^4\theta\cos^2\theta\,d\theta = \frac{a^2}{2}\int \sin^2\theta\cos^2\theta(\cos^2\theta + \sin^2\theta)d\theta$

$= \frac{a^2}{2}\left[\left(\frac{(2-1)(2-1)(1)}{(4-0)(4-2)(1)} \times \frac{\pi}{2}\right)\right] = \frac{a^2}{2} \times \frac{\pi}{2}\left[\frac{1 \cdot 1}{4 \cdot 2}\right]$ $\frac{\pi a^2}{32} \times 4 \times 3 = \frac{3\pi a^2}{8}$ //

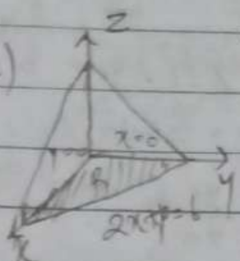
7. Let $\vec{F}$ be velocity of fluid (VPF) & $\iint \vec{F} \cdot \hat{n}\,ds = 0$ then VPF is solenoid ; flux across a surface
the whole rep. amt of fluid flowing per unit time thru surface → flux (phi)

6. Method eval: $\iint_S \vec{F}\cdot\hat{n}\,ds$

(i) If xy plane, $\iint_S \vec{F}\cdot\hat{n}\,ds = \iint_{R_1} \vec{F}\cdot\hat{n}\,\dfrac{dx\,dy}{|\hat{n}\cdot\hat{k}|}$ (c) if zx plane, $\iint_S \vec{F}\cdot\hat{n}\,ds$

(b) yz plane, $\iint_S \vec{F}\cdot\hat{n}\,ds = \iint_{R_2} \vec{F}\cdot\hat{n}\,\dfrac{dy\,dz}{|\hat{n}\cdot\hat{i}|}$ $= \iint_{R_3} \vec{F}\cdot\hat{n}\,\dfrac{dz\,dx}{|\hat{n}\cdot\hat{j}|}$

24.5. (a)



$\iint_S \vec{F}\cdot\hat{n}\,ds = \iint_S \vec{F}$

$n = \nabla\phi = 2\hat{i}+\hat{j}+2\hat{k}$

$\hat{n} = \dfrac{2\hat{i}+\hat{j}+2\hat{k}}{3}$

$\iint_S \vec{F}\cdot\hat{n}\,ds = \iint \vec{F}\cdot\hat{n}\,\dfrac{dx\,dy}{|\hat{n}\cdot\hat{k}|}$ ($2/3$)

$(4x\hat{i}+y\hat{j}+2\hat{k})\,(\hat{n})$

$\int_0^{3}\int_0^{6-2x}\left(\dfrac{8x}{3}+\dfrac{y}{3}+\dfrac{4}{3}\right)dx\,dy\,\dfrac{3}{2}$

$= \dfrac{1}{2}\int_0^3 \left(8x[6-2x] + \dfrac{1}{2}[y^2]_0^{6-2x} + 4[y]_0^{6-2x}\right)dx = \dfrac{1}{2}\int\left(48x-16x^2+\dfrac{(6-2x)^2}{2}+4(6-2x)\right)dx$

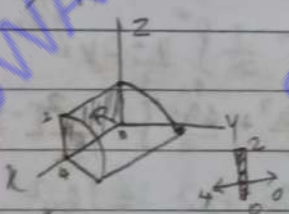
$= \dfrac{1}{2}\left[\dfrac{48}{2}[x^2]_0^3 - \dfrac{16}{3}[x^3]_0^3 + \dfrac{6}{2}[x]_0^3 - \dfrac{2}{3}[x^3]_0^3 + 24[x]_0^3 - \dfrac{8}{2}[x^2]_0^3\right]$

$= \dfrac{1}{2}\left[24(9) - \dfrac{16}{3}(27) + 3(3) - \dfrac{1}{3}(3^3) + 24(3) - 4(9)\right]$

$= \dfrac{1}{2}[216-144+9-9+72-36] = \dfrac{1}{2}[297-189] = 63$

(b) (vi) $\iint \vec{F}\cdot\hat{n}\,ds$ where $\vec{F} = z\hat{i}+x\hat{j}-3yz\hat{k}$ & S is the surface of cylinder $y^2+z^2=4$

$\iint \vec{F}\cdot\hat{n}\,ds = \iint_R \vec{F}\cdot\hat{n}\,\dfrac{dx\,dz}{|\hat{n}\cdot\hat{j}|}$ $\hat{n} = \dfrac{2y\hat{j}+2z\hat{k}}{2} = \hat{j}+\hat{k}+0\hat{i}$ $\hat{n}\cdot\hat{j}=1$



$\left\{\int_0^4\int_0^2\left(x-3y(\sqrt{4-y^2})\right)d\right\} = \int_0^4\int_0^2 \left(x-3z(\sqrt{4-z^2})\right)\dfrac{dx\,dz}{1}$

$= \int_0^2 \dfrac{1}{2}[x^2]_0^4 - 3z(\sqrt{4-z^2})[x]_0^4 = \int_0^2\left(8-12z\sqrt{4-z^2}\right)dz = 8[z]_0^2 - \int_0^2 12z\sqrt{4-z^2}\,dz$

$4-z^2=t^2$

$2t\,dt = -2z\,dz$

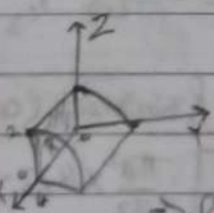
$t\,dt = -z\,dz$

$dz = \dfrac{t\,dt}{-z}$

$16-12\int_0^2 z t^2\,\dfrac{t\,dt}{-z} = 16+12\int_2^0 t^3\,dt = 16+\dfrac{12}{4}[t^4]_2^0$

$= 16+3[-16]$

$\iint \vec{F}\cdot\hat{n}\,ds = \iint_R \vec{F}\cdot\hat{n}\,\dfrac{dx\,dz}{|\hat{n}\cdot\hat{j}|}$ & $\hat{n} = \dfrac{2y\hat{j}+2z\hat{k}}{2}$ $y\hat{j}+z\hat{k}$



$\hat{n} = \dfrac{2y\hat{j}+2z\hat{k}}{2} = y\hat{j}+z\hat{k}+0\hat{i}$ $\hat{n} = \dfrac{2y\hat{j}+2z\hat{k}}{\sqrt{4(y^2+z^2)}} = \dfrac{y}{2}\hat{j}+\dfrac{z}{2}\hat{k}+0\hat{i}$

$\iint (xy-3yz^2)\,\dfrac{dx\,dz}{y}$ $|\hat{n}\cdot\hat{j}|$

$= \dfrac{1}{2}\iint (xy-3yz^2)\,\dfrac{dx\,dz}{y/2} = \int_0^2\int_0^4 (x-3z^2)\,dx\,dz = \int_0^2 \dfrac{1}{2}[x^2]_0^4 - 3z^2[x]_0^4\,dz$

$= \int_0^2 (8-12z^2)\,dz = 8[z]_0^2 - \dfrac{12}{3}[z^3]_0^2 = 16-4(8) = -16$

(c) $\vec{F} = (2x^2 - 3z)\hat{i} - 2xy\hat{j} - 4x\hat{k}$ (i) $\iiint \text{div}\,\vec{F}\,dV$ (ii) $\iiint \text{curl}\,\vec{F}\,dV$

where V is the volume bounded by $x = 0,\ y = 0,\ z = 0$ & $2x + 2y + z = 4$

(i) $\nabla \cdot \vec{F} = 4x - 2x = 2x$ so $\iiint 2x\,dx\,dy\,dz = \iint 2x[z]_0^{4-2x-2y}\,dy\,dx$

$= \iint 2x(4 - 2x - 2y)\,dy\,dx = \iint (8x - 4x^2 - 4xy) = \int \left(8x[y]_0^{2-x} - 4x^2[y]_0^{2-x} - \frac{4x}{2}[y^2]_0^{2-x}\right) dx$

$\int_0^2 16x - 8x^2 - 8x^2 + 4x^3 - 2x[4 + x^2 - 4x] = \int_0^2 (16x - 8x^2 - 8x^2 + 4x^3 - 8x - 2x^3 + 8x^2)\,dx$

$= \int_0^2 (8x - 8x^2 - 2x^3)\,dx = 2\int_0^2 (4x - 4x^2 - x^3)\,dx = 2\left[\frac{-20}{3}\right] = \frac{-40}{3}$

$\int_0^2 (16x - 8x^2 - 8x^2 + 4x^3 - 2x(4 + x^2 - 4x))\,dx = \int_0^2 16x - 8x^2 - 8x^2 + 4x^3 - 8x - 2x^3 + 8x^2$

$= \int_0^2 (-8x^2 + 8x + 2x^3)\,dx = \int_0^2 (-8x^2 + 8x + 2x^3)\,dx = \frac{8}{3}$ //

(ii) $\nabla \times \vec{F} = \hat{i}(0 - 0) - \hat{j}(-4 + 3) + \hat{k}(-2y - 0) = \hat{j} - 2y\hat{k}$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2x^2 - 3z & -2xy & -4x \end{vmatrix}$$

$\int_0^2\int_0^{2-x}\int_0^{4-2y-2x} (\hat{j} - 2y\hat{k})\,dz\,dy\,dx = \iint \hat{j}[z]_0^{4-2y-2x} - \iint 2y[z]_0^{4-2y-2x}\hat{k}$

$= \iint (4 - 2y - 2x)\hat{j} - \iint 2y(4 - 2y - 2x)\hat{k}$

$= \int \left(4[y]_0^{2-x} - \frac{2}{2}[y^2]_0^{2-x} - 2x[y]_0^{2-x}\right)\hat{j}\,dx - \int \left(\frac{8}{2}[y^2]_0^{2-x} - \frac{4}{2}[y^2]_0^{2-x} - \frac{4x}{2}[y^2]_0^{2-x}\right)\hat{k}\,dx$

$= \int_0^2 (8 - 4x - 4 - x^2 + 4x - 4x + 2x^2)\hat{j}\,dx - \int 16 + 4x^2 - 16x - 2(4 + x^2 - 4x) - 2x(4 + x^2 - 4x)\,\hat{k}\,dx$

$\int_0^2 (x^2 - 4x + 4)\,dx\,\hat{j} - \int_0^2 (8x^2 - 2x^3 - 16x + 8)\hat{k}\,dx$ $\quad 16 + 4x^2 - 16x - 8 - 2x^2 + 8x - 8x - 2x^3 + 8x^2$

$= \frac{8}{3}\hat{j} - \frac{8}{3}\hat{k} = \frac{8}{3}(\hat{j} - \hat{k})$ (?)

(A) GDT: If V is the vol bounded by a closed surface S & $\vec{F}$ VPF ~~possessing~~ 1st order PD of V, $\iiint \text{div}\,\vec{F}\,dV = \iint \vec{F} \cdot \hat{n}\,ds$ where $\hat{n}$ is the unit outward drawn normal vec to the surface S.

44. $\int_S (yz\hat{i} + zx\hat{j} + xy\hat{k}) \cdot \hat{n}\,ds$ S → surface of cube $x = 0, y = 0, z = 0, x = 1, y = 1, z = 1$.

So, $\iiint_V \text{div}\,\vec{F}\,dV = \iint_S \vec{F} \cdot \hat{n}\,ds$ so $\vec{F} = yz\hat{i} + zx\hat{j} + xy\hat{k}$ so LHS

$\iiint_0 0\hat{i} + 0\hat{j} + 0\hat{k} = \iiint 0 = \underline{0}$ so we get answer as 0 by GDT

45. $\int_S \vec{r} \cdot \hat{n}\,ds = ?$ So, GDT $\iiint_V \text{div}\,\vec{F}\,dV = \iint_S \vec{F} \cdot \hat{n}\,ds$ so $\iiint_V \nabla \cdot (x\hat{i} + y\hat{j} + z\hat{k})\,dV$

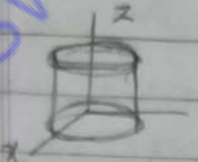
$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ $\quad \vec{r} = \vec{F}$ (LHS)

$= 3\iiint_V dV = \underline{3V}$ where V is volume bounded by S

46. $\int_S \nabla r^2 \cdot \hat{n}\,ds = 6V$ So GDT $\iiint_V \text{div}\,V\,dV = \iint_S \vec{F} \cdot \hat{n}\,ds$ $\iiint_V \nabla \cdot (2x\hat{i} + 2y\hat{j} + 2z\hat{k})$ LHS

$= \iiint 2 + 2 + 2 = 6\iiint dV = \underline{6V}$ ~~proved~~ where V is volume bounded by S

51. $\iint_S \vec{F}\cdot\hat{n}\,ds$ then By G.D.T. If V is the volume bound by closed surface S & $\vec{F}$ VPF having 1st order PD then $\iiint_V \nabla\cdot\vec{F}\,dV = \iint_S \vec{F}\cdot\hat{n}\,ds$ where $\hat{n}$ is the unit outward normal to the surface



$\iiint (4-4y+2z)\,dV$ 'Closed surface' V

Lets change to CPC $x=\rho\cos\phi$, $y=\rho\sin\phi$, $x^2+y^2=\rho^2$, $|J|=\rho$, $z=z$, $\phi: 0$ to 2π

$= \int_0^3\int_0^{2\pi}\int_0^2 (4-4\rho\sin\phi+2z)\,\rho\,d\rho\,d\phi\,dz$

$= \int_0^3\int_0^{2\pi} \left(\frac{4}{2}[\rho^2]_0^2 - \frac{4}{3}[\rho^3]_0^2\sin\phi + \frac{2z}{2}[\rho^2]_0^2\right) d\phi\,dz$

$= \iint 2(4) - \frac{4}{3}(8)\sin\phi + z(4)\,d\phi\,dz = \int 8[\phi]_0^{2\pi} + \frac{32}{3}[\cos\phi]_0^{2\pi} + 4z[\phi]_0^{2\pi}$

$= \int 16\pi + \frac{32}{3}[0] + 4z[2\pi]\,dz = 16\pi[z]_0^3 + 8\frac{\pi}{2}[z^2]_0^3 = 48\pi + 36\pi$

$= 84\pi$

53. $\vec{F} = 2xy\hat{i} + yz^2\hat{j} + xz\hat{k}$ By G.D.T, $\iiint \mathrm{div}\vec{F}\,dV = \iint_S \vec{F}\cdot\hat{n}\,ds$ So LHS

$2y + z^2 + x = \mathrm{div}\vec{F}$ & $\int_0^3\int_0^1\int_0^2 2y+z^2+x\,dx\,dy\,dz = \iint 2y[x]_0^2 + z^2[x]_0^2 + \left[\frac{x^2}{2}\right]_0^2$

$= \iint 4y + 2z^2 + 2 = \int \frac{4}{2}[y^2]_0^1 + \cancel{\frac{3}{3}[z^3]_0}\, 2z^2[y]_0^1 + 2[y]_0^1\,dz$

$= \int 2 + 2z^2 + 2 = 2\int 2 + z^2 = 2\left[2[z]_0^3 + \frac{1}{3}[z^3]_0^3\right]$

$= 2\left[6 + \frac{1}{3}\cancel{[27]}\ 9\right] = 2(15) = 30$

52. $\iint_S (x^3\hat{i} + y^3\hat{j} + z^3\hat{k})\cdot\hat{n}\,ds = ?$ By G.D.T, $\iiint \mathrm{div}\vec{F}\,dV = \iint_S \vec{F}\cdot\hat{n}\,ds$

So, $\vec{F} = x^3\hat{i} + y^3\hat{j} + z^3\hat{k}$ $x = r\sin\phi\cos\theta$ $y = r\sin\phi\sin\theta$ $z = r\cos\theta$

and $|J| = r^2\sin\theta$

$3\int_0^{a}\int_0^{2\pi}\int_0^{\pi} (x^2+y^2+z^2)\,r^2\sin\theta = 3\iiint r^2\sin^2\phi\cos^2\theta + r^2\sin^2\phi\sin^2\theta + r^2\cos^2\theta\ r^2\sin\theta\ dr\,d\theta\,d\phi$

$= 3r^2\iiint \sin^2\phi(1) + r^2\cos^2\theta\sin^2\theta\,d\theta$

$= \cancel{3r^2}\ 3\iint \frac{\sin^2\phi}{3}[r^3]_0^a + \frac{\cos^2\theta\sin^2\theta}{5}[r^5]_0^a\,d\theta$

$= \iint \frac{a^3\sin^2\phi}{3} + \frac{\cos^2\theta\sin^2\theta\,a^5}{5} = \int \frac{a^3}{3}\sin^2\phi\,[\theta]_0^{\pi} + \int \frac{a^5}{5}\left[\frac{(2-1)(2-1)}{(4-0)(4-2)}\frac{\pi}{2}\right]$

$= \frac{\pi a^3}{3}\int \sin^2\phi\,d\phi + \frac{a^5\pi}{80}\int d\phi = \frac{\pi a^3}{3}\times 4\times\left(\frac{2-1}{2}\frac{\pi}{2}\right) + \frac{a^5\pi}{40}[\phi]_0^{2\pi}$

$2\pi \cdot \frac{\pi}{2} = \pi^2$ $\}\times$ So,

$3\int_0^{a}\int\int r^4\sin\theta\,d\theta\,d\phi\,dr = 3\iint \frac{\sin\theta}{5}[r^5]_0^a = \frac{3a^5}{5}\int[\cos\theta]_0^{\pi} = \frac{6a^5}{5}[\phi]_0^{2\pi}$

$= \frac{12\pi a^5}{5}$

41. (A) Stokes theorem: If $\vec{F}$ is VPF and S is an open surface bounded by a S.C.C. & possess cont 1st OPD, then $\oint_C \vec{F}\cdot d\vec{r} = \iint_S \mathrm{curl}\,\vec{F}\cdot\hat{n}\,ds$ where C is traversed in +ve dir.

PAGE NO.
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42. If $\vec{F}$ is irrot, $\nabla\times\vec{F}$ or curl $\vec{F}=0$ so By S.T., $\oint_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot\hat{n}\,ds = 0$ (proved)

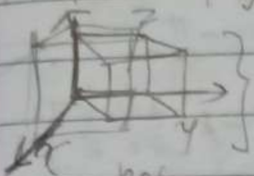
43. By S.T., $\oint (e^x dx + 2y\,dy - dz)$ $C: x^2+y^2=4, z=2$; $\oint_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot\hat{n}\,ds$

So, $e^x\hat{i} + 2y\hat{j} - \hat{k} = \vec{F}$; $= \iint \text{curl}\,\vec{F}\cdot\hat{n}\,ds$; curl $\vec{F}$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ e^x & 2y & -1 \end{vmatrix} = \hat{i}(0-0) - \hat{j}(0-0) + \hat{k}(0-0)$$

$= 0$ so $\oint_C \vec{F}\cdot d\vec{r} = 0$

54. $\vec{F} = 4xz\hat{i} - y^2\hat{j} + yz\hat{k}$; $\oint_S \vec{F}\cdot\hat{n}\,ds$ G.D.T. $x: 0\to1, y: 0\to1, z: 0\to1$

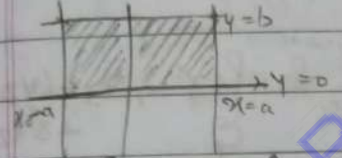


G.D.T.: $\iiint_V \text{div}\,\vec{F}\,dV = \iint_S \vec{F}\cdot\hat{n}\,ds$ RHS = LHS. div $\vec{F}$ =

$4z - 2y + y = 4z - y$ so $\int_0^1\int_0^1\int_0^1 (4z-y)\,dx\,dy\,dz$

$= \iint (4z[x]_0^1 - y[x]_0^1)\,dy\,dz = \int (4z[y]_0^1 - \frac{1}{2}[y^2]_0^1)\,dz = \frac{4}{2}[z^2]_0^1 - \frac{1}{2}[z]_0^1 = \frac{3}{2}$

47.



S.T. $\oint_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot\hat{n}\,ds$;
$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2+y^2 & -2xy & 0 \end{vmatrix}$$
$= \hat{i}(0-0) - \hat{j}(0-0) + \hat{k}(-2y+2y)$

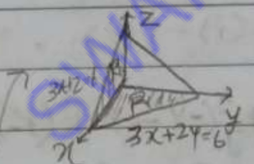
$\hat{n} = \hat{k}$ $\left[= -4y\hat{k}\right.$; $\int_0^b\int_{-a}^{a} (-4y)(\hat{k}\cdot\hat{k})\frac{dx\,dy}{|\hat{n}\cdot\hat{k}|} = \iint -4y\frac{dx\,dy}{|\hat{k}\cdot\hat{k}|} = \int -4y[x]_{-a}^{a} = -\frac{8a}{2}[y^2]_0^b$

$= -4a(b^2-0) = -4ab^2$

48. ~~Intercept form~~

$\frac{x}{2} + \frac{y}{3} + \frac{z}{6} = 1$

$3x + 2y + z = 6$



S.T. $\oint_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot\hat{n}\,ds$ so,

$(x+y)\hat{i} + (2x-z)\hat{j} + (y+z)\hat{k}$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x+y & 2x-z & y+z \end{vmatrix} = \hat{i}(1+1) - \hat{j}(0-0) + \hat{k}(2-1) = 2\hat{i} + \hat{k}$$

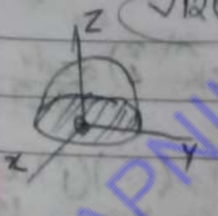
$= \int_0^2\int_0^{\frac{6-3x}{2}} (2\hat{i}+\hat{k}+0\hat{j})\left(\frac{3\hat{i}+2\hat{j}+\hat{k}}{\sqrt{14}}\right)\frac{dx\,dy}{|\hat{n}\cdot\hat{k}|}$; $\hat{n} = \frac{3\hat{i}+2\hat{j}+\hat{k}}{\sqrt{14}}$

$= \iint\left(\frac{6+1}{\sqrt{14}}\right)\left(\frac{dx\,dy}{1/\sqrt{14}}\right) = 7\int [y]_0^{\frac{6-3x}{2}}dx = 7\int 6 - \frac{3x}{2} = \frac{7}{2}\left[6[x]_0^2 - \frac{3}{2}[x^2]_0^2\right]$

$= 7/2\left[(12-0) - \frac{3}{2}(4)\right] = 7/2\,[12-6] = 21$

50. S.T., $\oint_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot\hat{n}\,ds$ so curl $\vec{F}$
$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & z & x \end{vmatrix} = \hat{i}(0-1) - \hat{j}(1-0) + \hat{k}(0-1) = -\hat{i} - \hat{j} - \hat{k}$$

$\hat{n} = \frac{2x\hat{i} + 2y\hat{j} + 2z\hat{k}}{\sqrt{12(x^2+y^2+z^2)}} = \left(\frac{x\hat{i}+y\hat{j}+z\hat{k}}{\sqrt{3}}\right) = \iint -(\hat{i}+\hat{j}+\hat{k})\cdot\left(\frac{x\hat{i}+y\hat{j}+z\hat{k}}{\sqrt{3}}\right)\frac{dx\,dy}{|\hat{n}\cdot\hat{k}|}$



$\iint \frac{-x-y-z}{\sqrt{3}}\frac{dx\,dy}{z/\sqrt{3}} = -\left[\iint \frac{x+y+z}{z}\,dx\,dy\right.$ X

S.T $\oint_c \vec{F}\cdot d\vec{r} = \iint_S (\text{curl}\,\vec{F})\cdot\hat{n}\, ds$ So $\begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & z & x \end{vmatrix} = \hat{i}(0-1) - \hat{j}(0-1) + \hat{k}(0-1) = -\hat{i}-\hat{j}-\hat{k}$

$\hat{n} = \dfrac{2x\hat{i}+2y\hat{j}+2z\hat{k}}{\sqrt{4x^2+4y^2+4z^2}} = \dfrac{2(\;)}{2(\sqrt{1})} = \hat{i}+\hat{j}+\hat{k}$ So $-\iint 1+1+1 = -3\iint \dfrac{dxdy}{|\hat{n}\cdot\hat{k}|}$

$x^2+y^2=1$

$= -3\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dxdy = -3\left[\int_{-1}^{1}[y]_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dx\right] = -6\left[\int_{-1}^{1}\sqrt{1-x^2}\right]$

(OR)

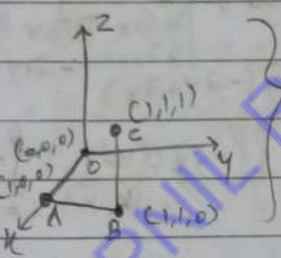
↓ (take LHS) $= -6\left[\dfrac{x}{2}\sqrt{1-x^2} + \dfrac{1}{2}\sin^{-1}\left(\dfrac{x}{1}\right)\right]_{-1}^{1} = -6\left[0+\frac{1}{2}\left(\frac{\pi}{2}\right)-0+\frac{\pi}{2}\right]$

$\oint_c (y\hat{i}+z\hat{j}+x\hat{k})(dx\hat{i}+dy\hat{j}+dz\hat{k})$ $-6\left[\dfrac{2\pi}{6}\right] = -\pi$

$= \oint_c y\,dx+z\,dy+x\,dz = \oint_0^{2\pi} -r^2\sin^2\theta = -4\left[\dfrac{2-1}{2\cdot 0}\,\dfrac{\pi}{2}\right] = -\pi$

$x = r\cos\theta$, $y = r\sin\theta$, $dx = -r\sin\theta$, $dy = r\cos\theta$ → (circulation type) $r=1$, $z=0$, $dz=0$ So by LHS

18.



$\int_C \vec{F}\cdot d\vec{r} \Rightarrow \int_c = \int_{c_1}+\int_{c_2}+\int_{c_3}+\int_{c_4}$ $(0,0,0)\to(1,0,0)\to(1,1,0)\to(1,1,1)$ So $\int_{c_1} (x^2+2y)dx - 6yz\,dy + 8xz^2dz$

$(0,0,0)\to(1,0,0)$: $y=0$, $z=0$, $dy=0$, $dz=0$ → $\int_{c_1} = \int_0^1 x^2 dx = \frac{1}{3}[x^3]_0^1 = \frac{1}{3}$ (i)

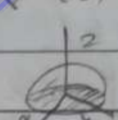
$\int_{c_2} 0-0+0 = 0$ (ii); $x=1$, $dx=0$, $y: 0\to1$, $z=0$, $dz=0$

$\int_{c_3} 0-0+8z^2dz = \frac{8}{3}[z^3]_0^1 = \frac{8}{3}$ (iii); $x=1$, $dx=0$, $y=1$, $dy=0$, $z: 0\to1$

$0+\frac{1}{3}+\frac{8}{3} = (ii)+(i)+(iii) = 9/3 = 3$

19. $\begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & x-2xz & -xy \end{vmatrix} = \hat{i}(-x+2x) - \hat{j}(-y-0) + \hat{k}(1-2z-1)$

curl $\vec{F} = x\hat{i}+y\hat{j}-2z\hat{k}$ $\dfrac{2(x\hat{i}+y\hat{j}+z\hat{k})}{2\sqrt{a}} = \hat{n}$



$\Rightarrow \int (x\hat{i}+y\hat{j}-2z\hat{k})\left(\dfrac{x\hat{i}+y\hat{j}+z\hat{k}}{\sqrt{a}}\right)\dfrac{dxdy}{|\hat{n}\cdot\hat{k}|}$, $\dfrac{z}{\sqrt{a}}$

$= \dfrac{1}{\sqrt{a}}\int (x^2+y^2-2z^2)\dfrac{dxdy}{z/\sqrt{a}} = \int_{2\pi}\left\{\dfrac{x^2+y^2}{z}-2z\right\}$ $x^2+y^2+z^2-3z^2\}$

So $\oint_c \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot\hat{n}\,ds$ LHS: $\oint_0 (y\,dx+(x-2xz)dy-xy\,dz)$

$x^2+y^2=a^2$

$x = r\cos\theta$, $dx = -r\sin\theta$, $y = r\sin\theta$, $dy = r\cos\theta$, $z=0$, $dz=0$ $= \oint_0^{2\pi}(-r^2\sin^2\theta + (r\cos\theta-0)(+r\cos\theta)-0)d\theta$

$= \oint_0^{2\pi} -r^2\sin^2\theta + r^2\cos^2\theta = a\int_0^{2\pi}\cos 2\theta = \dfrac{a}{2}[\sin 2\theta]_0^{2\pi} = \dfrac{a}{2}(0-0) = 0$

(55.)

20.

$\int_{S_1}+\int_{S_2}+\int_{S_3}+\int_{S_4}+\int_{S_5}+\int_{S_6}$ So $\int_{S_1} (4xz\hat{i}-y^2\hat{j}+yz\hat{k})(\hat{i})\dfrac{dzdy}{|\hat{n}\cdot\hat{i}|}$ $\iint 4xz\,dzdy = \frac{4}{2}$

$\int_{S_2} (\;)(-\hat{i})\dfrac{dzdy}{|\hat{n}\cdot\hat{i}|}$, $x=0$: $\iint 4xz\,dzdy = 0$; $\int_{S_3}(\;)(\hat{j})\dfrac{dxdz}{|\hat{n}\cdot\hat{j}|}$, $y=0$

PAGE NO.
DATE /

$S_1(\text{face})$ — $\hat{i}$ — dS — Plane

$S_2(\ldots)$ — $-\hat{k}$ — $\frac{dxdy}{|\hat{n}\cdot\hat{k}|}$ — $z=0$

$S_3(\ldots)$ — $\hat{k}$ — $\frac{dxdy}{|\hat{n}\cdot\hat{k}|}$ — $z=1$

$S_4(\ldots)$ — $\hat{j}$ — $\frac{dxdz}{|\hat{n}\cdot\hat{j}|}$ — $y=1$

$S_5(\ldots)$ — $-\hat{j}$ — $\frac{dxdz}{|\hat{n}\cdot\hat{j}|}$ — $y=0$

$S_5(ABD_1)$ $\hat{i}$ $\frac{dydz}{|\hat{n}\cdot\hat{i}|}$ $x=1$ $\iint -y^2\,dxdz = -\int\int_0^1 = -1$ (ii)

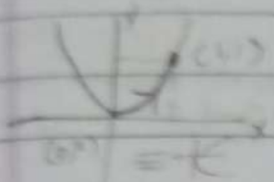
$S_6(OCEF)$ $-\hat{i}$ $dydz$ $x=0$ $\iint y^2\,dxdz = 0$ (iv)

$\iint 4xz = \iint 4z\,dydz = 2$ (v)

So $\iint -yz = 0$ (i) $\iint -4xz = 0$ (vi)

$\iint yz = \int_0^1\int_0^1 y\,dxdy = \frac{1}{2}$ (iii) $\Sigma = \frac{1}{2}+2-1 = 3/2$ //

21.



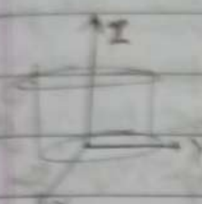
$\int_C \vec{F}\cdot d\vec{r} = \int(x^2-y^2+x)\,dx - (2xy+y)\,dy$ So $x=y^2$, $2y\,dy = dx$, $y: 0\to 1$

$= \int_0^1 (y^4 - y^2 + y^2)\,2y\,dy - (2y^3+y)\,dy$

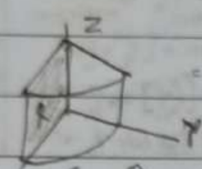
$= \int_0^1 2y^5\,dy - (2y^3+y)\,dy = \int_0^1 (2y^5 - 2y^3 - y)\,dy$

$= \int_0 = \frac{2}{6}[y^6]_0^1 - \frac{2}{4}[y^4]_0^1 - \frac{1}{2}[y^2]_0^1 = \frac{1}{3} - \frac{1}{2} - \frac{1}{2} = -2/3$ //

22.



X at 1st octant



$= \iint_S \left(\frac{zx}{4} + \frac{yx}{4}\right)\frac{dxdz}{|\hat{n}\cdot\hat{j}|}$ $t^2 = 4 - x^2$, $2t\,dt = -2x\,dx$

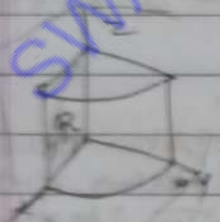
$\hat{n} = \frac{2x\hat{i}+2y\hat{j}}{2\cdot 4} = \frac{x\hat{i}+y\hat{j}}{4}$

$= \iint \left(\frac{zx}{\sqrt{16-x^2}} + x\right)dxdz$ $y/4$

$= \iint \frac{zx}{t}\frac{t\,dt}{x} + \iint x\,dxdz = \int z[t]_4^0\,dz + \frac{1}{2}\int[x^2]_0^4\,dz$ $\quad 16[z]_0^5 = 40$

$= 4\int z\,dz = \frac{4}{2}[z^2]_0^5 = 50$ $\quad 50+40 = 90$ //

23.

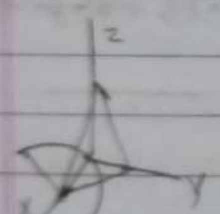


$\int_S \frac{zx}{4} + \frac{yx}{4}\,\frac{dxdz}{|\hat{n}\cdot\hat{j}|}$ $y/4$ $= \iint \frac{zx}{y} + x$ $\quad \left(\begin{array}{l} y = \sqrt{16-x^2} \\ t^2 = 16 - x^2 \\ 2t\,dt = -2x\,dx \\ dx = -\frac{t\,dt}{x}\end{array}\right)$ $= \int_0^5\int_0^4 \frac{zx}{t}\frac{t\,dt\,dz}{x} + \int_0^5\int_0^4 x\,dx\,dz$

$\hat{n} = \frac{2x\hat{i}+2y\hat{j}}{8} = \frac{x\hat{i}+y\hat{j}}{4}$ $\quad = \frac{4}{2}[25-0] + \int \frac{1}{2}16\,dz = 50$, $8[25-0] = 40$

$50+40 = 90$ //

24.



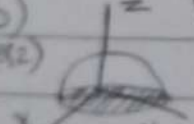
$\iint_S \vec{F}\cdot d\vec{s}$ by G.D.T. $\iiint_V \text{div}\vec{F}\,dV = \iint_S \vec{F}\cdot\hat{n}\,dS$, we can

Solve, $\int_0^2\int_0^6\int_4^{x^2} 2z\,dV \to dz\,dy\,dx = \iint \frac{2}{2}[z^2]_4^{x^2} = 16\int[y]_0^6 = 16\times 6\int_0^2 x$

$\text{div}F = 2z - 0 - 0 = 2z$ $= \iint [z^2]_4^{x^2} = \iint (x^4-16)\,dy\,dx$

$= \int(x^4[y]_0^6 - 16[y]_0^6)\,dx = \frac{6}{5}[x^5]_0^2 - 16\times 6[x]_0^2 = \frac{768}{5}$ //

24.5 (b) (ver2)



S.T. $\int_C \vec{F}\cdot d\vec{r} = \iint_S \text{Curl}\vec{F}\cdot\hat{n}\,dS$ & $\hat{n} = \hat{k}$ & $dS = \frac{dxdy}{|\hat{n}\cdot\hat{k}|}$ $\quad \hat{n} = \frac{2x\hat{i}+2y\hat{j}+2z\hat{k}}{2} = x\hat{i}+y\hat{j}+z\hat{k}$

$\text{Curl}\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x-y & -yz^2 & -y^2z \end{vmatrix}$ $= \iint_S \hat{k}(\;)\,\frac{dxdy}{|\hat{n}\cdot\hat{k}|} = \iint z\,\frac{dxdy}{z}$

$= \hat{i}(-2yz+2yz) - \hat{j}(0-0) + \hat{k}(0+1) = \hat{k}$ $= \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dx\,dy = \int[\sqrt{1-x^2}+\sqrt{1-x^2}] = 2\int_{-1}^{1}\sqrt{1-x^2}\,dx$

$= 2\left[\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}(x)\right]_{-1}^{1} = 2\left[0+\frac{\pi}{4} - 0 + \frac{\pi}{4}\right] = 2\left[\frac{\pi}{2}\right] = \pi$ //

UNIT 5

Unit V

1. GCD: The GCD of 2 no. (both not zero) is the largest +ve Z that divides both of them. Denoted by (a, b) or gcd(a, b). Ex (12, 18) = 6 & so on (12, 25) = 1

2. Proper divisor: Divisor of a no. that is smaller than the no. itself
Ex: 20 → 1, 2, 4, 5 are PDs

3. Div alg: If 'a' be any integer & 'b' any +ve Z (b > 0), there is unique pair of q, r existing (Z) such that $a = bq + r$, $0 \le r < b$
Ex: 42 ÷ 5 so $42 = 5 \times 8 + 2$ & $2 \in [0, 5)$ ✓

4. $a|b \Rightarrow b = ak_1$ & $b|c \Rightarrow c = k_2 b$ so $c = \underbrace{ak_1k_2}_{k_3}$ = $c = k_3 a \Rightarrow a|c$ proved

5. $a|b \Rightarrow b = ak_1$ & $a|c \Rightarrow c = ak_2$ so $bm + nc = ak_1m + nak_2 = a(mk_1 + nk_2)$
$\Rightarrow bm + nc = ak_3$ so $a | bm + nc$ proved where $k_3 = mk_1 + nk_2$

6. Euclid's lemma: $a|bc \Rightarrow bc = ak_1$ & $(a,b) = 1 \Rightarrow ax + by = 1$ so $cax + bcy = c$
$\Rightarrow cax + ak_1y = c \Rightarrow a(cx + yk_1) = c \Rightarrow ak_3 = c$ so } X instead
as $a | acx$ as $a|bc$, $a | acx + bcy$ too $\Rightarrow a|c$ proved

7. $a|c \Rightarrow c = ak_1$, $b|c \Rightarrow c = bk_2$ & $(a,b) = 1 \Rightarrow ax + by = 1$ so $acx + byc = c$
$bk_2ax + abk_1y = c \Rightarrow ab(k_2x + k_1y) = c \Rightarrow abk_3 = c$ so $ab|c$

8. $(a,b) = 1 \Rightarrow ax + by = 1 = au + cv = 1$ so $(1 - ax) = by$ & $cv = (1 - au)$ for now
$(1 - ax)(1 - au) = bc(uv) \Rightarrow bc(yv) = 1 - au - ax + a^2xu$
$a(u + x + axu) + bc(vy) = 1 \longleftarrow bc(vy) = 1 - a(u + x + axu)$
$\Rightarrow am + bc(n) = 1$ so $(a, bc) = 1$ proved where $m = u + x + axu$, $n = yv$

10. Same as 2

9. $(a,b) = 1 \Rightarrow ax + by = 1$ & $c|a \Rightarrow a = ck_1$ & $d|b \Rightarrow b = dk_2$ so subst.
a, b as $ck_1 + dk_2 = 1$ so $(c, d) = 1$ proved

{ $\in$ belongs to, $\exists$ there exists, $\Rightarrow$ such that } & $a|axc$ ... $b|c$ $(a,b)=1$ $a|c$ & $a|axc$, $bc = ak_1$, $ax+by=1$ & $byc = ak_1$, $axc+byc$ ... so $a|byc$

(11) $(a,b)=1 \Rightarrow ax+by=1$ so raise to $n+k-1$ b.s. $(ax+by)^{n+k-1} = 1$ so now,

$\binom{n+k-1}{0} a^{n+k-1}x^{n+k-1} + \binom{n+k-1}{1} a^{n+k-2}x^{n+k-2}by + \dots + \binom{n+k-1}{k-1} a^n x^n b^{k-1}y^{k-1}$
$+ \binom{n+k-1}{k} a^{n-1}x^{n-1}b^k y^k + \dots + b^{n+k-1}y^{n+k-1}$ so take common

$a^n\left(a^{k-1}x^{n+k-1} + \binom{n+k-1}{1} a^{k-2}x^{n+k-2}by + \dots + \binom{n+k-1}{k-1} x^n b^{k-1}y^{k-1}\right)$
$+ b^k\left(\binom{n+k-1}{k} a^{n-1}x^{n-1}y^k + \dots + b^{k-1}y^{n+k-1}\right) = a^n p + b^k q = 1$
$\Rightarrow (a^n, b^k) = 1$ proved

(12) $(a,b)=1 \Rightarrow ax+by=1$ so $(a+b, a-b) \Rightarrow (a+b, a-b) = d \Rightarrow d|(a+b)(1)+(a-b)(1)$
$\Rightarrow d|(a+b)(1)+(a-b)(-1)$ × $(a+b, a-b) = d$ so $d|a+b$ $= d|2a$
& $d|a-b$ so $a+b = dk_1$ & $a-b = dk_2$ so $2a = d(k_1+k_2)$ &
$2b = d(k_1-k_2)$ so $d|2a$ & $d|2b$ but as $(a,b)=1$, $d \nmid a$
and $d \nmid b$ so $d|2$, thus must be 1 or 2

(13.) Proof by MI: Consider $n=2$, div by prime as $2/2$,
Consider $n=3$, div by prime as $3/3$, Consider $n=4$, div by prime as $4/2$
so on as $n = 2, 3, 4, 5, \dots k$ is div by prime / PoP. Now we got to
prove that $k+1$ also div by P or PoP. so.
(Case i) $k+1$ is a prime: then $k+1 | k+1$ so $k+1$ div by prime
(Case ii): $k+1$ not prime: then $k+1 = rs$ such that $1 < r, s < k+1$
& as prod of prime, r & s div by prime, lets say, $p|r$ & $r|k+1$
& by transitivity of divisibility $p|r$ & $r|k+1 \Rightarrow p|k+1$
$\therefore$ every no >1 is p or PoP

(14) Assume finite prime no. Say N primes where $N \in \mathbb{Z}$
{ $\{p_1, p_2, \dots, p_N\}$, $p_i > 1$ $\forall i$ } × Consider $q = (p_1p_2p_3 \dots p_n)+1$
$\forall i$. As finite primes & $q >$ all, then q is not prime then $q/p_1 \in \mathbb{Z}$
say $p_1 \Rightarrow \frac{q}{p_1} = p_2p_3 \dots p_n + \frac{1}{p_1}$ so $p_2p_3 \dots \in \mathbb{Z}$ & $\frac{1}{p_1} \notin \mathbb{Z}$ & as $\frac{1}{p_1}$ is
rational, $q/p_1 \notin \mathbb{Z}$ thus, Contradiction. $\therefore \infty$ Primes.

(15) Let $d = (p, a)$ then $d|p$ & p is prime so $d = 1$ or p & also $d|a$ also given $p \nmid a$
$\therefore d \neq p$ so $d = 1$. Thus, $(p, a) = 1$

PAGE NO.
DATE

(16.) Suppose p/ab and $p \nmid a$ then proving p/b: Since $(p,a)=1$ then there exist x, y s.t. $px + ay = 1$ so $pxb + ayb = b$ & p/ab so p/bpx, p/ab & $p/ayb \Rightarrow p/b$

for more general stmt - proof by induction: when $n=1$, p/a_1; for $k=n$, $p/a_1a_2 \dots a_k$, p/a_i where $1 \le i \le k$ so true for $k+1$ here $p/a_1a_2 \dots a_ka_{k+1}$.

Case (i): $p/a_1a_2a_3 \dots a_k$ then p/a_i, $1 \le i \le k$
Case (ii): p/a_{k+1} then p/a_i, $i = k+1$ So p/a_i, $1 \le i \le k+1$ } ∴ proved

17. (i) $87 = 27 \times 3 + 6$, $27 = 6 \times 4 + \boxed{3}$ NZR, $6 = 3 \times 2 + 0$ } GCD: $\underline{\underline{3}}$

(ii) $418 = 165 \times 2 + 88$, $165 = 88 \times 1 + 77$, $88 = 77 \times 1 + \boxed{11}$ NZR

$77 = 11 \times 7 + 0$ } So GCD: $\underline{\underline{11}}$

(iv) $((50, 90), 240)$
$90 = 50 \times 1 + 40$, $50 = 40 \times 1 + \boxed{10}$ NZR, $40 = 4 \times 10 + 0$ → $(10, 240)$: $240 = 24 \times 10 + 10$

GCD: $\underline{\underline{10}}$

(v) $(((18, 24), 30), 38)$:
$24 = 18 \times 1 + \boxed{6}$ } $18 = 6 \times 3 + 0$ } $((6, 30), 38)$ $30 = 6 \times 5 + 0$ } $(6, 38)$ $38 = 6 \times 6 + \boxed{2}$ NZR } $6 = 2 \times 3 + 0$

GCD $\underline{\underline{2}}$

18. $1890 = 826 \times 2 + 238$ ① } $826 = 238 \times 3 + 112$ ② } $238 = 112 \times 2 + \boxed{14}$ NZR ③

$112 = 14 \times 8 + 0$ So $d = \text{GCD} = 14$. Now, $14 = 1890x + 826y$

③: $238 - 112 \times 2 + 14$ So on ⟵ $14 = 238 - 112(2) = (1890 - 2(826)) - 2(826 - 3(238))$

$= (1890 - 2(826)) - 2(826 - 3(1890 - 826(2)))$

$= 1890 - 2(826) - 2(826(2)) - 6(1890) + 6(826)$

$= 1890 + 2(826) \Rightarrow 7(1890) - 16(826)$ & $x = 7$, $y = \underline{-16}$ (= 14)

19. $g = (n, m)$ then $g = nx + my$ iff $\frac{n}{g}x + \frac{m}{g}y = 1 = \frac{1}{g}(g) = 1$ showed here

20. (i) (27, 87)

$27)87(3$, $\frac{81}{6}$, $87 = 27\times3+6$ ① } $6)27(4$, $\frac{24}{3}$, $27 = 6\times4+\boxed{3}$ ② } $3)6(2$, $\frac{6}{0}$, $6 = 3\times2+0$ } $3 = 27-6(4)$, $27-4(87-27(3))$, $27-4(87)+4(3)(27)$

$3 = 13(27) - 4(87)$ so $x = 13$ & $y = -4$

(ii) (165, 418)

$165)418(2$, $\frac{330}{88}$, $418 = 165\times2+88$ ① } $88)165(1$, $\frac{88}{77}$, $165 = 88\times1+77$ ② } $77)88(1$, $\frac{77}{11}$, $88 = 77\times1+\boxed{11}$ ③ } $11)77(7$, $\frac{77}{0}$, $77 = 11\times7+0$ ④

$11 = 88-77(1) = 418-2(165) - (165-88(1)) = 418-2(165) - [165-1(418-2(165))]$

$= 418 - 2(165) + (-165+1(418-2(165))) = 418-2(165)-165+418-2(165)$

$11 = 2(418) - 5(165)$ so $x = 2$ & $y = -5$

(iii) (252, 595)

$252)595(2$, $\frac{504}{91}$ ①, $595 = 252\times2+91$ | $91)252(2$, $\frac{182}{70}$ ②, $252 = 91\times2+70$ | $70)91(1$, $\frac{70}{21}$ ③, $91 = 70\times1+21$ | $21)70(3$, $\frac{63}{7}$ ④, $70 = 21\times3+\boxed{7}$ NZR

$7)21(3$, $\frac{21}{0}$ ⑤, $21 = 7\times3+0$ | $n-1$th $\Downarrow$ 4 } $7 = 70-21(3) = 252-91(2) - 3(91-70(1))$
$= 252 - 2[595-2(252)] - 3(595-2(252) - 252-2(91))$

$= 252-2[595-2(252)] - 3(595-2(252) - 252 - 2[595-2(252)])$

$= 252 - 2(595) + 4(252) - 3(595) + 6(252) + 3(252) + 6(595-2(252))$

$= 252-2(595)+4(252) - 3(595) + 6(252) + 3(252) + 6(595) - 12(252)$

$\frac{14}{2}$ $7 = 2(252)$ ✗ as $x = 26$, $y = -11$ so $26(252) + (-11)(595)$ (?)

21. Ⓐ Any eqn whose integer soln are sought ⇒ DEs and a DEs form $ax+by = c$ where $a, b, c \neq 0 \in \mathbb{Z}$ is LDE Ex $4x+3y = 2$

22. ⊛ Defn + Condn: $ax+by = c$ has soln iff c is multiple of $\gcd(a,b)$ (or d/c).

23. (i) $12x + 18y = 30 \longrightarrow$ $12)18(1$, $\frac{12}{6}$, $18 = 12\times1 + \boxed{6}$; $6)12(2$, $12 = 6\times2+0$ } $\gcd = 6$ Yes

So $\gcd(a,b) * k_1 = 30$ } So $6\times5 = 30$

(ii) $2x + 3y = 4 \longrightarrow$ $2)3(1$, $\frac{2}{1}$, $3 = 2\times1+\boxed{1}$ NZR; $1)2(2$, $\frac{2}{0}$, $2 = 2\times1+0$ } $\gcd(2,3)=1$ Yes

So $\gcd(2,3) * k_1 = 4$ } So $1 * 4 = 4$

(iii) $6x + 8y = 25 \longrightarrow$ $6)8(1$, $\frac{6}{2}$, $8 = 6\times1+2$ } $2)6(3$, $\frac{6}{0}$, $6 = 3\times2+0$ } No

So $\gcd(a,b) * k_1 = 25$ } $2 * x = 25$

(iv) $12x + 16y = 18 \longrightarrow$ $12)16(1$, $\frac{12}{4}$, $16 = 12\times1+\boxed{4}$ NZR } $4)12(3$, $\frac{12}{0}$, $12 = 3\times4+0$ } No

So $\gcd(a,b) * k_1 = 18$ } $(2\times2) * x = 18$

(v) $2x + 4y = 5 \longrightarrow$ $2)4(2$, $\frac{4}{0}$, $\boxed{2}\times2+0 = 4$ } $2 * x = 5$ So No

$\gcd(a,b) * k_1 = 5$

24. (i) $28x + 91y = 119$ $\longrightarrow$ ① $28)91(3$, $\frac{84}{7}$, $91 = 28\times3+\boxed{7}$ NZR ② $7)28(4$, $\frac{28}{0}$, $28 = 7\times4+0$ } $\gcd = 7$ Yes, $7\times17 = 119$

$\gcd(a,b) * k_1 = 119$

So $x = x_0 + \left(\frac{b}{d}\right)t$, $y = y_0 - \left(\frac{a}{d}\right)t$ } $\longrightarrow$ Eqn

$7 = 91 - 28(3)$

So $7 = 1(91) + (-3)28$ } $119 = 17(91) + (-51)28$

multiply by $(\times17)$; y_0, x_0

$\{ a = 28, b = 91, d = 7$

$x = x_0 + \left(\frac{b}{d}\right)t \longrightarrow x = -51 + 13t$

$y = y_0 - \left(\frac{a}{d}\right)t \longrightarrow y = 17 - 4t$

$(t \in \mathbb{Z})$

(ii) $63x - 23y = -7 \longrightarrow$ $23)63(2$, $\frac{46}{17}$ ①, $63 = 23\times2+17$ | $17)23(1$, $\frac{17}{6}$ ②, $23 = 17\times1+6$ | $6)17(2$, $\frac{12}{5}$ ③, $17 = 6\times2+5$

So $\gcd(a,b) * k_1 = -7$

$5)6(1$, $\frac{5}{1}$, $6 = 5\times1+\boxed{1}$ — ④ } exact $\gcd$ { $\gcd(a,b) = 1 * -7 = -7$ eqn

$1 = 6 - 5(1) = 23 - 17(1) - 1(17 - 2(6))$

$= 23 - 1(63 - 2(23)) - 1(63 - 2(23) - 2(23 - 1(63 - 2(23))))$

~~$23 - 1(63 - 2(23)) - 1(63 - 2(23) - 2(23 - 1(63 - 2(23)))) = 1$~~

~~$1 = (23 - 63(1) + 2(23) - 63(1) + 2(23) - 2(23 - 63(1) + 2(23)))$~~

~~$1 = (\quad - 2(23) + 2(63) - 4(23))$~~

$1 = 6 - 5(1) = 23 - 17(1) - 1[17 - 2(6)] = 23 - 2[63 - 2(23)] + 2[23 - 17(1)]$

$= 23 - 17(1) - 17 + 2(6)$

$= 23 - 2(17) + 2(6)$ $= 23 - 2(63) + 4(23) + 2(23) - 2(17)$

$= 7(23) - 2(63) - 2[63 - 2(23)] = 7(23) - 2(63) - 2(63) + 4(23)$

a b

63 −23 $\quad 1 = 11(23) - 4(63) \Rightarrow 1 = 23(11) + (-4)63$ $\times -7$ both

$-7 = (-77)23 + (28)63$ So $x_0 = 28$, $y_0 = -77$ } $x = 28 - 23t$, $y = -77 - 63t$

x_0 y_0

25. (i) $1485x + 1745y = 15$ $ax+by=c$ LDE, $\gcd(a,b)^* k_1 = c$

$1485)1745(1$, 1485, 260 — (1) — $1745 = 1485\times1 + 260$

$260)1485(5$, 1300, 185 — (2) — $1485 = 260\times5 + 185$

$185)260(1$, 185, 75 — (3) — $260 = 185\times1 + 75$

$75)185(2$, 150, 35 — (4) — $185 = 75\times2 + 35$

$35)75(2$, 70, 5 — (5) — $75 = 35\times2 + \boxed{5}$ NZR

$5)35(7$, 35, 0 — (6) — $35 = 7\times5 + 0$

$5 = 75 - 2(35)$ & $5^* k_1 = 15$, $k_1 = 3$

~~$5 = 260 - 1(185)$ $5 = 260 - 180(2)$~~

$5 = 260 - 185(1) - 2(185 - 2(75)) = 260 - 1(185) - 2(185) + 4(75)$

$\leftarrow 260 - 3(185) + 4(75)$

$1745 - 1(1485) - 3(1485 - 260(5)) + 4(260 - 1(185))$

$x = x_0 + \frac{b}{d}t$

$y = y_0 - \frac{a}{d}t$

$1745 - 1(1485) - (1485)3 + 15(260) + 4(260) - 4(185)$

$= 1745 - 4(1485) + 19(260) - 4(185) = 1745 - 4(1485) + 19(1745 - 1(1485)) - 4(1485 - \ldots)$

$= 1745 - 4(1485) + 19(1745) - 19(1485) - 4(1485) + 20(260)$

a, b — 1485, 1745

$= 1745(20) - 27(1485) + 20(1745 - 1485(1)) = 1745(20) - 27(1485) + 20(1745) - 20(1485)$

$5 = 40(1745) - 47(1485)$ $\times 3$ bs $\Rightarrow 15 = 1485(-141) + 1745(120)$

$x_0 = -141$, $y_0 = 120$

$x = -141 + 349t$ & $y = 120 - 297t$ $\{t \in \mathbb{Z}$

26. (A) Let $m = \mathbb{Z}^+$ & $m > 1$ & if 'a' & 'b' are any integers then a is said to be congruent to b modulo m iff $m/(a-b)$. Symbol $a \equiv b \pmod m$.

27. (A) A congruence of the form $ax \equiv b \pmod m$ where x is solvable $\Rightarrow$ LC in 1 variable

Rel^n [A linear diophantine eqn of form $ax + by = c$ can be written as $ax \equiv c \pmod b$] X $ax \equiv b \pmod m$ any LC, then $\left(\frac{ax-b}{m}\right)$, & $ax - my = b$ LDE

28. (A) Euler's totient fn denoted by $\phi(n)$ is defined as no. of $\mathbb{Z}^+$ less than or equal to n & relatively prime to n. $\phi(n) \leq n$

Ex $n = 4$ then $\phi(4) = \{1, 3\} = 2$

29. (A) Prop (i) If n is a prime no, $\phi(n) = n-1$ as Ex $\phi(7) = \{1,2,3,4,5,6\}$

(ii) If n is a Pop $\phi(n) = \phi(pq) = \phi(p) \cdot \phi(q)$

(iii) If ϕ is a multiplicative fn, i.e. $(m,n) = 1$ then $\phi(mn) = \phi(m) \cdot \phi(n)$

$\phi(n) = \phi(pq) = \phi(p)\,\phi(q) = (p-1)(q-1)$

35 (i) $27x \equiv 11 \pmod 7$: $ax \equiv c \pmod b$ & $(27,7) = 1 = d$ & d/c ✓

$27x + 7y = 11$ & $6x \equiv 4 \pmod 7$ — in the set $\{0,1,2,3,\ldots,6\}$ only 1 x will satisfy

$0x \equiv 14 \pmod 7$

$6x \equiv 6 \times 3 \pmod 7$ (Cancellation law)

$x \equiv 3 \pmod 7$. Thus, sol^n $x = 3 + 7k$, $k \in \mathbb{Z}$ {incongruent sol^n ...}

{$3 \in \{0,1,2,3,4,5,6\}$} So $\underline{x = 3}$ (IS) & GS $\underline{x = 3 + 7k}$

(ii) $36x \equiv 96 \pmod{156}$: LDE check $ax \equiv c \pmod b \rightarrow (a,b) = d/c$ $(36, 156) =$

$36) 156 ($

(ii) $37x \equiv 5 \pmod{11}$: LDE check $ax \equiv c \pmod b$ $(37,11) = d/c = 1/5$ ✓

$11) 37 (3$, 33, 4 — $11 \times 3 + 4$; $4) 11 (2$, 8, 3 — $4 \times 2 + 3$; $3) 4 (1$, 3, $\boxed{1}$ } So, incongruent sol^n exists

LDE conversion: $37x + 11y = 5$

and exactly one from $\{0,1,2,\ldots,10\}$ will satisfy the congruence

as $36, 5 \nmid 11$, dont divide 5 So $4x \equiv 5 \pmod{11}$

$0x \equiv 11 \pmod{11}$

$x \equiv 4 \pmod{11}$ ← $4x \equiv 16 \pmod{11}$ {$4 \in \{0,\ldots,10\}$}

So $x = 4 + 11k$, $k \in \mathbb{Z}$

incongruent sol^n = $\underline{4}$ { GS of LC: $\underline{x = 4 + 11t,\ t \in \mathbb{Z}}$

(iv) $36x \equiv 96 \pmod{156}$ [LDE] $(36, 156) = d$ and $d | c$

So $d = 12$ and $12 | 96$ ✓

Now, [Cong. Class] Sol^n from $\{0,1,\ldots,155\}$, So incongruent sol^n ✓

$\boxed{\text{one}}$ x will satisfy the congruence as an LDE: $36x + 156y = 96$

↳ 12 as $d = 12$ So $a, b \nmid 156$ So $\div 12$

$3x \equiv 8 \pmod{13}$ → $x \equiv 7 \pmod{13}$ So IS $x_0 = 7$

$0x \equiv 13 \pmod{13}$ So $x = x_0 + \frac{b}{d} t$

$\cancel{3}x \equiv \cancel{3} \times 7 \pmod{13}$ (CL) $x = 7 + 13t$ GS → $t \in \mathbb{Z}$ & put $0, 1, 2, \ldots 11$

Sol^n of congruence $x = 7 + 156k$, $x = 20 + 156k$ ← $x = 7, 20, 33, \ldots$

(iii) $15x \equiv 12 \pmod{36}$: [LDE] $ax \equiv c \pmod b$ & $(15, 36) = d/c$

$(15, 36) = 3$ and $d/12$ ✓ as $3/12$ and LDE $15x + 36y = 12$

[CC]: 3 sol^{n}s out of $\{0,1,2,\ldots,35\}$ So as $a, b \nmid 36$, $\div 3$ (IS)

(IS) $5x \equiv 4 \pmod{12}$, $0x \equiv 36 \pmod{12}$ } $x \equiv 8 \pmod{12}$ So $x_0 = 8$

$x = x_0 + \left(\frac{b}{d}\right)t$ so $x = 8 + 12t$; gs and $t = \{0, 1, 2\}$ so $x = 8, 20, 32$ thus IS $8, 20, 32$ and soln of congruence: $x = 8+36k$, $x = 20+36k$, $x = 32+36k$, $k \in Z$

36. (i) Wkt $ax + by = c \Rightarrow ax \equiv c \pmod b$ } ~~wkt $ax \equiv b \pmod m$ ax by c then $ax \equiv c \pmod{}$ but $m \mid ax - b$ so $ax - my = b$~~
(3, 0) d/c 1/4 ✓ so $ax \equiv b \pmod m$ then $ax - my = b$
thus, $3x \equiv 4 \pmod 5$
wkt iff $m \mid (a-b)$ so $5 \mid 3x - 4$ so $3x - 5y = 4 \rightarrow$ LDE $y \in Z$

(27, 7) d/c 1/11 ✓ (ii) $27x \equiv 11 \pmod 7$ — $a \equiv b \pmod m$ iff $m \mid (a-b)$; $m > 1$, $a, b \in Z$; $ax \equiv b \pmod m$ $x \rightarrow$ var
so $7 \mid 27x - 11$ so ~~$27x + 11$~~ $27x - 7y = 11 \rightarrow$ LDE $y \in Z$

(8, 11) d/c 1/5 ✓ (iii) $8x \equiv 5 \pmod{11}$ so $11 \mid 8x - 5$ so $8x - 11y = 5$ $y \in Z \rightarrow$ LDE

37. (a) $15x \equiv 12 \pmod{36}$:
[LDE:] $(15, 36) = 3$ and $3 \mid 12$ ✓ and LDE: $36 \mid 15x - 12$ so $15x - 36y = 12$ $y \in Z$
[CC:] There are 3 IS (d=3) in $\{0, 1, \ldots, 35\}$
and as $15, 12 \not> 36$, $\div 3$ — $5x \equiv 4 \pmod{12}$, $0x \equiv 36 \pmod{12}$, $x \equiv 8 \pmod{12}$ → PS $x_0 = 8$
$x = x_0 + \frac{b}{d}t$
$x = 8 + 12t$ gs $t \in Z$ $= \{0, 1, 2\}$
$x = 8, 20, 32$ IS
{ soln of congruence: $x = 8 + 36k$, $x = 20 + 36k$, $x = 32 + 36k$ } soln, $k \in Z$

(b) $8x \equiv 10 \pmod 6$:
[LDE]: $(8, 6) = 2$ and $2 \mid 10$ ✓ and LDE: $8x - 6y = 10$, $y \in Z$
[CC]: There are 2 Z sol IS in $\{0, 1, \ldots, 5\}$
and as $8, 10 > 6$, $\div 6$ — $2x \equiv 4 \pmod 6$, $0x \equiv 6 \pmod 6$, $x \equiv 5 \pmod 6$ → PS $x_0 = 5$ ~~$x = x_0 + \left(\frac{b}{d}\right)t$, $t \in \{0,1\}$, $x = 5 + 3t$, $x = 5$ and $x = 8$~~
as $d \neq 1$, $\div d$
$4x \equiv 5 \pmod 3$
$0x \equiv 3 \pmod 3$
$4x \equiv 8 \pmod 3$ → $x \equiv 2 \pmod 3$, $x_0 = 2$ PS
$x = x_0 + \frac{b}{d}t = 2 + 3t$ $t \in \{0, 1\}$
$x = 2, 5$ IS
$x = 2 + 3t$ gs $t \in \{0, 2\}$ $\in Z$ and soc: $x = 2 + 6k$ { $x = 5 + 6k$

33. Let $m_1, m_2, \ldots, m_n$ be $\in \mathbb{Z}^+$ & $\gcd(m_i, m_j)=1$ then the sys of l.c.

$x \equiv a_1 \pmod{m_1}$

$x \equiv a_2 \pmod{m_2}$ has unique solⁿ modulo M

$x \equiv a_3 \pmod{m_3} \ldots x \equiv a_n \pmod{m_n}$ $\quad M = m_1 m_2 \ldots m_n$

$$x \equiv (a_1 M_1 M_1^{-1} + a_2 M_2 M_2^{-1} + \ldots + a_n M_n M_n^{-1}) \pmod{M}$$

where $M_i = M/m_i$ & $M_i^{-1} M_i \equiv 1 \pmod{m_i}$

8. (i) $x \equiv 2 \pmod 5$, $x \equiv 3 \pmod 7$, $x \equiv 4 \pmod{11}$

$a_1 = 2;\ a_2 = 3;\ a_3 = 4$ & [$m_1, m_2, m_3 = 5, 7, 11$ & $(m_i, m_j) = 1$

then sys 3 eqns have unique solⁿ modulo M

$x \equiv (a_1 M_1 M_1^{-1} + a_2 M_2 M_2^{-1} + a_3 M_3 M_3^{-1}) \pmod M$ & $M = m_1 m_2 m_3$

$M_i = \frac{M}{m_i}$] So $m_1 = 5, m_2 = 7, m_3 = 11$ & $(m_1, m_2) = (m_2, m_3) = (m_1, m_3) = 1$

then sys (eqns) have unique solⁿ modulo $M = 5 \times 7 \times 11 = 385$

& $a_1 = 2, a_2 = 3, a_3 = 4$ & $M_1 = \frac{M}{m_1} = \frac{385}{5} = 77$, $M_2 = 55$, $M_3 = 35$

$M_i M_i^{-1} \equiv 1 \pmod{m_i} \Rightarrow M_1 M_1^{-1} \equiv 1 \pmod 5$ So $77 M_1^{-1} \equiv 1 \pmod 5$

$5 / 77x - 1$ So $77x - 5y = 1$ So $x = \frac{1 + 5y}{77}$ so $\frac{77x - 1}{5} = y \in \mathbb{Z}$

find $x \in \mathbb{Z}$ for $x \to 3 = M_1^{-1}$

& $M_2 M_2^{-1} \equiv 1 \pmod{m_2}$ So $M_2 \cdot M_2^{-1} \equiv 1 \pmod 7$ or $7/(55x - 1)$ so

$55x - 7y = 1$ or $\frac{55x - 1}{7} = y$ for M_2^{-1}; $x \to 6$ & $M_3 M_3^{-1} \equiv 1 \pmod m$

So $35 \cdot M_3^{-1} \equiv 1 \pmod{11}$

$11 / (35x - 1)$ So $35x - 11y = 1$ So $y = \left(\frac{35x - 1}{11}\right)$ $x \to 6$ So $M_3^{-1} = 6$

Now, $x = (2 \times 3 \times 77 + 55 \times 6 \times 3 + 4 \times 6 \times 35) \bmod (385)$

$x = 2292 \bmod (385)$ So $x_0 = 2292$ (common solⁿ)

and $x = 2292 + 385K$ $\quad K \in \mathbb{Z}$ G.S

(e1) & (e2) Same with sys of eqns in (ii)

(e3) $17x \equiv 9 \pmod{84}$ $84 = 3 \cdot 4 \cdot 7$ So $17x \equiv 9 \pmod 3$

$17x \equiv 9 \pmod 4$, $17x \equiv 9 \pmod 7$ [& for $m_1 m_2 \ldots$ are such that

$\gcd(m_i, m_j) = 1$ then sys have unique solⁿ $x \equiv (a_1 M_1 M_1^{-1} + \ldots$

$+ a_n M_n M_n^{-1}) \pmod{M}$ such that $M = m_1 m_2 \ldots m_n$ & $M_i = \frac{M}{m_i}$]

So $m_1, m_2, m_3 = 3, 4, 7$ & $\gcd(3,4) = \gcd(4,7) = \gcd(3,7) = 1$

then $a_1, a_2, a_3 = 9, 9, 9$ & $M = 84$ & $M_1 = \frac{84}{3}$ $M_2 = \frac{84}{4}$ $M_3 = \frac{84}{7}$

So $M_1 M_1^{-1} \equiv 1 \pmod 3$ ... M_1^{-1} ... M_2^{-1} ... $= 28$ $= 21$ $= 12$

$x = (9\times1\times ... + 9\times1\times ... + 9\times3\times2) \bmod (84) \Rightarrow x = 653 \pmod{84}$

So $x_0 = 653$ is soln of $17x$ ≡ not x

[So $m_1, m_2, \ldots, m_n$ ∃ $\gcd(m_i, m_j) = 1$ then for sys of eqn has unique soln $x = (a_1 M_1 M_1^{-1} + \ldots + a_n M_n M_n^{-1}) \pmod M$ ∋ $M = m_1 m_2 \ldots m_n$ & $M_i = \frac{M}{m_i}$ and $M_i M_i^{-1} \equiv 1 \pmod{m_i}$]

$17x \equiv 9 \pmod 3$ $17x \equiv 9 \pmod 4$ $17x \equiv 9 \pmod 7$ (× 17^{-1} b.s)

$x \equiv$ ... (?)

So new eqn: $2x \equiv 6 \pmod{14}$, $3x \equiv 9 \pmod{15}$, $5x \equiv 20 \pmod{60}$

$x \equiv 3 \pmod 7$, $x \equiv 3 \pmod 5$, $x \equiv 4 \pmod{12}$

[$m_1, m_2, \ldots, m_n$ & $(m_i, m_j) = 1$ then for sys of eqn has US $x \equiv (a_1 M_1 M_1^{-1} + \ldots + a_n M_n M_n^{-1}) \pmod M$, where $M = m_1 m_2 \ldots m_n$ & $M_i = \frac{M}{m_i}$ & $M_i M_i^{-1} \equiv 1 \pmod{m_i}$] So $7, 5, 12 = m_1, m_2, m_3$ & $(7,5) = (5,12) = (7,12) = 1$

& $(a_1, a_2, a_3) = (3, 3, 4)$ & $M_1 = \frac{420}{7} = 60$ $M_2 = \frac{420}{5} = 84$ $M_3 = \frac{420}{12} = 35$

$M = 5 \times 7 \times 12 = 420$

$M_1 M_1^{-1} \equiv 1 \pmod 7$ so $7 / M_1 x - 1$ so $M_1 x - 7y = 1$ or $\frac{M_1 x - 1}{7} = y$ so $M_1^{-1} = 2$

So $x = (3 \times 60 \times 2 + 4 \times 3 \times 84 + 35 \times 11 \times 4) \pmod{420}$ $M_2^{-1} = 4$ $M_3^{-1} = 1$

$x \equiv 2908 \pmod{420}$ $x_0 = 2908$ is & $x = 2908 + 420k$, $k \in \mathbb{Z}$

32. Stmt: If p is a prime, then $(p-1)! \equiv -1 \pmod p$

proof: Given p is prime. for $p = 2$, $(2-1)! \equiv -1 \pmod p$

So $2 / (1+1) = 2/2$ ✓ So true for $p = 2$. Suppose $p > 2$, then the regular residues RRR $\{1, 2, 3, \ldots, p-1\}$ are invertible mod p. Here 1 & $(p-1)$ are self invertible mod p. So rest $p-3$ residues which are $\{2, \ldots, p-2\}$ & form $\frac{p-3}{2}$ pairs of inverses a & $b \Rightarrow ab \equiv 1 \pmod p$ for every pair of a & b thus $2 \cdot 3 \ldots (p-2) \equiv 1 \pmod p \Rightarrow 1 \cdot 2 \cdot 3 \ldots (p-2)(p-1) \equiv 1 (p-1) \pmod p$

So $(p-1)! \equiv (p-1) \pmod p$

But we know $p - 1 \equiv -1 \pmod p$ So $(p-1)! \equiv -1 \pmod p$ proved

Ex: $p = 13$: $(13-1)! \equiv -1 \pmod{13}$ So $m / (12! + 1) = 13 / (12! + 1)$ ✓

proof: $p = 2$, ✓ for $p > 2$ here 13 then its RRR $\{1, 2, \ldots, 12\}$ is inv mp but 1 & 12 are self inv mp So $p - 3 = 10$ & $\frac{10}{2} = 5$ pairs are inverses ∋ $ab \equiv 1 \pmod p$ so $2, 3, 4, 5, 6, 7, 8, 9, 10, 11$ & $2 \cdot 7 \equiv 1 \pmod p$ & so on

$2 \cdot 3 \ldots 11 \cdot 12 \equiv -1 \pmod{13}$ proved.

PAGE NO
DATE

Ⓐ Pt a is self inv iff $a \equiv \pm 1 \pmod p$ & $a \in \mathbb{Z}$.
Suppose a is self invertible, $a^2 \equiv 1 \pmod p$ $p / a^2 - 1 = p / (a+1)(a-1)$
$\Leftrightarrow a \equiv 1 \pmod p$ or $a \equiv -1 \pmod p$ so $\Rightarrow a \equiv \pm 1 \pmod p$ Proved

Ⓐ Stmnt: Let p be a prime & a any $\mathbb{Z}$ such that $p \nmid a$ then $a^{p-1} \equiv 1 \pmod p$
Proof: By DA, $a = pq + r$ and $0 \le r \le p-1$ (as $0 \le r < p$) as
$p \nmid a$, $(p,a) = 1$ so, $1 \le r \le p-1$ (p is prime) (a is any $\mathbb{Z}$)
i.e. when $p \nmid a$, r is remainder b/w 1 & $p-1$ so Ka will leave from 1 to $p-1$
$1 \le K \le p-1$ i.e. $a, 2a, 3a, \dots (p-1)a / p$ will leave rem $1, 2, 3, \dots (p-1)$
So, $a \cdot 2a \cdot 3a \cdots (p-1)a \equiv 1 \cdot 2 \cdot 3 \cdots (p-1) \pmod p$ so $a^{p-1}(p-1)! \equiv (p-1)! \pmod p$;
By cancellation law, $a^{p-1} \equiv 1 \pmod p$ proved
Ex: $p = 7$ & $a = 10$ so $a(1 \cdot 2 \cdot 3 \cdots 6) \equiv 1 \cdot 2 \cdot 3 \cdots 6 \pmod 7$ so $a^6 6! \equiv 6! \bmod 7$
$a^6 \equiv 1 \pmod 7$ proved or $10^6 \equiv 1 \pmod p$

42. Ⓐ By FLT, $a^{p-1} \equiv 1 \pmod p$ so $a \cdot a^{p-2} \equiv 1 \pmod p$
so we can see that a^{p-2} is inverse of $a \pmod p$ Showed

43. Ⓐ $ax \equiv b \pmod p$ has a unique sol^n as $p \nmid a$ or $(a,p) = 1$ & $1 / b$ ✓ (a/c)
$a^{p-2}(ax) \equiv a^{p-2} \cdot b \pmod p \Rightarrow (a^{p-1})x \equiv a^{p-2} \cdot b \pmod p$ —(i)
& By FLT, $a^{p-1} \equiv 1 \pmod p$ so, $1 \cdot x \equiv a^{p-2} \cdot b \pmod p$
$\Rightarrow x \equiv a^{p-2} b \pmod p$ Showed

39. (iii) 3^{181} div by 17, $r = ?$
By FLT, wkt $a^{p-1} \equiv 1 \pmod p$ so, $3^{(180+2)-1} \equiv 1 \pmod{182}$
and wkt $a = 3$ here so $0 \le r \le 180$ and ak, $K \in [0, 180]$ too
$p = 17$ so $3 = a$ & $a^{p-1} \equiv 1 \pmod p$ so $3^{16} \equiv 1 \pmod{17}$
and $3^{181} = 3^{16 \cdot 11 + 5} = 3^{16 \cdot 11} \cdot 3^5$ so $\equiv (3^{16})^{11} \cdot 3^5 \equiv (1)^{11} \cdot 3^5 \pmod{17}$
$\equiv 243 \pmod{17} \equiv 5 \pmod{17}$ so Remainder $= 5 \in [0, p-1 = 16]$

(iv) 3^{289} div by 23, $r = ?$
$a = 3$, $p = 23$ so By FLT, $a^{p-1} \equiv 1 \pmod p$ so $3^{22} \equiv 1 \pmod{23}$ —(i)
and $3^{289} = 3^{22 \cdot 13 + 3} = 3^{22 \cdot 13} \cdot 3^3 \equiv (1)^{13} \cdot 3^3 \pmod{23} \equiv 27 \pmod{23}$
$\equiv 4 \pmod{23}$ so Remainder $= 4$; $0 \le r \le 22$ so $4 \in [0, 22]$

(7) $16^{53} \div 7$, $r = ?$

By FLT, $a^{p-1} \equiv 1 \pmod{p}$ & $a = 16$ & $p = 7$ so $a^6 \equiv 1 \pmod 7$ and

$16^{53} = 16^{6\cdot 8+5} = (16^6)^8 \cdot 16^3 \equiv (1)^8 \cdot 16^5 \pmod 7 \equiv 1048576 \pmod 7$

$\equiv 4 \pmod 7$ so $r = 4$

60. (a) $27x \equiv 11 \pmod 7$: so $a = 27$, $p = 7$ then

By FLT: $27^6 \equiv 1 \pmod 6$? ✗ First, $(27, 7) = 1 = d$ & $d/11$ ✓ so $\checkmark$ soln

$a^{p-1} \equiv 1 \pmod p$ $\div m$ & replace $6x \equiv 4 \pmod 7$

$\oplus x \equiv 14 \pmod 7$

$a = 6$ & $p = 7$ so $6^6 \equiv 1 \pmod 7$ ← $6x \equiv 18 \pmod 7$

& $6 \cdot 6^5 \equiv 36 \pmod 7$

$6^5 \equiv 6 \pmod 7$ $\quad$ $6x \equiv 4 \pmod 7$

$\hookrightarrow 27x \cdot 6 \equiv 11 \cdot 6 \pmod 7$ $\times 6$

1] $(27, 7) = 1 = d$ & d/c so $1/11$ ✓ soln

2] FLT, $a^{p-1} \equiv 1 \pmod p$

(ii) $12x \equiv 6 \pmod 7$:

1] $(12, 7) = 1$ so $1/6$ ✓ 1 soln exists

2] FLT: $a^{p-1} \equiv 1 \pmod p$ and $a = 12$ and $p = 7$

so $12^6 \equiv 1 \pmod 7$ $\quad$ b.s by x

$x \equiv x 12^6 \pmod 7 \longrightarrow x \equiv 12^5 \cdot 12x \pmod 7 \rightarrow x \equiv 12^5\, 6 \pmod 7$

$12^5 \equiv y \pmod 7$ & $7 / 12^5 - y$ so $12^5 - 7K = y$

or $\dfrac{12^5 - y}{7} = K$ $\quad$ or $5x \equiv 6 \pmod 7$

$\oplus\ x \equiv 14 \pmod 7$ $\quad$ $x \equiv 4 \pmod 7$

$\frac{x \times x}{7} = 4$ or 3 ✓ $\quad$ $5x \equiv 20 \pmod 7$

so $12^5 \equiv 3 \pmod 7$ $\quad$ Substitute, $x \equiv 3 \times 6 \pmod 7$

3] $x \equiv 18 \pmod 7$ $\quad \div 7$ & $r \Rightarrow x \equiv 4 \pmod 7$

(e) $4x \equiv 11 \pmod{19}$

1] $d = (4, 19) = 1$ so d/c, $1/11$ ✓ so soln ✓

2] FLT $a^{p-1} \equiv 1 \pmod p$ so $a = 4$, $p = 19$ now $4^{18} \equiv 1 \pmod{19}$

so, $x \equiv 4^{18} x \pmod{19} \Rightarrow x \equiv 4^{17} \cdot 4x \pmod{19} \Rightarrow x \equiv 4^{17} \cdot 11 \pmod{19}$

$4x \equiv 11 \pmod{19}$ $\quad$ $4^{17} \equiv 5 \pmod{19}$ Now,

$\oplus\ x \equiv 57 \pmod{19}$ $\quad x \equiv 17 \pmod{19}$ $\quad$ $x \equiv 5 \times 11 \pmod{19}$ & $\div 19$

$4x \equiv 68 \pmod{19}$ $\quad \frac{x \times 11}{19} = y$ $\quad$ $x \equiv 55 \pmod{19}$

$19 - y = 17$ $\quad$ or $x \equiv 17 \pmod{19}$

DATE / /

44. $n = pq = 19939$ & $\phi(n) = 19656$, p, q ?

So $\phi(n) = \phi(pq) = \phi(p) \cdot \phi(q) = (p-1)(q-1)$

If $n = pq$ ($p \ne p$) so $19656 = \phi(n) = \phi(pq) = \phi(p)\phi(q)$
$= (p-1)(q-1)$
$= pq - p - q + 1$
$= pq - (p+q) + 1$

Sum & product: $19656 = 19939 - (p+q) + 1$

$19655 - 19656 = p+q$

$p + q = 284$ So $(x-p)(x+q)$ $(p+q)^2 = p^2 + q^2 + 2pq$

$x^2 - (p+q)x + pq = 0$ So $x^2 - 284x + 19939 = 0$

$x = 157 = p$ and 127 So $p = 157$ & $q = 127$

33. Stmt: If $m \in Z^+$ & a any integer such that m/a to $a^{\phi(m)} \equiv 1 \pmod m$

Proof: Let $a_1, a_2, \ldots a_{\phi(m)}$ be least +ve $<$PR that are Rel. prime with m.

And as $(a, m) = 1$

$a \cdot a_1, a \cdot a_2, \ldots, a a_{\phi(m)}$ are congruent to $a_1, a_2, \ldots, a_{\phi(m)}$ in same order

$(a a_1)(a a_2)(\ldots)(a a_{\phi(m)}) \equiv (a_1 a_2 \ldots a_{\phi(m)}) \bmod (m)$

$a^{\phi(m)} \equiv 1 \pmod m$ By Canc. law

34. $d = \frac{k\phi(n)+1}{j}$ & $\phi(p) = p-1$ So $\frac{k \times 10146 + 1}{119} = d \in Z$ So $k = 23$, $d = 1961$ (?)

Public Key $= \langle n, j \rangle = \langle 10147, 119 \rangle$ $C \equiv M^e \pmod n$

C = cipher $D \equiv C^d \pmod{\phi}$

$\langle n, j \rangle = \langle 10147, 119 \rangle$ and $D \equiv C^d \pmod n$

for d: $d \cdot j = 1 \pmod{\phi(n)}$ and for $\phi(n) - \phi(10147)$

73
139

$d \cdot 119 = 1 \pmod{9936}$ $= \phi(pq) - \phi(p)\phi(q)$

$9936 / (119d - 1)$ $= (p-1)(q-1)$

$= \phi(73 \times 139) = 72 \times 138 = 9936$

or $119d - 9936y = 1$

a=097
A=065
Space=032

$y = \left(\frac{119d - 1}{9936}\right) (\in Z)$ or $\frac{9936y + 1}{119} = d \to 167$; $y \to 2$

So $D \equiv C^{167} \pmod{\cancel{9936}}$ $D \equiv C^{167} \pmod{10147}$ (?)

45. $\langle n, j \rangle = \langle 8633, 5 \rangle$ & $D = C^d \pmod n$ & $dj \equiv 1 \pmod{\phi(n)}$

$\phi(n) - \phi(pq) = \phi(p) \cdot \phi(q) = (p-1)(q-1)$

$C = 69^5 \pmod{8633}$
$= 8005$

ENSIGHT = {069, 078, 065, 083, 073, 071, 072, 084}

$\equiv^{14}$. Calculate c for every letter, c = { 8005 ——— 3

46. 4611 4658 1493 RSA $n = 8051 = 83 \times 97$ & $j = 5$

So. $D = C^d \pmod n$ & $d \cdot j \equiv 1 \pmod{\phi(n)}$ & $\phi(n) = \phi(8051)$

$= \phi(83 \times 97)$

$= \phi(83)\,\phi(97)$

$= 82 \times 96$

$= 7872$

So $d \cdot 5 \equiv 1 \pmod{7872}$

$7872 / 5d - 1$ so $5d - 7872y = 1$

$d = \frac{1 + 7872y}{5}$ so $d = 3149$ $(y=2)$

Now. $D \equiv C^{3149} \pmod{8051}$ & $C = \{4611\ 4658\ 1493\}$

$C \equiv r^j \pmod{8051} \Rightarrow 4611 \equiv r^5 \pmod{8051}$

$m = \{67\ 65\ 82\}$ CAR

My grade in MAC11: O grade / S grade

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Thanks :>