$= (\alpha|0\rangle + \beta|1\rangle)$

$\Downarrow$ ȳ written ay

$X \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$

$= \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \beta \\ \alpha \end{bmatrix}$

$= \beta|0\rangle + \alpha|1\rangle$

pauli Matrices

sigma $\sigma_0 = I = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$

$\sigma_1 = \sigma_x = X = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$

$\sigma_2 = \sigma_y = Y = \begin{vmatrix} 0 & -i \\ i & 0 \end{vmatrix}$

$\sigma_3 = \sigma_y$

Quantum gates applications

X – works like a classical NOT ~~Gat~~
– Bit flip (used to inverting qubit values
– Building block for universal gates

Z ~~P~~ = ~~flip the phase together of~~ ~~Bit~~ phase shift ~~together of~~ $|1\rangle$ flip the phase of $|1\rangle$
= ~~use~~ Important for phase estimation algo's, error correction
– (stabilizer codes) & building controlled operations

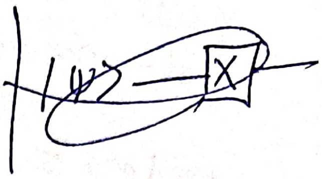
~~Y~~ = Bit & phase flip together
= used in certain error – correction codes & Block sphere rotation
–

H – ~~create~~ for create superposition from a classical state
– used ~~to~~ at the start of many algorithms
(Deutsch – Jozsa, Grover & Sho

Quantum X Gate (NOT Gate)

- ~~Analogous~~ Quantum Analogous to classical NOT gate (Equivalent to classical NOT gate)
- Quantum NOT gate takes the state $|0\rangle$ to $|1\rangle$ and vice versa

→ X = pauli-X ~~gate~~ matrix $= \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$



→ ~~If~~ If input is $|0\rangle$
~~X~~ output is $X|0\rangle = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \begin{vmatrix} 1 \\ 0 \end{vmatrix} = \begin{vmatrix} 0 \\ 1 \end{vmatrix} = |1\rangle$

→ ~~If~~ input is $|1\rangle$
~~X~~ output is $X|1\rangle = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \begin{vmatrix} 0 \\ 1 \end{vmatrix} = \begin{vmatrix} 1 \\ 0 \end{vmatrix} = |0\rangle$

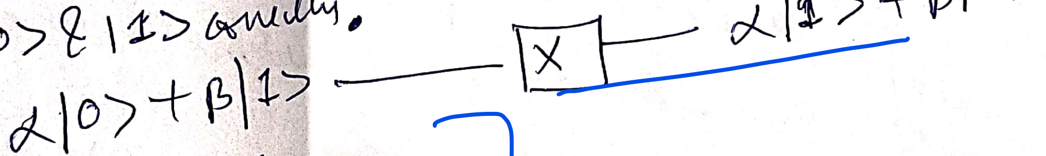
→ ~~If~~ Therefore X gate is called bit-flip gate because it inverts ~~either~~ each input bit.

~~Suppose two X gates are connected in series~~

→ If a superposed (superposition), the quantum NOT gate acts linearly in that state $\alpha|0\rangle + \beta|1\rangle$ is taken in to output is

$$X \begin{vmatrix} \alpha \\ \beta \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \begin{vmatrix} \alpha \\ \beta \end{vmatrix} = \begin{vmatrix} \beta \\ \alpha \end{vmatrix} \therefore \beta|0\rangle + \alpha|1\rangle$$

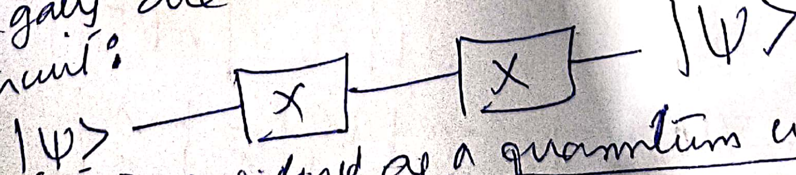
→ Here that the X gate 'negates' the computational basis state $|0\rangle$ & $|1\rangle$ ~~and~~ correctly.



Truth Table

Input	Output
$\|0\rangle$	$\|1\rangle$
$\|1\rangle$	$\|0\rangle$
$\alpha\|0\rangle+\beta\|1\rangle$	$\beta\|0\rangle+\alpha\|1\rangle$

→ Suppose two X gates are connected in series to form a quantum circuit:



→ A line in the circuit is considered as a quantum wire & basically represents a single bit.

The input $|\psi\rangle$ is transformed into $X|\psi\rangle$ and second X gate acts onto to form $X \cdot X \cdot |\psi\rangle$. $X \cdot X = I$, $I \cdot A = A$, $\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = I$

Quantum Z-Gate

- This gate maps input $|k\rangle$ to $(-1)^k|k\rangle$. Therefore $|0\rangle$ to $|0\rangle$ — only first not changed, $|1\rangle$ to $-|1\rangle$.
- phase flip ~~matrix~~ Gate
- Pauli Z- matrix $Z = \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix}$
- $Z|0\rangle = \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} \begin{vmatrix} 1 \\ 0 \end{vmatrix} = \begin{vmatrix} 1 \\ 0 \end{vmatrix} = |0\rangle$ — not changed
- $Z|1\rangle = \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} \begin{vmatrix} 0 \\ 1 \end{vmatrix} = \begin{vmatrix} 0 \\ -1 \end{vmatrix} = -|1\rangle$ — Inverts the sign of $|1\rangle$ (phaseshift)
- $Z(\alpha|0\rangle + \beta|1\rangle) = \alpha Z|0\rangle + \beta Z|1\rangle = \alpha|0\rangle - \beta|1\rangle$

The Input is Transformed to output

$\alpha|0\rangle+\beta|1\rangle \longrightarrow \boxed{Z} \longrightarrow \alpha|0\rangle-|1\rangle$

Truth Table

Input	output	
$\lvert 0\rangle$	$\lvert 0\rangle$	— no change
$\lvert 1\rangle$	$-\lvert 1\rangle$	— phaseshift of $\lvert 1\rangle$
$\alpha\lvert 0\rangle + \beta\lvert 1\rangle$	$\alpha\lvert 0\rangle - \beta\lvert 1\rangle$	— phase flip

Unitarity constraint

$\alpha|0\rangle+\beta|1\rangle$ the probabilities of $|\alpha|^2 + |\beta|^2 = 1$ because of probabilities be conserved to 1.

not necessary: This constraint is defined by the matrix U.

Then $\mathrm{adj}(U) \cdot U = I$

$\mathrm{adj}(Z) \cdot Z = \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = I$

Quantum Y Gate

$Y = \begin{vmatrix} 0 & -i \\ i & 0 \end{vmatrix}$

- It maps $|0\rangle$ to $i|1\rangle$ & $|1\rangle$ to $-i|0\rangle$

$\alpha|0\rangle+\beta|1\rangle \longrightarrow \boxed{Y} \longrightarrow i\alpha|1\rangle - i\beta|0\rangle$

- $Y|0\rangle = \begin{vmatrix} 0 & -i \\ i & 0 \end{vmatrix} \begin{vmatrix} 1 \\ 0 \end{vmatrix} = \begin{vmatrix} 0 \\ i \end{vmatrix} = i|1\rangle$
- $Y|1\rangle = \begin{vmatrix} 0 & -i \\ i & 0 \end{vmatrix} \begin{vmatrix} 0 \\ 1 \end{vmatrix} = \begin{vmatrix} -i \\ 0 \end{vmatrix} = -i|0\rangle$

$Y(\alpha|0\rangle + \beta|1\rangle) = \alpha Y|0\rangle + \beta Y|1\rangle$
$= \alpha(i|1\rangle) + \beta(-i|0\rangle)$
$= i\alpha|1\rangle - i\beta|0\rangle$

$\lvert 0\rangle$	$i\lvert 1\rangle$
$\lvert 1\rangle$	$-i\lvert 0\rangle$
$\alpha\lvert 0\rangle + \beta\lvert 1\rangle$	$-i\beta\lvert 0\rangle + i\alpha\lvert 1\rangle$

Hadamard Gate

- It is also known as a truly quantum gate
- It maps input $|m\rangle$ to $H|m\rangle$

$$H|m\rangle = \frac{1}{\sqrt{2}} \sum_{k=0,1} (-1)^{mk} |k\rangle$$

$$= \frac{|0\rangle + (-1)^m |1\rangle}{\sqrt{2}}$$

- $H = \frac{1}{\sqrt{2}} \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}$

$$H|0\rangle = \frac{1}{\sqrt{2}} \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} \begin{vmatrix} 1 \\ 0 \end{vmatrix} = \frac{\begin{vmatrix} 1 \\ 1 \end{vmatrix}}{\sqrt{2}} = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

- $\frac{1}{\sqrt{2}} \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} \begin{vmatrix} \alpha \\ \beta \end{vmatrix} = \frac{1}{\sqrt{2}} \begin{vmatrix} \alpha + \beta \\ \alpha - \beta \end{vmatrix} = \frac{\alpha + \beta}{\sqrt{2}} |0\rangle + \frac{\alpha - \beta}{\sqrt{2}} |1\rangle$

  or

  $\alpha \frac{|0\rangle + |1\rangle}{\sqrt{2}} + \beta \frac{|0\rangle - |1\rangle}{\sqrt{2}}$

$\alpha |0\rangle + \beta |1\rangle$

- $\frac{1}{\sqrt{2}} \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} \alpha$

$\lvert 0\rangle$	$\frac{1}{\sqrt{2}} \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} \begin{vmatrix} 1 \\ 0 \end{vmatrix} = \frac{1}{\sqrt{2}} \begin{vmatrix} 1 \\ 1 \end{vmatrix} = \frac{1}{\sqrt{2}} (\lvert 0\rangle + \lvert 1\rangle)$
$\lvert 1\rangle$	$\frac{1}{\sqrt{2}} \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} \begin{vmatrix} 0 \\ 1 \end{vmatrix} = \frac{1}{\sqrt{2}} \begin{vmatrix} 1 \\ -1 \end{vmatrix} = \frac{1}{\sqrt{2}} (\lvert 0\rangle - \lvert 1\rangle)$
$\alpha \lvert 0\rangle + \beta \lvert 1\rangle$	$\frac{1}{\sqrt{2}} \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} \begin{vmatrix} \alpha \\ \beta \end{vmatrix} = \frac{1}{\sqrt{2}} \begin{vmatrix} \alpha + \beta \\ \alpha - \beta \end{vmatrix}$
$\frac{\lvert 0\rangle + \lvert 1\rangle}{\sqrt{2}}$	$\frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} \begin{vmatrix} 1 \\ 1 \end{vmatrix} = \frac{1}{2} \begin{vmatrix} 2 \\ 0 \end{vmatrix} = \begin{vmatrix} 1 \\ 0 \end{vmatrix} = \lvert 0\rangle$ Reverse
$\frac{\lvert 0\rangle - \lvert 1\rangle}{\sqrt{2}}$	$\frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} \begin{vmatrix} 1 \\ -1 \end{vmatrix} = \frac{1}{2} \begin{vmatrix} 0 \\ 2 \end{vmatrix} = \begin{vmatrix} 0 \\ 1 \end{vmatrix} = \lvert 1\rangle$

$$\alpha|0\rangle + \beta|1\rangle \longrightarrow \boxed{H} \longrightarrow \alpha \frac{|0\rangle + |1\rangle}{\sqrt{2}} + \beta \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

Phase Gate or S Gate

- ~~S gate returns~~ $\alpha|0\rangle$ into $|0\rangle$ & $\alpha|1\rangle$ into $i|1\rangle$

- $S = \begin{vmatrix} 1 & 0 \\ 0 & i \end{vmatrix}$

$S|0\rangle = \begin{vmatrix} 1 & 0 \\ 0 & i \end{vmatrix}\begin{vmatrix} 1 \\ 0 \end{vmatrix} = \begin{vmatrix} 1 \\ 0 \end{vmatrix} = |0\rangle$

$S|1\rangle = \begin{vmatrix} 1 & 0 \\ 0 & i \end{vmatrix}\begin{vmatrix} 0 \\ 1 \end{vmatrix} = \begin{vmatrix} 0 \\ i \end{vmatrix} = i|1\rangle$

$S(\alpha|0\rangle + \beta|1\rangle) = \begin{vmatrix} 1 & 0 \\ 0 & i \end{vmatrix}\begin{vmatrix} \alpha \\ \beta \end{vmatrix} = \begin{vmatrix} \alpha \\ i\beta \end{vmatrix}$ means $\alpha|0\rangle + i\beta|1\rangle$

Truth table

I/O	O/P
$\lvert 0\rangle$	$\lvert 0\rangle$
$\lvert 1\rangle$	$i\lvert 1\rangle$
$\alpha\lvert 0\rangle+\beta\lvert 1\rangle$	$\alpha\lvert 0\rangle + i\beta\lvert 1\rangle$

$\alpha|0\rangle+\beta|1\rangle$ — $\boxed{S}$ — $\alpha|0\rangle + i\beta|1\rangle$

prove $Z = S \cdot S$.

— $\boxed{Z}$ — $=$ — $\boxed{S}$ — $\boxed{S}$ —

$S^2 = \begin{vmatrix} 1 & 0 \\ 0 & i \end{vmatrix}\begin{vmatrix} 1 & 0 \\ 0 & i \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} = Z$

T Gate

$T = \begin{vmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{vmatrix}$

T is also called $\pi/4$ gate.

$T|0\rangle = \begin{vmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{vmatrix}\begin{vmatrix} 1 \\ 0 \end{vmatrix} = \begin{vmatrix} 1 \\ 0 \end{vmatrix} = |0\rangle$

$T|1\rangle = \begin{vmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{vmatrix}\begin{vmatrix} 0 \\ 1 \end{vmatrix} = \begin{vmatrix} 0 \\ e^{i\pi/4} \end{vmatrix} = e^{i\pi/4}|1\rangle$

~~State~~ $\alpha|0\rangle + \beta|1\rangle$ transform to $\alpha|0\rangle + \beta e^{i\pi/4}|1\rangle$

$\begin{vmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{vmatrix}\begin{vmatrix} \alpha \\ \beta \end{vmatrix} = \begin{vmatrix} \alpha \\ \beta e^{i\pi/4} \end{vmatrix}$

prove $S = T^2$.

$\begin{vmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{vmatrix}\begin{vmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & i \end{vmatrix} = S$

$\alpha|0\rangle+\beta|1\rangle$ — $\boxed{S}$ — $\alpha|0\rangle + e^{i\pi/4}\beta|1\rangle$

Truth Table

I/O	O/P
$\lvert 0\rangle$	$\lvert 0\rangle$
$\lvert 1\rangle$	$e^{i\pi/4}\lvert 1\rangle$
$\alpha\lvert 0\rangle+\beta\lvert 1\rangle$	$\alpha\lvert 0\rangle + e^{i\pi/4}\beta\lvert 1\rangle$

CNOT gate (controlled NOT Gate)

- CNOT operates on two qubits at the same time.
- CNOT inverts the right qubit if the left qubit is 1
- CNOT gate basically implements a reversible EX-OR gate.

Can be used to generate entanglement.

$CNOT|00\rangle = |00\rangle,$
$CNOT|01\rangle = |01\rangle,$
$CNOT|10\rangle = |11\rangle,$
$CNOT|11\rangle = |10\rangle.$

Truth Table

a	b	x	y
0	0	0	0
0	1	0	1
1	0	1	1
1	1	1	0

or

I/p	O/P
$\|00\rangle$	$\|00\rangle$
$\|01\rangle$	$\|01\rangle$
$\|10\rangle$	$\|11\rangle$
$\|11\rangle$	$\|10\rangle$

~~If~~ The left bit is called control bit and the right bit is called the target qubit.

- The control bit is unchanged by CNOT, whereas the target qubit becomes the XOR of the inputs.
- CNOT is a quantum XOR gate.
- CNOT is also called CX gate or controlled-X gate.

Acting on a superposition

$CNOT(c_0|00\rangle + c_1|01\rangle + c_2|10\rangle + c_3|11\rangle)$
$= c_0 CNOT|00\rangle + c_1 CNOT|01\rangle + c_2 CNOT|10\rangle + c_3 CNOT|11\rangle$
$= c_0|00\rangle + c_1|0\cancel{0}\rangle + c_2|11\rangle + c_3|10\rangle$
$= c_0|00\rangle + c_1|01\rangle + c_3|10\rangle + c_2|11\rangle$

So, the amplitudes of $|10\rangle$ & $|11\rangle$ are swapped.

As a Matrix CNOT

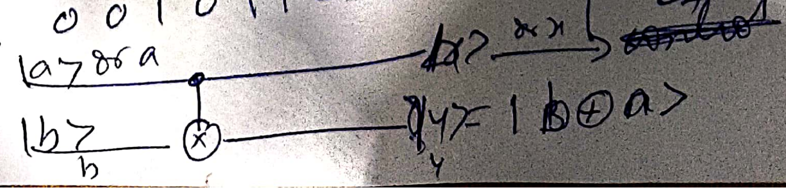
The columns corresponding to CNOT acting on $|00\rangle, |01\rangle, |10\rangle$ & $|11\rangle$

$$CNOT = \begin{array}{c|cccc} & 00 & 01 & 10 & 11 \\ \hline 00 & 1 & 0 & 0 & 0 \\ 01 & 0 & 1 & 0 & 0 \\ 10 & 0 & 0 & 0 & 1 \\ 11 & 0 & 0 & 1 & 0 \end{array}$$

$CNOT(c_0|00\rangle + c_1|01\rangle + c_2|10\rangle + c_3|11\rangle)$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} c_0 \\ c_1 \\ c_3 \\ c_2 \end{bmatrix}$$

CNOT is represented as



Multiple Qubits

Tensor product $\rightarrow \otimes$ $\boxed{\otimes}$

example Two qubits, both in the $|0\rangle$ state are written

$|0\rangle \otimes |0\rangle$ 'zero tensor zero'

we compress the notation both written & speech

$|0\rangle|0\rangle$ ~~further~~

further compress into $|00\rangle$

When two qubits, the z-basis is $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$

The superposition of these basis states are

$c_0|00\rangle + c_1|01\rangle + c_2|10\rangle + c_3|11\rangle$

If we measure these two qubits in the z-basis,

we get $|00\rangle$ with probability $|c_0|^2$, $|01\rangle$ with probability $|c_1|^2$, $|10\rangle$ with probability $|c_2|^2$, or $|11\rangle$ with $|c_3|^2$ probability.

So. The total probability is $|c_0|^2 + |c_1|^2 + |c_2|^2 + |c_3|^3 = 1$

1111y for 3 qubits are 8 z basis state $|000\rangle, |001\rangle, |010\rangle, |011\rangle$,

$|100\rangle, |101\rangle, |110\rangle$ & $|111\rangle$.

So These translate the don $|b_2 b_1 b_0\rangle$

decimal representation $2^2 b_2 + 2^1 b_1 + 2^0 b_0$

Like decimal representation.

called little ~~En~~ endian.

Superposition $\sum_{j=0}^{7} c_j|j\rangle = c_0|0\rangle + c_1|1\rangle + \dots c_7|7\rangle$.

$$\sum_{j=0}^{7} |c_j|^2 = 1$$

a is called source
b is called Target
output $x = a$ that is x takes the value of source s
The source is called the control input & controls the application of the NOT operation on the target input.

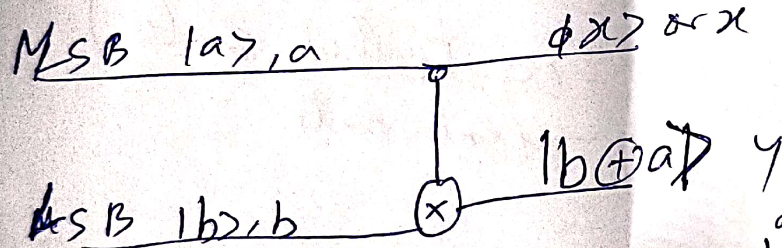
output $y = a \oplus b$

ie y is the inverse of the target b when source is 1 otherwise $y = b$.

In other words, whether y gets the inverted value of the target b is controlled by a source a.

Hence, CNOT is called controlled NOT gate.

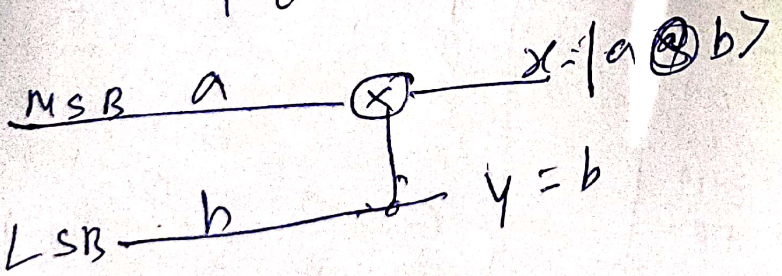
<entangled> & swap gate



ab	xy
$\|00\rangle$	$\|00\rangle$
$\|01\rangle$	$\|01\rangle$
$\|10\rangle$	$\|11\rangle$
$\|11\rangle$	$\|10\rangle$

$$\begin{array}{c|cccc} ab \backslash xy & 00 & 01 & 10 & 11 \\ \hline 00 & 1 & 0 & 0 & 0 \\ 01 & 0 & 1 & 0 & 0 \\ 10 & 0 & 0 & 0 & 1 \\ 11 & 0 & 0 & 1 & 0 \end{array}$$

$$CNOT = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{vmatrix}$$

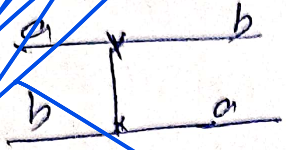


ab	xy
$\|00\rangle$	$\|00\rangle$
$\|01\rangle$	$\|11\rangle$
$\|10\rangle$	$\|10\rangle$
$\|11\rangle$	$\|01\rangle$

$$\begin{array}{c|cccc} ab & & & & \\ \hline 00 & 1 & 0 & 0 & 0 \\ 01 & 0 & 0 & 0 & 1 \\ 10 & 0 & 0 & 1 & 0 \\ 11 & 0 & 1 & 0 & 0 \end{array}$$

CNOT

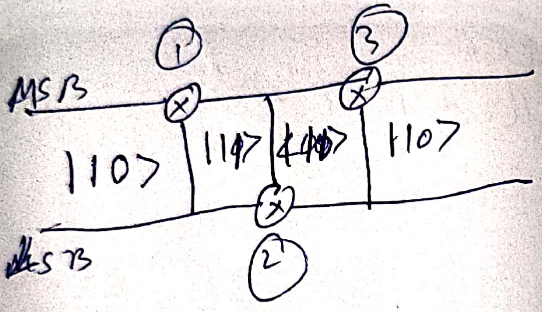
Implementing SWAP Gate using XOR



SWAP = 3 NOT Gates

00	00
01	10
10	01
11	11

$$= \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$



$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{vmatrix} \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{vmatrix} \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{vmatrix}$$

(3) (2) (1)

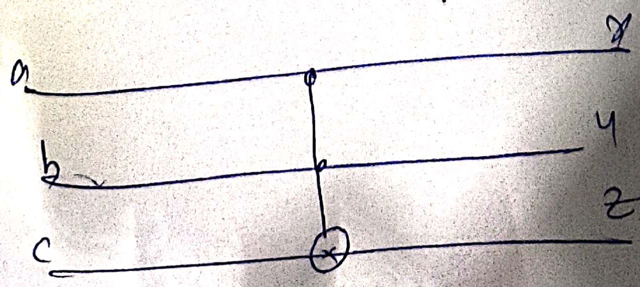
Toffoli Gate (controlled-controlled Gate)

CCNOT gate is called Toffoli Gate.

- CCNOT
- It has 3 inputs.
- The outputs are the same as the inputs except the third qubit which flips only if the first two qubits are both 1's.

Input	Output
000	000
001	001
010	010
011	011
100	100
101	101
110	111
111	110

$$\begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{vmatrix}$$



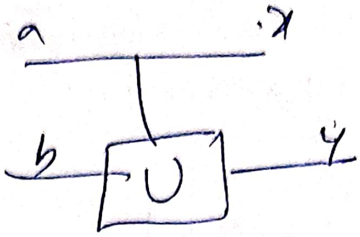
$x = a$

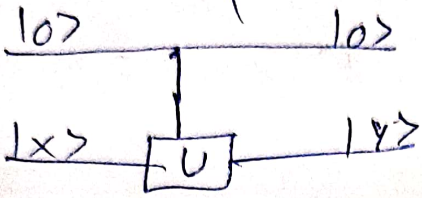
$y = b$

$z = \cancel{c \oplus ab}$ $ab \oplus c$

controlled U-gate

$$U = \begin{bmatrix} u_{00} & u_{01} \\ u_{10} & u_{11} \end{bmatrix}$$



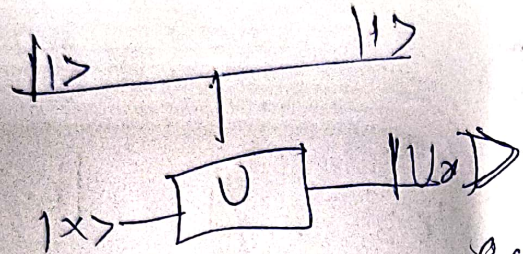


$00 \rightarrow 00$

$01 \rightarrow 01$

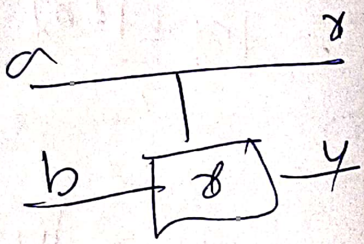
$10 \rightarrow |1\rangle(u_{00}|0\rangle + u_{10}|1\rangle)$

$11 \rightarrow |1\rangle(u_{01}|0\rangle + u_{11}|1\rangle)$



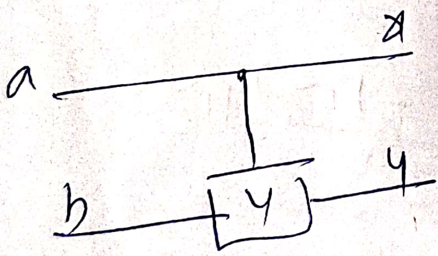
$$\begin{array}{c|cccc} & 00 & 01 & 10 & 11 \\ \hline 00 & 1 & 0 & 0 & 0 \\ 01 & 0 & 1 & 0 & 0 \\ 10 & 0 & 0 & u_{00} & u_{01} \\ 11 & 0 & 0 & u_{10} & u_{11} \end{array}$$

So



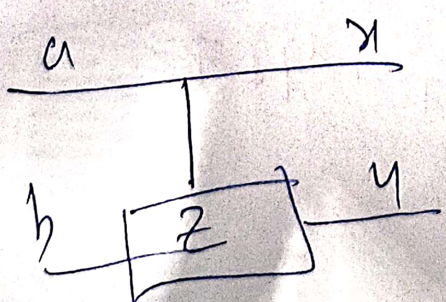
$$\Longrightarrow \begin{array}{c|cccc} ab \backslash xy & 00 & 01 & 10 & 00 \\ \hline 00 & 1 & 0 & 0 & 0 \\ 01 & 0 & 1 & 0 & 0 \\ 10 & 0 & 0 & 0 & 1 \\ 11 & 0 & 0 & 1 & 0 \end{array}$$

$\underbrace{\boxed{\begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix}}}_{X}$



$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{bmatrix}$$

$\underbrace{\boxed{\begin{matrix} 0 & -i \\ i & 0 \end{matrix}}}_{Y}$



$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$\underbrace{\boxed{\begin{matrix} 1 & 0 \\ 0 & -1 \end{matrix}}}_{Z}$

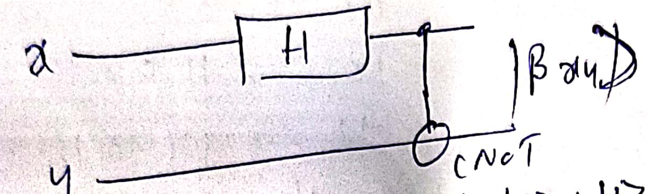
Bell states (EPR pairs)

↳ Einstein, Podolsky, & Rosen who first pointed out the strange properties.

phy+ $|\Phi^+\rangle = \dfrac{|00\rangle + |11\rangle}{\sqrt{2}}$ $\quad |\Phi^-\rangle = \dfrac{|00\rangle - |11\rangle}{\sqrt{2}}$ ...

psi $|\Psi^+\rangle = \dfrac{|01\rangle + |10\rangle}{\sqrt{2}}$ $\quad |\Psi^-\rangle = \dfrac{|01\rangle - |10\rangle}{\sqrt{2}}$

Hadamard gate



Tensor product

$H \otimes I \; |0\rangle$ P00

Input $|00\rangle$: output of H gate is $\dfrac{|0\rangle + |1\rangle}{\sqrt{2}}|0\rangle \rightarrow \left(\dfrac{|0\rangle + |1\rangle}{\sqrt{2}}\right)|0\rangle$

β_{00} CNOT $= \dfrac{|00\rangle + |11\rangle}{\sqrt{2}}$

Input $|01\rangle$ H output $\left(\dfrac{|0\rangle + |1\rangle}{\sqrt{2}}\right)|1\rangle = \dfrac{|01\rangle + |10\rangle}{\sqrt{2}}$

β_{01}

Input $|10\rangle$ H output $\dfrac{(|0\rangle - |1\rangle)}{\sqrt{2}}|0\rangle = \dfrac{|00\rangle - |11\rangle}{\sqrt{2}}$

β_{10}

Input $|11\rangle$ H gate output $\dfrac{(|0\rangle - |1\rangle)}{\sqrt{2}}|1\rangle$

CNOT

β_{11} $= \dfrac{|01\rangle - |10\rangle}{\sqrt{2}}$

$$|\beta_{xy}\rangle = \frac{|0,y\rangle + (-1)^x |1,\bar{y}\rangle}{\sqrt{2}}$$

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Deutsch-Jozsa Algorithm
(DOY-ch JO-zuh)

Algorithm steps

Step 1: Initialize the qubits
Start with $|0\rangle^{\otimes n} |1\rangle$
The last bit is set to $|1\rangle$

$n = 2$ $\quad |00\rangle|1\rangle$

Step 2: Apply Hadamard gates (H) to all qubits

$$|0\rangle \to \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$|1\rangle \to \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

So the state becomes $\frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n-1} |x\rangle \otimes \frac{|0\rangle - |1\rangle}{\sqrt{2}}$

That's a superposition of all possible inputs — the source of quantum parallelism.

Step 3: Apply oracle U_f
The oracle flips the phase depending on $f(x)$:

$$U_f |x\rangle \otimes \frac{|0\rangle - |1\rangle}{\sqrt{2}} = (-1)^{f(x)} |x\rangle \otimes \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

(The last qubit is unchanged — it's just used to mark the phase)

Now the new state is

$$\frac{1}{\sqrt{2^n}} \sum_x (-1)^{f(x)} |x\rangle \otimes \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

Step 4: Apply Hadamard on the first n qubits again
Each Hadamard on n-qubit input to perform a Fourier like transform

$$H^{\otimes n} |x\rangle = \frac{1}{\sqrt{2^n}} \sum_{z=0}^{2^n-1} (-1)^{x \cdot z} |z\rangle$$

where $x \cdot z$ is the bitwise dot product (mod 2)

So after the second Hadamard

$$\frac{1}{2^n} \sum_z \left[\sum_x (-1)^{f(x)} (-1)^{x \cdot z} \right] |z\rangle \otimes \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

(PTO)

Step 4: Measure the first n qubits.
lets analyze the amplitude for $|z=0\rangle = |00\ldots0\rangle$:

$$A(0) = \frac{1}{2^n} \sum_{x} (-1)^{f(x)}$$

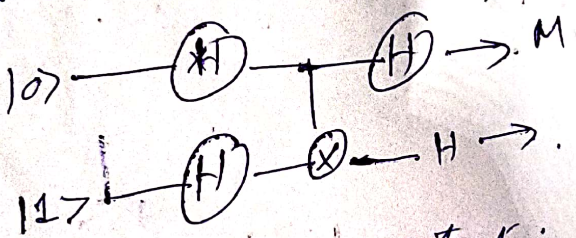
- If $f(x)$ is constant, all terms are the same:
  - If $f(x) = 0$: $A(0) = 1$
  - If $f(x) = 1$; $A(0) = -1$
  - $\Rightarrow$ probability $= 1$ — always measure $|0\ldots0\rangle$
- If $f(x)$ is balanced, half are $+1$, half are -1, so they cancel $A(0) = 0$

$=$ probability $= 0 \Rightarrow$ never get $|0\ldots0\rangle$.

So
- constant function $\Rightarrow$ measure all 0s
- Balanced function $\Rightarrow$ measure anything else.

One measurement = final Answer.

Example: $n = 1$



X represents oracle U_f

case 1: $f(x) = 0$ (constant):
After Hadamards: $\frac{1}{2}(|0\rangle + |1\rangle)(|0\rangle - |1\rangle)$
oracle adds phase $(-1)^{f(x)} = 1$ so no change.
After final H on the first qubit, we get $|0\rangle$
$\rightarrow$ measurement $= 0$ — constant

case 2: $f(x) = x$ (balanced)
oracle applies $(-1)^x$, flipping the phase of $|1\rangle$ term.
So we get $\frac{1}{2}(|0\rangle - |1\rangle)(|0\rangle - |1\rangle)$
After final H on the first qubit $\rightarrow |1\rangle$
measurement $= 1$ — balanced.