Assignment3

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Question:

Find the QR decomposition of
$A = \begin{pmatrix} 8 & 5 \\ 3 & 2 \end{pmatrix}$

Solution:

The QR decomposition of a matrix is a decomposition of the matrix into an orthogonal matrix and an upper triangular matrix. A QR decomposition of a real square matrix A is a decomposition of A as

$$\mathbf{A} = \mathbf{QR} \tag{1}$$

where Q is an orthogonal matrix and R is an upper triangular matrix
Given,

$$A = \begin{pmatrix} 8 & 5 \\ 3 & 2 \end{pmatrix} \tag{2}$$

Let a and b be the column vectors of the given matrix

$$a = \begin{pmatrix} 8 \\ 3 \end{pmatrix} \tag{3}$$

$$b = \begin{pmatrix} 5 \\ 2 \end{pmatrix} \tag{4}$$

The above column vectors can be expressed as

$$a = t_1 u_1 \tag{5}$$

$$b = s_1 u_1 + t_2 u_2 \tag{6}$$

where,

$$t_1 = ||a|| \tag{7}$$

$$u_1 = \frac{a}{t_1} \tag{8}$$

$$s_1 = \frac{u_1^T b}{||u_1||^2} \tag{9}$$

$$u_2 = \frac{b - s_1 u_1}{||b - s_1 u_1||} \tag{10}$$

$$t_2 = u_2^T b \tag{11}$$

the equation (5) and (6) can be written as

$$\begin{pmatrix} a & b \end{pmatrix} = \begin{pmatrix} u_1 & u_2 \end{pmatrix} \begin{pmatrix} t_1 & s_1 \\ 0 & t_2 \end{pmatrix} \tag{12}$$

$$\begin{pmatrix} a & b \end{pmatrix} = QR \tag{13}$$

Here, R is an upper triangular matrix and Q is an orthogonal matrix such that

$$Q^T Q = I \tag{14}$$

Now, using equation (7) to (11) we get,

$$t_1 = \sqrt{8^2 + 3^2} = \sqrt{73} \tag{15}$$

$$u_1 = \frac{1}{\sqrt{73}} \begin{pmatrix} 8 \\ 3 \end{pmatrix} \tag{16}$$

$$s_1 = \begin{pmatrix} \frac{8}{\sqrt{73}} & \frac{3}{\sqrt{73}} \end{pmatrix} \begin{pmatrix} 5 \\ 2 \end{pmatrix} = \frac{46}{\sqrt{73}} \tag{17}$$

$$u_2 = \frac{1}{\sqrt{73}} \begin{pmatrix} -3 \\ 8 \end{pmatrix} \tag{18}$$

$$t_2 = \begin{pmatrix} \frac{-3}{\sqrt{73}} & \frac{8}{\sqrt{73}} \end{pmatrix} \begin{pmatrix} 5 \\ 2 \end{pmatrix} = \frac{1}{\sqrt{73}} \tag{19}$$

Substituting all these values into equation (12),
We obtain the QR decomposition as

$$A = \begin{pmatrix} \frac{8}{\sqrt{73}} & \frac{-3}{\sqrt{73}} \\ \frac{3}{\sqrt{73}} & \frac{8}{\sqrt{73}} \end{pmatrix} \begin{pmatrix} \sqrt{73} & \frac{46}{\sqrt{73}} \\ 0 & \frac{1}{\sqrt{73}} \end{pmatrix} \tag{20}$$

Question:

Find the QR decomposition of
$A = \begin{pmatrix} 2 & 5 \\ 1 & 4 \end{pmatrix}$

Solution:

QR decomposition of a square matrix is given by,

$$\mathbf{A} = \mathbf{QR} \tag{21}$$

where Q is an orthogonal matrix and R is an upper triangular matrix.
Given matrix,

$$A = \begin{pmatrix} 2 & 5 \\ 1 & 4 \end{pmatrix} \tag{22}$$

The column vectors of the matrix is given by,

$$a = \begin{pmatrix} 2 \\ 1 \end{pmatrix} b = \begin{pmatrix} 5 \\ 4 \end{pmatrix} \tag{23}$$

Equation (22) can be written in form of (23) as,

$$\begin{pmatrix} a & b \end{pmatrix} = \begin{pmatrix} q_1 & q_2 \end{pmatrix} \begin{pmatrix} u_1 & u_3 \\ 0 & u_2 \end{pmatrix} = QR \tag{24}$$

where,

$$u_1 = ||a|| = \sqrt{2^2 + 1^2} = \sqrt{5} \tag{25}$$

$$q_1 = \frac{a}{u_1} = \begin{pmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix} \tag{26}$$

$$u_3 = \frac{q_1^T b}{||q_1||^2} = \begin{pmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} 5 \\ 4 \end{pmatrix} = \frac{14}{\sqrt{5}} \tag{27}$$

$$q_2 = \frac{b - u_3 q_1}{||b - u_3 q_1||} = \frac{1}{\sqrt{5}} \begin{pmatrix} -1 \\ 2 \end{pmatrix} \tag{28}$$

$$u_2 = q_2^T b = \frac{3}{\sqrt{5}} \tag{29}$$

Substituting equation (25) to (29) in (24),
we obtain the QR decomposition

$$A = \begin{pmatrix} \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} \sqrt{5} & \frac{14}{\sqrt{5}} \\ 0 & \frac{3}{\sqrt{5}} \end{pmatrix} \tag{30}$$