$$\sinh x = \frac{e^{x} - e^{-x}}{2}$$

$$\cosh x = \frac{e^{x} + e^{-x}}{2}.$$

1. A partial differential equation involves

a) 1 independent variable

b) 2 independent variables

c) 2 or more independent variables

d) None of these

Q solve the Laplace equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

subject to the conditions

$$u(0, y) = u(l, y) = u(x, 0) = 0$$

and $u(x, \alpha) = \sin \frac{n\pi x}{l}$

Sol The three possible solutions of

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{——} \quad (1)$$

are :

$$u = (c_1 e^{px} + c_2 e^{-px})(c_3 \cos py + c_4 \sin py) \quad \text{--- (2)}$$

$$u = (c_5 \cos px + c_6 \sin px)(c_7 e^{py} + c_8 e^{-py}) \quad \text{--- (3)}$$

$$u = (c_9 x + c_{10})(c_{11} y + c_{12}) \quad \text{--- (4)}$$

We have to solve (1) which satisfies the following boundary conditions:

$$u(0,y) = 0 \quad \text{—— (5)}$$

$$u(l,y) = 0 \quad \text{—— (6)}$$

$$u(x,0) = 0 \quad \text{—— (7)}$$

$$u(x,\alpha) = \sin\frac{n\pi x}{l} \quad \text{—— (8)}$$

using (5) and (6) in (2),

$$u = (c_1 e^{px} + c_2 e^{-px})(c_3 \cos py + c_4 \sin py) \quad \text{--- (2)}$$

$u(0, y) = 0$ --- (5) $\qquad u(l, y) = 0$ --- (6)

Using (5) in (2), we get

$$0 = (c_1 + c_2)(c_3 \cos py + c_4 \sin py)$$

$$\Rightarrow c_1 + c_2 = 0$$

Using (6) in (2), we get

$$0 = (c_1 e^{pl} + c_2 e^{-pl})(c_3 \cos py + c_4 \sin py)$$

$$\Rightarrow c_1 e^{pl} + c_2 e^{-pl} = 0$$

$(c_1 + c_2) = 0$ and $c_1 e^{pl} + c_2 e^{-pl} = 0$

On solving these equations, we get

$$c_1 = c_2 = 0$$

$\because c_1 = -c_2$ and $c_1 e^{pl} - c_1 e^{-pl} = 0$

$\Rightarrow c_1 [e^{pl} - e^{-pl}] = 0 \Rightarrow c_1 = 0$

$\Rightarrow c_2 = 0$

which leads to trivial solution.

Now using 5 and 6 in 4

where 4 is :→

$$u = (c_9 x + c_{10})(c_{11} y + c_{12}) \quad \text{—— } (4)$$

$$u(0, y) = 0 \quad \text{—— } (5) \qquad u(l, y) = 0 \quad \text{—— } (6)$$

$$\therefore (4) \Rightarrow (c_{10})(c_{11} y + c_{12}) = 0$$

$$\Rightarrow c_{10} = 0$$

$\therefore$ (4) $\Rightarrow u = c_9 x \,(c_{11} y + c_{12})$

and $u(l, y) = 0$

$\Rightarrow c_9 l \,(c_{11} y + c_{12}) = 0$

$\Rightarrow c_9 = 0$

$\Rightarrow u = 0$

which is a trivial solution.

Hence the suitable solution for the present problem is solution (3).

Using (5) in (3) we get

Here (5) is $u(0,y) = 0$

(3) is $u = (C_5 \cos px + C_6 \sin px)(C_7 e^{py} + C_8 e^{-py})$

Now $u(0,y) = 0$

$\Rightarrow \quad C_5 (C_7 e^{py} + C_8 e^{-py}) = 0$

$\Rightarrow \quad \boxed{C_5 = 0}$

$\therefore$ (3) $\Rightarrow$ $u = c_6 \sin px \left(c_7 e^{py} + c_8 e^{-py}\right)$ ——— (9)

Using (6), i.e. $u(l, y) = 0$

(9) $\Rightarrow$ $c_6 \sin pl \left(c_7 e^{py} + c_8 e^{-py}\right) = 0$

$\Rightarrow$ $c_6 = 0$ or $\sin pl = 0$

If we take $c_6 = 0$, we get trivial solution.

Thus $\sin pl = 0$

$\Rightarrow pl = n\pi$

$\Rightarrow p = \frac{n\pi}{l}$, $n \to$ integer

$\therefore$ (9) becomes.

$$u = c_6 \sin \frac{n\pi}{l}x \left[c_7 e^{\frac{n\pi}{l}y} + c_8 e^{-\frac{n\pi}{l}y} \right] \quad \text{--- (10).}$$

using (7) i.e. $u(x,0) = 0$

(10) $\Rightarrow \quad c_6 \sin \frac{n\pi}{l} x \left[c_7 + c_8 \right] = 0$

$\Rightarrow \quad c_7 + c_8 = 0 \qquad \because \; c_6 \neq 0$ since $c_6 = 0$

$\Rightarrow \quad u = 0$ (trivial sol)

$\Rightarrow c_8 = -c_7.$

$$\therefore u(x,y) = c_6 \sin \frac{n\pi x}{l} \left[c_7 e^{\frac{n\pi y}{l}} - c_7 e^{-\frac{n\pi y}{l}} \right]$$

$$= c_6 c_7 \sin \frac{n\pi x}{l} \left[e^{\frac{n\pi y}{l}} - e^{-\frac{n\pi y}{l}} \right]$$

$$u(x,y) = b_n \sin\frac{n\pi x}{l}\left[e^{\frac{n\pi y}{l}} - e^{-\frac{n\pi y}{l}}\right] \qquad \text{——} \; (11)$$

Using condition (8), i.e. $u(x,a) = \sin\frac{n\pi x}{l}$

$$(11) \Rightarrow \quad b_n \sin\frac{n\pi x}{l}\left[e^{\frac{n\pi a}{l}} - e^{-\frac{n\pi a}{l}}\right] = \sin\frac{n\pi x}{l}$$

$$\Rightarrow \quad b_n \left[e^{\frac{n\pi a}{l}} - e^{-\frac{n\pi a}{l}}\right] = 1$$

$$\Rightarrow \quad b_n = \frac{1}{e^{\frac{n\pi a}{l}} - e^{-\frac{n\pi a}{l}}}$$

$\therefore$ required solution is

$$u(x,y) = \frac{\left[e^{\frac{n\pi y}{l}} - e^{-\frac{n\pi y}{l}}\right]}{\left[e^{\frac{n\pi a}{l}} - e^{-\frac{n\pi a}{l}}\right]} \sin\frac{n\pi x}{l}.$$

$$u(x,y) = \frac{\sinh\left(\frac{n\pi y}{l}\right)}{\sinh\left(\frac{n\pi a}{l}\right)} \sin \frac{n\pi x}{l}$$

which is the required solution.