9.2.7

EE24BTECH11018 - Durgi Swaraj Sharma

Exercise 9.2 In each of the Exercises 1 to 10 verify that the given functions (explicit or implicit) is a solution of the corresponding differential equation:

$$7. \, xy = logy + C : y' = \frac{y^2}{1-xy} \, (xy \neq 1) \tag{1}$$

Theroretical solution

$$\frac{dy}{dx} = \frac{y^2}{1-xy} \, (xy \neq 1) \tag{2}$$

Taking reciprocal,

$$\frac{dx}{dy} = \frac{1-xy}{y^2} \tag{3}$$

$$\frac{dx}{dy} = \frac{1}{y^2} - \frac{x}{y} \tag{4}$$

Rearranging the terms,

$$\frac{dx}{dy} + x\left(\frac{1}{y}\right) = \frac{1}{y^2} \tag{5}$$

We can apply the Integrating Factor method to solve this differential equation,

$$\text{where } IF = e^{\int \frac{1}{y} dy} = e^{ln(y)} = y \tag{6}$$

$$IF \cdot x = \int IF \cdot \frac{1}{y^2} dy \tag{7}$$

$$yx = \int y \frac{1}{y^2} dy \tag{8}$$

$$xy = ln\,(y) + C \tag{9}$$

We have theoretically verified that the given function is a solution to the given differential equation.

Numerical Solution

We shall write computer programs that will give us an approximate solution to the differential equation, if we know the solution's starting point.

The following algorithm is the Finite Differences algorithm.

$$\text{Let initial point be} \left(x_i, y_i\right)$$
$$\text{Set } x_0 \leftarrow x_i, y_0 \leftarrow y_i.$$
$$\text{Let} \left(x_1, y_1\right), \cdots, \left(x_n, y_n\right) \text{ be approximate points on the solution curve.}$$
$$\text{Iterate:}$$
$$y_{n+1} = y_n + h \cdot y'_n$$
$$x_{n+1} = x_n + h.$$

The iteration step is derived from the Difference Equation which is given by,

$$\frac{dy}{dx} \approx \frac{f(x+h) - f(x)}{h} \tag{10}$$
$$\frac{dy}{dx} \approx \frac{y_{i+1} - y_i}{h} \tag{11}$$

Rearranging,

$$y_{n+1} = y_n + \frac{dy}{dx} h \tag{12}$$

Due to the nature of our differential equation's finite difference, the Finite Differences algorithm is the same as Euler's method.
Applying this algorithm to our differential equation,

$$\text{Taking initial point (without loss of generality) as} \left(5, 1\right) \tag{13}$$
$$x_0 \leftarrow 5, y_0 \leftarrow 1 \tag{14}$$
$$\text{Iterate } x_{n+1} = x_n + h, y_{n+1} = y_n + h \cdot \frac{y^2}{1 - xy} \tag{15}$$

Verifying this algorithm,

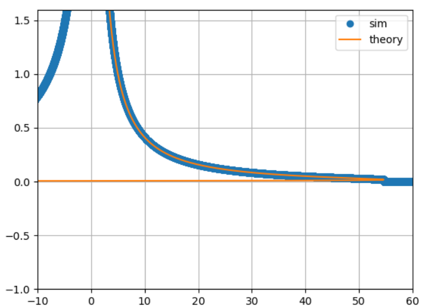

Fig. 0: Verification of our Finite Differences algorithm

we can see that the simulation graph shows unusual behaviour. To fix this, we will apply the Finite Differences algorithm as follows,

$$\text{Let initial point be}\left(x_i, y_i\right)$$
$$\text{Set } x_0 \leftarrow x_i, y_0 \leftarrow y_i.$$
$$\text{Let}\left(x_1, y_1\right), \cdots, \left(x_n, y_n\right) \text{be approximate points on the solution curve.}$$
$$\text{Iterate:}$$
$$x_{n+1} = x_n + h \cdot x_n'$$
$$y_{n+1} = y_n + h.$$

This prevents our code from producing inaccurate results when $\frac{dy}{dx} \to \infty$.

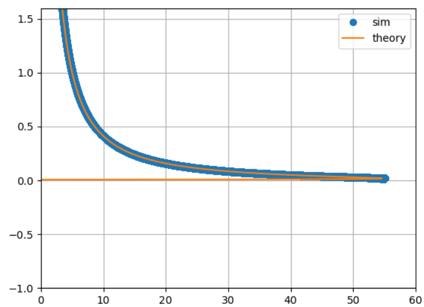

Fig. 0: Verification after interchanging x and y

However, on oberving at a closer scale, we notice that the simulation strays away too quickly from the true values at crucial regions of the graph.

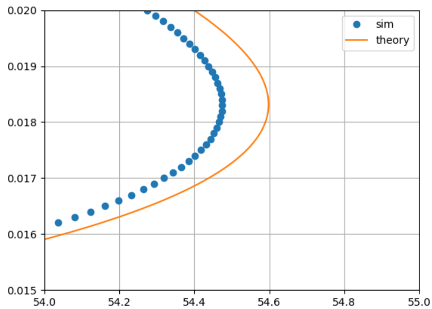

Fig. 0: Zoomed in section of the previous graph

We will have to use an algorithm that converges fast enough to be useful in our case. One of these methods is the Improved Euler's method, which is as follows

$$\text{Let initial point be} \left(x_i, y_i\right)$$
$$\text{Set } x_0 \leftarrow x_i, y_0 \leftarrow y_i.$$
$$\text{Let} \left(x_1, y_1\right), \cdots, \left(x_n, y_n\right) \text{be approximate points on the solution curve.}$$
$$\text{Iterate:}$$
$$k_1 = h \cdot y_n', \quad \text{where } y_n' = f(x_n, y_n)$$
$$k_2 = h \cdot f\left(x_n + h, y_n + k_1\right)$$
$$y_{n+1} = y_n + \frac{1}{2} \cdot (k_1 + k_2)$$
$$x_{n+1} = x_n + h$$

Applying the same x to y switch we made to the previous approach and plotting, there result is

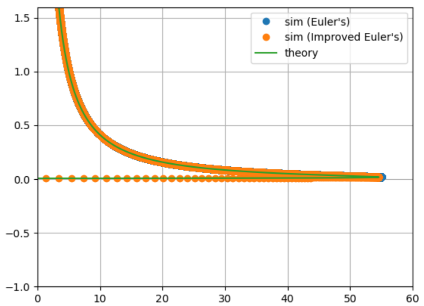

Fig. 0: Verification of Improved Euler's method

And a closer look:

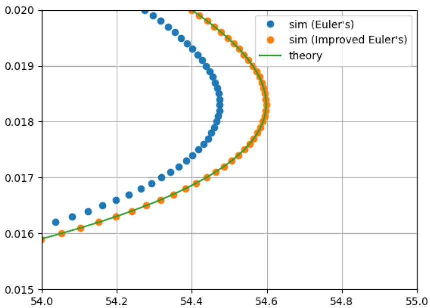

Fig. 0: Zoomed in section of the previous graph

Thus, we have numerically verified that $xy = log\left(y\right) + C$ is a solution to the differential equation $\frac{dy}{dx} = \frac{y^2}{1-xy}$.