12.8.1.7

EE24BTECH11018 - Durgi Swaraj Sharma

Exercise 8.1.7 Find the area of the smaller part of the circle $x^2 + y^2 = a^2$ cut off by the line $x = \frac{a}{\sqrt{2}}$.
Theroretical solution

$$x^2 + y^2 = a^2 \tag{0.1}$$
$$y^2 = a^2 - x^2 \tag{0.2}$$

Taking square root on both sides,

$$y = \pm\sqrt{a^2 - x^2} \tag{0.3}$$
$$\tag{0.4}$$

Exploiting symmetry of our problem along the x-axis,

$$y = \sqrt{a^2 - x^2} \tag{0.5}$$

The smaller part of the circle cut by $x = \frac{a}{\sqrt{2}}$ is the region between $x = \frac{a}{\sqrt{2}} and x = a$. So we find the area of this region as follows.

$$\text{New area } = \cdot \int_{\frac{a}{\sqrt{2}}}^{a} \sqrt{a^2 - x^2} \tag{0.6}$$

Changing to polar coordinates,

$$x = a\cos\theta \tag{0.7}$$
$$dx = -a\sin\theta d\theta \tag{0.8}$$

$$\text{Plugging into our integral, } \int_{\frac{\pi}{4}}^{0} \sqrt{a^2 - a^2\cos^2\theta}\,(-a\sin\theta)\,d\theta \tag{0.9}$$

$$= \int_{\frac{\pi}{4}}^{0} \sqrt{a^2\sin^2\theta}\,(-a\sin\theta)\,d\theta \tag{0.10}$$

$$= \int_{\frac{\pi}{4}}^{0} a\sin\theta\,(-a\sin\theta)\,d\theta \tag{0.11}$$

$$= -\int_{\frac{\pi}{4}}^{0} a^2 \sin^2 \theta d\theta \tag{0.12}$$

$$= -a^2 \left[\frac{x}{2} - \frac{\sin 2x}{4}\right]_{\frac{\pi}{4}}^{0} \tag{0.13}$$

$$= -a^2 \left[0 - \frac{\pi}{8} - 0 + \frac{1}{4}\right] \tag{0.14}$$

$$\text{Required area } = 2 \cdot \text{New area } = 2 \cdot a^2 \cdot \frac{\pi - 2}{8} = a^2 \cdot \frac{\pi - 2}{4} \tag{0.15}$$

We have theoretically found the area of the smaller part of the circle cut by the line to be $a^2 \frac{\pi-2}{4}$.

Computational Solution

To find the desired area computationally, we'll be utilising the Trapezoidal Rule.
Following from 0.5

$$y = \sqrt{a^2 - x^2}$$

$$\int_{x_n}^{x_{n+1}} y\, dx = \int_{x_n}^{x_{n+1}} \sqrt{a^2 - x^2} dx \tag{0.16}$$

We can solve the integral on the R.H.S. using the Trapezoidal Rule as follows.

$$A_{n+1} - A_n = \int_{x_n}^{x_{n+1}} y\, dx = h \left[\frac{\sqrt{a^2 - x_{n+1}^2} + \sqrt{a^2 - x_n^2}}{2}\right] \tag{0.17}$$

Where n is the number of iterations we want to calculate in, $h = \frac{a - \frac{a}{\sqrt{2}}}{n}$, A_n is the area calculated till the n^{th} iteration, and $x_0 = \frac{a}{\sqrt{2}}$. The update equation for our area will be:

$$A_{n+1} = A_n + h \left[\frac{\sqrt{a^2 - x_{n+1}^2} + \sqrt{a^2 - x_n^2}}{2}\right] \tag{0.18}$$

And the total area will be computed to be:

$$A_{total} = \sum_{i=0}^{n-1} h \left[\frac{\sqrt{a^2 - x_{i+1}^2} + \sqrt{a^2 - x_i^2}}{2}\right] \tag{0.19}$$

The smaller our step-size, h, is, the more accurate our area calulation will be.
And the required area is twice of the calculated area, as the calculated region reflects

about the x-axis in the total region.

$$\begin{aligned} \text{Area calulcated when } a = 13 \text{ from:} \\ \text{Theoretical Solution is } 48.232289 \\ \text{Computational Solution is } 48.232288 \end{aligned}$$