

12.8.1.7

EE24BTECH11018 - Durgi Swaraj Sharma

Exercise 8.1.7 Find the area of the smaller part of the circle $x^2 + y^2 = a^2$ cut off by the line $x = \frac{a}{\sqrt{2}}$.

Theoretical solution

$$x^2 + y^2 = a^2 \quad (0.1)$$

$$y^2 = a^2 - x^2 \quad (0.2)$$

Taking square root on both sides,

$$y = \pm \sqrt{a^2 - x^2} \quad (0.3)$$

$$(0.4)$$

Exploiting symmetry of our problem along the x -axis,

$$y = \sqrt{a^2 - x^2} \quad (0.5)$$

The smaller part of the circle cut by $x = \frac{a}{\sqrt{2}}$ is the region between $x = \frac{a}{\sqrt{2}}$ and $x = a$. So we find the area of this region as follows.

$$\text{New area} = \int_{\frac{a}{\sqrt{2}}}^a \sqrt{a^2 - x^2} \, dx \quad (0.6)$$

Changing to polar coordinates,

$$x = a \cos \theta \quad (0.7)$$

$$dx = -a \sin \theta \, d\theta \quad (0.8)$$

$$\text{Plugging into our integral, } \int_{\frac{\pi}{4}}^0 \sqrt{a^2 - a^2 \cos^2 \theta} (-a \sin \theta) \, d\theta \quad (0.9)$$

$$= \int_{\frac{\pi}{4}}^0 \sqrt{a^2 \sin^2 \theta} (-a \sin \theta) \, d\theta \quad (0.10)$$

$$= \int_{\frac{\pi}{4}}^0 a \sin \theta (-a \sin \theta) \, d\theta \quad (0.11)$$

$$= - \int_{\frac{\pi}{4}}^0 a^2 \sin^2 \theta d\theta \quad (0.12)$$

$$= -a^2 \left[\frac{x}{2} - \frac{\sin 2x}{4} \right]_{\frac{\pi}{4}}^0 \quad (0.13)$$

$$= -a^2 \left[0 - \frac{\pi}{8} - 0 + \frac{1}{4} \right] \quad (0.14)$$

$$\text{Required area} = 2 \cdot \text{New area} = 2 \cdot a^2 \cdot \frac{\pi - 2}{8} = a^2 \cdot \frac{\pi - 2}{4} \quad (0.15)$$

We have theoretically found the area of the smaller part of the circle cut by the line to be $a^2 \frac{\pi-2}{4}$.

Computational Solution

To find the desired area computationally, we'll be utilising the Trapezoidal Rule.

Following from 0.5

$$y = \sqrt{a^2 - x^2}$$

$$\int_{x_n}^{x_{n+1}} y dx = \int_{x_n}^{x_{n+1}} \sqrt{a^2 - x^2} dx \quad (0.16)$$

We can solve the integral on the R.H.S. using the Trapezoidal Rule as follows.

$$A_{n+1} - A_n = \int_{x_n}^{x_{n+1}} y dx = h \left[\frac{\sqrt{a^2 - x_{n+1}^2} + \sqrt{a^2 - x_n^2}}{2} \right] \quad (0.17)$$

Where n is the number of iterations we want to calculate in, $h = \frac{a - \frac{a}{\sqrt{2}}}{n}$, and A_n is there area calculated till the n^{th} iteration, and $x_0 = \frac{a}{\sqrt{2}}$. The update equation for our area will be:

$$A_{n+1} = A_n + h \left[\frac{\sqrt{a^2 - x_{n+1}^2} + \sqrt{a^2 - x_n^2}}{2} \right] \quad (0.18)$$

And the total area will be computed to be:

$$A_{total} = \sum_{i=0}^{n-1} h \left[\frac{\sqrt{a^2 - x_{i+1}^2} + \sqrt{a^2 - x_i^2}}{2} \right] \quad (0.19)$$

The smaller our step-size, h , is, the more accurate our area calculation will be.

And the required area is twice of the calculated area, as the calculated region reflects

about the x -axis in the total region.

Area calculated when $a = 13$ from:

Theoretical Solution is 48.232289

Computational Solution is 48.232288