10.4.ex.13.1: Finding Roots of a Quadratic

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Problem Statement

Find the roots of the quadratic equation:

$$3x^2 - 5x + 2 = 0 \tag{0.1}$$

We will use two methods:

Newton-Raphson Method

Eigenvalue Approach via QR Decomposition
(using Householder Reflections)

Newton-Raphson Method: Overview

Idea: Iteratively improve an initial guess to find a root of $f(x) = 0$.
Iteration Formula:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{0.2}$$

For our equation:

$$f(x) = 3x^2 - 5x + 2, \quad f'(x) = 6x - 5 \tag{0.3}$$

$$\Rightarrow \quad x_{n+1} = x_n - \frac{3x_n^2 - 5x_n + 2}{6x_n - 5}. \tag{0.4}$$

Newton-Raphson Method: How It Works

- Start with an initial guess x_0.
- Compute the function value $f(x_n)$ and its derivative $f'(x_n)$.
- Update the guess using:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{0.5}$$

- Repeat until the guess converges to a root.

Key Points:

- Requires a good initial guess.
- Convergence is rapid if the guess is close to the actual root.
- May fail or converge slowly if the derivative is small or the guess is poor.

QR Decomposition via Householder Reflections: Overview

Goal: Decompose a matrix A into $A = QR$, where:

- Q is an orthogonal matrix.
- R is an upper triangular matrix.

This is useful for finding eigenvalues
(the roots of the characteristic equation) using the QR algorithm.

Householder Reflections: The Concept

Householder Reflection: A transformation that reflects a vector about a hyperplane.
Reflection Matrix:

$$H = I - 2\frac{vv^T}{v^T v}, \tag{0.6}$$

where v is a vector chosen such that when H is applied to a vector, it zeros out all but the first component.

Using Householder Reflections for QR Decomposition

1. **Select a column:** For the kth column of A starting from row k, choose vector a_k.
2. **Construct vector v:**
$$v = a_k + \operatorname{sign}(a_{k1}) \|a_k\| e_1, \tag{0.7}$$
where e_1 is the standard basis vector.
3. **Form the Householder matrix:**
$$H_k = I - 2\frac{vv^T}{v^T v}. \tag{0.8}$$
4. **Apply transformation:** Compute
$$A^{(k)} = H_k A^{(k-1)}, \tag{0.9}$$
so that the subdiagonal elements in the kth column are zeroed.
5. **Accumulate Q:** The product of the Householder matrices gives
$$Q^T = H_m H_{m-1} \cdots H_1, \tag{0.10}$$
so that $Q = H_1 H_2 \cdots H_m$.

QR Algorithm for Eigenvalue Computation

- Once A is decomposed as $A = QR$, form a new matrix:

$$A_1 = RQ. \tag{0.11}$$

- A_1 is similar to A (same eigenvalues).
- Repeat the QR decomposition on A_1 to obtain A_2, and so on.
- After sufficient iterations, A_k converges to an upper triangular (*Schur*) form.
- The eigenvalues are the diagonal entries of the converged matrix.

Application to the Quadratic Equation

Companion Matrix: For $3x^2 - 5x + 2 = 0$, first normalize to get a monic polynomial:

$$x^2 - \frac{5}{3}x + \frac{2}{3} = 0. \tag{0.12}$$

The companion matrix is:

$$C = \begin{bmatrix} 0 & \frac{2}{3} \\ 1 & -\frac{5}{3} \end{bmatrix}. \tag{0.13}$$

- Applying the QR algorithm (using Householder reflections) on C yields its eigenvalues.
- The eigenvalues are the roots of the quadratic: approximately 0.666667 and 1.000000.

Summary

Newton-Raphson Method

- Iterative scheme:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}. \tag{0.14}$$

- Converges rapidly with a good initial guess.

QR Decomposition via Householder Reflections

- Uses reflections to zero out subdiagonal entries.
- Decomposes A into Q (orthogonal) and R (upper triangular).
- The QR algorithm applied to the companion matrix finds the eigenvalues.

Both methods confirm the roots:

$$x \approx 0.666667 \quad \text{and} \quad x \approx 1.000000. \tag{0.15}$$

Plot

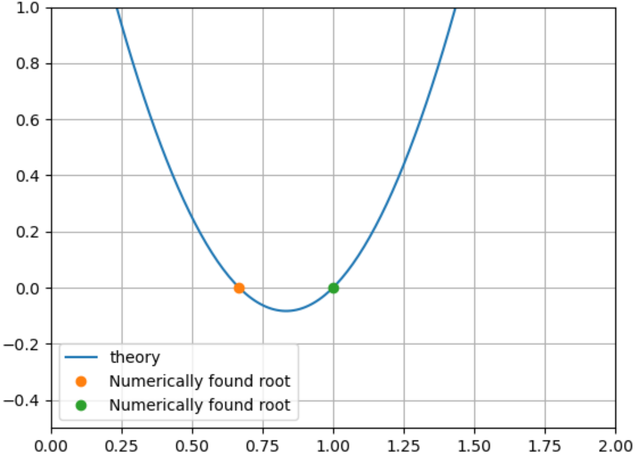

Figure: Plotting the result from Newton-Raphson method