10.3.5.4.3

EE24BTECH11018 - Durgi Swaraj Sharma

Question: Yash scored 40 marks in a test, getting 3 marks for each right answer and losing 1 mark for each wrong answer. Had 4 marks been awarded for each correct answer and 2 marks been deducted for each incorrect answer, then Yash would have scored 50 marks. How many questions were there in the test?

Solution:

Writing the problem in mathematical equations,

$$3x - 1y = 40 \tag{1}$$

$$4x - 2y = 50 \tag{2}$$

$$x + y = ? \tag{3}$$

where x represents the number of correct answers, y represents the number of incorrect answers. $x + y$ gives us the total number of questions in the test.

Representing using matrices,

$$\begin{pmatrix} 3 & -1 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 40 \\ 50 \end{pmatrix} \tag{4}$$

LU Decomposition

We shall solve this system of equations by LU Decomposition. Any non-sigular matrix can be represented as a product of a lower triangular matrix L and an upper triangular matrix U

$$A\mathbf{x} = LU\mathbf{x} = \mathbf{b} \tag{5}$$

Applying row reduction on A to find U,

$$\begin{pmatrix} 3 & -1 \\ 4 & -2 \end{pmatrix} \xrightarrow{R_2 \leftrightarrow R_2 - \frac{4}{3}R_1} \begin{pmatrix} 3 & -1 \\ 0 & -\frac{2}{3} \end{pmatrix} \tag{6}$$

Let

$$L = \begin{pmatrix} 1 & 0 \\ l_{21} & 1 \end{pmatrix} \tag{7}$$

l_{21} is the multiplier used to zero a_{21}, so $l_{21} = \frac{4}{3}$.

Now,

$$A = \begin{pmatrix} 3 & -1 \\ 4 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{4}{3} & 1 \end{pmatrix} \begin{pmatrix} 3 & -1 \\ 0 & -\frac{2}{3} \end{pmatrix} \tag{8}$$

We have thus obtained LU Decomposition of the matrix A.

The LU Decomposition of matrix A can also be obtained by Doolittle's Algorithm. This gives us update equations to construct the L and U matrix.

Elements of U matrix:
For each column j,

$$U_{ij} = \begin{cases} A_{ij} & i = 0 \\ A_{ij} - \sum_{k=0}^{i-1} L_{ik} U_{kj} & i > 0 \end{cases} \tag{9}$$

Elements of L matrix:
For each row i,

$$L_{ij} = \begin{cases} \frac{A_{ij}}{U_{ij}} & j = 0 \\ \frac{A_{ij} - \sum_{k=0}^{j-1} L_{ik} U_{kj}}{U_{ij}} & j > 0 \end{cases} \tag{10}$$

The above proccess decomposes any non-sigular matrix A into an upper-triangular matrix U and a lower-triangular matrix L.
Now, let

$$U\mathbf{x} = \mathbf{y} \tag{11}$$

$$L\mathbf{y} = \mathbf{b} \tag{12}$$

Substituting the values,

$$\begin{pmatrix} 1 & 0 \\ \frac{4}{3} & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 40 \\ 50 \end{pmatrix} \tag{13}$$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 40 \\ -\frac{10}{3} \end{pmatrix} \tag{14}$$

Backsubstituting,

$$\begin{pmatrix} 3 & -1 \\ 0 & -\frac{2}{3} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 40 \\ -\frac{10}{3} \end{pmatrix} \tag{15}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 15 \\ 5 \end{pmatrix} \tag{16}$$

Thus, the system of equations is solved at $\mathbf{x} = \begin{pmatrix} 15 \\ 5 \end{pmatrix}$. The total number of questions in the test is **20**.

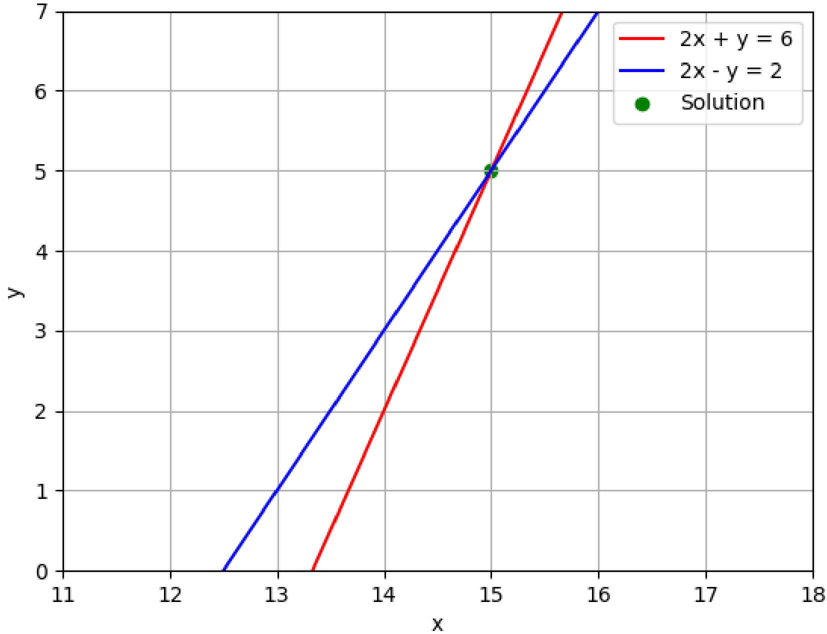

Fig. 1: Solving the system of equations, $3x - 1y = 40, 4x - 2y = 50$