Problem: A box contains 1 red and 3 identical white balls. Two balls are drawn at random in succession without replacement. Write the sample space for this experiment.
Additional: Find the probability mass function too.

Solution:
The sample space σ is given by:

$$\sigma = \{WW, RW, WR\} \tag{0.1}$$

We shall define random variable X, which denotes the position at which the Red ball is drawn.
Then X takes the values

$$X = \begin{cases} 0, & \text{red ball never drawn} \\ 1, & \text{red ball drawn first} \\ 2, & \text{red ball drawn second} \end{cases} \tag{0.2}$$

There is a total of $4 \cdot 3$ outcomes that can happen, with 4 in first draw and 3 in second. These outcomes relate with X as:

- $X = 0$ when both balls drawn are white, and there's a total of $3 \cdot 2$ these outcomes.
- $X = 1$ when the first ball is red, the second ball has to white. Giving a total of $1 \cdot 3$.
- $X = 2$ when the second ball is red, the first has to be white. Giving a total of $3 \cdot 1$.

This translates to the Probability Mass Function as

$$p_X(k) = \begin{cases} \frac{6}{12} = 0.50, & k = 0 \\ \frac{3}{12} = 0.25, & k = 1 \\ \frac{3}{12} = 0.25, & k = 2 \end{cases} \tag{0.3}$$

Computing the Probabilities

We perform a Monte Carlo simulation to verify these probabilities by running a large number of trials. The algorithm proceeds as follows:

1. Randomly select the first ball: Red with probability $\frac{1}{4}$, White with probability $\frac{3}{4}$.
2. If the first ball is White, randomly select the second ball: Red with probability $\frac{1}{3}$, White with probability $\frac{2}{3}$.
3. Count occurrences of each event (WW, RW, WR).
4. Compute empirical probabilities as the relative frequencies of these events.

This is how the code works:

- We generate X_1 by selecting a random number between 0 and 3. We set $X_1 = 1$ if the chosen number is 0, which occurs with probability 0.25.
- If $X_1 = 1$, we set $X_2 = 0$ since the only red ball has already been drawn.
- Otherwise, we generate X_2 by selecting a random number between 0 and 2, setting $X_2 = 1$ if the chosen number is 0, which occurs with probability 0.33.

Outcome Definitions:

$$\begin{aligned} RW &\quad X_1 = 1, X_2 = 0 \\ WR &\quad X_1 = 0, X_2 = 1 \\ WW &\quad X_1 = 0, X_2 = 0 \end{aligned}$$

The estimated probabilities from the simulation are:

$$P(WW) \approx 0.50 \tag{0.4}$$

$$P(RW) \approx 0.25 \tag{0.5}$$

$$P(WR) \approx 0.25 \tag{0.6}$$