10.3.5.4.3

EE24BTECH11018 - Durgi Swaraj Sharma

Question

Problem Statement:
Yash scored 40 marks in a test, getting 3 marks for each right answer and losing 1 mark for each wrong answer. Had 4 marks been awarded for each correct answer and 2 marks been deducted for each incorrect answer, then Yash would have scored 50 marks. How many questions were there in the test?

Mathematical Formulation

Let

- $x =$ number of correct answers,
- $y =$ number of incorrect answers.

Then, we can write the two scenarios as:

$$3x - y = 40, \tag{0.1}$$

$$4x - 2y = 50. \tag{0.2}$$

Our goal is to find $x + y$, the total number of questions.

Matrix Representation

We represent the system above in matrix form:

$$\begin{pmatrix} 3 & -1 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 40 \\ 50 \end{pmatrix}. \tag{0.3}$$

LU Decomposition Overview

Any non-singular matrix A can be decomposed as:

$$A = LU, \tag{0.4}$$

where:

- L is a lower triangular matrix,
- U is an upper triangular matrix.

Thus, we have:

$$A \begin{pmatrix} x \\ y \end{pmatrix} = LU \begin{pmatrix} x \\ y \end{pmatrix} = \vec{b}. \tag{0.5}$$

Finding U via Row Reduction

Consider the coefficient matrix

$$A = \begin{pmatrix} 3 & -1 \\ 4 & -2 \end{pmatrix}. \tag{0.6}$$

To form U, we eliminate the subdiagonal element in the second row by performing the row operation:

$$R_2 \to R_2 - \frac{4}{3}R_1. \tag{0.7}$$

This yields:

$$U = \begin{pmatrix} 3 & -1 \\ 0 & -\frac{2}{3} \end{pmatrix}. \tag{0.8}$$

Constructing L

The lower triangular matrix L captures the multipliers used in the row operations. Since the multiplier is

$$l_{21} = \frac{4}{3}, \tag{0.9}$$

we set

$$L = \begin{pmatrix} 1 & 0 \\ \frac{4}{3} & 1 \end{pmatrix}. \tag{0.10}$$

Thus, the LU decomposition of A is:

$$A = \begin{pmatrix} 1 & 0 \\ \frac{4}{3} & 1 \end{pmatrix} \begin{pmatrix} 3 & -1 \\ 0 & -\frac{2}{3} \end{pmatrix}. \tag{0.11}$$

Doolittle's Algorithm (Overview)

An alternative method is Doolittle's Algorithm which provides update equations:

For U: For each column j,

$$U_{ij} = \begin{cases} A_{ij} & \text{if } i = 1, \\ A_{ij} - \displaystyle\sum_{k=1}^{i-1} L_{ik} U_{kj} & \text{if } i > 1. \end{cases} \tag{0.12}$$

For L: For each row i,

$$L_{ij} = \begin{cases} \frac{A_{ij}}{U_{jj}} & \text{if } j = i, \\ \frac{A_{ij} - \displaystyle\sum_{k=1}^{j-1} L_{ik} U_{kj}}{U_{jj}} & \text{if } j < i. \end{cases} \tag{0.13}$$

This process decomposes any non-singular matrix A into L and U.

Solving the System via LU Decomposition

We now solve:

$$LU \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 40 \\ 50 \end{pmatrix}. \tag{0.14}$$

This involves two steps:

1. Solve $L\,\vec{y} = \vec{b}$ for $\vec{y}$,
2. Solve $U\,\vec{x} = \vec{y}$ for $\vec{x}$ by back-substitution.

Substitute the known values:

$$L \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{4}{3} & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 40 \\ 50 \end{pmatrix}. \tag{0.15}$$

This gives:

$$y_1 = 40, \quad \frac{4}{3}(40) + y_2 \quad = 50 \quad \Rightarrow \quad y_2 = 50 - \frac{160}{3} = -\frac{10}{3}. \tag{0.16}$$

Next, solve:

$$U \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ 0 & -\frac{2}{3} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 40 \\ -\frac{10}{3} \end{pmatrix}. \tag{0.17}$$

Back-substitution yields:

$$-\frac{2}{3}x_2 = -\frac{10}{3} \quad \Rightarrow \quad x_2 = 5, \tag{0.18}$$

$$3x_1 - x_2 = 40 \quad \Rightarrow \quad 3x_1 = 45 \quad \Rightarrow \quad x_1 = 15. \tag{0.19}$$

Final Answer

We have obtained:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 15 \\ 5 \end{pmatrix}. \tag{0.20}$$

Thus, the total number of questions is:

$$x + y = 15 + 5 = 20. \tag{0.21}$$

Answer: The test consisted of **20 questions**.

Illustration

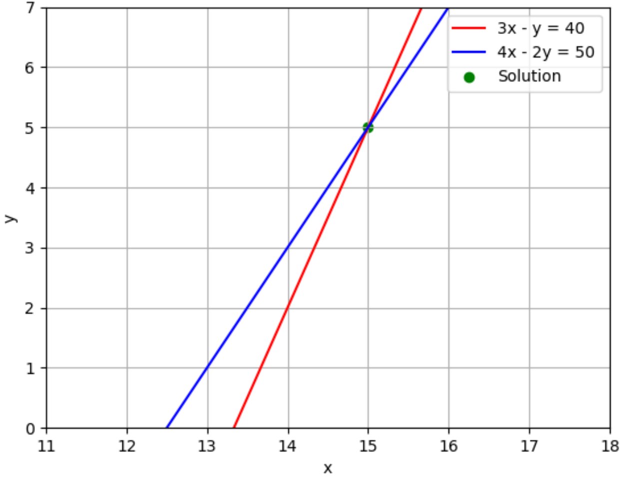

Figure: Solving the system: $3x - y = 40, \quad 4x - 2y = 50.$