9.5.6

EE24BTECH11018 - Durgi Swaraj Sharma

Question:

Find the area of the region lying above the X axis and included between the circle $x^2+y^2 = 8x$ and inside of the parabola $y^2 = 4x$. (12, 2018)

Variable	Description
e	Eccentricity of conic
F	Focus of conic
I	Identity matrix
$\mathbf{n}^\top \mathbf{x} = c$	Equation of directrix
n	Slope of normal to directrix
f	$\|\mathbf{n}\|^2 \|\mathbf{F}\|^2 - c^2e^2$
V	A symmetric matrix given by eigenvalue decomposition
u	Vertex of conic with same directrix

TABLE 0: Variables Used

Solution:

Equation of a circle is of form $\mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0$ with

$$\mathbf{u} = \begin{pmatrix} -4 \\ 0 \end{pmatrix} \tag{0.1}$$

$$f = \|\mathbf{u}\|^2 - r^2 \tag{0.2}$$

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{0.3}$$

Equation of parabola with directrix $\mathbf{n}^\top \mathbf{x} = c$ is given by,

$$g(\mathbf{x}) = \mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{0.4}$$

$$\mathbf{V} = \|\mathbf{n}\|^2 \mathbf{I} - e^2 \mathbf{n}\mathbf{n}^\top \tag{0.5}$$

$$\mathbf{u} = ce^2\mathbf{n} - \|\mathbf{n}\|^2 \mathbf{F} \tag{0.6}$$

$$f = \|\mathbf{n}\|^2 \|\mathbf{F}\|^2 - c^2e^2 \tag{0.7}$$

and for the parabola $y^2 = 4x$, equation of directrix is, $\begin{pmatrix} -1 & 0 \end{pmatrix} \mathbf{x} = 1$

$$\mathbf{V} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{0.8}$$

$$\mathbf{u} = \begin{pmatrix} -2 \\ 0 \end{pmatrix} \tag{0.9}$$

$$f = 0 \tag{0.10}$$

The intersection of two conics with parameters $\mathbf{V}_i, \mathbf{u}_i, f_i, i = 1, 2$ is defined as,

$$\mathbf{x}^\top (\mathbf{V}_1 - \mathbf{V}_2)\,\mathbf{x} + 2\,(\mathbf{u}_1 - \mathbf{u_2})^\top \mathbf{x} + (f_1 - f_2) = 0 \tag{0.11}$$

On solving we get the points of intersection to be $\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 4 \end{pmatrix}$.

Area between the two curves above X axis is,

$$\int_0^4 2\sqrt{x}dx - \int_0^4 \sqrt{x^2 - 8x}dx = \frac{12\pi - 32}{3} = 1.899\text{approx.} \tag{0.12}$$

The area between the curves $y^2 = 4x, x^2 + y^2 = 8x$ above the X axis is around 1.899 units

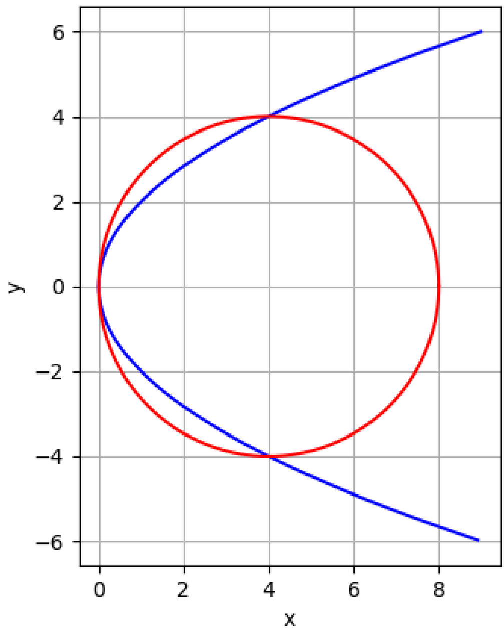

Fig. 0.1: Required parabola and circle