EE24BTECH11018 - Durgi Swaraj Sharma

Question:

Find the area of the region lying above the *X* axis and included between the circle $x^2 + y^2 = 8x$ and inside of the parabola $y^2 = 4x$. (12, 2018)

Variable	Description
e	Eccentricity of conic
F	Focus of conic
I	Identity matrix
$\mathbf{n}^{T}\mathbf{x} = c$	Equation of directrix
n	Slope of normal to directrix
f	$\ \mathbf{n}\ ^2 \ \mathbf{F}\ ^2 - c^2 e^2$
V	A symmetric matrix given by eigenvalue decomposition
u	Vertex of conic with same directrix

TABLE 0: Variables Used

Solution:

Equation of a circle is of form $\mathbf{x}^{\mathsf{T}}\mathbf{V}\mathbf{x} + 2\mathbf{u}^{\mathsf{T}}\mathbf{x} + f = 0$ with

$$\mathbf{u} = \begin{pmatrix} -4\\0 \end{pmatrix} \tag{0.1}$$

$$f = \|\mathbf{u}\|^2 - r^2 \tag{0.2}$$

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{0.3}$$

Equation of parabola with directrix $\mathbf{n}^{\mathsf{T}}\mathbf{x} = c$ is given by,

$$g(\mathbf{x}) = \mathbf{x}^{\mathsf{T}} \mathbf{V} \mathbf{x} + 2 \mathbf{u}^{\mathsf{T}} \mathbf{x} + f = 0 \tag{0.4}$$

$$\mathbf{V} = \|\mathbf{n}\|^2 \mathbf{I} - e^2 \mathbf{n} \mathbf{n}^{\mathsf{T}} \tag{0.5}$$

$$\mathbf{u} = ce^2 \mathbf{n} - ||\mathbf{n}||^2 \mathbf{F} \tag{0.6}$$

$$f = \|\mathbf{n}\|^2 \|\mathbf{F}\|^2 - c^2 e^2 \tag{0.7}$$

and for the parabola $y^2 = 4x$, equation of directrix is, $\begin{pmatrix} -1 & 0 \end{pmatrix} \mathbf{x} = 1$

$$\mathbf{V} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{0.8}$$

$$\mathbf{u} = \begin{pmatrix} -2\\0 \end{pmatrix} \tag{0.9}$$

$$f = 0 \tag{0.10}$$

The intersection of two conics with parameters V_i , u_i , f_i , i = 1, 2 is defined as,

$$\mathbf{x}^{\mathsf{T}} (\mathbf{V}_1 - \mathbf{V}_2) \mathbf{x} + 2 (\mathbf{u}_1 - \mathbf{u}_2)^{\mathsf{T}} \mathbf{x} + (f_1 - f_2) = 0$$
 (0.11)

On solving we get the points of intersection to be $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 4 \\ 4 \end{pmatrix}$.

Area between the two curves above X axis is,

$$\int_0^4 2\sqrt{x}dx - \int_0^4 \sqrt{x^2 - 8x}dx = \frac{12\pi - 32}{3} = 1.899approx.$$
 (0.12)

The area between the curves $y^2 = 4x$, $x^2 + y^2 = 8x$ above the X axis is around 1.899 units

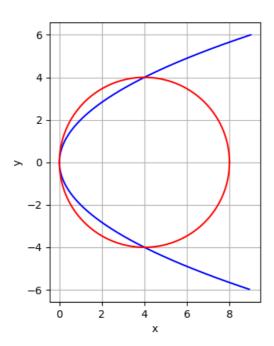


Fig. 0.1: Required parabola and circle