PROBLEM SET - 4

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PROBLEM 1:

Given $\dot{x}(t) = \frac{1}{12}\begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix} x(t) + e^{t/2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t)$

To determine if system is controllable

Solution

The state space is represented by

$$\dot{x}(t) = A\vec{x}(t) + B\vec{u}(t) \quad \ldots\ldots (1)$$

where A and B are the dynamics of the system

Comparing (1) with the equation given in the problem statement, we get

$$A = \begin{bmatrix} \frac{5}{12} & \frac{1}{12} \\ \frac{1}{12} & \frac{5}{12} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} e^{t/2} \\ e^{t/2} \end{bmatrix}$$

Now to determine if the given system is controllable or not, According to the controllability law, the system is controllable if the grammian of controllability $W_c(0,t^*)$ is positive definite (invertible)

We also know that the grammian of controllability is invertible if the $n \times nm$ matrix of controllability satisfies rank$(CB_K \; AB_K \; A^2B_K \; ---- \; A^{n-1}B_K) = n$

To find rank $(CB_K \; AB_K \; --- \; A^{n-1}B_K)$

$$B_K = \begin{bmatrix} e^{t/2} \\ e^{t/2} \end{bmatrix} \qquad AB_K = \begin{bmatrix} \frac{5}{12} & \frac{1}{12} \\ \frac{1}{12} & \frac{5}{12} \end{bmatrix} \begin{bmatrix} e^{t/2} \\ e^{t/2} \end{bmatrix}$$

$$AB_K = \begin{bmatrix} \frac{5}{12}e^{t/2} + \frac{1}{12}e^{t/2} \\ \frac{1}{12}e^{t/2} + \frac{5}{12}e^{t/2} \end{bmatrix}$$

$$AB_K = \begin{bmatrix} \frac{6}{12}e^{t/2} \\ \frac{6}{12}e^{t/2} \end{bmatrix}$$

(2)

$$AB_c = \begin{bmatrix} \frac{1}{2}e^{t/2} \\ \frac{1}{2}e^{t/2} \end{bmatrix}$$

$$\text{rank}\,[B \quad AB] = \text{rank}\left[\begin{bmatrix} e^{t/2} & \frac{1}{2}e^{t/2} \\ e^{t/2} & \frac{1}{2}e^{t/2} \end{bmatrix}\right]$$

So, as we can observe that the second colum is a multiple of the first column (If we multiply column 1 by $1/2$, it is the same as the 2^{nd} column)

So the rank of the matrix is 1

Here $n = 2$ and rank $\neq 2$

So, we can deduce that the grammian of the system is not invertible

Thus, the system is Uncontrollable $\square$

(3)

PROBLEM 2 : Given

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} \gamma_1(t) & \gamma_2(t) \\ -\gamma_2(t) & \gamma_1(t) \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

$\gamma_1(t)$ and $\gamma_2(t)$ are continious function of t

To compute the state transition matrix of the system and the solution to the state equation

Solution : Comparing the given equation to

$$\dot{X}(t) = A\vec{X}(t) + B\vec{U}(t)$$

$$\text{So, } A = \begin{bmatrix} \gamma_1(t) & \gamma_2(t) \\ -\gamma_2(t) & \gamma_1(t) \end{bmatrix} \quad \& \quad B\vec{U}(t) = 0$$

We know that

$$\Phi(t,t_0) = M\, e^{A(t-t_0)}\, M^{-1}$$ (where M is the modal matrix)

To find Eigen Vectors : first we find the eigen values

$$|A - \lambda I| = \begin{vmatrix} \gamma_1(t) & \gamma_2(t) \\ -\gamma_2(t) & \gamma_1(t) \end{vmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} \gamma_1(t) - \lambda & \gamma_2(t) \\ -\gamma_2(t) & \gamma_1(t) - \lambda \end{vmatrix} = 0$$

$$\Rightarrow (\gamma_1(t) - \lambda)^2 + (\gamma_2(t))^2 = 0$$

$$(\gamma_1(t))^2 + (\lambda)^2 - 2\gamma_1(t)\lambda + (\gamma_2(t))^2 = 0$$

$$\lambda^2 - 2\lambda\gamma_1(t) + ((\gamma_1(t))^2 + (\gamma_2(t))^2) = 0$$

The roots of the equation are :-

$$= \frac{-2\gamma_1(t) \pm \sqrt{4(\gamma_1(t))^2 - 4(1)((\gamma_1(t))^2 + (\gamma_2(t))^2}}{2\times 1}$$

$$= \frac{-2\gamma_1(t) \pm \sqrt{-4(\gamma_2(t))^2}}{2} = -\gamma_1(t) \pm \frac{\gamma_2(t)\sqrt{-4}}{2}$$

$$: -\gamma_1(t) \pm \frac{\gamma_2 t\, 2i}{2}$$

Roots are $\Rightarrow -\gamma_1(t) \pm i\gamma_2(t)$

$\lambda = -\gamma_1(t) + i\gamma_2(t)$ and $\lambda = -\gamma_1(t) - i\gamma_2(t)$

<u>for, $\lambda = -\gamma_1(t) + i\gamma_2(t)$</u>

$$\begin{bmatrix} -i\gamma_2 & \gamma_2 \\ -\gamma_2 & -i\gamma_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\gamma_2\begin{bmatrix} -i & 1 \\ -1 & i \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

<u>for, $\lambda = -\gamma_1(t) - i\gamma_2(t)$</u>

$$\begin{bmatrix} i\gamma_2 & \gamma_2 \\ -\gamma_2 & i\gamma_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\gamma_2\begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\begin{bmatrix} -i & 1 \\ -1 & i \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$-ix_1 + x_2 = 0$

$-x_1 + ix_2 = 0$

$X_1 = \begin{bmatrix} 1 \\ i \end{bmatrix}$

$$\begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$ix_1 + x_2 = 0$

$-x_1 + ix_2 = 0$

$X_2 = \begin{bmatrix} 1 \\ -i \end{bmatrix}$

We know that $M = [X_1 \ \ X_2]$, Substituting value of X_1 & X_2

$$M = \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \qquad \boxed{|P| = -i - i = -2i}$$

$$M^{-1} = \frac{\text{Adj}\, M}{|M|} = -\frac{1}{2i}\begin{bmatrix} -i & -1 \\ -i & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} & \frac{-i}{2} \\ \frac{1}{2} & \frac{i}{2} \end{bmatrix}$$

$$\Phi(t_0, t_6) = M e^{A(t_f - t_0)} M^{-1}$$

Where $e^{A(t_f - t_0)}$ can be computed via Taylor Series expansion, this can be written as

$$e^{A(t_6 - t_0)} = \begin{bmatrix} e^{\int_t^{t_6} \gamma_1(t) + i\gamma_2(t)\,dt} & 0 \\ 0 & e^{\int_{t_0}^{t_6} \gamma_1(t) - i\gamma_2(t)\,dt} \end{bmatrix}$$

So, $\Phi(t_0, t_f) =$

$$= \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \begin{bmatrix} e^{\int_t^{t_f} \gamma_1(t) + i\gamma_2(t)\,dt} & 0 \\ 0 & e^{\int_{t_0}^{t_f} \gamma_1(t) - i\gamma_2(t)\,dt} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{i}{2} \\ \frac{1}{2} & \frac{i}{2} \end{bmatrix}$$

$$= \begin{bmatrix} e^{\int_t^{t_f} \gamma_1(t) + i\gamma_2(t)\,dt} & e^{\int_{t_0}^{t_f} \gamma_1(t) - i\gamma_2(t)\,dt} \\ i\, e^{\int_t^{t_f} \gamma_1(t) + i\gamma_2(t)\,dt} & -i\, e^{\int_{t_0}^{t_f} \gamma_1(t) - i\gamma_2(t)\,dt} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{i}{2} \\ \frac{1}{2} & \frac{i}{2} \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} \frac{1}{2}\left[e^{\int_t^{t_f} \gamma_1(t) + i\gamma_2(t)\,dt} + e^{\int_{t_0}^{t_f} \gamma_1(t) - i\gamma_2(t)\,dt} \right] & \frac{i}{2}\left[e^{\int_t^{t_f} \gamma_1(t) + i\gamma_2(t)\,dt} + e^{\int_{t_0}^{t_f} \gamma_1(t) - i\gamma_2(t)\,dt} \right] \\ \frac{i}{2}\left[e^{\int_t^{t_f} \gamma_1(t) + i\gamma_2(t)\,dt} - e^{\int_t^{t_f} \gamma_1(t) + i\gamma_2(t)\,dt} \right] & \frac{1}{2}\left[e^{\int_t^{t_f} \gamma_1(t) + i\gamma_2(t)\,dt} + e^{\int_{t_0}^{t_f} \gamma_1(t) - i\gamma_2(t)\,dt} \right] \end{bmatrix}}_{\text{This matrix is the state matrix of the system. } \Phi(t_0, t_f)}$$

The solution to the state equation is given by

$$x(t) = \Phi(t_0, t_f)\; x\; t_0$$

PROBLEM 3

Given, J_1 - represents the inertia of the space shuttle robot arm

J_2 - represents the inertia of the space shuttle

and Equations of motion are,

$$J_1 \ddot{q}_1 = \tau$$

$$J_2 \ddot{q}_2 = \tau$$

To write the equation in state space form and show that it is uncontrollable & to discuss the implications and suggest possible solutions

Solution From the given data

$$J_1 \ddot{q}_1 = J_2 \ddot{q}_2 = \tau \Rightarrow \boxed{\ddot{q}_1 = \frac{\tau}{J_1} \ \& \ \ddot{q}_2 = \frac{\tau}{J_2}}$$

To compute the state space form, lets consider the following

state variables $\begin{cases} x_1 = q_1 & x_2 = \dot{q}_1 & x_3 = q_2 & x_4 = \dot{q}_2 \\ \dot{x}_1 = \dot{q}_1 & \dot{x}_2 = \ddot{q}_1 & \dot{x}_3 = \dot{q}_2 & \dot{x}_4 = \ddot{q}_2 \end{cases}$

as the state variables.

state space equation is,

$$\dot{X} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} \dot{q}_1 \\ \ddot{q}_1 \\ \dot{q}_2 \\ \ddot{q}_2 \end{bmatrix}$$

$$Ax + BU = [\dot{q}_1 \; \ddot{q}_1 \; q_2 \; \ddot{q}_2]^T$$

So, $$Ax + BU = \left[\dot{q}_1 \; \frac{\tau}{J_1} \; \dot{q}_2 \; \frac{\tau}{J_2}\right]$$

$$\begin{bmatrix} \dot{q}_1 \\ 0 \\ \dot{q}_2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \tau/J_1 \\ 0 \\ \tau/J_2 \end{bmatrix} \Rightarrow \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} q_1 \\ \dot{q}_1 \\ q_2 \\ \dot{q}_2 \end{bmatrix}}_{X} + \underbrace{\begin{bmatrix} 0 \\ 1/J_1 \\ 0 \\ 1/J_2 \end{bmatrix}}_{B} \underbrace{\tau}_{U}$$

Comparing to $AX+BU$

Then in state space, we have

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1/J_1 \\ 0 \\ 1/J_2 \end{bmatrix} [\tau]$$

To show that the system is uncontrollable, the grammian should not be invertible that is the rank of $(B \; AB_k \cdots A^{n-1}B_k)$ should not be equal to n (4 in this case)

$$\text{Rank of} = \begin{bmatrix} 0 & J_1 & 0 & 0 \\ 1/J_1 & 0 & 0 & 0 \\ 0 & J_2 & 0 & 0 \\ 1/J_2 & 0 & 0 & 0 \end{bmatrix}$$

$$AB = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1/J_1 \\ 0 \\ 1/J_2 \end{bmatrix} = \begin{bmatrix} J_1 \\ 0 \\ J_2 \\ 0 \end{bmatrix}$$

So the rank of the matrix = 2

Thus the system is uncontrollable

Implications of uncontrollability →

A uncontrollable system is basically a system whose certain states cannot be controlled through a control input. Thus, it will hinder the maneuverability. This is a significant drawback since space applications require precise controls.

Possible solutions to this

→ Augmenting the system input with a control input vector u such that the grammian of controllability is invertible

PROBLEM 4:

Given, Linear 2nd order system.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ -2 \end{bmatrix} u$$

To find linear state feedback control, $u = k_1 x_1 + k_2 x_2$ so that closed loop system has poles at $s = -2, 2$

Solution,

Comparing the equation with $\dot{\vec{x}}(t) = A\vec{x}(t) + B\vec{u}(t)$

$$A = \begin{bmatrix} 1 & -3 \\ 1 & -2 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

First, we check if the system is controllable or not, for that lets check if rank of $(B \; AB)$ is equal to n or not

$$AB = \begin{bmatrix} 1 & -3 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \end{bmatrix}$$

$$\text{rank}\begin{pmatrix} B & AB \end{pmatrix} \Rightarrow \text{rank}\begin{bmatrix} 1 & 7 \\ -2 & 5 \end{bmatrix}$$

Rank = 2, and n is also equal to 2

Thus the system is controllable

For a state feedback $u = Kx$, the closed loop system matrix A_c will be

$$A_c = A + BK$$

$$A_c = \begin{bmatrix} 1 & -3 \\ 1 & -2 \end{bmatrix} + \begin{bmatrix} 1 \\ -2 \end{bmatrix} \begin{bmatrix} K_1 & K_2 \end{bmatrix}$$

$$A_c = \begin{bmatrix} 1+K_1 & -3+K_2 \\ 1-2K_1 & -2-2K_2 \end{bmatrix}$$

The characteristic equation of closed loop system is given by $\det(\lambda I - A_c) = 0$

$$\det\left(\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1+K_1 & -3+K_2 \\ 1-2K_1 & -2-2K_2 \end{bmatrix}\right) = 0$$

$$\det\left(\begin{bmatrix} \lambda-1-K_1 & 3-K_2 \\ -1+2K_1 & \lambda+2+2K_2 \end{bmatrix}\right) = 0.$$

$$\left[(\lambda-1-k_1)(\lambda+2+k_2)\right]-\left[(-1+2k_1)(3-k_2)\right]=0$$

$$(\lambda^2+2\lambda+k_2\lambda-\lambda-2-k_2-k_1\lambda-2k_1-k_1k_2)$$
$$-(-3+k_2+6k_1-2k_1k_2)=0$$

$$\lambda^2+\lambda+\lambda(2k_2-k_1)+1-8k_1-3k_2=0$$
$$\lambda^2+\lambda(1+2k_2-k_1)+(1-8k_1-3k_2)=0 \quad -(1)$$

Now for the given poles, $s=-2,2$
the characteristic equation is given by

$$(\lambda+2)(\lambda-2)=0$$
$$\lambda^2-4=0 \quad -(2)$$

Comparing co-efficients of (1) and (2), we get

$$(1+2k_2-k_1)=0$$
$$(1-8k_1-3k_2)=-4$$

Solving for k_1 and k_2, we get.

$$k_2 = \frac{k_1 - 1}{2}$$

So, $$1 - 8k_1 - 3\left[\frac{k_1 - 1}{2}\right] = -4$$

$$-8k_1 - 3\left[\frac{k_1 - 1}{2}\right] = -5$$

$$-16k_1 - 3k_1 + 3 = -10$$

$$-19k_1 = -13$$

$$k_1 = \frac{-13}{-19}$$

$$k_1 = \frac{13}{19}$$

$$k_2 = \frac{\frac{13}{19} - 1}{2} = \frac{13 - 19}{19 \times 2} = \frac{-\cancel{6}^{3}}{19 \times \cancel{2}} = \frac{-3}{19}$$

So, $u = k_1 x_1 + k_2 x_2$

$$\boxed{u = \frac{13}{19} x_1 - \frac{3}{19} x_2}$$

PROBLEM 5:

Given $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$

To find → If poles can be placed at -2

If system can be stabilized

Solution,

from given data, the equation is of the form $\dot{x}(t) = A\vec{x}(t) + B\vec{u}(t)$

So, $A = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$ $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

first, lets check for stability

$\text{rank}(B \ AB \ A^2B \dots) = n$?

$B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $AB = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$

$\text{rank}(B \ AB) = \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix}$

$\text{rank}(B \ AB) = 1$ which is not equal to $n = 2$

Thus the system is uncontrollable

The closed loop system matrix can be defined as

$$A_c = A + BK$$

$$= \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} K_1 & K_2 \end{bmatrix}$$

$$A_c = \begin{bmatrix} -1 & 0 \\ K_1 & 2+K_2 \end{bmatrix}$$

Characteristic equation is given by

$$\det(\lambda I - A_c) = 0$$

$$\det\left(\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} -1 & 0 \\ K_1 & 2+K_2 \end{bmatrix}\right)$$

$$\det\left(\begin{bmatrix} \lambda+1 & 0 \\ -K_1 & \lambda-2-K_2 \end{bmatrix}\right) = 0$$

$$(\lambda+1)(\lambda-2-K_2) = 0$$

$$\lambda = -1 \quad \text{or} \quad K_2+2$$

→ Can the closed loop poles be placed at -2?

If K_2 value, in K_2+2, is -4, one pole can be placed at -2, but for any value of K_1 and K_2 it is simply not possible to place closed loop poles at $(-2, -2)$

→ Can the system be stabilized?

Since one of the poles as obtained is on the left hand plane (-1) and the other pole K_2+2 can be on the left half of the plane if K_2 is appropriately chosen, Thus the system can be stabilizable

PROBLEM 6 :

Given $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$

Solution first lets check for controllability

rank (B AB)

$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$AB = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$

rank of $\begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix} \Rightarrow 1$

rank $\neq 2$, thus the system is not controllable

The matrix of the closed loop system is

$A_c = A + BK$

$= \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} [K_1 \; K_2]$

$$A_c = \begin{bmatrix} 1 & 0 \\ k_1 & k_2+2 \end{bmatrix}$$

characteristic equation is given by

$$\det(\lambda I - A_c) = 0$$

$$\det\left(\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ k_1 & k_2+2 \end{bmatrix}\right) = 0$$

$$\det\left(\begin{bmatrix} \lambda-1 & 0 \\ -k_1 & \lambda-k_2-2 \end{bmatrix}\right) = 0$$

$$(\lambda-1)(\lambda-k_2-2) = 0$$

$$\lambda = 1 \text{ or } k_2+2$$

The poles for the given closed loop system are 1 and k_2+2

→ Can the poles be placed at -2 ?

It is not possible for both poles to be placed at (-2, -2) because one of the pole is always at 1 and only the other pole can be varied by varying the value of k_2 (when $K_2 = -4$, one pole is at -2)

→ Can this system be stabilized?

One of the poles of the system is always on the right hand plane, thus the system cannot be stabilized $\square$

PROBLEM 7: Given, $y(t)$ is the output of a linear time invariant system corresponding to input $u(t)$ then the output of the system corresponding to the input $\tilde{u}(t) = \dot{y}(t)$ and initial state, $x(t_0) = 0$

Solution: The LTI system for a corresponding input of $u(t)$ is

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

Taking laplace transform of both the equations gives

$$L(\dot{x}(t)) = L(Ax(t)) + L(Bu(t))$$
$$L(y(t)) = L(Cx(t)) + L(Du(t))$$
$$\Downarrow$$
$$\Rightarrow \begin{cases} sX(s) = AX(s) + BU(s) & -(1) \\ Y(s) = CX(s) + DU(s) & -(2) \end{cases}$$

Rearranging the terms we get in (1)

$$X(s)(sI - A) = BU(s)$$
$$\Rightarrow \underline{\underline{X(s) = (sI - A)^{-1} BU(s)}}$$

Substituting value of $X(s)$ in equation(2)

$$Y(S) = C(SI-A)^{-1}BU(S) + DU(S)$$

$$Y(S) = \left(C(SI-A)^{-1}B + D\right)U(S) \quad \text{...(eqn A)}$$

Now, to show that the output of the system for an input $\dot{u}(t)$ is given by $\dot{y}(t)$.

Consider an output $\tilde{y}(t) = CX(t) + D\dot{U}(t)$, replace $U(t)$ with the input $\dot{U}(t)$.

we get $\Rightarrow$
$$\left.\begin{cases} \dot{X}(t) = AX(t) + B\dot{U}(t) \\ \tilde{y}(t) = CX(t) + D\dot{U}(t) \end{cases}\right\} \leftarrow (3)$$

Taking laplace transform of the above equations

$$\rightarrow \quad SX(S) = AX(S) + BSU(S)$$

$$X(S)(SI-A) = BSU(S)$$

$$X(S) = (SI-A)^{-1}BSU(S)$$

and $\rightarrow$ $\tilde{Y}(S) = CX(S) + DSU(S)$

Substituting $X(S)$ value in (3) we get

$$\tilde{Y}(S) = C(SI-A)^{-1} BSU(S) + DSU(S)$$

$$= (C\ (SI-A)^{-1} BS + DS)\ U(S)$$

$$\tilde{y}(S) = S\,[C(SI-A)^{-1}B + D]\,U(S)$$

from eqn A, we know that

$$Y(S) = (C(SI-A)^{-1} B + D)\,U(S)$$

thus $\tilde{Y}(S) = S\,Y(S)$ — eqn (B)

Taking inverse laplace of eqn(B)

$$\tilde{Y}(t) = \dot{y}(t)$$

So, we can deduce that the output is $\dot{y}(t)$ for an input $\dot{u}(t)$ for a linear Time invariant system with initial state, $x(t_0) = 0$ $\square$

PROBLEM 8, Given

$$\dot{x}(t) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -4 & 4 \\ 0 & -1 & 0 \end{bmatrix} x(t)$$

$$x(t_0) = x_0$$

To find, solution of the linear-state equation

Solution Comparing the equation to the standard form

$\dot{x}(t) = A\vec{x}(t) + B\vec{U}(t)$, We get $A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -4 & 4 \\ 0 & -1 & 0 \end{bmatrix}$

To find the solution to the linear state equation, the given equation which is of the form $\dot{X}(t) = AX(t)$ is taken and laplace transform is applied

This gives:⇒ $SX(S) - x(t_0) = AX(S)$

Rearranging we get

$X(S)(SI - A) = x(t_0) \Rightarrow X(S) = (SI-A)^{-1} x(t_0)$

Take Inverse laplace transform of the above equation, we get

$$x(t) = \mathcal{L}^{-1}\{(SI - A)^{-1} x(t_0)\}$$

$$x(t) = \mathcal{L}^{-1}\{\underbrace{(SI - A)^{-1}}_{\text{calculating this part of the equation}} x_0\} \quad \text{—(Eqn 1)}$$

↓

$$(SI - A) = \begin{bmatrix} S & 0 & 0 \\ 0 & S & 0 \\ 0 & 0 & S \end{bmatrix} - \begin{bmatrix} -1 & 0 & 0 \\ 0 & -4 & 4 \\ 0 & -1 & 0 \end{bmatrix}$$

$$SI - A = \begin{bmatrix} S+1 & 0 & 0 \\ 0 & S+4 & -4 \\ 0 & 1 & S \end{bmatrix}$$

$$(SI - A)^{-1} = \frac{Adj\,(SI - A)}{|SI - A)}$$

Now, $\operatorname{adj}(SI - A) = \begin{bmatrix} (S+2)^2 & 0 & 0 \\ 0 & S(S+1) & 4(S+1) \\ 0 & -1(S+1) & (S+4)(S+1) \end{bmatrix}$

and $|SI - A| = (S+1)\left[(S+4)(S) + 4\right] \Rightarrow (S+1)(S^2+4S+4)$

$\Rightarrow (S+1)(S+2)^2$

So, we have,

$$(SI - A)^{-1} = \frac{1}{(S+1)(S+2)^2} \begin{bmatrix} (S+2)^2 & 0 & 0 \\ 0 & S(S+1) & 4(S+1) \\ 0 & -1(S+1) & (S+4)(S+1) \end{bmatrix}$$

$$(SI - A)^{-1} = \begin{bmatrix} \frac{1}{S+1} & 0 & 0 \\ 0 & \frac{S}{(S+2)^2} & \frac{4}{(S+2)^2} \\ 0 & \frac{-1}{(S+2)^2} & \frac{S+4}{(S+2)^2} \end{bmatrix}$$

Substituting $(SI-A)$ in eqn 1, we get.

$$\mathcal{L}^{-1}\left[(SI-A)^{-1} X_0\right] \quad \begin{bmatrix} \mathcal{L}^{-1}\left[\frac{1}{S+1}\right] & 0 & 0 \\ 0 & \mathcal{L}^{-1}\left[\frac{S}{(S+2)^2}\right] & \mathcal{L}^{-1}\left[\frac{4}{(S+2)^2}\right] \\ 0 & \mathcal{L}^{-1}\left[\frac{-1}{(S+2)^2}\right] & \mathcal{L}^{-1}\left[\frac{S+4}{(S+2)^2}\right] \end{bmatrix} X_0$$

$$X(t) = \begin{bmatrix} e^{-t} & 0 & 0 \\ 0 & e^{-2t}(1-2t) & -4te^{-2t} \\ 0 & -te^{-2t} & (1+2t)e^{-2t} \end{bmatrix} X_0$$

$$X(t) = \begin{bmatrix} e^{-t} & 0 & 0 \\ 0 & e^{-2t}(1-2t) & -4te^{-2t} \\ 0 & -te^{-2t} & (1+2t)e^{-2t} \end{bmatrix} X_0$$