PROBLEM - SET - 5

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PROBLEM 3

Given State Space Equation

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

Cost function

$$J = \int_0^{\infty} (x^T Q x + u^2)\, du$$

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & \gamma \end{bmatrix} \qquad \gamma > 0$$

Solution

Comparing the given state with $\dot{x}(t) = Ax(t) + Bu(t)$, we get

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Given $Q = \begin{pmatrix} 1 & 0 \\ 0 & \gamma \end{pmatrix}$, which is a symmetric positive definite matrix

Optimal solution is given by

$$K = -R^{-1} B_K^T P$$

where P is the symmetric positive definite solution of the following stationary Ricatti Equation

$$A^T P + PA - PBR^{-1}B^T P = 0$$

Using this equation first we find P

$$A^T = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \qquad B^T = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

and R =

Comparing cost function $J = \int_0^\infty (x^T Q x + u^2)\, du$ with the standard cost function of an LQR controller $J(K, \vec{x}(0)) = \int_0^\infty x^T(t) Q X(t) + U_K^T(t) R U_K(t)\, dt$. , we get R as

$$R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ or } I$$

Substituting these into the Ricatti Equation we get,

$$\begin{bmatrix}0&0\\1&0\end{bmatrix}P + P\begin{bmatrix}0&1\\0&0\end{bmatrix} - P\begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{bmatrix}0&1\end{bmatrix}P = -Q$$

for matrix multiplications we can consider P (a 2×2 matrix in this case) as

$$P = \begin{bmatrix}P_{1,1} & P_{1,2}\\ P_{1,2} & P_{2,2}\end{bmatrix}$$

{ Note:- $P_{1,2}$ is present twice as the matrix is a symmetric matrix }

So the above equation becomes

$$\begin{bmatrix}0&0\\P_{1,1}&P_{1,2}\end{bmatrix} + \begin{bmatrix}0&P_{1,1}\\0&P_{1,2}\end{bmatrix} - \begin{bmatrix}P_{1,2}\\P_{2,2}\end{bmatrix}\begin{bmatrix}0&1\end{bmatrix}\begin{bmatrix}P_{1,1}&P_{1,2}\\P_{1,2}&P_{2,2}\end{bmatrix} = -Q$$

$$\begin{bmatrix}0&0\\P_{1,1}&P_{1,2}\end{bmatrix} + \begin{bmatrix}0&P_{1,1}\\0&P_{1,2}\end{bmatrix} - \begin{bmatrix}P_{1,2}\\P_{2,2}\end{bmatrix}\begin{bmatrix}P_{1,2}&P_{2,2}\end{bmatrix} = -Q$$

$$\begin{bmatrix}0&0\\P_{1,1}&P_{1,2}\end{bmatrix} + \begin{bmatrix}0&P_{1,1}\\0&P_{2,1}\end{bmatrix} - \begin{bmatrix}(P_{1,2})^2 & P_{1,2}P_{2,2}\\ P_{1,2}P_{2,2} & (P_{2,2})^2\end{bmatrix}$$

$$\begin{bmatrix}-P_{1,2}^2 & P_{1,1}-P_{1,2}P_{2,2}\\ P_{1,1}-P_{1,2}P_{2,2} & 2P_{1,2}-P_{2,2}^2\end{bmatrix} = \begin{bmatrix}-1&0\\0&-\gamma\end{bmatrix}$$

$P_{1,2} = \pm 1$

$P_{1,1} = P_{2,2}$

$(P_{2,2})^2 = +\gamma + 2P_{1,2}$

When $P_{1,2} = 1$

$(P_{2,2})^2 = +\gamma + 2$

$(P_{2,2}) = \pm\sqrt{\gamma + 2}$

(1st Part)

when $P_{1,2} = -1$

$(P_{2,2})^2 = \gamma - 2$

$(P_{2,2}) = \pm\sqrt{\gamma - 2}$

(2nd Part)

Considering both, we can deduce that even when $\gamma > 0$, the 2nd part can be complex, ~~so~~ such a system is undesirable, so we only consider 1st part.

So, $P_{2,2} = \sqrt{\gamma+2}$

and

$P_{2,2} = -\sqrt{\gamma+2}$

So, we can now write P as. (for $P_{2,2} = \sqrt{\gamma+2}$)

$$P = \begin{bmatrix} \sqrt{\gamma+2} & 1 \\ 1 & \sqrt{\gamma+2} \end{bmatrix}$$

The eigen values of P are :-

$$\det|(P-\lambda I)| = \det\begin{vmatrix} \sqrt{\gamma+2}-\lambda & 1 \\ 1 & \sqrt{\gamma+2}-\lambda \end{vmatrix}$$

$$\Rightarrow \quad (\sqrt{\gamma+2}-\lambda)^2 - 1 = 0$$

$$(\sqrt{\gamma+2}-\lambda)^2 = 1$$

$$(\sqrt{\gamma+2}-\lambda) = \pm 1$$

$$\lambda = \sqrt{\gamma+2} \pm 1 \quad \text{for } P_{2,2} = \sqrt{\gamma+2}$$

(1)

for $P_{2,2} = -\sqrt{\gamma+2}$

$$P = \begin{bmatrix} -\sqrt{\gamma+2} & 1 \\ 1 & -\sqrt{\gamma+2} \end{bmatrix}$$

Eigen values are

$$\det(P-\lambda I) = \begin{bmatrix} -\sqrt{\gamma+2}-\lambda & 1 \\ 1 & -\sqrt{\gamma+2}-\lambda \end{bmatrix}$$

$$(-\sqrt{\gamma+2}-\lambda)^2 - 1 = 0$$

$$\left(-\sqrt{\gamma+2}-\lambda\right)^2 - 1 = 0$$

$$\left(-\sqrt{\gamma+2}-\lambda\right)^2 = 1$$

$$\left(-\sqrt{\gamma+2}-\lambda\right) = \pm 1$$

$$\lambda = \pm 1 - \sqrt{\gamma+2} \quad - (2)$$

Closelly observing (1) and (2) we can infer from (2) that for selected values, P can still be negative and this cannot be used as P is a positive matrix, So we can use (1) only further

$$P = \begin{bmatrix} \sqrt{\gamma+2} & 1 \\ 1 & \sqrt{\gamma+2} \end{bmatrix}$$

As we have now calculated P, the gain matrix K for the LQR can be calculated

$$K = -R^{-1} B_k^{T} P$$

$$K = -I^{-1} \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{\gamma+2} & 1 \\ 1 & \sqrt{\gamma+2} \end{bmatrix}$$

$$= -\begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{\gamma+2} & 1 \\ 1 & \sqrt{\gamma+2} \end{bmatrix}$$

$$K = -\begin{bmatrix} 1 & \sqrt{\gamma+2} \end{bmatrix}$$

State feedback, given, $u = Kx$,

$$\dot{x} = A + BK$$

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} -1 & -\sqrt{\gamma+2} \end{bmatrix} x$$

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ -1 & -\sqrt{\gamma+2} \end{bmatrix} x$$

This is the required state equation

$$\left\{ \dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & -\sqrt{\gamma+2} \end{bmatrix} \right\}$$

PROBLEM 4

Given, $\dot{x}(t) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} U(t)$

Solution

Controllability matrix $C = [B \;\; AB \;\; A^2B]$

because the dimension of C are $n \times nm$

$3 \times (3 \times 2) \Rightarrow 3 \times 6$ matrix

Comparing given state equation to

$\dot{x} = Ax + BU$

$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$AB = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$A^2B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \\ 2 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} B & AB & A^2B \end{bmatrix} \Rightarrow$$

$$C = \begin{bmatrix} 0 & 1 & 1 & 1 & 2 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 2 & 1 \end{bmatrix}$$

To check if the system is controllable or not rank of C must be equal to n

Rank $(C) = 2$, which is less than n $(n=3)$, thus the system is uncontrollable

Applying Similarity Transforms

$$S_{n-n_z} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow S = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

For (A, B) uncontrollable, there is a non singular matrix S such that $\hat{A} = S^{-1}AS$ and $\hat{B} = S^{-1}B$

To find $S^{-1} \Rightarrow$

$$S^{-1} = \frac{1}{|S|} \operatorname{Adj}(S) \Rightarrow \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & -1 \end{bmatrix}$$

$$A = \overset{(S^{-1})}{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix}} \overset{(A)}{\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}} \overset{(S)}{\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}}$$

$$\hat{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

$$\hat{A} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\hat{B} = S^{-1} B_{\cancel{c}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Comparing this to standard form

$$\hat{A} = S^{-1} A S = \begin{bmatrix} A_1 & A_{12} \\ 0 & A_2 \end{bmatrix}$$

and $B = S^{-1} B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}$

$A_{11} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ and $B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Standard form of uncontrollable system is given by

$$\dot{x} = \hat{A}x + \hat{B}u$$

$$\dot{x} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} u$$

and the controllable part is given by (A_{11}, B_1) so the corresponding state space form is

$$\dot{x} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} u$$

Problem 5

Given system, $\dot{x} = \begin{bmatrix} -3 & 2 \\ -1 & -1 \end{bmatrix} x$

To investigate stability using Lyapunov equation and $Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Solution

The Lyapunov equation

$A^T P + P A = -Q$ is a continious time algebraic matrix equation where A is a square matrix, P is a symmetric positive matrix and Q is a symmetric positive definite matrix

$$P = \begin{bmatrix} P_{1,1} & P_{1,2} \\ P_{1,2} & P_{2,2} \end{bmatrix}$$

Comparing given equation to $\dot{x} = Ax + BU$

$$A = \begin{bmatrix} -3 & 2 \\ -1 & -1 \end{bmatrix} \qquad A^T = \begin{bmatrix} -3 & -1 \\ 2 & -1 \end{bmatrix}$$

So, Lyapunov equation is

$$\begin{bmatrix} -3 & -1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} P_{1,1} & P_{1,2} \\ P_{1,2} & P_{2,2} \end{bmatrix} + \begin{bmatrix} P_{1,1} & P_{1,2} \\ P_{1,2} & P_{2,2} \end{bmatrix} \begin{bmatrix} -3 & 2 \\ -1 & -1 \end{bmatrix} = -Q$$

$$\begin{bmatrix} -3P_{1,1} - P_{1,2} & -3P_{1,2} - P_{2,2} \\ 2P_{1,1} - P_{1,2} & 2P_{1,2} - P_{2,2} \end{bmatrix} + \begin{bmatrix} -3P_{1,1} - P_{1,2} & 2P_{1,1} - P_{1,2} \\ -3P_{1,2} - P_{2,2} & 2P_{1,2} - P_{2,2} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

Equating, we get

$$-6P_{1,1} - 2P_{1,2} = -1$$

$$2P_{1,1} - 4P_{1,2} - P_{2,2} = 0$$

$$-3P_{1,2} - P_{2,2} + 2P_{1,1} - P_{1,2} = 0$$

$$4P_{1,2} - 2P_{2,2} = -1$$

$$P_{1,1} = \frac{-1+2P_{1,2}}{6}$$

$$4P_{1,2} - 2P_{2,2} = -1$$

$$-P_{2,2} + \frac{1-14P_{1,2}}{3} = 0$$

Simplifying

$$\frac{-20P_{2,2}+9}{6} = 0$$

$$\boxed{P_{2,2} = \frac{9}{20}}$$

$$P_{1,2} = -\frac{1+2P_{2,2}}{4}$$

$$P_{1,2} = \frac{-1+9/10}{4}$$

$$P_{1,2} = -\frac{1}{40}$$

$$P_{1,1} = \frac{-1+2(-1/40)}{6}$$

$$P_{1,1} = \frac{7}{40}$$

$$P = \begin{bmatrix} \frac{7}{40} & \frac{-1}{40} \\ \frac{-1}{40} & \frac{9\times 2}{20\times 2} \end{bmatrix}$$

$$P = \frac{1}{40}\begin{bmatrix} 7 & -1 \\ -1 & 18 \end{bmatrix}$$

$$P-\lambda I = \begin{bmatrix} 7-\lambda & -1 \\ -1 & 18-\lambda \end{bmatrix}$$

$$\det(P-\lambda I) = (7-\lambda)(18-\lambda) - 1 = 0$$

$$126 - 7\lambda - 18\lambda + \lambda^2 - 1 = 0$$

$$125 - 25\lambda + \lambda^2 = 0$$

$$\lambda = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$\lambda = \frac{25 \pm \sqrt{625-500}}{2}$$

$$\frac{25 \pm \sqrt{125}}{2}$$

$$\lambda = \frac{25}{2} \pm \frac{5\sqrt{5}}{2}$$

So $\lambda_1 = \frac{25}{2} + \frac{5\sqrt{5}}{2}$ and $\lambda_2 = \frac{25}{2} - \frac{5\sqrt{5}}{2}$

So both the eigen values are greater than zero

So, the given system is stable $\square$

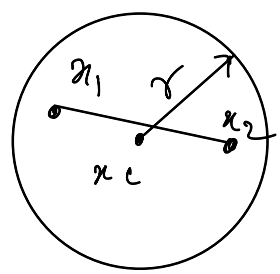
PROBLEM 2

Given, a euclidean ball $B(x_c, r)$ in R^n

$$B(x_c, r) = \{x \in R^n \mid \|x - x_c\| \leq r\}$$

$$r > 0, \quad \|x\| = \sqrt{x^T x}$$

Solution



A set C is convex if the line segment between any two points in C lies in C

if for any $x_1, x_2 \in C$ and for any theta (θ) $\{0 \leq \theta \leq 1\}$

Convex combination of x_1, x_2 is given by $\theta x_1 + (1-\theta)x_2 \in C$

from the figure and the definition of a norm

$\|x_1\|$ is always less than r

and $\|x_2\|$ is also always less than r

$\|\theta x_1 + (1-\theta)x_2\|$ can be represented by using triangular inequality as

$$\|\theta x_1 + (1-\theta)x_2\| \le \|\theta x_1\| + \|(1-\theta)x_2\|$$

Using $\|x_1\| \le r$ and $\|x_2\| \le r$

$$\theta\|x_1\| + (1-\theta)x_2\| \le \theta r + (1-\theta)r$$

$$\theta_1 \|x_1\| + (1-\theta)\|\vec{x_2}\| \le r$$

we know that

$$\|\theta x_1 + (1-\theta) x_2\| \leq \theta \|x_1\| + (1-\theta) \|x_2\|$$

So $\|\theta x_1 + (1-\theta) x_2\| \leq r$

This shows that the convex combination of x_1 and x_2 is in $B(x_c, r)$ and hence we can conclude that $B(x_c, r)$ is a convex set.

PROBLEM 1

Given $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{m \times m}$

To show $X(t) = e^{At} X(0) e^{Bt}$ is the solution to the equation

Solution :- We know from the definition of matrix exponential that

$$e^{At} = \sum_{i=0}^{\infty} \frac{1}{i!}(At)^i$$

It is clear that for any matrix A

$$e^{At} = I \text{ when } t=0$$

and $$\frac{de^{At}}{dt} = Ae^{At} = e^{At}A$$

Thus, $\frac{d}{dt}(e^{At}x(0)e^{Bt})$, Using product rule

$$= \left(\frac{d}{dt}e^{At}\right)x(0)e^{Bt} + e^{At}\left(\frac{d}{dt}x(0)\right)e^{Bt} + e^{At}x(0)\left(\frac{d}{dt}e^{Bt}\right)$$

$$= A(e^{At}x(0)e^{Bt}) + 0 + (e^{At}x(0)e^{Bt})B$$

$=$

for $T=0$

$$(e^{At}x(0)e^{Bt})\Big|_{t=0} = x(0) \quad \square$$

Hence, it can be concluded that the proposed function is in fact the solution to the initial value problem under consideration