ENPM 667
PROBLEM SET #2
Directory Id: swarajmr

PROBLEM 1: Evaluate determinants

a) $\begin{vmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & -2 & 1 \\ 3 & -3 & 4 & -2 \\ -2 & 1 & -2 & 1 \end{vmatrix}$

Solution: To find determinant of given matrix we use Laplace Expansion, so we have

$A_{11}C_{11} + A_{12}C_{12} + A_{13}C_{13} + A_{14}C_{14}$

we can perform some operations on the given matrix to find the determinant easily

Note: R_3 means row 3 and similarly R_1 is row 1

C_1 is column 1

$\begin{vmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & -2 & 1 \\ 3 & -3 & 4 & -2 \\ -2 & 1 & -2 & 1 \end{vmatrix}$ Reduce further by $\Rightarrow R_3 \to R_3 - 3R_1$

$\Downarrow$

$\begin{vmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & -2 & 1 \\ 0 & -3 & -2 & -11 \\ -2 & 1 & -2 & 1 \end{vmatrix}$

Reduce further by $R_4 \to R_4 + 2R_1$

$\Rightarrow \begin{vmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & -2 & 1 \\ 0 & -3 & -2 & -11 \\ 0 & 1 & 2 & 7 \end{vmatrix}$

$\Rightarrow$ Reduce further by $C_3 \to C_3 - 2C_1$

$\Downarrow$

$\begin{vmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & -2 & 1 \\ 0 & -3 & -2 & -11 \\ 0 & 1 & 2 & 7 \end{vmatrix}$

Reduce further by

$\Rightarrow C_4 \rightarrow C_4 - 3C_1$, we get

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & -3 & -2 & -11 \\ 0 & 1 & 2 & 7 \end{vmatrix}$$

So, we have $A_{11} = 1$, $A_{12} = 0$, $A_{13} = 0$, $A_{14} = 0$

substituting values of $A_{11}, A_{12}, A_{13}, A_{14}$ in the equation, we get

$A_{11} C_{11}$ as the determinant

$$\underset{(A_{11})}{1} \cdot \underset{(C_{11})}{\begin{vmatrix} 1 & -2 & 1 \\ -3 & -2 & -11 \\ 1 & 2 & 7 \end{vmatrix}} \Rightarrow 1(-14+22) + 2(-21+11) + 1(-6+2)$$

$$= 8 - 20 - 4$$

$$\boxed{\text{Determinant} = \underline{\underline{-16}}}$$

Problem 16

$$\begin{vmatrix} gc & ge & a+ge & gb+ge \\ 0 & b & b & b \\ c & e & e & b+e \\ a & b & b+f & b+d \end{vmatrix}$$

Solution:- For reduction, row 3 and row 1 are similar

$R_1 \to R_1 - g \times (R_3)$

$$\begin{vmatrix} 0 & 0 & a & 0 \\ 0 & b & b & b \\ c & e & e & b+e \\ a & b & b+f & b+d \end{vmatrix}$$

Reduce further by

$R_4 \to R_4 - R_2$

$$\begin{vmatrix} 0 & 0 & a & 0 \\ 0 & b & b & b \\ c & e & e & b+e \\ a & 0 & f & d \end{vmatrix}$$

So, determinant of above matrix is given by

$A_{11}C_{11} + A_{12}C_{12} + A_{13}C_{13} + A_{14}C_{14}$

We have A_{11}, A_{12}, A_{14} as zero

So, To calculate determinant we have

$$A_{13}\ C_{13}$$

where, $A_{13} = a$ $\qquad C_{13} = \begin{vmatrix} 0 & b & 0 \\ c & e & b+e \\ a & 0 & d \end{vmatrix}$

$$a \begin{vmatrix} 0 & b & b \\ c & e & b+e \\ a & 0 & d \end{vmatrix} \Rightarrow$$

we can reduce this further by subtracting Column 3 from Column 2

$$\Downarrow$$

$$a \begin{vmatrix} 0 & b & 0 \\ c & e & b \\ a & 0 & d \end{vmatrix}$$

$$= a\left[(0) + (-b)(cd - ab) + 0\right)$$

$$= -ab(cd - ab)$$

$$\boxed{\text{Determinant} = -abcd + a^2b^2}$$

PROBLEM 2:

$$\begin{vmatrix} x & a & a & 1 \\ a & x & b & 1 \\ a & b & x & 1 \\ a & b & c & 1 \end{vmatrix}$$

Solution:

The given matrix when expanded yields a cubic equation in terms of x (can be observed if $A_{11}C_{12}$ is obtained)

i.e. $x\begin{vmatrix} x & b & 1 \\ b & x & 1 \\ b & c & 1 \end{vmatrix}$), this means there are 3 roots.

Situation 1 :- When $x=a$ the first column is a scalar multiple of column 4, so according to properties of determinants the determinant of the resulting matrix is zero.

Situation 2: when $x=b$, the second and third row are equal, hence again the determinant $=0$

Situation 3 :- When $x=C$, row 3 will be equal to row 4, so again determinant is equal to zero.

$\boxed{\text{So, the roots are, } x=a;\ x=b \text{ and } x=C}$

PROBLEM 3 : Solution

$$A = \begin{bmatrix} 1 & \gamma_1 & 0 \\ \gamma_2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Let λ represent Eigen values and Identity matrix of order 3×3 is $\rightarrow$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Then, $\lambda I = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$

$(A-\lambda I) = \begin{bmatrix} 1-\lambda & \gamma_1 & 0 \\ \gamma_2 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{bmatrix}$

$\boxed{\text{Characteristic Equation}}$

Eigen values λ of matrix A are the roots of $A-\lambda I$

$$\begin{vmatrix} 1-\lambda & \gamma_1 & 0 \\ \gamma_2 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix}$$

From basic linear algebra, we know that the equation $(A-\lambda I)x = 0$, has a nontrivial solution when $|A-\lambda I| = 0$

$$(1-\lambda)\left[(1-\lambda)^2 - \gamma_1\gamma_2\right] = 0$$

From the above equation, we have $(1-\lambda) = 0$ and

$$(1-\lambda)^2 - \gamma_1\gamma_2 = 0$$

So, $\underline{\lambda = 1}$ and $(1-\lambda)^2 = \gamma_1\gamma_2$

$$(1-\lambda) = \pm\sqrt{\gamma_1\gamma_2}$$

$$\underline{\lambda = 1 \pm \sqrt{\gamma_1\gamma_2}}$$

Case 1 :- when $\lambda = 1$

$$Ax' = 1x'$$

$$\begin{bmatrix} 1 & \gamma_1 & 0 \\ \gamma_2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \Rightarrow$$

$\gamma_1 x_2 = 0$

$\boxed{x_2 = 0}$

$\gamma_2 x_1 + x_2 = x_2$

$\gamma_2 x_1 = 0$

$\boxed{x_1 = 0}$

$\boxed{x_3 = x_3}$

For $\lambda = 1$, the corresponding eigen vector we get is

$$x^1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Similarly for other two values of λ

Case 2: $\lambda = 1 + \sqrt{\gamma_1 \gamma_2}$

$$\begin{bmatrix} 1 & \gamma_1 & 0 \\ \gamma_2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 1 + \sqrt{\gamma_1 \gamma_2} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\cancel{x_1} + \gamma_1 x_2 = \cancel{x_1} + x_1 \sqrt{\gamma_1 \gamma_2}$$

$$\gamma_1 x_2 = x_1 \sqrt{\gamma_1 \gamma_2} \rightarrow \frac{x_2}{x_1} = \frac{\sqrt{\gamma_2}}{\sqrt{\gamma_1}}$$

$$\rightarrow \gamma_2 x_1 + x_2 = (1 + \sqrt{\gamma_1 \gamma_2}) x_2 \Rightarrow \gamma_2 x_1 + \cancel{x_2} = \cancel{x_2} + x_2 \sqrt{\gamma_1 \gamma_2}$$

$$\boxed{\frac{x_1}{x_2} = \frac{\sqrt{\gamma_1}}{\sqrt{\gamma_2}}} \Leftarrow \gamma_2 x_1 = x_2 \sqrt{\gamma_1 \gamma_2}$$

$$x_3 = (1 + \sqrt{\gamma_1 \gamma_2}) x_3$$

$$x_3 = 0$$

$x_1 = \sqrt{\gamma_1}$ $\quad x_2 = \sqrt{\gamma_2}$

So $x^2 = \begin{bmatrix} \sqrt{\gamma_1} \\ \sqrt{\gamma_2} \\ 0 \end{bmatrix}$

$\underline{\underline{\text{Case 3}}}$ when $\lambda = 1 - \sqrt{\gamma_1 \gamma_2}$

$$\begin{bmatrix} 1 & \gamma_1 & 0 \\ \gamma_2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 1 - \sqrt{\gamma_1 \gamma_2} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\cancel{x_1} + \gamma_1 x_2 = \cancel{x_1} - x_1 \sqrt{\gamma_1 \gamma_2}$$

$$\frac{x_2}{x_1} = \frac{-\sqrt{\gamma_2}}{\sqrt{\gamma_1}}$$

$$x_1 = \sqrt{\gamma_1}$$

$$x_3 = x_3$$

$$x_2 = -\sqrt{\gamma_2}$$

$$\boxed{x_3 = 0}$$

$$\text{So, } x^3 = \begin{bmatrix} \sqrt{\gamma_1} \\ -\sqrt{\gamma_2} \\ 0 \end{bmatrix}$$

$\underline{\text{Solution}}$ $\underline{\text{3 b)}}$ Conditions for Eigen values to be real

$\Rightarrow$ For eigen values to be real the product of γ_1 and γ_2 has to be positive and real

$\underline{\text{Solution}}$ $\underline{\underline{\text{3C)}}}$ for Eigenvectors to be orthogonal

$$0 = (x^2)^T \cdot x^3 \Rightarrow \sqrt{\gamma_1}\sqrt{\gamma_1} - \sqrt{\gamma_2}\sqrt{\gamma_2} = 0$$

$$|\gamma_1| = |\gamma_2|$$

similarly, dot product of x^1 and x^3

$$(x^1)^+(x^3) = [0\ 0\ 1]\begin{bmatrix}\sqrt{\gamma_1}\\ -\sqrt{\gamma_2}\\ 0\end{bmatrix}$$

$$\underline{\underline{=0}}$$

So eigen vectors are orthogonal

Similarly, x^1 & x^2

$$(x^1)^+ x^2 = [0\ 0\ 1]\cdot\begin{bmatrix}\sqrt{\gamma_1}\\ \sqrt{\gamma_2}\\ 0\end{bmatrix}$$

$\underline{\underline{=0}}$ (orthogonal)

$\boxed{|\gamma_1| = |\gamma_2| \text{ is the condition}}$

Problem 4

LU decomposition of $A = \begin{pmatrix} 2 & -3 & 1 & 3 \\ 1 & 4 & -3 & -3 \\ 5 & 3 & -1 & -1 \\ 3 & -6 & -3 & 1 \end{pmatrix}$

and solve $Ax = b$ for (i) $b = (-4 \ \ 1 \ \ 8 \ \ -5)^T$

(ii) $b = (-10 \ \ 0 \ \ -3 \ \ -24)^T$

Solution

LU decomposition for a 4×4 matrix is

$$A = LU$$

where $L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ L_{21} & 1 & 0 & 0 \\ L_{31} & L_{32} & 1 & 0 \\ L_{41} & L_{42} & L_{43} & 1 \end{pmatrix}$

and $U = \begin{pmatrix} U_{11} & U_{12} & U_{13} & U_{14} \\ 0 & U_{22} & U_{23} & U_{24} \\ 0 & 0 & U_{33} & U_{34} \\ 0 & 0 & 0 & U_{44} \end{pmatrix}$

So, $A = \begin{pmatrix} U_{11} & U_{12} & U_{13} & U_{14} \\ (L_{21}U_{11}) & (L_{21}U_{12} + U_{22}) & (L_{21}U_{13} + U_{23}) & (L_{21}U_{14} + U_{24}) \\ (L_{31}U_{11}) & (L_{31}U_{12} + L_{32}U_{22}) & (L_{31}U_{13} + L_{32}U_{23} + U_{33}) & (L_{31}U_{14} + L_{32}U_{24} + U_{34}) \\ (L_{41}U_{11}) & (L_{41}U_{12} + L_{42}U_{22}) & (L_{41}U_{13} + L_{42}U_{23} + L_{43}U_{33}) & (L_{41}U_{14} + L_{42}U_{24} + L_{43}U_{34} + U_{44}) \end{pmatrix}$

Comparing with A, we get,

$$\boxed{U_{11} = 2 \;,\; U_{12} = -3 \;,\; U_{13} = 1 \;,\; U_{14} = 3}$$

$L_{21} U_{11} = 1 \Rightarrow \boxed{L_{21} = 1/2}$

$L_{31} U_{11} = 5 \Rightarrow \boxed{L_{31} = 5/2}$

$L_{41} U_{11} = 3 \Rightarrow \boxed{L_{41} = 3/2}$

$L_{21} U_{12} + U_{22} = 4 \Rightarrow \boxed{U_{22} = 11/2}$

$L_{21} U_{13} + U_{23} = -3 \Rightarrow \boxed{U_{23} = -7/2}$

$L_{21} U_{14} + U_{24} = -3 \Rightarrow \boxed{U_{24} = -9/2}$

$L_{31} U_{12} + L_{32} U_{22} = 3 \Rightarrow \boxed{L_{32} = 21/11}$

$L_{31} U_{13} + L_{32} U_{23} + U_{33} = -1 \Rightarrow \boxed{U_{33} = 35/11}$

$L_{31} U_{14} + L_{32} U_{24} + U_{34} = -1 \Rightarrow \boxed{U_{34} = 1/11}$

$L_{41} U_{12} + L_{42} U_{22} = -6 \Rightarrow \boxed{L_{42} = 3/2}$

$L_{41}U_{13} + L_{42}U_{23} + L_{43}U_{33} = -3$

$$\boxed{L_{43} = \frac{-12}{7}}$$

$L_{41}U_{14} + L_{42}U_{24} + L_{43}U_{34} + U_{44} = 1$

$$\boxed{U_{44} = \frac{-32}{7}}$$

$A = LU$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & 0 \\ 5/2 & \frac{21}{11} & 1 & 0 \\ 3/2 & -3/11 & \frac{-12}{7} & 1 \end{bmatrix} \begin{bmatrix} 2 & -3 & 1 & 3 \\ 0 & 11/2 & -7/2 & -9/2 \\ 0 & 0 & \frac{35}{11} & 1/11 \\ 0 & 0 & 0 & \frac{-32}{7} \end{bmatrix}$$

Solution for 4(i)

$Ax = b$

$LUx = b$, this can

be written as

$$\underline{Ly = b} \qquad \underline{Ux = y}$$

$\underline{Ly = b} \Rightarrow$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & 0 \\ 5/2 & \frac{21}{11} & 1 & 0 \\ 3/2 & -3/11 & \frac{-12}{7} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} -4 \\ 1 \\ 8 \\ 5 \end{bmatrix}$$

$$\boxed{y_1 = -4}$$

$$\frac{y_1}{2} + y_2 = 1 \Rightarrow \boxed{y_2 = 3}$$

$$\frac{5}{2} y_1 + \frac{21}{2} y_2 + y_3 = 8$$

$$\boxed{y_3 = \frac{135}{11}}$$

$$\frac{3}{2}y_1 + -\frac{3}{11}y_2 - \frac{12}{7}y_3 + y_4 = 5$$

$$\boxed{y_4 = \frac{160}{7}} \quad \text{so, } y = \begin{bmatrix} -4 \\ 3 \\ 135/11 \\ 160/7 \end{bmatrix}$$

Solving $Ux = y$

$$\begin{bmatrix} 2 & -3 & 1 & 3 \\ 0 & 11/2 & -7/2 & -9/2 \\ 0 & 0 & 35/11 & 1/11 \\ 0 & 0 & 0 & -32/7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -4 \\ 3 \\ 135/11 \\ 160/7 \end{bmatrix}$$

$$\Rightarrow -\frac{32}{7}x_4 = \frac{160}{7} \Rightarrow \boxed{x_4 = -5}$$

$$\Rightarrow \frac{35}{11}x_3 + \frac{x_4}{11} = \frac{135}{11}$$

$$x_3 = \left(\frac{135}{11} + \frac{5}{11}\right) \times \frac{11}{35}$$

$$\boxed{x_3 = 4}$$

$\Rightarrow \frac{11}{2}x_2 + \left(-\frac{7}{2}x_3\right) + \left(-\frac{9}{2}x_4\right) = 3$

$$x_2 = \left(3 + \frac{28}{2} - \frac{45}{2}\right) \times \frac{2}{11}$$

$$\boxed{x_2 = -1}$$

$\Rightarrow 2x_1 + 3 + 4 - 15 = -4$

$$2x_1 = 4$$

$$\boxed{x_1 = 2}$$

$$x = \begin{bmatrix} 2 \\ -1 \\ 4 \\ -5 \end{bmatrix}$$

Solution for 4(iii)

Taking $Ly = b$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & 0 \\ 5/2 & \frac{21}{11} & 1 & 0 \\ 3/2 & -3/11 & -\frac{12}{7} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} -10 \\ 0 \\ -3 \\ -24 \end{bmatrix}$$

$$\boxed{y_1 = -10}$$

$$\frac{1}{2}y_1 + y_2 = 0 \quad \Rightarrow \quad \boxed{y_2 = 5}$$

$$\frac{5}{2}y_1 + \frac{21}{11}y_2 + y_3 = -3 \quad \Rightarrow \quad y_3 = -3 + 25 - \frac{105}{11}$$

$$\boxed{y_3 = \frac{137}{11}}$$

$$\frac{3}{2}y_1 + \frac{-3}{11}y_2 - \frac{12}{7}y_3 + y_4 = -24$$

$$y_4 = -24 + 15 + \frac{15}{11} + \frac{12}{7}\times\frac{137}{11}$$

$$\boxed{y_4 = \frac{96}{7}} \qquad y = \begin{bmatrix} -10 \\ 5 \\ 137/11 \\ 96/7 \end{bmatrix}$$

Now, solving $Ux = y$

$$\begin{bmatrix} 2 & -3 & 1 & 3 \\ 0 & 11/2 & -7/2 & -9/2 \\ 0 & 0 & \frac{35}{11} & 1/11 \\ 0 & 0 & 0 & \frac{-32}{7} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -10 \\ 5 \\ 137/11 \\ 96/7 \end{bmatrix}$$

$\boxed{x_4 = -3}$

$$\frac{35}{11}x_3 + \frac{x_4}{11} = \frac{137}{11}$$

$$x_4 = \frac{140}{11} \times \frac{11}{35} \Rightarrow \boxed{x_3 = 4}$$

$$\frac{11}{2}x_2 - \frac{7}{2}x_3 - \frac{9}{2}x_4 = 5$$

$$x_2 = \left(5 - \frac{27}{2} + 14\right) \times \frac{2}{11}$$

$$\boxed{x_2 = 1}$$

$$2x_1 - 3x_2 + x_3 + 3x_4 = -10$$

$$x_1 = -10 + 9 - 4 + 3$$

$$2x_1 = -2 \qquad \boxed{x_1 = -1}$$

$$x = \begin{bmatrix} -1 \\ 1 \\ 4 \\ 3 \end{bmatrix}$$

Determinant of matrix A is product of diagonals of the matrix

$$|A| = 2 \times \frac{11}{2} \times \frac{35}{11} \times -\frac{32}{7}$$

$$\boxed{|A| = -160}$$

Direct Calculation Confirmation of determinant

$$\begin{bmatrix} 2 & -3 & 1 & 3 \\ 1 & 4 & -3 & -3 \\ 5 & 3 & -1 & -1 \\ 3 & -6 & -3 & 1 \end{bmatrix}$$

$C_1 \to C_1 - 2C_3$

$C_2 \to C_2 + 3C_3$

will give

$$\begin{bmatrix} 0 & 0 & 1 & 3 \\ 7 & -5 & -3 & -3 \\ 7 & 0 & -1 & -1 \\ 9 & -15 & -3 & 1 \end{bmatrix}$$

$R_4 \to R_4 - 3R_3$

$$\begin{vmatrix} 0 & 0 & 1 & 0 \\ 7 & -5 & -3 & 6 \\ 7 & 0 & -1 & 2 \\ 9 & -15 & -3 & 10 \end{vmatrix}$$

$$\text{Determinant} = 1 \begin{vmatrix} 7 & -5 & 6 \\ 7 & 0 & 2 \\ 9 & -15 & 10 \end{vmatrix} \Rightarrow 7(30) + 5(70-18) + 6(-105)$$

$$\boxed{|A| = -160}$$

PROBLEM 5 given matrix $\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}$

Solution

(i) Yes, this matrix is diagonizable because the matrix is symmetric

(ii) The Eigen values of matrix is

$$(A - \lambda I) = \begin{vmatrix} -1-\lambda & 2 & 2 \\ 2 & -1-\lambda & 2 \\ 2 & 2 & -1-\lambda \end{vmatrix}$$

Determinant is $\Rightarrow$

$$(-1-\lambda)\left((-1-\lambda)^2-4\right) - 2(-2\lambda-6) + 2(2\lambda+6)$$

$$\Rightarrow (-1-\lambda)(\lambda^2+2\lambda-3) + 4\lambda + 12 + 4\lambda + 12$$

$$\Rightarrow -\lambda^3 - 3\lambda^2 + \lambda + 3 + 8\lambda + 24$$

$$\Rightarrow -\lambda^3 - 3\lambda^2 + 9\lambda + 27$$

Solving the equation for values of λ

$$-\lambda^3 - 3\lambda^2 + 9\lambda + 27 = 0$$

Roots are 3 and -3

$$\lambda = 3 \quad , \lambda = -3$$

To find Eigen Vectors

$$Ax = \lambda x'$$

$$\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & -1 \\ 2 & 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 3 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$-x_1 + 2x_2 + 2x_3 = 3x_1$$
$$2x_1 - x_2 - x_3 = 3x_2$$
$$2x_1 + 2x_2 - x_3 = 3x_3$$

$$2x_2 + 2x_3 = 4x_1$$
$$2x_1 - x_3 = 4x_2$$
$$2x_1 + 2x_2 = 4x_3 \rightarrow \begin{matrix}(2x_3 = x_1 + x_2) \\ (2x_1 = 4x_3 - 2x_2)\end{matrix}$$

So $2x_2 + x_1 + x_2 = 4x_1$

$$3x_2 = 3x_1 \quad \underline{x_1 = x_2}$$

$$2x_1 + 2x_2 = 4x_3 \quad \text{, if } x_1 = x_2$$

$$4x_1 = 4x_3$$

$$\boxed{x_1 = x_3}$$

$$\boxed{x_1 = x_2 = x_3 = K}$$

So, eigenvector for $\lambda = 3$ is

$$x' = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Eigen vector for $\lambda=-3$

$$\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = -3\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$-x_1+2x_2+2x_3=-3x_1 \rightarrow 2x_1+2x_2+2x_3=0$$
$$2x_1-x_2+2x_3=-3x_2 \rightarrow 2x_1+2x_2+2x_3=0$$
$$2x_1+2x_2-x_3=-3x_3 \rightarrow 2x_1+2x_2+2x_3=0$$

Solving this using matrix format, we have

$$\begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Performing the following Row operations $\left\{\begin{array}{l} R_1 \rightarrow \frac{1}{2}R_1 \\ R_2 \rightarrow R_2-2R_1 \\ R_3 \rightarrow R_3-2R_1 \end{array}\right.$

we get

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 = -x_2 - x_3$$

$$\begin{bmatrix} -x_2-x_3 \\ x_2 \\ x_3 \end{bmatrix} \Rightarrow \begin{bmatrix} -x_2 \\ x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} -x_3 \\ 0 \\ x_3 \end{bmatrix}$$

let $x_2 = 1$ $x_3 = 1$

we get $x^2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ & $x^3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

i.e, $A = SDS^{-1}$ where D is diagonal matrix

Now to get diaganalized matrix, D

$D = S^{-1}AS$

Where S is, Eigen vectors corresponding to eigenvalues compose columns of matrix S

$$S = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

To find S^{-1} $\Rightarrow$ Reduce matrix to Identity matrix

$$\left[\begin{array}{ccc|ccc} -1 & -1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

Performing following operations on the matrix S

$R_1 \to -1 \times R_1$, $R_2 \to R_2 - R_1$ and $R_2 \to (-1 \times R_2)$

and $R_1 - R_2$, we get

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

Perform $R_3 \to R_3 - R_2$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & -1 & -1 & 0 \\ 0 & 0 & 3 & 1 & 1 & 1 \end{array}\right]$$

$R_3 \to \frac{R_3}{3}$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & -1 & -1 & 0 \\ 0 & 0 & 1 & 1/3 & 1/3 & 1/3 \end{array}\right]$$

$R_1 \to R_1 - R_3$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1/3 & +2/3 & -1/3 \\ 0 & 1 & -2 & -1 & -1 & 0 \\ 0 & 0 & 1 & 1/3 & 1/3 & 1/3 \end{array}\right]$$

$R_2 \to R_2 + 2R_3$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1/3 & 2/3 & -1/3 \\ 0 & 1 & 0 & -1/3 & -1/3 & 2/3 \\ 0 & 0 & 1 & 1/3 & 1/3 & 1/3 \end{array}\right]$$

So, $S^{-1} = \begin{bmatrix} -1/3 & 2/3 & -1/3 \\ -1/3 & -1/3 & 2/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$

We know that $D = S^{-1}AS$

$$D = S^{-1}AS$$

$$= \underbrace{\begin{bmatrix} -1/3 & 2/3 & -1/3 \\ -1/3 & -1/3 & 2/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}}\begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 1 \\ 1 & 1 & -2 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \Longrightarrow$$ This is the diagonalized matrix

PROBLEM 6 given $A \in R^{n \times n}$
$U \in R^{n \times K}$
$V \in R^{n \times K}$

& A, $A+UV^T$ & $1+V^TA^{-1}U$ are non singular matrices

Prove that $(A+UV^T)^{-1} = A^{-1} - A^{-1}U(1+V^TA^{-1}U)^{-1}V^TA^{-1}$

Solution : Considering $A+UV^T \Rightarrow$

Let I be an Identity matrix

Considering left hand side of the equation

$(A+UV^T)^{-1} \Rightarrow$ A can be represented also as IA and UV^T can be represented as IUV^T

So we can write $(A+UV^T)^{-1}$ as

$(IA+IUV^T)^{-1}$

This can also be written as

$(IA + AA^{-1}UV^T)^{-1}$ because I can also be expanded as $I = AA^{-1}$

The equation can be written as

$$\left(A(I + A^{-1}UV^T)\right)^{-1}$$

$$= \underbrace{(I + A^{-1}UV^T)^{-1}}_{\text{Part 1}} A^{-1}$$

Part 1 of equation can be further expanded by using Carl Numann series expansion.

$$(I + A^{-1}UV^T)^{-1} = \sum_{n=0}^{\infty} (-1)^n (A^{-1}UV^T)^n$$

$$\Rightarrow I \times 1 - (A^{-1}UV^T) + (A^{-1}UV^T)^2 \dots$$

Substituting the expansion, we get

$$(I + A^{-1}UV^T)^{-1}A^{-1} \Rightarrow \left(I - (A^{-1}UV^T) + (A^{-1}UV^T)^2 \dots\right)A^{-1}$$

$$\Rightarrow \left(IA^{-1} - A^{-1}UV^TA^{-1} + (A^{-1}UV^T)^2A^{-1} \dots\right)$$

$$\Rightarrow A^{-1} - (A^{-1}U)(V^TA^{-1}) + (A^{-1}U)(V^TA^{-1}U)(V^TA^{-1}) \dots$$

$$\Rightarrow A^{-1} - A^{-1}U(I - V^TA^{-1}U + (V^TA^{-1}U)^2 + \dots)V^TA^{-1}$$

So, $(A + UV^T)^{-1} = A^{-1} - A^{-1}U\underbrace{(I - V^TA^{-1}U + (V^TA^{-1}U)^2 + \dots)}_{\text{Part I}}V^TA^{-1}$

(Equation 1)

Part I of the above equation when compared to a geometric series, i.e,

Geometric Series = $a + ar + ar^2 + \dots$

Comparing $I - V^T A^{-1} U + (V^T A^{-1} U)^2 \dots$ to $a + ar + ar^2 \dots$

we get $a = I$ and $r = -V^T A^{-1} U$

We know that sum of first S_n terms of a geometric series is given by $S_n = \dfrac{a(1-r^n)}{1-r}$

So sum of first n terms of Part I is $S_n = \dfrac{I(I - (-V^T A^{-1} U)^n)}{I - (-V^T A^{-1} U)}$

$$= \frac{I(I - (-V^T A^{-1} U)^n)}{I + V^T A^{-1} U}$$

Considering the term $(-V^T A^{-1} U)^n$, the value of this term approaches zero as the sequence progresses

So, the sum of first n terms of the series now is $\frac{I(I-0)}{I+V^T A^{-1} U}$

$$\Rightarrow \frac{I}{I+V^T A^{-1} U} \Rightarrow (I+V^T A^{-1} U)^{-1}$$

So, substituting the value obtained in equation I, we get

$$\boxed{A^{-1} - A^{-1} U (I + V^T A^{-1} U)^{-1} V^T A^{-1}}$$

This obtained solution is equal to Right hand side, that is

$$(A + UV^T)^{-1} = A^{-1} - A^{-1} U (I + V^T A^{-1} U)^{-1} V^T A^{-1}$$