PROBLEMS SET 3 SOLUTIONS

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PROBLEM 1, Solution

given, spectral norm of a mxn matrix A is

$$\|A\| = \max_{\|x\|=1} \|Ax\|$$

here, the term $\|x\|=1$ means that x is a unit vector

To prove that,

$$\|A\| = \max_{x \neq 0} \frac{\|Ax\|}{\|x\|}$$

-> Here x is a non-zero vector

The vector x can also be written as

$$x = a\, x_1$$

This means that the vector x is a times of a unit vector

Substituting the above in $\max_{x \neq 0} \frac{\|Ax\|}{\|x\|}$

we get

$$\max_{x \neq 0} \frac{\|Ax\|}{\|x\|} = \max_{x=1} \frac{\|Ax\|}{\|ax\|}$$

Using the homogeneous property of norms.

we can write

$$= \max_{x=1} \frac{|a|\,\|Ax\|}{|a|\,\|x\|}$$

$$= \max_{x=1} \frac{\|Ax\|}{\|X\|}$$

Here, we know that $\|X\| = 1$, since it is a unit vector

So, we hax $\max_{x=1} \|AX\|$

From the given definition of spectral norm of a matrix we know that $\max_{x=1} \|AX\|$ is equal to $\|A\|$

So we get $\boxed{\|A\| = \|A\|}$

To conclude that for any $n \times 1$ vector

$\|Ax\| \leq \|A\| \|x\|$

We know that the max possible value of $\frac{\|Ax\|}{\|x\|}$ is equal to or greater than any value of $\frac{\|Ax\|}{\|x\|}$

This is because the max gives a value which is always equal to or greater than the inputs considered

thus, we have $\max\limits_{x \neq 0} \frac{\|Ax\|}{\|x\|} \geq \frac{\|Ax\|}{\|x\|}$

$$\|A\| \geq \frac{\|Ax\|}{\|x\|}$$

So we have

$\boxed{\|Ax\| \leq \|A\| \|x\|}$

For non conformable matrices A and B

$\|AB\| \leq \|A\| \, \|B\|$

From the spectral norm notation, we have

$$\|AB\| = \max_{x \neq 0} \frac{\|ABx\|}{\|x\|}$$

we also know that

$$\|ABx\| \leq \|A\| \, \|Bx\|$$

$$\|AB\| \leq \max_{x \neq 0} \frac{\|A\| \, \|Bx\|}{\|x\|}$$

Using submultiplicative property, we can write

$$\|AB\| \leq \max_{x \neq 0} \frac{\|A\| \, \|B\| \, \|X\|}{\|X\|}$$

$\boxed{\text{So we have } \|AB\| \leq \|A\| \, \|B\|}$ $\square$

PROBLEM 2:

Solution: given, the spectral radius of square matrix A is given by $\sigma(A)$, which is the largest absolute value of eigen values of A.

To show, that for $n \times n$ matrix A

$$\sigma(A) \leq \|A\|$$

By definition,

$$\sigma(A) = \max \{|\lambda_1|, |\lambda_2| \dots |\lambda_n|\}$$

where λ_l is the largest eigen value of A, so we can write

$$\sigma(A) = |\lambda_l|$$

Let λ_l be the eigen value of square matrix A, and let $x \neq 0$ be the corresponding Eigen vector

So, we have

$$Ax = \lambda_l x$$

$$|\lambda_l| \, \|x\| = \|\lambda_l x\| = \|Ax\|$$

$$\|Ax\| \le \|A\| \, \|x\|$$

x is not a zero matrix, so $\|A\|$ is greater than or equal to zero.

we get $$|\lambda_l| \le \|A\|$$

$$\sigma(A) \le \|A\| \quad \square$$

PROBLEM 3

Solution

given, $A(t)$ is continiously differenti-able square matrix that is invertible at each t

We know that $A(t)\, A^{-1}(t) = I$

Differentiating the above equation with respect to t, we get

$$\frac{d}{dt}\left(A(t)\, A^{-1}(t)\right) = \frac{d}{dt}(I)$$

Using product rule, i.e

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

we get,

$$A(t)\frac{dA^{-1}(t)}{dt} + \frac{dA(t)}{dt}A^{-1}(t) = 0$$

Re-arranging the terms, we get.

$$A(t)\frac{dA^{-1}(t)}{dt} = -\frac{dA(t)}{dt}A^{-1}(t)$$

$$\Rightarrow \frac{d}{dt}A^{-1}(t) = -A^{-1}(t)\frac{dA(t)}{dt}A^{-1}(t)$$

$$\boxed{\frac{d}{dt}A^{-1}(t) = -A^{-1}(t)\dot{A}(t)A^{-1}(t)}$$

$\square$

PROBLEM 4

Solution:

Given, using laplace transform solve $\dot{x} = a x(t) + b(t) u(t)$, using initial condition. $x(0)$, a is a constant and $x(t)$, $b(t)$ and $u(t)$ are real valued functions.

Taking laplace transform of the given equation $\dot{x} = a x(t) + b(t) u(t)$, we have $L(\dot{x}) = L[a x(t) + b(t) u(t)]$

(From the table of laplace transforms, we know that $L(f'(t)) = SF(S) - f(0)$)

So, we get...

$$s X(s) - X(0) = a X(s) + \mathcal{L}\{b(t)u(t)\}$$

Rearranging the terms, we get

$$X(s)(s-a) = X(0) + \mathcal{L}\{b(t)u(t)\}$$

dividing by $(s-a)$ throughout

$$\frac{X(s)(s-a)}{(s-a)} = \frac{X(0)}{(s-a)} + \frac{\mathcal{L}\{b(t)u(t)\}}{(s-a)}$$

$$X(s) = \frac{X(0)}{(s-a)} + \frac{\mathcal{L}\{b(t)u(t)\}}{(s-a)}$$

Taking Inverse laplace transform, we get

$$\mathcal{L}^{-1}\{\mathcal{L}\{X(t)\}\} = \mathcal{L}^{-1}\left\{\frac{X\{0\}}{s-a} + \frac{\mathcal{L}\{b(t)u(t)\}}{s-a}\right\}$$

$$\boxed{\Rightarrow X(t) = X\{0\} e^{at} + b(t)u(t)e^{at}}$$

PROBLEM 5, Solution

given, an n^{th} order differential equation

$$y^{n}(t) + a_{n-1}t^{-1}y^{n-1}(t) + a_{n-2}t^{-2}y^{n-2}(t) \cdots$$
$$+ a_1 t^{-n+1} y^{(1)}(t) + a_0 t^{-n} y(t) = 0.$$

where $y^{(n)}(t) = \dfrac{d^n y(t)}{dt^n}$

Considering state variables as x_1 and x_2

$$x_1 = t^{-n} y(t)$$
$$x_2 = t^{-n+1} y'(t)$$
$$x_3 = t^{-n+2} y''(t)$$
$$x_{n-1} = t^{-n+(n-1-1)} y^{(n-2)}(t)$$
$$= t^{-2} y^{(n-2)}(t)$$

and $x_n = t^{-1} y^{n-1}(t)$

$$x_1' = y'(t)\,t^{-n} + y(t)(-n)\,t^{-n-1}$$

$$x_1' = t^{-1}\left[y'\,t^{-n+1} + (-n)\,y(t)\,t^{-n}\right]$$

$$\underline{x_1' = t^{-1}\left[x_2 - n x_1\right]}$$

$$x_2' = y''(t)\,t^{-n+1} + y'(t)(-n+1)\,t^{-n}$$

$$x_2' = t^{-1}\left[y''(t)\,t^{-n+2} + y'(t)(-n+1)\,t^{-n+1}\right]$$

$$\underline{x_2' = t^{-1}\left[x_3 + x_2(-n+1)\right]}$$

$$x_3' = y'''(t)\,t^{-n+2} + y''(t)(-n+2)\,t^{-n+1}$$

$$x_3' = t^{-1}\left[y'''(t)\,t^{-n+3} + y''(t)(-n+2)\,t^{-n+2}\right]$$

$$\underline{x_3' = t^{-1}\left[x_4 + (-n+2)\,x_3\right)}$$

$$x_{n-1}' = y^{n-1}(t)\,t^{-2} - 2\cdot\left(y^{n-2}\,t^{-3}\right)$$

$$= t^{-1}\left[y^{n-1}(t)\,t^{-1} + (-2)\,y^{n-2}\,t^{-2}\right]$$

$$= t^{-1}\left[x_n + (-2)\,x_{n-1}\right]$$

$$x_n' = y^n(t)\,t^{-1} - 1\left[y^{n-1} t^{-2}\right]$$

$$= t^{-1}\left[y^n(t) - 1\,y^{n-1} t^{-1}\right]$$

$$\underline{x_n' = t^{-1}\left[y^n(t) - x_{n-1}\right]}$$

So
$$\dot{\vec{x}} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} t^{-1}(x_2 - nx_1) \\ t^{-1}\left[x_3 + x_2(-n+1)\right] \\ t^{-1}\left[x_4 + (-n+2)x_3\right) \\ \vdots \\ t^{-1}\left(x_n + (-2)x_{n-1}\right] \\ t^{-1}\left[y^n(t) - x_{n-1}\right] \end{bmatrix}$$

$$\dot{\vec{x}} = t^{-1}\begin{bmatrix} (x_2 - nx_1) \\ \left[x_3 + x_2(-n+1)\right] \\ \left[x_4 + (-n+2)x_3\right) \\ \vdots \\ \left[x_n + (-2)x_{n-1}\right] \\ \left[y^n(t) - x_{n-1}\right] \end{bmatrix}$$

$$\dot{\vec{x}} = t^{-1}\begin{bmatrix} -n & 1 & 0 & 0 & \cdots & 0 \\ 0 & (-n+1) & 1 & 0 & \cdots & 0 \\ 0 & 0 & (-n+2) & 1 & \cdots & 0 \\ \vdots & & & \vdots & & \\ 0 & 0 & \cdots & \cdots & (-2) & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & (-a_{n-2}) & (-a_{n-1}) \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix}$$

$$\dot{\vec{X}} = t^{-1} A \vec{X}(t)$$

where A from the above matrix is

$$A = \begin{bmatrix} -n & 1 & 0 & 0 & \cdots & 0 \\ 0 & (-n+1) & 1 & 0 & \cdots & 0 \\ 0 & 0 & (-n+2) & 1 & \cdots & 0 \\ \vdots & & & \vdots & & \\ 0 & 0 & \cdots & \cdots & (-2) & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & (-a_{n-2}) & (-a_{n-1}) \end{bmatrix}$$

PROBLEM 6

Prove that

$$\frac{\partial}{\partial \tau} \Phi(t,\tau) = -\Phi(t,\tau) A(\tau)$$

We know that the transition matrix as a function of two variables is given by Peano Baker series

So, we have $\Phi(t,\tau) =$

$$I + \int_{\tau}^{t} A(\sigma_1)\,d\sigma_1 + \int_{\tau}^{t} A\sigma_1 \int_{\tau}^{\sigma_1} A(\sigma_2)\,d\sigma_2\,d\sigma_1$$

$$+ \int_{\tau}^{t} A(\sigma_1) \int_{\tau}^{\sigma_1} A(\sigma_2) \int_{\tau}^{\sigma_2} A(\sigma_3)\,d\sigma_3\,d\sigma_2\,d\sigma_1 + \ldots\ldots$$

Taking partial derivative with respect to τ of the $(n+1)^{th}$ term of Peano-Baker series using Leibniz rule

$$\frac{\partial}{\partial \tau}\left[\int_{\tau}^{t} A(\sigma_1)\int_{\tau}^{\sigma_1} A(\sigma_2)\int_{\tau}^{\sigma_2}\cdots\int_{\tau}^{\sigma_n} A(\sigma_{n+1})\, d\sigma_{n+1}\cdots d\sigma_1\right]$$

$$\Rightarrow \left[A(t)\int_{\tau}^{t} A(\sigma_2)\int_{\tau}^{\sigma_2}\cdots\int_{\tau}^{\sigma_n} A(\sigma_{n+1})\, d\sigma_{n+1}\cdots d\sigma_2\right]\cdots$$

$$\cdots \frac{d}{d\tau} t - \left[A(\tau)\int_{\tau}^{\tau} A(\sigma_2)\int_{\tau}^{\tau}\cdots d\sigma_{n+1}\cdots d\sigma_2\right]\cdots$$

$$\cdots \frac{d}{d\tau}\tau + \int_{\tau}^{t} A(\sigma_1)\frac{\partial}{\partial \tau}\left[\int_{\tau}^{\sigma_1} A(\sigma_2)\int_{\tau}^{\sigma_2}\cdots\int_{\tau}^{\sigma_n} A(\sigma_{n+1})\right] d\sigma_{n+1}\cdots d\sigma_1$$

$$\Rightarrow \int_{\tau}^{t} A(\sigma_1)\frac{\partial}{\partial \tau}\left[\int_{\tau}^{\sigma_1} A(\sigma_2)\int_{\tau}^{\sigma_2} A(\sigma_3)\cdots\int_{\tau}^{\sigma_n} A(\sigma_{n+1})\right] d\sigma_{n+1}\cdots d\sigma_1$$

If the process is repeated n times, we obtain the following.

$$\frac{\partial}{\partial\tau}\left[\int_\tau^t A(\sigma_1)\,\frac{\partial}{\partial\tau}\left[\int_\tau^{\sigma_1}A(\sigma_2)\int_\tau^{\sigma_2}A(\sigma_3)\cdots\int_\tau^{\sigma_n}A(\sigma_{n+1})\right]d\sigma_{n+1}\cdots d\sigma_1\right]$$

$$=\int_\tau^t A(\sigma_1)\int_\tau^{\sigma_1}A(\sigma_2)\cdots\int_\tau^{\sigma_{n-1}}A(\sigma_n)\,\frac{\partial}{\partial\tau}\left[\int_\tau^{\sigma_2}A(\sigma_{n+1})\,d\sigma_{n+1}\right]d\sigma_n\cdots d\sigma_1$$

$$=\int_\tau^t A(\sigma_1)\int_\tau^{\sigma_1}A(\sigma_2)\cdots\int_\tau^{\sigma_{n-1}}A(\sigma_n)\left[0-A(\tau)+\int_\tau^{\sigma_2}0\cdot d\sigma_{n+1}\right]d\sigma_n\cdots d\sigma_1$$

$$=\underbrace{\left[\int_\tau^t A(\sigma_1)\int_\tau^{\sigma_1}A(\sigma_2)\int_\tau^{\sigma_2}\cdots\int_\tau^{\sigma_{n-1}}A(\sigma_k)\,d\sigma_k\cdots d\sigma_1\right]\left[-A(\tau)\right]}_{\left[-\Phi(t,\tau)\,A(\tau)\right]}$$

So we can write

$$\boxed{\frac{\partial}{\partial\tau}\Phi(t,\tau)=-\Phi(t,\tau)\,A(\tau)}\quad\square$$

PROBLEM 8

<u>Solution</u>

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & 5 & -2 \end{bmatrix}$$

To find Eigen Values of matrix A

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 6 & 5 & -2-\lambda \end{vmatrix} = 0$$

$$-\lambda(-\lambda(-2-\lambda)+6 = 6$$

$$-\lambda(2\lambda + \lambda^2)+6 = 0$$

$$-\lambda^3 - 2\lambda^2 + 5\lambda + 6 = 0$$

$$-(\lambda+1)(\lambda-2)(\lambda+3) = 0$$

So, $\lambda = -1$, 2 and -3 are the eigen values

Eigen Vectors are:-

<u>for $\lambda = -1$</u>

$x_2 = -x_1$

$x_3 = -x_2$

$6x_1 + 5x_2 - 2x_3 = -x_3$

$\rightarrow 6x_1 = 6x_3$

$x_1 = x_3$

$x_1 = x_3 = -x_2$

Eigenvector for $\lambda = -1$ is $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

<u>for $\lambda = 2$</u>

$x_2 = 2x_1$

$x_3 = 2x_2$

$6x_1 + 5x_2 - 2x_3 = 2x_3$

$x_2 = \frac{x_3}{2} = 2x_1$

So eigenvector is

$\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$

<u>for $\lambda = -3$</u>

$x_2 = -3x_1$

$x_3 = -3x_2$

$6x_1 + 5x_2 - 2x_3 = -3x_3$

$x_2 = -3x_1$

$x_3 = 9x_1$

So, eigenvector is

$\begin{bmatrix} 1 \\ -3 \\ 9 \end{bmatrix}$

We know that $e^{At} = V e^{\Lambda t} V^{-1}$

where $e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & 0 & 0 \\ 0 & e^{\lambda_2 t} & 0 \\ 0 & 0 & e^{\lambda_3 t} \end{bmatrix}$

where $\lambda_1 = -1$, $\lambda_2 = 2$, $\lambda_3 = -3$

V is the eigen vector matrix

$$V = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 2 & -3 \\ 1 & 4 & 9 \end{bmatrix}$$

$$V^{-1} = \frac{1}{|V|} \begin{bmatrix} \begin{vmatrix} v_{22} & v_{23} \\ v_{32} & v_{33} \end{vmatrix} & \begin{vmatrix} v_{13} & v_{12} \\ v_{33} & v_{32} \end{vmatrix} & \begin{vmatrix} v_{12} & v_{13} \\ v_{22} & v_{23} \end{vmatrix} \\ \begin{vmatrix} v_{23} & v_{21} \\ v_{33} & v_{31} \end{vmatrix} & \begin{vmatrix} v_{11} & v_{13} \\ v_{31} & v_{33} \end{vmatrix} & \begin{vmatrix} v_{13} & v_{11} \\ v_{23} & v_{21} \end{vmatrix} \\ \begin{vmatrix} v_{21} & v_{22} \\ v_{31} & v_{32} \end{vmatrix} & \begin{vmatrix} v_{12} & v_{11} \\ v_{32} & v_{31} \end{vmatrix} & \begin{vmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{vmatrix} \end{bmatrix}$$

where $|V|$ is 30

$$V^{-1} = \frac{1}{|30|} \begin{bmatrix} 30 & -5 & -5 \\ 6 & 8 & 2 \\ -6 & -3 & 3 \end{bmatrix}$$

$$V^{-1} = \begin{bmatrix} 1 & \frac{-1}{6} & \frac{-1}{6} \\ \frac{1}{5} & \frac{4}{15} & \frac{1}{15} \\ \frac{-1}{5} & \frac{-1}{10} & \frac{1}{10} \end{bmatrix}$$

$$e^{At} = V e^{\Lambda t} V^{-1}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ -1 & 2 & -3 \\ 1 & 4 & 9 \end{bmatrix} \begin{bmatrix} e^{t} & 0 & 0 \\ 0 & e^{t} & 0 \\ 0 & 0 & e^{t} \end{bmatrix} \begin{bmatrix} 1 & \frac{-1}{6} & \frac{-1}{6} \\ \frac{1}{5} & \frac{4}{15} & \frac{1}{15} \\ \frac{-1}{5} & \frac{-1}{10} & \frac{1}{10} \end{bmatrix}$$

$$= \begin{bmatrix} e^{-t} & e^{2t} & e^{-3t} \\ -e^{-t} & 2e^{2t} & -3e^{-3t} \\ e^{-t} & 4e^{2t} & 9e^{-3t} \end{bmatrix} \begin{bmatrix} 1 & \frac{-1}{6} & \frac{-1}{6} \\ \frac{1}{5} & \frac{4}{15} & \frac{1}{15} \\ \frac{-1}{5} & \frac{-1}{10} & \frac{1}{10} \end{bmatrix}$$

$$e^{At} = \begin{bmatrix} \dfrac{\frac{5}{e^{t}} + e^{2t} - \frac{1}{e^{3t}}}{5} & \dfrac{-5e^{-t} + 8e^{2t} - 3e^{-3t}}{30} & \dfrac{-5e^{-t} + 2e^{2t} + 3e^{-3t}}{30} \\ \dfrac{-5 + 2e^{3t} + 3e^{-2t}}{5e^{t}} & \dfrac{5e^{-t} + 16e^{2t} + 9e^{-3t}}{30} & \dfrac{5e^{-t} + 4e^{2t} - 9e^{-3t}}{30} \\ \dfrac{5 + 4e^{3t} - 9e^{-2t}}{5e^{t}} & \dfrac{-5e^{-t} + 32e^{2t} - 27e^{-3t}}{30} & \dfrac{-5e^{-t} + 8e^{2t} \ldots \ldots 27e^{-3t}}{30} \end{bmatrix}$$

PROBLEM 7

Solution

To compute state transition matrix $\Phi(t, t_0)$

for $A(t) = \begin{bmatrix} 1 & 0 \\ 1 & \eta(t) \end{bmatrix}$ where $\eta(t)$ is bounded and continuous function of t

We know that

$$\dot{x}(t) = A(t)\, x(t) + B(t)\, u(t)$$

we can write the above equation as

$$\dot{x}(t) = A(t)\, x(t) + \underbrace{B(t)\, u(t)}_{\text{This term} = 0, \text{ because } u(t) = 0}$$

We can write the above equation in state

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & \eta(t) \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

This gives

$$\dot{x}_1(t) = x_1(t) \quad \text{—— (1)}$$

$$\dot{x}_2(t) = x_1(t) + \eta(t)\, x_2(t) \quad \text{— (2)}$$

We can find value of x_1 using Eqn. (1) by solving the ODE

We know that $\dot{x}_1(t) = \frac{dx_1}{dt}$ & similarly

From (1)

$$\frac{dx_1}{dt} = x_1 \Rightarrow \frac{dx_1}{x_1} = dt$$

Integrating the above equation we have

$$\int_{x_{10}}^{x_1} \frac{1}{x_1}\, dx_1 = \int_{t_0}^{t} dt$$

$$\ln(x_1) - \ln(x_{10}) = (t - t_0)$$

$$\ln\left(\frac{x_1}{x_{10}}\right) = (t - t_0)$$

$$\frac{x_1}{x_{10}} = e^{(t-t_0)}$$

$$\boxed{x_1 = x_{10}\, e^{t-t_0}}$$

Considering equation (2)

$$\dot{x}_2(t) - \eta(t)\, x_2(t) = x_1(t)$$

To find if equation is exact.

$$\frac{dx_2}{dt} - \eta(t)\, x_2(t) = x_1(t)$$

$$dx_2 = (x_1 + \eta x_2)\,dt$$

$$\underbrace{(x_1 + \eta x_2)}_{A} dt - dx_2 = 0$$

A and $B = -1$

$$\frac{\partial A}{\partial x_2} = 1 \qquad \frac{\partial B}{\partial t} = 0$$

Not exact !! , So we have to find integrating factor to make it exact

$$I.F = \exp\left\{ \int f(x)\,dx \right\}$$

where $f(x) = \frac{1}{B}\left(\frac{\partial A}{\partial x_2} - \frac{\partial B}{\partial t}\right)$

$$f(x) = -1\left[\eta(t) - 0\right]$$

$$f(x) = -\eta(t)$$

$$I.F = \exp\left\{ \int_{t_0}^{t} -\eta(t)\, dt \right.$$

$$= e^{-\int_{t_0}^{t} \eta(t)\, dt}.$$

Multiplying this Integrating factor with the equation, we get.

$$e^{\int_{t_0}^{t} -\eta(t)\, dt} \left[x_{10} e^{(t-t_0)} + \eta x_2 \right) dt - dx_2 \Big] = 0$$

Solution to this equation is given by

$$U(x,y) = \int A + F(y) \quad \text{- General Solution}$$

$$U(t, x_2) = \int_{t_0}^{t} x_{10}\, e^{t - t_0 - t\int_{t_0}^{t} \eta(t) dt} + \eta x_2 e^{-\int_{t_0}^{t} \eta(t) dt} + F(x_2)$$

We can integrate this using the sum rule

$$\underbrace{\int_{t_0}^{t} x_{10}\, e^{t-t_0-t\int_{t_0}^{t}\eta(t)dt}}_{\text{Part 1}} + \underbrace{\int_{t_0}^{t} \eta x_2 e^{-\int_{t_0}^{t}\eta(t)dt}}_{\text{Part 2}} + F(x_2)$$

<u>Considering part 1</u>

lets take $t-t_0-t\int_{t_0}^{t}\eta(t)\,dt = g$

$$\frac{dg}{dt} = 1-\eta(t)$$

$$dt = \frac{dg}{1-\eta(t)}$$

Also integration limits change

@ $t=t$, $g = t-t_0-t\int_{t_0}^{t}\eta(t)\,dt$

$g = g$

@ $t = t_0$ $\quad g = t_0 - t_0 - t_0 \int_{t_0}^{t_0} \eta(t)\, dt$

$g = 0$

So part 1 can be written as

$$\int_0^g x_{10}\, e^g \frac{dg}{1-\eta(t)}$$

$$\frac{x_{10}}{1-\eta(t)} \int_0^g e^g\, dg = \frac{x_{10}}{1-\eta(t)} \left[e^g \Big|_0^g \right]$$

$$= \frac{x_{10}}{1-\eta(t)} \left[e^g - e^0 \right]$$

$$= \frac{x_{10}}{1-\eta(t)} \left[e^g - 1 \right]$$

<u>Part 1</u> $= \dfrac{x_{10}}{1-\eta(t)} \left[e^{t - t_0 - t_0 \int_{t_0}^{t_0} \eta(t)\, dt} - 1 \right]$

Now considering part 2

$$\int_{t_0}^{t} \eta x_2 e^{-\int_{t_0}^{t} \eta(t)\,dt}$$

Consider $\int_{t_0}^{t} -\eta(t)\,dt = h$

$$\frac{dh}{dt} = \frac{d}{dt}\left[-\int_{t_0}^{t} \eta(t)\,dt\right]$$

$$\frac{dh}{dt} = -\eta(t)\,dt$$

$$dt = \frac{dh}{-\eta(t)}$$

The integration limits are

@ $t = t$ $\quad h = \int_{t_0}^{t} -\eta(t)\,dt.$

$h = h$

@ $t = t_0$ $\qquad h = \int_{t_0}^{t_0} -\eta(t)\,dt$

$$h = 0$$

So we have

$$-\int_0^h \cancel{\eta(t)}\, x_2 e^h \frac{dh}{\cancel{\eta(t)}}$$

$$-\int_0^h x_2 e^h\,dh = -x_2\left[e^h\Big|_0^h\right]$$

$$= -x_2\left[e^h - 1\right]$$

<u>Part 2</u> $\quad = -x_2\left[e^{-\int_{t_0}^{t}\eta(t)} - 1\right]$

Combining Part 1 and part 2 we get

$$U(t, x_2) = \frac{x_{10}}{1-\eta(t)}\left[e^{t - t_0 - t_0\int_{t_0}^{t_0}\eta(t)\,dt} - 1\right] - x_2\left[e^{-\int_{t_0}^{t}\eta(t)} - 1\right] + F(x_2)$$

and

$\hookrightarrow$ (3)

To solve this equation, we can differentiate with respect to x_2 and equate it with $B(t, x_2)$

$$\frac{d}{dx_2}\left[\frac{x_{10}}{1-\eta(t)}\left[e^{t-t_0-t_0\int_{t_0}^{t_0}\eta(t)\,dt}-1\right]-x_2\left[e^{-\int_{t_0}^{t}\eta(t)}-1\right]+F(x_2)\right]=B(t, x_2)$$

and

$$B(t, x_2) = 0-\left[e^{-\int_{t_0}^{t}\eta(t)\,dt}-1\right]+\frac{dF(x_2)}{dx_2}$$

$$-e^{-\int_{t_0}^{t}\cancel{\eta(t)}\,dt} = -\left[e^{-\int_{t_0}^{t}\cancel{\eta(t)}\,dt}-1\right]+\frac{dF(x_2)}{dx_2}$$

$$\frac{dF(x_2)}{dx_2} = -1$$

Integrating we get $F(x_2) = -x_2 + C$

Substituting $F(x_2) = -x_2 + C$ in the eqn(3) we get.

$$\frac{x_{10}}{1-\eta(t)}\left[e^{t-t_0-t_0\int_{t_0}^{t_0}\eta(t)\,dt}-1\right]-x_2\left[e^{-\int_{t_0}^{t}\eta(t)}-1\right]+(-x_2+C) \quad -(4)$$

To find C,

at time $t = t_0$, $x_2 = x_{20}$

$$\frac{x_{10}}{1-\eta(t)}\left[e^{\cancelto{0}{t_0-t_0}-t\int_{t_0}^{t_0}\eta(t)\,dt}-1\right]-x_{20}\left[e^{\cancelto{0}{-\int_{t_0}^{t_0}\eta(t)}}-1\right]-x_{20}+C=0$$

$$\underline{C = x_{20}}$$

So, eqn $C(t)$ is

$$\frac{x_{10}}{1-\eta(t)}\left[e^{t-t_0-t_0\int_{t_0}^{t_0}\eta(t)\,dt}-1\right]-x_2\left[e^{-\int_{t_0}^{t}\eta(t)}-1\right]+(-x_2+x_{20})$$

$$U(t,x_0)=\frac{x_{10}}{1-\eta(t)}\left[e^{t-t_0-t_0\int_{t_0}^{t_0}\eta(t)\,dt}-1\right]-x_2\left[e^{-\int_{t_0}^{t}\eta(t)}-1\right]+(-x_2+x_{20})=0$$

Rearranging the terms, we get

$$x_2=\frac{x_{10}}{1-\eta(t)}\,\frac{\left[e^{t-t_0-\int_{t_0}^{t}\eta(t)\,dt}-1\right]}{e^{-\int_{t_0}^{t}\eta(t)\,dt}}+\frac{x_{10}}{e^{-\int_{t_0}^{t}\eta(t)\,dt}}.$$

So eqn (1) and eqn (2) can be written as

$$\dot{x}_1(t) = x_{10} e^{(t-t_0)}$$

$$\dot{x}_2(t) = x_{10} e^{t-t_0} - \eta(t)\left[\frac{x_{10}}{1-\eta(t)} \frac{\left[e^{t-t_0 - \int_{t_0}^{t} \eta(t)\,dt} - 1\right]}{e^{-\int_{t_0}^{t} \eta(t)\,dt}} + \frac{x_{10}}{e^{-\int_{t_0}^{t} \eta(t)\,dt}}\right]$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} e^{t-t_0} & 0 \\ \frac{\eta(t)}{1-\eta(t)} \frac{\left[e^{t-t_0 - \int_{t_0}^{t} \eta(t)\,dt} - 1\right]}{e^{-\int_{t_0}^{t} \eta(t)\,dt}} & \frac{\eta(t)}{e^{-\int_{t_0}^{t} \eta(t)\,dt}} \end{bmatrix} \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix}$$

$$\boxed{\Phi(t,t_0) = \begin{bmatrix} e^{t-t_0} & 0 \\ \frac{\eta(t)}{1-\eta(t)} \frac{\left[e^{t-t_0 - \int_{t_0}^{t} \eta(t)\,dt} - 1\right]}{e^{-\int_{t_0}^{t} \eta(t)\,dt}} & \frac{\eta(t)}{e^{-\int_{t_0}^{t} \eta(t)\,dt}} \end{bmatrix}}$$