HOMEWORK - 1

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PROBLEM 1:

$\alpha, \beta = ?$

$$dF(x,y) = \left(\frac{1}{x^2+2} + \frac{\alpha}{y}\right)dx + (xy^\beta + 1)dy$$

Soln.

Comparing the above equation to

$$A(x,y)\,dx + B(x,y)\,dy = 0$$

$$A(x,y) = \left(\frac{1}{x^2+2} + \frac{\alpha}{y}\right) \quad \text{and} \quad B(x,y) = (xy^\beta + 1)$$

for which, $\frac{\partial A}{\partial y} = \frac{\partial B}{\partial x}$. because it is an Exact Differential

Assumption:- α is considered not to be a function of x or y

$$\frac{\partial A}{\partial y} \Rightarrow \frac{\partial}{\partial y}\left(\frac{1}{x^2+2} + \frac{\alpha}{y}\right) \Rightarrow \left(-\alpha y^{-2}\right)$$

$$\frac{\partial B}{\partial x} = \frac{\partial}{\partial x}(xy^\beta + 1) \Rightarrow (y^\beta + 0)$$

Since $\frac{\partial A}{\partial y} = \frac{\partial B}{\partial x}$

$$\frac{-\alpha}{y^2} = y^\beta$$

comparing, we get values of α and β as

$$\alpha = -1 \quad \text{and} \quad \beta = -2$$

Substituting values of α and β in $dF(x,y) = \left(\frac{1}{x^2+2} + \frac{\alpha}{y}\right)dx + (xy^{\beta}+1)dy$

$$dF(x,y) = \left(\frac{1}{x^2+2} - \frac{1}{y}\right)dx + \left(\frac{x}{y^2}+1\right)dy$$

Integrating the equation,

$$F(x,y) = \int A(x,y) + F(y) = C_1$$

$$\int\left(\frac{1}{x^2+2} - \frac{1}{y}\right)dx + F(y) = C_1$$

Applying Sum Rule,

$$\int \frac{1}{x^2+2}\,dx - \int \frac{dx}{y}$$

$$F(x,y) = \frac{1}{\sqrt{2}} \arctan\left(\frac{x}{\sqrt{2}}\right) - \frac{x}{y} + y = C$$

Problem 2 :- Given, $R\frac{dq}{dt} + \frac{q}{C} = V(t)$ and $V(t) = V_0 \sin(\omega t)$ and $q(0) = 0$. ②

Soln. The standard form of the given equation is

$$\frac{dq}{dt} + \frac{q}{RC} = \frac{V_0}{R}\sin\omega t \quad \text{— (Equation 1)}$$

$$dq = \left(\frac{V_0 \sin(\omega t)}{R} - \frac{q}{RC}\right) dt$$

$$A = \frac{q}{RC} - \frac{V_0 \sin(\omega t)}{R} \qquad B = +1$$

So, $\frac{\partial A}{\partial y} \neq \frac{\partial B}{\partial x}$, ODE is not exact.

$$f(t) = \frac{1}{B}\left(\frac{\partial A}{\partial q} - \frac{\partial B}{\partial t}\right)$$

$$f(t) = \frac{+1}{+1}\left(\frac{1}{RC} - 0\right)$$

$$f(t) = +\frac{1}{RC}$$

Therefore, an integrating factor exists that is function of x alone

$$q(t) = \exp\left\{\int \frac{1}{RC}\,dt\right\}$$

$$q(t) = \exp\left(\frac{t}{RC}\right) \Rightarrow q(t) = e^{t/RC}$$

Multiplying this integrating factor to Equation 1

$$e^{t/RC}\left(\frac{dq}{dt} + \frac{q}{RC}\right) = e^{t/RC}\,\frac{V_0 \sin(\omega t)}{R}$$

$$e^{t/RC}\frac{dq}{dt} + e^{t/RC}\frac{q}{RC} = e^{t/RC}\,\frac{V_0 \sin(\omega t)}{R} \quad \text{— (Equation (2))}$$

the equation is of the form

$$\frac{d}{dt}(u.v) = u\frac{dv}{dt} + v\frac{du}{dt}.$$

Comparing the equation

$$u = e^{t/RC} \quad \text{and} \quad v = q$$

So, equation (2) is ...

$$\frac{d}{dt}\left(e^{t/RC} q(t)\right) = \frac{V_0 e^{t/RC}}{R}\sin(\omega t)$$

given that $q(0)=0$ and charge on capacitor is a function of time, t, we can apply the limits 0 to t and integrate the total derivative

$$q(t) = \left(\int \frac{V_0 e^{t/RC}}{R}\sin(\omega t)\,dt\right) \times \left(\frac{1}{e^{t/RC}}\right)$$

$$q(t) = \frac{V_0 e^{-t/RC}}{R}\int_0^t e^{t/RC}\sin\omega t\,dt$$

$$\int f g' = fg - \int f' g$$

$$f = \sin(\omega t) \quad \text{and} \quad g = \frac{e^{-t/RC}}{-1/RC}$$

$$f' = \omega\cos(\omega t) \qquad g' = \frac{e^{-t/RC}}{-1/RC}$$

$$\frac{\sin(\omega t)\times e^{-t/RC}}{-1/RC} - \int \frac{\omega\, e^{+t/RC}\cos(\omega t)\,dt}{-1/RC}$$

$$f = \omega\cos(\omega t) \qquad g' = e^{-t/RC}$$

$$\Rightarrow \frac{e^{+t/RC}\sin(\omega t)}{+1/RC} - \left(\frac{\omega e^{+t/RC}\cos(\omega t)}{(+1/RC)^2}\right) - \int \frac{-\omega^2 e^{+t/RC}\sin(\omega t)\,dt}{(+1/RC)^2}$$

$$\Rightarrow \frac{e^{+t/RC}\sin(\omega t)}{+1/RC} - \left(\frac{\omega e^{+t/RC}\cos(\omega t)}{(+1/RC)^2}\right) + \frac{\omega^2}{(+1/RC)^2}\int e^{+t/RC}\sin(\omega t)\,dt.$$

$$q(t) = \frac{V_0 e^{-t/RC}}{R}$$

$$\cancel{\frac{\sin(\omega t)\, e^{t/RC}}{+1/RC}}$$

$$e^{-t/RC}\left[\frac{\frac{+1}{RC}\sin(\omega t) - \frac{\omega\cos(\omega t)}{(+1/RC)^2}}{\omega^2 + (-1/RC)^2}\right]_0^t$$

$$e^{-t/RC}\left[\frac{\frac{+1}{RC}\left[\sin(\omega t) - \omega\cos(\omega t)\right]}{\omega^2 + (-1/RC)^2}\right]_0^t$$

$$= \frac{\frac{+1}{RC}\, e^{+t/RC}\sin(\omega t) - e^{+t/RC}\omega\cos(\omega t)}{\omega^2 + (+1/RC)^2} + \frac{\omega}{\omega^2 + (+1/RC)^2}$$

$$= \frac{V_0 e^{-t/RC}}{R}\left[\left[\frac{\frac{+1}{RC}\, e^{+t/RC}\left(\sin(\omega t) - \omega\cos(\omega t)\times e^{+t/RC}\right)}{\omega^2 + (+1/RC)^2}\right] + \frac{\omega}{\omega^2 + (+1/RC)^2}\right]$$

$$= \frac{V_0 e^{-t/RC}}{R(\omega^2 + (+1/RC)^2)}\left[e^{t/RC}\left[\frac{+1}{RC}\sin\omega t - \omega\cos(\omega t)\right] + \omega\right]$$

$$= \frac{V_0 \times R^2C^2}{R(\omega^2R^2C^2 + 1)}\left[\frac{e^{t/RC}}{e^{t/RC}}\left[\frac{\sin\omega t}{RC} - \omega\cos(\omega t)\right] + \frac{\omega}{e^{t/RC}}\right]$$

$$= \frac{V_0 RC^2}{\omega^2R^2C^2 + 1}\left[\frac{\sin(\omega t) - \omega\cos(\omega t)RC + \omega e^{-t/RC}\cdot RC}{RC}\right]$$

$$\boxed{q(t) = \frac{V_0 C}{\omega^2R^2C^2 + 1}\left[\sin\omega t - \omega\cos(\omega t)RC + \omega e^{-t/RC}\cdot RC\right]}$$

③ Problem 3 :-

Solution :-

$$\frac{dy}{dx} = -\frac{2x^2+y^2+x}{xy}$$

$\cancel{xy\,dy + 2x^2+y^2+}$

$$xy\,dy + (2x^2+y^2+x)dx = 0$$

Comparing to the form $B(x,y)dy + A\,dx = 0$

$B = xy \qquad A = (2x^2+y^2+x)$

$\frac{\partial B}{\partial x} = y \qquad \frac{\partial A}{\partial y} = 2y$, Not equal, so not exact

$$f(x) = \frac{1}{xy}(2y - y) \Rightarrow \frac{1}{xy} \times y = \frac{1}{x}$$

Integrating factor exists that is function of x alone

$$\mu(x) = \exp\left\{\int \frac{dx}{x}\right\} = \exp(\ln x) = x$$

Exact equation is - - .

$$x \times (x y\,dy + (2x^2+y^2+x)dx) = 0$$

$$x^2y\,dy + (2x^3+xy^2+x^2)dx = 0$$

$$\frac{\partial F}{\partial x} = 2x^3 + xy^2 + x^2$$

Integrating to $f(x,y) = C$

$$f(x,y) = \frac{2x^4}{4} + \frac{x^2y^2}{2} + \frac{x^3}{3} + g(y)$$

$$= \frac{x^4}{2} + \frac{x^2y^2}{2} + \frac{x^3}{3}$$

$$\frac{\partial u}{\partial y} = x^2 y \quad , \text{ so } f(y) = 0$$

So, Solution is $\dfrac{x^4}{2} + \dfrac{x^2y^2}{2} + \dfrac{x^3}{3} = C$

Problem (4a)

Soln. given $\dfrac{d^2f}{dt^2} + \dfrac{2df}{dt} + 5f = 0$, $f(0) = 1$ $f'(0) = 0$

Auxillary equation is

$\lambda^2 + 2\lambda + 5 = 0$, where $\lambda^n = \dfrac{d^nf}{dt^n}$, $\lambda^2 = \dfrac{d^2f}{dt}$.

Roots are $\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$-\frac{2 \pm \sqrt{4 - (4 \times 1 \times 5)}}{2}$$

$$= -1 \pm \sqrt{1-5}$$

$$\lambda = -1 \pm \sqrt{-4}$$

$$\lambda = -1 + 2i \quad , -1 - 2i$$

Roots are unequal, complex and distinct., so

Complementary function is given by

$$c_1 e^{(\alpha + i\beta)x} + c_2 e^{(\alpha - i\beta)x}$$

$$f(t) = e^{-t}(A\cos 2t + B\sin 2t)$$

Since RHS of eqn is zero, no Particular Integral required

given $f(0)=1$ and $f'(0)=0$

Applying $f(0)=1$, we get.

$$1 = e^{-0}(A\cos(0) + B\sin(0))$$

$$\boxed{A=1}$$

Applying $f'(0)=0$

$$f'(t) = -2e^{-t}\sin(2t) - e^{-t}\cos(2t) - Be^{-t}\sin(2t) + 2Be^{-t}\cos(2t)$$

$$0 = (-2\times0) - (1\times1) - (B\times0) + (2B)$$

$$2B-1=0$$

$$B=1/2$$

Substituting values of A & B, we get

$$\boxed{f(t) = e^{-t}\left(\cos(2t) + \frac{1}{2}\sin(2t)\right)}$$

Problem 4(b)

Solution

$$\lambda^2 + 2\lambda + 5 = 0$$

$\lambda = -1 \pm 2i$, So $CF = e^{-t}(A\cos 2t + B\sin 2t)$

$$y_c(x) = e^{-t}(A\cos 2t + B\sin 2t)$$

$f(x) \neq 0$

So general solution is given by

$$y(x) = y_c(x) + y_p(x)$$

y_p is particular integral linearly independent of $y_c(x)$

Assuming a parameterised form for $f(t) = e^{-t}\cos(3t)$ - ①

If $f(t) = a_1 \sin rx + a_2 \cos rx$ - ②

comparing ① & ②

$$y_p(t) = e^{-t}(b_1 \cos 3t + b_2 \sin 3t)$$

$\frac{d}{dt}(e^{-t}) = -e^{-t}$, Applying product rule.

$$f' = -e^{-t}(b_1 \cos 3t + b_2 \sin 3t) + e^{-t}(-3b_1 \sin 3t + 3b_2 \cos 3t)$$

$$f'' = \frac{d}{dt}\left(-e^{-t}(b_1 \cos 3t + b_2 \sin(3t))\right) + \frac{d}{dt}\left(e^{-t}(-3b_1 \sin 3t + 3b_2 \cos 3t)\right)$$

$$= +e^{-t}(b_1 \cos 3t + b_2 \sin(3t)) + -e^{-t}(3b_1 \sin 3t + 3b_2 \cos(3t))$$

$$+ -e^{-t}(-3b_1 \sin(3t) + 3b_2 \cos 3t) + e^{-t}(-9b_1 \cos(3t) - 9b_2 \sin(3t))$$

$$= e^{-t}(b_1 \cos 3t + b_2 \sin 3t + 3b_1 \sin 3t - 3b_2 \cos 3t + 3b_1 \sin 3t - 3b_2 \cos 3t$$
$$- 9b_1 \cos 3t - 9b_2 \sin(3t))$$

$$= e^{-t}(\sin(3t)(b_2 + 3b_1 + 3b_1 - 9b_2) + \cos(3t)(b_1 - 3b_2 - 3b_2 - 9b_1)$$

$$= e^{-t}(\sin(3t)(6b_1 - 8b_2) + \cos(3t)(-6b_2 - 8b_1))$$

Substituting f' and f''

$$e^{-t}(\sin 3t(6b_1 - 8b_2) + \cos(3t)(-6b_2 - 8b_1)) + -e^{-t}(b_1 \cos 3t + b_2 \sin 3t)$$
$$+ e^{-t}(-3b_1 \sin 3t + 3b_2 \cos 3t) + 5(e^{-t}(b_1 \cos 3t + b_2 \sin 3t) = e^{-t}\cos(3t)$$

Since $\sin(3t)$ related term $= 0$, (comparing LHS & RHS in above equation

$$\sin(3t) \times (b_2 + \cancel{6}b_1 - 9b_2 - 2b_2 - \cancel{6}b_1 + 5b_2) = 0.$$

$$-5b_2 = 0$$

$$\boxed{b_2 = 0}$$

Equating LHS and RHS for $\cos(3t)e^{-t}$

$$e^{-t}\cos(3t)\,(b_1 - 9b_1 - 2b_1 + 0 + 5b_1) = e^{-t}\cos(3t)$$

$$6b_1 - 11b_2 = 1 \quad \Rightarrow \quad -5b_2 = 1 \Rightarrow b_1 = -1/5$$

Substituting value of b_1 and b_2

$$f(t) = e^{-t}\left(b_1 \sin(3t) + b_2 \cos(3t)\right)$$

$$= e^{-t}\left(-\frac{1}{5}\cos(3t)\right)$$

$$f(t) = e^{-t}\left(A\cos(2t) + B\sin(2t)\right) - \frac{1}{5}\left(e^{-t}\cdot\cos(3t)\right)$$

Boundary conditions $f(0)=0$ $f'(0)=0$

$$e^{-0}(A\times 1 + B\times 0) - \frac{1}{5}e^{-0}\times 1 = 0$$

$$A - \frac{1}{5} = 0 \qquad A = \frac{1}{5}$$

$$f'(t) = -e^{-t}\left(A\cos 2t + B\sin(2t)\right) + e^{-t}\left(2A\sin(2t) + 2B\cos(2t)\right)$$
$$+ \frac{1}{5}e^{-t}\cos(3t) + \frac{e^{-t}}{5}\sin(3t)$$

~~$0 = -e^{-t}(A\cos(2t) + B\sin(2t) - \frac{1}{5}\cos(3t)) + e^{-t}($~~

Substituting value of A to the above equation...

$$f'(t) = -e^{-t}\left(\frac{\cos(2t)}{5} + B\sin(2t)\right) + e^{-t}\left(\frac{2}{5}\sin(2t) + 2B\cos 2t\right)$$
$$+ \frac{1}{5}e^{-t}\cos(3t) - \frac{e^{-t}}{5}\sin(3t)$$

(Multiply by 5)

we get ...$\Rightarrow -e^{-t}\left(\cos(2t) + 5B\sin 2t + (-\cos 3t)\right) + e^{-t}\left(-2\sin 2t + 10B\cos 2t + 3\sin 3t\right)$

at $t=0$,

$$0 = -(1+0-1) + (0 + 10B + 0)$$

$$10B = 0 \quad \boxed{B = 0}$$

Substituting values of A & B in f(t)

$$f(t) = \frac{1}{5}e^{-t}(\cos 2t - \cos 3t)$$

x —— x —— x

Problem 5 : Given $y''(t) + y(t) = \sin(2t)$

$y(0) = 2$, $y'(0) = 1$

Solution

$$\frac{d^2y}{dt^2} + \frac{dy}{dt} = \sin(2t)$$

finding laplace transform of $f(t) = \sin 2t$

$$L\{\sin 2t\} = \int_0^\infty e^{-st} \sin(2t)\,dt$$

Refering to table of standard results for $\sin(at)$ ~~eq~~ solution is $L(\sin(at)) = \frac{a}{a^2+s^2}$, also laplace transform of n^{th} derivative

$$y''(s) = s^2 y(s) - sy(0) - y'(0) + y(s) = \frac{2}{s^2+4}$$

$$(s^2+1)y(s) - 2s - 1 = \frac{2}{s^2+4}$$

$$(s^2+1)y(s) = \frac{2}{s^2+4} + 2s + 1$$

$$(s^2+1)y(s) = \frac{2 + 2s^3 + 8s + s^2 + 4}{s^2+4}$$

$$y(s) = \frac{2s^3+s^2+8s+6}{(s^2+4)(s^2+1)}$$

Using partial fractions

$$\frac{2s^3+s^2+8s+6}{(s^2+4)(s^2+1)} = \frac{a_1s+a_0}{s^2+4} + \frac{a_3s+a_2}{s^2+1} \quad \text{--- Equation (1)}$$

Multiply by denominator

$$2s^3+s^2+8s+6 = (a_1s+a_0)(s^2+1) + (a_3s+a_2)(s^2+4)$$

$$2s^3+s^2+8s+6 = a_1s^3 + a_1s + a_0s^2 + a_0 + a_3s^3 + 4a_3s + a_2s^2 + 4a_2$$

$$= s^3(a_1+a_3) + s^2(a_0+a_2) + s(a_1+4a_3) + 4a_2 + a_0$$

Equating, we get

$$a_1+a_3 = 2 \quad (1)$$
$$a_0+a_2 = 1 \quad (2)$$
$$a_1+4a_3 = 8 \quad (3)$$
$$a_0+4a_2 = 6 \quad (4)$$

$\Rightarrow$ ~~Using (1) & (3)~~

$3a_3 = 6$ $\boxed{a_3 = 2}$ $\boxed{a_1 = 0}$

$3a_2 = 5$ $\boxed{a_2 = 5/3}$ $a_0 = 1 - \frac{5}{3}$ $\boxed{a_0 = -2/3}$

Substituting the values in Equation (1), we get...

$$\frac{0.s + (-2/3)}{s^2+4} + \frac{2s+\frac{5}{3}}{s^2+1} \Rightarrow \frac{-2}{3(s^2+4)} + \frac{6s+5}{3(s^2+1)} \quad \text{--- Equation (2)}$$

Taking Inverse laplace by using table of transform pairs for equation (2)

$$2\frac{s}{s^2+1} + \frac{5}{3}\left(\frac{1}{s^2+1}\right) - \frac{1}{3}\left(\frac{2}{s^2+4}\right)$$

$$= \boxed{2(\cos t) + \frac{5}{3}(\sin t) - \frac{1}{3}\sin 2t}$$

$\boxed{\text{Problem (6)}}$

Solution.

$$\frac{d^3y}{dx^3} + 3\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + y = 30e^{-x}$$

consider $\frac{d^n}{dx^n} = D$

$$(D^3+3D^2+3D+1)y = 30e^{-x} \quad \text{— Equation (1)}$$

To find complementary equation

$$\lambda^3 + 3\lambda^2 + 3\lambda + 1 = 0$$

$$(\lambda+1)^3 = 0 \qquad \lambda = -1, -1, -1$$

Complementary function for the roots obtained are

$$y_c(x) = (c_1 + c_2 x + c_3 x^2)e^{-x}$$

To find Particular Solution, $y_p(x)$

~~$f(x) = 30e^{-x}$~~

from Equation (1)

$$y = \frac{30e^{-x}}{(D^3+3D^2+3D+1)} = \frac{30e^{-x}}{(D+1)^3}$$

$$\frac{1}{f(D)} e^{ax} \cdot u = e^{ax} \cdot \frac{1}{f(D+a)} \cdot u$$

$$= \frac{30e^{-x}}{(D+(-1)+1)^3}$$

$$= \frac{30e^{-x}}{D^3}$$

Splitting this,

$$\Rightarrow \frac{30e^{-x}}{D^2} \cdot \left[\frac{1}{D}\right]$$

Integrating, we get

$$\Rightarrow \frac{30e^{-x}}{D^2} \cdot x \quad \Rightarrow 30e^{-x} \frac{1}{D} \int x\,dx$$

Integrating again, we get

$$\frac{30e^{-x}}{} \times \frac{1}{D} \cdot \frac{x^2}{2}$$

Particular integral $y_p(x) = \frac{30e^{-x}}{2} \int x^2 dx$

$$= \frac{30e^{-x}}{2} \cdot \frac{x^3}{3}$$

$$y_p(x) = 5x^3 e^{-x}$$

general Solution $y(x) = y_c(x) + y_p(x)$

$$y = (C_1 + C_2 x + C_3 x^2)e^{-x} + 5x^3 e^{-x}$$

Problem 7

a) $y'' - y = x^n$

b) $y'' + y = \tan(x)$, $0 < x < \pi/2$

$\boxed{\text{Solution 7a)}}$ $y'' - y = x^n$

general solution is $y = y_c + y_p$

$\Rightarrow$ To find complementary function, y_c

Auxillary equation is

$$m^2 - 1 = 0$$

$$m^2 = 1$$

So $m = 1$ and $m = -1$

complementary solution for real distinct roots is.

$$y_c = c_1 e^{1x} + c_2 e^{-1 \times x}$$

$$\boxed{y_c = c_1 e^{x} + c_2 e^{-x}}$$

To find particular integral, we assume a solution of the form

$$y_p = k_1 e^{x} + k_2 e^{-x}$$

Imposing additional constraints

$$k_1' e^{x} + k_2' e^{-x} = 0 .$$

$\rightarrow$ Differentiating y_p equation once, we get

$$y_p'(x) = k_1' e^{x} + e^{x} k_1 + k_2' e^{-x} - e^{-x} k_2$$

$$= e^{x} k_1 - e^{-x} k_2 + e^{x} k_1' + k_2' e^{-x}$$

this is equal to zero.

$$y_p' = k_1 e^{x} - k_2 e^{-x}$$

$\rightarrow$ Differentiating y_p' once again we get

$$y_p'' = k_1 e^{x} + k_1' e^{x} - k_2' e^{-x} + k_2 e^{-x}$$

Substituting values of y'' and y' in $y''-y=x^n$

$$y_p''-y_p = k_1e^x + k_1'e^x - k_2'e^{-x} + k_2e^{-x} - k_1e^x - k_2e^{-x} = x^n$$

$$y_p''-y_p = \boxed{k_1'e^x + k_2'e^{-x} = x^n} \quad -①$$

Also, we have $\quad k_1'e^x + k_2'e^{-x} = 0 \quad -②$

Solving ① and ②

$$2k_1'e^x = x^n$$

$$k_1' = \frac{x^n}{2e^x}$$

$$k_1' = \frac{x^ne^{-x}}{2}$$

Integrating k_1' to find k, we get.

$$\int k_1' = \int \frac{x^ne^{-x}}{2}$$

$$k_1 = \frac{1}{2}\left[e^{-x}\, n! \sum_{m=0}^{n} \frac{x^m}{m!}\right]$$

Solving Equations ① and ② to find k_2', we get.

$$2k_2'e^{-x} = -x^n$$

$$k_2' = -\frac{x^ne^x}{2}$$

Integrating k_2' to find k_2, we get

$$\int k_2' = -\frac{1}{2}\int x^ne^x$$

$$k_2 = -\frac{e^x}{2}\left[\sum_{k=0}^{n}(-1)^{n-k}\frac{n!}{k!}x^k\right]$$

Substituting values of K_1 and K_2 in

$$\cancel{y_p = \left[\frac{-e^{x}}{2} n! \sum_{m=0}^{n} \frac{x^m}{m!}\right] e^{x} + \left[\frac{-e^{x}}{2} n! \sum_{m=0}^{n} \frac{x^m}{m!}\right] e^{-x}}$$

$$y_p = K_1 e^{x} + K_2 e^{-x}$$

$$y_p = \frac{1}{2}\left[e^{-x} n! \sum_{m=0}^{n} \frac{x^m}{m!}\right] e^{x} + -\frac{e^{x}}{2}\left[\sum_{k=0}^{n} (-1)^{n-k} \frac{n!}{k!} x^k\right] e^{-x}$$

With values of y_c and y_p, the general solution is

$$y = c_1 e^{x} + c_2 e^{-x} + \frac{1}{2}\left[n! \sum_{m=0}^{n} \frac{x^m}{m!}\right] + -\frac{1}{2}\left[\sum_{k=0}^{n} (-1)^{n-k} \frac{n!}{k!} x^k\right]$$

$$\boxed{y = c_1 e^{x} + c_2 e^{-x} + \frac{1}{2}\left[n! \sum_{m=0}^{n} \frac{x^m}{m!}\right] - \frac{1}{2}\left[\sum_{k=0}^{n} (-1)^{n-k} \frac{n!}{k!} x^k\right]}$$

Problem 7b $y'' + y = \tan x$

Solution

General solution is $y = y_c + y_p$

To find complementary function y_c, the auxillary equation is

$$m^2 + 1 = 0$$

$$m^2 = -1$$

$$m = -i, i$$

The complementary equation for distinct, imaginary roots is

$$y_c = c_1 e^{\alpha x} \cos \beta x + c_2 e^{\alpha x} \sin \beta x$$

$$\boxed{y_c = c_1 \cos x + c_2 \sin x}$$

To find particular integral, we can assume a solution as, $y_p = K_1 \cos x + K_2 \sin x$

$\Rightarrow$ Additional conditions imposed

$$k_1' \cos(x) + k_2' \sin x = 0.$$

$\Rightarrow$ Differentiating y_p to find y_p', we get.

$$y_p' = -k_1 \sin x + k_1' \cos x + k_2 \cos x + k_2' \sin x$$

$$-k_1 \sin x + k_2 \cos x + k_1' \cos x + k_2' \sin x$$

$\rightarrow$ this is equal to zero according to the constraints

So, $y_p' = -k_1 \sin(x) + k_2 \cos x$

$\Rightarrow$ Differentiating y_p' to find y_p'' we get

$$y_p'' = -k_1 \cos x - k_1' \sin x - k_2 \sin x + k_2' \cos x$$

$\Rightarrow$ Substituting values of y_p'' and y_p in $y'' + y = \tan x$, we get.

$$-\cancel{k_1 \cos x} - k_1' \sin x - \cancel{k_2 \sin x} + k_2' \cos x + \cancel{k_1 \cos x} + \cancel{k_2 \sin x} = \tan x$$

$$-k_1' \sin x + k_2' \cos x = \tan x \quad - (1)$$

also $\quad k_1' \cos x + k_2' \sin x = 0. \quad - (2)$

To solve (1) and (2), multiply equation (1) by $\cos x$ and equation (2) by $\sin x$

$$-k_1' \sin x \cos x + k_2' \cos^2 x = \tan x \times \cos x$$

$$k_1' \sin x \cos x + k_2' \sin^2 x = 0$$

Adding the equations

$$k_2' \sin^2 x + k_2' \cos^2 x = \tan x \times \cos x$$

$$k_2' \times (\sin^2 x + \cos^2 x) = \sin x$$

$$k_2' = \sin x$$

Integrating k_2', we get $k_2 = \int \sin x \Rightarrow \boxed{k_2 = -\cos x}$

From $k_1' \cos x + k_2' \sin x = 0$

$$k_1' \cos x = -k_2' \sin x$$

$$k_1' = -k_2' \tan x$$

$$k_1' = -\sin x \times \frac{\sin x}{\cos x}$$

$$k_1' = -\sin x \tan x$$

Integrating k_1', we get k_1

$$\int k_1' = \int -\sin x \tan x$$

$$k_1 = -\int \frac{\sin^2 x}{\cos x}$$

$$k_1 = -\int \frac{1-\cos^2 x}{\cos x} \Rightarrow k_1 = -\int \frac{1}{\cos x} + \int \cos x$$

$$k_1 = -\int \sec x + \int \cos x$$

$$\boxed{k_1 = -\ln|\sec x + \tan x| + \sin x}$$

Substituting the value of k_1 and k_2 in y_p, we get

$$y_p = k_1 \cos x + k_2 \sin x$$

$$y_p = [-\ln|\sec x + \tan x| + \sin x]\cos x + (-\cos x \times \sin x)$$

$$y_p = -\ln|\sec x + \tan x|\cos x + \sin x \cos x - \sin x \cos x$$

$$\boxed{y_p = -\ln|\sec x + \tan x| . \cos x}$$

general solution of $y'' + y = \tan x$ is

$$\boxed{y = y_c + y_p = c_1 \cos x + c_2 \sin x - \ln|\sec x + \tan x| \cos x}$$