# ENPM 667 - Control Of Robotic Systems Project 1 - Technical Report on Feedback Linearization vs. Adaptive Sliding Mode Control for a Quadrotor Helicopter

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***Abstract*—This is a technical report on the Journal paper: Daewon Lee, H. Jin Kim, and Shankar Sastry, "Feedback Linearization vs. Adaptive Sliding Mode Control for a Quadrotor Helicopter", International Journal of Control, Automation, and Systems (2009) 7(3):419-428. The report goes over the concepts of Feedback Linearization and Adaptive Sliding Mode Control, two nonlinear controllers commonly used to control the trajectory of a quadrotor helicopter. Section I of this report goes over the introduction of the quadrotor model considered and the control methods selected. This is followed by Section II which explains the modelling of the quadrotor helicopter. Feedback Linearization controller design is explained in Section III followed by Section IV which examines the Adaptive Sliding Mode controller design. Section V discusses the simulation and results and the final section, section VI, gives the conclusions.**

***Index Terms*—Feedback Linearization, Sliding Mode Control, Quadrotor Helicopter, Simulink**

## I. INTRODUCTION

Currently, quadrotors are being utilized in various domains, ranging from the military field where they are used for reconnaissance and combat, to the commercial and medical field where they are used for package and medicine delivery. Therefore, current scholarly investigations are mostly centered on the advancement of resilient control techniques capable of precisely guiding the trajectory and course of quadrotors, which are characterized by a complex and interconnected nonlinear system.

Several control techniques, including backstepping control, feedback linearization control, model predictive control, sliding mode control, fuzzy logic control, control based on neural networks are employed to effectively control nonlinear systems. Every control method possesses both advantages and disadvantages. Some methods are straightforward to adopt but are susceptible to noise and disturbances, while others are more intricate but yield robust control of the system. The analyzed quadrotor model is characterized by six output variables $(x, y, z, \phi, \theta, \psi)$and four input terms ($F_1$, $F_2$, $F_3$, and $F_4$) representing the thrust of the four rotors. Consequently, this model is classified as an underactuated system. The quadrotor's output variables exhibit strong coupling, whereby alterations in one state variable can impact the values of other variables within the system, resulting in a nonlinear dynamic system.

Feedback linearization is a control technique used for controlling non-linear systems by transforming the nonlinear dynamics of the system into a linear one so that linear control methods can be employed[6]. This concept involves using the inverse system dynamics to generate the control input, to cancel out the nonlinear terms. In order to establish a correlation between the input and output terms, the dynamic equations of the systems are systematically differentiated until the input terms are present in the output equations. This makes the system unstable and can result in performance degradation. Also, the controller is highly sensitive to external and sensor noises, as it involves higher derivative terms of dynamic variables.

Another technique included in the study is adaptive sliding mode control. This involves using the concept of sliding surface to control the behavior of the system. Sliding mode controllers provide more robustness in comparison to feedback linearization controllers, rendering them suitable for implementation in the presence of uncertainties and sensor noise. Hence when the copter is hovering, the upward thrust will be equal to the weight of the quadrotor

Traditional sliding mode controllers often require significant gains to counteract disturbances and noise, rendering them less suitable for systems with limited tolerance for high input gains, such as quadrotors. In order to address this limitation, a modification is incorporated into the sliding mode controller, resulting in reduced input gains.

## II. MODELLING OF FIXED-PITCH ANGLE BLADED QUADCOPTER

The quadrotor has 4 fixed-pitch angle blades. These blades produce an upward thrust and the amount of thrust produced depends on the angular velocity of the rotor. Hence, by controlling the rotational speed of the rotors, the quadrotor model can be controlled.

Figure 1 illustrates a basic quadrotor with motors attached to the ends of its four arms, each arm having the same length, l. Rotors 1 and 3 rotate in the opposite direction to rotors 2 and 4. This setup is designed to counteract angular momentum effects, allowing the quadrotor to maintain a fixed position in its equilibrium state. The rotors labeled 1, 2, 3, and 4 generate vertical forces denoted as $F_1$, $F_2$, $F_3$, and $F_4$ along the z-axis. Therefore, when the quadrotor is hovering, the upward force from the rotors equals the gravitational force acting on the quadrotor. To initiate motion in the quadrotor, it's necessary to adjust the thrust produced by each rotor based on calculated values aligned with the intended trajectory..

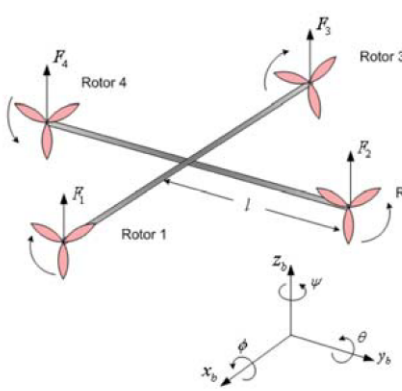


Fig. 1: Quadrotor Configuratin with Roll-Pitch-yaw Euler Angles

Depending on the speed of rotation of each propeller, it is possible to identify the movements of the quadrotor, this can be seen in figure (2)

The rotors, which are at the end of the arms, generate an upward thrust of $F_1$, $F_2$, $F_3$, and $F_4$ along the z-axis. $J_x, J_y, and J_z$ are the moment of inertia about the x-axis, y-axis, and z-axis respectively, and $\rho$ is the force to moment scaling factor. Assuming that the quadrotor is rigid, we can use the Newton Euler equations to describe the dynamics of the quadrotor.
Let the position of the quadrotor with respect to the inertial frame be,

$$\mathbf{P} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{1}$$

where $\mathbf{P}$ is the linear position vector.

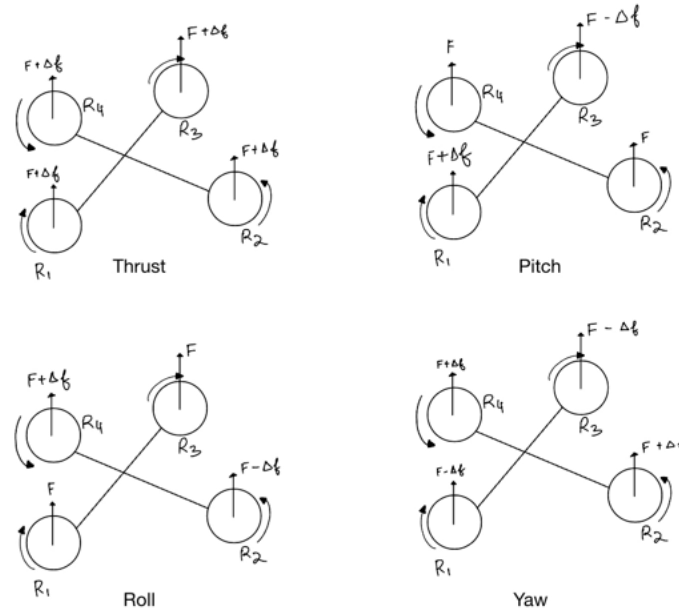


Fig. 2: Thrust and changes in rotor angular speed required to achieve Pitch, Roll and Yaw motion of a Quadrotor ($R_i$ is the Rotor numbering)

The rotation of the quadrotor is described with respect to the body frame. The roll angle ($\phi$) is along the x-axis, the pitch angle ($\theta$) is along the y-axis and the yaw angle ($\psi$) is along the z-axis. The attitude of the quadrotor can be described by using the Euler angles as

$$\mathbf{A} = \begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix} \tag{2}$$

The basic rotation matrices about the x, y and z axis can be represented as

$$R_{x,\phi} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix}$$

$$R_{y,\theta} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$R_{z,\psi} = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The transformation matrix (R) from the body frame to the initial frame is calculated by multiplying the Rotation matrices.

$$R = R_{x,\phi} \cdot R_{y,\theta} \cdot R_{z,\psi}$$

$$R = \begin{bmatrix} c\psi c\theta & s\phi s\theta c\psi - c_\phi s_\psi s_\phi & c_\phi s_\theta c_\psi + s_\phi s_\psi \\ s_\psi c_\theta & s_\psi s_\theta s_\phi + c_\psi c_\phi & s_\psi s_\theta c_\phi - c_\psi s_\phi \\ -s_\theta & c_\theta s_\phi & c_\theta c_\phi \end{bmatrix} \tag{3}$$

where $c_\phi = \cos\phi$, $s_\phi = \sin\phi$, $c_\theta = \cos\theta$, $s_\theta = \sin\theta$, $c_\psi = \cos\psi$, and $s_\psi = \sin\psi$.

Resolving the force acting on the quadrotor with respect to the body frame yields

$$\text{ma} + \text{CF} = \text{Thrust} - \mathbf{G} + \mathbf{G_r} \tag{4}$$

Where, CF is the centrifugal force vector defined by the product of the mass of the quadrotor (m), the velocity vector

$$\text{CF} = m \cdot \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} \times \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \tag{5}$$

The acceleration vector ($\mathbf{a}$) is given by:

$$\mathbf{a} = \begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} \tag{6}$$

G is the gravitational force in the z direction and $g_r(z)$ is the grounding effect[7]. The total force exerted along the z-axis, is the combination of all the thrust forces generated by each of the four rotors, i.e., $F_1 + F_2 + F_3 + F_4$

Equation (4) in the expaned form is

$$m \cdot \begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} + \omega \cdot m \cdot \mathbf{V} = \sum_{i=1}^{4} F_i - \begin{bmatrix} 0 \\ 0 \\ m * g \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ m * g_r(z) \end{bmatrix} \tag{7}$$

To obtain the same equation with respect to the inertial frame, the thrust vector is multiplied by the Z component of the rotational matrix. Also, in the inertial frame the centrifugal force gets nullified, the equation for the inertial frame becomes

$$m \cdot \begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} = \sum_{i=1}^{4} F_i \cdot R \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ m \cdot g \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ m \cdot g_r(z) \end{bmatrix} \tag{8}$$

Dividing throughout by m, the equation becomes

$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} = \frac{1}{m} \sum_{i=1}^{4} F_i \cdot R \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ g \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ g_r(z) \end{bmatrix} \tag{9}$$

The torque around x, y and z axis is given by the product of moment of inertia and angular acceleration and the equation is given as,

$$\begin{bmatrix} J_1 \\ J_2 \\ J_3 \end{bmatrix} \begin{bmatrix} \ddot{\phi} \\ \ddot{\theta} \\ \ddot{\psi} \end{bmatrix} = \begin{bmatrix} F_\phi \\ F_\theta \\ F_\psi \end{bmatrix} \begin{bmatrix} l \\ l \\ \rho \end{bmatrix} = \begin{bmatrix} F2 - F4 \\ F3 - F1 \\ F1 - F2 + F3 - F4 \end{bmatrix} \begin{bmatrix} l \\ l \\ \rho \end{bmatrix} \tag{10}$$

From these equations the terms can be separated and written individually as,

$$\ddot{\phi} = \frac{1}{J_1} \cdot (F_2 - F_4) \cdot l \tag{11}$$

$$\ddot{\theta} = \frac{1}{J_2} \cdot (F_3 - F_1) \cdot l \tag{12}$$

$$\ddot{\psi} = \frac{\rho}{J_3} \cdot (F_1 - F_2 + F_3 - F_4) \cdot \rho \tag{13}$$

To simplify the equation (8), the input terms can be defined as,

$$u_1 = \frac{(F_1 - F_2 + F_3 - F_4)}{m} \tag{14}$$

$$u_2 = \frac{(F_2 - F_4)}{J_1} \tag{15}$$

$$u_3 = \frac{(-F_1 + F_3)}{J_2} \tag{16}$$

$$u_4 = \frac{\rho \cdot (F_1 - F_2 + F_3 - F_4)}{m} \tag{17}$$

Combining equations (9) to (17) we obtain the following equations,

$$\ddot{x} = u_1 \cdot (\cos(\phi)\sin(\theta)\cos(\psi) + \sin(\phi)\sin(\psi)) \tag{18}$$

$$\ddot{y} = u_1 \cdot (\cos(\phi)\sin(\theta)\sin(\psi) - \sin(\phi)\cos(\psi)) \tag{19}$$

$$\ddot{z} = u_1 \cdot (\cos(\psi)\cos(\theta)) - g - g_r(z) \tag{20}$$

$$\ddot{\phi} = u_2 \cdot l \tag{21}$$

$$\ddot{\theta} = u_3 \cdot l \tag{22}$$

$$\ddot{\psi} = u_4 \tag{23}$$

These equations of motion can be represented in an equivalent vector form as,

$$\ddot{x} = f(x) + g(x)u + f_r(x), \tag{24}$$

where,

$$f(x) = \begin{bmatrix} 0 \\ 0 \\ -g \\ 0 \\ 0 \\ 0 \end{bmatrix}, f_r(x) = \begin{bmatrix} 0 \\ 0 \\ -g_r(z) \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

$$g(x) = \begin{bmatrix} \cos(\phi)\sin(\theta)\cos(\psi) + \sin(\phi)\sin(\psi) & 0 & 0 & 0 \\ \cos(\phi)\sin(\theta)\sin(\psi) - \sin(\phi)\cos(\psi) & 0 & 0 & 0 \\ \cos(\psi)\cos(\theta) & 0 & 0 & 0 \\ 0 & l & 0 & 0 \\ 0 & 0 & l & 0 \\ 0 & 0 & 0 & l \end{bmatrix},$$

$$\text{and} \quad u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}.$$

The variable $g_r$(z) denotes the phenomenon of ground effect that occurs during the landing phase. It is postulated that the ground impacts of the quadrotor become significant when the quadrotor descends below a specific altitude. $z_0$[8]

$$g_r(z) = \begin{cases} \frac{A}{(z+z_{\text{cg}})^2} - \frac{A}{(z_0+z_{\text{cg}})^2}, & \text{if } 0 < z \leq z_0 \\ 0 & \text{else} \end{cases} \tag{25}$$

In the context of ground effect analysis, the parameter A signifies the ground effect constant, while $z_{\text{cg}}$ represents the vertical component of the center of gravity. Due to the intricate nature of deriving precise equations for ground effect, the term $g_r$(z) is acknowledged as an unidentified perturbation when formulating a controller. This perturbation necessitates compensation or adaptation strategies for effective controller design.

## III. FEEDBACK LINEARIZATION CONTROL

The system of the quadrotor is a non-linear model and is difficult to control, so the non-linear model is transformed into an equivalent linear system which makes the system easier to handle.
Feedback linearization is a widely studied method in recent years for the design of nonlinear control systems. The primary concept revolves around the algebraic transformation of nonlinear systems dynamics into either fully or partially linear ones, enabling the application of linear control techniques.

### A. Selecting the Design of the Controller

For feedback linearization, if the terms $z$, $\phi$, $\theta$, $\psi$ are considered as outputs,i.e. they are used for feedback linearization. The zero dynamics of the system based on the selected terms can be represented as,

$$\begin{gathered} \ddot{z} = u_1 \cdot (\cos(\phi)\cos(\theta)) - g - g_r(z) \\ 0 = u_1 \cdot (\cos(\phi)\cos(\theta)) - g - g_r(z) \\ u_1 \cdot (\cos(\phi)\cos(\theta)) = g + g_r(z) \end{gathered} \tag{26}$$

When $z = 0$, $g_r(z) = 0$. Substituting the value of $g_r(z) = 0$ in the above equations and re-arranging the terms, $u_1$ can be represented as

$$u_1 = \frac{g}{\cos(\phi)\cos(\theta)} \tag{27}$$

Substituting the value of $u_1$ in $\ddot{x}$ and $\ddot{y}$, the equations can be represented as

$$\ddot{x} = \frac{g}{\cos\phi\cos\theta} \cdot (\cos\phi\sin\theta\cos\psi + \sin\phi\sin\psi)$$

$$\ddot{x} = g \cdot \left(\frac{\cos\phi\sin\theta\cos\psi}{\cos\phi\cos\theta} + \frac{\sin\phi\sin\psi}{\cos\phi\cos\theta}\right)$$

$$\ddot{x} = g \cdot \left(\tan\theta\cos\psi + \frac{\tan\phi\sin\psi}{\cos\theta}\right)$$

$$\ddot{x} =\approx -g \cdot \tan\theta$$

and,

$$\ddot{y} = \frac{g}{\cos\phi\cos\theta} \cdot (\cos\phi\sin\theta\cos\psi - \sin\phi\cos\psi)$$

$$\ddot{y} = g \cdot \left(\frac{\cos\phi\sin\theta\cos\psi}{\cos\phi\cos\theta} - \frac{\sin\phi\cos\psi}{\cos\phi\cos\theta}\right)$$

$$\ddot{y} = g \cdot \left(\tan\theta\sin\psi - \frac{\tan\psi\cos\psi}{\cos\theta}\right)$$

$$\ddot{y} \approx -g \cdot \frac{\tan\phi}{\cos\theta}$$

In the scenario when $|\psi| \ll 1$, the zero dynamics of the system demonstrate characteristics of instability and complexity. As a result, the management of such a system presents difficulties, and maintaining stability, while feasible, requires thorough examination and the implementation of specialized control procedures to achieve specific control goals. The same can be inferred from the paper[2]

Considering the outputs of the system as $x, y, z$, and $\psi$, the system of equations under consideration is:

$$\begin{aligned} \ddot{x} &= u_1\,(\cos(\phi)\sin(\theta)\cos(\psi) + \sin(\phi)\sin(\psi)) \\ \ddot{y} &= u_1\,(\cos(\phi)\sin(\theta)\cos(\psi) - \sin(\phi)\sin(\psi)) \\ \ddot{z} &= u_1\cos(\phi)\cos(\theta) - g - g_r(z) \\ \ddot{\psi} &= u_4 \end{aligned}$$

From the system of equations above, it can be inferred that the control variables $u_2$ and $u_3$ are not present as part of the control system. The system of equations contains the terms $\phi$ and $\theta$, and if $\ddot{x}$, $\ddot{y}$, and $\ddot{z}$ are differentiated twice, $\ddot{\phi}$ and $\ddot{\theta}$ will appear in the equations. These can be substituted to introduce the control variables $u_2$ and $u_3$ in the controller design.

Differentiating $\ddot{x}$, $\ddot{y}$ and $\ddot{z}$ once gain we obtain,

$$\begin{aligned}\dddot{x} = {} & (\sin(\phi(t))\sin(\psi(t)) + \sin(\theta(t))\cos(\phi(t))\cos(\psi(t)))\,\dot{u}_1 \\ & + \big(-\sin(\phi(t))\sin(\theta(t))\cos(\psi(t))\dot{\phi} + \sin(\phi(t))\cos(\psi(t))\dot{\psi} \\ & - \sin(\psi(t))\sin(\theta(t))\cos(\phi(t))\dot{\psi} + \sin(\psi(t))\cos(\phi(t))\dot{\phi} \\ & + \cos(\phi(t))\cos(\psi(t))\cos(\theta(t))\dot{\theta}\big)\,u_1\end{aligned} \tag{28}$$

$$\begin{aligned}\dddot{y} = {} & (\sin(\phi(t))\sin(\theta(t))\cos(\psi(t)) - \sin(\psi(t))\cos(\phi(t)))\dot{u}_1 \\ & + (-\sin(\phi(t))\sin(\psi(t))\sin(\theta(t))\dot{\psi} + \sin(\phi(t))\sin(\psi(t))\dot{\phi} \\ & + \sin(\phi(t))\cos(\psi(t))\cos(\theta(t))\dot{\theta} + \sin(\theta(t))\cos(\phi(t)) \\ & \cos(\psi(t))\dot{\phi} - \cos(\phi(t))\cos(\psi(t))\dot{\psi})u_1(t)\end{aligned} \tag{29}$$

$$\begin{aligned}\dddot{z} = {} & -u_1(t)\sin(\psi(t))\cos(\theta(t))\dot{\psi} - u_1(t)\sin(\theta(t))\cos(\psi(t))\dot{\theta} \\ & + \cos(\psi(t))\cos(\theta(t))\dot{u}_1(t)\end{aligned} \tag{30}$$

Differentiating all the three equations ($\dddot{x}$,$\dddot{y}$,$\dddot{z}$) once again, the following is obtained

$$\begin{aligned}\ddddot{x} = {} & \sin(\phi)\sin(\psi) + \sin(\theta)\cos(\phi)\cos(\psi)\cdot\ddot{u}_1 \\ & + 2\Big(-\sin(\phi)\sin(\theta)\cos(\psi)\dot{\phi} + \sin(\phi)\cos(\psi)\dot{\psi} \\ & - \big(\sin(\psi)\sin(\theta)\cos(\phi)\dot{\psi} + \sin(\psi)\cos(\phi)\dot{\phi} \\ & + \cos(\phi)\cos(\psi)\cos(\theta)\dot{\theta}\big)\dot{u}_1 \\ & + \big(2\sin(\phi)\sin(\psi)\sin(\theta)\dot{\phi}\dot{\psi} - \sin(\phi)\sin(\psi)\dot{\phi}^2 \\ & - \sin(\phi)\sin(\psi)\dot{\psi}^2 - \sin(\phi)\sin(\theta)\cos(\psi)\ddot{\phi} \\ & - 2\sin(\phi)\cos(\psi)\cos(\theta)\dot{\phi}\dot{\theta} + \sin(\phi)\cos(\psi)\ddot{\psi} \\ & - \big(\sin(\psi)\sin(\theta)\cos(\phi)\ddot{\psi} + 2\sin(\psi)\cos(\phi)\cos(\theta)\dot{\psi}\dot{\theta} \\ & + \sin(\psi)\cos(\phi)\ddot{\phi} - \sin(\theta)\cos(\phi)\cos(\psi)\dot{\phi}^2 \\ & - \sin(\theta)\cos(\phi)\cos(\psi)\dot{\psi}^2 \\ & - \sin(\theta)\cos(\phi)\cos(\psi)\ddot{\theta} + 2\cos(\phi)\cos(\psi)\dot{\phi}\dot{\psi}\big)u_1\Big)\end{aligned} \tag{31}$$

$$\begin{aligned}\ddddot{y} = {} & (\sin\phi\sin\theta\cos\psi - \sin\psi\cos\phi)\,\ddot{u}_1 \\ & + 2\Big(-\sin\phi\sin\psi\sin\theta\dot{\psi} + \sin\phi\sin\psi\dot{\phi} \\ & + \sin\phi\cos\psi\cos\theta\dot{\theta} + \sin\theta\cos\phi\cos\psi\dot{\phi} \\ & - \cos\phi\cos\psi\dot{\psi}\Big)\dot{u}_1 \\ & + \Big(-\sin\phi\sin\psi\sin\theta\ddot{\psi} - 2\sin\phi\sin\psi\cos\theta\dot{\psi}\dot{\theta} \\ & + \sin\phi\sin\psi\ddot{\phi} - \sin\phi\sin\theta\cos\psi\dot{\phi}^2 \\ & - \sin\phi\sin\theta\cos\psi\dot{\psi}^2 - \sin\phi\sin\theta\cos\psi\dot{\theta}^2 \\ & + \sin\phi\cos\psi\cos\theta\ddot{\theta} + 2\sin\phi\cos\psi\dot{\phi}\dot{\psi} \\ & - 2\sin\psi\sin\theta\cos\phi\dot{\phi}\dot{\psi} + \sin\psi\cos\phi\dot{\phi}^2 \\ & + \sin\psi\cos\phi\dot{\psi}^2 + \sin\theta\cos\phi\cos\psi\ddot{\phi} \\ & + 2\cos\phi\cos\psi\cos\theta\dot{\phi}\dot{\theta} - \cos\phi\cos\psi\ddot{\psi}\Big)\dot{u}_1\end{aligned} \tag{32}$$

$$\begin{aligned}\ddddot{z} = {} & 2u_1\sin\psi\sin\theta\dot{\psi}\dot{\theta} - u_1\sin\psi\cos\theta\ddot{\psi} \\ & - u_1\sin\theta\cos\psi\ddot{\theta} - u_1\cos\psi\cos\theta\dot{\psi}^2 \\ & - u_1\cos\psi\cos\theta\dot{\theta}^2 - 2\sin\psi\cos\theta\dot{\psi}\dot{u}_1 \\ & - 2\sin\theta\cos\psi\dot{\theta}\dot{u}_1 + \cos\psi\cos\theta\ddot{u}_1\end{aligned} \tag{33}$$

### *B. Choosing x-y-z for the Controller Design With Small Angle Assumption*

From the obtained equations (31), (32) and (33), we can infer that the control system will involve complex computation and several derivative terms that are sensitive to noise.
Thus, to reduce the complications, the equations of motions can be differentiated by considering the small angle assumption and ignoring the ground effect coefficient $g_r(z)$

The equations of motion when considering small angle assumption and ignoring the ground effect are

$$\ddot{x} = u_1\sin(\theta) \tag{34}$$
$$\ddot{y} = -u_1\sin(\phi) \tag{35}$$
$$\ddot{z} = u_1\cos(\theta)\cos(\phi) - g \tag{36}$$
$$\ddot{\phi} = u_2\cdot l \tag{37}$$
$$\ddot{\theta} = u_3\cdot l \tag{38}$$
$$\ddot{\psi} = u_4 \tag{39}$$

Again, the controller does not have terms $u_2$ and $u_3$ in the controller equations,

The workaround for this would be to obtain the second derivatives the terms $\ddot{x}$,$\ddot{y}$, $\ddot{z}$ to make the terms $\ddot{\phi}$ and $\ddot{\theta}$ appear in the controller equations. The input terms, $u_2 = \frac{\ddot{\phi}}{l}$ and $u_3 =$

$\frac{\ddot{\theta}}{l}$ can be substituted in place of the second derivatives of $\ddot{\phi}$ and $\ddot{\theta}$ to introduces the required input terms in the controller design.
The step by step procedure to do the same is as follows,

Differentiating $\ddot{x}$,$\ddot{y}$ and $\ddot{z}$ once the following set of equations are obtained.

$$\dddot{x} = \dot{u}_1\dot{\theta}\cos\theta + \dot{u}_1\sin\theta \tag{40}$$

$$\dddot{y} = -\dot{u}_1\dot{\phi}\cos\phi - \dot{u}_1\sin\phi \tag{41}$$

$$\dddot{z} = -\dot{u}_1\sin\phi\cos\theta\dot{\phi} - \dot{u}_1\sin\theta\cos\phi\dot{\theta} + \cos\phi\cos\theta\dot{u}_1 \tag{42}$$

Differentiating $\dddot{x}$,$\dddot{y}$ and $\dddot{z}$once again, the following equations are obtained

$$\ddddot{x} = -\dot{u}_1\sin(\dot{\theta})^2 + \dot{u}_1\cos(\dot{\theta})\ddot{\theta} \quad + \sin(\dot{\theta})\ddot{u}_1 + 2\cos(\dot{\theta})\dot{\theta}\dot{u}_1 \tag{43}$$

$$\ddddot{y} = \dot{u}_1\sin(\dot{\phi})^2 - \dot{u}_1\cos(\dot{\phi})\ddot{\phi} \quad - \sin(\dot{\phi})\ddot{u}_1 - 2\cos(\dot{\phi})\dot{\phi}\dot{u}_1 \tag{44}$$

$$\begin{aligned}\ddddot{z} = {} & 2\dot{u}_1\sin(\dot{\phi})\sin(\dot{\theta})\dot{\phi}\dot{\theta} \\ & - \dot{u}_1\sin(\dot{\phi})\cos(\dot{\theta})\ddot{\phi} - \dot{u}_1\sin(\dot{\theta})\cos(\dot{\phi})\ddot{\theta} \\ & - \dot{u}_1\cos(\dot{\phi})\cos(\dot{\theta})\dot{\phi}^2 - \dot{u}_1\cos(\dot{\phi})\cos(\dot{\theta})\dot{\theta}^2 \\ & - 2\sin(\dot{\phi})\cos(\dot{\theta})\dot{u}_1\end{aligned} \tag{45}$$

In the equations 31, 32 and 33, substituting $\ddot{\phi}$ and $\ddot{\theta}$ with $\ddot{\phi}$ = $u_2 \cdot l$ and $\ddot{\theta}$ with $\ddot{\theta} = u_3 \cdot l$, the equations obtained are,

$$\begin{aligned}\ddddot{x} = {} & -\dot{u}_1\sin(\dot{\theta})^2 + \dot{u}_1\cos(\dot{\theta})(u_3 \cdot l) \\ & + \sin(\dot{\theta})\ddot{u}_1 + 2\cos(\dot{\theta})\dot{\theta}\dot{u}_1\end{aligned} \tag{46}$$

$$\begin{aligned}\ddddot{y} = {} & \dot{u}_1\sin(\dot{\phi})^2 - \dot{u}_1\cos(\dot{\phi})(u_2 \cdot l) \\ & - \sin(\dot{\phi})\ddot{u}_1 - 2\cos(\dot{\phi})\dot{\phi}\dot{u}_1\end{aligned} \tag{47}$$

$$\begin{aligned}\ddddot{z} = {} & 2\dot{u}_1\sin(\dot{\phi})\sin(\dot{\theta})(u_2 \cdot l)(u_3 \cdot l) \\ & - \dot{u}_1\sin(\dot{\phi})\cos(\dot{\theta})(u_2 \cdot l) - \dot{u}_1\sin(\dot{\theta})\cos(\dot{\phi})(u_3 \cdot l) \\ & - \dot{u}_1\cos(\dot{\phi})\cos(\dot{\theta})(u_2 \cdot l)^2 - \dot{u}_1\cos(\dot{\phi})\cos(\dot{\theta})(u_3 \cdot l)^2 \\ & - 2\sin(\dot{\phi})\cos(\dot{\theta})\dot{u}_1\end{aligned} \tag{48}$$

From the equations 34, 35 and 36 we can deduce that the controller inputs generated are $\ddot{u}_1$, $u_2$ and $u_3$ The equations can be represented in an equivalent matrix format as

$$\begin{aligned}\begin{bmatrix}\ddddot{x}\\ \ddddot{y}\\ \ddddot{z}\end{bmatrix} = {} & \begin{bmatrix}\sin\theta & 0 & u_1 l\cos\theta \\ -\sin\phi & -u_1 l\cos\phi & 0 \\ \cos\theta\cos\phi & -u_1 l\cos\theta\sin\phi & -u_1 l\cos\phi\sin\theta\end{bmatrix} \cdot \begin{bmatrix}\ddot{u}_1\\ u_2\\ u_3\end{bmatrix} \\ & + \begin{bmatrix}2\dot{u}_1\dot{\theta}\cos\theta - u_1\dot{\theta}^2\sin\theta \\ -2\dot{u}_1\dot{\phi}\cos\phi + u_1\dot{\phi}^2\sin\phi \\ (-2\dot{u}_1\dot{\theta}\sin\theta\cos\phi - 2\dot{u}_1\dot{\phi}\sin\phi\cos\theta + 2u_1\dot{\theta}\dot{\phi}\sin\theta\sin\phi \\ -u_1((\dot{\theta})^2 + (\dot{\phi})^2)\cos\theta\cos\phi)\end{bmatrix}\end{aligned} \tag{49}$$

Rearranging the terms, the controller inputs are

$$\begin{aligned}\begin{bmatrix}\ddot{u}_1\\ u_2\\ u_3\end{bmatrix} = {} & \begin{bmatrix}\sin\theta & 0 & u_1\cos\theta \\ -\sin\phi & u_1\cos\phi & 0 \\ \cos\theta\cos\phi & -u_1\cos\theta\sin\phi & u_1\sin\theta\cos\phi\end{bmatrix}^{-1} \cdot \\ & \begin{bmatrix}-2\dot{u}_1\dot{\theta}\cos\theta + u_1(\dot{\theta})^2\sin\theta + v_1 \\ 2\dot{u}_1\dot{\phi}\cos\phi - u_1(\dot{\phi})^2\sin\phi + v_2 \\ 2\dot{u}_1\dot{\theta}\sin\theta\cos\phi + 2\dot{u}_1\dot{\phi}\cos\theta\sin\phi - 2u_1\dot{\theta}\dot{\phi}\sin\theta\sin\phi \\ +u_1((\dot{\theta})^2 + (\dot{\phi})^2)\cos\theta\cos\phi + v_3\end{bmatrix}\end{aligned} \tag{50}$$

Where $v_1$,$v_2$ and $v_3$ are the pseduo input terms
Setting the psedo input terms as

$$v_1 = \ddddot{x}_d - k_{x1}\dddot{e}_x - k_{x2}\ddot{e}_x - k_{x3}\dot{e}_x - k_{x4}e_x \tag{51}$$

$$v_2 = \ddddot{y}_d - k_{y1}\dddot{e}_y - k_{y2}\ddot{e}_y - k_{y3}\dot{e}_y - k_{y4}e_y \tag{52}$$

$$v_3 = \ddddot{z}_d - k_{z1}\dddot{e}_z - k_{z2}\ddot{e}_z - k_{z3}\dot{e}_z - k_{z4}e_z \tag{53}$$

yields,

$$\ddddot{e}_x + k_{x1}\dddot{e}_x + k_{x2}\ddot{e}_x + k_{x3}\dot{e}_x + k_{x4}e_x = 0 \tag{54}$$

$$\ddddot{e}_y + k_{y1}\dddot{e}_y + k_{y2}\ddot{e}_y + k_{y3}\dot{e}_y + k_{y4}e_y = 0 \tag{55}$$

$$\ddddot{e}_z + k_{z1}\dddot{e}_z + k_{z2}\ddot{e}_z + k_{z3}\dot{e}_z + k_{z4}e_z = 0 \tag{56}$$

Where $e_x = x - x_d$, $e_y = y - y_d$, and $e_z = z - z_d$. The parameters $[k_{x1}, \ldots, k_{x4}]$, $[k_{y1}, \ldots, k_{y4}]$, and $[k_{z1}, \ldots, k_{z4}]$ represent the gain values assigned to achieve stable error dynamics for the simplified system.
And the $\psi$ controller is a Proportional - Derivative controller and $u_4$ is given by,

$$u_4 = \ddot{\psi}_d + k_{\psi 1}(\dot{\psi}_d - \dot{\psi}) + k_{\psi 2}(\psi_d - \psi) \tag{57}$$

Where $k_{\psi 1}$ and $k_{\psi 2}$ are the derivative and proportional gains

## IV. ADAPTIVE SLIDING MODE CONTROL

Sliding Mode Control is frequently used in handling non-linear uncertain systems due to its robustness and finite time convergence. Besides its capacity to handle fluctuations and disruptions, this method also allows for model order reduction. It finds application in various dynamic systems such as power systems, airplanes, autonomous cars, power converters, electric drives, mechanical systems, and industrial processes. The efficacy of Sliding Mode Control is attributed to its use of a discontinuous control law. Nonetheless, it faces the "chattering problem." Exploring the chattering phenomena, including higher order Sliding Mode Control, Adaptive Sliding Mode Control, and chattering-free Sliding Mode Control, is a significant area of research in this field.
This section provides a description of an adaptive sliding mode controller. A suitable sliding surface is defined, together with adaptation methods, in order to ensure that the trajectory of the system adheres to intended references in the presence of

ground effects and noisy sensors.
The formulation of a sliding mode controller necessitates the careful selection of a sliding surface to ensure that the ensuing sliding motion adheres to specified performance criteria.
Additionally, it requires the identification of a control law that renders the switching surface an attractor for the system state [2].

### A. Adaptive Sliding Mode Control With Augmented Inputs

From the vector representation of the equations of motion, the $g(x)$ term in $\ddot{x} = f(x) + g(x)u - fr(x), x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ where the value of $m$ is less than $n$ is the control vector of the system, and $f_r(x) \in \mathbb{R}^n$ is the vector which describes the external disturbances.
In the equation $\ddot{x} = f(x) + g(x)u - fr(x)$, $g(x)$ is

$$g(x) = \begin{bmatrix} \cos(\phi)\cdot\sin(\theta)\cdot\cos(\psi)+\sin(\phi)\cdot\sin(\psi) & 0 & 0 & 0 \\ \cos(\phi)\cdot\sin(\theta)\cdot\sin(\psi)-\sin(\phi)\cdot\cos(\psi) & 0 & 0 & 0 \\ \cos(\phi)\cdot\cos(\theta) & 0 & 0 & 0 \\ 0 & l & 0 & 0 \\ 0 & 0 & l & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Since g(x) is a 6 x 4 matrix because the system under consideration is an underactuated system, the g(x) matrix is augmented with slack variables to overcome the underactuated properties, but the slack variables[5] need to be generated properly to guarantee the stability of the controller.
Augmenting the slack variables $g_s$ to g(x) and $u_s$ to u to form a square system, where $u_5$ and $g_s$ is, $u_5 = [u_5, u_6]^T$ and,

$$\mathbf{g_s} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{58}$$

$g_s$ is set to be constant. It is defined in advance so as to make the matrix G(x) invertible and v= $g_s u_s$, so

$$\mathbf{v} = \begin{bmatrix} u_5 & u_6 & 0 & 0 & 0 & 0 \end{bmatrix}^\top \tag{59}$$

Thus, the system dynamics can be rewritten as follows,

$$\ddot{x} \;=\; f(x) \;+\; G(x)\;U \;-\; v \;+\; f_r(x) \tag{60}$$

where G(x) is

$$\mathbf{G(x)} = \begin{bmatrix} \cos\phi\sin\theta\cos\psi+\sin\phi\sin\psi & 0 & 0 & 0 & 1 & 0 \\ \cos\phi\sin\theta\sin\psi\sin\phi\cos\psi & 0 & 0 & 0 & 0 & 1 \\ \cos\phi\cos\theta & 0 & 0 & 0 & 0 & 0 \\ 0 & l & 0 & 0 & 0 & 0 \\ 0 & 0 & l & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \tag{61}$$

If the slack variables and the adaptation laws are appropriately used with the sliding mode control law, the error states of a control affine underactuated nonlinear system converges to zero as the time increases.

To estimate the slack variables,
Our control goal is to stabilize all states to zero from any initial conditions. First, the error vector, e, can be defined as the difference between the instantaneous state (x) and the desired state ($x_d$), the mathematical representation of the same is,e = x − $x_d$ ,
The desired state variables are $\mathbf{x}_d = [x_d, y_d, z_d, \phi_d, \theta_d, \psi_d]$. and the first derivative of the error term is given by,

$$\dot{e} = \dot{x} - \dot{x}_d \tag{62}$$

and its second derivative is

$$\ddot{e} = \ddot{x} - \ddot{x}_d \tag{63}$$

The sliding surface for control is defined as

$$S = \dot{e} + ke$$

Where,

$$k = \begin{bmatrix} k_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & k_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & k_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & k_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & k_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & k_6 \end{bmatrix}$$

Here, k is a diagonal matrix with positive values that ensures the trajectory of the system follows the desired references, which also means to control the inputs so as to make S converge to the manifold S = 0.
Along with this, to cancel the nonlinear terms in the system dynamics equation, the terms v and fr(x) are estimated and their respective errors is given by,

$$\widetilde{v} = v - \hat{v} \tag{64}$$

Where, $\hat{v}$ is the the estimated value and $\widetilde{v}$ is the error between the actual and estimated values.

$$\widetilde{f_r}(x) = f_r(x) - \widehat{f_r}(x) \tag{65}$$

Where, $\widehat{f_r}(x)$ is the estimated value and $\widetilde{f_r}(x)$ is the error between the actual and estimated values.
Using these, the Lyapunov function can be defined. The Lyapunov function type chosen by the journal is the quadratic type. The quadratic function is chosen because of its simplicity and its effectiveness.
The Quadratic function is of the form V(x) = $\hat{x} \cdot P \cdot x$, where P is a symmetric matrix.
The Lyapunov function for the system can be written as,

$$L = \frac{1}{2}S^T S + \frac{1}{2}\widetilde{v}^T \Gamma \widetilde{v} + \frac{1}{2}\widetilde{f_r}(x)^T \Omega \; \widetilde{f_r}(x) \tag{66}$$

The stability of the system can be analyzed by showing that the derivative of the Lyapunov function along the trajectories of the system is negative or non-positive. The derivative of

the Lyapunov function represents the rate of change of the system along the trajectory of the system. To ensure stability or asymptotic stability, the derivative needs to show a decrease in value of the Lyapunov function over time. Making the Lyapunov function negative definite ensures a non-increasing behavior of the system, i.e., stability.
The first derivative of the Lyapunov function can be derived as,

$$\dot{L} = S^T \dot{S} + \widetilde{v}^T \Gamma \ (\dot{\widetilde{v}}) + \widetilde{f_r} \ (x)^T \Omega \ \left(\dot{\widetilde{f_r}} \ (x)\right) \tag{67}$$

We can find the first derivative of S and substitute it in the equation.

$$\dot{S} = \ddot{e} + k\dot{e} \tag{68}$$

We know that

$$\ddot{e} = \ddot{x} - \ddot{x}_d, \quad \text{where } \ddot{x} = f(x) + G(x)U - v + f_r(x) \tag{69}$$

Substituting $\ddot{x}$ into the equation,

$$\ddot{e} = f(x) + G(x)U - v + f_r(x) - \ddot{x}_d \tag{70}$$

So,

$$\dot{S} = f(x) + G(x)U - v + f_r(x) - \ddot{x}_d + k\dot{e} \tag{71}$$

Substituting $\dot{S}$ in $\dot{L}$, the resulting equation obtained is,

$$\begin{aligned} \dot{L} = S^T [f(x) + G(x)U - v + f_r(x) - \ddot{x}_d + k\dot{e}] \\ + \widetilde{v}^T \Gamma \dot{\widetilde{v}} + \widetilde{f_r}(x)^T \Omega \dot{\widetilde{f_r}}(x) \end{aligned} \tag{72}$$

If the assumption is made that $v$ and $f_r(x)$ have a slow rate of change, then the expressions $(\dot{\widetilde{v}})$ is nearly equal to $-\dot{\hat{v}}$ and $(-\dot{\widetilde{f_r}}(x))$ is nearly equal to $(-\dot{\hat{f_r}}(x))$.

Mathematically, it can be represented as

$$(\dot{\widetilde{v}}) \approx -\dot{\hat{v}} \quad \text{and} \quad (-\dot{\widetilde{f_r}}(x)) \approx (-\dot{\hat{f_r}}(x)) \tag{73}$$

For the slow rate of change, the first-order derivative of the Lyapunov function can be represented as,

$$\begin{aligned} \dot{L} = S^T \ [f(x) + G(x)U - v + f_r(x) - \ddot{x}_d + k\dot{e}] \\ + \widetilde{v}^T \Gamma - \dot{\hat{v}} + \widetilde{f_r} \ (x)^T \Omega \ (-\dot{\hat{f_r}}(x)) \end{aligned} \tag{74}$$

To make sure that the Lyapunov function's first derivative satisfies the negative definite criteria, introduction of an augmented input is one of the strategies to modify the stability analysis which leads to achieving the desired stability properties.
The choice of the augmented input depends on its effect on the system dynamics to achieve the intended stability properties, the choice made in the journal is to use the constant rate reaching law[3].
This characteristic of constant rate reaching law is its simplicity. Provided the gain value is too small, reaching the sliding surface can take a long time. Or, if the gain value of the gain is too large, chattering is generated.

The standard form of the constant rate reaching law [1] is given by $\dot{S}$ =$-K_1$ sign(s) , where $K_1$ is the gain value

The augmented input U is selected as

$$\begin{aligned} U = G^{-1}(x)\big[ - f(x) + \hat{v} - \widehat{f_r}(x) \\ + \ddot{x}_d - k\dot{e} - \text{diag}[c_1, c_2, c_3, c_4, c_5, c_6] \, \text{sign}(S^T)\big] \end{aligned} \tag{75}$$

Where ci (i=1 to 6) is the gain values
The augmented input $U$ is obtained from the system dynamics equation. The constant rate reaching law $-\text{diag}[c_1, c_2, c_3, c_4, c_5, c_6]\text{sign}(S^T)$ is added to the system dynamics equation, and the terms are rearranged to obtain the equation for $U$, the augmented input. It has been derived as follows:

$$\ddot{x} = f(x) + G(x)U - v + f_r(x) + \text{diag}[c_1, c_2, c_3, c_4, c_5, c_6] \, \text{sign}(S^T) \tag{76}$$

from the double derivative of error equation we can write $\ddot{x}$ as $\ddot{e}$ + $\ddot{x_d}$

$$\begin{aligned} G(x)U = \ddot{e} + \ddot{x_d} - f(x) + v \\ - f_r(x) - \text{diag}[c_1, c_2, c_3, c_4, c_5, c_6] \, \text{sign}(S^T) \end{aligned} \tag{77}$$

So,

$$\begin{aligned} U = G^{-1}(x) \, [\ddot{x_d} - f(x) + v - f_r(x) - k\dot{e} \\ -\text{diag}[c_1, c_2, c_3, c_4, c_5, c_6] \, \text{sign}(S^T)\big] \end{aligned} \tag{78}$$

Substituting U in $\dot{L}$, the equation $\dot{L}$ thus obtained is,

$$\begin{aligned} \dot{L} =& S^T \big[ f(x) + G(x) \big[ G^{-1}(x) \big[ -f(x) + \hat{v} - \widehat{f_r}(x) \\ &+ \ddot{x}_d - k\dot{e} - \text{diag}[c_1, c_2, c_3, c_4, c_5, c_6] \cdot \text{sign}(S^T) \big] \big] \\ &- v + f_r(x) - \ddot{x}_d + k\dot{e} \big] + [\widetilde{v}^T \Gamma (-\dot{\hat{v}})] + \widetilde{f_r}(x)^T \Omega (-\dot{\hat{f_r}}(x)) \end{aligned} \tag{79}$$

G(x) and $G^{-1}(x)$ is equal to I, thus equation becomes

$$\begin{aligned} \dot{L} =& S^T \big[ f(x) + [I \big[ -f(x) + \hat{v} - \widehat{f_r}(x) \\ &+ \ddot{x}_d - k\dot{e} - \text{diag}[c_1, c_2, c_3, c_4, c_5, c_6] \cdot \text{sign}(S^T) \big] \big] \\ &- v + f_r(x) - \ddot{x}_d + k\dot{e} \big] + [\widetilde{v}^T \Gamma (-\dot{\hat{v}})] + \widetilde{f_r}(x)^T \Omega (-\dot{\hat{f_r}}(x)) \end{aligned} \tag{80}$$

$$\dot{L} = \widetilde{v}^T [-S - \Gamma \dot{\hat{v}}] + \widetilde{f_r} \ (x)^T [S - \Omega(-\dot{\hat{f_r}}(x)] - C^T |S| \tag{81}$$

C in the equation (62) is the gain vector. The gain vector is

$$C = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \end{bmatrix} \tag{82}$$

To estimate the values of $\hat{v}$ and $\hat{f_r}(x)$, the first order derivative of the Lyapunov function can be utilized to do so. Equation (62) in the expanded form is,

$$\dot{L} = \begin{bmatrix} u_5-\hat{u}_5 & u_6-\hat{u}_6 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -\dot{x}+\dot{x}_d-k_1[x-x_d]-\hat{u}_5 \\ -\dot{y}+\dot{y}_d-k_2[y-y_d]-\hat{u}_6 \\ -\dot{z}+\dot{z}_d-k_3[z-z_d] \\ -\dot{\phi}+\dot{\phi}_d-k_4[\phi-\phi_d] \\ -\dot{\theta}+\dot{\theta}_d-k_5[\theta-\theta_d] \\ -\dot{\psi}+\dot{\psi}_d-k_5[\psi-\psi_d] \end{bmatrix}$$
$$+\begin{bmatrix} 0 & 0 & -\widetilde{g}_r(x) & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} \dot{x}-\dot{x}_d+k_1[x-x_d] \\ \dot{y}-\dot{y}_d+k_2[y-y_d] \\ \dot{z}-\dot{z}_d+k_3[z-z_d] \\ \dot{\phi}-\dot{\phi}_d+k_4[\phi-\phi_d] \\ \dot{\theta}-\dot{\theta}_d+k_5[\theta-\theta_d] \\ \dot{\psi}-\dot{\psi}_d+k_5[\psi-\psi_d] \end{bmatrix}$$
$$+\begin{bmatrix} c_1 & c_2 & c_3 & c_4 & c_5 & c_6 \end{bmatrix} \cdot \begin{bmatrix} \dot{x}-\dot{x}_d+k_1[x-x_d] \\ \dot{y}-\dot{y}_d+k_2[y-y_d] \\ \dot{z}-\dot{z}_d+k_3[z-z_d] \\ \dot{\phi}-\dot{\phi}_d+k_4[\phi-\phi_d] \\ \dot{\theta}-\dot{\theta}_d+k_5[\theta-\theta_d] \\ \dot{\psi}-\dot{\psi}_d+k_5[\psi-\psi_d] \end{bmatrix} \tag{83}$$

From the above equation, we can update $\hat{v}$ and $\hat{f}_r(x)$ as

$$\dot{\hat{v}} = -\begin{bmatrix} \dot{e}_x+k_1e_x, & \dot{e}_x+k_2e_y, & 0, & 0, & 0, & 0 \end{bmatrix}^T \tag{84}$$

and

$$\dot{\hat{f}}_r = -\begin{bmatrix} 0 & 0, & \dot{e}_z+k_1e_z, & 0, & 0, & 0 \end{bmatrix}^T \tag{85}$$

The first derivative of the Lyapunov function thus is,

$$\dot{L} = \begin{bmatrix} c_1 & c_2 & c_3 & c_4 & c_5 & c_6 \end{bmatrix} \cdot \begin{bmatrix} \dot{x}-\dot{x}_d+k_1[x-x_d] \\ \dot{y}-\dot{y}_d+k_2[y-y_d] \\ \dot{z}-\dot{z}_d+k_3[z-z_d] \\ \dot{\phi}-\dot{\phi}_d+k_4[\phi-\phi_d] \\ \dot{\theta}-\dot{\theta}_d+k_5[\theta-\theta_d] \\ \dot{\psi}-\dot{\psi}_d+k_5[\psi-\psi_d] \end{bmatrix} \tag{86}$$

Which can also be written shortly as,

$$\dot{L} = -C^T \cdot |S| \tag{87}$$

When all the values of $C_i$ (where $i = 1, 2, \ldots, 6$) are positive. The first order derivative of the lyapunov function can be written as,

$$\dot{L} = -C^T \cdot |S| < 0 \tag{88}$$

$\dot{L} < 0$ means that the system is approaching zero, this shows that as the state of the system evolves or moves within the state space, the Lyapunov function approaches zero. This indicates that the change of L with respect to time is always negative, which co-relates that the system is moving towards stability or a specific equilibrium point. This can also be denoted as $S \rightarrow 0$.

Combining S $\approx$0, $\dot{S}$ $\approx$0, the system dynamics equation and the augmented input U, the steady state of the estimated variables assumed for the adaptive sliding mode controller is,

$$v - \hat{v} \approx f_r - \hat{f}_r$$

This also can be represnted as,

$$\begin{bmatrix} u_5-\hat{u}_5 \\ u_6-\hat{u}_6 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ g_r-\hat{g}_r \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

From this, the inference that the estimated values of $\hat{u_5}$, $\hat{u_6}$, and $\hat{g}_r$ will reach its actual or true values in steady state. From the small angle assumption, the G(x) is

$$G(x) = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & l & 0 & 0 & 0 & 0 \\ 0 & 0 & l & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \tag{89}$$

And its inverse is

$$G^{-1}(x) = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{l} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{l} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{90}$$

Thus the Augmented input U thus can be written as,

$$U = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{l} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{l} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} * \begin{bmatrix} u_5-\hat{u}_5+\ddot{x}_d+K_1(\dot{x}-\dot{x}_d) \\ u_6-\hat{u}_6+\ddot{y}_d+K_2(\dot{y}-\dot{y}_d) \\ g+g_r(z)+z_d+K_3(\dot{z}-\dot{z}_d) \\ \ddot{\phi}_d+K_4(\dot{\phi}-\dot{\phi}_d) \\ \ddot{\theta}_d+K_5(\dot{\theta}-\dot{\theta}_d) \\ \ddot{\psi}_d+K_6(\dot{\psi}-\dot{\psi}_d) \end{bmatrix}$$

$$U = \begin{bmatrix} g+g_r(z)+z_d+K_3(\dot{z}-\dot{z}_d) \\ \frac{1}{l}*(\ddot{\phi}_d+K_4(\dot{\phi}-\dot{\phi}_d)) \\ \frac{1}{l}*(\ddot{\theta}_d+K_5(\dot{\theta}-\dot{\theta}_d)) \\ \ddot{\psi}_d+K_6(\dot{\psi}-\dot{\psi}_d) \\ u_5-\hat{u}_5+\ddot{x}_d+K_1(\dot{x}-\dot{x}_d) \\ u_6-\hat{u}_6+\ddot{y}_d+K_2(\dot{y}-\dot{y}_d) \end{bmatrix} \tag{91}$$

From the above equation, it can be inferred that the dominant terms for $u_1$, $u_2$, $u_3$, and $u_4$ are $[\dot{e}_z, e_z]$, $[\dot{e}_\phi, e_\phi]$, $[\dot{e}_\theta, e_\theta]$, and $[\dot{e}_\psi, e_\psi]$. Thus, the $x$ and $y$ states cannot be directly controlled from the inputs.

So, the desired values for $\phi$ and $\theta$ can be defined as:

$$\begin{aligned} \phi_d &= \dot{e}_y + k_\phi e_y \\ \theta_d &= \dot{e}_x + k_\theta e_x \end{aligned} \tag{92}$$

where $k_\phi$ and $k_\theta$ are the proportional gains.

### B. To demonstrate the advantage of Adaptive Sliding Mode Control over Sliding Mode Control

To compare the advantages of the adaptive sliding mode control method to the standard sliding mode control, the first derivative of the Lyapunov function without adaptation can be compared to the first derivative of the Lyapunov function with adaptation.

From the equation (63), without considering the adaptation for the slack variables v and grounding effect fr(x) , the first order derivative of the Lyapunov function will be as follows,

With positive entries of c, i.e., when ci(i =1 to 6) most of the terms in the first order derivative of the Lyapunov equation cancel each other and the only terms that will remain are

$$\dot{L}_{non-adaptive} = \begin{bmatrix} \dot{x} + \dot{x}_d + k_1[x + x_d] \\ \dot{y} + \dot{y}_d + k_2[y + y_d] \\ \dot{z} + \dot{z}_d + k_3[z + z_d] \\ \dot{\phi} + \dot{\phi}_d + k_4[\phi + \phi_d] \\ \dot{\theta} + \dot{\theta}_d + k_5[\theta + \theta_d] \\ \dot{\psi} + \dot{\psi}_d + k_5[\psi + \psi_d] \end{bmatrix} \cdot \begin{bmatrix} -u_5 - c_1\,\text{sign}(|S_1|) \\ -u_6 - c_2\,\text{sign}(|S_2|) \\ -g_r - c_3\,\text{sign}(|S_3|) \\ -c_4\,\text{sign}(|S_4|) \\ -c_5\,\text{sign}(|S_5|) \\ -c_6\,\text{sign}(|S_6|) \end{bmatrix} \tag{93}$$

Which can be written as,

$$\dot{L}_{non-adaptive} = S^T \cdot \begin{bmatrix} -u_5 - c_1\,\text{sign}\,|S_1| \\ -u_6 - c_2\,\text{sign}\,|S_2| \\ -g_r - c_3\,\text{sign}\,|S_3| \\ -c_4\,\text{sign}\,|S_4| \\ -c_5\,\text{sign}\,|S_5| \\ -c_6\,\text{sign}\,|S_6| \end{bmatrix} \tag{94}$$

Comparing $\dot{L}_{non-adaptive}$ with $\dot{L}_{adaptive}$ , the inference can be made that the adaptations enhance the performance of Sliding mode control, especially in the presence of uncertainties and disturbances. It can also be highlighted that in adaptive control method there is reduction in the chattering effect, and this can lead to smoother control inputs and improved tracking performance and follow the desired trajectory. Also, the adaptive controller requires relatively small input magnitude, which also is the major advantage in reducing the chattering and improved power efficiency of the system.

### C. Sensor Noise Analysis in Adaptive Sliding Mode Control

According to the journal paper, this technical report is based on, the experimental setup uses a vision sensor to estimate the position and attitude information. The information relayed also carries sensor noises and calibration errors which affect the system performance.
In our simulation setup, we have mimicked the sensor noise and calibration errors synthetically. This enables us to assess the impact of noise and calibration errors on the overall performance of the system. The mimicked noise and calibration errors, although may not be exactly same when compared with a physical sensor, it serves as an alternative to ensure that the implemented Adaptive Sliding Mode Controller is robust.

The Error vector, E can be defined as follows,

$$\mathbf{E} = \begin{bmatrix} \varepsilon_x & \varepsilon_y & \varepsilon_z & \varepsilon_\phi & \varepsilon_\theta & \varepsilon_\psi \end{bmatrix}^T \tag{95}$$

So the measured state variables, $\hat{X}$ can be defined as the sum of actual values and the sensor error. This can be mathematically represented as,

$$\hat{X} = X + E \tag{96}$$

The calibration error terms only affect the desired value of $\phi_d$ and $\theta_d$. Including the noise and calibration error terms in the equations (68), we have

$$\phi_d = (\dot{y} + \dot{\epsilon_y} - \dot{y_d} + K_\phi(y + \epsilon_y - y_d)) \tag{97}$$

$$\theta_d = (\dot{x} + \dot{\epsilon_x} - \dot{x_d} + K_\theta(x + \epsilon_x - x_d)) \tag{98}$$

The desired state vector including error terms is,

$$\hat{X}_d = \begin{bmatrix} x_d \\ y_d \\ z_d \\ (\dot{y} + \dot{\epsilon_y} - \dot{y_d} + K_\phi(y + \epsilon_y - y_d)) \\ (\dot{x} + \dot{\epsilon_x} - \dot{x_d} + K_\theta(x + \epsilon_x - x_d)) \\ \psi_d \end{bmatrix} \tag{99}$$

Wrting the error terms included in equation (75) separately in matrix format we have,

$$\eta x_d = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \dot{\epsilon_y} + K_\phi \epsilon_y \\ \dot{\epsilon_x} + K_\theta \epsilon_x \\ 0 \end{bmatrix} \tag{100}$$

Differentiating this twice, we get,

$$\eta \ddot{x}_d = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \dddot{\epsilon_y} + K_\phi \ddot{\epsilon}_y \\ \dddot{\epsilon_x} + K_\theta \ddot{\epsilon}_x \\ 0 \end{bmatrix} \tag{101}$$

Including this in the input term U, the term U becomes,

$$\eta_U = (G(X))^{-1} \cdot \begin{bmatrix} -k_1\dot{\epsilon}_x \\ -k_2\dot{\epsilon}_y \\ -k_3\dot{\epsilon}_z \\ \dddot{\epsilon}_y + K_\phi \ddot{\epsilon}_y - k_4\dot{\epsilon}_\phi \\ \dddot{\epsilon}_x + K_\theta \ddot{\epsilon}_x k_5 \dot{\epsilon}_\theta \\ -k_6 \dot{\epsilon}_p si \end{bmatrix} \tag{102}$$

### *D. Error Dynamics of Adaptive Sliding Mode Controller*

$$\ddot{x} = f(x) + G(x)U - v + f_r(x) \tag{103}$$

$$\begin{aligned} U = G^{-1}(x)\,[\ddot{x}_d - f(x) + v - f_r(x) - k\dot{e} - \\ \mathrm{diag}[c_1, c_2, c_3, c_4, c_5, c_6]\,\mathrm{sign}(S^T)\big] \end{aligned} \tag{104}$$

Substituting $U$ in $\ddot{x}$,

$$\begin{aligned} \ddot{x} &= f(x) + G(x)\left[G^{-1}(x)\,[\ddot{x}_d - f(x) + v - f_r(x)\right. \\ &\quad \left. -k\dot{e} - \mathrm{diag}[c_1, c_2, c_3, c_4, c_5, c_6]\,\mathrm{sign}(S^T)\big] - v + f_r(x)\right] \\ \ddot{x} &= \ddot{x}_d - k\dot{e} - \mathrm{diag}[c_1, c_2, c_3, c_4, c_5, c_6]\,(S^T) \end{aligned} \tag{105}$$

Substituting $\mathrm{sign}(S^T)$ as $(S^T)$ and equating it to $S = \dot{e} + ke$,

$$\begin{aligned} \ddot{x} &= \ddot{x}_d - k\dot{e} - C(\dot{e} + ke) \\ \ddot{x} - \ddot{x}_d &= -k\dot{e} - C(\dot{e} + ke) \\ \ddot{e} &= -k\dot{e} - C(\dot{e} + ke) \\ \ddot{e} + k\dot{e} + C(\dot{e} + ke) &= 0 \end{aligned} \tag{106}$$

The above equations represent the error dynamics of the system controlled by the adaptive sliding mode controller.

## V. SIMULATION

This section goes over the simulations carried out on MATLAB Simulink software to test the adaptive sliding mode and feedback linearization controllers. For testing the adaptive sliding mode controller, the simulations were carried out initially without considering the sensor noise and later by considering the sensor noise and the results were compared. Also, in simulations, the S term was used instead of Sign(S) to reduce the chattering caused by zero crossing. Zero Crossing is a phenomenon where a small change in the value of S around the curve S=0 will cause significant change in the value of sign(S) as sign(S) returns 1 if S is positive and -1 if S is negative. For the feedback linearization controller, the ground effect and sensor noises were not considered as the controller is highly sensitive to external disturbances and noises.

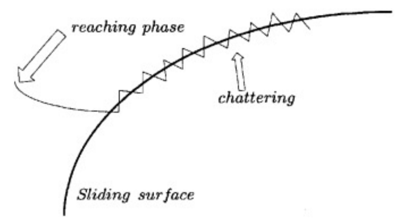


Fig. 3: Zero Crossing and Chattering[4]

### *A. System Parameters*

1) Moment of Inertia about X axis, $J_1 = 2\,\mathrm{Ns^2/rad}$
2) Moment of Inertia about Y axis, $J_2 = 2\,\mathrm{Ns^2/rad}$
3) Moment of Inertia about Z axis, $J_3 = 3\,\mathrm{Ns^2/rad}$
4) Mass of the quadrotor, $m = 2.5\,\mathrm{kg}$
5) Length of the quadrotor arms, $l = 1\,\mathrm{m}$
6) Acceleration due to gravity, $g = 9.81\,\mathrm{m/s^2}$
7) Ground effect constant, $A = 0.4668$
8) Height below which quadrotor experiences ground effect, $z_0 = 2\,\mathrm{m}$
9) Semi definite positive weighing matrix, $\Gamma$:
$$\Gamma = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$
10) Semi definite positive weighing matrix, $\Omega$:
$$\Omega = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$
11) Input Gain Vector, $C = \begin{bmatrix} 5 \\ 5 \\ 5 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

12) Diagonal matrix with positive entries to make system trajectory follow the desired reference on sliding surface $S = 0$,

$$K = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.7 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 10 \end{bmatrix}$$

13) Proportional gains, $K_\phi = K_\theta = 5$

14) Initial conditions: $x_0 = 10\,\mathrm{m}$, $y_0 = 10\,\mathrm{m}$, $z_0 = 20\,\mathrm{m}$, $\phi_0 = 300°$, $\theta_0 = 300°$, and $\psi_0 = 300°$

15) Gains of FL controller:

$$\begin{bmatrix} k_{x1} & k_{x2} & k_{x3} & k_{x4} \\ k_{y1} & k_{y2} & k_{y3} & k_{y4} \\ k_{z1} & k_{z2} & k_{z3} & k_{z4} \end{bmatrix} = \begin{bmatrix} 10.00 & 42.49 & 40.27 & 13.43 \\ 10.00 & 42.49 & 40.27 & 13.43 \\ 10.00 & 42.49 & 40.27 & 13.43 \end{bmatrix}$$

Values of the gains are obtained from a linear quadratic regulator method with the cost function:

$$J(u) = \int_0^\infty (x^T Q x + u^T R u)\, dt$$

where $Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ and $R = 0.01$

**The quadrotor is required to move from an initial point of** $(10, 10, 20)$ **to the final point of** $(0, 0, 0)$ **through a waypoint** $(20, -10, 10)$**.**

*B. Simulation Results*

The Fig.4 and Fig.5 represents the 3D and XYZ trajectory of the quadrotor obtained from the error dynamics of the Feedback Linearization System(54,55,56), whereas the Fig.9 and Fig.10 shows the trajectory of the quadrotor generated from the error dynamics of the Adaptive Sliding Mode Controller System(106). This trajectory is the desired trajectory that the quadrotor should follow in the absence of external disturbances and sensor noises.

The Fig.6 shows the trajectory of the quadrotor under the control of the Feedback Linearization Controller. The position and attitude graphs of the quadrotor is included under Fig.7. Fig.8 gives the inputs generated by the Feedback Linearization Controller without uncertainties and sensor noise.

Fig.11 and Fig.12 gives the trajectory, position and attitude values of the quadrotor under the control of adaptive sliding mode controller, without sensor noise. Fig.13 and Fig.14 displays the inputs generated by the adaptive sliding mode controller, without sensor noise.

Fig.15 to Fig.18 gives the behavior of the quadrotor system under the control of the adaptive sliding mode controller, with sensor noise. The sensor noise was mimicked in simulation environment by considering a sinusoidal input of 5hz. Fig.19 shows the comparison of the ground effect with and without sensor noise.

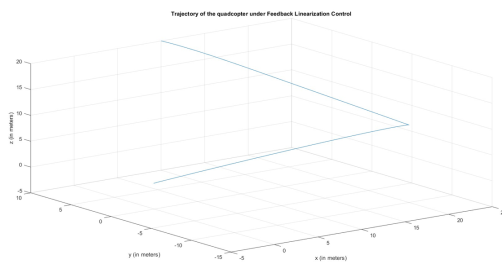


Fig. 4: Trajectory of UAV in 3-D axes with the Feedback Linearization from error dynamics

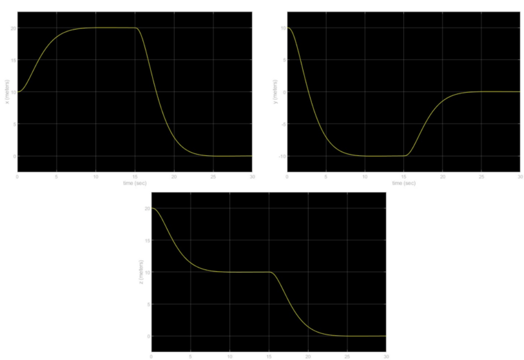

Fig. 5: Feedback Linearization: XYZ trajectory from error dynamics

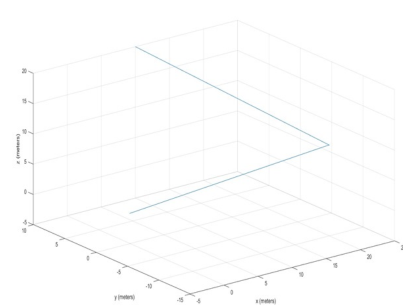

Fig. 6: Trajectory of UAV in 3-D axes with the Feedback Linearization

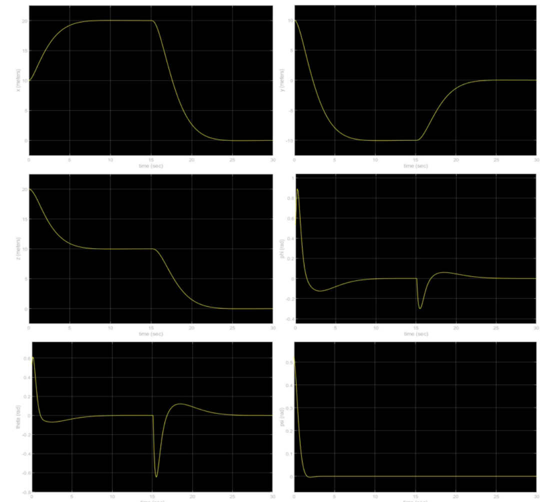

Fig. 7: Feedback Linearization controller results without sensor noise. x,y,z positions and roll, pitch and yaw angles

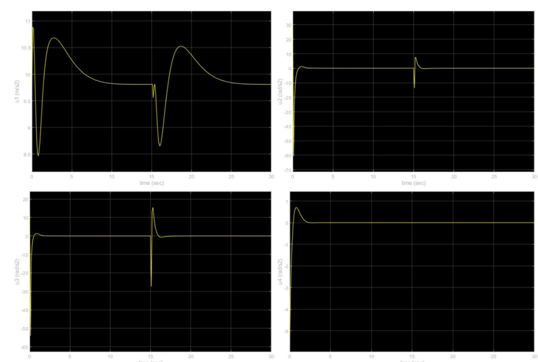

Fig. 8: Inputs generated by the Feedback Linearizatoin controller without sensor noise

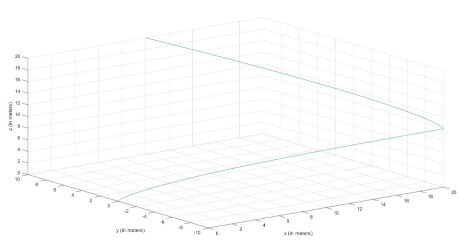


Fig. 9: Trajectory of UAV in 3-D axes with the adaptive sliding mode controller without sensor noise from error dynamics

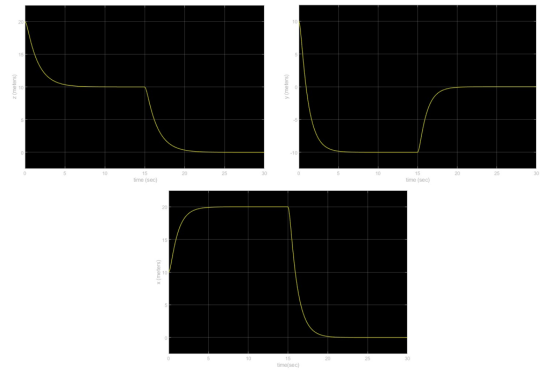

Fig. 10: XYZ of UAV with the adaptive sliding mode controller without sensor noise from error dynamics

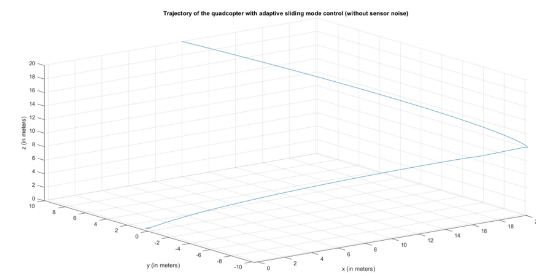


Fig. 11: Trajectory of UAV in 3-D axes with the adaptive sliding mode controller without sensor noise

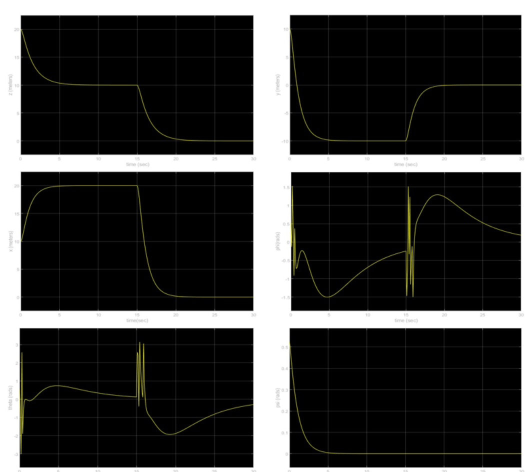

Fig. 12: Positions and Attitudes using the Adaptive Sliding Mode Controller without sensor noise

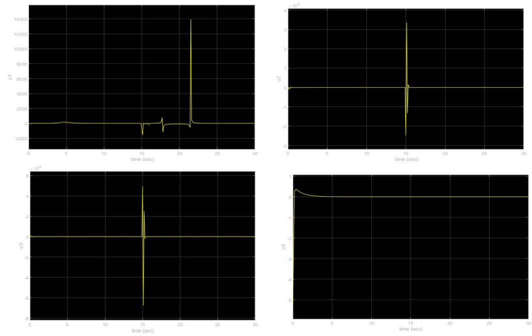

Fig. 13: Inputs generated by the adaptive sliding mode controller without sensor noise

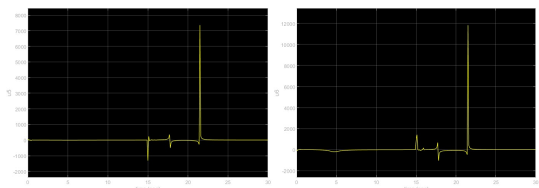

Fig. 14: Augmented inputs of the adaptive sliding mode controller without sensor noise.

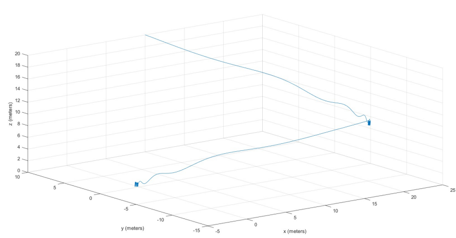

Fig. 15: Trajectory of UAV in 3-D axes with the adaptive sliding mode controller with sensor noise.

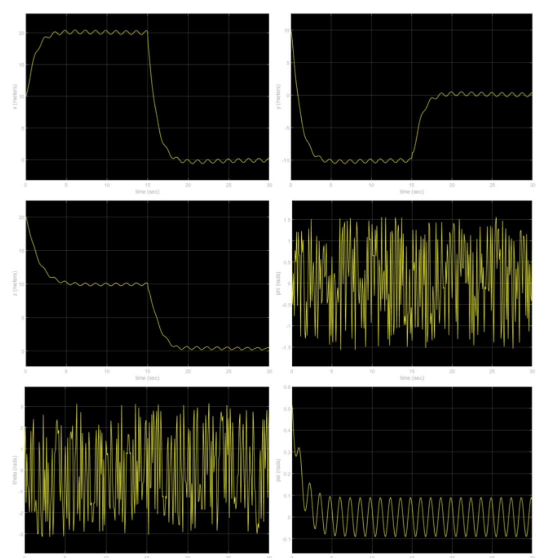

Fig. 16: Positions and Attitudes using the Adaptive Sliding Mode Controller with sensor noise

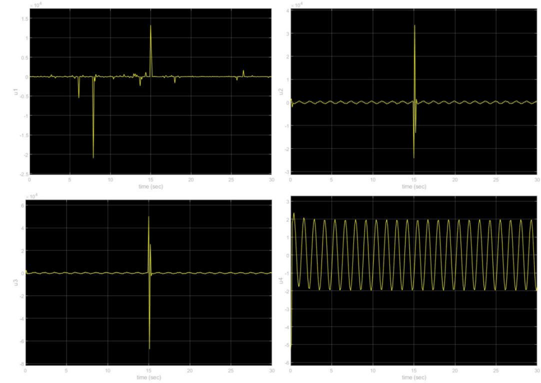

Fig. 17: Inputs generated by the adaptive sliding mode controller with sensor noise

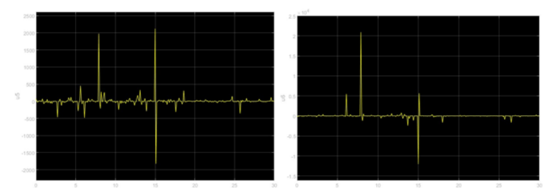

Fig. 18: Augmented inputs of the adaptive sliding mode controller with sensor noise.

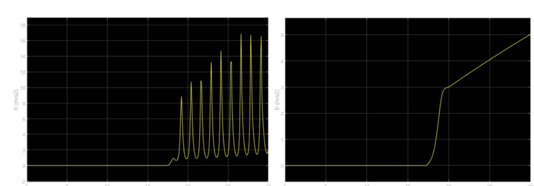

Fig. 19: Ground effect in the adaptive sliding mode controller with and without sensor noise

### C. Inferences and Deviation

Since we have chosen to control the output variables x,y,z and $\psi$ of the system, the other variables of roll($\phi$) and pitch($\theta$) are considered as the internal dynamics of the system. Fig.7 shows that the internal dynamics of the Feedback Linearization Controller is stable.

We can observe in figure 12 that chattering occurs in the system even without sensor noise. This is because the desired values of the roll and pitch angles are estimated from the error terms of y and x positions respectively (92). Hence, we can infer that Feedback Linearization controller uses more efficient inputs without any chattering.

Although both Feedback Linearization Controller and Adaptive Sliding Mode Controller gives similar results without sensor noise (Fig.7 vs Fig.12), only Adaptive Sliding mode controller will be able to give satisfactory results when sensor noise is considered (Fig.15 to Fig.18) . This is because Feedback Linearization contains higher order differential terms of the system state, which will make the system highly sensitive to disturbances. In contrast, the Adaptive sliding mode controller only involves $2^{nd}$ order terms of the system states. This makes the controller more robust when compared to Feedback Linearization Controller.

The sliding mode controller on the other has proven its robustness in the simulations. The controller gives stable trajectory even under the influence of uncertainties. It can be seen from the graphs that the input terms tend to spike at different time stamps, but the controller immediately corrects itself and stabilizes the input, giving us robust control of the system. In the simulations conducted with the consideration of sensor noise, we can see that chattering exists around the desired trajectory, but in the end, the system reaches the desired waypoints in the desired time. This, once again, proves the robustness of the system.

The simulations obtained for both the Feedback Linearization controller and the Adaptive Sliding Mode Controller match with the results obtained in the journal paper that is being studied. Also, it is evident from the simulations that Adaptive Sliding mode controllers are more robust and will give better control of the system under external disturbances. But, without any disturbances, both the controllers give satisfactory results, with Feedback Linearization Controller being the one without chatter. So, depending upon the application, the selection of the right controller can be made.

## VI. CONCLUSION

This technical document is a report on the Journal paper: Daewon Lee, H. Jin Kim, and Shankar Sastry, "Feedback Linearization vs. Adaptive Sliding Mode Control for a Quadrotor Helicopter", International Journal of Control, Automation, and Systems (2009) 7(3):419-428. This report compares the robustness of two different controllers, a Feedback Linearization Controller and an Adaptive Sliding Mode Controller, to control the trajectory of a quadrotor. For designing the feedback linearization controller, the dynamics of the quadrotor was simplified by small angle assumption and the state terms were differentiated twice to get all the input terms in the output equations. This method is straightforward one, but will make the system highly sensitive to uncertainties and disturbances. To overcome this drawback, a sliding mode controller with adaptations has been proposed and implemented. The sliding mode controller is based on the concept of a devising a sliding surface and selection of a control law to follow the defined or desired trajectory. This controller is highly robust to uncertainties and noises as the controller input contain only the first order derivatives of the system states. An adaptation has been made to a conventional sliding controller by considering the uncertain terms in the input equation. The robustness of this controller was tested both with and without sensor noise, where the controller performs exceptionally well when compared to the feedback linearization controller under the influence of sensor noise.

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