3.3

$y_i = \beta_0 + \beta_1\bar{x} + \beta_1 x_i - \beta_1\bar{x} + e_i$
$y_i = (\beta_0 + \beta_1\bar{x}) + \beta_1(x_i - \bar{x}) + e_i$
$y_i = \alpha + \beta_1(x_i - \bar{x}) + e_i$ where $\alpha = \beta_0 + \beta_1\bar{x}$

- If we normalize our x around the mean, then α would be the new intercept of our linear regression line.

- $y_i = \alpha + \beta_1(x_i - \bar{x}) + e_i$
  RSS $= \sum_1^N [y_i - \alpha - \beta_1(x_i - \bar{x})]^2$
  $\frac{\partial RSS}{\partial \alpha} = 0$
  $\implies \sum_1^N 2[y_i - \alpha - \beta_1(x_i - \bar{x})][0 - 1 - 0] = 0$
  $\implies \sum_1^N [y_i - \alpha - \beta_1(x_i - \bar{x})] = 0$
  as we know $\sum_1^N \beta_1(x_i - \bar{x}) = 0$
  $\implies \sum_1^N [y_i - \alpha = 0$
  $\implies \sum_1^N y_i - N\alpha = 0$
  $\implies \hat{\alpha} = \frac{\sum_1^N y_i}{N} = \bar{y}$

  Now we take derivative against β_1 RSS $= \sum_1^N [y_i - \alpha - \beta_1(x_i - \bar{x})]^2$
  $\frac{\partial RSS}{\partial \beta_1} = 0$
  $\implies \sum_1^N 2[y_i - \alpha - \beta_1(x_i - \bar{x})][-(x_i - \bar{x})] = 0$
  $\implies \sum_1^N -2[y_i - \alpha - \beta_1(x_i - \bar{x})][(x_i - \bar{x})] = 0$
  $\implies \sum_1^N [y_i - \alpha - \beta_1(x_i - \bar{x})][(x_i - \bar{x})] = 0$
  Now we can replace α with $\hat{\alpha}$ i.e. $\bar{y}$
  So,
  $\implies \sum_1^N [y_i - \bar{y} - \beta_1(x_i - \bar{x})][(x_i - \bar{x})] = 0$
  $\implies \sum_1^N (y_i - \bar{y})(x_i - \bar{x}) - \beta_1 \sum_1^N (x_i - \bar{x})(x_i - \bar{x}) = 0$
  $\implies S_{xy} - \beta_1 S_{xx} = 0$
  $\implies \hat{\beta}_1 = \frac{S_{xy}}{S_{xx}}$

- $\hat{\beta}_1 = \frac{S_{xy}}{S_{xx}}$

  $\implies \hat{\beta}_1 = \frac{\sum_1^N (y_i - \bar{y})(x_i - \bar{x})}{S_{xx}}$
  $\implies \hat{\beta}_1 = \frac{\sum_1^N (y_i(x_i - \bar{x}) - \bar{y}(x_i - \bar{x}))}{S_{xx}}$
  2nd part is 0 and $\frac{\sum_1^N (y_i(x_i - \bar{x})}{S_{xx}}$ is constant
  $\implies \hat{\beta}_1 = \sum_1^N c_i y_i$
  $\implies Var(\hat{\beta}_1) = Var(\sum_1^N c_i y_i) = \sum_1^N c_i^2 \sigma^2$
  $\sum_1^N c_i^2 = \frac{1}{Sxx} \implies Var(\hat{\beta}_1) = \frac{\sigma^2}{S_{xx}}$

  $Var(\hat{\alpha}) = Var(\bar{y}) = \frac{\sigma^2}{N}$

$Cov(\hat{\alpha}, \hat{\beta}_1) = Cov(\frac{\sum_1^N y_i}{N}, \sum_1^N c_i)$
$\implies Cov(\hat{\alpha}, \hat{\beta}_1) = \frac{1}{N} \sum_1^N \sum_1^N c_j Cov(y_i, y_j)$
for $i <> j$ $Cov(y_i, y_j == 0) = 0$
$\implies Cov(\hat{\alpha}, \hat{\beta}_1) = \frac{1}{N} \sum_1^N c_i \implies Cov(\hat{\alpha}, \hat{\beta}_1) = 0$