Physics Cup Problem 3

Łukasz Basiński

1 Introduction

Assume that x axis goes down along the inclined plane and y axis is perpendicular to x axis. By using a slated coordinate system we can see that the component of acceleration perpendicular to the inclined plane is constant as well as component parallel to the plane. They are equal respectively:

$a_y = g \times cos(\alpha)$

$a_x = g \times sin(\alpha)$

v_0 is initial velocity when ball touches plane. We can decompose it into components (respectively along x and y axis):

$v_{0\mathrm{x}} = v_0 \times sin(\alpha)$

$v_{0\mathrm{y}} = v_0 \times cos(\alpha)$

The ball's motion perpendicular to the plane is a series of bounces which have parabolic shape. Each parabola has the same maximum height because the motion along y axis is either uniformly accelerated or decelerated with initial velocity. Moreover, for that reason the time between subsequent bounces is constant.

2 Motion equations

Let us write the equation of motion for the first bounce (when $x = 0$ and $y = 0$ in our slated coordinate system):

$x(t) = v_x t + \frac{1}{2} a_x t^2$
$v_x(t) = v_{0\mathrm{x}} + a_x t$

$y(t) = v_y t - \frac{1}{2} a_y t^2$
$v_y(t) = v_{0\mathrm{y}} + a_y t$

Rewriting these equations in terms of v_0 and g:

$x(t) = v_0 sin(\alpha) t + \frac{1}{2} g sin(\alpha) t^2$
$v_x(t) = v_0 sin(\alpha) + g sin(\alpha) t$

$y(t) = v_0 cos(\alpha) t - \frac{1}{2} g cos(\alpha) t^2$
$v_y(t) = v_0 cos(\alpha) - g cos(\alpha) t$

Time of single bounce is constant and equal: $T = 2t_0$, where t_0 is an argument for which $v_y(t_0) = 0$ so:
$t_0 = \frac{v_0}{g}$
$T = 2\frac{v_0}{g}$

Thus the n-th bounce has initial parameters (for appropriate time i.e. $t \in \langle nT; (n+1)T \rangle$ where $n \in \mathbb{N}$):

$v_{x_n} = v_0 sin(\alpha) + g sin(\alpha) n \cdot 2 \cdot \frac{v_0}{g} = v_0 sin(\alpha) + 2v_0 n sin(\alpha) = v_0 sin(\alpha)(2n+1)$

$x_n = x(nT) = 2v_0 sin(\alpha) n \frac{v_0}{g} + \frac{1}{2} g sin(\alpha) n^2 \cdot 4 \cdot \frac{{v_0}^2}{g^2} = \frac{2{v_0}^2}{g} sin(\alpha) n(n+1)$

$v_{y_n} = v_{y_0} = v_0 cos(\alpha)$

$y_n = 0$

So the motion equations of n-th bounce has coordinates (for appropriate t):

$x_n(t) = \frac{2{v_0}^2}{g} sin(\alpha) n(n+1) + v_0 sin(\alpha)(2n+1) + \frac{1}{2} g sin(\alpha) t^2$

$y_n(t) = v_0 cos(\alpha) t - \frac{1}{2} g cos(\alpha) t^2$

3 Shape of each parabola

From the last section we can see that each parabola has roots for:

$x_n = \frac{2{v_0}^2}{g} sin(\alpha) n(n+1)$ and

$x_{n+1} = \frac{2{v_0}^2}{g} sin(\alpha)(n+1)(n+2)$

Each also reaches its maximum (there is the vertex of the parabola) when:

$t_v = \frac{1}{2} nT = (2n+1)\frac{v_0}{g}$.

The maximum height is constant so we can count the value only for $n = 0$:

$y_v = v_0 cos(\alpha) \frac{v_0}{g} - \frac{1}{2} g cos(\alpha) \frac{{v_0}^2}{g^2} = \frac{{v_0}^2}{2g} cos(\alpha)$

The x coordinate of vertex is equal:

$x_v = \frac{x_n + x_{n+1}}{2} = \frac{2{v_0}^2}{g} sin(\alpha) n(n+1) + \frac{{v_0}^2}{g} sin(\alpha)(2n+1) + \frac{{v_0}^2}{2g} sin(\alpha)$

Because we have three characteristic points we can write a detailed equation of n-th parabola.

$y = a(x - x_n)^2 + y_v$

$y = a(x - (\frac{2{v_0}^2}{g}sin(\alpha)n(n+1) + \frac{{v_0}^2}{g}sin(\alpha)(2n+1) + \frac{{v_0}^2}{2g}sin(\alpha)))^2 + \frac{{v_0}^2}{2g}cos(\alpha)$

After simplification we get:

$y_n = a(x - \frac{2{v_0}^2}{g}sin(\alpha) \cdot (n+1)^2)^2 + \frac{{v_0}^2}{2g}cos(\alpha)$

We know that for x_n $y_n = 0$ so we can write (substitute x_n to this formula) so:

$a(\frac{2{v_0}^2}{g}sin(\alpha) \cdot (n(n+1) - \frac{2{v_0}^2}{g}sin(\alpha) \cdot (n+1)^2))^2 + \frac{{v_0}^2}{2g}cos(\alpha) = 0$

$a(\frac{2{v_0}^2}{g}sin(\alpha))^2 \cdot (n^2 + n - n^2 - 2n - 1)^2 + \frac{{v_0}^2}{2g}cos(\alpha) = 0$

Thus a is equal:

$a = -\frac{1}{(n+1)^2} \cdot \frac{g}{8{v_0}^2} \cdot \frac{1}{tan(\alpha)sin(\alpha)}$

Let us rewrite the parabola's equation:

$y = -\frac{1}{(n+1)^2} \cdot \frac{g}{8{v_0}^2} \cdot \frac{1}{tan(\alpha)sin(\alpha)}(x - \frac{2{v_0}^2}{g}sin(\alpha) \cdot (n+1)^2)^2 + \frac{{v_0}^2}{2g}cos(\alpha)$

$(x - \frac{2{v_0}^2}{g}sin(\alpha) \cdot (n+1)^2)^2 = -\frac{1}{(n+1)^2} \cdot \frac{g}{8{v_0}^2} \cdot \frac{1}{tan(\alpha)sin(\alpha)} \cdot (y - \frac{{v_0}^2}{2g}cos(\alpha))$

4 Foci of parabolas

Let $4p = a$, therefore:

$p = \frac{-2{v_0}^2 tan(\alpha)sin(\alpha)(n+1)^2}{g}$

The focus of n-th parabola has coordinates:

$x_f = x_v$

$y_f = y_v + p = \frac{{v_0}^2}{2g}cos(\alpha) - \frac{2{v_0}^2 sin^2(\alpha)(n+1)^2}{gcos(\alpha)} = \frac{{v_0}^2}{g}(\frac{cos^2(\alpha) - 4sin^2(\alpha)(n+1)^2}{2cos(\alpha)})$

Let $k = \frac{{v_0}^2}{g}$:

$x_f = 2ksin(\alpha)n(n+1) + ksin(\alpha)(2n+1) + \frac{k}{2}sin(\alpha)$

$y_f = k(\frac{cos^2(\alpha) - 4sin^2(\alpha)(n+1)^2}{2cos(\alpha)})$

In order to get equation of this curve in horizontal frame of reference we have to rotate each point by angle 360°- α (counterclockwise) around (0,0).
$sin(360°-\alpha) = -sin(\alpha)$
$cos(360°-\alpha) = cos(\alpha)$

Thus:

$x\prime = x_f cos(\alpha) + y_f sin(\alpha)$

$y\prime = y_f cos(\alpha) - x_f sin(\alpha)$

After substitution and simplification we get:

$x\prime = 2ktan(\alpha)cos(2\alpha)(n+1)^2$

$y\prime = -k(4sin^2(\alpha)n^2 + 8sin^2(\alpha)n + 3sin^2(\alpha) + \frac{cos(2\alpha)}{2})$

Let us get n from the former equation:

$n = -\sqrt{\frac{x\prime}{2ktan(\alpha)cos(2\alpha)}} - 1$, or
$n = \sqrt{\frac{x\prime}{2ktan(\alpha)cos(2\alpha)}} - 1$

After both substitutions to the latter equation we get:

$y\prime = -\frac{2sin^2(\alpha)}{tan(\alpha)cos(2\alpha)}x\prime + k(sin^2(\alpha) + \frac{cos(2\alpha)}{2})$

$y\prime = -\frac{2sin^2(\alpha)}{tan(\alpha)cos(2\alpha)}x\prime + \frac{{v_0}^2}{g}(sin^2(\alpha) + \frac{cos(2\alpha)}{2})$

Moreover $sin^2(\alpha) + \frac{cos(2\alpha)}{2} = \frac{1}{2}$ (after some simplification - using that $\cos(2\alpha) = cos^2(\alpha) - sin^2(\alpha)$) and $\frac{sin^2(\alpha)}{tan(\alpha)} = sin(\alpha)cos(\alpha) = \frac{1}{2}sin(2\alpha)$ so:

$y\prime = -tan(2\alpha)x\prime + \frac{k}{2}$

Because: $cos(\alpha) = \frac{d}{h}$

Thus: $h = \frac{d}{cos(\alpha)}$

Using energy conservation law:

$k = \frac{2gh}{g} = \frac{2d}{cos(\alpha)}$

Hence:

$y\prime = -tan(2\alpha)x\prime + \frac{d}{cos(\alpha)}$

5 Summary

As we can see the foci of parabolas lie on a straight line which parameters depends on angle α and initial distance between the plane and ball. We can show (as it was done in section 4) that the directrix of all the parabolas is the same horizontal straight line, passing initial position of the ball. Therefore the parabolas will get more and more flattened. (The source code which generates this figure is available here: https://github.com/lvikasz/Physics) It is based on that angle of incidence and angle of reflection are the same (when collision is perfectly elastic).