Taylor's serier for functions of two Variables:

$$f(x,y) = f(a,b) + \left[(x-a) f_x(a,b) + (y-b) f_y(a,b)\right]$$
$$+ \frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b) f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)\right]$$
$$+ \frac{1}{3!}\left[(x-a)^3 f_{xxx}(a,b) + (y-b)^3 f_{yyy}(a,b) + 3(x-a)^2(y-b) f_{xxy}(a,b) + 3(x-a)(y-b)^2 f_{xyy}(a,b)\right]$$
$$+ \cdots$$

This is called the Taylor's Series expansion of $f(x,y)$ in powers of $(x-a)$ and $(y-b)$ (or, about $x=a$ and $y=b$)

<u>Note:</u> (1) Taking $x = a+h$ and $y = b+k$, then

$$f(a+h, b+k) = f(a,b) + \left[h f_x(a,b) + k f_y(a,b)\right]$$
$$+ \frac{1}{2!}\left[h^2 f_{xx}(a,b) + k^2 f_{yy}(a,b) + 2hk f_{xy}(a,b)\right]$$
$$+ \frac{1}{3!}\left[h^3 f_{xxx}(a,b) + k^3 f_{yyy}(a,b) + 3h^2k f_{xxy}(a,b) + 3hk^2 f_{xyy}(a,b)\right]$$
$$+ \cdots$$

$$\boxed{\begin{aligned} &\text{ere} \\ &x-a=h \\ &y-b=k \end{aligned}}$$

(2) Above series is called Maclaurin's series if $a=0$ and $b=0$.

Problems:

(1) Expand $f(x,y)=e^x.\log(1+y)$ in power of x and y upto terms of third degree.

Sol: Given $f(x,y)=e^x \log(1+y)$

here $(a,b)=(0,0)$ (or, $a=0$ and $b=0$)

Now, $f_x(x,y)=e^x \log(1+y) \Rightarrow f_x(0,0)=0$

$f_y=e^x.\frac{1}{1+y} \Rightarrow f_y(0,0)=1$

$f_{xx}(x,y)=e^x \log(1+y) \Rightarrow f_{xx}(0,0)=0$

$f_{xy}(x,y)=e^x.\frac{1}{1+y} \Rightarrow f_{xy}(0,0)=1$

$f_{yy}(x,y)=-e^x.\frac{1}{(1+y)^2} \Rightarrow f_{yy}(0,0)=-1$

$f_{xxx}(x,y)=e^x \log(1+y) \Rightarrow f_{xxx}(0,0)=0$

$f_{xxy}(x,y)=e^x.\frac{1}{1+y} \Rightarrow f_{xxy}(0,0)=1$

$f_{xyy}(x,y)=-e^x.\frac{1}{(1+y)^2} \Rightarrow f_{xyy}(0,0)=-1$

$f_{yyy}(x,y)=2e^x\frac{1}{(1+y)^3} \Rightarrow f_{yyy}(0,0)=2$

Therefore, by the Taylor's Series, we have

$$f(x,y) = f(0,0) + [x f_x(0,0) + y f_y(0,0)]$$
$$+ \frac{1}{2!}[x^2 f_{xx}(0,0) + y^2 f_{yy}(0,0) + 2xy f_{xy}(0,0)]$$
$$+ \frac{1}{3!}[x^3 f_{xxx}(0,0) + y^3 f_{yyy}(0,0) + 3x^2y f_{xxy}(0,0)$$
$$+ 3xy^2 f_{xyy}(0,0)]$$
$$+ \cdots$$

$$= y + xy - \frac{1}{2}y^2 + \frac{1}{2}(x^2y - xy^2) + \frac{1}{3}y^3 + \cdots$$

(2) Expand the ~~(fobo)~~ following functions inpowers of x and y as far as terms of third degree.

(i) $f(x,y) = e^y \sin x$ (ii) $f(x,y) = e^x \cos y$
(iii) $f(x,y) = e^y \log(1+x)$ (iv) $f(x,y) = \sin x \cos y$

(3) Expand $f(x,y) = \cos xy$ in powers of $(x-1)$ and $(y - \frac{\pi}{2})$.

(4) Expand $f(x,y) = x^2y + 3y - 2$ in powers of $(x-1)$ and $(y+2)$

(5) If $f(x,y) = \tan^{-1} xy$, compute $f(0.9, 1.2)$ approximately using Taylor's series

Sol: Given $f(x,y) = \tan^{-1}(xy)$.

We have to compute $f(0.9, -1.2)$

Using Taylor series.

Here $x = 0.9$ and $y = -1.2$

i.e., $x = 1 - 0.1$ and $y = -1 - 0.2$

or, $x = 1 + (-0.1)$ and $y = -1 + (-0.2)$

So, $a = 1$, $b = -1$, $h = -0.1$

and $k = -0.2$.

By Taylor series, we have

$$f(x,y) = f(a,b) + \left[(x-a) f_x(a,b) + (y-b) f_y(a,b)\right]$$

$$+ \cdots$$

Now, $f(a,b) = f(1,-1) = -0.7854$

$f_x(x,y) = \frac{y}{1+x^2y^2} \Rightarrow f_x(1,-1) = -\frac{1}{2} = -0.5$

$$f_{xx}(x,y) = \frac{\partial^2 f(x,y)}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{y}{1+x^2y^2}\right)$$

$$= -\frac{2xy^3}{(1+x^2y^2)^2}$$

$\Rightarrow f_{xx}(1,-1) = \frac{2}{4} = 0.5$, $f_y = \frac{x}{1+x^2y^2}$

$$f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{x}{1+x^2y^2}\right)$$

$f_y(1,-1) = 0.5$

$$= -\frac{2x^3y}{(1+x^2y^2)^2}$$

$\Rightarrow f_{yy}(1,-1) = \frac{2}{4} = 0.5$

$$f_{xy} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{x}{(1+x^2y^2)}\right)$$

$$= \frac{1-x^2y^2}{(1+x^2y^2)^2}$$

$\Rightarrow f_{xy}(1,-1) = 0$

Therefore,

$$f(1+(-0.1), -1+(-0.2))$$

$$= -0.7854 + \left[(-0.1)(-0.5) + (-0.2)(0.5)\right]$$

$$+ \frac{1}{2!}\left[(-0.1)^2(0.5) + (-0.2)^2(0.5) + 2(-0.1)(-0.2)(0)\right]$$

$$+ \cdots$$

$$= -0.8029$$

Maxima and Minima for functions of two Variables:

The necessary conditions for $f(x,y)$ to have a maximum or minimum at (a,b) are $f_x(a,b)=0$ and $f_y(a,b)=0$.

__Sufficient conditions:__

suppose $f_x(a,b)=0$ and $f_y(a,b)=0$.

let $r=\frac{\partial^2 f}{\partial x^2}$, $s=\frac{\partial^2 f}{\partial x \partial y}$ and $t=\frac{\partial^2 f}{\partial y^2}$

Then
(i) $f(a,b)$ is a maximum value of $f(x,y)$ if $rt-s^2>0$ and $r<0$ at (a,b).

(ii) $f(a,b)$ is a minimum value of $f(x,y)$ if $rt-s^2>0$ and $r>0$ at (a,b).

(iii) $f(a,b)$ is not an extreme value if $rt-s^2<0$ at (a,b). In this case (a,b) is called a saddle point.

(iv) If $rt-s^2=0$ at (a,b), then $f(x,y)$ fails to have extreme value and it needs further investigation.

Note: $f(a,b)$ is said to be a stationary value of $f(x,y)$ if $f_x(a,b)=0$ and $f_y(a,b)=0$

Thus every extreme value is a stationary value.

1. A rectangular box open at the top is to have volume of 32 cubic feet. Find the dimensions of the box requiring least material for its construction.

Sol: Let the dimensions of the rectangular box be x ft, y ft and z ft.

Given that Volume = 32 cubic ft.

i.e., $xyz = 32 \longrightarrow (1)$

So, $z = \frac{32}{xy}$. Let S be the surface area of the box. Then $S = xy + 2yz + 2zx \longrightarrow (2)$

(Since box is open at top)

Therefore, from (1) and (2), we have

$$S = xy + \frac{64}{x} + \frac{64}{y}.$$

So, $\frac{\partial S}{\partial x} = y - \frac{64}{x^2}$ and $\frac{\partial S}{\partial y} = x - \frac{64}{y^2}$

Now, $\frac{\partial S}{\partial x} = 0 \Rightarrow y - \frac{64}{x^2} = 0 \Rightarrow y = \frac{64}{x^2} \rightarrow (3)$

and $\frac{\partial S}{\partial y} = 0 \Rightarrow x - \frac{64}{y^2} = 0 \Rightarrow x = \frac{64}{y^2} \rightarrow (4)$

Solving (3) and (4), we get

$x = 4$ and $y = 4$.

Now, $r = \frac{\partial^2 S}{\partial x^2} = \frac{128}{x^3}$; $s = \frac{\partial^2 S}{\partial x \partial y} = 1$ and $t = \frac{\partial^2 S}{\partial y^2} = \frac{128}{y^3}$

At $(4,4)$, $rt - s^2 = 3 > 0$ and $r = 2 > 0$

$\therefore$ S has minimum at $x = 4$, $y = 4$

and hence from ①, we get $z = 2$. Therefore, the required dimensions are 4ft, 4ft and 2 ft to construst the box with least material.

Problems:

1. Find the extremum of the function

$$f(x,y) = x^2 + y^2 + 6x - 12.$$

Sol: Given $f(x,y) = x^2 + y^2 + 6x - 12$

so, $\dfrac{\partial f}{\partial x} = 2x + 6$ and $\dfrac{\partial f}{\partial y} = 2y$

Now, $\dfrac{\partial f}{\partial x} = 0 \Rightarrow 2x + 6 = 0$

$\Rightarrow x + 3 = 0$

and $\dfrac{\partial f}{\partial y} = 0 \Rightarrow 2y = 0$

$\Rightarrow y = 0$

Therefore, $(-3, 0)$ is an extremum point.

Next,

$r = \dfrac{\partial^2 f}{\partial x^2} = 2$, $s = \dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial y}\right) = 0$

and $t = \dfrac{\partial^2 f}{\partial y^2} = 2$.

At the point $(-3, 0)$, we have

$r > 0$ and $rt - s^2 = 4 > 0$

and hence the given function $f(x, y)$ is minimum at $(-3, 0)$ and the minimum value is $\boxed{-21}$

(2) obtain the extremum value of the function $f(x,y) = x^4 + y^4 - 2x^2 - 2y^2 + 4xy$

<u>Sol:</u> Here, $\frac{\partial f}{\partial x} = 4x^3 - 4x + 4y$

and $\frac{\partial f}{\partial y} = 4y^3 - 4y + 4x$

Now, $\frac{\partial f}{\partial x} = 0 \Rightarrow x^3 - x + y = 0 \quad \rightarrow (1)$

and $\frac{\partial f}{\partial y} = 0 \Rightarrow y^3 - y + x = 0 \quad \rightarrow (2)$

Adding (1) and (2), we get

$$x^3 + y^3 = 0$$

$$\Rightarrow x = -y$$

From (1), we have

$$x^3 - 2x = 0 \Rightarrow x(x^2 - 2) = 0$$

$$\Rightarrow x = 0, x = \pm\sqrt{2}$$

and hence $y = 0, \pm\sqrt{2}$.

Therefore, $(0,0)$, $(\sqrt{2}, -\sqrt{2})$ and $(-\sqrt{2}, \sqrt{2})$

are the possible extreme points

Now,

$$r = \frac{\partial^2 f}{\partial x^2} = 12x^2 - 4,$$

$$s = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = 4$$

and $t = \dfrac{\partial^2 f}{\partial y^2} = 12y^2 - 4$.

At $(0,0)$, $rt - s^2 = 16 - 16 = 0$

So, we cannot say anything about the nature of $f(x,y)$ at $(0,0)$.

At $(\sqrt{2}, -\sqrt{2})$:

$r = 20 > 0$ and $rt - s^2 > 0$

Therefore, $f(x,y)$ is minimum at $(\sqrt{2}, -\sqrt{2})$

At $(-\sqrt{2}, \sqrt{2})$: $r = 20 > 0$

and $rt - s^2 > 0$

Also, $f(x,y)$ is minimum at $(-\sqrt{2}, \sqrt{2})$.

Lagrange's method of Undetermined multipliers :

To find the extremum for $f(x,y,z)$ subject to the condition $\phi(x,y,z)=0$

consider the Lagrangean function

$$F(x,y,z) = f(x,y,z) + \lambda\phi(x,y,z)$$

Now, $\frac{\partial F}{\partial x} = 0$; $\frac{\partial F}{\partial y} = 0$ and $\frac{\partial F}{\partial z} = 0$

solving these three equations, we get points at which f has maxima or minima.

① Find the point on the plane $x+2y+3z=4$ that is closed to the origin.

Sol: Let $P(x,y,z)$ be a point on the ~~(origin)~~ given plane

Then $OP = \sqrt{x^2+y^2+z^2}$.

let $f = x^2+y^2+z^2 \longrightarrow (1)$

we have to minimize f subject to the condition $\phi(x,y,z) = x+2y+3z-4 = 0 \longrightarrow (2)$

Consider Lagrangean function

$$F(x, y, z) = f(x, y, z) + \lambda\phi(x, y, z).$$

i.e., $F = x^2 + y^2 + z^2 + \lambda(x + 2y + 3z - 4)$

For f to be minimum,

$$\frac{\partial F}{\partial x} = 0 \; ; \; \frac{\partial F}{\partial y} = 0 \text{ and } \frac{\partial F}{\partial z} = 0$$

Now,

$$\frac{\partial F}{\partial x} = 0 \Rightarrow 2x + \lambda = 0 \Rightarrow x = \frac{-\lambda}{2}$$

$$\frac{\partial F}{\partial y} = 0 \Rightarrow 2y + 2\lambda = 0 \Rightarrow y = -\lambda$$

$$\frac{\partial F}{\partial z} = 0 \Rightarrow 2z + 3\lambda = 0 \Rightarrow z = \frac{-3\lambda}{2}.$$

substitute these values in (2), we get

$$\frac{-\lambda}{2} + 2(-\lambda) + 3\left(\frac{-3\lambda}{2}\right) = 4$$

$$\Rightarrow \lambda = \frac{-4}{7}$$

and hence $x = \frac{2}{7}$, $y = \frac{4}{7}$ and $z = \frac{6}{7}$

Therefore, the point on the given plane closest to the origin is $\left(\frac{2}{7}, \frac{4}{7}, \frac{6}{7}\right)$.

—x—

Problems

① Find the volume of greatest rectangular parallelopiped that can be inscribed in the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

Hint: Volume $=(2x)(2y)(2z)=8xyz \rightarrow ①$

and $\phi = \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1=0 \rightarrow ②$

② Find the three positive numbers whose sum is 100 and whose product is maximum.

③ obtain the maximum value of $\cos A \cdot \cos B \cdot \cos C$ in a plane ΔABC.

Hint: Let $f=\cos A \cdot \cos B \cdot \cos C \rightarrow ①$

In ΔABC, $A+B+C=\pi$

Let $\phi = A+B+C-\pi=0 \rightarrow ②$

④ Obtain the shortest distance from origin to the surface $xyz^2=2$.

Problems:

1. obtain the shortest distance from origin to the surface $xyz^2 = 2$.

Sol: The distance d from the origin to any point $P(x, y, z)$ on the surface $xyz^2 = 2$ is given by

$$d^2 = x^2 + y^2 + z^2 = f(x, y, z) \text{ say.}$$

let $\phi(x, y, z) \equiv xyz^2 - 2 = 0 \longrightarrow (*)$

let $F = f + \lambda\phi$.

Then $\frac{\partial F}{\partial x} = 2x + \lambda yz^2$,

$\frac{\partial F}{\partial y} = 2y + \lambda xz^2$,

and $\frac{\partial F}{\partial z} = 2z + \lambda 2xyz$

Now, $\frac{\partial F}{\partial x} = 0 \Rightarrow 2x + \lambda yz^2 = 0$

$$\Rightarrow \frac{x}{yz^2} = \frac{-\lambda}{2} \longrightarrow (1)$$

$\frac{\partial F}{\partial y} = 0 \Rightarrow 2y + \lambda xz^2 = 0$

$$\Rightarrow \frac{y}{xz^2} = \frac{-\lambda}{2} \longrightarrow (2)$$

and $\frac{\partial F}{\partial z} = 0 \Rightarrow 2z + 2\lambda xyz = 0$

$\Rightarrow \frac{1}{xy} = -\lambda \longrightarrow$ (3)

From (1) and (2), we have

$$\frac{x}{yz^2} = \frac{y}{xz^2}.$$

$$\Rightarrow x^2 = y^2 \Rightarrow x = y$$

From (2) and (3), we have

$$\frac{y}{xz^2} = \frac{1}{2xy}$$

$$\Rightarrow y = \frac{z}{\sqrt{2}}$$

Threfore, from (*), we get

$$x(x)(2x^2) = 2$$

$$\Rightarrow x = 1$$

and hence $y = 1$ and $z = \sqrt{2}$

So, f has minimum at $(1, 1, \sqrt{2})$

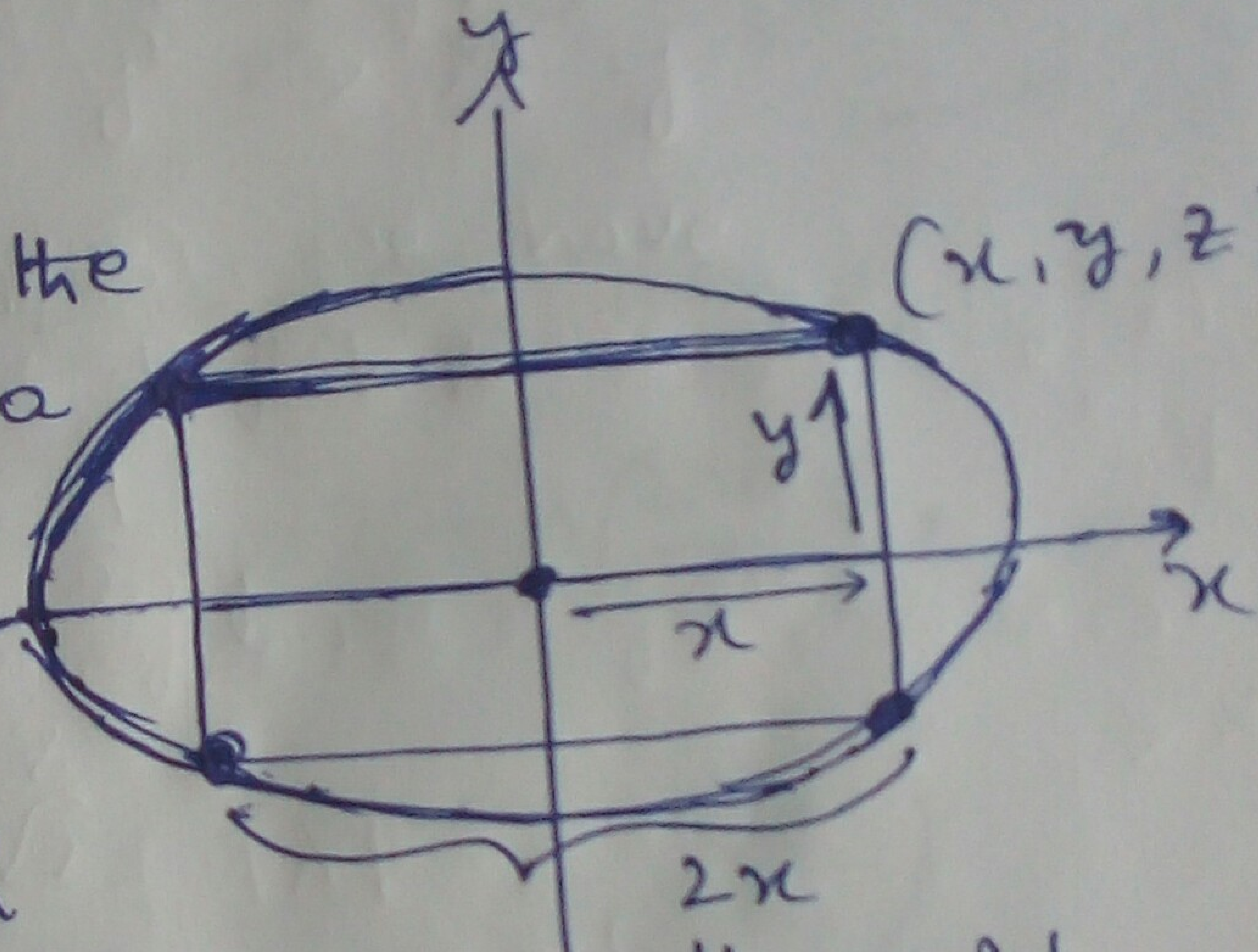
Thus the shortest distance from the origin ~~to~~ to the given surface is $d = \sqrt{1^2 + 1^2 + (\sqrt{2})^2} = 2.$

(2) Obtain the volume of the greatest rectangular parallelopiped that can be inscribed in the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

<u>Sol:</u>

Let (x,y,z) be the coordinates of a corner of a rectangular parallelopiped



inscribed in the given ellipsoid.

Then (x,y,z) satisfy the equation of the ellipsoid, so we have

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \longrightarrow (1)$$

Also, in this case $2x$, $2y$ and $2z$ are the dimensions of the parallelopiped

Therefore, its volume

$$V = 2x \cdot 2y \cdot 2z = 8xyz = \underset{\text{say}}{f(x,y,z)} \rightarrow (2)$$

we have to maximize $f(x,y,z)$ subject to the condition (1).

let $\phi(x,y,z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0$

$\rightarrow (3)$

let $F = f + \lambda \phi$. Then

$$\frac{\partial F}{\partial x} = 8yz + \lambda\left(\frac{2x}{a^2}\right).$$

$$\frac{\partial F}{\partial y} = 8zx + \lambda\left(\frac{2y}{b^2}\right)$$

and $$\frac{\partial F}{\partial z} = 8xy + \lambda\left(\frac{2z}{c^2}\right)$$

Now, $\frac{\partial F}{\partial x} = 0 \Rightarrow a^2\left(\frac{yz}{x}\right) = -\frac{\lambda}{4}$

$\frac{\partial F}{\partial y} = 0 \Rightarrow b^2\left(\frac{zx}{y}\right) = \frac{-\lambda}{4}$

$\frac{\partial F}{\partial z} = 0 \Rightarrow c^2\left(\frac{xy}{z}\right) = \frac{-\lambda}{4}$

By solving, we get

$$\lambda = -\frac{3xyz}{2}$$

and hence $x = \frac{a}{\sqrt{3}}$, $y = \frac{b}{\sqrt{3}}$

and $z = \frac{c}{\sqrt{3}}$.

Thus, the maximum volume $= \frac{8abc}{3\sqrt{3}}$.

(3) Find the maximum value of $x^2 + y^2 + z^2$ under the condition $x + y + z = 3$.

Ans: $3a^2$.