1. A 20 m ladder leans against a wall. The top slides down at a rate of 4 m/s. How fast is the bottom of the ladder moving when it is 16 m from the wall?

Solution: The top of the ladder is descending at a constant, $v_y = \frac{dy}{dt} = -4\ m/s$

(This is not very realistic, of course - this would normally accelerate due to gravity). At the point where the bottom of the ladder is $x = 16m$ from the wall (as required in the question).

At that point, observe the horizontal velocity, $v_x = \frac{dx}{dt} = 3\ m/s$

2. A stone is dropped into a pond, the ripples forming concentric circles which expand. At what rate is the area of one of these circles increasing when the radius is 4 m and increasing at the rate of 0.5 m/s.

Solution: The area of a circle with radius r is $A = \pi r^2$

Differentiate w.r.t. time, and then substitute known values

$$\frac{dA}{dt} = \frac{d}{dt}\left(\pi r^2\right)$$

$$= 2\pi r \frac{dr}{dt}$$

$$= 2\pi(4)(0.5)$$

$$= 4\pi \quad \simeq 12.56\ m^2/s$$

3. For a certain rectangle the length of one side is always three times the length of the other side.

(i) If the shorter side is decreasing at a rate of 2 inches/minute at what rate is the longer side decreasing?

(ii) At what rate is the enclosed area decreasing when the shorter side is 6 inches long and is decreasing at a rate of 2 inches per minute?

Solution: Let us denote the shorter side with x and the longer side with y.

Given y=3x.

(i) Here $\frac{dx}{dt} = -2$ and hence $\frac{dy}{dt} = -6$.

Therefore, the longer side decreasing at a rate of 6 inches per minute.

(ii) Area $A = x\,y$ and hence $\frac{dA}{dt} = -72$.

Therefore, area decreasing at a rate of 72 inches per minute.

4. A thin sheet of ice is in the form of a circle. If the ice is melting in such a way that the area of the sheet is decreasing at a rate of 0.5 m^2/sec at what rate is the radius decreasing when the area of the sheet is 12 m^2?

Answer: $\frac{dr}{dt} = -0.040717$.

Determine the absolute extrema of the following functions

1. $f(x)=8x^3+81x^2-42x-8$ on $[-8,\ 2]$

Answer: Absolute maximum is 1511 at x= −7

and Absolute minimum is −13.3125 at x=1/4

2. $f(x)=3x^4-26x^3+60x^2-11$ on $[1,\ 5]$

Answer: Absolute maximum is 114 at x=5

and Absolute minimum is 26 at x=1.

3. $f(x)=\dfrac{x+4}{2x^2+x+8}$ on $[-10,\ 0]$

Answer: Absolute maximum is 0.5 at x=0

and Absolute minimum is −0.03128 at x=$-4-3\sqrt{2}$.

4. $f(x)=\log_e(x^2+4x+14)$ on $[-4,\ \ 2]$

Answer: Absolute maximum is 3.2581at x=2

and Absolute minimum is 2.3026 at x= −2 .

1. Determine c if $f(x) = x^2 - x$ on $[0,\ 1]$ satisfies Rolle 's Theorem.

Solution:

Clearly f is continuous on $[0,\ 1]$ and differentiable on $(0,\ 1)$.

Also $f(0) = f(1)$.

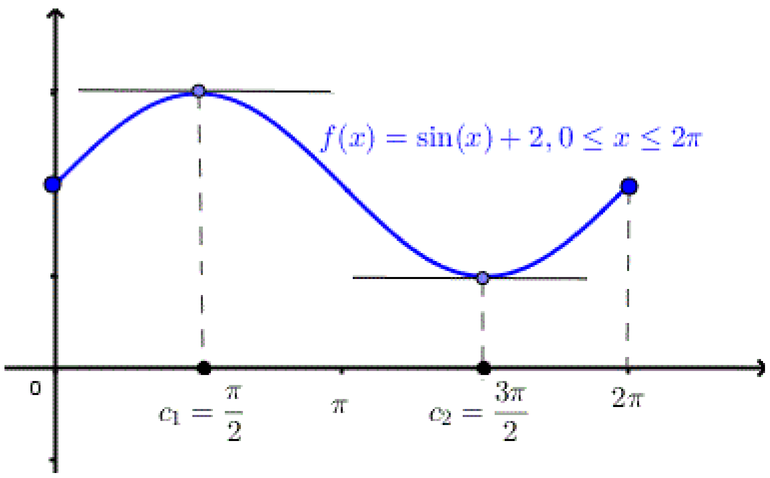
Therefore f satisfies all the conditions of Rolle's theorem.

Hence there exists $c \in (0,\ 1)$ such that $f'(c) = 0$.

Which implies $2c - 1 = 0 \Rightarrow c = \frac{1}{2} \in (0,\ 1)$.

2. Determine all the number(s) c if $f(x) = \sin x + 2$ on $[0,\ 2\pi]$ satisfies Rolle's Theorem.

Answer: $c = \frac{\pi}{2}, \frac{3\pi}{2}$.

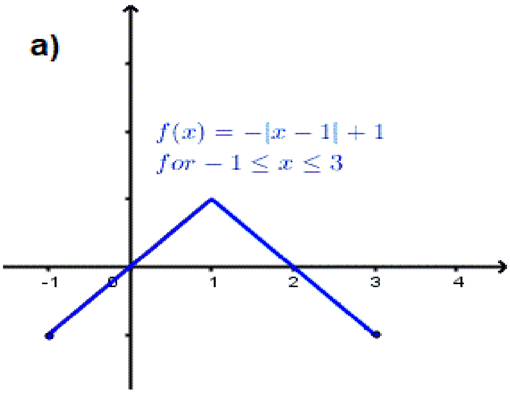


3. Determine all the number(s) c if the following functions satisfies Rolle's Theorem

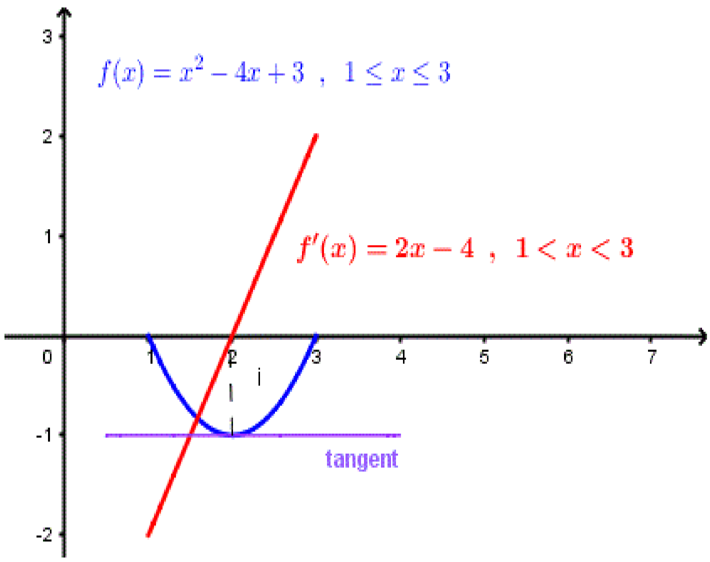
(a) $f(x) = -|x-1|+1$ on $[-1,\ 3]$

(b) $f(x) = x^2 - 4x + 3$ on $[1,\ 3]$

Hint:



b)



4. Determine all the number(s) c if $A(t) = 8t + e^{-3t}$ on $[-2,\ 3]$ satisfies Lagrange's Mean Value Theorem.

Answer: Cleary $A(t)$ is continuous on $[-2,\ 3]$ and differentiable on $(-2,\ 3)$.

Therefore, $A(t)$ satisfies Lagrange's mean value theorem.

Here, $c = -1.0973$.

5. Suppose we know that $f(x)$ is continuous and differentiable on the interval $[-7,\ 0]$, that $f(-7) = -3$ and that $f'(x) \le 2$. What is the largest possible value for $f(0)$?

Answer: Since $f(x)$ satisfies Lagrange's Mean Value Theorem, there exists

$c \in (-7,\ 0)$ such that $f'(c) = \dfrac{f(0) - f(-7)}{0+7} \Rightarrow f(0) = 7f'(c) - 3 \le 7(2) - 3 = 11.$

Therefore, the largest possible value for $f(0)$ is 11.

6. Suppose we know that $f(x)$ is continuous and differentiable on the interval $[-2,\ 5]$, that $f(5) = 14$ and that $f'(x) \le 10$. What is the smallest possible value for $f(-2)$?

1.4: Increasing and Decreasing functions

(i) A function $f(x)$ is increasing over an interval I, if for every $x_1, x_2 \in I$,

$x_1 \leq x_2 \Rightarrow f(x_1) \leq f(x_2)$.

Note: If $x_1 \leq x_2 \Rightarrow f(x_1) < f(x_2)$, then $f(x)$ is strictly increasing.

(ii) A function $f(x)$ is decreasing over an interval I, if for every $x_1, x_2 \in I$,

$x_1 \leq x_2 \Rightarrow f(x_1) \geq f(x_2)$.

Note: If $x_1 \leq x_2 \Rightarrow f(x_1) > f(x_2)$, then $f(x)$ strictly decreasing.

Using derivatives

Let $f(x)$ be Continuous on $[a, b]$ and differentiable on (a, b). Then

(i) If $f'(x) > 0$ for every $x \in (a, b)$, then $f(x)$ is increasing in the interval (a, b)

(ii) If $f'(x) < 0$ for every $x \in (a, b)$, then $f(x)$ is decreasing on (a, b).

(iii) If $f'(x) = 0$ for every $x \in (a, b)$, then $f(x)$ is constant on (a, b).

Calculation of intervals of increasing (or) decreasing :

Example Problems

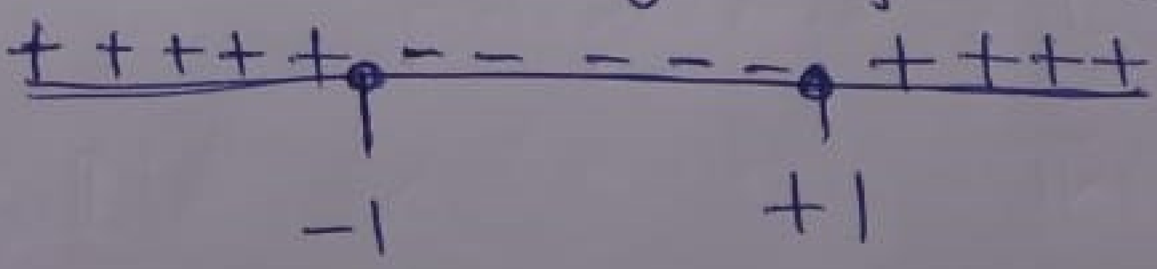
① Find the interval of increasing or decreasing of the function $f(x) = x^3 - 3x + 2$.

Sol: Given $f(x) = x^3 - 3x + 2$.

Clearly, $f'(x) = 3x^2 - 3$

Now, $f'(x) = 0 \Rightarrow 3x^2 - 3 = 0 \Rightarrow x = -1, 1$ (critical points)

Sign of $f'(x)$:



It is clear that (i) $f'(x) > 0$ for all $x \in (-\infty, -1)$

(ii) $f'(x) < 0$ for all $x \in (-1, 1)$

and (iii) $f'(x) > 0$ for all $x \in (1, \infty)$.

Therefore, $f(x)$ is increasing on $(-\infty, -1) \cup (1, \infty)$ and decreasing on $(-1, 1)$.

(2) $f(x) = 3x^4 - 4x^3 - 12x^2 + 3$.

clearly, $f'(x) = 12x^3 - 12x^2 - 24x$

Now, $f'(x) = 0 \Rightarrow x = 0, -1, 2$

(critical points)

sign of $f'(x)$:

$$\underset{}{- - - - -}\ \underset{-1}{\bullet}\ + + + + +\ \underset{0}{\bullet}\ - - - - - - -\ \underset{2}{\bullet}\ + + +$$

So, $f(x)$ is increasing on $(-1, 0) \cup (2, \infty)$ and decreasing on $(-\infty, -1) \cup (0, 2)$

1.5: First Derivative Test

Suppose $f(x)$ is continuous at a critical point x_0.

i) If $f'(x) > 0$ on an open interval extending left from x_0 and $f'(x) < 0$ on an open interval extending right from x_0, then $f(x)$ has local maximum at x_0.

ii) If $f'(x) < 0$ on an open interval extending left from x_0 and $f'(x) > 0$ on an open interval extending right from x_0, then $f(x)$ has a local minimum at x_0.

iii) If $f'(x)$ has same sign on an open interval extending left from x_0 and on an open interval extending right from x_0, then $f(x)$ has an inflection point at x_0.

Note: Relative extrema occur where $f'(x)$ changes sign.

Example: ① Let $f(x) = 3x^4 - 4x^3 - 12x^2 + 3$ on $[-2, 3]$.

Then $f'(x) = 0 \Rightarrow x = -1, 0, 2$

(critical points)

Sign of $f'(x)$

$$\begin{array}{ccccccc} ---- & \bullet & ++++ & \bullet & ------ & \bullet & +++ \\ & -1 & & 0 & & 2 & \end{array}$$

By the First derivative test, f has a relative maximum at $x=0$ and relative minimum at $x=-1$ and $x=2$

② If $f(x) = \sin x + \cos x$ on $[0, 2\pi]$, then determine all local extrema for the function.

Sol: Suppose $f(x) = \sin x + \cos x$ on $[0, 2\pi]$.

Then, $f'(x) = 0 \Rightarrow x = \frac{\pi}{4}$ and $\frac{5\pi}{4}$

critical points which are in the interval $[0, 2\pi]$

By the First derivative test, f has

Sign of $f'(x)$

$$\begin{array}{ccccc} ++++ & \bullet & ---- & \bullet & ++++ \\ & \frac{\pi}{4} & & \frac{5\pi}{4} & \end{array}$$

local minimum at $x = \frac{5\pi}{4}$

and local maximum at $x = \frac{\pi}{4}$.

local maximum is $\sqrt{2}$ and local minimum is $-\sqrt{2}$.

③ A ball is thrown in the air. Its height (meters) at any time t sec is given by

$$h = 3 + 14t - 5t^2.$$

What is its maximum height?

Sol: Given $h = 3 + 14t - 5t^2$

So, $\dfrac{dh}{dt} = 14 - 10t$

Now, $\dfrac{dh}{dt} = 0 \Rightarrow t = 1.4$

Sign of $h'(t)$

$+ + + \bullet - - -$

1.4

Clearly, $h(t)$ has maximum at $t = 1.4$ and the maximum height is 12.8 m.

SECOND DERIVATIVE TEST

Let c be a critical point at which $f'(c)=0$. Suppose $f''(c)$ exists. Then

(i) f has a relative maximum at c if $f''(c)<0$

(ii) f has a relative minimum at c if $f''(c)>0$

and (iii) the test is inconclusive if $f''(c)=0$.

Example Problems

① Use the second derivative test to find the local maximum and local minimum values of the function $f(x)=x^4-2x^2+3$.

Sol: Clearly $f'(x)=4x^3-4x$

Now, $f'(x)=0 \Rightarrow 4x(x^2-1)=0$

$\Rightarrow x=0,-1,1$ (critical points)

Here, the second derivative of $f(x)$ is $12x^2-4$

i.e., $f''(x)=12x^2-4$

Clearly, (i) $f''(-1)=8>0$, so $f(-1)=2$ is the local minimum value

(ii) $f''(0)=-4<0$, so $f(0)=3$ is the local maximum value.

(iii) $f''(1)=8>0$, so $f(1)=2$ is the local minimum value.

(2) $f(x) = x^3 - 3x^2 + x - 2$

Now,

$f'(x) = 0 \Rightarrow 3x^2 - 6x + 1 = 0$

$\Rightarrow x = \frac{3-\sqrt{6}}{3}, \frac{3+\sqrt{6}}{3}$

clearly, $f''(x) = 6x - 6$

and (i) $f''\left(\frac{3-\sqrt{6}}{3}\right) < 0$, so f has relative maximum at $x = \frac{3-\sqrt{6}}{3}$

(ii) $f''\left(\frac{3+\sqrt{6}}{3}\right) > 0$, so f has relative minimum at $x = \frac{3+\sqrt{6}}{3}$.

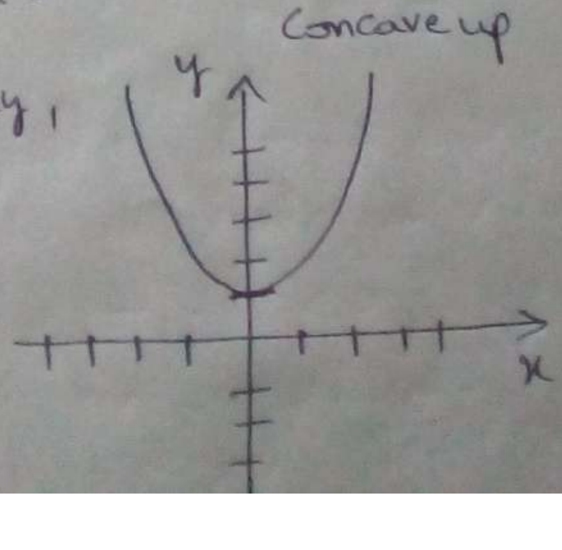
1.6: Concavity

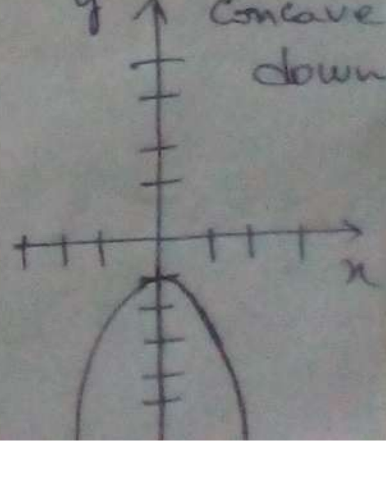
Concavity relates to the rate of change of a function's derivative. A function f is concave up (or upwards) where the ~~deriti~~ derivative f' is increasing (or equivalently, f'' is positive). Similarly, f is concave down (or, downwards) where the derivative f' is decreasing (or equivalently, f'' is negative).

OR

If the function $f(x)$ is twice differentiable at $x = c$, then the graph of f is concave upwards at $(c, f(c))$ if $f''(c) > 0$ and concave downwards at $(c, f(c))$ if $f''(c) < 0$.

Graphically,





Example Problems:

① Discuss the concavity of the function $f(x) = x^3 - 3x^2 + x - 2$.

Sol: Given $f(x) = x^3 - 3x^2 + x - 2$.

So, $f'(x) = 3x^2 - 6x + 1$ and $f''(x) = 6x - 6$.

Clearly $f''(x) > 0$ for $x > 1$,

and $f''(x) < 0$ for $x < 1$.

Hence, $f(x)$ is concave up on $(1, \infty)$

and concave down on $(-\infty, 1)$.

② Discuss the concavity of $f(x) = 5x^3 + 2x^2 - 3x$.

Note: An inflection is where a curve changes from concave upward to concave downward (or, vice versa).

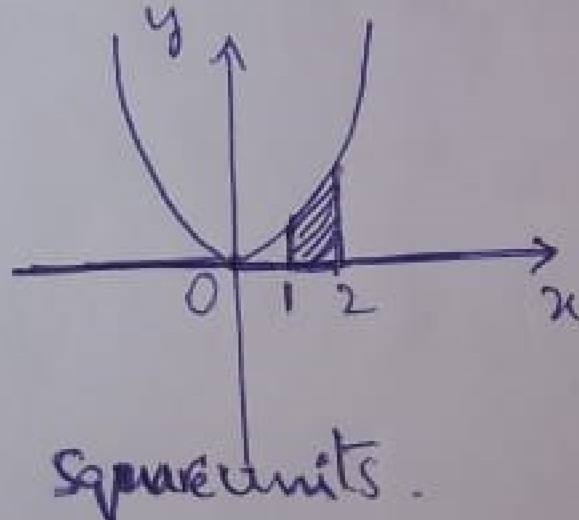
1.7: Areas (Area between the curves)

The area which is bounded by the curve $y=f(x)$ and the lines $x=a$, $x=b$ is denoted by $\int_a^b y\,dx$

Example Problems

① obtain the area bounded by the curve $y=x^2$, the lines $x=1$, $x=2$ and the x-axis.

Sol: Required Area $= \int_1^2 x^2\,dx$

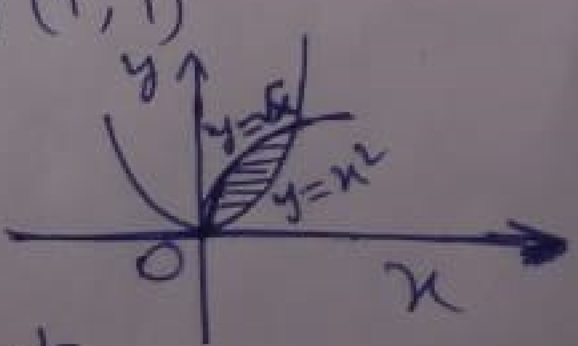


$= \left[\frac{x^3}{3}\right]_1^2 = 7/3$ square units.

② Find the area of the region enclosed by $y=x^2$ and $y=\sqrt{x}$

Sol: The given curves $y=x^2$ and $y=\sqrt{x}$ are intersecting at $(0,0)$ and $(1,1)$

Required Area $= \int_0^1 (\sqrt{x}-x^2)\,dx$



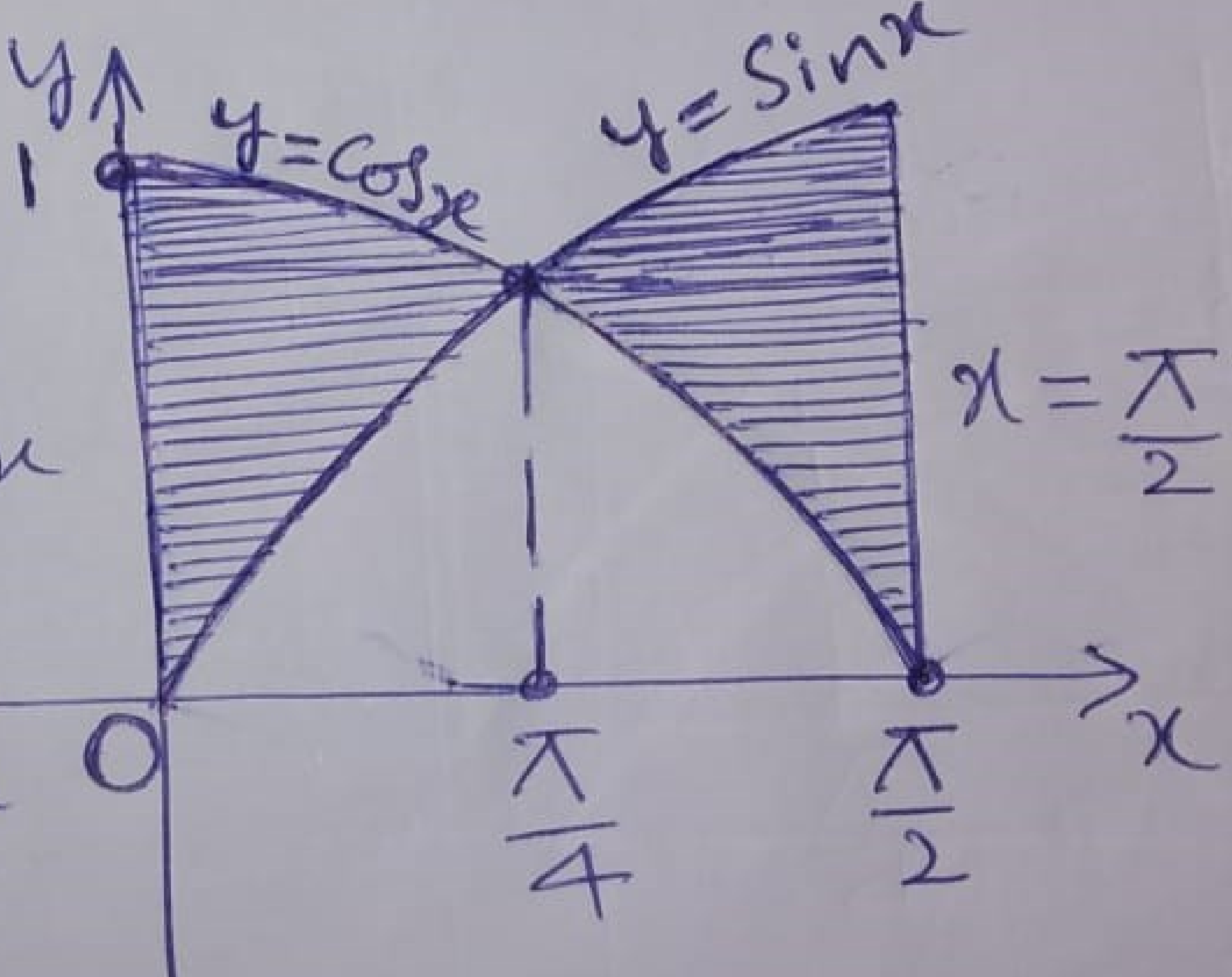
$= \frac{1}{3}$ sq. units.

(3) Find the area of the region enclosed by $y = \sin x$, $y = \cos x$, $x = \frac{\pi}{2}$ and the y-axis.

Sol:

Required

$$\text{Area} = \int_0^{\pi/4} (\cos x - \sin x)\,dx + \int_{\pi/4}^{\pi/2} (\sin x - \cos x)\,dx$$



$$= 2\sqrt{2} - 2 \text{ sq. units.}$$

(4) Determine the area of the region enclosed by $x = \frac{y^2}{2} - 3$ and $y = x - 1$

Hint: The given curves are intersecting at the points $(-1, -2)$ and $(5, 4)$

Volumes of Solids of revolution:

Volumes using Cross-sections:

Definition: The volume of a solid of integrable cross-section area $A(x)$ from $x=a$ to $x=b$ is $V=\int_a^b A(x)\,dx$.

Definition: The volume of a solid of integrable cross-section area $A(y)$ from $y=c$ to $y=d$ is $V=\int_c^d A(y)\,dy$.

To get the cross-sectional area, it is required to cut the object perpen-dicular ($\perp$) to the axis of rotation.

Doing this, the crossection will be either a solid disk if the object is solid or a ring if we have hollowed out a portion of the solid.

In the case of solid disk, area is

$\pi(\text{radius})^2$ where the radius will depend upon the function and the axis of ~~the~~) rotation

In the case of ring, the area is

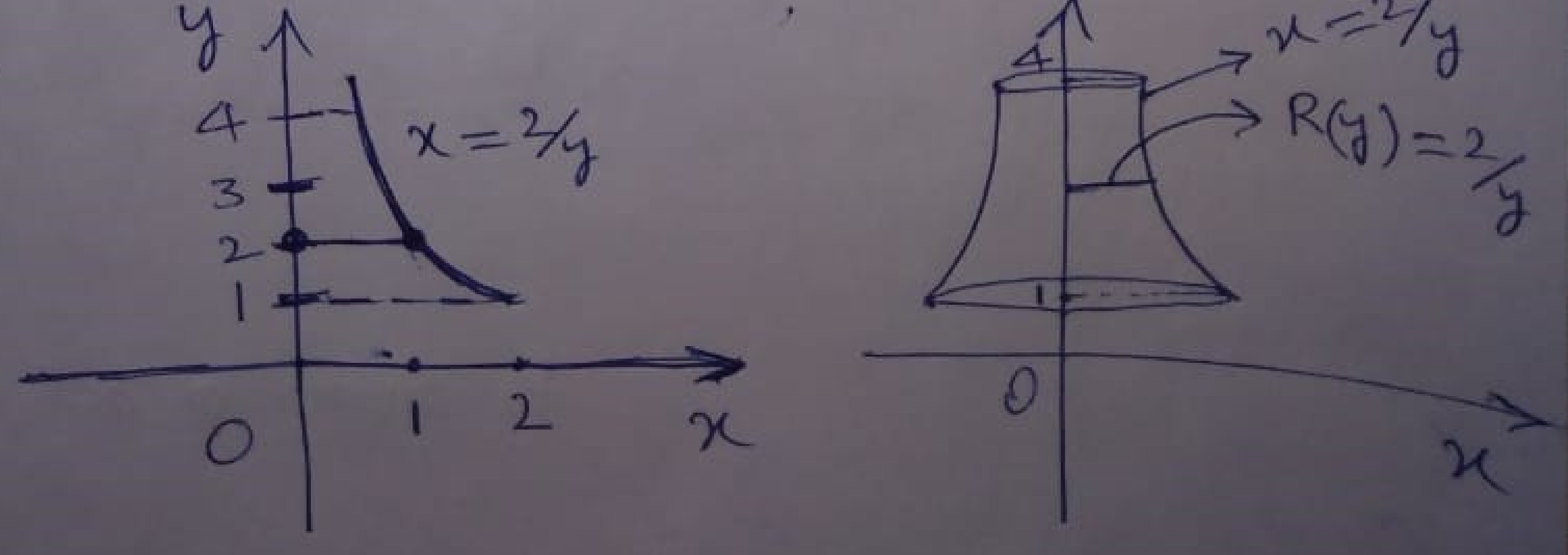
$$\pi\left[(\text{Outer radius})^2 - (\text{Inner radius})^2\right]$$

where again both of the radii depends on the functions given and the axis of rotation.

Example Problems:

① Find the volume of the solid generated by revolving the region between the y-axis and the curve $x = 2/y$, $1 \le y \le 4$ about the y-axis.

Sol:



Required Volume is $V = \int_1^4 A(y)\,dy$,

where $A(y) =$ the cross-sectional area

$= \pi [R(y)]^2$ (since the cross-section is circle)

$= \pi \left(\frac{2}{y}\right)^2$

$R(y) =$ radius of the circle

$=$ distance between the point $(0, y)$ on y-axis and the point $(2/y, y)$ on $x = 2/y$.

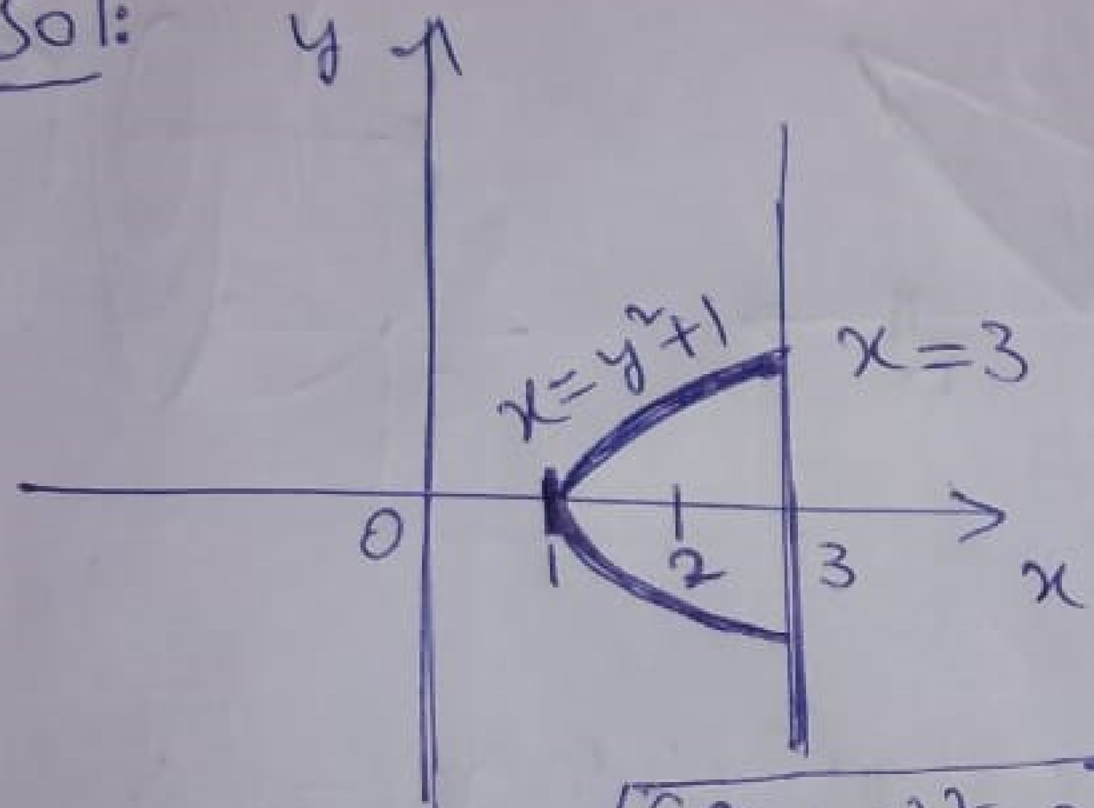
$= \sqrt{\left(\frac{2}{y} - 0\right)^2 + (y-y)^2} = \frac{2}{y}$

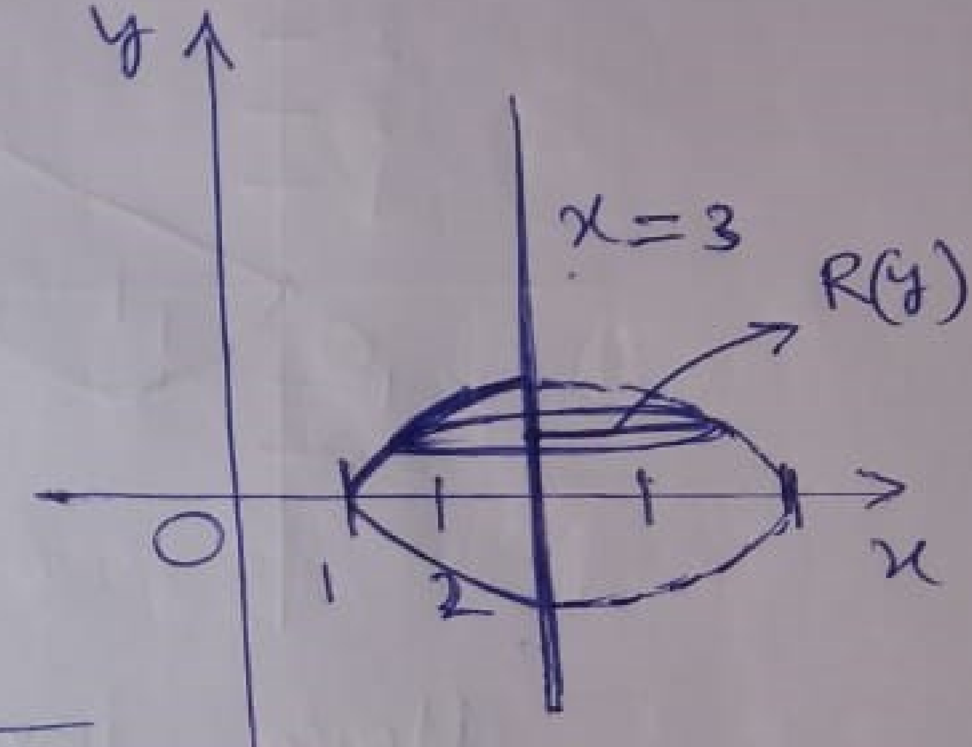
Hence, volume $= \pi \int_1^4 \left(\frac{2}{y}\right)^2 dy$

$= 3\pi$ cubic units.

(2) Find the volume of the solid generated by revolving the region between the parabola $x = y^2 + 1$ and the line $x = 3$ about the line $x = 3$

Sol:





Here $R(y) = \sqrt{(y^2+1-3)^2 + (y-y)^2} = y^2 - 2$

given curve $x = y^2 + 1$ and the line $x = 3$

are intersecting at $(3, -\sqrt{2})$ and $(3, \sqrt{2})$

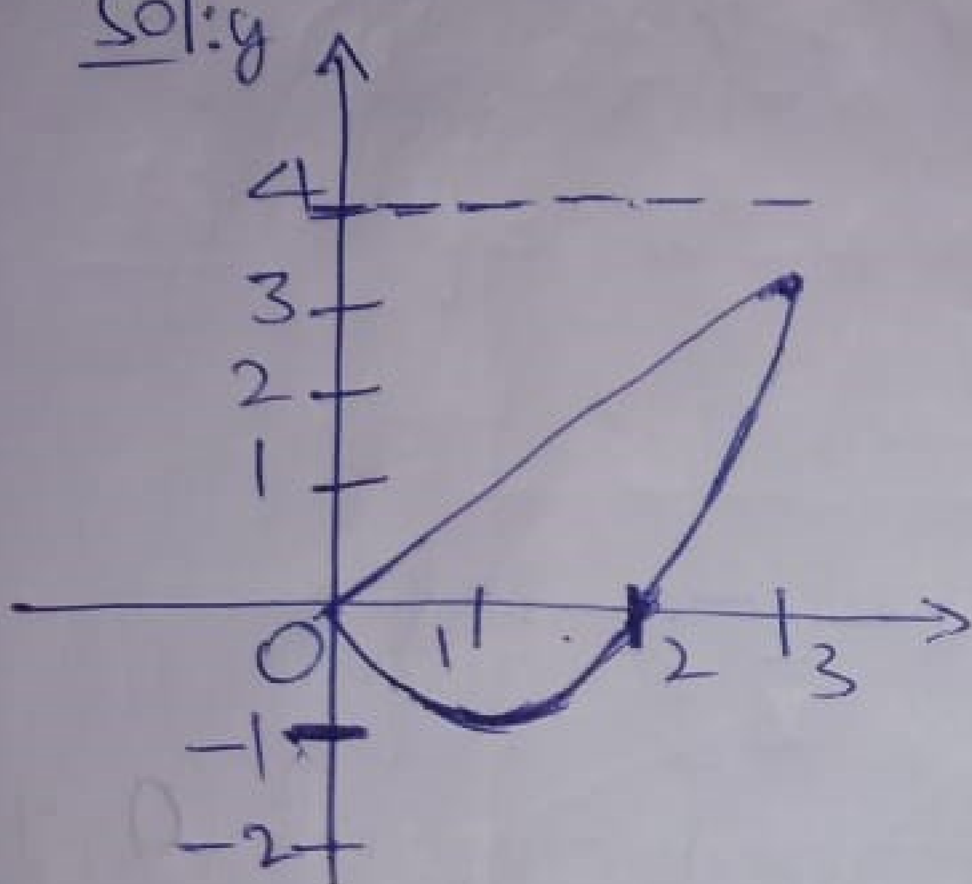
therefore, Volume $= \int_{-\sqrt{2}}^{2} \pi (R(y))^2 \, dy$

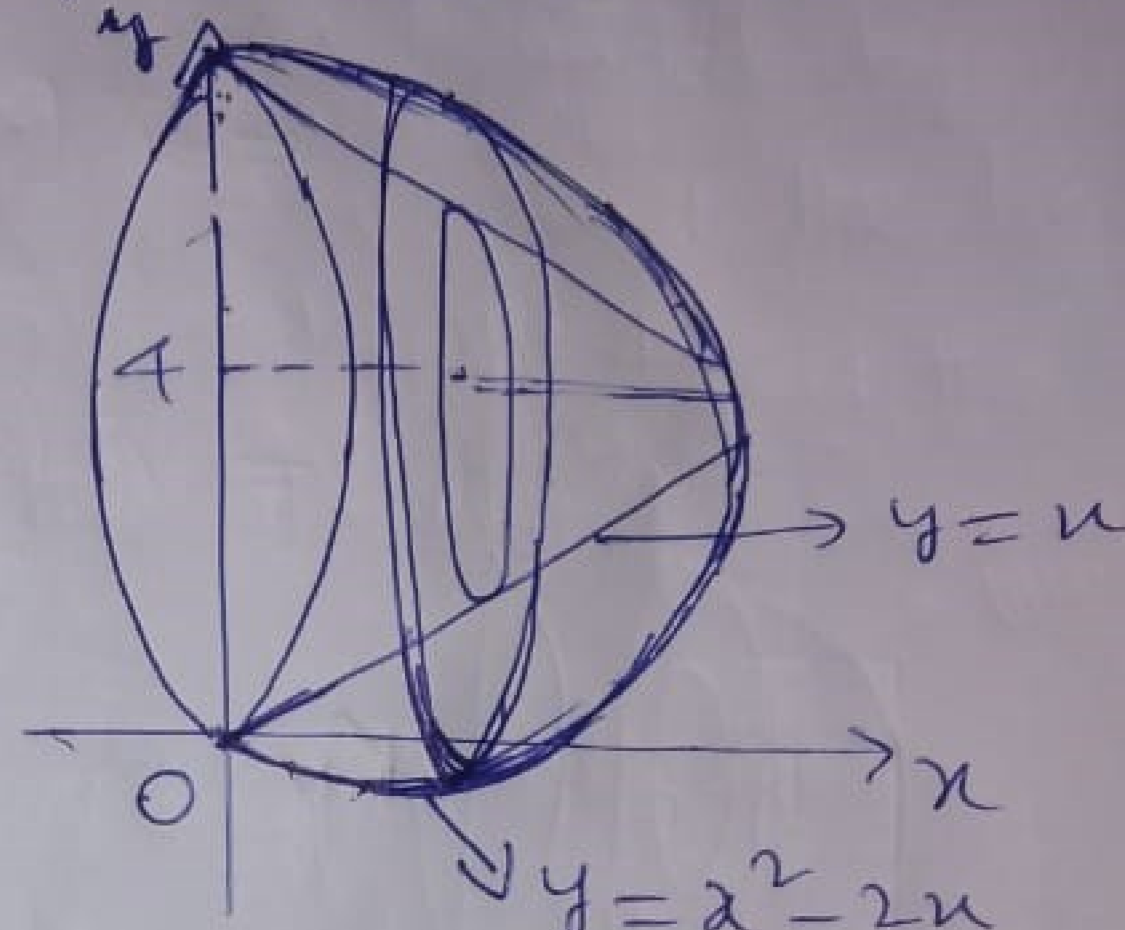
$= \pi \int_{-\sqrt{2}}^{2} (y^2 - 2)^2 \, dy$

$= \dfrac{64 \pi \sqrt{2}}{15}$ cubic units.

③ Determine the volume of the solid generated by rotating the region bounded by $y = x^2 - 2x$ and $y = x$ about the line $y = 4$.

Sol:





Inner radius $r(x) = \sqrt{(x-x)^2 + (4-x)^2}$

$= 4 - x$

Outer radius $R(x) = \sqrt{(x-x)^2 + (4-(x^2-2x))^2}$

$= 4 - (x^2 - 2x)$

Therefore, volume $= \int_0^4 \pi \left([4-(x^2-2x)]^2 - [4-x]^2 \right) dx$

$= \pi \int_0^4 (x^4 - 4x^3 - 5x^2 + 24x)\, dx$

$= \dfrac{153\pi}{6}$ cubic units.

(4) Determine the volume of the solid obtained by rotating the region bounded by

$y = x^2 - 4x + 5$, $x = 1$, $x = 4$ and the x-axis about the x-axis

Answer: Volume $= 78\pi/5$.

(5) Determine the volume of the solid generated by rotating the portion of the region bounded by $y = \sqrt[3]{x}$ and $y = \frac{x}{4}$, that lies in the first quadrant about the y-axis.

Answer: $\frac{512\pi}{21}$