Mar. 11, 2022

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$

$$\frac{1}{v^2} = \mu_0 \epsilon_0$$

$$v = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$$

$$\mu_0 = 4\pi \times 10^{-7}\ H/m$$

$$\epsilon_0 = 8.82 \times 10^{-12}\ F/m$$

$$v = \frac{1}{\sqrt{4\pi \times 10^{-7} \times 8.82 \times 10^{-12}}}$$

$v = 3 \times 10^8$ m/sec = Speed of light

$$\nabla \times \vec{B} = \mu \vec{J} + \mu \epsilon \frac{\partial \vec{E}}{\partial t} \quad ; \quad J = \sigma E$$

$$\nabla \times \vec{B} = \sigma \mu \vec{E} + \mu \epsilon \frac{\partial \vec{E}}{\partial t}$$

$$\nabla \times (\nabla \times \bar{B}) = \nabla \times (\sigma \mu \bar{E}) + \nabla \times \left(\mu \epsilon \frac{\partial \bar{E}}{\partial t}\right)$$

$$\nabla (\cancel{\nabla \cdot B})^{0} - \nabla^2 B = \sigma \mu \underbrace{(\nabla \times \bar{E})}_{-\frac{\partial \bar{B}}{\partial t}} + \mu \epsilon \frac{\partial}{\partial t} \underbrace{(\nabla \times \bar{E})}_{-\frac{\partial \bar{B}}{\partial t}}$$

$$-\nabla^2 B = -\sigma \mu \frac{\partial \bar{B}}{\partial t} - \mu \epsilon \frac{\partial^2 \bar{B}}{\partial t^2}$$

$$\boxed{\nabla^2 B = \sigma \mu \frac{\partial \bar{B}}{\partial t} + \mu \epsilon \frac{\partial^2 B}{\partial t^2}}$$

for free space $\sigma \to 0$

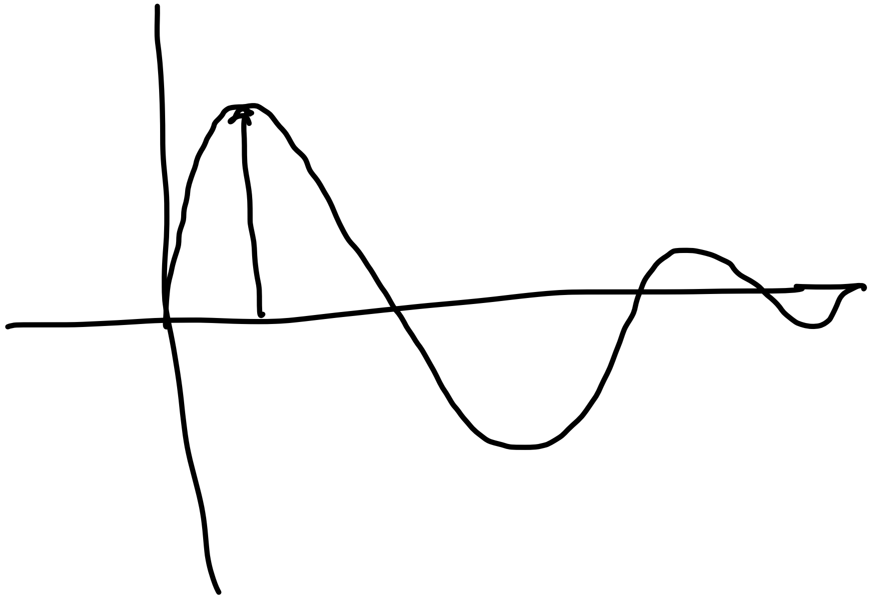
$$\boxed{\nabla^2 B = \mu_0 \epsilon_0 \frac{\partial^2 B}{\partial t^2}} \leftarrow$$

$$B = B_0 e^{i(\omega t - \vec{K}\cdot\vec{r})}$$

$$E = E_0 e^{i(\omega t - \boxed{\bar{K}\cdot\bar{r}})}$$

$\downarrow$

Scalar



$$\nabla \cdot \bar{E} = \frac{\rho}{\epsilon}$$

$$\nabla \cdot \bar{B} = 0$$

$$\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t}$$

$$\nabla \times \bar{B} = \mu \bar{J} + \mu\epsilon \frac{\partial \bar{E}}{\partial t}$$

for static case. $\frac{\partial \bar{B}}{\partial t} \to 0, \frac{\partial E}{\partial t} \to 0$

$$\nabla \cdot E = \frac{\rho}{\epsilon} \quad ; \quad \nabla \times E = 0$$

$$\nabla \cdot B = 0 \quad ; \quad \nabla \times \bar{B} = \mu \vec{J}$$

For free space

$\rho \to 0, \quad \sigma \to 0$

$\mu \to \mu_0 \quad ; \quad \epsilon \to \epsilon_0$

$$\nabla \cdot \bar{E} = 0$$

$$\nabla \cdot \bar{B} = 0$$

$$\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t}$$

$$\nabla \times \bar{B} = \mu_0 \epsilon_0 \frac{\partial E}{\partial t}$$

Maxwell correction in Ampere's Law

From Ampere's law

$$\nabla \times \vec{B} = \mu \vec{J}$$

Taking divergence on both sides

$$\nabla \cdot (\nabla \times \vec{B}) = \nabla \cdot (\mu \vec{J}) = \mu (\vec{\nabla} \cdot \vec{J})$$

Since divergence of curl is always zero.

$$\Rightarrow \quad \nabla \cdot (\vec{\nabla} \times \vec{B}) = 0$$

$$\Rightarrow \quad \mu (\nabla \cdot \vec{J}) = 0$$

$$\nabla \cdot \vec{J} = 0$$

Since $\nabla \cdot \vec{J}$ can not be zero except for the steady state. i.e. Ampere's Law can not be used outside the magnetostatics.

Maxwell's correction to the Ampere's Law:

Using Continuity equation

$$\nabla \cdot \vec{J} = -\frac{\partial \rho}{\partial t}$$

From the Gauss law in differentical form

$$\nabla \cdot E = \frac{\rho}{\epsilon}$$

$$\Rightarrow \quad \rho = \epsilon \, \nabla \cdot \vec{E}$$

$\Rightarrow$ equation of continuity will become

$$\nabla \cdot \vec{J} = -\frac{\partial}{\partial t} (\epsilon \nabla \cdot \vec{E})$$

$$= -\nabla \cdot \left(\epsilon \frac{\partial \vec{E}}{\partial t} \right)$$

$$\vec{\nabla} \cdot \left(\vec{J} + \epsilon \frac{\partial \vec{E}}{\partial t} \right) = 0$$

If one replace $\vec{J} \longrightarrow \vec{J} + \epsilon \dfrac{\partial \vec{E}}{\partial t}$

in Ampere's Law.

$$\boxed{\vec{\nabla} \times \vec{B} = \mu \vec{J} + \mu \epsilon \frac{\partial \vec{E}}{\partial t}}$$

Wave equation

In the region where no charge and current.
i.e. $\rho = 0$, $\vec{J} = 0$

Maxwell's equation will reduce to –

i) $\nabla \cdot \vec{E} = 0$

ii) $\nabla \cdot \vec{B} = 0$

iii) $\nabla \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t}$

iv) $\nabla \times \vec{B} = \mu_0 \epsilon_0 \dfrac{\partial \vec{E}}{\partial t}$

Wave equation in terms of E :

Using Maxwell's 3rd eqn

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

Taking curl both sides:

$$\vec{\nabla} \times (\nabla \times \vec{E}) = -\nabla \times \frac{\partial \vec{B}}{\partial t} = -\frac{\partial}{\partial t}(\nabla \times \vec{B})$$

using vector identity $A \times (B \times C) = B(A \cdot C) - (A \cdot B)C$

$$\nabla(\vec{\nabla} \cdot E) - (\nabla \cdot \nabla)\vec{E} = -\frac{\partial}{\partial t}(\nabla \times \vec{B})$$

Using Maxwell's 1st and 4th eqn putting $\nabla \cdot \vec{E}$ and $\nabla \times \vec{B}$

$$\cancel{-}\nabla^2 \vec{E} = \cancel{-}\frac{\partial}{\partial t}\left(\mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}\right)$$

$$\Rightarrow \boxed{\nabla^2 \vec{E} = \mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}}$$

Comparing with the standard wave eqn

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$

$$\boxed{v = \frac{1}{\sqrt{\mu_0 \epsilon_0}}} = c \quad \text{(for E.M. waves)}$$

$\mu_0 = 4\pi \times 10^{-7} NA^{-2}$
$\epsilon_0 = 8.85 \times 10^{-12} Fm^{-1}$

Wave equation in terms of B :

Using Maxwell's 4th eqn in free space

$$\nabla \times \vec{B} = \mu_0 \epsilon_0 \frac{\partial E}{\partial t}$$

taking curl both the sides:

$$\nabla \times (\nabla \times \vec{B}) = \nabla \times \left(\mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}\right)$$

$$\cancel{\nabla(\nabla \cdot \vec{B})}^{\,0} - \nabla^2 B = \mu_0 \epsilon_0 \frac{\partial}{\partial t}(\nabla \times \vec{E})$$

$$-\nabla^2 B = \mu_0 \epsilon_0 \frac{\partial}{\partial t}\underbrace{\left(-\frac{\partial \vec{B}}{\partial t}\right)}_{\text{(from Maxwell's 3rd eqn)}}$$

Q1: An electromagnetic wave, propagating with the speed of light in free space is incident normally on a perfect dielectric medium for which the relative permittivity is 5.0. By what factor will the velocity of the wave will get reduced?

Q2: Determine the conduction current and displacement current densities in a material having $\sigma = 10^{-3}$ mhos/m and relative permittivity $\epsilon_r = 2.45$. The electric field in the material is given by $E = 4 \times 10^{-6} \sin(9 \times 10^{9}\, t)$ V/m.

Q3: Electric field in free space is given by $\vec{E} = E_0 cos(10^8 t + kx)\hat{y}$ V/m, find out (i) Direction of wave propagation and magnetic field, (ii) calculate k (iii) Magnetic field vector.