# Module - 3

## Multivariable Calculus

### 1. Limits and Continuity of a function of two variables

**Def:** Let $f$ be a function of two variables $x$ and $y$. The limit of $f(x,y)$ as $(x,y)$ approaches $(a,b)$ is $l$, written

$$\lim_{(x,y)\to(a,b)} f(x,y) = l, \quad \text{if } \lim_{x\to a}\left\{\lim_{y\to b} f(x,y)\right\}$$

and $\lim_{y\to b}\left\{\lim_{x\to a} f(x,y)\right\}$ exists and elqual to $l$.

i.e., $$\lim_{x\to a}\left\{\lim_{y\to b} f(x,y)\right\} = l = \lim_{y\to b}\left\{\lim_{x\to a} f(x,y)\right\}$$

**Note:** At the origin $(0,0)$, we have to verify the limit along the path $y = mx^n$ also if both the limits are equal.

**Examples**

(i) Evaluate $\displaystyle\lim_{(x,y)\to(1,2)} \left(\frac{2x^2 y}{x^2+y^2+1}\right)$.

**Sol:** $$\lim_{x\to 1}\left\{\lim_{y\to 2}\left(\frac{2x^2 y}{x^2+y^2+1}\right)\right\} = \lim_{x\to 1} \frac{4x^2}{x^2+5}$$

$$= \frac{2}{3}$$

and $\underset{y\to 2}{Lt}\left\{\underset{x\to 1}{Lt}\left(\frac{2x^2y}{x^2+y^2+1}\right)\right\} = \underset{y\to 2}{Lt}\ \frac{2y}{y^2+2}\Big\}$

$= \frac{2}{3}$

Therefore, both $\underset{x\to 1}{Lt}\left\{\underset{y\to 2}{Lt}\left(\frac{2x^2y}{x^2+y^2+1}\right)\right\}$ and

$\underset{y\to 2}{Lt}\left\{\underset{x\to 1}{Lt}\left(\frac{2x^2y}{x^2+y^2+1}\right)\right\}$ exists and equal

to $\frac{2}{3}$. Hence $\underset{(x,y)\to(1,2)}{Lt}\left(\frac{2x^2y}{x^2+y^2+1}\right)$

exists and equal to $\frac{2}{3}$

i.e., $\underset{(x,y)\to(1,2)}{Lt}\left(\frac{2x^2y}{x^2+y^2+1}\right) = \frac{2}{3}$

(ii) Evaluate $\underset{(x,y)\to(0,0)}{Lt}\left(\frac{xy}{x^2+y^2}\right)$

Sol: $\underset{x\to 0}{Lt}\left\{\underset{y\to 0}{Lt}\left(\frac{xy}{x^2+y^2}\right)\right\} = \underset{x\to 0}{Lt}\ 0 = 0$

and $\underset{y\to 0}{Lt}\left\{\underset{x\to 0}{Lt}\left(\frac{xy}{x^2+y^2}\right)\right\} = \underset{y\to 0}{Lt}\ 0 = 0$

Along the path $y = mx$,

$\underset{\to(0,0)}{}\left(\frac{xy}{x^2+y^2}\right) = \underset{x\to 0}{Lt}\left(\frac{x\,mx}{x^2+m^2x^2}\right) = \underset{x\to 0}{Lt}\ \frac{m}{m^2+1} = \frac{m}{m^2+1}$

which depends upon the value of m.
Hence the limit does not exist.

(iii) Evaluate
$$\lim_{(x,y)\to(1,1)} \left(\frac{x(y-1)}{y(x-1)}\right).$$

Sol: $\lim_{x\to1} \left\{ \lim_{y\to1} \left(\frac{x(y-1)}{y(x-1)}\right) \right\} = \lim_{x\to1} 0 = 0$

and $\lim_{y\to1} \left\{ \lim_{x\to1} \left(\frac{x(y-1)}{y(x-1)}\right) \right\} = \lim_{y\to0} \frac{1}{0} = \infty.$

Therefore, the limit does not exist.

(iv) Evaluate $\lim_{(x,y)\to(0,0)} \left(\frac{x-y}{2x+y}\right)$

$\boxed{\text{Ans}}$: does not exist

(v) Evaluate $\lim_{(x,y)\to(1,1)} \left(\frac{(x-1)^2 \log_e x}{(x-1)^2+y^2}\right)$

Ans: 0

(vi) ~~#~~ Evaluate $\lim_{(x,y)\to(0,0)} \left(\frac{xy}{\sqrt{2x^2+y^2}}\right).$

Ans: ~~[illegible]~~ 0

## Continuity:

**Def:** A function $f(x,y)$ is said to be continuous at the point $(a,b)$ if

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$$

## Examples:

(i) Show that the function

$$f(x,y) = \begin{cases} x^2+2y \;; & (x,y)\neq(1,2) \\ 0 \;; & (x,y)=(1,2) \end{cases}$$

is discontinuous at $(1,2)$.

**Sol:** $$\lim_{x\to1}\left\{\lim_{y\to2} f(x,y)\right\} = \lim_{x\to1}\left\{\lim_{y\to2}(x^2+2y)\right\}$$

$$= \lim_{x\to1}(x^2+4) = 5$$

and $$\lim_{y\to2}\left\{\lim_{x\to1}(x^2+2y)\right\} = \lim_{y\to2}(1+2y) = 5$$

Therefore, $$\lim_{(x,y)\to(1,2)} f(x,y) = 5$$

But $f(1,2) = 0$

So, $$\lim_{(x,y)\to(1,2)} f(x,y) \neq f(1,2)$$

Hence $f$ is discontinuous at $(1,2)$

(ii) Investigate the continuity of the function $f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & ; (x,y) \neq (0,0) \\ 0 & ; (x,y) = (0,0) \end{cases}$

at the origin.

Sol: Along $y = mx$,

$$\lim_{\substack{x \to 0 \\ y \to 0}} \left( \frac{xy}{x^2+y^2} \right) = \lim_{x \to 0} \left( \frac{x\,mx}{x^2+m^2x^2} \right)$$

$$= \lim_{x \to 0} \left( \frac{m}{m^2+1} \right)$$

$$= \frac{m}{m^2+1}.$$

which depends on the value of $m$.

Hence $f(x,y)$ is not continuous at $(0,0)$.

Exercise:

1. verify the continuity of the function

$$f(x,y) = \begin{cases} \dfrac{x^2 y}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

at the origin. Ans: f is continuous at (0,0).

2. verify the continuity of the function

$$f(x,y) = \begin{cases} \dfrac{x^2}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

at the origin Ans: f is discontinuous at (0,0)

(3) verify the continuity of the function

$$f(x,y) = \begin{cases} \dfrac{x+y}{x^2+y^2+1}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

at the origin.

Ans: It is continuous at $(0,0)$

(4) Find $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2-y^2}{x^2+y^2}$, if it exists

Ans: does not exist.

(5) Find $\lim\limits_{(x,y)\to(1,2)} \left(\dfrac{x}{\sqrt{2x+y}}\right)$, if it exists.

Ans: $\dfrac{1}{2}$

(6) Find $\lim\limits_{(x,y)\to(0,0)} \left(\dfrac{3x^2y}{x^2+y^2}\right)$, if it exists.

Ans: $0$

## 3.2: Partial Differentiation

**Def:** Let $z = f(x,y)$. Keeping $y$ constant and varying only $x$, the partial derivative of $z$ with respect to $x$ is denoted by $\frac{\partial z}{\partial x}$ or, $z_x$ and is defined as

$$\frac{\partial z}{\partial x} = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

Partial derivative of $z$ with respect to $y$ is denoted by $\frac{\partial z}{\partial y}$ or $z_y$ and defined as

$$\frac{\partial z}{\partial y} = \lim_{h \to 0} \frac{f(x,y+h) - f(x,y)}{h}$$

**Note:** (i) $\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 f}{\partial x^2} = f_{xx}$

(ii) $\frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 f}{\partial y^2} = f_{yy}$

(iii) $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 f}{\partial x \partial y} = f_{xy}$.

Examples:

1. If $u(x,y) = \log_e\left(\dfrac{x^2+y^2}{xy}\right)$, then verify $\dfrac{\partial^2 u}{\partial x \partial y} = \dfrac{\partial^2 u}{\partial y \partial x}$.

Sol: Suppose $u(x,y) = \log_e\left(\dfrac{x^2+y^2}{xy}\right)$

or, $u(x,y) = \log(x^2+y^2) - \log x - \log y$.

Then,
$$\frac{\partial u}{\partial x} = \frac{1}{x^2+y^2}(2x) - \frac{1}{x}$$
$$= \frac{2x}{x^2+y^2} - \frac{1}{x}$$

So,
$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)$$
$$= \frac{\partial}{\partial y}\left(\frac{2x}{x^2+y^2} - \frac{1}{x}\right)$$
$$= \frac{-4xy}{(x^2+y^2)^2}.$$

And,
$$\frac{\partial u}{\partial y} = \frac{1}{x^2+y^2}\cdot(2y) - \frac{1}{y}$$
$$= \frac{2y}{x^2+y^2} - \frac{1}{y}.$$

So, $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)$

$= \frac{\partial}{\partial x}\left(\frac{2y}{x^2+y^2} - \frac{1}{y}\right)$

$= -\frac{4xy}{(x^2+y^2)^2}$

Therefore, $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$.

(2) If $u(x,y) = \tan^{-1}\left(\frac{y}{x}\right)$, then find (i) $u_x$ (ii) $u_y$.

Sol: (i) $u_x = \frac{1}{1+\left(\frac{y}{x}\right)^2} \frac{\partial}{\partial x}\left(\frac{y}{x}\right)$

$= \frac{x^2}{x^2+y^2} \cdot \left(\frac{-y}{x^2}\right)$

$= -\frac{y}{x^2+y^2}$

and, (ii) $u_y = \frac{1}{1+\left(\frac{y}{x}\right)^2} \frac{\partial}{\partial y}\left(\frac{y}{x}\right)$

$= \frac{x^2}{x^2+y^2} \cdot \frac{1}{x} = \frac{x}{x^2+y^2}$

(3) If $u = \sin^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y}{x}\right)$, then find $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$.

Sol: Suppose $u = \sin^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y}{x}\right)$

Then

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}} \cdot \frac{\partial}{\partial x}\left(\frac{x}{y}\right)$$

$$+ \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{\partial}{\partial x}\left(\frac{y}{x}\right)$$

$$= \frac{y}{\sqrt{y^2-x^2}} \cdot \frac{1}{y} + \frac{x^2}{x^2+y^2}\left(\frac{-y}{x^2}\right)$$

$$= \frac{1}{\sqrt{y^2-x^2}} - \frac{y}{x^2+y^2}$$

and $$\frac{\partial u}{\partial y} = \frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}} \cdot \frac{\partial}{\partial y}\left(\frac{x}{y}\right) + \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{\partial}{\partial y}\left(\frac{y}{x}\right)$$

$$= \frac{y}{\sqrt{y^2-x^2}} \cdot \left(\frac{-x}{y^2}\right) + \frac{x^2}{x^2+y^2} \cdot \left(\frac{1}{x}\right)$$

$$= \frac{-x}{y\sqrt{y^2-x^2}} + \frac{x}{x^2+y^2}$$

Therefore,

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{x}{\sqrt{y^2-x^2}}-\frac{xy}{x^2+y^2}-\frac{x}{\sqrt{y^2-x^2}}+\frac{xy}{x^2+y^2}$$

$$=0$$

④ If $u=\log(x^3+y^3+z^3-3xyz)$,

then S.T. $\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u=-\frac{9}{(x+y+z)^2}$

Sol: We have

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u$$

$$=\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)u$$

$$=\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right) \longrightarrow (*)$$

Now,

$$u_x=\frac{\partial u}{\partial x}=\frac{3x^2-3yz}{x^3+y^3+z^3-3xyz}$$

$$\frac{\partial u}{\partial y} = \frac{3y^2 - 3xz}{x^3+y^3+z^3-3xyz}$$

and $$\frac{\partial u}{\partial z} = \frac{3z^2 - 3xy}{x^3+y^3+z^3-3xyz}$$

So, $$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}$$

$$= \frac{3(x^2+y^2+z^2-xy-yz-zx)}{x^3+y^3+z^3-3xyz}$$

$$= \frac{3(x^2+y^2+z^2-xy-yz-zx)}{(x+y+z)(x^2+y^2+z^2-xy-yz-zx)}$$

$$= \frac{3}{x+y+z}.$$

From (*), we have

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{3}{x+y+z}\right)$$

$$= 3\left(\frac{\partial}{\partial x}\left(\frac{1}{x+y+z}\right)+\frac{\partial}{\partial y}\left(\frac{1}{x+y+z}\right) + \frac{\partial}{\partial z}\left(\frac{1}{x+y+z}\right)\right)$$

$$= 3\left(\frac{-1}{(x+y+z)^2} - \frac{1}{(x+y+z)^2} - \frac{1}{(x+y+z)^2}\right)$$

$$= \frac{-9}{(x+y+z)^2}$$

5. If $u = \log(x^2+y^2)$, then find $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$.

<u>Ans:</u> 0

6. If $u = \log(x^3+y^3-x^2y-xy^2)$, then S.T. $\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2}$

$$= \frac{-4}{(x+y)^2}$$

<u>Hint:</u>

$$\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2}$$

$$= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^2 u$$

$$= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) u$$

7. If $r^2 = x^2+y^2+z^2$ and $u = r^m$,

then prove that

$$u_{xx} + u_{yy} + u_{zz} = m(m+1)\, r^{m-2}$$

8. If $u(x,y) = \log(x^2+y^2) + \tan^{-1}\left(\frac{y}{x}\right)$,

then find $u_{xx} + u_{yy}$.

Ans: $0$

9. If $u = x^2 \tan^{-1}\left(\frac{y}{x}\right) - y^2 \tan^{-1}\left(\frac{x}{y}\right)$,

then find $\dfrac{\partial^2 u}{\partial y \partial x}$.

Ans: $\dfrac{x^2-y^2}{x^2+y^2}$

10. If $u = e^x(x\cos y - y\sin y)$,

then find $u_{xx} + u_{yy}$.

Ans: $0$

## 3.3: Total Differential, Total derivative and Chain rule:

Def: Let $z = f(x, y)$, where $x = \phi(t)$ and $y = \psi(t)$.

Then (i) The total differential of $z$ is defined as

$$\boxed{dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy}$$

(ii) The total derivative of $z$ with ~~de~~ respect to $t$ (or, the total differential coefficient of $z$ w.r.t. $t$) is denoted by $\frac{dz}{dt}$ and defined as

$$\boxed{\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}}$$

Let $z = f(x, y)$ where $x = \phi(s, t)$ and $y = \psi(s, t)$, then

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial s} \longrightarrow (1)$$

and $$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial t} \longrightarrow (2)$$

The equations (1) and (2) are known as <u>chain rule</u> for partial differentiation.

<u>Examples:</u>

1. Find the total differential co-efficient of $x^2 y$ with respect to $x$ when $x$ and $y$ are connected by $x^2 + xy + y^2 = 1$

Sol: let $z = x^2 y$.

Then the total derivative of $z$ w.r.t. $x$ is

$$\frac{dz}{dx} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dx} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dx}$$

$$= (2xy)\,1 + x^2 \frac{dy}{dx} \longrightarrow (*)$$

we have

$$x^2 + xy + y^2 = 1 \longrightarrow (1)$$

~~S~~ Differentiating (1) w.r.t. $x$, we get

$$2x + \left(x\frac{dy}{dx} + y\right) + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow (2x+y) + (2y+x)\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{-(2x+y)}{2y+x}$$

Therefore, from (*), we get

$$\frac{dz}{dx} = 2xy + x^2\left(-\frac{(2x+y)}{2y+x}\right)$$

$$= 2xy - x^2\frac{(2x+y)}{(2y+x)}$$

(2). If $u = x\log xy$, where $x^3+y^3+3xy=1$, find $\frac{du}{dx}$.

Sol: Suppose $u = x\log(xy)$.

we have

$$\frac{du}{dx} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dx} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dx}$$

$$= (1+\log xy) + \frac{x}{y}\cdot\frac{dy}{dx} \longrightarrow (*)$$

Now, differentiating $x^3+y^3+3xy=1$ w.r.t. $x$, we get

$$\frac{dy}{dx} = -\left(\frac{x^2+y}{x+y^2}\right)$$

Therefore, from (*), we get

$$\frac{du}{dx} = (1+\log(xy)) - \frac{x}{y}\left(\frac{x^2+y}{x+y^2}\right)$$

(3) Find $\frac{dy}{dx}$ if $x^y + y^x = c$, constant

Ans: $\frac{dy}{dx} = \frac{yx^{y-1} + y^x . \log y}{x^y . \log x + xy^{x-1}}$

(4) If $z = x^2y + xy^2$, then find the total differential of $z$.

Sol: we have $dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$

$$\boxed{dz = 2xy\, dx + 2xy\, dy}$$

(5) If $u = f(r, s, t)$, where $r = x + y$, $s = y + z$ and $t = z +$ find $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$ and $\frac{\partial u}{\partial z}$.

Sol: Suppose $u = f(r, s, t)$, where $r = x + y$, $s = y + z$ and $t = z + x$.

By Chain rule, we have

(i) $$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial x} + \frac{\partial u}{\partial s} \cdot \frac{\partial s}{\partial x} + \frac{\partial u}{\partial t} \cdot \frac{\partial t}{\partial x}$$

$$= \frac{\partial u}{\partial r}(1) + \frac{\partial u}{\partial s}(0) + \frac{\partial u}{\partial t}(1)$$

$$= \frac{\partial u}{\partial r} + \frac{\partial u}{\partial t}.$$

(ii) $$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial y} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial y}$$

$$= \frac{\partial u}{\partial r}(1) + \frac{\partial u}{\partial s}(1) + \frac{\partial u}{\partial t}(0)$$

$$= \frac{\partial u}{\partial r} + \frac{\partial u}{\partial s}$$

iii)

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial z} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial z} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial z}$$

$$= \frac{\partial u}{\partial r}(0) + \frac{\partial u}{\partial s}(1) + \frac{\partial u}{\partial t}(1)$$

$$= \frac{\partial u}{\partial s} + \frac{\partial u}{\partial t}.$$

(6) If $u = f(y-z, z-x, x-y)$, find

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}$$

Hint: Let $r = y-z$, $s = z-x$, $t = x-y$.

Then $u = f(r, s, t)$:

Ans: 0

(7) The radius '$r$' of ~~the~~ a right circular cone is decreasing at a rate of 3 cm per minute and the height '$h$' is increasing at a rate of 2 cm per minute. When $r = 9$ cm and $h = 6$ cm, find the rate of change of its volume.

Sol: Volume of right circular cone is $V = \frac{\pi r^2 h}{3}$.

So, $\frac{dv}{dt} = \frac{\partial v}{\partial r} \cdot \frac{dr}{dt} + \frac{\partial v}{\partial h} \cdot \frac{dh}{dt}$

(by Total derivative v w.r.to t.)

$= \frac{\pi}{3}\left\{2rh\frac{dr}{dt} + r^2\frac{dh}{dt}\right\}$

given $r = 9$ cm, $h = 6$ cm, $\frac{dr}{dt} = -3$ cm/min

and $\frac{dh}{dt} = 2$ cm/min.

Therefore, $\frac{dv}{dt} = \frac{\pi}{3}\left\{108(-3) + 81(2)\right\}$

$= -54\pi$ cm$^3$/min.

Hence, volume is decreasing at the rate of $54\pi$ cm$^3$/min.

(8) The radius of the cylinder is 2 ft and the height is 4 ft and they are increasing at the rate of 0.5 ft/sec and 1.5 ft/sec respectively. Find the rate of change of volume w.r.t. time.

<u>Ans:</u> $14\pi\ ft^3/sec$.

<u>Hint:</u>
Volume $V = \pi r^2 h$

So, $$\frac{dv}{dt} = \frac{\partial V}{\partial r}\cdot\frac{dr}{dt} + \frac{\partial V}{\partial h}\cdot\frac{dh}{dt}$$

(9) If $u = f\left(\frac{x}{y}, \frac{y}{z}, \frac{z}{x}\right)$, find

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}$$

<u>Ans:</u> 0

(10) If $z = f(x, y)$ where $x = e^u + e^{-v}$ and $y = e^{-u} - e^{v}$, then prove that $$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$$

sol:

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}$$

$$= \frac{\partial z}{\partial x} e^{u} + \frac{\partial z}{\partial y}(-e^{-u})$$

and $$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}$$

$$= \frac{\partial z}{\partial x}(-e^{-v}) + \frac{\partial z}{\partial y}(-e^{v})$$

Therefore,

$$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = (e^{u}+e^{-v})\frac{\partial z}{\partial x} - (e^{-u}-e^{v})\frac{\partial z}{\partial y}$$

$$= x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}.$$

(11) If $z = f(u,v)$ where $u = e^{x}\cos y$ and $v = e^{x}\sin y$, then prove that $\frac{\partial z}{\partial x} = u\frac{\partial z}{\partial u} + v\frac{\partial z}{\partial v}$

## 3.4: Jacobian

**Def:** If $u$ and $v$ are two differentiable functions of two independent variables $x$ and $y$. Then the Jacobian of $u$ and $v$ with respect to $x$ and $y$ is denoted by $J\left(\frac{u, v}{x, y}\right)$ or, $\frac{\partial(u,v)}{\partial(x,y)}$ and defined as

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$$

Similarly, we cand define

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}$$

**Note:** $\frac{\partial(u,v)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(u,v)} = 1$

Problems:

1. If $u = x^2 + y^2$ and $v = xy$, find $\frac{\partial(u,v)}{\partial(x,y)}$

Sol: Suppose $u = x^2 + y^2$ and $v = xy$.

Then, $u_x = \frac{\partial u}{\partial x} = 2x$

$u_y = \frac{\partial u}{\partial y} = 2y$

$v_x = \frac{\partial v}{\partial x} = y$

$v_y = \frac{\partial v}{\partial y} = x$.

Therefore, $\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$

$= \begin{vmatrix} 2x & 2y \\ y & x \end{vmatrix}$

$= 2x^2 - 2y^2$

$= 2(x^2 - y^2)$

2. If $x = r\cos\theta$, $y = r\sin\theta$, find $\frac{\partial(r,\theta)}{\partial(x,y)}$.

Sol: Suppose $x = r\cos\theta$ and $y = r\sin\theta$. $\longrightarrow$ (1)

we have

$$\frac{\partial(r,\theta)}{\partial(x,y)} = \begin{vmatrix} r_x & r_y \\ \theta_x & \theta_y \end{vmatrix}$$

From (1), we have

$$x^2+y^2 = r^2(\cos^2\theta + \sin^2\theta) \Rightarrow x^2+y^2 = r^2$$

$$\Rightarrow \boxed{r = \sqrt{x^2+y^2}}$$

and

$$\frac{y}{x} = \frac{\sin\theta}{\cos\theta} \Rightarrow \tan\theta = \frac{y}{x}$$

$$\Rightarrow \boxed{\theta = \tan^{-1}\left(\frac{y}{x}\right)}$$

Now,

$$r_x = \frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2+y^2}} = \frac{x}{r}$$

$$r_y = \frac{\partial r}{\partial y} = \frac{y}{\sqrt{x^2+y^2}} = \frac{y}{r}$$

$$\theta_x = \frac{\partial \theta}{\partial x}$$

$$= \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{\partial}{\partial x}\left(\frac{y}{x}\right)$$

$$= \frac{x^2}{x^2+y^2} \cdot \left(-\frac{y}{x^2}\right)$$

$$= \frac{-y}{x^2+y^2}$$

$$= \frac{-y}{r^2}$$

$$\theta_y = \frac{\partial \theta}{\partial y}$$

$$= \frac{1}{1+\left(\frac{y}{x}\right)^2} \frac{\partial}{\partial y}\left(\frac{y}{x}\right)$$

$$= \frac{x^2}{x^2+y^2}\left(\frac{1}{x}\right)$$

$$= \frac{x}{x^2+y^2}$$

$$= \frac{x}{r^2}$$

Therefore, $\dfrac{\partial(r,\theta)}{\partial(x,y)} = \begin{vmatrix} \frac{x}{r} & \frac{y}{r} \\ \frac{-y}{r^2} & \frac{x}{r^2} \end{vmatrix}$

$$= \frac{x^2}{r^3} + \frac{y^2}{r^3}$$

$$= \frac{1}{r^3}\left(x^2+y^2\right)$$

$$= \frac{1}{r^3} \cdot r^2 = \frac{1}{r}$$

$$\boxed{\text{So, } \frac{\partial(x,y)}{\partial(r,\theta)} = r}$$

3. If $x = r\cos\theta$, $y = r\sin\theta$ and $z = z$,

find $\dfrac{\partial(x,y,z)}{\partial(r,\theta,z)}$.

Hint: $\dfrac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} x_r & x_\theta & x_z \\ y_r & y_\theta & y_z \\ z_r & z_\theta & z_z \end{vmatrix}$

Ans: $\boxed{r}$

4. In spherical polar coordinates,

$x = r.\sin\theta.\cos\phi$, $y = r\sin\theta.\sin\phi$

and $z = r\cos\theta$, find $\dfrac{\partial(x,y,z)}{\partial(r,\theta,\phi)}$

Ans: $\boxed{r^2\sin\theta}$

## 3.5: Functional Dependence

If $J\left(\frac{u,v}{x,y}\right)=0$, then we say that $u$ and $v$ are functionally dependent.

Otherwise (i.e., $J\left(\frac{u,v}{x,y}\right)\neq 0$), $u$ and $v$ are functionally independent.

Problems: 1. Verify $u=x(1-y)$ and $v=xy$ are functionally dependent or not.

If $u$ and $v$ are functionally dependent, find the relation between $u$ and $v$.

Sol: Given $u=x(1-y)$ and $v=xy$

So, $u_x = 1-y$ | $v_x = y$

$u_y = -x$ | $v_y = x$

Therefore, $\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$

$$= \begin{vmatrix} 1-y & -x \\ y & x \end{vmatrix} = x(1-y)+xy = x \neq 0$$

Hence $u$ and $V$ are not functionally dependent. They are functionally independent.

2. If $u = \tan^{-1}x + \tan^{-1}y$ and $V = \frac{x+y}{1-xy}$, prove that $u$ and $V$ are functionally dependent and hence find the relation between $u$ and $V$.

Sol: Suppose $u = \tan^{-1}x + \tan^{-1}y$

and $V = \frac{x+y}{1-xy}$.

Then,

$$u_x = \frac{\partial u}{\partial x} = \frac{1}{1+x^2}, \quad u_y = \frac{1}{1+y^2}$$

$$V_x = \frac{\partial V}{\partial x} = \frac{1+y^2}{(1-xy)^2}, \quad V_y = \frac{1+x^2}{(1-xy)^2}$$

Therefore,

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$$

$$= \begin{vmatrix} \frac{1}{1+x^2} & \frac{1}{1+y^2} \\ \frac{1+y^2}{(1-xy)^2} & \frac{1+x^2}{(1-xy)^2} \end{vmatrix}$$

$$= \frac{1}{(1-xy)^2} \begin{vmatrix} \frac{1}{1+x^2} & \frac{1}{1+y^2} \\ 1+y^2 & 1+x^2 \end{vmatrix}$$

$$= 0$$

Hence $u$ and $v$ are functionally dependent.

Now,

$$u = \tan^{-1} x + \tan^{-1} y$$

$$= \tan^{-1}\left(\frac{x+y}{1-xy}\right)$$

$$= \tan^{-1} v$$

$$\boxed{\text{i.e., } u = \tan^{-1} v}$$

3. Prove that $u = x+y+z$,

$v = x^2+y^2+z^2-xy-yz-zx$

and $w = x^3+y^3+z^3-3xyz$ are functionally dependent. Also find the relation between $u$, $v$ and $w$.

Sol: $u_x = 1$, $u_y = 1$, $u_z = 1$

$v_x = 2x-y-z$, $v_y = 2y-x-z$

$v_z = 2z-y-x$

$w_x = 3x^2-3yz$, $w_y = 3y^2-3xz$

$w_z = 3z^2-3xy$

Clearly, $\dfrac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}$

$= 0$

Therefore, $u$, $v$ and $w$ are functionally dependent.

And,

$$w = x^3+y^3+z^3-3xyz$$

$$= (x^2+y^2+z^2-xy-yz-xy)(x+y+z)$$

$$= vu$$

$$\boxed{\text{i.e, } w = uv}$$

4. If $u = \frac{x}{y}$ and $v = \frac{y}{x}$, verify whether $u$ and $v$ are functionally dependent and if so find the relation between $u$ and $v$.

Sol: Suppose $u = \frac{x}{y}$ and $v = \frac{y}{x}$

Then $u_x = \frac{1}{y}$, $u_y = -\frac{x}{y^2}$

$v_x = -\frac{y}{x^2}$, $v_y = \frac{1}{x}$

Now, $$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$$

$$= \begin{vmatrix} \frac{1}{y} & -\frac{x}{y^2} \\ -\frac{y}{x^2} & \frac{1}{x} \end{vmatrix} = \frac{1}{xy} - \frac{1}{xy} = 0$$

Therefore, $u$ and $v$ are functionally dependent. And $u = \frac{x}{y}$

$$= \frac{1}{\left(\frac{y}{x}\right)} = \frac{1}{v}$$

Hence $\boxed{uv = 1}$

5. If $u = x^2+y^2+z^2$, $v = xy+yz+zx$ and $w = x+y+z$, verify whether $u$, $v$ and $w$ are functionally dependent, and if so find the relation between them. <u>Ans:</u> $\boxed{w^2 = u+2v}$

6. If $u = \frac{x-y}{x+y}$ and $v = \frac{xy}{(x+y)^2}$, verify whether $u$ and $v$ are functionally dependent and if so find the relation between $u$ and $v$

Ans: $u$ and $v$ are functionally dependent.

And $u^2 = \frac{(x-y)^2}{(x+y)^2} = \frac{(x+y)^2 - 4xy}{(x+y)^2}$

$$= 1 - 4\frac{xy}{(x+y)^2}$$

$$= 1 - 4v$$

Therefore, $\boxed{u^2 + 4v = 1}$

7. If $u = e^{x+y}$ and $v = x^2 + y^2 + (2x + 2y + 2xy)$, verify whether $u$ and $v$ are functionally dependent and if so find the relation between $u$ and $v$.

Ans: $u$ and $v$ are functionally dependent

And $\cancel{v} = (x+y)^2 + 2(x+y) = [\log e^{x+y}]^2 + 2[\log e^{x+y}]$

Therefore, $V = [\log u]^2 + 2[\log u]$

8. If $u = \frac{x}{y-z}$, $v = \frac{y}{z-x}$ and $w = \frac{z}{x-y}$,

find the Jacobian of u, v and w with respect to x, y and z $\left(\text{ie., } \frac{\partial(u,v,w)}{\partial(x,y,z)}\right.$

Find the relational between u, v and w if they are functionally dependent.