Module-5:

Multiple Integrals

1. Evaluation of Double Integrals

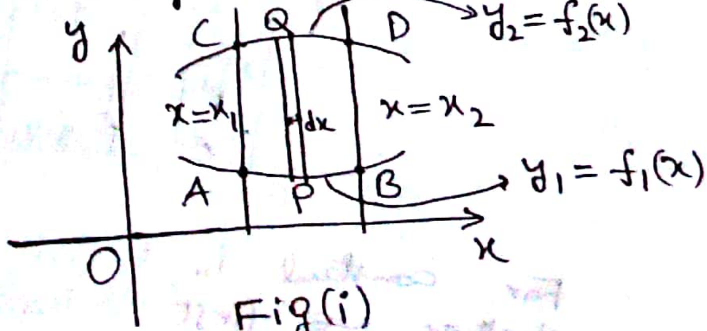
The double integral of $f(x,y)$ over the region A of the xy-plane is written as $\iint_A f(x,y)\, dA$ and expressed as

$$\int_{x_1}^{x_2}\int_{y_1}^{y_2} f(x,y)\, dx\, dy.$$

(i) when y_1, y_2 are functions of x and x_1, x_2 are constants; $f(x,y)$ is first integrated w.r.t. y keeping x fixed between limits y_1, y_2 and then the resulting expression is integrated w.r.t. x within the limits x_1, x_2

$$\text{i.e., } I_1 = \int_{x_1}^{x_2}\left(\int_{y_1}^{y_2} f(x,y)\, dy\right) dx$$

Fig (i) illustrates this process (geometrically illustrated)

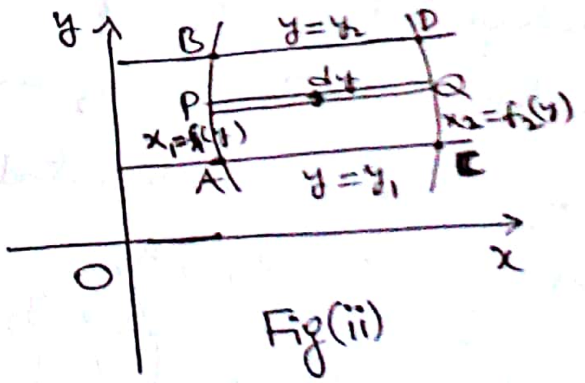


Fig(i)

(ii) when x_1, x_2 are functions of y and y_1, y_2 are constants, $f(x,y)$ is first integrated w.r.t. x keeping y fixed, within the limits x_1, x_2 and the resulting expression is integrated w.r.t. y between the limits y_1, y_2.

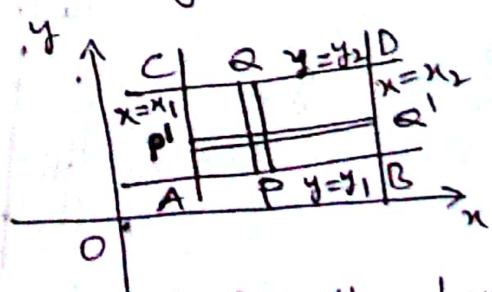
$$\text{i.e., } I_2 = \int_{y_1}^{y_2}\left(\int_{x_1}^{x_2} f(x,y)\,dx\right)dy$$

Fig (ii) illustrates this process geometrically



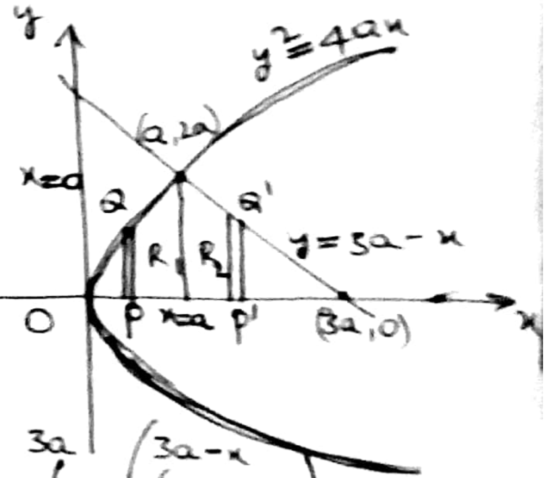
Fig(ii)

(iii) when both pairs of limits are constants, the region of integration is the rectangle ABCD



For constant limits, it hardly matters whether we first integrate w.r.t. x and then w.r.t. y or vice versa.

1. Using double integral Evaluate the area of the region bounded by the curves $y^2 = 4ax$, $x+y=3a$ and $y=0$



Sol: $\text{Area} = \iint\limits_R dx\,dy$

$$= \iint\limits_{R_1} dx\,dy + \iint\limits_{R_2} dx\,dy$$

$$= \int_{x=0}^{a} \left(\int_{y=0}^{2\sqrt{ax}} dy \right) dx + \int_{x=a}^{3a} \left(\int_{y=0}^{3a-x} dy \right) dx$$

$$= \frac{4a^2}{3} + 2a^2 = \frac{10a^2}{3}$$

2. Evaluate $\iint\limits_R xy\,dx\,dy$, where R is the domain bounded by x-axis , $x=2a$ and the curve $x^2=4ay$

3. Evaluate $\iint\limits_R (4xy-y^2)\,dx\,dy$ where **R** is the region (rectangle) bounded by $x=1$, $x=2$, $y=0$ and $y=3$.

Problems

1. Evaluate the following integrals

i) $\int_0^1 \int_0^{x^2} e^{y/x}\, dy\, dx$ ii) $\int_0^5 \int_0^{y^2} x(x^2+y^2)\, dx\, dy$

Sol. i) $\int_0^1 \int_0^{x^2} e^{y/x}\, dy\, dx = \int_{x=0}^1 \left[\int_{y=0}^{x^2} e^{y/x}\, dy \right] dx$

$= \int_0^1 \left[x e^{y/x} \right]_0^{x^2} dx$

$= \int_0^1 x(e^x - 1)\, dx$

$= \int_0^1 (x e^x - x)\, dx$

$= \left[e^x(x-1) - \frac{x^2}{2} \right]_0^1$

$= -\frac{1}{2} - (-1) = \frac{1}{2}$

ii) $\int_0^5 \int_0^{y^2} x(x^2+y^2)\, dx\, dy = \int_{y=0}^5 \left[\int_{x=0}^{y^2} (x^3 + xy^2)\, dx \right] dy$

$= \int_0^5 \left[\frac{x^4}{4} + \frac{x^2 y^2}{2} \right]_0^{y^2} dy$

$= \int_0^5 \left[\frac{(y^2)^4}{4} + \frac{(y^2)^2 y^2}{2} \right] dx$

$$= \int_0^5 \left(\frac{y^8}{4} + \frac{y^6}{2}\right) dy$$

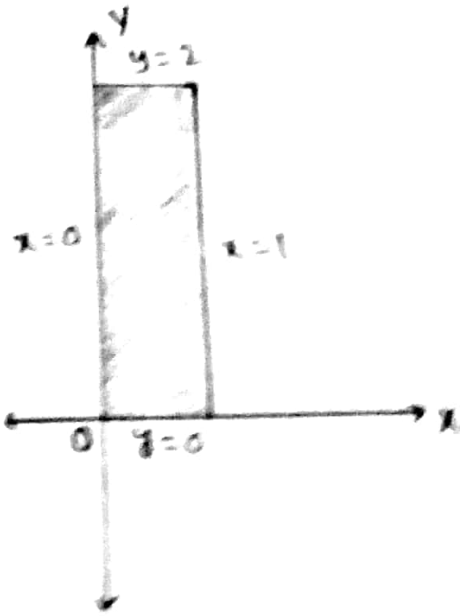
$$= \left(\frac{y^9}{36} + \frac{y^7}{14}\right)_0^5$$

$$= \frac{5^9}{36} + \frac{5^7}{14}$$

$$= \frac{5^7}{4}\left(\frac{25}{9} + \frac{2}{7}\right)$$

2. If A is the area of the rectangular region bounded by the lines $x=0$, $x=1$, $y=0$, $y=2$, evaluate $\int_A (x^2+y)\,dA$

Sol. Here, x varies from 0 to 1 and for each x, y varies from 0 to 2.



$$\therefore \int_A (x^2+y)\,dA = \int_{x=0}^{1}\left[\int_{y=0}^{2}(x^2+y)\,dy\right]dx$$

$$= \int_0^1 \left[x^2 y + \frac{y^2}{2}\right]_0^2 dx$$

$$= \int_0^1 \left(2x^2 + \frac{8}{3}\right) dx$$

$$= \left(2\frac{x^3}{3} + \frac{8}{3}x\right)_0^1$$

$$= \frac{2}{3} + \frac{8}{3}$$

$$= \frac{10}{3}$$

3. Evaluate $\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dy\,dx}{1+x^2+y^2}$ ③

Sol. Given integral $= \int_{x=0}^{1} \left[\int_{y=0}^{\sqrt{1+x^2}} \frac{dy}{(1+x^2)+y^2} \right] dx$

$$= \int_0^1 \left[\int_0^P \frac{dy}{P^2+y^2} \right] dx \text{ where } P = \sqrt{1+x^2}$$

$$= \int_0^1 \left[\frac{1}{P} \mathrm{Tan}^{-1}\left(\frac{y}{P} \right) \right]_0^P dx$$

$$= \int_0^1 \frac{1}{P} \left[\mathrm{Tan}^{-1}(1) - \mathrm{Tan}^{-1}(0) \right] dx$$

$$= \int_0^1 \frac{1}{P} \left[\frac{\pi}{4} - 0 \right] dx$$

$$= \frac{\pi}{4} \int_0^1 \frac{dx}{\sqrt{1+x^2}} \qquad (\because P = \sqrt{1+x^2}$$

$$= \frac{\pi}{4} \left[\log\left(x + \sqrt{x^2+1}\right) \right]_0^1$$

$$= \frac{\pi}{4} \log(1+\sqrt{2})$$

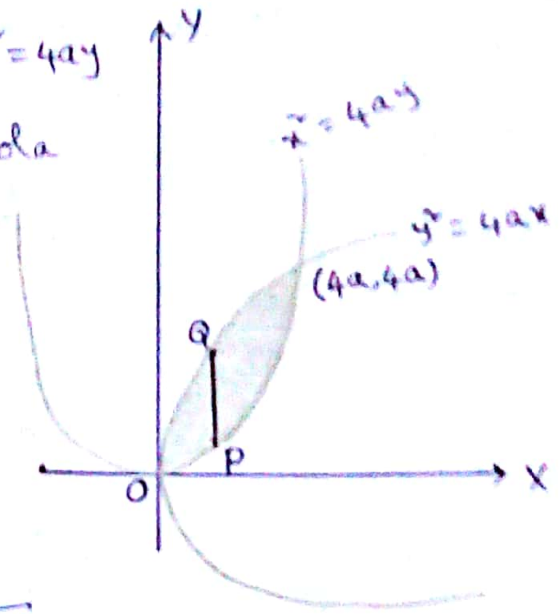
4. Evaluate $\iint_R y\,dx\,dy$, where R is the region bounded by $y^2 = 4ax$ and $x^2 = 4ay$, $a > 0$.

sol. Given parabolas $y^2 = 4ax$ and $x^2 = 4ay$ intersect at $(0,0)$ and $(4a,4a)$. The region bounded by these parabolas is shown in the figure.

In this region, y increases from a point P on the parabola $x^2 = 4ay$ to a point Q on the parabola $y^2 = 4ax$.

At P, $y = \frac{x^2}{4a}$ and at Q $y = \sqrt{4ax}$.

x increases from 0 to $4a$.



$$\therefore \iint_R y\,dx\,dy = \int_{x=0}^{4a} \left[\int_{y=\frac{x^2}{4a}}^{\sqrt{4ax}} y\,dy \right] dx$$

$$= \int_0^{4a} \left[\frac{y^2}{2} \right]_{\frac{x^2}{4a}}^{\sqrt{4ax}} dx$$

$$= \int_0^{4a} \frac{1}{2} \left[4ax - \frac{x^4}{16a^2} \right] dx$$

$$= \frac{1}{2} \left[2ax^2 - \frac{1}{16a^2} \left(\frac{x^5}{5} \right) \right]_0^{4a}$$

$$= \frac{1}{2} \left[32a^3 - \frac{1}{16a^2} \frac{(4a)^5}{5} \right]$$

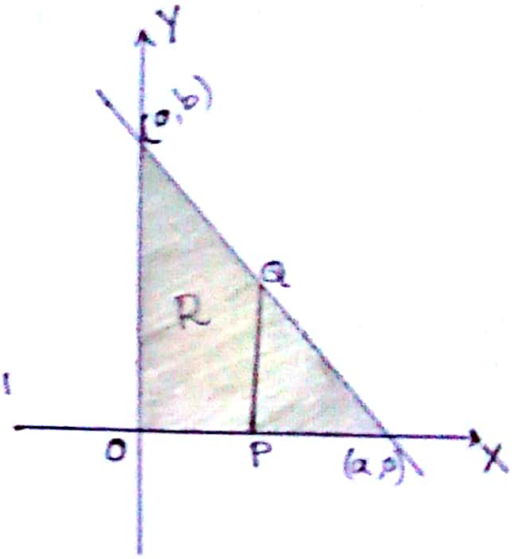
$$= \frac{1}{2} \left[32a^3 - \frac{64a^3}{5} \right]$$

$$= \frac{1}{2} \left[\frac{96a^3}{5} \right] = \frac{48}{5} a^3$$

5. Evaluate $\iint xy\,dx\,dy$ over the region in the positive quadrant for which $\frac{x}{a} + \frac{y}{b} \le 1$ (5)

Sol. The shaded region is the region of integration.

In this region, for a fixed x, y varies from 0 to $b\left(1 - \frac{x}{a}\right)$ $\left(\because \frac{x}{a} + \frac{y}{b} = 1\right)$ and then x varies from 0 to a.



$$\iint_R xy\,dx\,dy = \int_{x=0}^{a} \left[\int_{y=0}^{b(1-\frac{x}{a})} y\,dy \right] x\,dx$$

$$= \int_0^a \left[\frac{y^2}{2} \right]_0^{b(1-\frac{x}{a})} x\,dx$$

$$= \frac{1}{2} \int_0^a b^2 \left(1 - \frac{x}{a}\right)^2 x\,dx$$

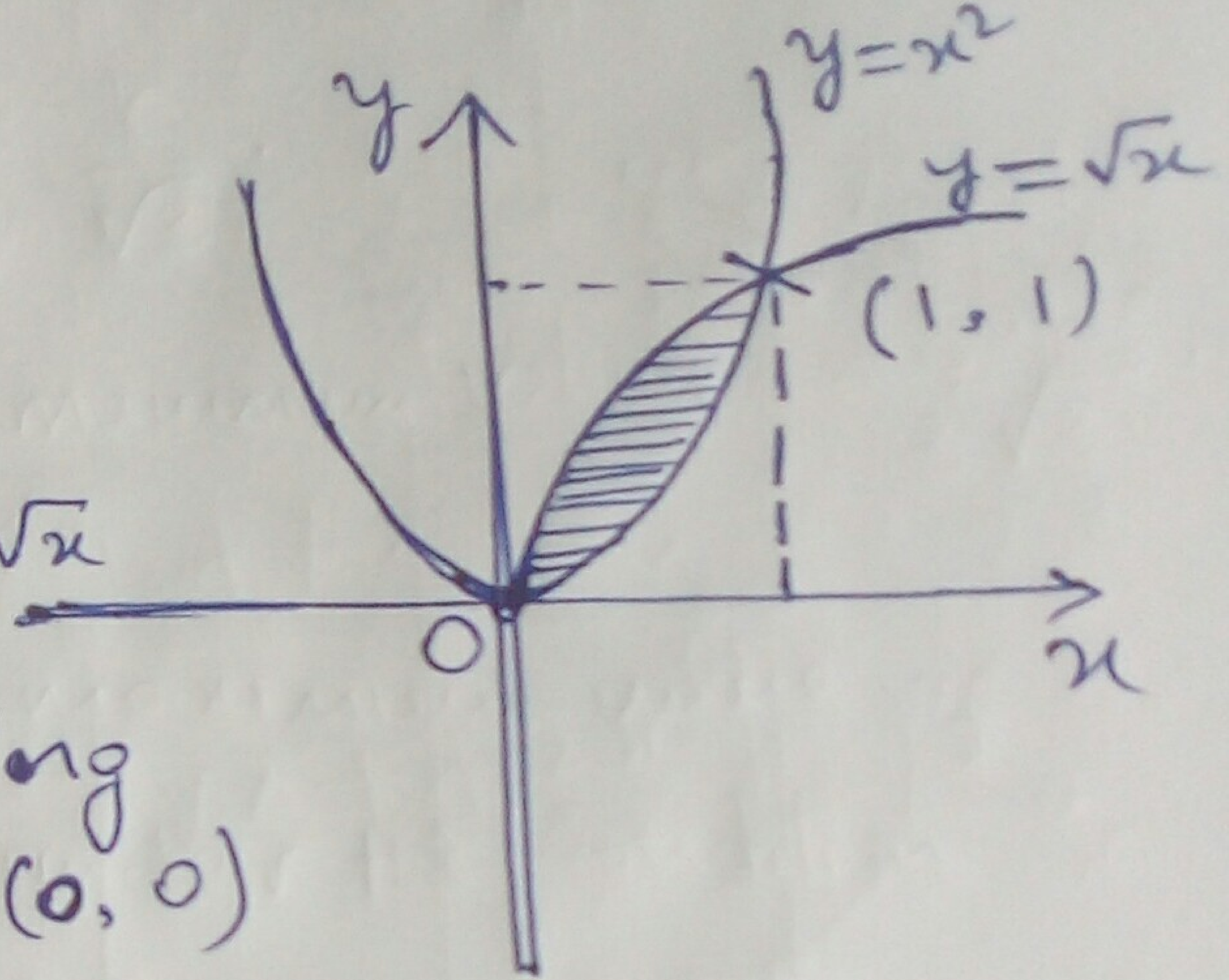
$$= \frac{b^2}{2} \int_0^a \left(1 - \frac{2x}{a} + \frac{x^2}{a^2}\right) x\,dx$$

$$= \frac{b^2}{2} \int_0^a \left(x - \frac{2x^2}{a} + \frac{x^3}{a^2}\right) dx$$

$$= \frac{b^2}{2} \left[\frac{x^2}{2} - \frac{2x^3}{3a} + \frac{x^4}{4a^2} \right]_0^a$$

$$= \frac{b^2}{2} \left[\frac{a^2}{2} - \frac{2a^3}{3a} + \frac{a^4}{4a^2} \right] = \frac{a^2 b^2}{24}$$

① Obtain the Area of the region bounded by $y = x^2$ and $y = \sqrt{x}$ using double integral.



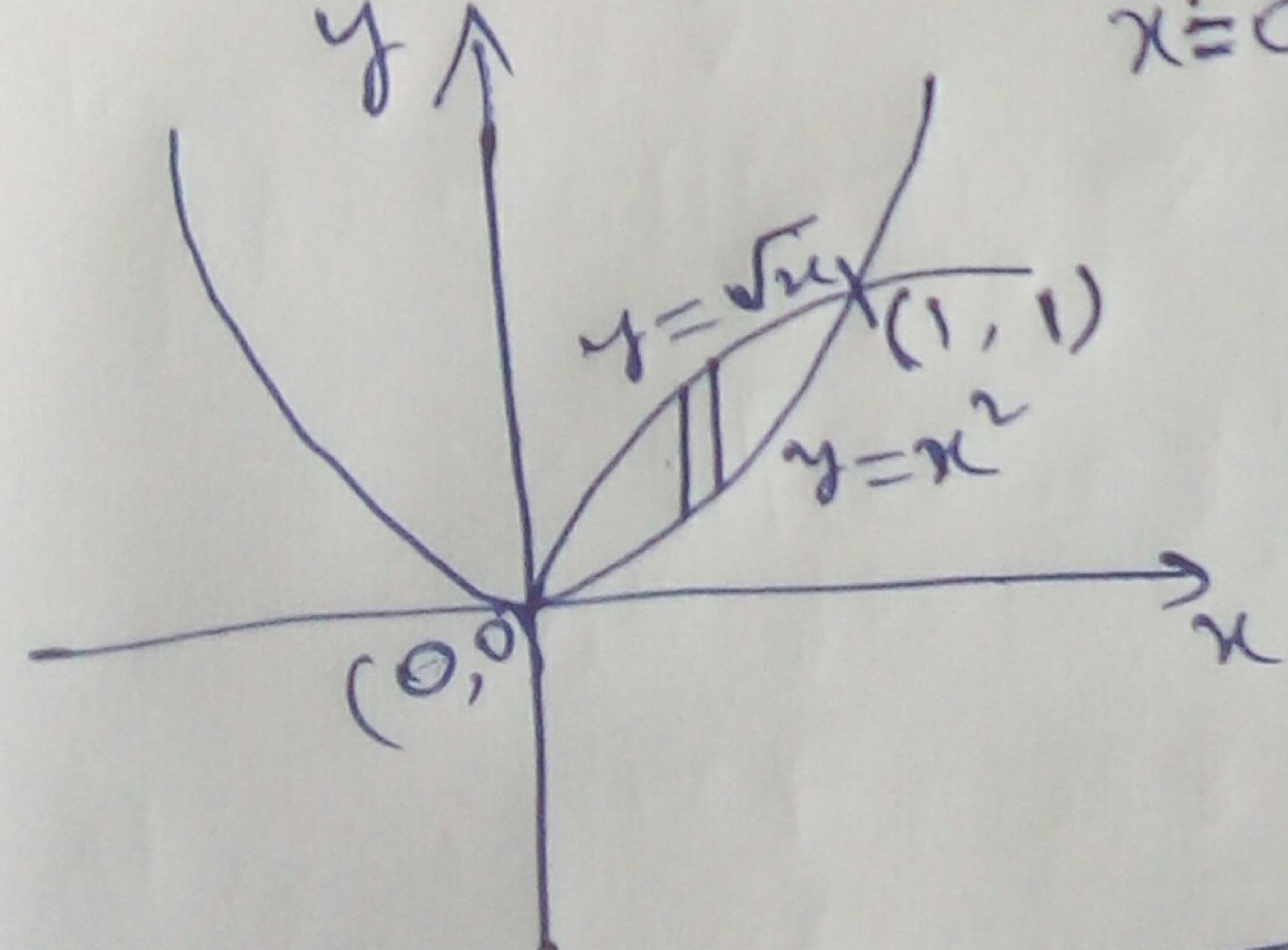
Sol:

Given curves

$y = x^2$ and $y = \sqrt{x}$

are intersecting at the points $(0, 0)$ and $(1, 1)$.

$$\text{Required Area} = \int_{x=0}^{1}\left(\int_{y=x^2}^{\sqrt{x}} dy\right)dx$$



$$= \int_{x=0}^{1} (\sqrt{x} - x^2)\,dx$$

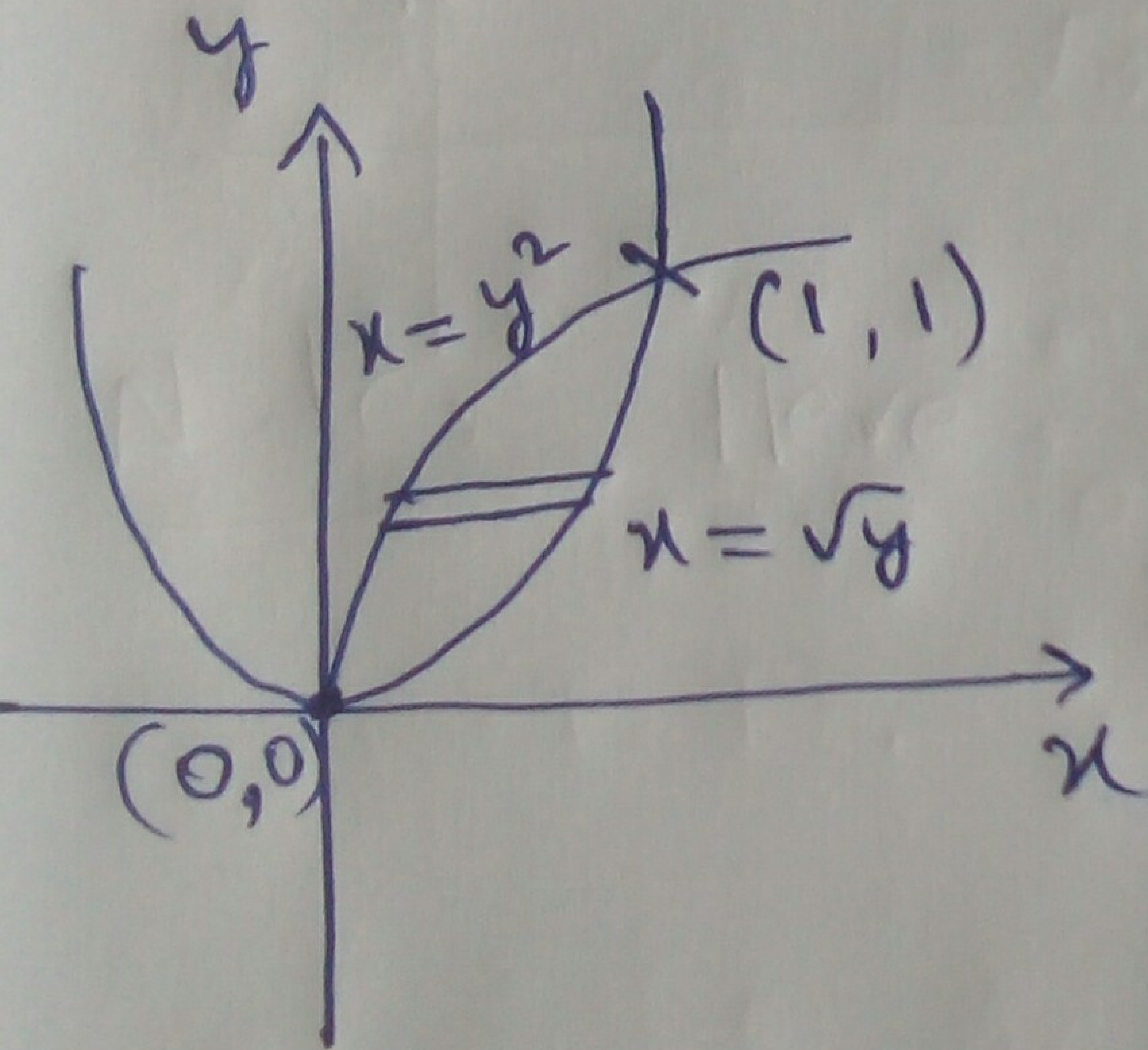
$$= \frac{2}{3}\left(x^{3/2}\right)_0^1 - \left(\frac{x^3}{3}\right)_0^1$$

$$= \frac{2}{3} - \frac{1}{3} = \frac{1}{3} \text{ sq. units}$$

Note: Infact, $\int_{x=0}^{1} (\sqrt{x} - x^2)\, dx$

$$= \int_{x=0}^{1} \left(\int_{y=x^2}^{\sqrt{x}} dy \right) dx$$

OR, Required Area $= \int_{y=0}^{1} \left(\int_{x=y^2}^{\sqrt{y}} dx \right) dy$

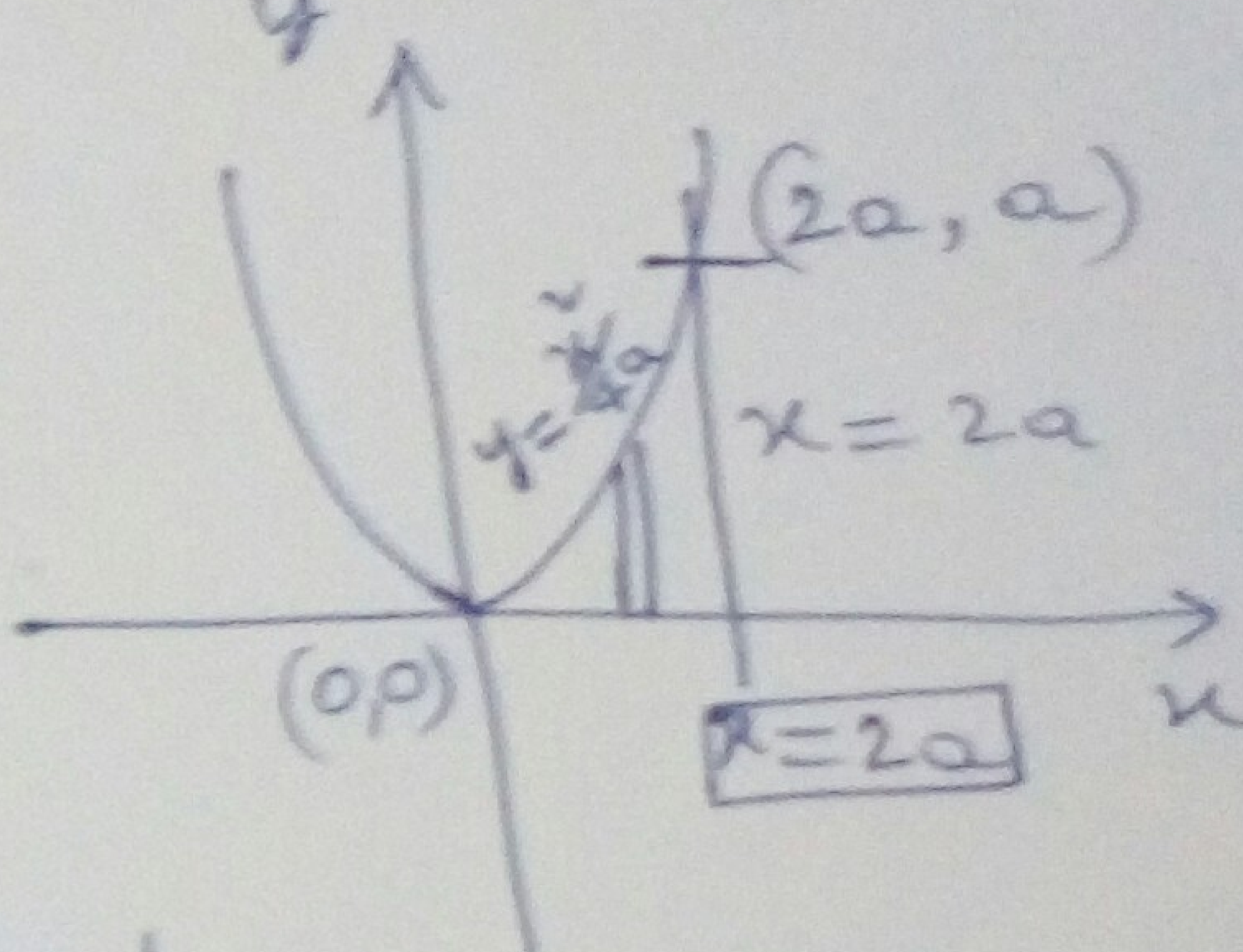


$$= \int_{y=0}^{1} (\sqrt{y} - y^2)\, dy$$

$$= \frac{1}{3} \text{ Squ units.}$$

② Evaluate $\iint_R xy\,dx\,dy$, where R is the region bounded by x-axis, $x=2a$ and the curve $x^2=4ay$.

Sol:



$$\iint_R xy\,dx\,dy = \int_{x=0}^{2a}\left(\int_{y=0}^{y=x^2/4a} xy\,dy\right)dx$$

$$= \int_{x=0}^{2a}\left(x\left[\frac{y^2}{2}\right]_0^{x^2/4a}\right)dx$$

$$= \frac{1}{32a^2}\int_{x=0}^{2a} x^5\,dx = \frac{a^4}{3}$$

Change of Variable in Double integrals

Let $x = f(u,v)$ and $y = g(u,v)$ be the relations between the old variables and the new variables u, v of the new coordinate system.

Then $$\iint_R F(x,y)\,dx\,dy = \iint_R F(f,g)\,|J|\,du\,dv$$

where $J = \dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}$, which is called the Jacobian of the coordinate transformation.

Change of variables from cartesian to polar coordinates:

In this case $u = r$, $v = \theta$ and $x = r\cos\theta$, $y = r\sin\theta$

and $J = \dfrac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$

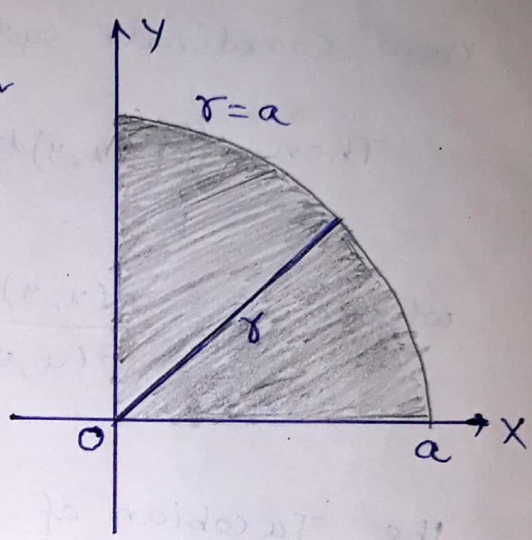
$$= r(\cos^2\theta + \sin^2\theta) = r$$

$$\therefore \iint_R F(x,y)\,dx\,dy = \iint_R F(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

Problems.

1. Evaluate the double integral $\iint xy\,dx\,dy$ over the positive quadrant bounded by the circle $x^2+y^2=a^2$

Sol. In the positive quadrant bounded by the circle $x^2+y^2=a^2$ the radial distance r varies from 0 to a and the polar angle θ varies from 0 to $\frac{\pi}{2}$



$$\iint xy\,dx\,dy = \int_{r=0}^{a}\int_{\theta=0}^{\frac{\pi}{2}} (r\cos\theta)(r\sin\theta)(r\,dr\,d\theta)$$

$$= \int_0^a r^3\,dr \times \int_0^{\frac{\pi}{2}} \sin\theta\cos\theta\,d\theta$$

$$= \left(\frac{r^4}{4}\right)_0^a \times \left(\frac{\sin^2\theta}{2}\right)_0^{\frac{\pi}{2}}$$

$$= \frac{a^4}{4} \times \frac{1}{2}$$

$$= \frac{a^4}{8}.$$

2. Evaluate $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx\,dy$ by changing into polar coordinates. (8)

Sol. In the given integral, both x and y increase from 0 to ∞. Therefore the region of integration is the whole of the first quadrant of the XY-plane. In this quadrant r varies from 0 to ∞ and θ varies from 0 to $\frac{\pi}{2}$.

$$\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx\,dy = \int_{\theta=0}^{\frac{\pi}{2}} \int_{r=0}^{\infty} e^{-r^2} r\,dr\,d\theta$$

$$= \int_{\theta=0}^{\frac{\pi}{2}} \left[\int_{r=0}^{\infty} e^{-r^2} r\,dr\right] d\theta$$

$$= \int_{\theta=0}^{\frac{\pi}{2}} \left[\int_{t=0}^{\infty} \frac{1}{2} e^{-t}\,dt\right] d\theta \quad (\text{put } r^2 = t$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \left(-e^{-t}\right)_0^\infty d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} d\theta = \frac{\pi}{4}$$

Deduction: Using the above result,

$$\frac{\pi}{4} = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx\,dy = \int_0^\infty e^{-x^2} dx \times \int_0^\infty e^{-y^2} dy$$

$$= \left[\int_0^\infty e^{-x^2} dx\right]^2$$

Therefore $\int_0^{\infty} e^{-x^2} dx = \sqrt{\frac{\pi}{4}} = \frac{\sqrt{\pi}}{2}$.

3. Evaluate the integral $I = \int_0^a \int_0^{\sqrt{a^2-x^2}} y^2 \sqrt{x^2+y^2}\, dy\, dx$ by transforming to polar coordinates.

Sol. In this integral, x increases from 0 to a and for each x, y varies from 0 to $\sqrt{a^2-x^2}$.

Thus, the lower value of y lies on X axis and the upper value of y lies on the curve $y = \sqrt{a^2-x^2}$ i.e $x^2+y^2 = a^2$ which is a circle of radius a centred at the origin.

The region of integration is the region in the first quadrant bounded by the circle $x^2+y^2=a^2$.

Here θ varies from 0 to $\frac{\pi}{2}$ and r varies from 0 to a.

$$I = \int_0^a \int_0^{\sqrt{a^2-x^2}} y^2 \sqrt{x^2+y^2}\, dx\, dy = \int_{\theta=0}^{\frac{\pi}{2}} \int_{r=0}^{a} (r^2 \sin^2\theta)\, r\, (r\, dr\, d\theta)$$

$$= \int_{r=0}^{a} r^4\, dr \times \int_{\theta=0}^{\frac{\pi}{2}} \sin^2\theta\, d\theta$$

$$= \left(\frac{r^5}{5}\right)_0^a \times \frac{1}{2} \cdot \frac{\pi}{2}$$

$$= \frac{a^5}{5} \times \frac{\pi}{4}$$

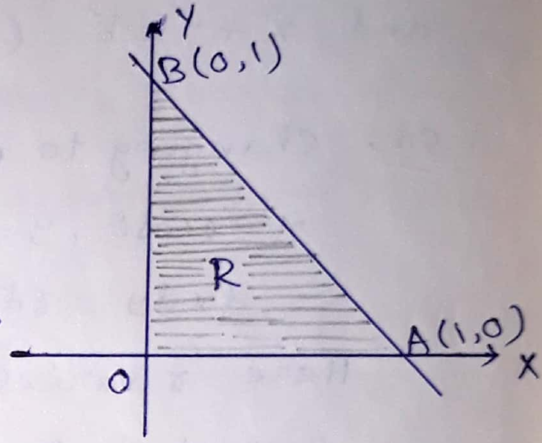
$$= \frac{\pi}{20} a^5.$$

4. By using the transformation $x+y=u$, $y=uv$ (10)

show that $\int_0^1 \int_0^{1-x} e^{y/x+y}\, dy\,dx = \frac{1}{2}(e-1)$

Sol: The region of integration in the given integral is $y=0$, $y=1-x$, $x=0$ and $x=1$ i.e the triangle OAB.

Given transformation is



$x+y=u \xrightarrow{} 1)$ and $y=uv \longrightarrow 2)$

Substitue 2) in 1), we get

$x+uv=u$ i.e $x=u(1-v) \longrightarrow 3$

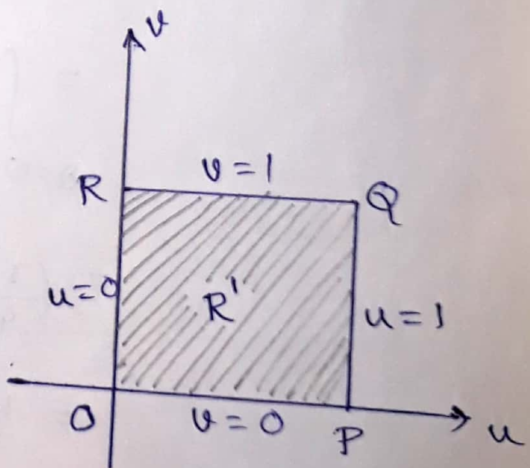
Now $y=0 \Rightarrow uv=0 \Rightarrow u=0$ or $v=0$

$y=1-x \Rightarrow x+y=1 \Rightarrow u=1$

$x=0 \Rightarrow u=0$ or $v=1$ [using 3)

∴ The region R is transformed to R' where R' is the square OPQR in uv plane

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$



$$= \begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix}$$

$$= u-uv+uv = u$$

$$\therefore \int_0^1 \int_0^{1-x} e^{y/x+y}\, dx\,dy = \iint_{R'} e^{uv/u}\, |J|\, du\,dv$$

$$= \int_{v=0}^{1} \int_{u=0}^{1} e^{v}\, u\, du\,dv$$

$$= \int_0^1 e^v dv \int_0^1 u\, du$$

$$= (e^v)_0^1 \left(\frac{u^2}{2}\right)_0^1$$

$$= \frac{1}{2}(e-1)$$

5. By changing into polar coordinates evaluate $\iint \frac{x^2y^2}{x^2+y^2}dx\,dy$ over the annular region between the circles $x^2+y^2=a^2$ and $x^2+y^2=b^2$ $(b>a)$.

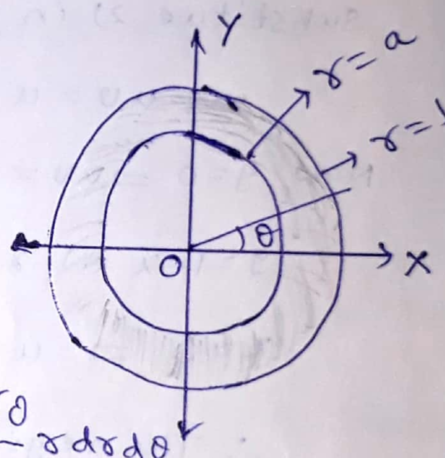
Sol. Changing to polar coordinates

$x = r\cos\theta$, $y = r\sin\theta$

$dx\,dy = r\,dr\,d\theta$

Here r varies from a to b

θ varies from 0 to 2π.



$$\iint \frac{x^2y^2}{x^2+y^2}dx\,dy = \int_{\theta=0}^{2\pi}\int_{r=a}^{b} \frac{r^2\sin^2\theta\, r^2\cos^2\theta}{r^2} r\,dr\,d\theta$$

$(\because x^2+y^2=r^2$

$$= \int_{\theta=0}^{2\pi}\int_{r=a}^{b} r^3\cos^2\theta\sin^2\theta\, dr\,d\theta$$

$$= \left(\frac{r^4}{4}\right)_a^b \int_0^{2\pi}\sin^2\theta\cos^2\theta\, d\theta = \frac{b^4-a^4}{4}\int_0^{2\pi}\frac{(\sin 2\theta)^2}{4}d\theta$$

$$= \frac{b^4-a^4}{16}\int_0^{2\pi}\sin^2 2\theta\, d\theta = \frac{b^4-a^4}{32}\int_0^{2\pi}(1-\cos 4\theta)d\theta$$

$$= \frac{b^4-a^4}{32}\left(\theta - \frac{\sin 4\theta}{4}\right) = \frac{b^4-a^4}{32}[2\pi - 0]$$

$$= \frac{\pi}{16}(b^4-a^4)$$

1. i)

$$I = \int_1^4 \int_0^{\sqrt{4-x}} xy \cdot dy \cdot dx$$

$$I = \int_{x=1}^{4} \left(\int_{y=0}^{\sqrt{4-x}} xy \cdot dy \right) \cdot dx$$

$$I = \int_{x=1}^{4} \left(x \left[\frac{y^2}{2} \right]_0^{\sqrt{4-x}} \right) \cdot dx$$

$$I = \int_{x=1}^{4} \left[x \left(\frac{4-x}{2} \right) \right] \cdot dx$$

$$I = \int_{x=1}^{4} \left(\frac{4x - x^2}{2} \right) \cdot dx$$

$$I = \frac{1}{2} \int_{x=1}^{4} (4x - x^2) \cdot dx$$

$$I = \frac{1}{2} \left[\frac{4x^2}{2} - \frac{x^3}{3} \right]_1^4$$

$$I = \frac{1}{2} \left[2x^2 - \frac{x^3}{3} \right]_1^4$$

$$I = \frac{1}{2} \left[2(16-1) - \frac{1}{3}(64-1) \right]$$

$$I = \frac{1}{2} [30 - 21]$$

$$I = \boxed{\frac{9}{2}}$$

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ii) Let $y/x = t$

$dy = x\,dt$

UL:- $y = x^2 \Rightarrow t = x$

LL:- $y = 0 \Rightarrow t = 0$

$$I = \int_{x=0}^{1} \left(\int_{t=0}^{x} e^t \times x\,dt \right) dx$$

$$I = \int_{x=0}^{1} x \left(\int_{t=0}^{x} e^t \cdot dt \right) \cdot dx$$

$$I = \int_{x=0}^{1} x \left[e^t \right]_0^x \cdot dx$$

$$I = \int_{x=0}^{1} x \cdot e^x \cdot dx - \int_{x=0}^{1} x \cdot dx \Rightarrow (1)$$

Let $I_1 = \int_{x=0}^{1} x \cdot e^x \cdot dx$

Using ILATE rule, i.e;

$$\int_{x=a}^{b} u \cdot v' \cdot dx = [uv]_{x=a}^{b} - \int_{x=a}^{b} u' \cdot v \cdot dx$$

$v' = e^x \qquad u = x$

$v = e^x \qquad u' = 1$

$$I_1 = \left[x \cdot e^x \right]_0^1 - \int_{x=0}^{1} e^x \cdot dx$$

$$I_1 = e - \left[e^x \right]_0^1$$

$$I_1 = e - (e - 1) \Rightarrow I_1 = 1$$

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Subs: I_1 in equn ①

$$\Rightarrow I = 1 - \left[\frac{x^2}{2}\right]_0^1$$

$$I = 1 - \frac{1}{2}$$

$$I = \boxed{\frac{1}{2}}$$

iii) $$I = \int_{y=0}^{1}\int_{x=0}^{\sqrt{1-y^2}} x^3 y \cdot dx \cdot dy$$

$$I = \int_{y=0}^{1} y\left(\int_{x=0}^{\sqrt{1-y^2}} x^3 \cdot dx\right)\cdot dy$$

$$I = \int_{y=0}^{1} y\left[\frac{x^4}{4}\right]_0^{\sqrt{1-y^2}} \cdot dy$$

$$I = \frac{1}{4}\int_{y=0}^{1} y\,(1-y^2)^2 \cdot dy$$

$$4I = \int_{y=0}^{1} y(1+y^4-2y^2)\cdot dy$$

$$4I = \int_{y=0}^{1} (y+y^5-2y^3)\cdot dy$$

$$4I = \left[\frac{y^2}{2} + \frac{y^6}{6} - \frac{y^4}{2}\right]_0^1$$

$$4I = \cancel{\frac{1}{2}} + \frac{1}{6} - \cancel{\frac{1}{2}}$$

$$I = \boxed{\frac{1}{24}}$$

iv) $I = \int_{x=0}^{1}\left(\int_{y=0}^{1} \frac{1}{\sqrt{1-x^2}} \times \frac{1}{\sqrt{1-y^2}} \cdot dy\right) dx$

$I = \int_{x=0}^{1} \frac{dx}{\sqrt{1-x^2}} \times \int_{y=0}^{1} \frac{dy}{\sqrt{1-y^2}}$

$I = \left(\int_{x=0}^{1} \frac{dx}{\sqrt{1-x^2}}\right)^2$

$\sqrt{I} = \int_{x=0}^{1} \frac{dx}{\sqrt{1-x^2}}$

Let $x = \sin\theta$
$dx = \cos\theta \cdot d\theta$

UL:- $x = 1 \rightarrow \theta = \pi/2$

LL:- $x = 0 \rightarrow \theta = 0$

$\sqrt{I} = \int_{\theta=0}^{\pi/2} \frac{\cancel{\cos\theta}}{\cancel{\cos\theta}} \cdot d\theta$

$\sqrt{I} = \int_{\theta=0}^{\pi/2} d\theta$

$\sqrt{I} = [\theta]_0^{\pi/2}$

$\sqrt{I} = \frac{\pi}{2}$

$I = \boxed{\frac{\pi^2}{4}}$

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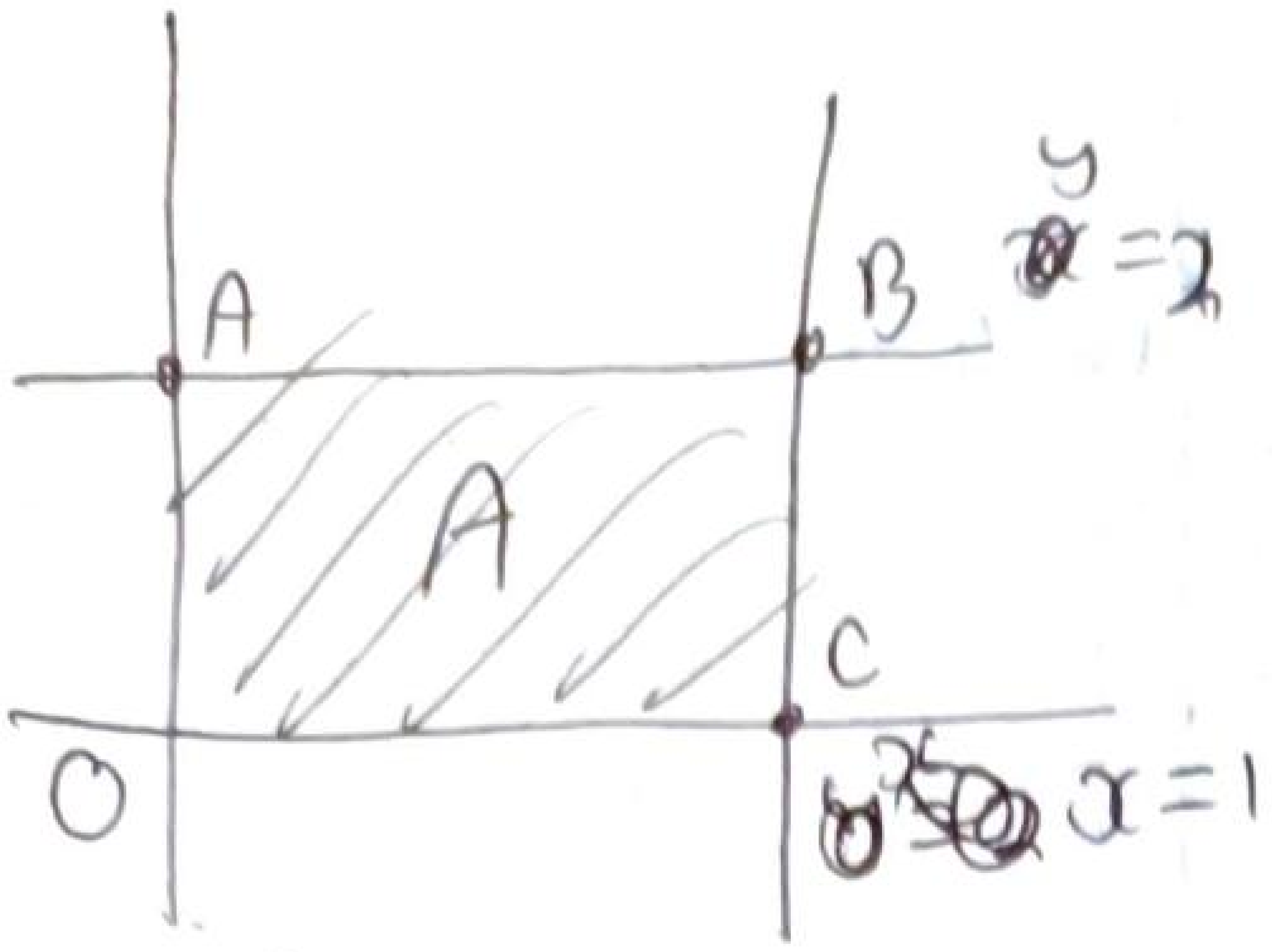
v)



A = Area of the region bounded by the rectangle OABC

$$I = \int_{x=0}^{1} \int_{y=0}^{2} (x^2+y^2) \cdot dx \cdot dy$$

$$I = \int_{x=0}^{1} \left(\int_{y=0}^{2} (x^2+y^2) \cdot dy \right) \cdot dx$$

$$I = \int_{x=0}^{1} \left[x^2 y + \frac{y^3}{3} \right]_0^2 \cdot dx$$

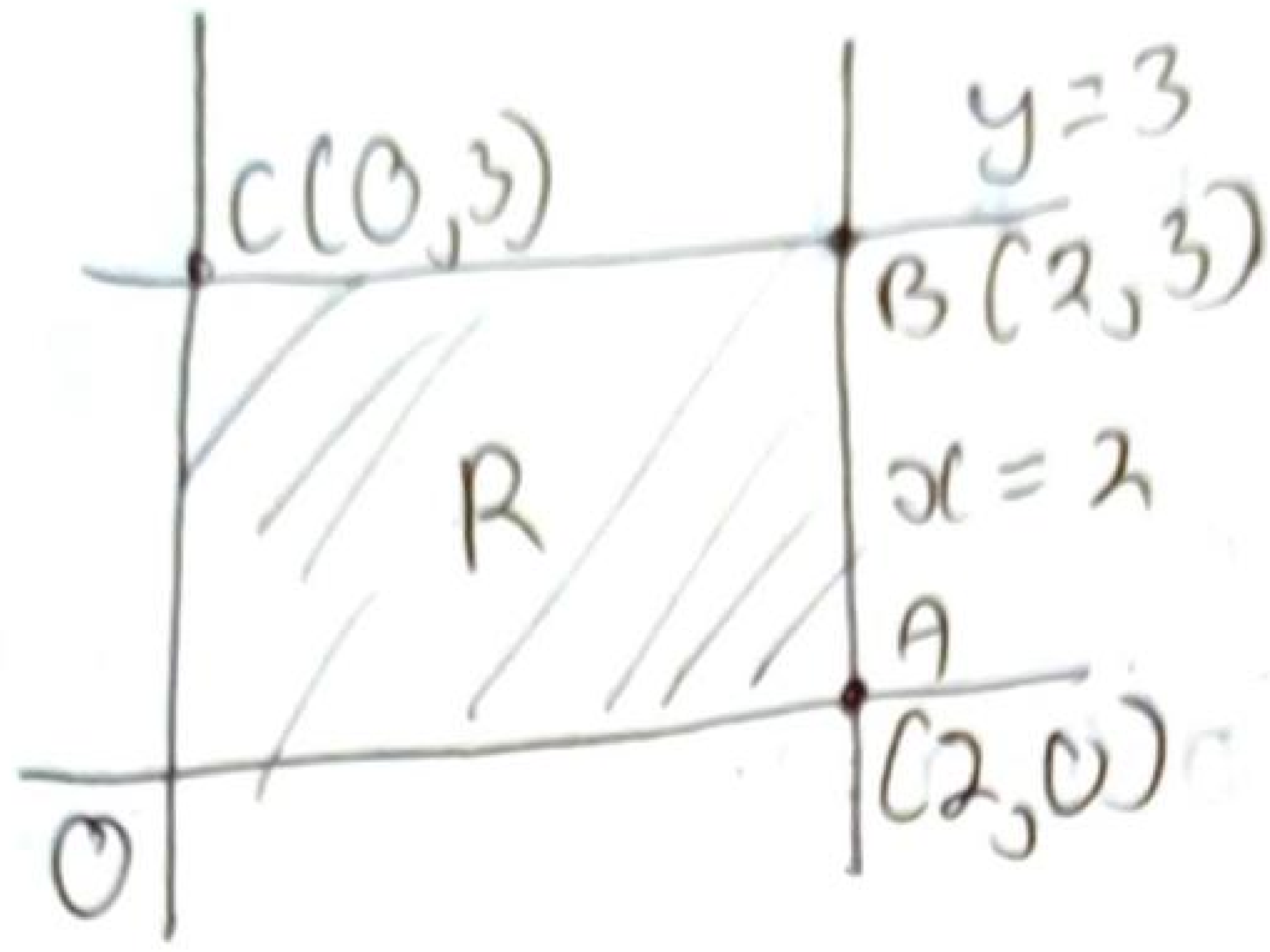
$$I = \int_{x=0}^{1} \left(2x^2 + \frac{8}{3} \right) \cdot dx$$

$$I = \left[\frac{2x^3}{3} + \frac{8x}{3} \right]_0^1$$

$$I = \frac{2}{3} + \frac{8}{3}$$

$$I = \boxed{\frac{10}{3}}$$

vi)



$$I = \iint_R x^2 y^2 \cdot dx \cdot dy$$

$$I = \int_{x=0}^{2} \left(\int_{y=0}^{3} x^2 y^2 \cdot dy \right) \cdot dx$$

$$I = \int_{x=0}^{2} \left[x^2 \frac{y^3}{3} \right]_0^3 \cdot dx$$

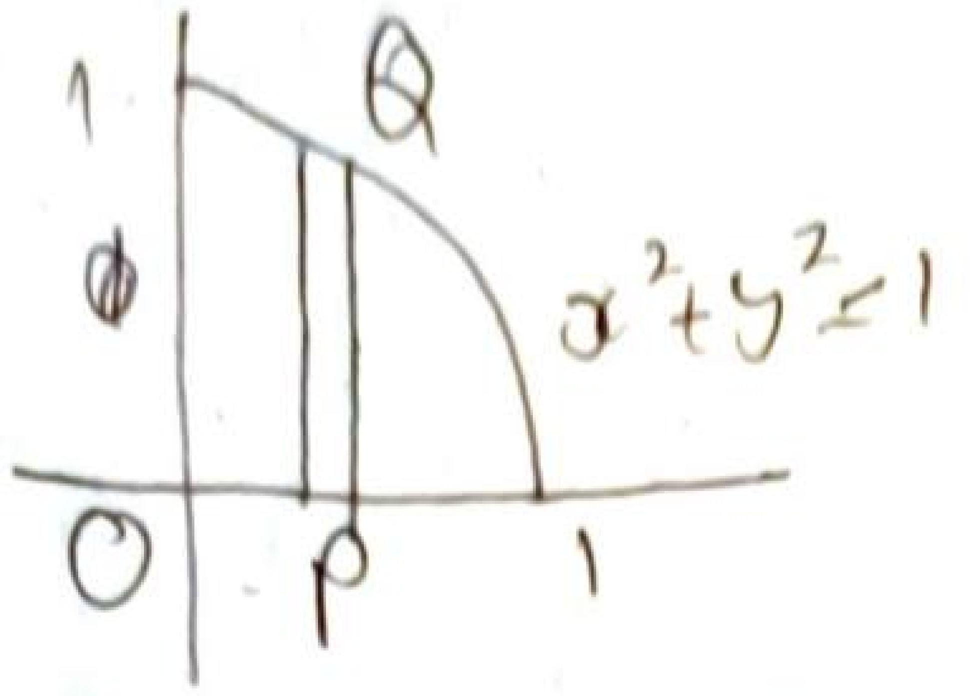
$$I = \int_{x=0}^{2} 9x^2 \cdot dx$$

$$I = \left[\frac{9x^3}{3} \right]_0^2$$

$$I = \left[3x^3 \right]_0^2$$

$$I = 24 \text{ sq. units}$$

3.



$$I = \int_{x=0}^{1} \left(\int_{y=0}^{\sqrt{1-x^2}} \frac{xy}{\sqrt{1-y^2}} \cdot dy \right) \cdot dx$$

Let $\sqrt{1-y^2} = t$

$t^2 = 1 - y^2$

$2t\,dt = -2y \cdot dy$

$t\,dt = y\,dy$

UL:- $y = \sqrt{1-x^2} \rightarrow t = \sqrt{1-(\sqrt{1-x^2})^2}$

$t = \sqrt{\cancel{1} - (\cancel{1} - x^2)}$

$t = \sqrt{x^2}$

$t = x$

LL:- $y = 0 \rightarrow t = 1$

$$I = \int_{x=0}^{1} \left(\int_{t=1}^{x} -\frac{x \cdot t\,dt}{\cancel{t}} \right) \cdot dx$$

$$I = \int_{x=0}^{1} \left(\int_{t=x}^{1} x\,dt \right) dx$$

$$I = \int_{x=0}^{1} \left(x\,[t]_x^1 \right) \cdot dx$$

$$I = \int_{x=0}^{1} x(1-x) \cdot dx$$

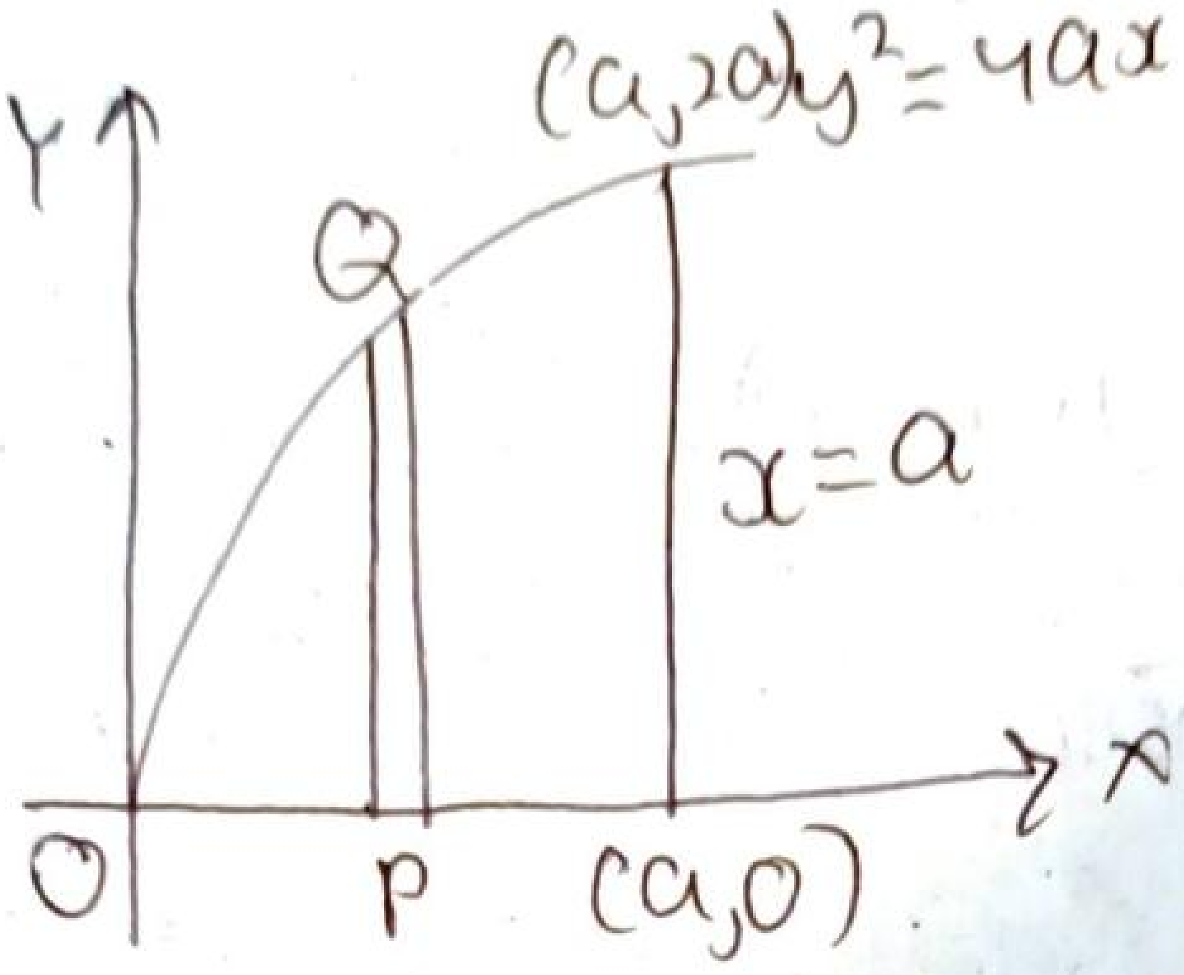
$$I = \int_{x=0}^{1} (x - x^2) \cdot dx$$

$$I = \left[x - \frac{x^3}{3} \right]_0^1$$

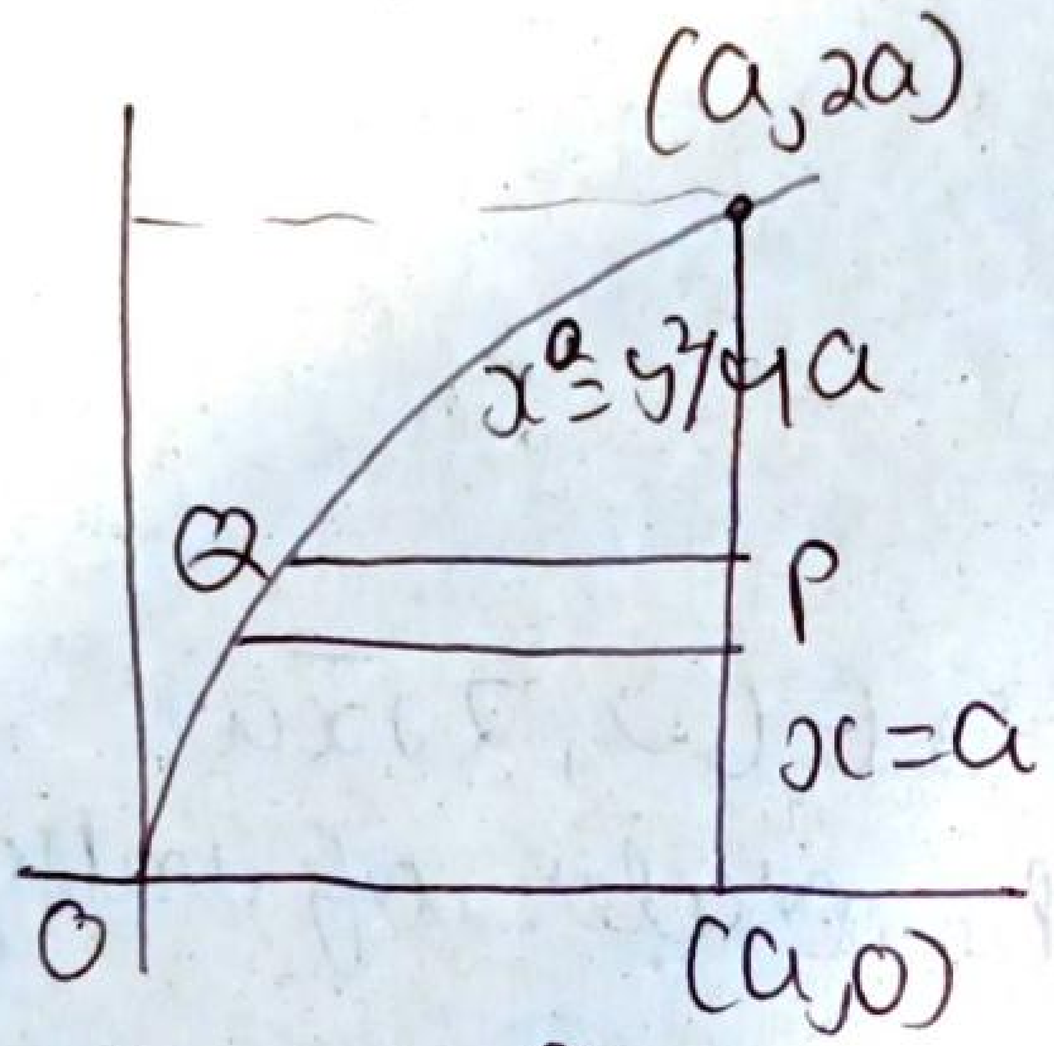
$$I = 1 - \frac{1}{3} = \boxed{\frac{2}{3}}$$

4.

$$I = \int_{x=0}^{a} \left(\int_{y=0}^{2\sqrt{xa}} x^2 \, dy \right) dx$$



Given



Required

$$I = \int_{y=0}^{2a} \left(\int_{x=y^2/4a}^{a} x^2 \cdot dx \right) \cdot dy$$

$$I = \int_{y=0}^{2a} \left[\frac{x^3}{3} \right]_{y^2/4a}^{a} \cdot dy$$

$$I = \frac{1}{3} \int_{y=0}^{2a} \left(a^3 - \frac{y^6}{2^6 a^3} \right) \cdot dy$$

$$I = \frac{1}{3} \left[a^3 y - \frac{y^7}{7 \cdot 2^6 a^3} \right]_0^{2a}$$

$$I = \frac{1}{3} \left[2a^4 - \frac{2^7 \cdot a^7}{7 \cdot 2^6 \cdot a^3} \right] = \frac{1}{3} \left[2a^4 - \frac{2a^4}{7} \right]$$

$$I = \frac{2a^4}{\cancel{3}} \times \frac{\cancel{6}^{\,2}}{7}$$

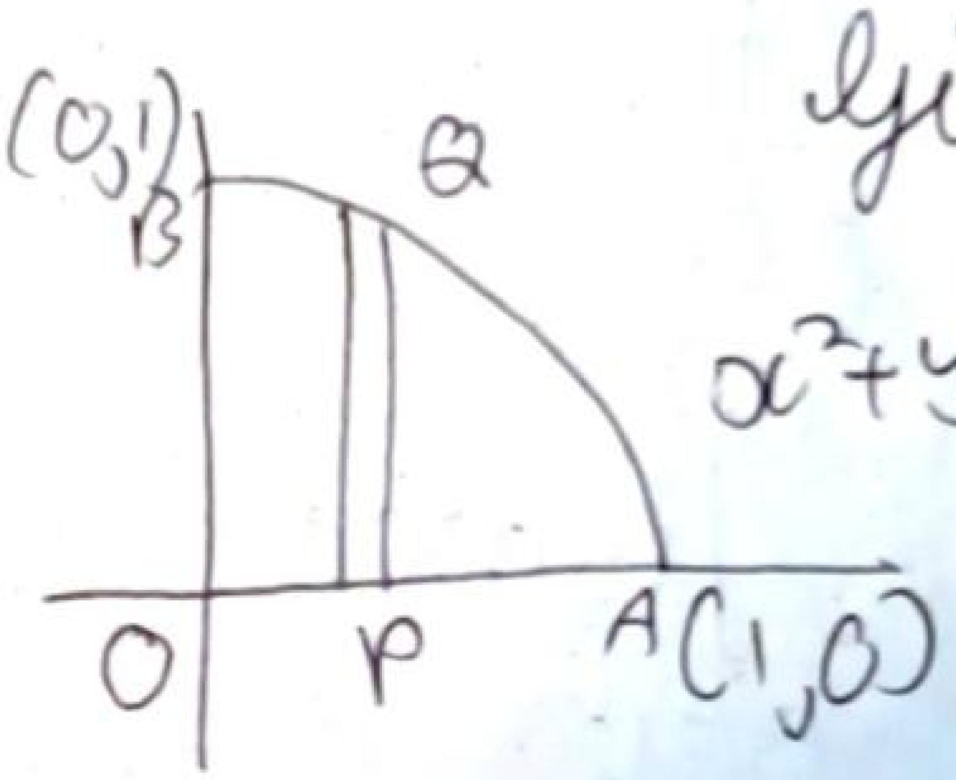
$$\boxed{I = \frac{4a^4}{7}}$$

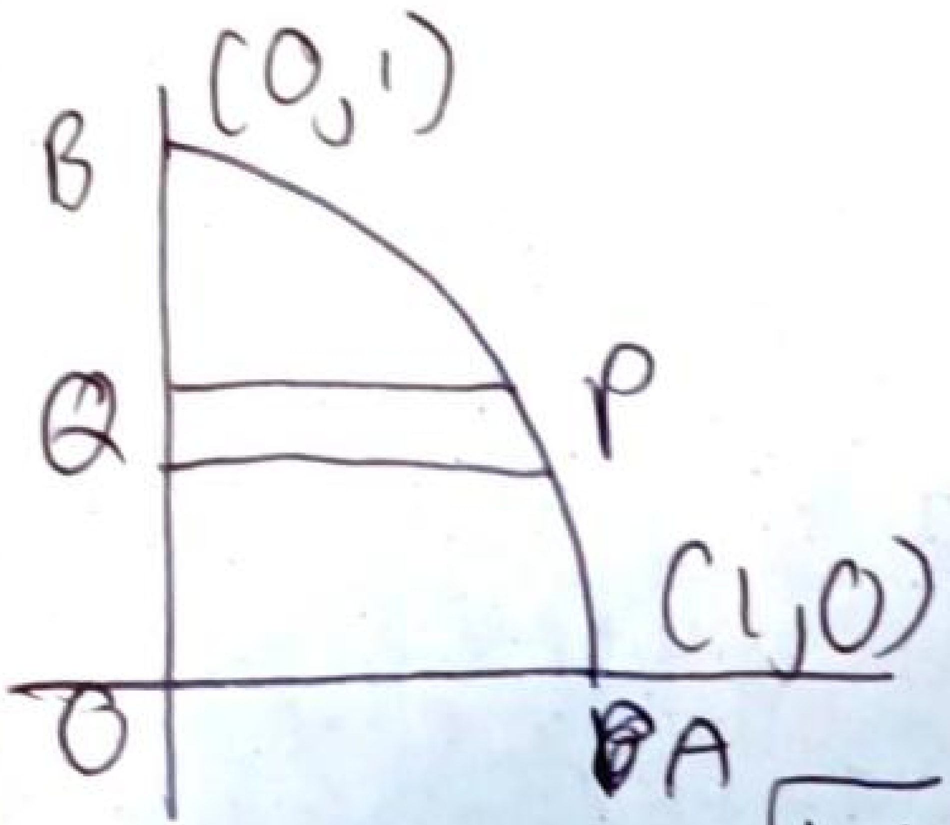
5. $I = \int_{x=0}^{1} \int_{y=0}^{\sqrt{1-y^2}} y^2 \cdot dy \cdot dx$

Sol

$y = \sqrt{1-x^2}$

$y^2 + x^2 = 1$





$I = \int_{y=0}^{1} \left(\int_{x=0}^{\sqrt{1-y^2}} y^2 \cdot dx \cdot \right) dy$

$I = \int_{y=0}^{1} y^2 \left(\int_{x=0}^{\sqrt{1-y^2}} dx \right) \cdot dy$

$I = \int_{y=0}^{1} \left(y^2 \left[x \right]_0^{\sqrt{1-y^2}} \right) \cdot dy$

$I = \int_{y=0}^{1} \left(y^2 \times \sqrt{1-y^2} \right) \cdot dy$

Let $y = \cos\theta$ $\quad dy = -\sin\theta \cdot d\theta$

UL:- $y = 1 \rightarrow \theta = 0$

LL:- $y = 0 \rightarrow \theta = \pi/2$

$\cancel{I = -\int_{y=1}^{0} \cos^2\theta \times \sin^2\theta}$

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$$I = -\int_{\theta=\pi/2}^{0} \sin^2\theta \cdot \cos^2\theta \cdot d\theta$$

$$4I = \int_{\theta=0}^{\pi/2} 4\sin^2\theta \cdot \cos^2\theta \cdot d\theta$$

$$4I = \int_{\theta=0}^{\pi/2} (\sin 2\theta)^2 \cdot d\theta$$

$$4I = \int_{0}^{\pi/2} \left[\frac{1-\cos 4\theta}{2}\right] d\theta$$

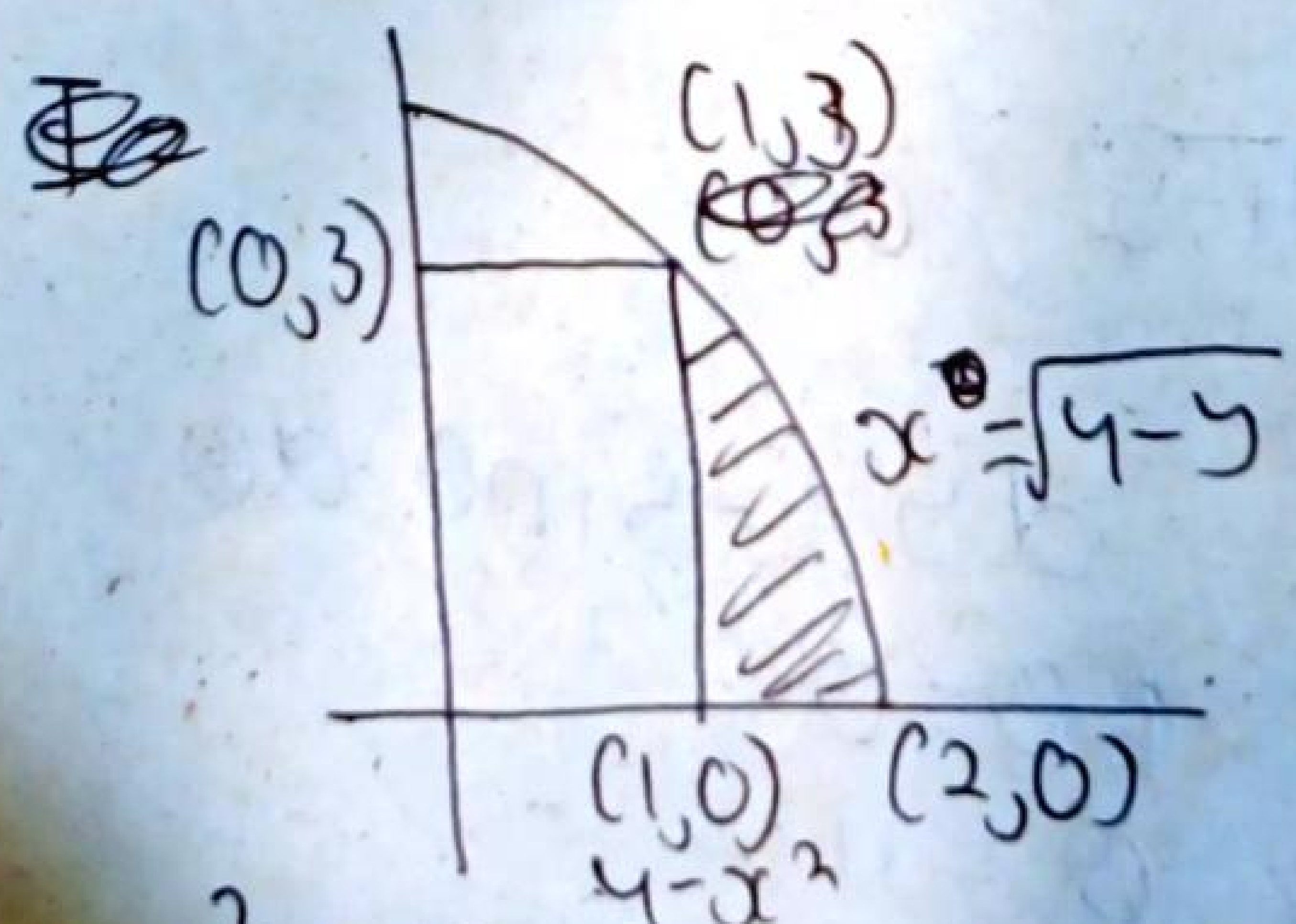
$$8I = \left[\theta - \frac{\sin 4\theta}{4}\right]_0^{\pi/2}$$

$$8I = \frac{\pi}{2}$$

$$I = \boxed{\frac{\pi}{16}}$$

6.

$$I = \int_{y=0}^{3} \int_{x=1}^{\sqrt{4-y}} (x+y) \cdot dx \cdot dy$$



$$I = \int_{x=1}^{2} \left(\int_{y=0}^{4-x^2} (x+y) \cdot dy \right) dx$$

$$I = \int_{x=1}^{2} \left(\left[xy + \frac{y^2}{2} \right]_0^{4-x^2} \right) \cdot dx$$

$$I = \int_{x=1}^{2} \left(x(4-x^2) + \frac{(4-x^2)^2}{2} \right) \cdot dx$$

$$I = \int_{x=1}^{2} \left(4x - x^3 + \frac{16 + x^4 - 8x^2}{2} \right) \cdot dx$$

$$I = \int_{1}^{2} \left(\frac{x^4 - 2x^3 - 8x^2 + 8x + 16}{2} \right) \cdot dx$$

$$\cancel{2I = \left(2^4 - 2^4 - 2^3 \times 2^2 + 2^3 \times 2 + 2^4 \right) - \left(1 - 2 - 8 + 8 + 16 \right)}$$

$$\cancel{2I = \left(-2^5 + 2^4 + 2^4 \right) - (15)}$$

$$2I =$$

$$2I = \left[\frac{x^5}{5} - 2\frac{x^4}{2} - \frac{8x^3}{3} + 4x^2 + 16x \right]_1^2$$

$$2I = \left(\frac{32}{5} - 8 - \frac{64}{3} + 16 + 32 \right) - \left(\frac{1}{5} - \frac{1}{2} - \frac{8}{3} + 4 + 16 \right)$$

$$\cancel{2I = \left(\frac{32}{5} - \frac{64}{3} + 40 \right) - \left(20 - \frac{7}{5} - \frac{1}{2} \right)}$$

$$\cancel{2I = \left(\frac{96 - 320}{15} + 40 \right) -}$$

$$2I = 20 + \frac{93 - 280}{15} + \frac{1}{2}$$

$$2I = 20 - \frac{187}{15} + \frac{1}{2}$$

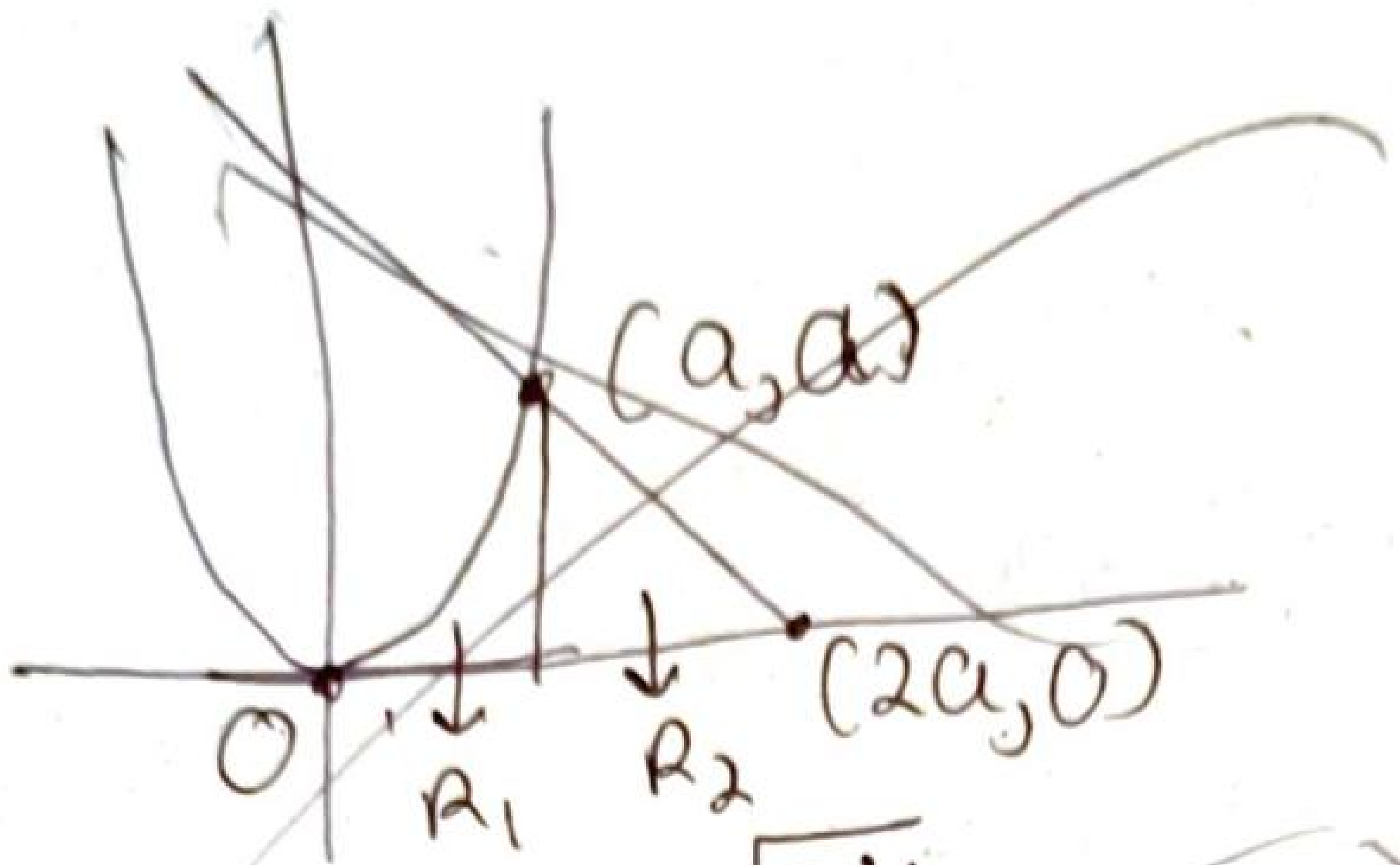
$$2I = \frac{31}{5} + 40 - 20 - \frac{64}{3} + \frac{8}{3} + \frac{1}{2}$$

$$2I = \frac{41}{2} - \frac{185}{15}$$

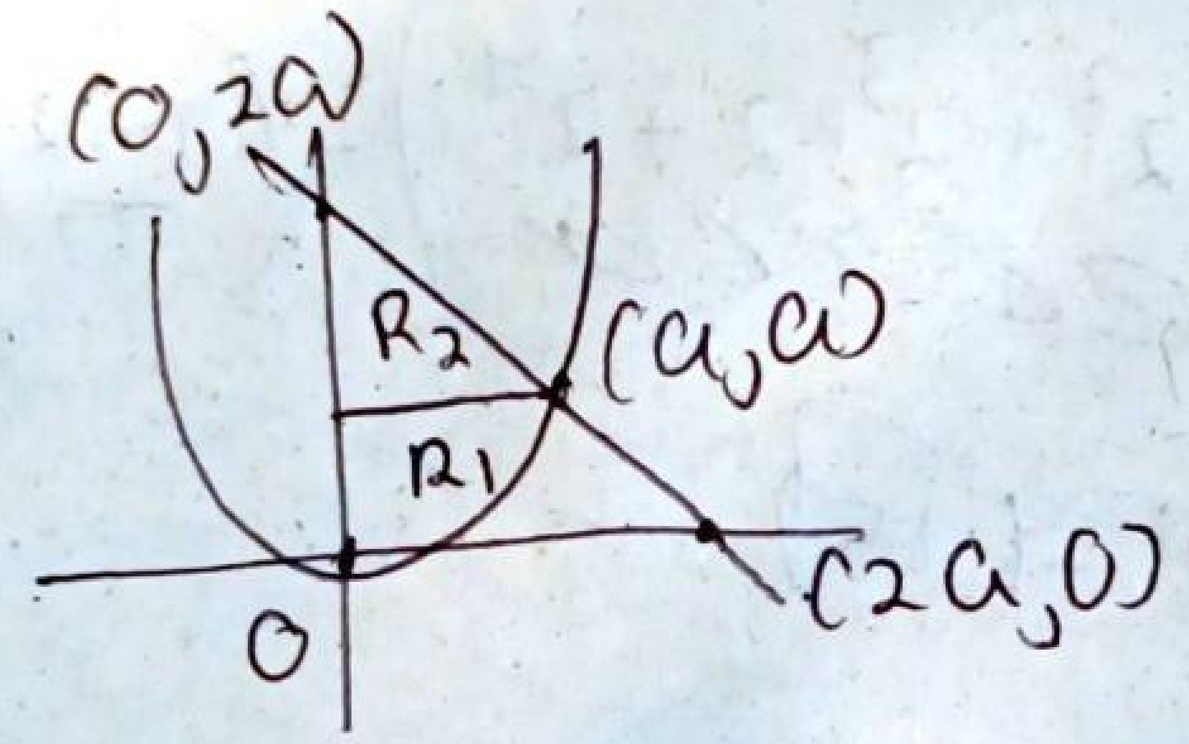
$$2I = 20 + \frac{31}{5} - \frac{56}{3} + \frac{1}{2}$$

$$2I = \frac{245}{30} \quad I = \frac{245}{60}$$

7. $I = \int_{x=0}^{a} \int_{y=x^2/a}^{2a-x} xy \cdot dy \cdot dx$



$\cancel{I = \int_{y=0}^{a} \left(\int_{x=0}^{\sqrt{ay}} xy \cdot dx \right) \cdot dy + \int_{y=0}^{a} \left(\int_{x=0} \right.}$



$I = \int_{y=0}^{a} \left(\int_{x=0}^{\sqrt{ay}} xy \cdot dx \right) dy + \int_{y=a}^{2a} \left(\int_{x=0}^{2a-y} xy \cdot dx \right) d$

8. $I = \int_{x=0}^{\infty} \int_{y=0}^{x} x \cdot e^{-x^2/2} \cdot dy \cdot dx$

~~Let~~ $\cancel{\int_{y=0}}$

$I = \int_{x=0}^{\infty} x^2 \cdot e^{-x^2/2} \cdot dx$

~~Let~~ Using ILATE rule

$v' = e^{-x^2/2}$ $\quad u = x^2$

$v = -\frac{x^3}{3} e^{-x^2/2}$ $\quad u' = 2x$

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$$I = \left[\frac{-x^4}{2} \cdot e^{-x^2/2}\right]_0^{\infty} + \int_{x=0}^{\infty} x^3 \cdot e^{-x^2/2}$$

$I = 0 + 0$

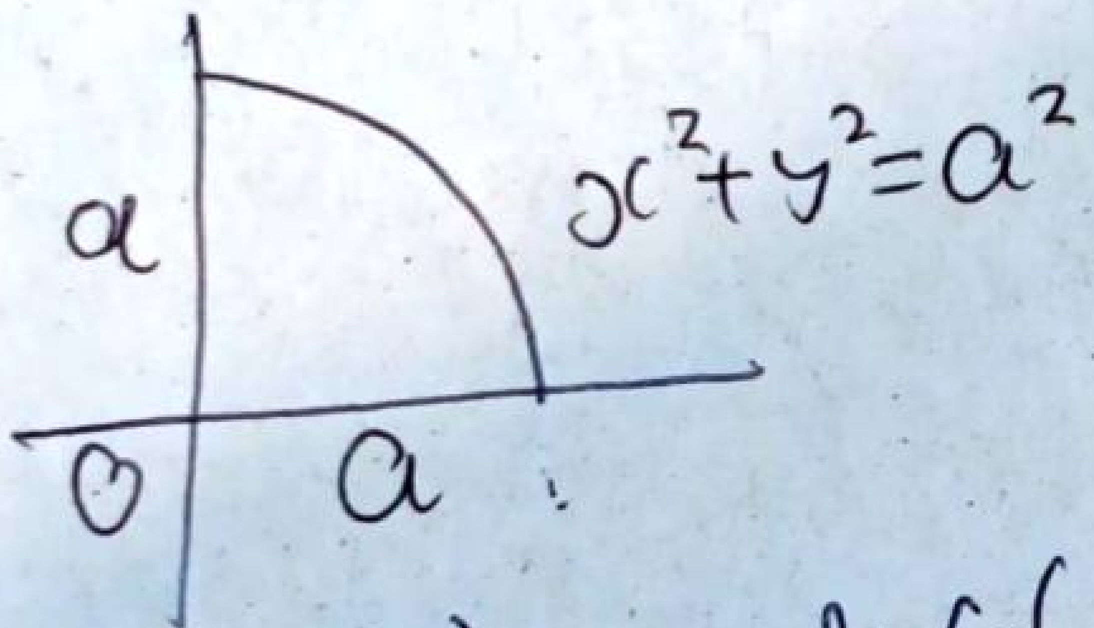
$\underline{I = 0}$

9. $$I = \int_{x=0}^{a} \left(\int_{y=0}^{\sqrt{a^2-x^2}} y^2 \sqrt{x^2+y^2} \cdot dy \right) \cdot dx$$

Let $x = r\cos\theta \quad y = r\sin\theta$

$J = r$

$x \in (0, a) \; ; \; y \in (0, \sqrt{a^2-x^2})$



$\Rightarrow \theta \in (0, \pi/2) \quad r \in (0, a)$

$$I = \int_{\theta=0}^{\pi/2} \left(\int_{r=0}^{a} r^2 \sin^2\theta \times r \cdot r \cdot dr \right) d\theta$$

$$I = \int_{\theta=0}^{\pi/2} \sin^2\theta \cdot d\theta \times \int_{r=0}^{a} r^4 \cdot dr$$

$$I = \int_{\theta=0}^{\pi/2} \sin^2\theta \times d\theta \times \frac{a^5}{5}$$

$$I = \frac{a^5}{5} \times \int_{\theta=0}^{\pi/2} \left(\frac{1-\cos 2\theta}{2} \right) \cdot d\theta$$

$$I = \frac{a^5}{10} \times \left[\theta - \frac{\sin 2\theta}{2} \right]_0^{\pi/2}$$

$$I = \frac{a^5}{10} \times \frac{\pi}{2}$$

$$I = \boxed{\frac{a^5 \pi}{20}}$$

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Change of order of integration in Double integral.

Problems (12)

1. Change the order of integration and evaluate

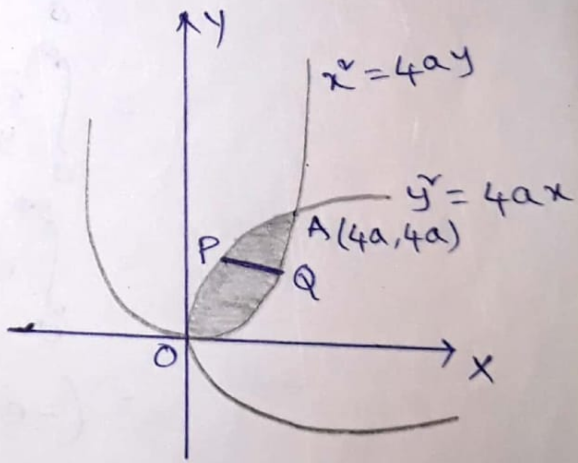
$$\int_0^{4a}\int_{\frac{x^2}{4a}}^{2\sqrt{ax}}$$

Sol. In this integral for a fixed x, y varies from $\frac{x^2}{4a}$ to $2\sqrt{ax}$ and then x varies from 0 to $4a$.

Let us draw the curves $y = \frac{x^2}{4a}$ i.e $x^2 = 4ay$ and $y = 2\sqrt{ax}$ i.e $y^2 = 4ax$.

These two parabolas intersect at $(0,0)$ and $(4a,4a)$

In changing the order of integration, for a fixed y, x varies from $y^2/4a$ to $\sqrt{4ay}$ and then y varies from 0 to $4a$.



$$\int_{y=0}^{4a}\int_{x=\frac{y^2}{4a}}^{2\sqrt{ay}} dx\,dy = \int_{y=0}^{4a}\left[\int_{x=\frac{y^2}{4a}}^{2\sqrt{ay}} dx\right]dy$$

$$= \int_0^{4a} [x]_{\frac{y^2}{4a}}^{2\sqrt{ay}}\, dy$$

$$= \int_0^{4a}\left(2\sqrt{ay} - \frac{y^2}{4a}\right)dy = \left(2\sqrt{a}\,\frac{y^{3/2}}{3/2} - \frac{y^3}{12a}\right)_0^{4a}$$

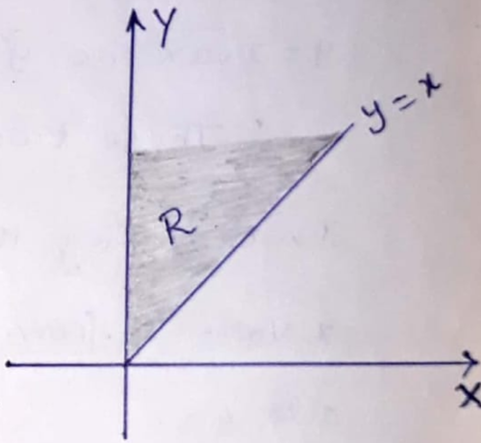
$$= 2\sqrt{a}\,\frac{4a\sqrt{4a}}{3/2} - \frac{64a^3}{12a} = \frac{32}{3}a^2 - \frac{16}{3}a^2 = \frac{16}{3}a^2$$

2. Evaluate $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\, dy\, dx$, by changing the order of integration. (13)

Sol. In the given integral x increases from 0 to ∞ and for each x, y increases from x to ∞. Thus, the lower value of y lies on the line $y = x$.

Therefore, the region of integration is the region in the first quadrant that lies above the line $y = x$.

In changing the order of integration, for a fixed y, x varies from 0 to y and then y varies from 0 to ∞



$$\therefore \int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\, dy\, dx = \int_{y=0}^{\infty} \left[\int_{x=0}^{y} \frac{e^{-y}}{y}\, dx \right] dy$$

$$= \int_0^\infty \frac{e^{-y}}{y} \left[\int_0^y dx \right] dy$$

$$= \int_0^\infty \frac{e^{-y}}{y} \cdot y\, dy$$

$$= \int_0^\infty e^{-y}\, dy$$

$$= \left(-e^{-y} \right)_0^\infty$$

$$= -(0 - 1)$$

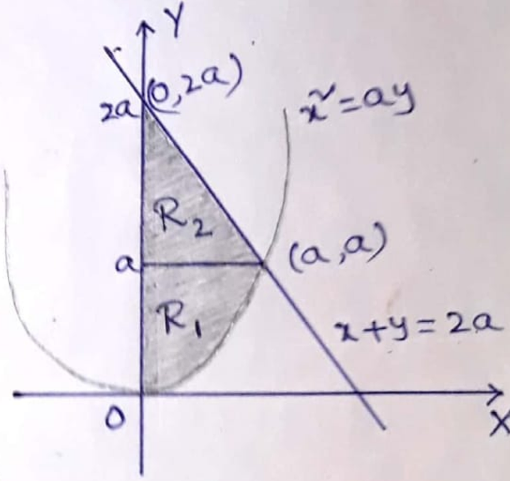
$$= 1$$

3. Evaluate the following integral by changing the order of integration: $\int_0^a \int_{x^2/a}^{2a-x} xy\,dy\,dx$. (14)

Sol. In the given integral, for a fixed x, y varies from x^2/a to $2a-x$ and then x varies from 0 to a.

Let us draw the curves $y = \frac{x^2}{a}$ i.e $x^2 = ay$ and $y = 2a - x$ i.e the line $x + y = 2a$.

The parabola $x^2 = ay$ and the line $x+y=2a$ intersect at (a,a).



The shaded region R is the region of integration. Observe that R is made up of two parts R_1 and R_2.

In R_1, for a fixed y, x varies from 0 to $\sqrt{ay}$ and then y varies from 0 to a.

In R_2, for a fixed y, x varies from 0 to $2a-y$ and then y varies from a to 2a.

$$\therefore \int_0^a \int_{x^2/a}^{2a-x} xy\,dy\,dx = \int_{y=0}^{a} \left[\int_{x=0}^{\sqrt{ay}} xy\,dx\right] dy + \int_{y=a}^{2a} \left[\int_{x=0}^{2a-y} xy\,dx\right] dy$$

$$= \int_0^a y\left[\frac{x^2}{2}\right]_0^{\sqrt{ay}} dy + \int_a^{2a} y\left[\frac{x^2}{2}\right]_0^{2a-y} dy$$

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$$= \int_0^a y\left(\frac{ay}{2}\right)dy + \int_a^{2a} y\,\frac{(2a-y)^2}{2}\,dy$$ (15)

$$= \frac{a}{2}\int_0^a y^2\,dy + \int_a^{2a} \frac{y}{2}\left(4a^2-4ay+y^2\right)dy$$

$$= \frac{a}{2}\left[\frac{y^3}{3}\right]_0^a + \frac{1}{2}\int_a^{2a}\left(4a^2y-4ay^2+y^3\right)dy$$

$$= \frac{a}{2}\left(\frac{a^3}{3}\right) + \frac{1}{2}\left[2a^2y^2-4a\,\frac{y^3}{3}+\frac{y^4}{4}\right]_a^{2a}$$

$$= \frac{a^4}{6} + \frac{1}{2}\Big[2a^2(4a^2-a^2) - \frac{4a}{3}(8a^3-a^3)$$
$$+ \frac{1}{4}(16a^4-a^4)\Big]$$

$$= \frac{a^4}{6} + \frac{1}{2}\left[6a^4 - \frac{28}{3}a^4 + \frac{15}{4}a^4\right]$$

$$= \frac{a^4}{6} + \frac{1}{2}\left(\frac{5a^4}{12}\right)$$

$$= \frac{9a^4}{24}$$

$$= \frac{3}{8}a^4$$

4. Change the order of integration in the integral

$$\int_0^1 \int_{\sqrt{y}}^{2-y} xy\,dx\,dy$$ and hence evaluate it.

Sol. In the given integral, for a fixed ~~x~~ y, x varies from $\sqrt{y}$ to $2-y$ and then y varies from 0 to 1.

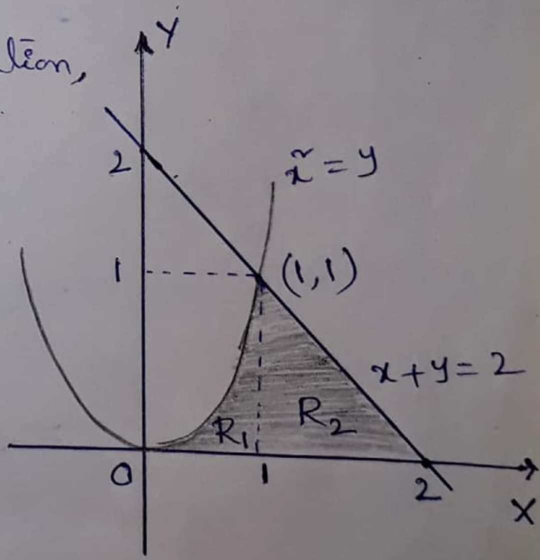
Let us draw the curves $x = \sqrt{y}$ (i.e the parabola $x^2 = y$) and $x = 2-y$ (i.e the line $x+y=2$)

The parabola $x^2 = y$ and the line $x+y=2$ intersect at $(1,1)$.

The shaded region R is the region of integration.

observe that R is made up of two parts R_1 and R_2.

To change the order of integration,



in R_1, for a fixed x, y varies from 0 to x^2 and then x varies from 0 to 1.

In R_2, for a fixed x, y varies from 0 to $2-x$ and then x varies from 1 to 2.

$$\therefore \int_0^1 \int_{\sqrt{y}}^{2-y} xy\,dx\,dy = \int_{x=0}^{1}\left[\int_{y=0}^{x^2} xy\,dy\right]dx + \int_{x=1}^{2}\left[\int_{y=0}^{2-x} xy\,dy\right]dx$$ (17)

$$= \int_0^1 x\left[\frac{y^2}{2}\right]_0^{x^2} dx + \int_1^2 x\left[\frac{y^2}{2}\right]_0^{2-x} dx$$

$$= \int_0^1 \frac{x^5}{2}\,dx + \int_1^2 \frac{x}{2}\left[(2-x)^2\right]dx$$

$$= \int_0^1 \frac{x^5}{2}\,dx + \frac{1}{2}\int_1^2 x(4-4x+x^2)\,dx$$

$$= \left(\frac{x^6}{12}\right)_0^1 + \frac{1}{2}\int_1^2 (4x-4x^2+x^3)\,dx$$

$$= \frac{1}{12} + \frac{1}{2}\left[8 - \frac{32}{3} + 4 - \left(2 - \frac{4}{3} + \frac{1}{4}\right)\right]$$

$$= \frac{1}{12} + \frac{1}{2}\left[\frac{4}{3} - \frac{11}{12}\right]$$

$$= \frac{1}{12} + \frac{1}{2}\left(\frac{5}{12}\right)$$

$$= \frac{1}{12} + \frac{5}{24}$$

$$= \frac{7}{24}$$

Problems

① Evaluate the following

(i) $$\int_1^4 \int_0^{\sqrt{4-x}} xy \, dy \, dx$$

(ii) $$\int_0^1 \int_0^{x^2} e^{y/x} \, dy \, dx$$

(iii) $$\int_0^1 \int_0^{\sqrt{1-y^2}} x^3 y \, dx \, dy$$

(iv) $$\int_0^1 \int_0^1 \frac{1}{\sqrt{1-x^2}\sqrt{1-y^2}} \, dx \, dy$$

(v) If A is the area of the region bounded by the lines $x=0$, $x=1$, $y=0$, $y=2$, then evaluate

$$\iint_A (x^2+y^2) \, dx \, dy$$

(vi) If R is the rectangular region with vertices $(0,0), (2,0), (2,3)$, evaluate

$$\iint_R x^2 y^2 \, dx \, dy.$$

(2) Evaluate $\iint_R (x+y)^2\,dx\,dy$, where R is the region bounded by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

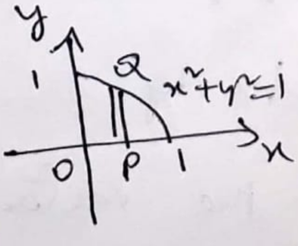
Hint: $\iint_R (x+y)^2\,dx\,dy = \int_{x=-a}^{a}\left(\int_{y=-b\sqrt{1-\left(\frac{x^2}{a^2}\right)}}^{b\sqrt{1-\left(\frac{x^2}{a^2}\right)}} (x+y)^2\,dy\right)dx$

$$= \frac{\pi}{4}\,ab\,(a^2+b^2)$$

(3) If R is the region bounded by the ~~curve~~ circle $x^2+y^2=1$ in the first quadrant, evaluate

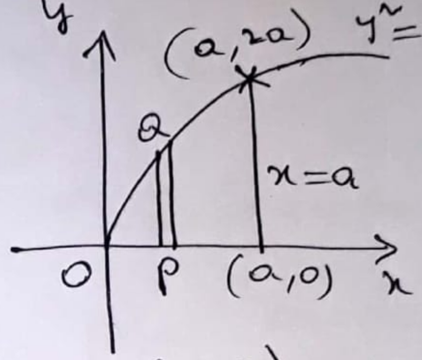
$$\iint_R \frac{xy}{\sqrt{1-y^2}}\,dx\,dy$$

Hint: $\iint_R \frac{xy}{\sqrt{1-y^2}}\,dx\,dy = \int_{x=0}\left(\int_{y=0}^{\sqrt{1-x^2}} \frac{xy}{\sqrt{1-y^2}}\,dy\right)dx$

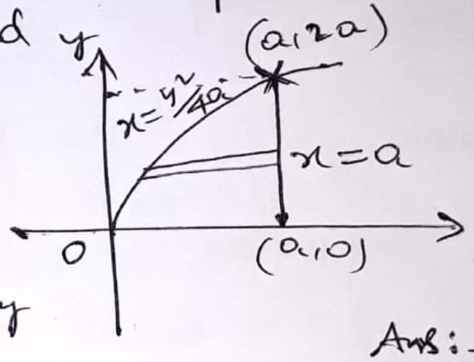


(4) Evaluate $\int_0^a \int_0^{2\sqrt{xa}} x^2 \, dy \, dx \; (a>0)$, by changing the order of integration.

[Hint: Given



required



$$\therefore \int_{x=0}^{a} \int_{y=0}^{2\sqrt{ax}} x^2 \, dy \, dx$$

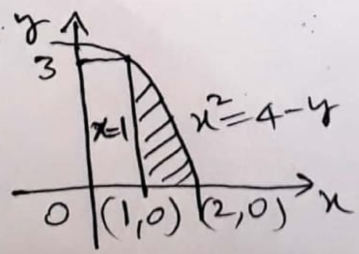
$$= \int_{y=0}^{2a} \left(\int_{x=\frac{y^2}{4a}}^{a} x^2 \, dx \right) dy$$

Ans: $\frac{4}{7} a^4$.]

(5) Evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} y^2 \, dy \, dx$ by changing the order of integration. $\boxed{\text{Ans: } \frac{\pi}{16}}$

(6) Evaluate $\int_0^3 \int_1^{\sqrt{4-y}} (x+y) \, dx \, dy$, by changing the order of integration.

$\boxed{\text{Ans: } \frac{241}{60}}$



(7) Evaluate $\int_0^a \int_{x^2/a}^{2a-x} xy\,dy\,dx$, by changing the order of the integration.

(8) Evaluate $\int_0^\infty \int_0^x x\, e^{-\frac{x^2}{2}}\,dy\,dx$

(9) Evaluate $\int_0^a \int_0^{\sqrt{a^2-x^2}} y^2 \sqrt{x^2+y^2}\,dy\,dx$ by transforming to polar co-ordinates.

(10) Evaluate $\iint_R \frac{x^2y^2}{x^2+y^2}\,dx\,dy$ over the annular region R between the circles $x^2+y^2=a^2$ and $x^2+y^2=b^2$ with $b>a$, by transforming to polar co-ordinates.

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1.i) $$I = \int_{x=0}^{1} \int_{y=0}^{1-x} \int_{z=0}^{1-x-y} dz \cdot dy \cdot dx$$

$$I = \int_{x=0}^{1} \int_{y=0}^{1-x} [z]_0^{1-x-y} \cdot dy \cdot dx$$

$$I = \int_{x=0}^{1} \int_{y=0}^{1-x} (1-x-y) \cdot dy \cdot dx$$

$$I = \int_{x=0}^{1} \left[y - xy - \frac{y^2}{2}\right]_0^{1-x} \cdot dx$$

$$I = \int_{x=0}^{1} \left((1-x) - x(1-x) - \frac{(1-x)^2}{2}\right) \cdot dx$$

$$I = \int_{0}^{1} \left[1 - x - x + x^2 - \frac{(1+x^2-2x)}{2}\right] \cdot dx$$

$$I = \int_0^1 \left[1 - 2x + x^2 - \frac{1}{2} - \frac{x^2}{2} + x\right] \cdot dx$$

$$I = \int_0^1 \left[\frac{1}{2} + \frac{x^2}{2} - x\right] \cdot dx$$

$$I = \int_0^1 \frac{(x^2 - 2x + 1)}{2} \cdot dx$$

$$2I = \left[\frac{x^3}{3} - x^2 + x\right]_0^1$$

$$2I = \frac{1}{3} - \cancel{1} + \cancel{1}$$

$$I = \boxed{\frac{1}{6}}$$

ii)
$$I = \int_{y=0}^{1} \int_{x=y^2}^{1} \int_{z=0}^{1-x} x \cdot dz \cdot dx \cdot dy$$

$$I = \int_{y=0}^{1} \int_{x=y^2}^{1} x[z]_0^{1-x} \cdot dx \cdot dy$$

$$I = \int_{y=0}^{1} \int_{x=y^2}^{1} (x - x^2) \cdot dx \cdot dy$$

$$I = \int_{y=0}^{1} \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_{y^2}^{1} \cdot dy$$

$$I = \int_{y=0}^{1} \left[\left(\frac{1}{2} - \frac{1}{3}\right) - \left(\frac{y^4}{2} - \frac{y^6}{3}\right)\right] \cdot dy$$

$$I = \int_{y=0}^{1} \left(\frac{1}{6} - \frac{y^4}{2} + \frac{y^6}{3}\right) \cdot dy$$

$$I = \left[\frac{y}{6} - \frac{y^5}{10} + \frac{y^7}{21}\right]_0^1$$

$$I = \frac{1}{6} - \frac{1}{10} + \frac{1}{21}$$

$$I = \frac{4}{60} + \frac{1}{21}$$

$$I = \frac{1}{3}\left[\frac{4}{20} + \frac{1}{7}\right]$$

$$I = \frac{\cancel{48}\ \cancel{16}\ 8}{\cancel{3}\ 1} \times \frac{1}{\cancel{140}\ 75}$$

$$I = \boxed{\frac{8}{75}}$$

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iii) $$I = \int_{z=-1}^{1}\int_{x=0}^{z}\int_{y=x-z}^{x+z}(x+y+z)\cdot dy\cdot dx\cdot dz$$

$$I = \int_{z=-1}^{1}\int_{x=0}^{z}\left[xy+\frac{y^2}{2}+yz\right]_{x-z}^{x+z}\cdot dx\cdot dz$$

$$I = \int_{z=-1}^{1}\int_{x=0}^{z}\left(x(x+z)+\frac{(x+z)^2}{2}+z(x+z)\right)-\left(x(x-z)+\frac{(x-z)^2}{2}+z(x-z)\right)\cdot dx\cdot dz$$

$$I = \int_{z=-1}^{1}\int_{x=0}^{z}\left(\cancel{x^2}+xz+\frac{\cancel{x^2}+\cancel{z^2}+2xz}{2}+xz+z^2\right)-\left(\cancel{x^2}-\cancel{xz}+\frac{\cancel{x^2}+\cancel{z^2}-2xz}{2}+\cancel{xz}-z^2\right)\cdot dx\cdot dz$$

$$I = \int_{z=-1}^{1}\int_{x=0}^{z}\left(2z^2+2xz+2xz\right)\cdot dx\cdot dz$$

$$\cancel{I = \int_{z=-1}^{1} 2xz^2+x^2z+x^2z}$$

$$I = \int_{z=-1}^{1}\int_{x=0}^{2}\left(2z^2+4xz\right)\cdot dx\cdot dz$$

$$I = \int_{z=-1}^{1}\left[2xz^2+2x^2z\right]_0^2\cdot dz$$

$$I = \int_{z=-1}^{1}(4z^2+8z)\cdot dz$$

$$I = \left[\frac{4z^3}{3}+4z^2\right]_{-1}^{1}$$

$$I = \left(\frac{4}{3}+4\right)-\left(-\frac{4}{3}+4\right)$$

$$I = \boxed{\frac{8}{3}}$$

iv) $I = \int_{x=-a}^{a} \int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_{z=0}^{x^2+y^2} z^2 \cdot dz \cdot dy \cdot dx$

$I = \int_{x=-a}^{a} \int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \left[\frac{z^3}{3}\right]_0^{x^2+y^2} \cdot dy \cdot dx$

$3I = \int_{x=-a}^{a} \int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} (x^2+y^2)^3 \cdot dy \cdot dx$

$\frac{3I}{2} =$

$\frac{3I}{4} =$

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~~$y = \sqrt{a^2 - x^2}$~~

$$\frac{3I}{2} = \int_{x=-a}^{a} \int_{y=0}^{\sqrt{a^2-x^2}} (x^2+y^2)^3 \cdot dy \cdot dx$$

$$\frac{3I}{4} = \int_{x=0}^{a} \int_{y=0}^{\sqrt{a^2-x^2}} (x^2+y^2)^3 \cdot dy \cdot dx$$

Let ~~$r = \cos\theta$~~ $y = r\sin\theta$

$x = r\cos\theta$

$$x^2+y^2 = r^2$$

$$J = r$$

$r \in (0, a)$ $\theta \in (0, \pi/2)$

$$\frac{3I}{4} = \int_{\theta=0}^{\pi/2} \int_{r=0}^{a} (r^2)^3 \times r \cdot dr \cdot d\theta$$

$$\frac{3I}{4} = \int_{\theta=0}^{\pi/2} \left[\frac{r^8}{8}\right]_0^a \cdot d\theta$$

$$\frac{3I}{\cancel{4}_1} \times \cancel{8}^2 = a^8 \cdot \int_{\theta=0}^{\pi/2} \cdot d\theta$$

$$6I = a^8 \times \frac{\pi}{2}$$

$$I = \boxed{a^8 \times \frac{\pi}{12}}$$

2. ~~Here $x \in (0, 12)$ $y \in (0,$~~

Here $z \in \left[0, \frac{12-2x-3y}{4}\right)$

$y \in \left(0, \frac{12-2x}{3}\right)$

$x \in (0, 6)$

$$I = \int_{x=0}^{6} \int_{y=0}^{\frac{12-2x}{3}} \int_{z=0}^{\frac{12-2x-3y}{4}} dz \cdot dy \cdot dx$$

$$I = \int_{x=0}^{6} \int_{y=0}^{\frac{12-2x}{3}} \left(\frac{12-2x-3y}{4}\right) \cdot dy \cdot dx$$

$$4I = \int_{x=0}^{6} \int_{y=0}^{\frac{12-2x}{3}} (12-2x-3y) \cdot dy \cdot dx$$

$$4I = \int_{x=0}^{6} \left[12y - 2xy - \frac{3y^2}{2}\right]^{\frac{12-2x}{3}} \cdot dx$$

$$4I = \int_{x=0}^{6} \left[\frac{24y - 4xy - 3y^2}{2}\right]^{\frac{12-2x}{3}} \cdot dx$$

$$8I = \int_{x=0}^{6} \left[24y - 4xy - 3y^2\right]^{\frac{12-2x}{3}} \cdot dx$$

$$8I = \int_{x=0}^{6} \left(24 \times \frac{(12-2x)}{3} - 4x\left(\frac{12-2x}{3}\right) - 3\left(\frac{12-2x}{3}\right)^2\right) \cdot dx$$

24
×12
48
24·
288
144
66
864
36
−24
4

$$24I = \int_{x=0}^{6} \left[288 - 48x - 48x + 8x^2 - (144 + 4x^2 - 48x)\right] dx$$

$$24I = \int_{x=0}^{6} (144 - 48x + 4x^2) \cdot dx$$

$$24I = \left[144x - 24x^2 + \frac{4}{3}x^3\right]_0^6$$

$$24I = \cancel{864} - \cancel{864} + \frac{4}{3} \times \overset{72}{\cancel{216}}$$

$$24I = \overset{3}{\cancel{72}} \times 4$$

$$I = \underline{12}$$

3. $$I = \int_{x=0}^{a} \int_{y=0}^{x} \int_{z=0}^{x+y} e^{(x+y+z)} \cdot dz \cdot dy \cdot dx$$

Let $x+y+z = t$ UL:- $z = x+y \rightarrow t = 2(x+y)$

$dz = dt$ LL:- $z = 0 \rightarrow t = (x+y)$

$$I = \int_{x=0}^{a} \int_{y=0}^{x} \int_{t=(x+y)}^{2(x+y)} e^t \cdot dt \cdot dy \cdot dx$$

$$I = \int_{x=0}^{a} \int_{y=0}^{x} \left[e^t\right]_{x+y}^{2(x+y)} \cdot dy \cdot dx$$

$$I = \int_{x=0}^{a} \int_{y=0}^{x} \left(e^{2(x+y)} - e^{(x+y)}\right) \cdot dy \cdot dx$$

$$\cancel{I = \int_{x=0}^{a} \int_{y=0}^{x} e^{(x+y)} \left(e^{x+y} - 1\right) \cdot dy \cdot dx}$$

Let $x+y = u$ UL:- $y = x \rightarrow u = 2x$

$du = dy$ LL:- $y = 0 \rightarrow u = x$

$$I = \int_{x=0}^{a} \int_{u=x}^{2x} (e^{2u} - e^{u}) \cdot du \cdot dx$$

$$I = \int_{x=0}^{a} \left[2 \cdot e^{2u} - e^{u}\right]_{x}^{2x} \cdot dx$$

$$I = \int_{x=0}^{a} \left[2 \cdot e^{4x} - e^{2x} - 2 \cdot e^{2x} + e^{x}\right] \cdot dx$$

$$I = \int_{x=0}^{a} \left[2 \cdot e^{4x} - 3 \cdot e^{2x} + e^{x}\right] \cdot dx$$

$$I = \left[8 \cdot e^{4x} - 6 \cdot e^{2x} + e^{x}\right]_{0}^{a}$$

$$I = (8 \cdot e^{4a} - 6 \cdot e^{2a} + e^{a}) - (8 - 6 + 1)$$

$$I = 8 \cdot e^{4a} - 6 \cdot e^{2a} + e^{a} - 3$$

Triple Integrals

The volume integral of $f(x,y,z)$ over the region R is denoted by $\iiint_R f(x,y,z)\,dx\,dy\,dz$ (or, $\int_V f(x,y,z)\,dv$; here V stands for the volume of R).

Thus, $$\int_V f(x,y,z)\,dv = \iiint_R f(x,y,z)\,dx\,dy\,dz$$

$$= \int_{x=a}^{b} \int_{y=y_1(x)}^{y_2(x)} \int_{z=z_1(x,y)}^{z_2(x,y)} f(x,y,z)\,dz\,dy\,dx.$$

Note: A volume integral is also called as a Triple Integral.

Problems:

1) Evaluate $\displaystyle\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} xyz\,dz\,dy\,dx$

Sol: $$\int_{x=0}^{1} \left(\int_{y=0}^{\sqrt{1-x^2}} \left(\int_{z=0}^{\sqrt{1-x^2-y^2}} xyz\,dz \right) dy \right) dx$$

$$= \int_{x=0}^{1} \int_{y=0}^{\sqrt{1-x^2}} xy \left[\frac{z^2}{2}\right]_0^{\sqrt{1-x^2-y^2}} dy\,dx$$

$$= \frac{1}{2}\int_{x=0}^{1}\left(\int_{y=0}^{\sqrt{1-x^2}} xy(1-x^2-y^2)\,dy\right)dx$$

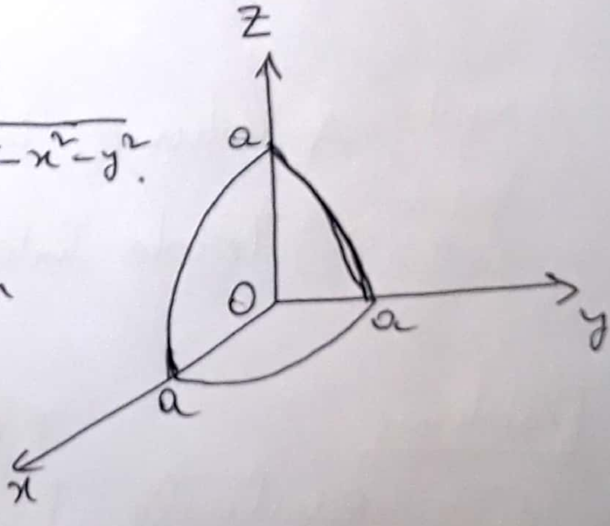
$$= \frac{1}{2}\int_{x=0}^{1} x\left\{(1-x^2)\frac{y^2}{2}-\frac{y^4}{4}\right\}_0^{\sqrt{1-x^2}} dx$$

$$= \frac{1}{8}\int_{x=0}^{1} x(1-2x^2+x^4)\,dx$$

$$= \frac{1}{8}\left(\frac{1}{2}-\frac{1}{2}+\frac{1}{6}\right) = \frac{1}{48}$$

(2) Evaluate $\iiint_R xy\,dx\,dy\,dz$, where R is the positive octant of the sphere $x^2+y^2+z^2=a^2$

Sol: In the given region,



z varies from 0 to $\sqrt{a^2-x^2-y^2}$.

For $z=0$, y varies from 0 to $\sqrt{a^2-x^2}$.

For $z=0$, $y=0$,

x varies from 0 to a.

Therefore, $\iiint_R xy\,dx\,dy\,dz = \int_{x=0}^{a}\left(\int_{y=0}^{\sqrt{a^2-x^2}}\left(\int_{z=0}^{\sqrt{a^2-x^2-y^2}} xy\,dz\right)dy\right)dx$

$$= \int_{x=0}^{a} x \left(\int_{y=0}^{\sqrt{a^2-x^2}} y \, [z]_0^{\sqrt{a^2-x^2-y^2}} \, dy \right) dx$$

$$= \int_{x=0}^{a} x \left(\int_{y=0}^{\sqrt{a^2-x^2}} y \sqrt{a^2-x^2-y^2} \, dy \right) dx$$

$$= \int_{x=0}^{a} x \left(\int_{t=a^2-x^2}^{0} \left(\frac{-1}{2}\right) \sqrt{t} \, dt \right) dx$$

$$= \frac{1}{2} \int_{x=0}^{a} x \left(\int_{0}^{a^2-x^2} \sqrt{t} \, dt \right) dx$$

$$= \frac{1}{2} \cdot \frac{2}{3} \int_{x=0}^{a} x \left[t^{3/2} \right]_0^{a^2-x^2} dx$$

Taking $t = (a^2 - x^2) - y^2$

then $dt = -2y\,dy$

and $y \to 0,\ t \to a^2 - x^2$

$y \to \sqrt{a^2-x^2},\ t \to 0$

$$= \frac{1}{3} \int_{x=0}^{a} x \left(a^2 - x^2\right)^{3/2} dx$$

$$= \frac{1}{3} \int_{u=a^2}^{0} u^{3/2} \cdot \left(\frac{-1}{2}\right) du$$

$$= \frac{1}{6} \int_{u=0}^{a^2} u^{3/2} \, du$$

$$= \frac{1}{6} \left[\frac{u^{5/2}}{\left(\frac{5}{2}\right)} \right]_0^{a^2} = \frac{1}{15} \cdot a^5$$

Taking $a^2 - x^2 = u$

then $-2x\,dx = du$

$x\,dx = -\frac{1}{2} du$

and $x \to 0,\ u \to a^2$

$x \to a,\ u \to 0$

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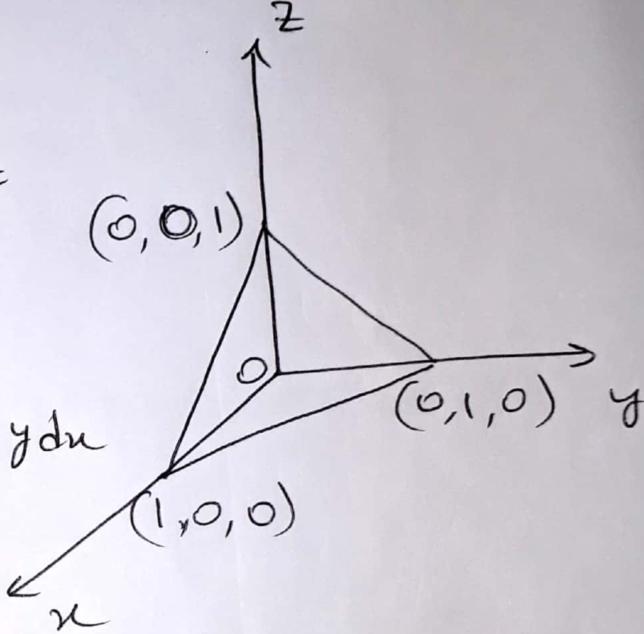
③ Evaluate $\iiint_R (x+y+z)\,dx\,dy\,dz$, where R is the region bounded by the planes $x=0$, $y=0$, $z=0$ and $x+y+z=1$.

Sol:



$$\iiint_R (x+y+z)\,dx\,dy\,dz$$

$$= \int_{x=0}^{1}\int_{y=0}^{1-x}\int_{z=0}^{1-x-y} (x+y+z)\,dz\,dy\,dx$$

$$= \int_{x=0}^{1}\left(\int_{y=0}^{1-x} \left\{(x+y)[z]_0^{1-x-y} + \left[\frac{z^2}{2}\right]_0^{1-x-y}\right\} dy\right)dx$$

$$= \frac{1}{2}\int_{x=0}^{1}\left(\int_{y=0}^{1-x} \left\{1-(x+y)^2\right\} dy\right)dx$$

$$= \frac{1}{2}\int_{x=0}^{1} \left[y - \frac{1}{3}(x+y)^3\right]_0^{1-x} dx$$

$$= \frac{1}{6}\cdot\int_{x=0}^{1} (2-3x+x^3)\,dx = \frac{1}{8}.$$

Problems

(1) Evaluate the following

(i) $$\int_0^1 \int_0^{1-x} \int_0^{1-x-y} dz\, dy\, dx$$

(ii) $$\int_0^1 \int_{y^2}^1 \int_0^{1-x} x\, dz\, dx\, dy$$

(iii) $$\int_{-1}^1 \int_0^z \int_{x-z}^{x+z} (x+y+z)\, dy\, dx\, dz$$

(iv) $$\int_{-a}^a \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_0^{x^2+y^2} z^2\, dz\, dy\, dx$$

(2) Evaluate $\iiint_R dx\,dy\,dz$, where R is the finite region of space formed by the planes $x=0$, $y=0$, $z=0$ and $2x+3y+4z=12$

(3) Evaluate (i) $$\int_0^a \int_0^x \int_0^{x+y} e^{x+y+z}\, dz\, dy\, dx$$

(4) Evaluate $$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{1}{\sqrt{1-x^2-y^2-z^2}}\, dz\, dy\, dx$$