3 Detection

We apply theory from Ch 2 to detection problems.

<u>Given</u> :
- data $\underline{x}$
- likelihood

} mandatory

- prior
- loss

} optional

<u>Desired</u> :
- Detector
- Detector performance
- How to optimize system parameters for an optimized system

<u>Comments</u> :
- Only binary detection
- Use detection notation H_0, H_1 instead of ω_1, ω_2

<u>Slide 3.1</u>

A general Log Likelihood Ratio Test (LRT) detector:

$$\ln\left(\frac{P(\underline{x}\mid H_1)}{P(\underline{x}\mid H_0)}\right) = -\frac{1}{2\sigma^2}\left[\sum_{n=0}^{d-1}(x(n)-A)^2 - \sum_{n=0}^{d-1}(x(n))^2\right]$$

$$= -\frac{1}{2\sigma^2}\left[-2A\sum_n x(n) + dA^2\right]\cdot\frac{-2dA^2}{-2dA^2}$$

• SNR is best for comparision in detection of noise in AWGN

$$= \underbrace{\frac{dA^2}{\sigma^2}}_{\rho>0}\underbrace{\left[\frac{1}{A}\cdot\underbrace{\frac{1}{d}\sum_n(x(n))}_{\bar{x}}\right.}_{t(\underline{x})}\left. - \frac{1}{2}\right]$$

$$= \rho\left(t(\underline{x})-\frac{1}{2}\right) \underset{H_0}{\overset{H_1}{\gtrless}} \ln\tilde{\gamma}$$

original threshold for LR

$$= t(\underline{x}) \underset{H_0}{\overset{H_1}{\gtrless}} \frac{1}{\rho}\ln\tilde{\gamma} + \frac{1}{2} := \gamma \text{ : threshold for } t(\underline{x})$$

with

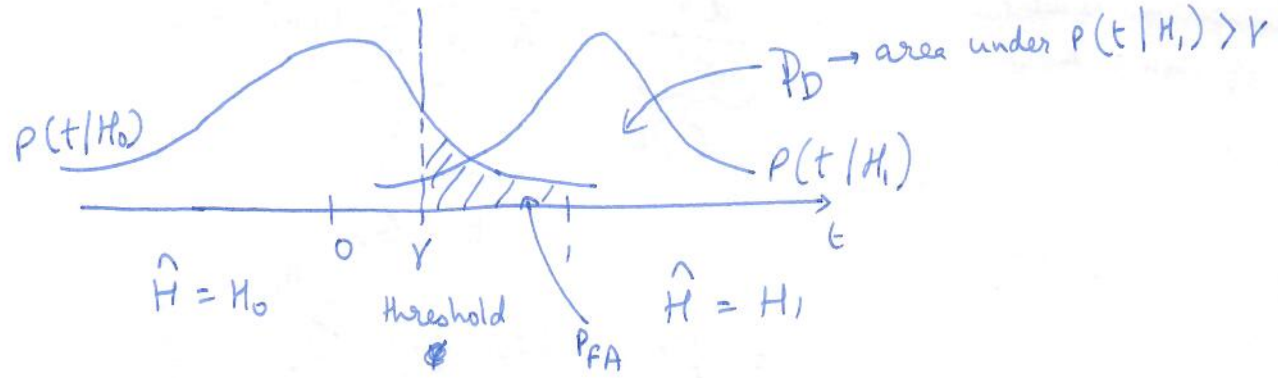
- test $t(\underline{x}) = \frac{\bar{x}}{A}$
- $\frac{A^2}{\sigma^2}$: Signal to noise ratio (SNR) of $x(n)\mid H_1 = A + z(n) \sim N(A, \sigma^2)$
- $\rho = d\cdot\frac{A^2}{\sigma^2}$: SNR of $\bar{x}\mid H_1 = A + \bar{z} \sim N\left(A, \frac{\sigma^2}{d}\right)$

Distribution of $t(x)$

$$H_0: t(x) = \frac{1}{A}\bar{z} \sim N\left(0, \frac{1}{A}\frac{\sigma^2}{d}\right) = N\left(0, \frac{1}{\rho}\right)$$

$$H_1: t(x) = \frac{1}{A}(A+\bar{z}) \sim N\left(1, \frac{1}{A}\frac{\sigma^2}{d}\right) \cong N\left(1, \frac{1}{\rho}\right)$$

Detailed performance measures P_{FA}, P_D :



Thus we can't only reduce P_{FA} and increase P_D as there is an overlap of the area of operations. Hence we need to make some compromise b/w them.

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False alarm rate

$$P_{FA}(\gamma) = P(\hat{H} = H_1 \mid H_0 = H_0) = P(t > \gamma \mid H_0) \; ; \; t \sim N\left(0, \frac{1}{\rho}\right)$$

$$= P(\underbrace{\sqrt{\rho}\, t}_{\sim N(0,1)} > \sqrt{\rho}\,\gamma)$$

$$= Q(\sqrt{\rho}\,\gamma)$$

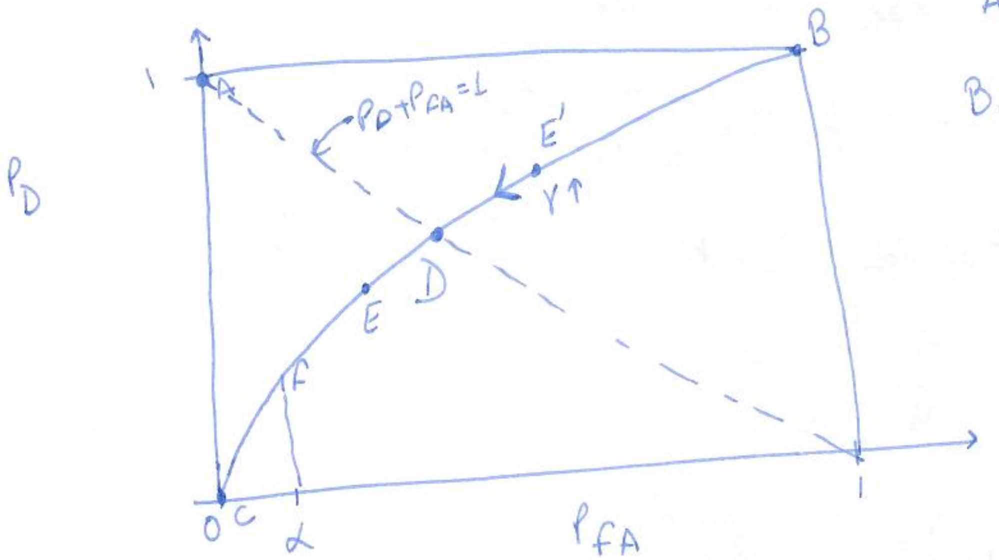
Detection rate

$$P_D(\gamma) = P(\hat{H} = H_1 \mid H = H_1) = P(t > \gamma \mid H_1) \; ; \; t \sim N\left(1, \frac{1}{\rho}\right)$$

$$= P(\underbrace{\sqrt{\rho}\,(t-1)}_{\sim N(0,1)} > \sqrt{\rho}\,(\gamma-1) \mid H_1) = Q(\sqrt{\rho}\,(\gamma-1))$$

$\gamma \uparrow$: $P_{FA} \downarrow$ $P_D \downarrow$ } max P_D and min P_{FA} impossible, only compro...

Receiver Operating Characteristics (ROC):



A : $P_{FA} = 0$, $P_D = 1$; dream

B: $\gamma = -\infty$ $\hat{H}(\underline{x}) = H_1 \ \forall \underline{x}$
$P_{FA} = P_D = 1$

C: $\gamma = +\infty$ $\hat{H}(\underline{x}) = H_0 \ \forall \underline{x}$
$P_{FA} = P_D = 0$

D: ML, $\gamma = \frac{1}{2}$,
$P_{FA} + P_D = 1$

E : MAP, $P_0 > P_1$, $\gamma > \frac{1}{2}$

E' : MAP, $P_0 < P_1$, $\gamma < \frac{1}{2}$

F: Neyman Pearson CFAR
$P_{FA} = \alpha$

ROC : $\left(P_{FA}(\gamma), P_D(\gamma) \right) \ \forall \gamma \cong$ all possible compromises.

Average Performance (2.2.1):

A single number for easier comparision.

- Bayesian Risk (BR) : $l_{00} \times (1 - P_{FA}) \cdot P_0 + l_{10} (1 - P_D) P_1 + l_{01} P_{FA} \cdot P_0 + l_{11} P_D P_1$

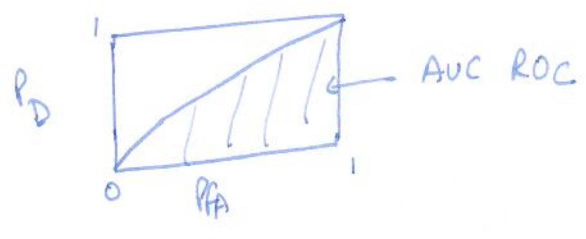
- Error rate : 0/1 loss i.e. $l_{00} = l_{11} = 0$, $l_{01} = l_{10} = 1$

$$ER = P_{FA} P_0 + (1 - P_D) P_1$$

- Balanced ER : $P_0 = P_1 = \frac{1}{2}$

$$BER = \frac{1}{2}\left(P_{FA} + (1 - P_D) \right)$$

- AUC - Area Under Curve - ROC



Special optimum detectors

(a) MBR: $\gamma_{MBR} = \frac{1}{\rho} \ln \tilde{\gamma}_{MBR} + \frac{1}{2}$

$$= \frac{1}{\rho} \ln \left(\frac{\lambda_{10} - \lambda_{00}}{\lambda_{01} - \lambda_{11}} \cdot \frac{p_0}{p_1} \right) + \frac{1}{2}$$

(b) ML : 0/1 loss, $p_0 = p_1 = \frac{1}{2}$ $\tilde{\gamma}_{ML} = 1$, $\gamma_{ML} = \frac{1}{2}$

i.e. $\frac{\bar{x}}{A} \underset{H_0}{\overset{H_1}{\gtrless}} \frac{1}{2}$

$$P_{FA}(\gamma_{ML}) = Q\left(\frac{1}{2}\sqrt{\rho}\right)$$

$$P_D(\gamma_{ML}) = Q\left(-\frac{1}{2}\sqrt{\rho}\right)$$

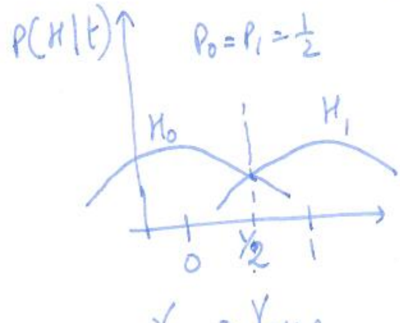
$$= 1 - Q\left(\frac{1}{2}\sqrt{\rho}\right)$$

$$P_{FA} + P_D = 1$$

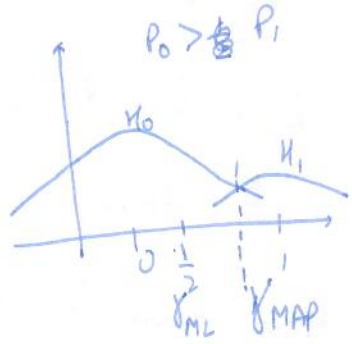
(c) MAP : 0/1 loss $p_0 \neq \frac{1}{2}$

$$\gamma_{MAP} = \frac{1}{\rho} \log\left(\frac{p_0}{p_1}\right) + \frac{1}{2} \underset{p_0 < p_1 - \epsilon'}{\overset{p_0 > p_1 - \epsilon}{\gtrless}} \gamma_{ML} = \frac{1}{2}$$

MAP decision: $\max_H P(H/t) \sim P(t/H) \cdot P(H)$



$\gamma_{ML} = \gamma_{MAP}$



(d) Neyman - Pearson : no priors , no loss.

$$P_{FA}(\gamma) = Q\left(\sqrt{\rho}\,\gamma_{NP}\right) \stackrel{!}{=} \alpha \leftarrow \text{system threshold set by us.}$$

$$\Rightarrow \gamma_{NP} = \frac{1}{\sqrt{\rho}} Q^{-1}(\alpha)$$

How to improve the detector performance ?

increase $\rho = d \cdot \frac{A^2}{\sigma^2}$

- increase signal power A^2: not always possible.
- increase the number of measurements d : not always possible
- reduce noise power σ^2

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E 3.2 : Bit detection

H_0 : $x(n) = -1 + z(n)$ $\quad z(n)$ iid $N(0, \sigma^2)$

H_1 : $x(n) = 1 + z(n)$

Let $y(n) = x(n) + 1$ $\left\{ \begin{array}{l} H_0 = z(n) \\ H_1 = A + z(n) \end{array} \right.$ $\quad A = 2$

Same sol^n as E 3.1

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