Detection and Pattern Recognition

Case B $\underline{\underline{C}}_i = \underline{\underline{C}} \neq \sigma^2 \underline{\underline{I}}$

$$f_i(\underline{x}) = -\frac{1}{2} \underbrace{(\underline{x} - \underline{\mu}_i)^T \underline{\underline{C}}^{-1} (\underline{x} - \underline{\mu}_i)}_{\text{Mahalanobis dist}^n} + \ln P_i$$

$$= -\frac{1}{2} \left[\underline{x}^T \underline{\underline{C}}^{-1} \underline{x} - 2 \underline{\mu}_i^T \underline{\underline{C}}^{-1} \underline{x} + \underline{\mu}_i^T \underline{\underline{C}}^{-1} \underline{\mu}_i \right] + \ln P_i$$

ignore as
independent of class

$$= \underline{w}_i^T \underline{x} + w_{io} \quad \text{affine function of } \underline{x}$$

$$\underline{w}_i = \underline{\underline{C}}^{-1} \underline{\mu}_i$$

$$w_{io} = \ln P_i - \frac{1}{2} \underline{\mu}_i^T \underline{\underline{C}}^{-1} \underline{\mu}_i$$

Slide - 2.13, 2.14

Shape of circles around Mahalanobis circles would be determined by the eigenvalue and eigen vectors.

2.14.1

Decision boundary β_{ij}

- $(\underline{w}_i - \underline{w}_j)^T \underline{x} + (w_{i0} - w_{j0}) = 0$

  Results in a hyper plane

- no longer orthogonal to the connecting line b/w μ_i, μ_j

  $\underline{\mu}_i - \underline{\mu}_j = \underline{\underline{C}} \cdot (\underline{w}_i - \underline{w}_j)$, due to $\underline{\underline{C}} \neq \sigma^2 \underline{\underline{I}}$

Case C $\quad \underline{\underline{C}}_i \neq \underline{\underline{C}}_j$

$f_i(\underline{x})$

$= -\frac{1}{2} \ln\left|(\underline{\underline{C}}_i)\right| - \frac{1}{2} (\underline{x} - \underline{\mu}_i)^T \underline{\underline{C}}_i^{-1} (x - \underline{\mu}_i) + \ln P_i$

a quadratic function in $\underline{x}$

Decision boundary β_{ij} :

quadratic equation in $\underline{x}$

2.3 Minimax decision Slide 2.17

2.2 → idea no. 1

$$\left.\begin{array}{ll} \text{data} & \underline{x} \\ \text{likelihood} & p(\underline{x} \mid w_j) \\ \text{prior} & p(w_j) \\ \text{loss} & l_{ij} \end{array}\right\} \text{all are well known in 2.2}$$

	2.3 idea 2	2.4 idea 3 for decision
data	✓	✓
likelihood	✓	✓
prior	X (unknown or varying)	X
loss	✓	X

No prior → no averaged loss / error rate
→ MBR, MAP impossible

2nd idea to design $\hat{\omega}(\underline{x})$: minimax

$\min_{\hat{\omega}}$ () the $\max_{p(w_j)}$ BR

$\max_{p(w_j)}$ ⟶ worst case prior

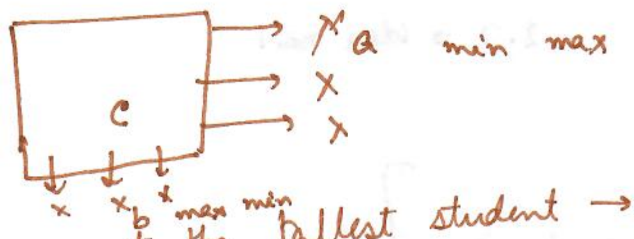
mini max

The inner operation has a larger impact.

E 2.8 cont

(c)

Students — the tallest ones - basket ball team



for each row get the tallest student → that is the basket ball team

for each column get the shortest student - weight lifter team

a → minimum ~~weight~~ height of tallest players

b → maximum height of shortest players

shortest basketball player a $\geq$ tallest weight lifter b

Proof $\quad a \geq c \quad b \leq c \Rightarrow a \geq b$

Bayesian Risk:

$BR = f($ likelihood, prior $P(w_j)$, $loss_{ij}$, $\hat{\omega}(\underline{x}))$

✓ $\quad$? $\quad$ ✓ $\quad$ ✓

$\hat{=} y \qquad \hat{=} x$

(a) $\min\limits_{\hat{\omega}()} \max\limits_{P(w_j)} BR$ $\quad$ 2·3

(b) $\max\limits_{p(w_j)} \min\limits_{\hat{\omega}()} BR$

2.2· MBR where $p(w_j)$ is known

In general : $a \geq b$; $\hat{\omega}()$, $P(w_j)$ of $a \neq \hat{\omega}()$, $P(w_j)$ of b

In case of binary class, they are equal.

Proof in books.

i.e. we determine the minimax soln by maximin:

- determine the soln of MBR analytically for any $P(\omega_1) = P_1$

$P(\omega_2) = P_2 = 1 - P_1$

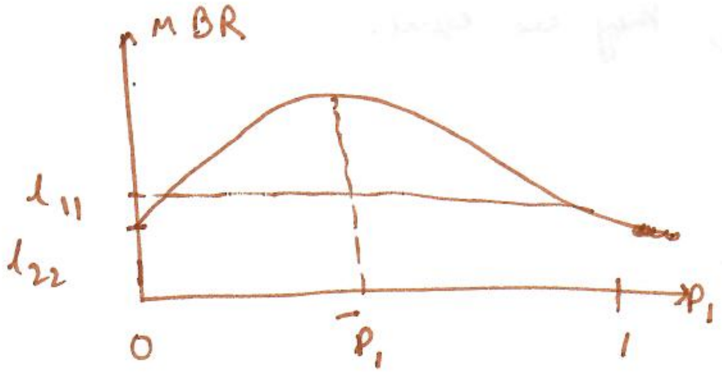
$$\hat{\omega}(\underline{x}, P_1) : \frac{P(\underline{x} \mid \omega_2)}{P(\underline{x} \mid \omega_1)} \underset{\omega_1}{\overset{\omega_2}{\gtrless}} \gamma(P_1) = \frac{l_{21} - l_{11}}{l_{12} - l_{22}} \cdot \frac{P_1}{1 - P_1}$$

$$MBR(\ ) = \underset{\underline{x}, \omega}{E}\left[l\left(\hat{\omega}(\underline{x}, P_1), \omega\right)\right]$$

- determine the worst case of P_1 :-

$$\max_{P_1} MBR(P_1)$$

$$\left.\frac{d}{dP_1} MBR(P_1)\right|_{P_1 = \bar{P}_1} \stackrel{!}{=} 0 \implies \text{worst case } \bar{P}_1 \implies \gamma(\bar{P}_1)$$



Again, for 2 class problem, min max is a ~~log~~ LRT:

$$LR \gtrless \gamma(\bar{P}_1)$$

2.4 Neyman-Pearson decision

- 2-class decision problem
- $p(\underline{x} / \omega_j)$ known
- $P(\omega_j)$ no prior
- no loss l_{ij}

$p(\hat{\omega} / \omega)$		ω_1	ω_2
$\hat{\omega}$	$\hat{\omega}_1$	$1 - P_{FA}$	$1 - P_D$
	$\hat{\omega}_2$	P_{FA}	P_D

Performance measures:

detection rate $P_D = TPR = P(\hat{\omega} = \omega_2 \mid \omega = \omega_2)$

false alarm rate $P_{FA} = FPR = P(\hat{\omega} = \omega_2 \mid \omega = \omega_1)$

Instead of BR, $\max_{\hat{\omega}()} P_D$ subject to (st) $P_{FA} \leq \alpha$.

α is the threshold different according to application.

eg $\alpha = 5\%$, 1% ... depending on app^n^

No need for any loss or priors.

A constraint optimisation problem:

Neyman - Pearson theorem:

- optimum decision rule = LRT.

$$LR = \frac{p(\underline{x} | \omega_2)}{p(\underline{x} | \omega_1)} \underset{\omega_1}{\overset{\omega_2}{\gtrless}} \gamma \text{ (defined by upperbound of } \alpha)$$

$\Rightarrow$ decision region ω_2 : $R_2(\gamma) = \{ \underline{x} \mid LR(\underline{x}) > \gamma \}$

γ is determined from

$$P_{FA} = \int_{R_2(\gamma)} p(\underline{x} | \omega_1) \, d\underline{x} \doteq \alpha$$

NP detector / Constant false alarm rate (CFAR) detector.