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Differential Equations

Assignment

824EE1067

Sai Swaroop

(B5)

(1.) $y' = 2xy$ $y(0) = 1$; $x_0 = 0$; $y_0 = 1$, $f(x, y) = 2xy$

Using Picards iteration method.

$$y = y_0 + \int_{x_0}^{x} f(x, y_0)\, dx$$

1st iteration

$$y_1 = y_0 + \int_{x_0}^{x} f(x, y_0)\, dx$$

$$y_1 = 1 + \int_0^x 2x\, dx = 1 + x^2$$

2nd iteration

$$y_2 = y_0 + \int_{x_0}^{x} f(x, y_1)\, dx$$

$$= 1 + \int_0^x 2x(1+x^2)\, dx = 1 + x^2 + \frac{x^4}{2}$$

3rd iteration

$$y_3 = y_0 + \int_{x_0}^{x} f(x, y_2)\, dx$$

$$= 1 + \int_0^x x\left(1 + x^2 + \frac{x^4}{2}\right) = 1 + x^2 + \frac{x^4}{2} + \frac{x^6}{6}$$

||ly $y_n = 1 + \frac{x^2}{1!} + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots \frac{(x^n)^2}{n!}$

→ y_n is expression of e^{x^2}

→ Solution of this D.E is e^{x^2}

(2.)

$$y''' - 3y'' - 6y' + 8y = x$$

$$(D^3 - 3D^2 - 6D + 8)\, y = x$$

$$y = \frac{x}{(D^3 - 3D^2 - 6D + 8)} = \frac{x}{8\left(\frac{D^3}{8} - \frac{3D^2}{8} - \frac{6D}{8} + 1\right)}$$

$$y = \frac{x}{8}\left(1 + \frac{D^3}{8} - \frac{3D^2}{8} - \frac{6D}{8}\right)^{-1}$$

$$y = \frac{x}{8}\left(1-\left(\frac{D^3}{8}-\frac{3D^2}{8}-\frac{6D}{8}\right)\right)$$

$$y = \frac{1}{8}\left(x-0-0+\frac{6}{8}\right)$$

$$y = \frac{x}{8}+\frac{3}{32}$$

$$\boxed{y_p = \frac{x}{8}+\frac{3}{32}}$$

General sol for homogeneous eqn is

$$y''' - 3y'' - 6y' + 8y = 0$$

$$r^3 - 3r^2 - 6r + 8 = 0$$

$$(r-1)(r-4)(r+2) = 0 \;\}\; r = 1, 4, -2$$

$$Sol^n = c_1 e^{x} + c_2 e^{4x} + c_3 e^{-2x}$$

Complete solution $\}$ $c_1 e^{x} + c_2 e^{4x} + c_3 e^{-2x} + \frac{x}{8} + \frac{3}{32}$

(3.) $(1+x)^2 y'' - 3xy' + 4y - x^2 = 3y'$

$$(x+1)^2 y'' - 3(x+1)y' + 4y = x^2$$

Let $x+1$ be X

$$X^2 y'' - 3Xy' + 4y = (X-1)^2$$

This is euler cauchy eqn.

Let $X = e^t$; $t = \log X$; $\frac{dX}{dt} = e^t$; $\frac{dy}{dX} = e^{-t}\frac{dy}{dt}$

$$\frac{d^2y}{dX^2} = e^{-2t}\left[\frac{d^2y}{dt^2} - \frac{dy}{dt}\right]$$

$$\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)\cancel{e^{-2t}}\cdot\cancel{e^{2t}} - 3\cancel{e^{t}}\left(\cancel{e^{-t}}\cdot\frac{dy}{dt}\right) + 4y = (e^t-1)^2$$

$$\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 4y = (e^t-1)^2$$

$$(D^2 - 4D + 4)\,y = (e^t-1)^2$$

corresponding homogeneous equation :-

$$y'' - 4y' + 4y = 0$$

$$r^2 - 4r + 4 = 0 \;\}\; r = 2, 2$$

sol of homogeneous eqn $\}$ $c_1 e^{2t} + c_2 t e^{2t}$

for non-homogeneous.

$$y = \frac{(e^t-1)^2}{D^2-4D+4} \;\Big\}\; \frac{1}{D^2-4D+4}e^{2t} + \frac{e^{0t}}{D^2-4D+4} - \frac{2e^{t}}{D^2+4D+4}$$

$$= \frac{1}{(D-2)^2}e^{2t} + \frac{1}{(D-2)^2}e^{0t} - \frac{2e^{t}}{(D-2)^2}$$

$$= \frac{t^2 e^{2t}}{2!} + \frac{e^{0t}}{(-2)^2} - \frac{2e^{t}}{(-1)^2}$$

$$= \frac{t^2 e^{2t}}{2} + \frac{1}{4} - 2e^{t}$$

Complete solution $\}$ $\frac{t^2 e^{2t}}{2} + \frac{1}{4} - 2e^{t} + c_1 e^{2t} + c_2 t e^{2t}$

$$e^t \to x+1$$

$$t \to \log(x+1)$$

$$\frac{\log(1+x)^2 (x+1)^2}{2} - 2(x+1) + \frac{1}{4} + c_1(x+1)^2 + c_2 \log(x+1)(x+1)^2$$

$$y = c_1(x+1)^2 + c_2\log(x+1)\cdot(x+1)^2 + \frac{(\log(x+1))^2(x+1)^2}{2} - 2(x+1) + \frac{1}{4}$$

(4.) $y'' - y = \frac{2}{1+e^{x}}$

corresponding homogeneous equation

$$y'' - y = 0$$

$$r^2 - 1 = 0;\; r = +1, -1$$

solution $\}$ $c_1 e^{x} + c_2 e^{-x}$ $\}$ $y_1 = e^{x}$, $y_2 = e^{-x}$

for non-homogeneous equation, by method of variation of parameters:

$$y_p = -y_1\int \frac{y_2\, r(x)}{W}\,dx + y_2\int \frac{y_1\, r(x)}{W}\,dx$$

$$w(y_1, y_2) = \begin{vmatrix} e^{x} & e^{-x} \\ e^{x} & -e^{-x} \end{vmatrix} = -1 - 1 = -2$$

$$y_p(x) \Rightarrow e^{x}\int \frac{e^{-x}}{1+e^{x}}\,dx - e^{-x}\int \frac{e^{x}}{1+e^{x}}\,dx \quad \Big\}\ 1+e^{x} = t,\ e^{x}dx = dt,\ dx = \frac{dt}{e^{x}}$$

$$e^{x}\int \frac{dt}{t^2(1+t)} - e^{-x}\int \frac{dt}{t}$$

$$\downarrow \qquad -e^{-x}\log|1+e^{x}|$$

$$e^{x}\int \frac{At+B}{t^2} + \frac{C}{(1+t)} \quad \Big\}\ At + B + At^2 + Bt + t^2C = 1$$

on comparing we get

$B = 1;\ A = -1,\ C = 1$

$$y_p(x) = e^{x}\left[-e^{-x} - \log e^{x} + \log|1+e^{x}|\right] - e^{-x}\log|1+e^{x}|$$

$$\hookrightarrow (e^{x} - e^{-x})\log(1+e^{x}) - e^{x}\log e^{x} - 1$$

final solution $\Big\}$ $\boxed{y = y_p + y_1 = c_1 e^{-x} + c_2 e^{x} + (e^{x} - e^{-x})\log(1+e^{x}) - x e^{x} - 1}$

5. Let x be the population of bacteria at time t, initial population $x_0 = 300$.

$$\frac{dx}{dt} \propto x,\quad \frac{dx}{dt} = kx$$

$$\int \frac{dx}{x} = k\int dt$$

$$\ln x = kt + c$$

at $t = 0$, $x = x_0 = 300$; $\ln(300) = c$

at $t = 12$ hr, $x = x_0 = \frac{300 \times 20}{100} + 300$, $x_{12} = 360$

$$\ln(360) = k(12) + \ln(300)$$

$$k = \frac{1}{12}\ln\left(\frac{6}{5}\right)$$

$$\ln x = \frac{1}{12}\ln\left(\frac{6}{5}\right) \times 48 + \ln(300)$$

$$\ln x = \ln\left(\frac{6}{5}\right)^4 + \ln(300)$$

$$\ln x = \ln\left(\left(\frac{6}{5}\right)^4 \times 300\right)$$

$$x = \left(\frac{6}{5}\right)^4 \times 300 \ \Big\}\ \boxed{x = 622.08}$$

The population of the bacteria after 2 days would be 622.

6. $(D^3 + 2D^2 - D - 2)y = e^{x} + x^2$

$$y''' + 2y'' - y' - 2y = e^{x} + x^2$$

corresponding homogeneous equation

$$y''' + 2y'' - y' - 2y = 0$$

$$r^3 + 2r^2 - r - 2 = 0$$

$$(r-1)(r+1)(r+2) = 0$$

$$r = 1, -1, -2$$

$$y_{homo} = c_1 e^{x} + c_2 e^{-x} + c_3 e^{-2x}$$

non-homogeneous equation

$$y''' + 2y'' - y' - 2y = e^{x} + x^2$$

trial solution is $Axe^{x} + Bx^2 + (Cx + D) = y$

$$y' = A(xe^{x} + e^{x}) + 2Bx + C$$

$$y'' = A(2e^{x} + xe^{x}) + 2B$$

$$y''' = A(3e^{x} + xe^{x})$$

Put in eqn

$$e^{x}(3A + 4A - A) + xe^{x}(A + 2A - A - 2A) + x(-2B - 2C) + x^2(-2B) + (4B - C - 2D) = e^{x} + x^2$$

$$6Ae^{x} + 0 + x(-2B - 2C) + x^2(-2B) + 4B - C - 2D = e^{x} + x^2$$

by comparing coefficients

$6A = 1$; $\boxed{A = \frac{1}{6}}$ $-2B = 1$; $\boxed{B = -1/2}$

$-2B - 2C = 0$; $\boxed{C = 1/2}$; $4B - C - 2D = 0$; $\boxed{D = -5/4}$

$$y_{non\,homo} = \frac{1}{6}xe^{x} - \frac{1}{2}x^2 + \frac{1}{2}x - \frac{5}{4}$$

Complete solution $\Big\}$

$$y = c_1 e^{x} + c_2 e^{-x} + c_3 e^{-2x} + \frac{1}{6}xe^{x} - \frac{1}{2}x^2 + \frac{1}{2}x - \frac{5}{4}$$

7. $(5D+4)y-(2D+1)z=e^{-x}$ — (1)

$(D+8)y-\cancel{3}z=5e^{-x}$ — (2)

$(D+8)y-5e^{-x}=3z$

$z=\frac{1}{3}\left((D+8)y-5e^{-x}\right)$

Put z in (1)

$(5D+4)y-(2D+1)\left(\frac{1}{3}\left[(D+8)y-5e^{-x}\right]\right)=e^{-x}$

$(5D+4)y-\frac{1}{3}(2D^2+17D+8)y+\frac{5}{3}(2D+1)e^{-x}=e^{-x}$

$5y'+4y-\frac{2}{3}y''-\frac{17}{3}y'-\frac{8}{3}y-\frac{10e^{-x}+5e^{-x}}{3}=e^{-x}$

$15y'+12y-2y''-17y'-8y=8e^{-x}$

$y''+y'-2y=-4e^{-x}$

corresponding homogeneous eqn

$y''+y'-2y=0$

$r^2+1-2=0\}\ r=1,-2\}\ \text{sol}\}\ C_1e^{x}+C_2e^{-2x}$

non-homogeneous eqn

$y''+y'-2y=-4\cancel{x}e^{-x}$ — (3)

trial solution be $y=Ae^{-x}$, $y'=-Ae^{-x}$; $y''=Ae^{-x}$

put in (3)

$\cancel{Ae^{-x}}+(-\cancel{Ae^{-x}})-\cancel{2}(A\cancel{e^{-x}})=\frac{2}{\cancel{-4e^{-x}}}$

$\boxed{A=2}$

$y_{\text{non-homo}}=2Ae^{-x}$

Complete solution} $C_1e^{x}+C_2e^{-2x}+2e^{-x}$

$z=\frac{1}{3}\left[(D+8)y-5e^{-x}\right]$

$=\frac{1}{3}\left[(D+8)(C_1e^{x}+C_2e^{-2x}+2e^{-x})-5e^{-x}\right]$

$=\frac{1}{3}\left[9C_1e^{x}+6C_2e^{-2x}+14e^{-x}\right)-5e^{-x}]$

$=\frac{1}{3}\left[9C_1e^{x}+6C_2e^{-2x}+9e^{-x}\right]$

$z=3C_1e^{x}+2C_2e^{-2x}+3e^{-x}$

$y=C_1e^{x}+C_2e^{-2x}+2e^{-x}$

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8. $(x^2-1)y''+xy'-y=0$ — (1)

$y''+\left(\frac{x}{x^2-1}\right)y'-\left(\frac{1}{x^2-1}\right)y=0$

$\lim_{x\to0}p(x)=\lim_{x\to0}\frac{x}{x^2-1}=0$; $\lim_{x\to0}q(x)=\lim_{x\to0}-\frac{1}{x^2-1}=1$

Let $y=\sum_{m=0}^{\infty}a_mx^m$ be sol of (1)

$y'=\sum_{m=0}^{\infty}ma_mx^{m-1}$; $y''=\sum_{m=0}^{\infty}m(m-1)a_mx^{m-2}$

$(x^2-1)\left(\sum_{m=0}^{\infty}m(m-1)a_mx^{m-2}\right)+x\left(\sum_{m=0}^{\infty}ma_mx^{m-1}\right)-\sum_{m=0}^{\infty}a_mx^m=0$

$\sum_{m=0}^{\infty}(m)(m-1)a_mx^m-\sum_{m=0}^{\infty}(m+2)(m+1)a_{m+2}x^m+\sum_{m=0}^{\infty}ma_mx^m-\sum_{m=0}^{\infty}a_mx^m=0$

$x^m\left(\sum_{m=0}^{\infty}m(m-1)a_m-(m+2)(m+1)a_{m+2}+ma_m-a_m\right)=0$

$m(m-1)a_m-(m+2)(m+1)a_{m+2}+ma_m-a_m=0$

$(m+2)\cancel{(m+1)}a_{m+2}=a_m((m-1)\cancel{(m+1)})$

$a_{m+2}=\frac{a_m(m-1)}{m+2}$

$m=0 \Rightarrow a_2 = \frac{-a_0}{2}$ $m=3 \Rightarrow a_5 = \dots = 0$

$m=1 \Rightarrow a_3 = 0$ $m=4 \Rightarrow a_6 = \frac{a_4 3}{6} = -\frac{a_0}{16}$

$m=2 \Rightarrow a_4 = \frac{a_2}{4} = -\frac{a_0}{8}$

$Sol^n \Rightarrow y = \sum_{m=0}^{\infty} a_m x^m = a_0 + x a_1 + x^2 a_2 + a_3 x^3 + \dots$

$$y = a_0\left(1 - \frac{x^2}{2} - \frac{x^4}{8} - \frac{x^6}{16} + \dots\right) + a_1 x$$

$$\boxed{\therefore \text{final ans} \} \; y = a_0\left(1 - \frac{x^2}{2} - \frac{x^4}{8} - \frac{x^6}{16} + \dots\right) + a_1 x}$$

(Q6) $2x^2 y'' + (x^2 - x) y' + y = 0$ —(1)

$$y'' + \left(\frac{x^2 - x}{2x^2}\right) y' + \left(\frac{1}{2x^2}\right) y = 0$$

$\lim_{x\to 0} P(x) = \lim_{x\to 0} \frac{x^2 - x}{x^2} = DNE$, $\lim_{x\to 0} q(x) = \lim_{x\to 0} \frac{1}{2x^2} = DNE$

$x=0$ is not a ordinary point

$\lim_{x\to 0} x P(x) = \lim_{x\to 0} \frac{x^3 - x^2}{2x^2} = -\frac{1}{2} = P_0$

$\lim_{x\to 0} x^2 q(x) = \lim_{x\to 0} \frac{x^2}{2x^2} = \frac{1}{2} = q_0$

$\Rightarrow x$ is a regular singular solution

Indicial equation

$$r^2 - \frac{3}{2} r + \frac{1}{2} = 0$$

$$2r^2 - 3r + 1 = 0 \Rightarrow r = \frac{1}{2}, 1$$

Now,

$$y = x^r \sum_{m=0}^{\infty} a_m x^m$$

$$y = \sum_{m=0}^{\infty} a_m x^{m+r}$$

$$y' = \sum_{m=0}^{\infty} (m+r) a_m x^{(m+r-1)}$$

$$y'' = \sum_{m=0}^{\infty} (m+r)(m+r-1) a_m x^{(m+r-2)}$$

Put in (1)

$$2\sum_{m=0}^{\infty} (m+r)(m+r-1) a_m x^{m+r} + \sum_{m=0}^{\infty} (m+r) a_m x^{m+r+1} - \sum_{m=0}^{\infty} a_m (m+r) x^{m+r}$$

$$+ \sum_{m=0}^{\infty} a_m x^{m+r} = 0$$

$$2(r)(r-1) a_0 x^r + 2\sum_{m=1}^{\infty} (m+r)(m+r-1) a_m x^{m+r} + \sum_{m=1}^{\infty} (m+r-1) a_{m-1} x^{m+r}$$

$$- r a_0 x^r - \sum_{m=1}^{\infty} (m+r) a_m x^{m+r} + a_0 x^r + \sum_{m=1}^{\infty} a_m x^{m+r} = 0$$

$$(2r^2 - 3r + 1) a_0 x^r + \sum_{m=1}^{\infty} \left(2(m+r)(m+r-1) + (m+r-1) a_{m-1} - (m+r) a_m + a_m\right] x^{m+r} = 0$$

Indicial eqn so "0"

$$a_m = \frac{-(m+r-1) a_{m-1}}{2(m+r)(m+r-1) - (m+r) + 1}$$

Put $r = \frac{1}{2}$

$$a_m = \frac{-(m - 1/2) a_{m-1}}{(m - 1/2)(2(m+1/2) - 1)} \Bigg\} a_m = \frac{-a_{m-1}}{2m}$$

$a_1 = -\frac{a_0}{2}$, $a_2 = \frac{a_0}{8}$, $a_3 = \frac{-a_0}{48}$

$$y_1 = a_0 x^r + a_1 x^{1+r} + a_2 x^{2+r} \dots$$

$$\boxed{y_1 = x^{1/2}\left[1 - \frac{x}{2} + \frac{x^2}{8} - \frac{x^3}{48} \dots\right] a_0}$$

for $r = 1$

$$a_m = \frac{-a_{m-1}}{2m+1}$$

$a_1 = -\frac{1}{3} a_0$; $a_2 = \frac{a_0}{15}$; $a_3 = -\frac{a_0}{105}$

$$y_2 = x a_0\left[1 - \frac{x}{3} + \frac{x^2}{15} - \frac{x^3}{105} \cdots\right]$$

Complete solution

$$y = C_1 y_1 + C_2 y_2$$

$$\boxed{\text{final solution}\Big\}\; y = C_1\left[x^{1/2} a_0\left(1 - \frac{x}{2} + \frac{x^2}{8} - \frac{x^3}{48} \cdots\right)\right] + C_2\left[x a_0\left(1 - \frac{x}{8} + \frac{x^2}{15} \cdots\right)\right]}$$

(10) $x^2 y'' + x y' + (x^2 - 1/4) y = 0$

$$y'' + \frac{1}{x} y' + \left(1 - \frac{1}{4x^2}\right) y = 0 \quad \text{—(1)}$$

$\lim_{x\to 0} p(x) = \lim_{x\to 0} \frac{1}{x} = \text{DNE}$, $\lim_{x\to 0} q(x) = \lim_{x\to 0}\left(1 - \frac{1}{4x^2}\right) = \text{DNE}$

$x = 0$ is not a ordinary point

$\lim_{x\to 0} x\,p(x) = \lim_{x\to 0} x \cdot \frac{1}{x} = 1 = p_0$

$\lim_{x\to 0} x^2 q(x) = \lim_{x\to 0} x^2 - \lim_{x\to 0} \frac{1}{4} = -\frac{1}{4} = q_0$

$x = 0$ is a regular singular point

Indicial equation,

$$r^2 - \frac{1}{4} = 0; \quad \boxed{r = \frac{1}{2}, -\frac{1}{2}}$$

$$y = x^r \sum_{m=0}^{\infty} a_m x^m$$

$$\left.\begin{aligned} y' &= \sum_{m=0}^{\infty} (m+r) a_m x^{m+r-1} \\ y'' &= \sum_{m=0}^{\infty} (m+r)(m+r-1) a_0 x^{m+r-2} \end{aligned}\right\} \text{Put in (1)}$$

$$\sum_{m=0}^{\infty} (m+r)(m+r-1) a_m x^{m+r} + \sum_{m=0}^{\infty} (m+r) a_m x^{m+r} + \sum_{m=2}^{\infty} a_{m-2} x^{m+r} - \frac{1}{4}\sum_{m=0}^{\infty} a_m x^{m+r} = 0$$

Coefficient of x^{m+r}

$$(m+r)(m+r-1) a_m + (m+r) a_m + a_{m-2} - \frac{a_m}{4} = 0$$

$$a_m = \frac{-a_{m-2}}{(m+r)(m+r-1) + (m+r) - \frac{1}{4}}$$

for $r = 1/2$

$$a_m = \frac{-a_{m-2}}{m^2 + m}$$

$$a_2 = \frac{-a_0}{6};\; a_3 = \frac{-a_1}{12};\; a_4 = \frac{-a_2}{20} = \frac{a_0}{120};\; a_5 = \frac{-a_3}{36} = \frac{a_1}{360};$$

$a_0 \neq 1/2$

$$y_1 = a_0 x^{1/2}\left[1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} \cdots\right] + a_1 x^{1/2}\left[x - \frac{2x^3}{4!} + \frac{2x^5}{6!} - \frac{2x^7}{8!} \cdots\right]$$

for $r = -1/2$

$$a_m = \frac{-a_{m-2}}{m(m-1)}$$

$$a_2 = \frac{-a_0}{2!};\; a_3 = \frac{-a_1}{3!};\; a_4 = \frac{a_0}{4!};\; a_5 = \frac{a_1}{5!};\; a_6 = \frac{-a_0}{6!};\; a_7 = \frac{-a_1}{7!}$$

$$y_2 = a_0 x^{-1/2}\left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \cdots\right] + a_1 x^{-1/2}\left[1 - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}\right]$$

Complete solution :-

$$\boxed{\begin{aligned} y = C_1&\left[a_0 x^{1/2}\left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} \cdots\right) + a_1 x^{1/2}\left(x - \frac{2x^3}{4!} + \frac{2x^5}{6!} - \frac{2x^7}{8!} \cdots\right)\right] \\ + C_2&\left[a_0 x^{-1/2}\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \cdots\right) + a_1 x^{-1/2}\left(1 - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \cdots\right)\right] \end{aligned}}$$