Proof by Induction

Outline

This topic gives an overview of the mathematical technique of a proof by induction

- We will describe the inductive principle
- Look at a few different examples
- A few examples where the technique is incorrectly applied
- Well-ordering of the natural numbers
- Strong induction

Definition

Suppose we have a formula $F(n)$ which we wish to show is true for all values $n \geq n_0$

- Usually $n_0 = 0$ or $n_0 = 1$

For example, we may wish to show that

$$F(n) = \sum_{k=0}^{n} k = \frac{n(n+1)}{2}$$

for all $n \geq 0$

Definition

We then proceed by:

- Demonstrating that $F(n_0)$ is true
- Assuming that the formula $F(n)$ is true for an arbitrary n
- If we are able to demonstrate that this assumption allows us to also show that the formula is true for $F(n + 1)$, the *inductive principle* allows us to conclude that the formula is true for all $n \geq n$

Definition

Thus, if $F(n_0)$ is true, $F(n_0 + 1)$ is true

and, if $F(n_0 + 1)$ is true, $F(n_0 + 2)$ is true

and, if $F(n_0 + 2)$ is true, $F(n_0 + 3)$ is true

and so on, and so on, for all $n \geq n$

Formulation

- Often $F(n)$ is an equation:
  - For example, $F(n)$ may be an equation such as:

$$\sum_{k=0}^{n} k = \frac{n(n+1)}{2} \quad \text{for } n \geq 0$$

$$\sum_{k=1}^{n} 2k - 1 = n^2 \quad \text{for } n \geq 1$$

$$\sum_{k=0}^{n} 2^k = 2^{n+1} - 1 \quad \text{for } n \geq 0$$

- It may also be a statement:
  - The integer $n^3 - n$ is divisible by 3 for all $n \geq 1$

Examples

We will now look at a few examples

At each case, we will show the inductive process...

Example 1

Prove that $\sum_{k=0}^{n} k = \frac{n(n+1)}{2}$ is true for $n \geq 0$

- When $n = 0$: $\sum_{k=0}^{0} k = 0 = \frac{0(0+1)}{2}$
- Assume that the statement is true for a given n: $\sum_{k=0}^{n} k = \frac{n(n+1)}{2}$
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$$\sum_{k=0}^{n+1} k = (n+1) + \sum_{k=0}^{n} k$$

Example 1

Prove that $\sum_{k=0}^{n} k = \frac{n(n+1)}{2}$ is true for $n \geq 0$

- When $n = 0$: $\sum_{k=0}^{0} k = 0 = \frac{0(0+1)}{2}$
- Assume that the statement is true for a given n: $\sum_{k=0}^{n} k = \frac{n(n+1)}{2}$
- We now show:

$$\sum_{k=0}^{n+1} k = (n+1) + \sum_{k=0}^{n} k$$
$$= (n+1) + \frac{n(n+1)}{2}$$
$$= \frac{2(n+1) + n(n+1)}{2}$$
$$= \frac{(n+2)(n+1)}{2} = \frac{(n+1)(n+2)}{2}$$

Example 2

Prove that the sum of the first n odd integers is n^2:

$$\sum_{k=1}^{n} 2k - 1 = n^2 \quad \text{for } n \geq 1$$

- When $n = 1$: $\sum_{k=1}^{1} 2k - 1 = 1 = 1^2$
- Assume that the statement is true for a given n: $\sum_{k=1}^{n} 2k - 1 = n^2$
-

$$\sum_{k=1}^{n+1} 2k - 1 = 2(n+1) - 1 + \sum_{k=0}^{n} 2k - 1$$

Example 2

Prove that the sum of the first n odd integers is n^2:

$$\sum_{k=1}^{n} 2k - 1 = n^2 \quad \text{for } n \geq 1$$

- When $n = 1$: $\sum_{k=1}^{1} 2k - 1 = 1 = 1^2$
- Assume that the statement is true for a given n: $\sum_{k=1}^{n} 2k - 1 = n^2$
- We now show:

$$\begin{aligned}
\sum_{k=1}^{n+1} 2k - 1 &= 2(n+1) - 1 + \sum_{k=1}^{n} 2k - 1 \\
&= 2(n+1) - 1 + n^2 \\
&= 2n + 2 - 1 + n^2 \\
&= n^2 + 2n + 1 \\
&= (n+1)^2
\end{aligned}$$

Example 3

Prove that $\sum_{k=0}^{n} 2^k = 2^{n+1} - 1$ for $n \geq 0$

- When $n = 0$: $\sum_{k=0}^{0} 2^k = 2^0 = 1 = 2^{0+1} - 1$
- Assume that the statement is true for a given n: $\sum_{k=0}^{n} 2^k = 2^{n+1} - 1$
-

$$
\begin{aligned}
\sum_{k=0}^{n+1} 2^k &= 2^{n+1} + \sum_{k=0}^{n} 2^k \\
&= 2^{n+1} + 2^{n+1} - 1 \\
&= 2 \cdot 2^{n+1} - 1 \\
&= 2^{n+2} - 1
\end{aligned}
$$

Example 4

Prove that $\sum_{k=0}^{n} r^k = \frac{1-r^{n+1}}{1-r}$

- When $n = 0$: $\sum_{k=0}^{0} r^k = 1 = \frac{1-r^1}{1-r}$
- Assume that the statement is true for a given n: $\sum_{k=0}^{n} r^k = \frac{1-r^{n+1}}{1-r}$
- We now show:

$$\sum_{k=0}^{n+1} r^k = r^{n+1} + \sum_{k=0}^{n} r^k$$

Example 4

Prove that $\sum_{k=0}^{n} r^k = \frac{1-r^{n+1}}{1-r}$

- When $n = 0$: $\sum_{k=0}^{0} r^k = 1 = \frac{1-r^1}{1-r}$
- Assume that the statement is true for a given n: $\sum_{k=0}^{n} r^k = \frac{1-r^{n+1}}{1-r}$
- We now show:

$$\begin{aligned}
\sum_{k=0}^{n+1} r^k &= r^{n+1} + \sum_{k=0}^{n} r^k \\
&= r^{n+1} + \frac{1-r^{n+1}}{1-r} \\
&= \frac{(1-r)\,r^{n+1}}{1-r} + \frac{1-r^{n+1}}{1-r} \\
&= \frac{(1-r)\,r^{n+1} + 1 - r^{n+1}}{1-r} \\
&= \frac{r^{n+1} - r \cdot r^{n+1} + 1 - r^{n+1}}{1-r} = \frac{1-r^{n+2}}{1-r}
\end{aligned}$$

Example 5

Prove by induction that $n^3 - n$ is divisible by 3 for all integers

- This is slightly different
  - Choose a base case, say $n = 0$
  - Next, prove that the truth of the formula for n implies the truth for $n + 1$
  - Also, prove that the truth of the formula for n also implies the truth for $n - 1$
- When $n = 0$: $0^3 - 0 = 0$ is divisible by 3
- Assume that the statement is true for a given n: $n^3 - \mathrm{n}$ is divisible by 3
- We now show:

$$\begin{aligned}(n+1)^3 - (n+1) &= n^3+3n^2+3n+1-n-1 \\ &= 3(n^2+n) + (n^3-n)\end{aligned}$$

$$\begin{aligned}(n-1)^3-(n-1) &= n^3 -3n^2+3n-1-n+1 \\ &= 3(n-n^2) + (n^3-n)\end{aligned}$$

In both cases, the first term is divisible by 3,
the second term is divisible by 3 by assumption;
Consequently, their sum must also be divisible by 3

Example 5

Of course, proof-by-induction may not always be the only approach:

Proving that $n^3 - n$ is divisible by 3 for all integers

- An alternative proof could follow by observing that all integers may be written as either

$$3m \qquad 3m+1 \qquad 3m+2$$

$$n^3 - n = (3m)^3 - (3m) = 3m(3m-1)(3m+1)$$

$$n^3 - n = (3m+1)^3 - (3m+1) = 3m(3m+1)(3m+2)$$

$$n^3 - n = (3m+2)^3 - (3m+2) = 3(m+1)(3m+1)(3m+2)$$

Example 6

Prove that $\ln(n!) \leq n\ln(n)$ for integer values of $n \geq 1$

- When $n = 1$: $\ln(1!) = \ln(1) = 0 = 1 \cdot \ln(1)$
- Assume that $\ln(n!) \leq n\ln(n)$
- We now show:

$$\ln((n+1)!) = \ln(n+1) + \ln(n!)$$

Example 6

Prove that $\ln(n!) \le n\ln(n)$ for integer values of $n \ge 1$

- When $n = 1$: $\ln(1!) = \ln(1) = 0 = 1 \cdot \ln(1)$
- Assume that $\ln(n!) \le n\ln(n)$
- We now show:

$$\begin{aligned}
\ln((n+1)!) &= \ln(n+1) + \ln(n!) \\
&\le \ln(n+1) + n\ln(n) \\
&< \ln(n+1) + n\ln(n+1) \\
&= (n+1) \cdot \ln(n+1)
\end{aligned}$$

Non-Examples

All of these examples have been examples where proof by induction satisfies the desired result

- What happens if it fails?
- We will look at two cases that makes the proof invalid:
  - The inductive step fails
  - The base case is false

Non-Example 1

Suppose we have an incorrect formula—what happens?

Recall Fibonacci numbers:

$$F(n) = \begin{cases} 1 & n = 0,1 \\ F(n-1) + F(n-2) & n \geq 2 \end{cases}$$

Suppose you saw that $F(2) = 2$ and $F(3) = 3$ and ask:

Is $F(n) = n$ for $n \geq 1$?

- When $n = 1$: $F(1) = 1$ by definition
- Assume that the statement is true for all $k = 1, 2, \ldots, n$: $F(k) = k$
- $$\begin{aligned} F(n+1) &= F(n) + F(n-1) \\ &= n + (n-1) \\ &= 2n - 1 \\ &\neq n+1 \end{aligned}$$

Thus, the formula is wrong!

Non-Example 2

In opposition to the statement that “x is a horse of a different color”, prove all horses are the same color

- A single horse has the same color as itself
- Assume that all horses in a set of size n have the same color
- Then, given a set of $n + 1$ horses $\{h_1, h_2, \ldots, h_n, h_{n+1}\}$, we may group them into two groups of size n:

  $$\{h_1, h_2, \ldots, h_n\} \text{ and } \{h_2, \ldots, h_n, h_{n+1}\}$$

- By assumption, all the horses in both sets of size n are the same color
- Therefore, all the horses in the set of $n + 1$ horses must be the same color

Problem: the inductive step fails if $n + 1 = 2$

- Given a set of two horses $\{\text{Sea Horse, Barbaro}\}$, there is no overlap in the two subsets $\{\text{Sea Horse}\}$ and $\{\text{Barbaro}$

Justification

The induction principle can either be assumed in a mathematical system as an axiom, or it can be derived from other axioms

Alternatively, it can be deduced from other axioms, such as:

- The natural numbers (0, 1, 2, 3, …) are linearly ordered
- Every natural number is either 0 or the successor of another natural number
- The successor is by definition greater than what it succeeds

Justification

You may ask: "Suppose you've proved some formula $F(n)$ by induction to be true for all $n = 0, 1, 2, \ldots$; all we really showed was that $F(0)$ is true—could it not fail for some larger value $F(k)$?

Suppose that there is a k such that $F(k)$ is false

- There must be a smallest value $k > 0$ such that
  $F(k - 1)$ is true but
  $F(k)$ is false
- But the inductive step showed that if $F(k - 1)$ is true, it must also be true that $F(k)$ is true!
- Thus, no such smallest k can exist
- Thus, no subset of $\mathbf{N}$ can have $F(n)$ be false

Strong Induction

A similar technique is *strong induction* where we replace the statement

- Assume that $F(n)$ true

with

- Assume that $F(n_0), F(n_0+1), F(n_0+2), \ldots, F(n)$ are all true

For example:

Prove that with 3 and 7 cent coins, it is possible to make exact change for any amount greater than or equal to 12 cents