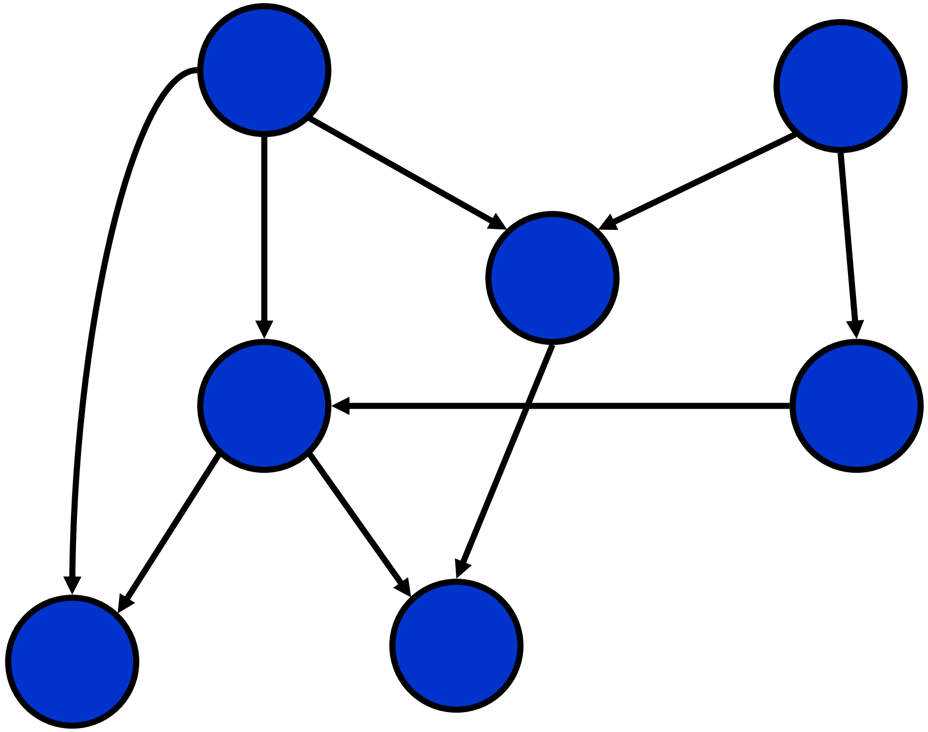
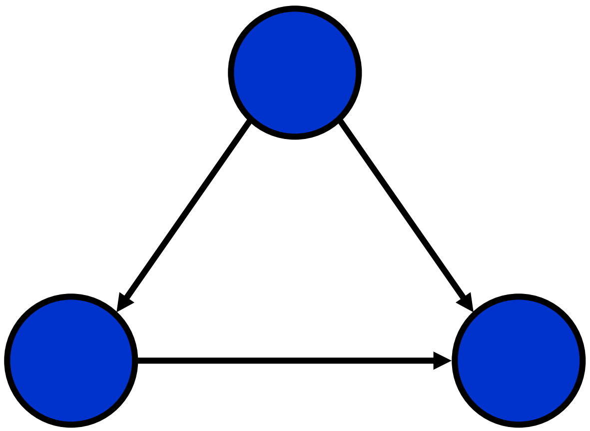
Graph Algorithms

Directed Acyclic Graphs

- A *directed acyclic graph* or *DAG* is a directed graph with no directed cycles:

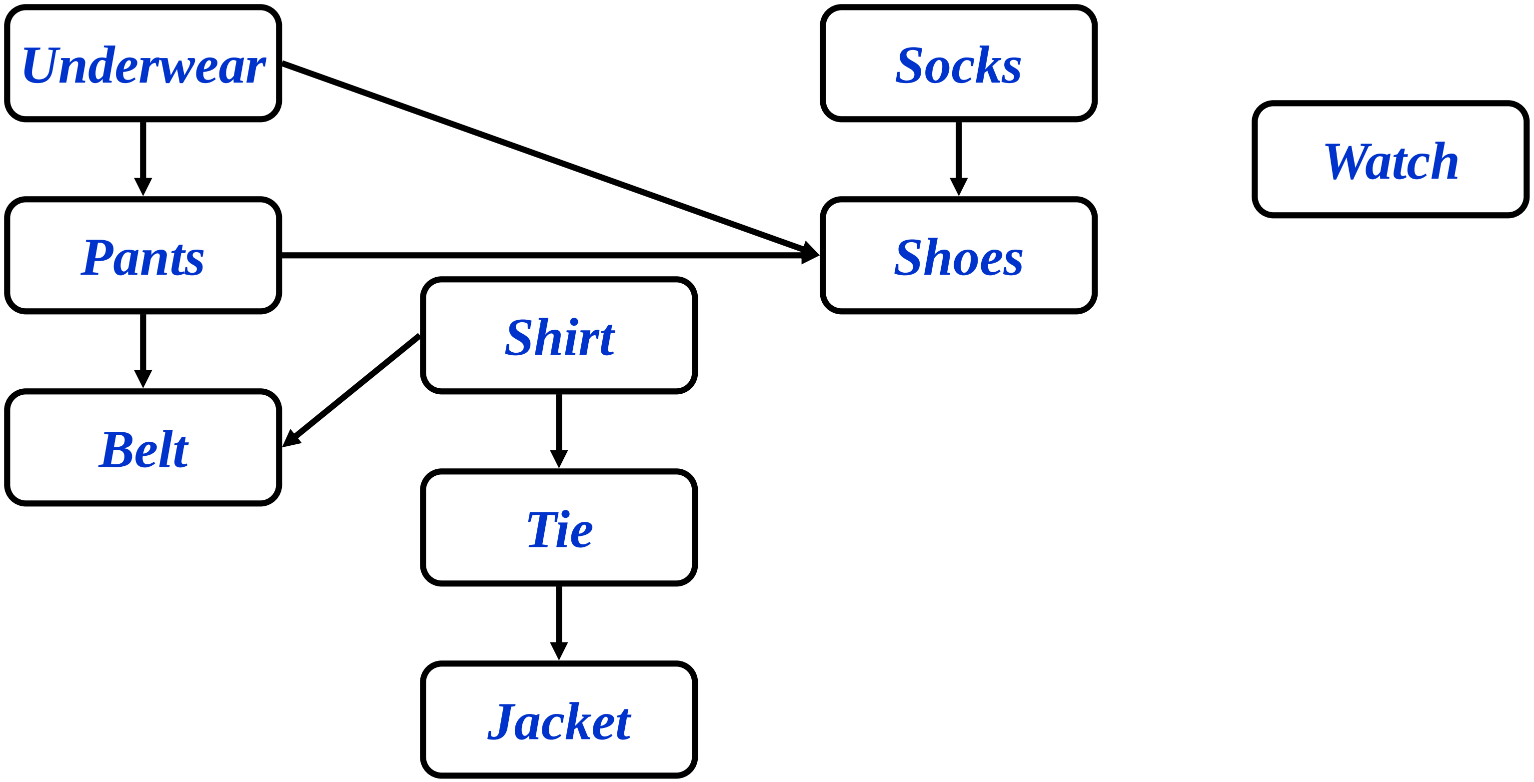




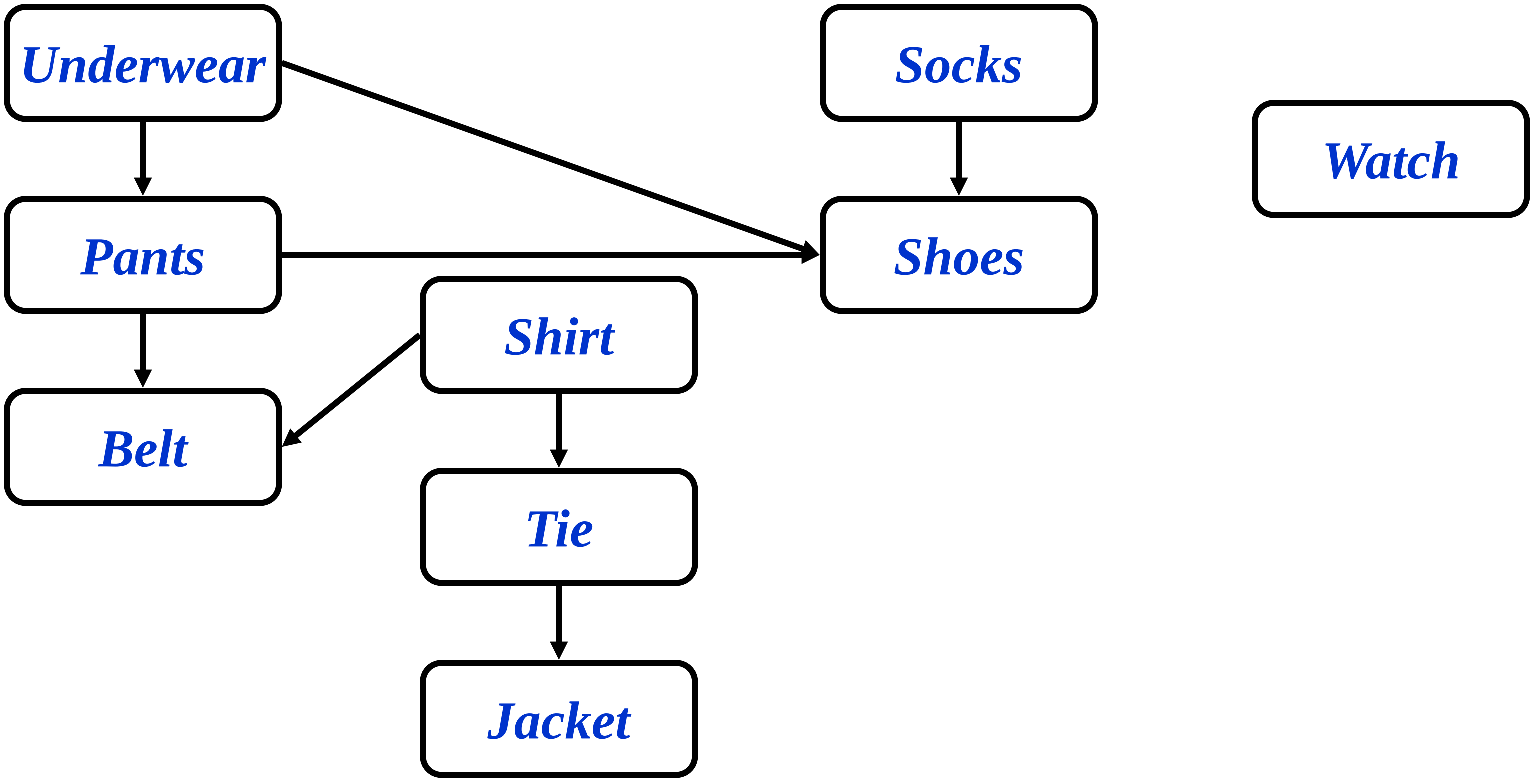
Topological Sort

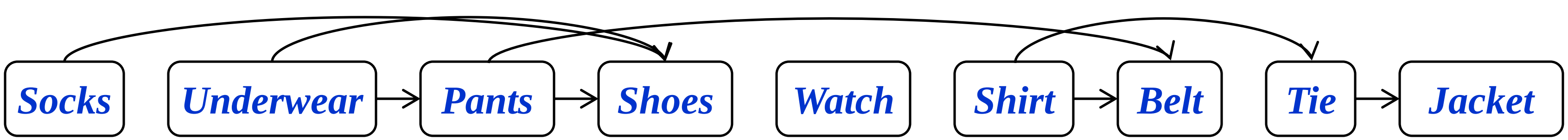
- *Topological sort* of a DAG:
  - Linear ordering of all vertices in graph G such that vertex u comes before vertex v if edge $(u, v) \in G$
- Real-world example: getting dressed

Getting Dressed



Getting Dressed





Topological Sort Algorithm

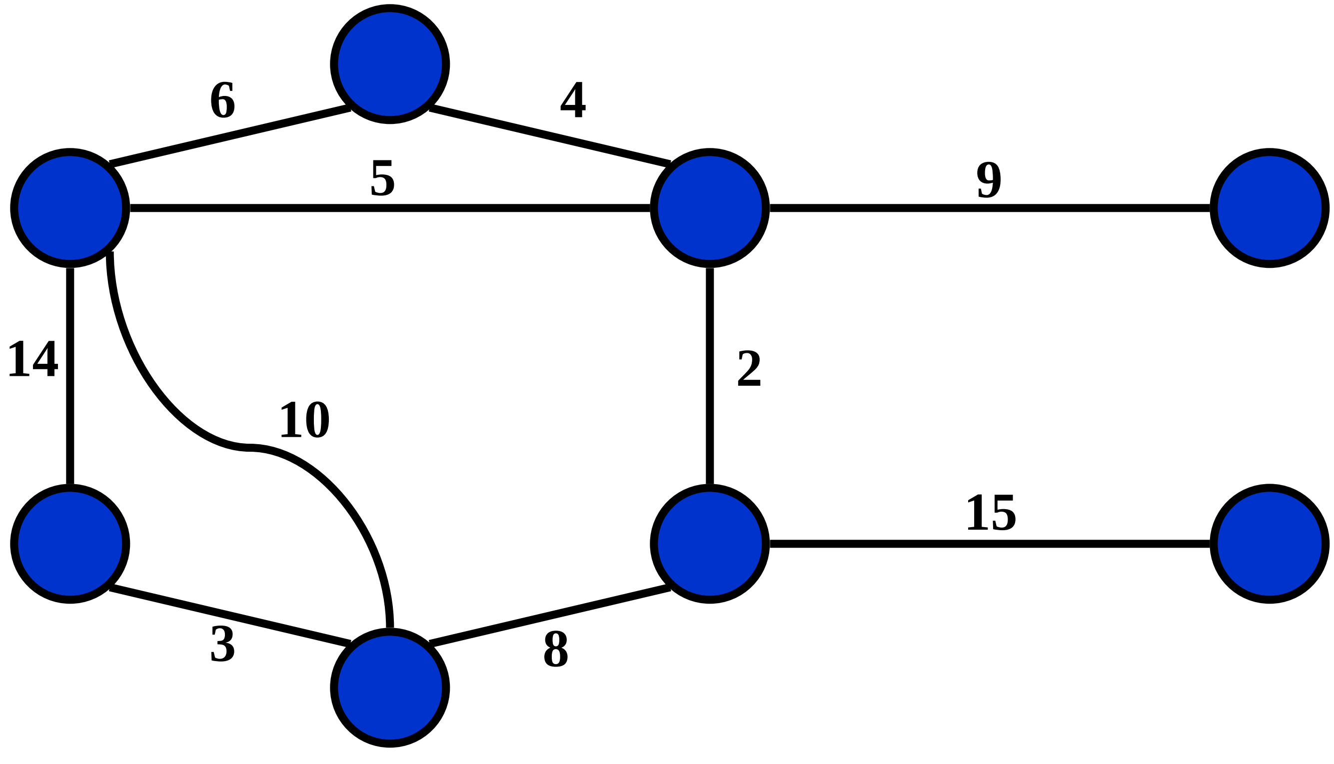
```
Topological-Sort()
{
   Run DFS
   When a vertex is finished, output it
   Vertices are output in reverse
     topological order
}
```

- Time: O(V+E)
- Correctness: Want to prove that

$$(u,v) \in G \Rightarrow u{\rightarrow}f > v{\rightarrow}f$$

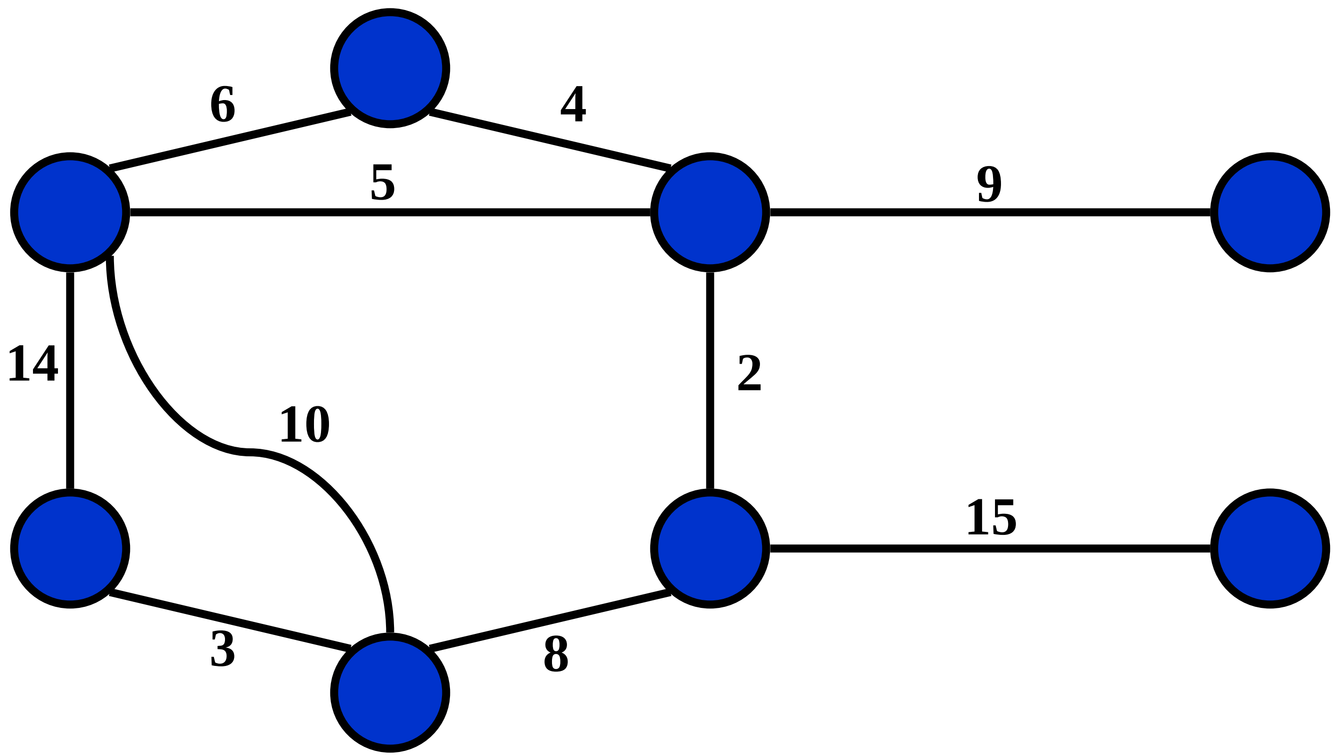
Minimum Spanning Tree

- Problem: given a connected, undirected, weighted graph:



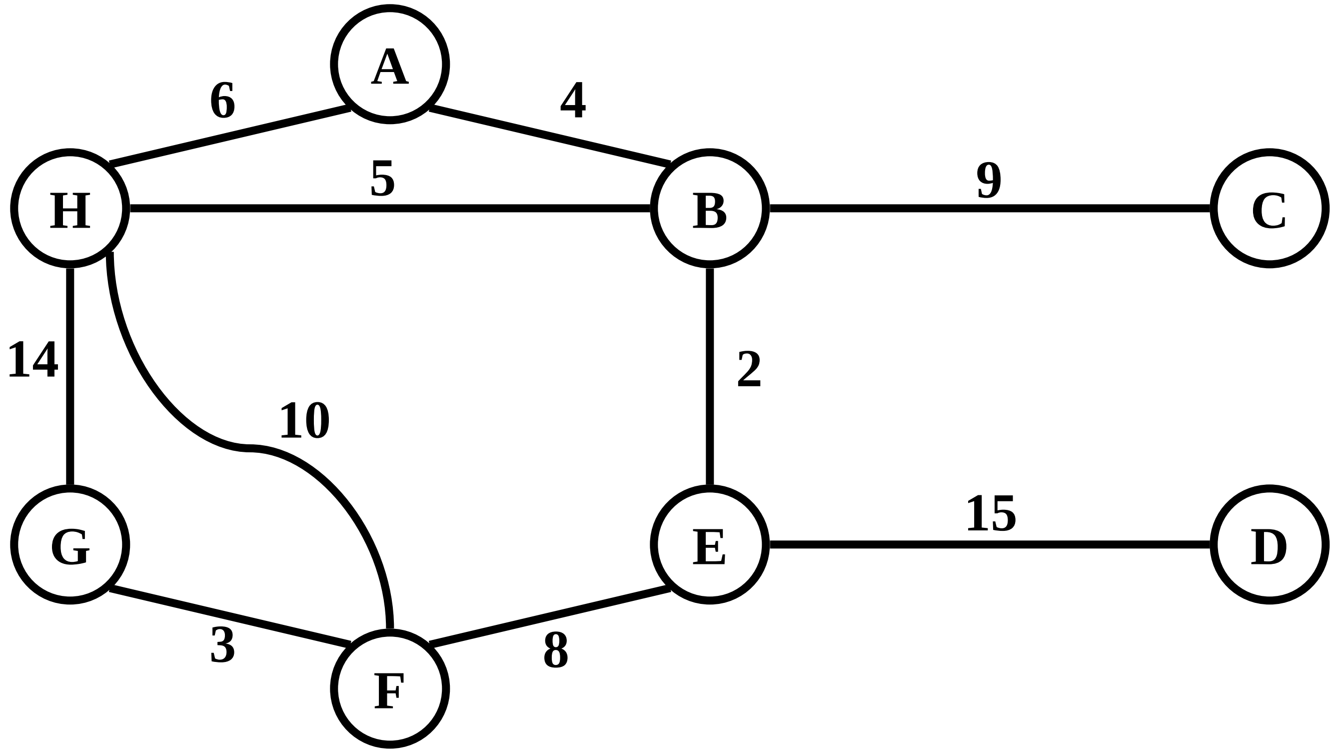
Minimum Spanning Tree

- Problem: given a connected, undirected, weighted graph, find a *spanning tree* using edges that minimize the total weight



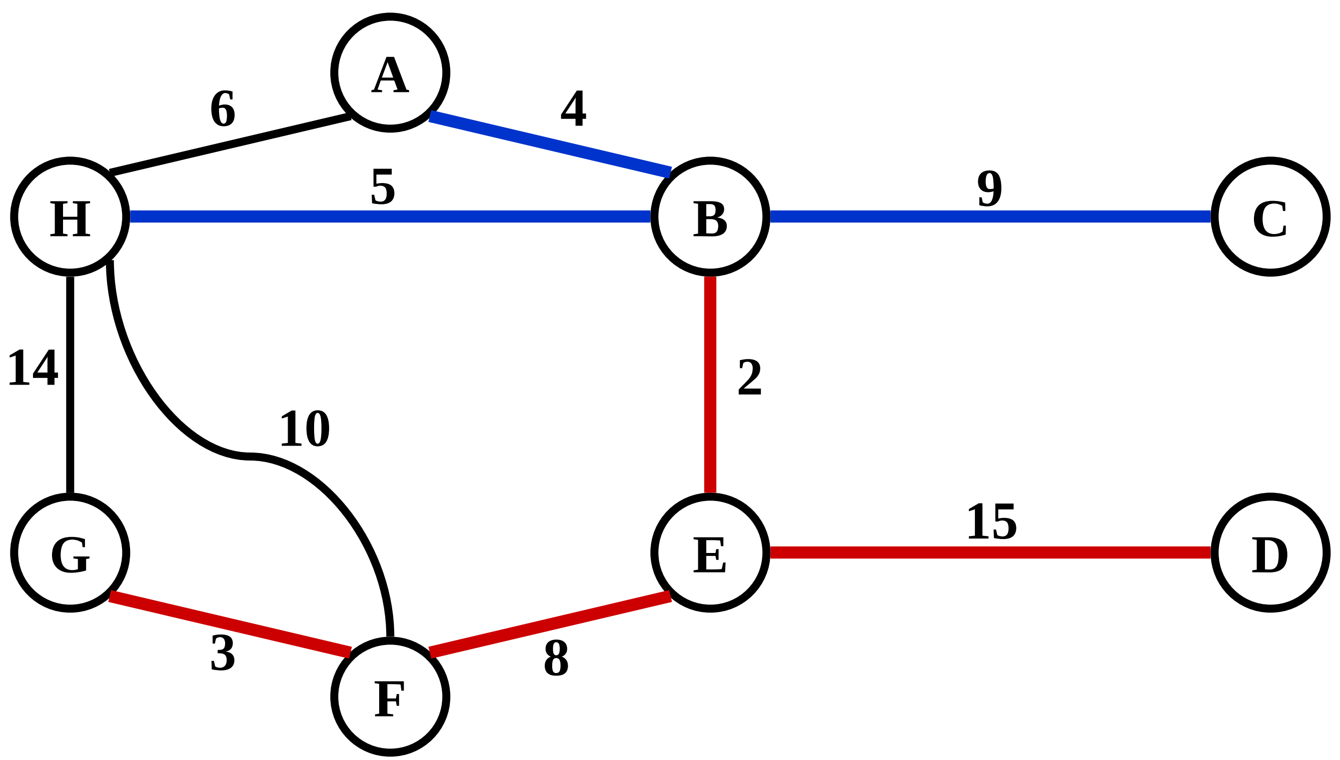
Minimum Spanning Tree

- *Which edges form the minimum spanning tree (MST) of the below graph?*



Minimum Spanning Tree

- Answer:



Minimum Spanning Tree

- MSTs satisfy the *optimal substructure* property: an optimal tree is composed of optimal subtrees
  - Let T be an MST of G with an edge (u,v) in the middle
  - Removing (u,v) partitions T into two trees T_1 and T_2
  - Claim: T_1 is an MST of $G_1 = (V_1,E_1)$, and T_2 is an MST of $G_2 = (V_2,E_2)$ (*Do V_1 and V_2 share vertices? Why?*)
  - Proof: $w(T) = w(u,v) + w(T_1) + w(T_2)$
    (There can't be a better tree than T_1 or T_2, or T would be suboptimal)

Minimum Spanning Tree

- Thm:
  - Let T be MST of G, and let A $\subseteq$ T be subtree of T
  - Let (u,v) be min-weight edge connecting A to V-A
  - Then $(u,v) \in$ T

Minimum Spanning Tree

- Thm:
  - Let T be MST of G, and let A $\subseteq$ T be subtree of T
  - Let (u,v) be min-weight edge connecting A to V-A
  - Then $(u,v) \in$ T
- Proof: in book (see Thm 24.1)

Minimum Spanning Tree

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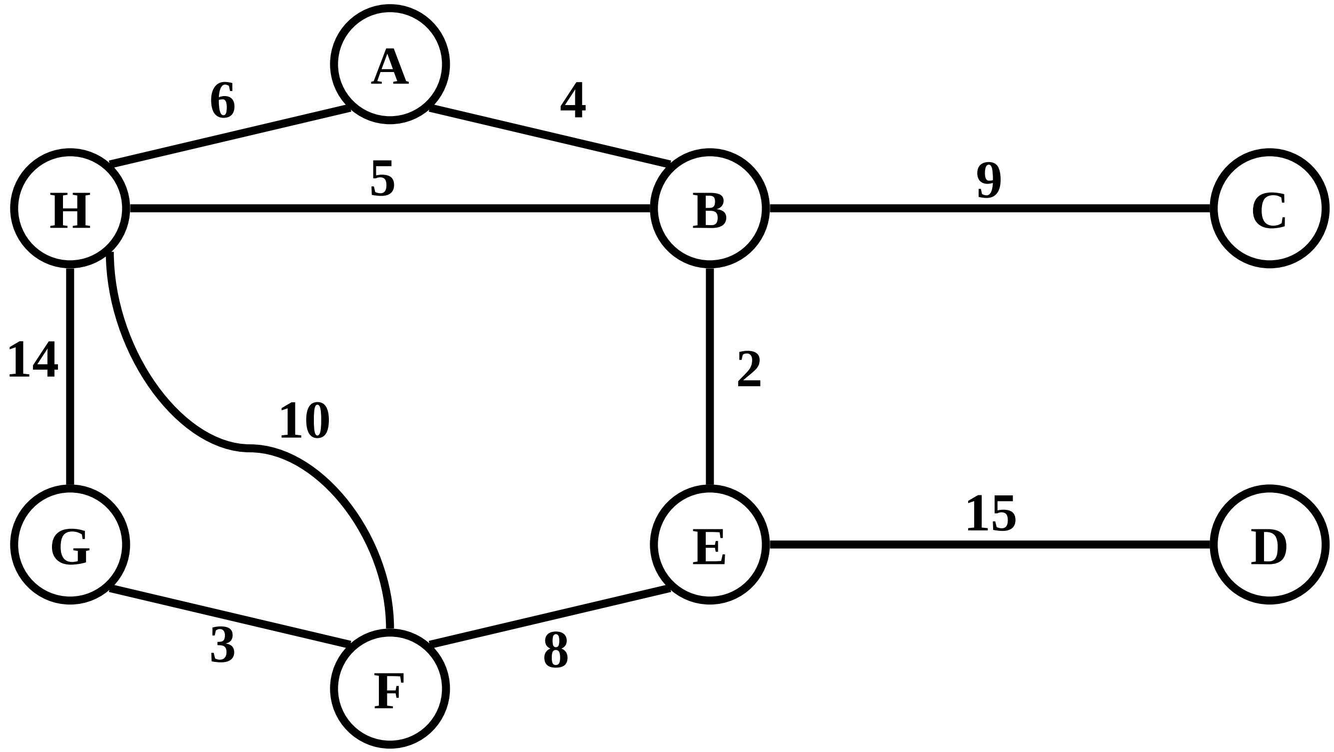

Prim's Algorithm

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MST-Prim(G, w, r)
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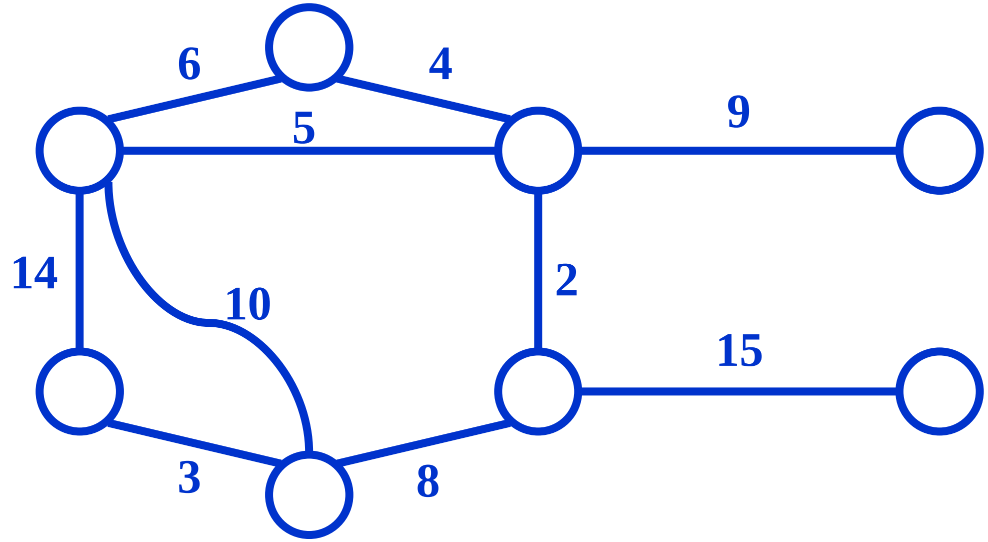

Run on example graph

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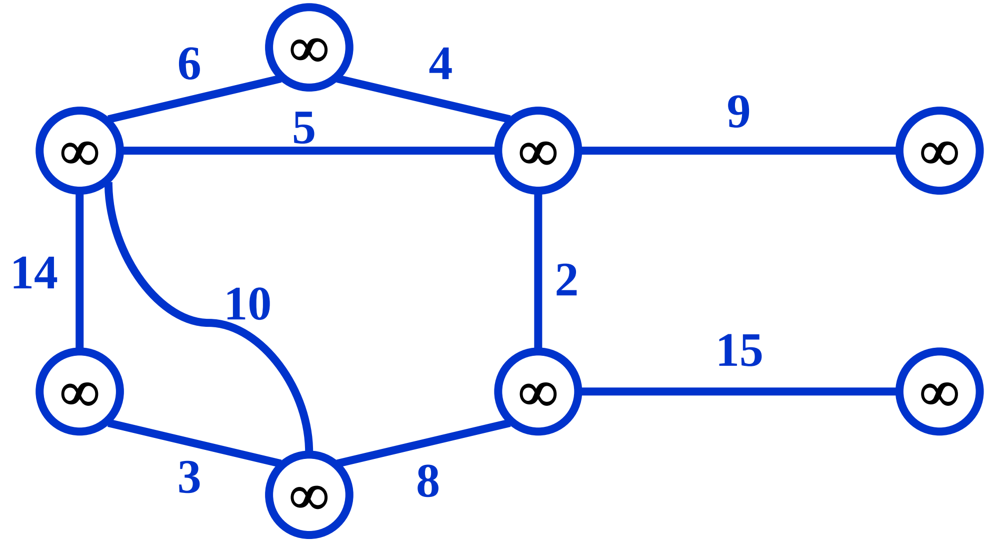

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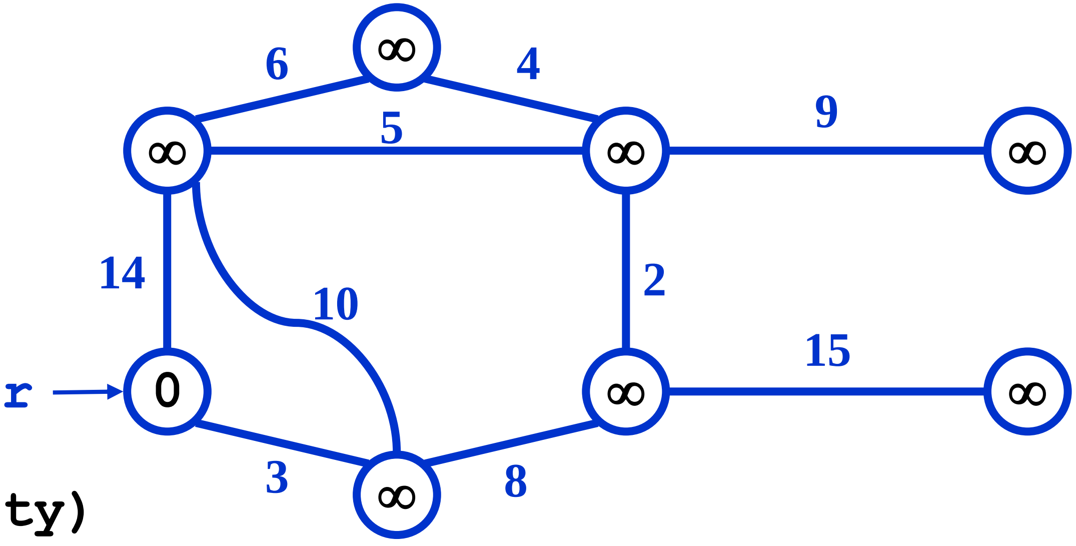

Pick a start vertex r

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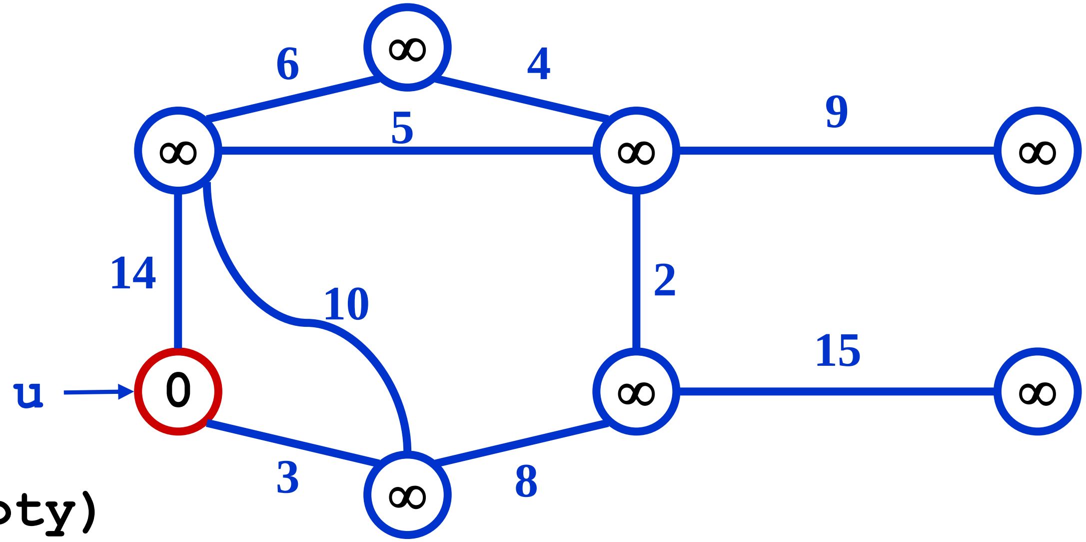

Red vertices have been removed from Q

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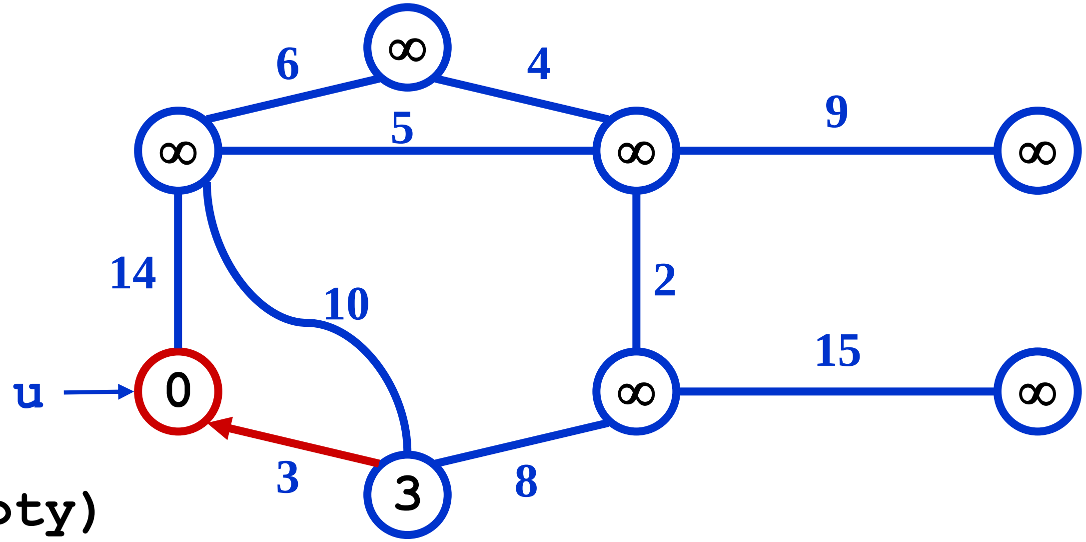

Red arrows indicate parent pointers

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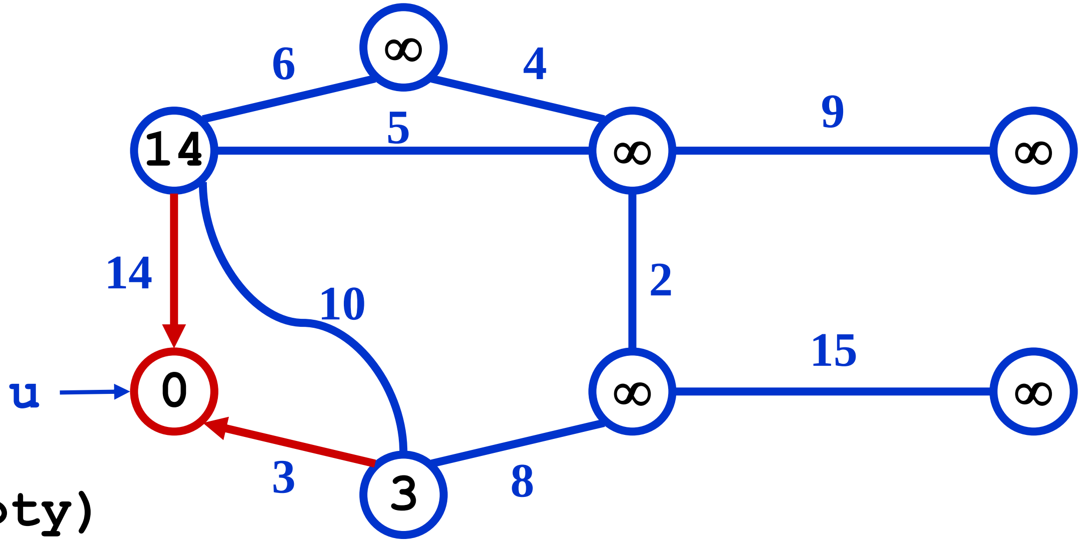

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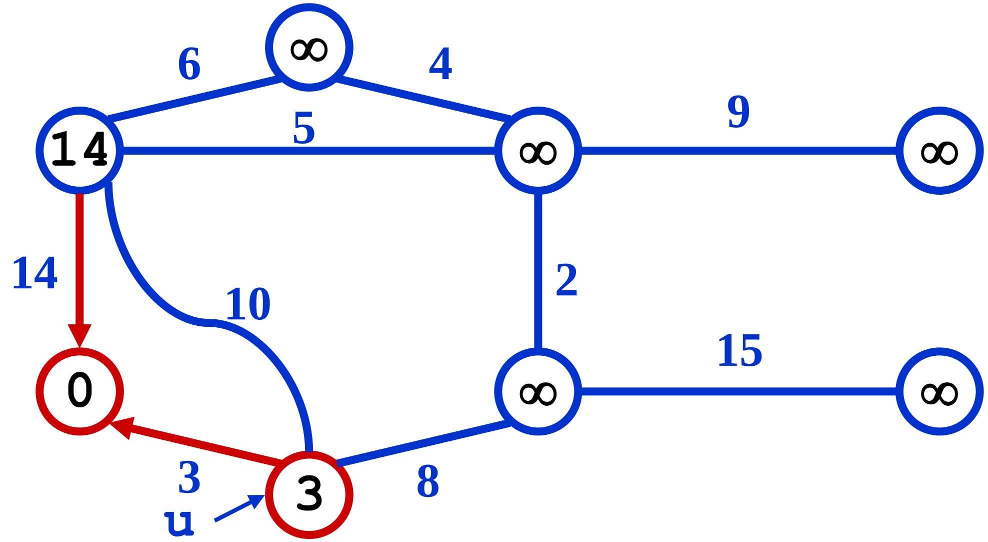

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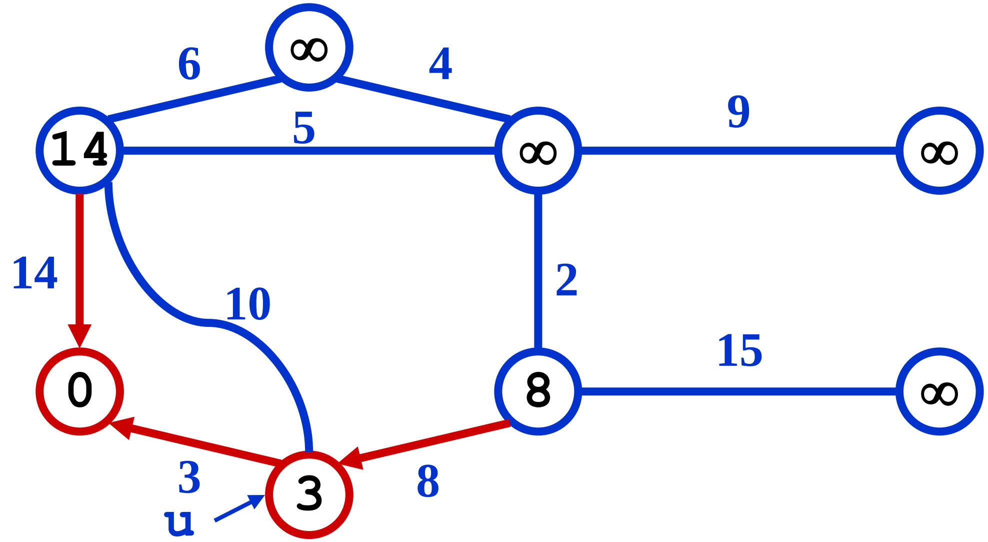

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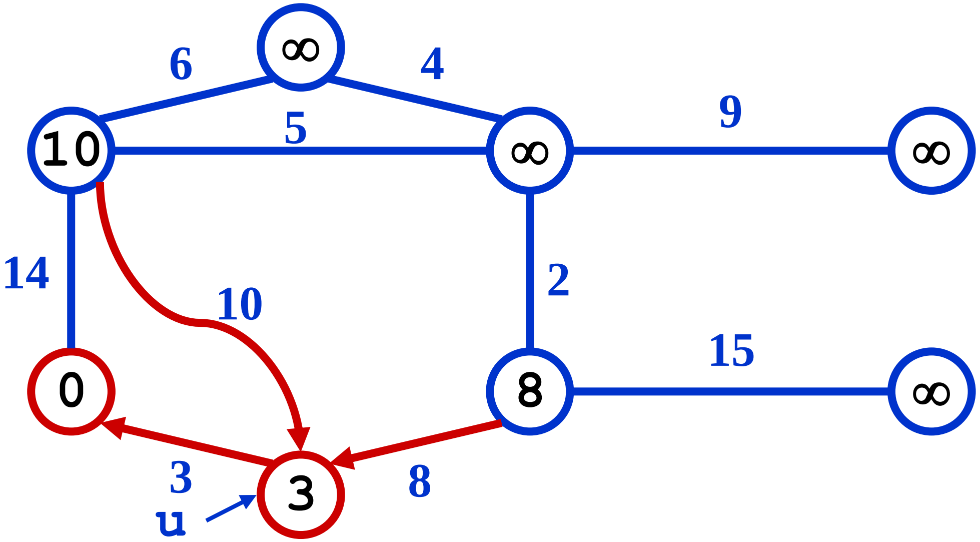

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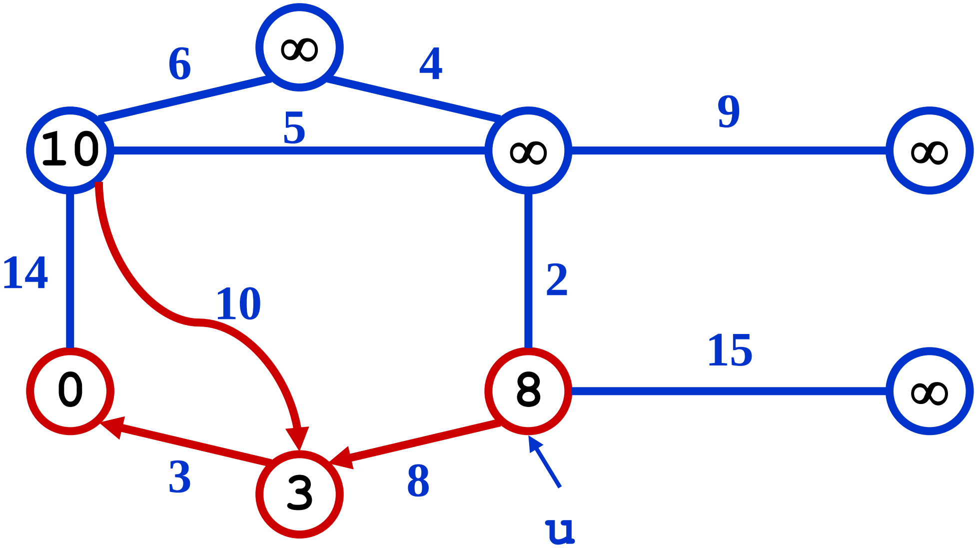

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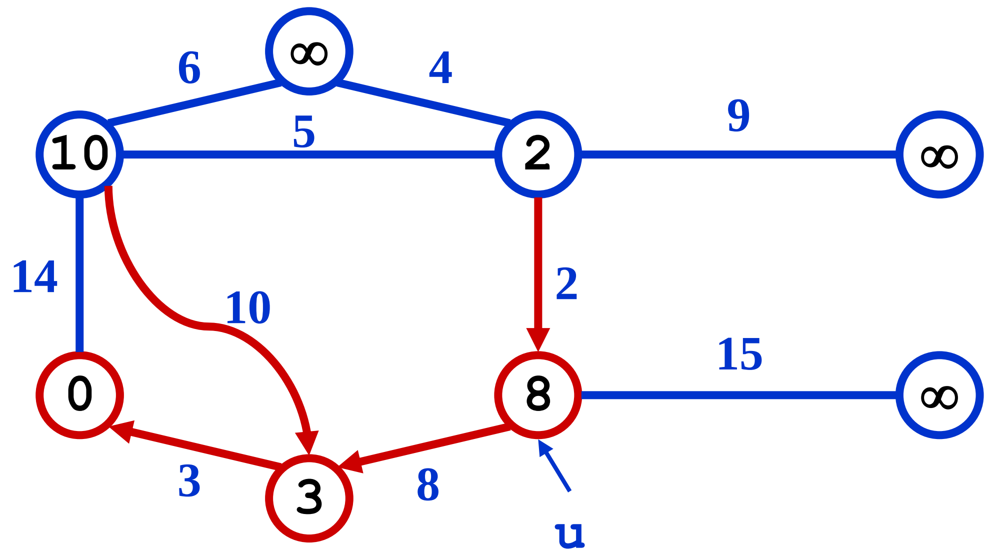

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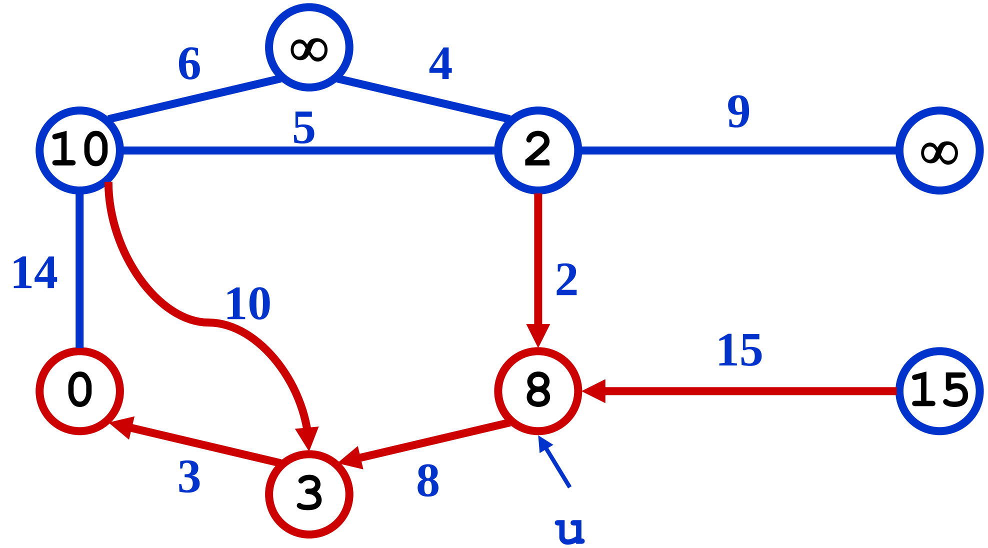

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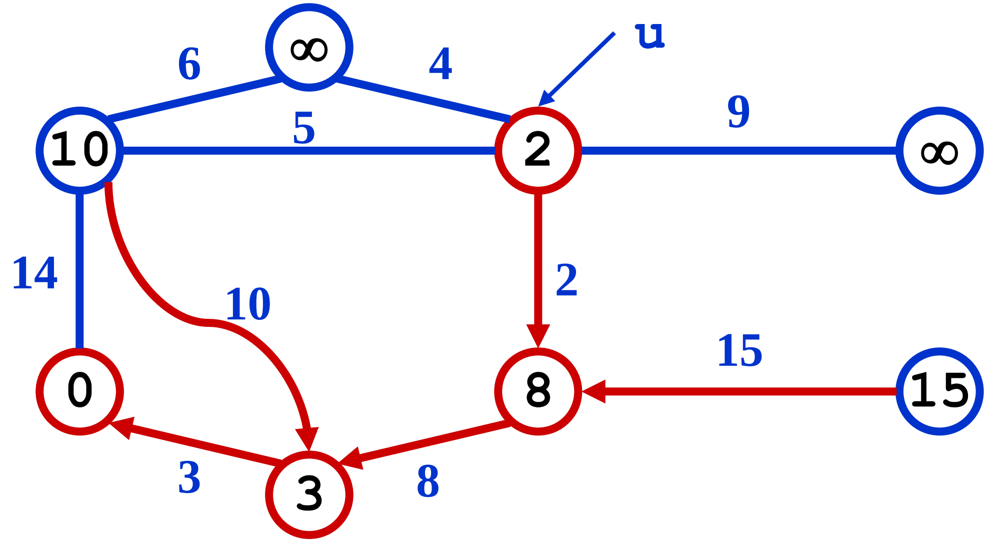

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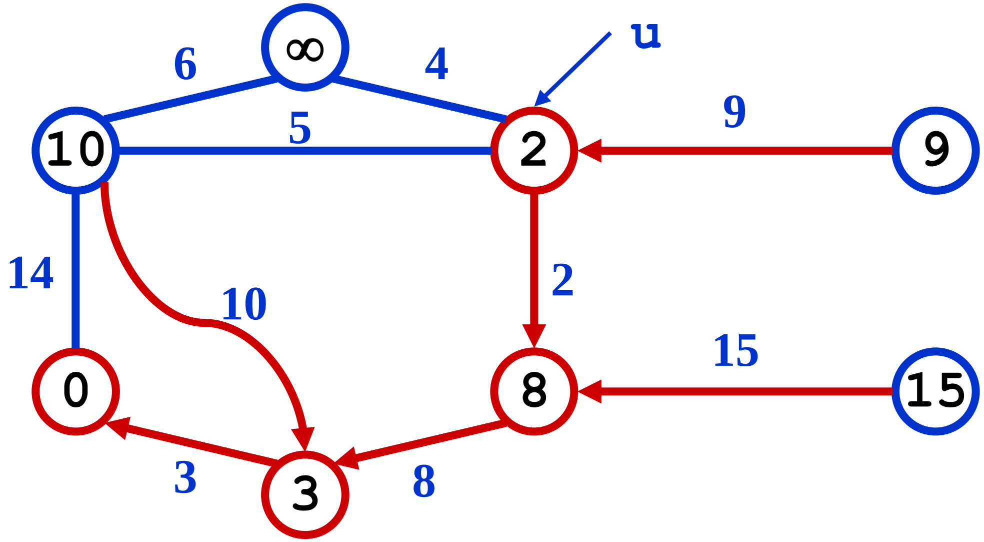

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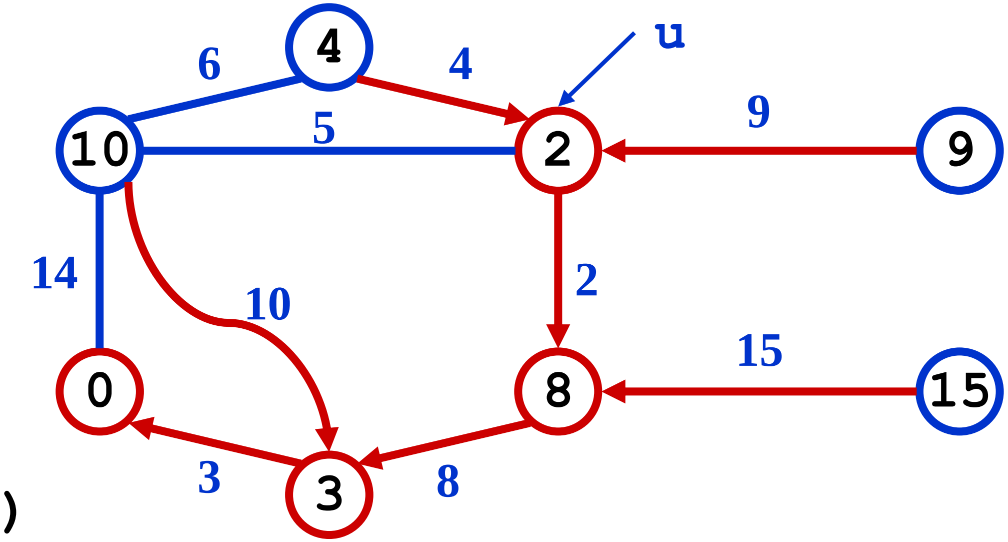

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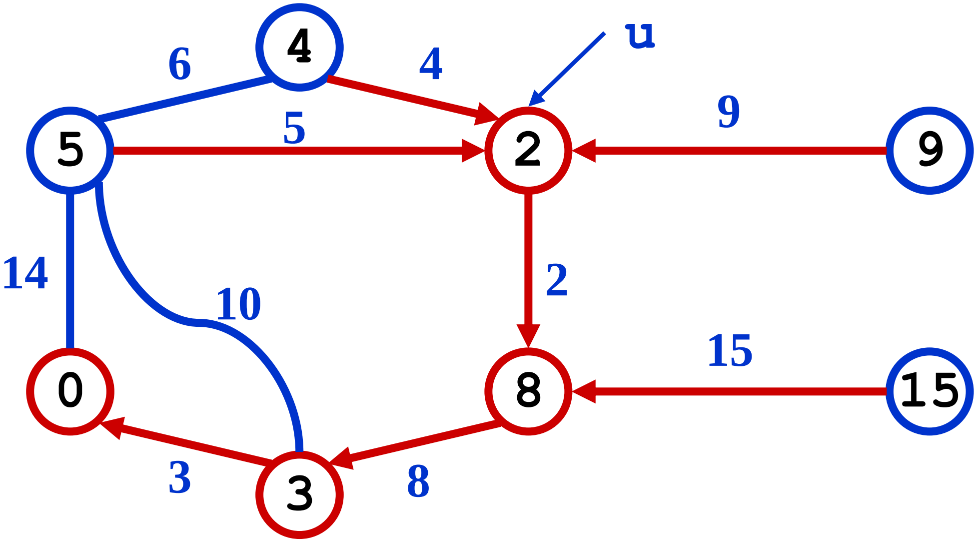

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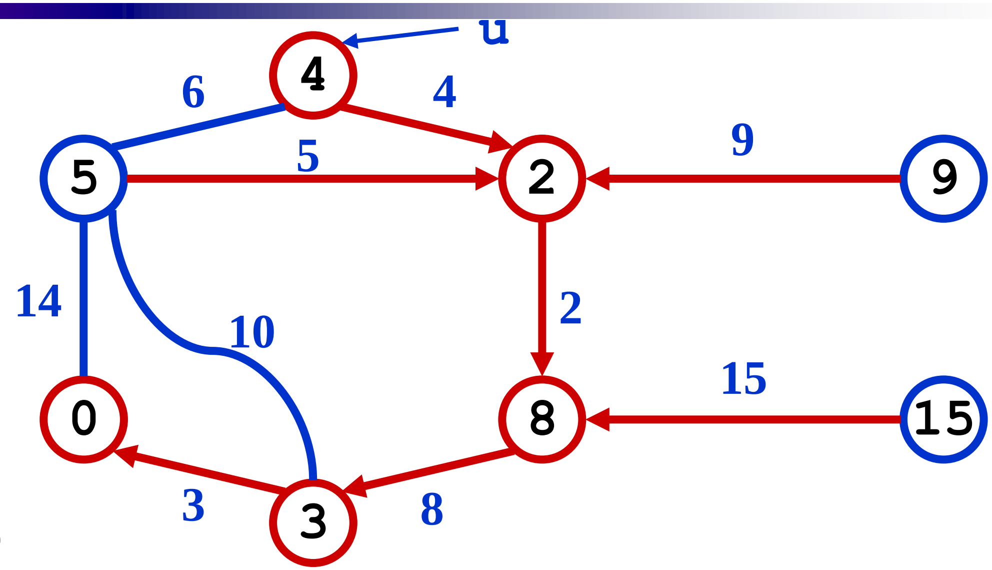

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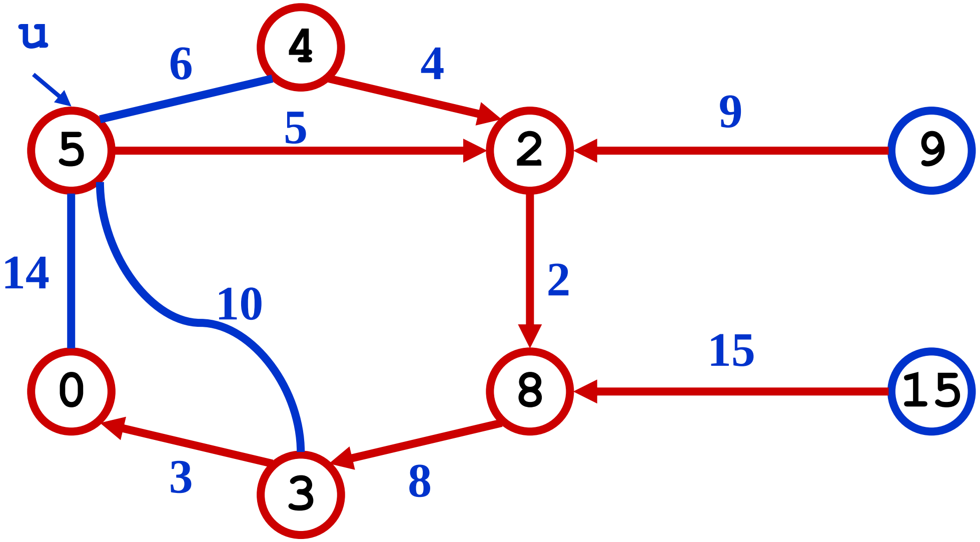

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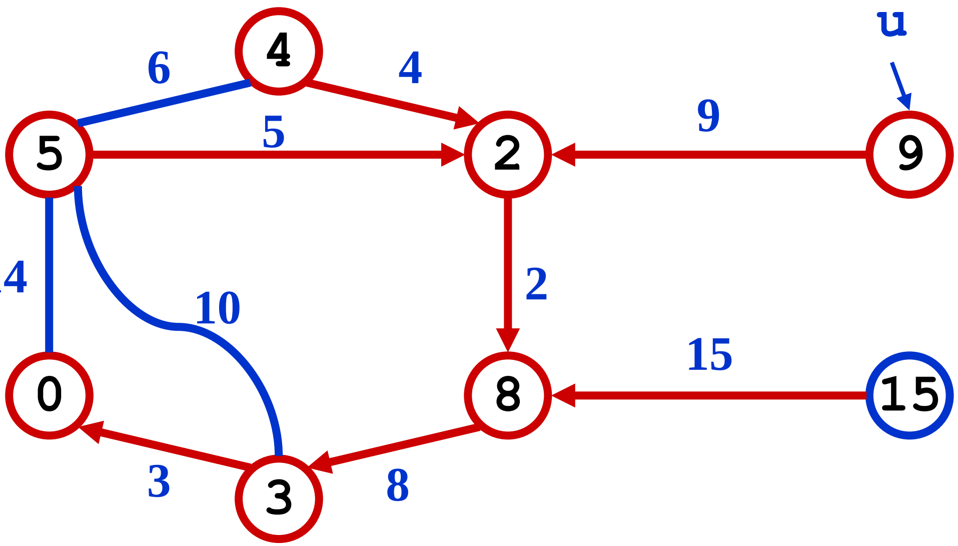

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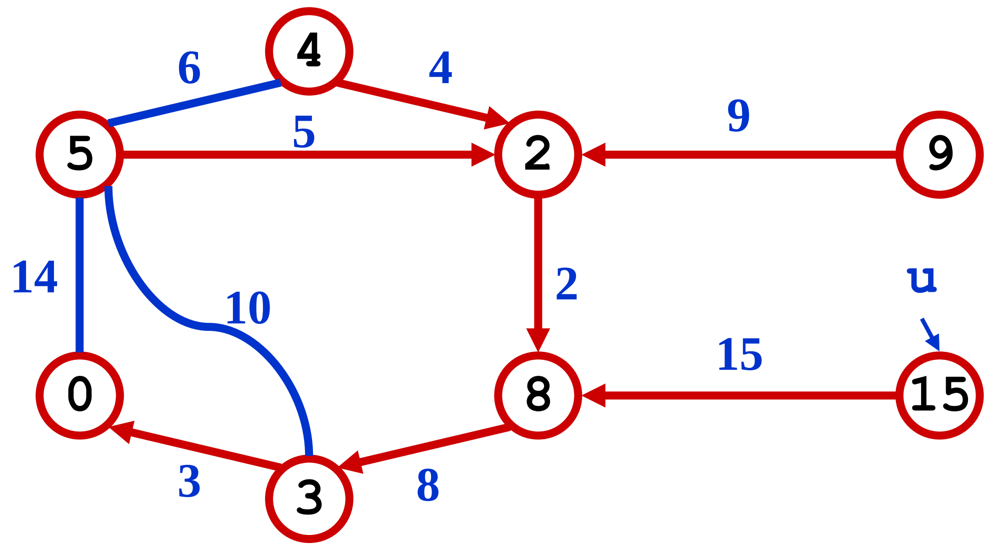

Review: Prim's Algorithm

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What will be the running time?

A: Depends on queue

binary heap: O(E lg V)

Fibonacci heap: O(V lg V + E)

Disjoint-Set Union Problem

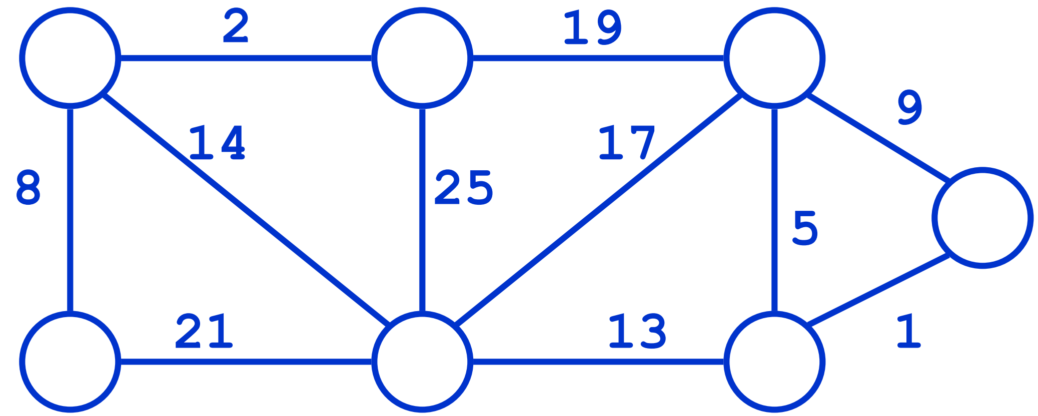
- Want a data structure to support disjoint sets
  - Collection of disjoint sets $\mathcal{S} = \{S_i\}$, $S_i \cap S_j = \emptyset$
- Need to support following operations:
  - MakeSet(x): $\mathcal{S} = \mathcal{S} \cup \{\{x\}\}$
  - Union(S_i, S_j): $\mathcal{S} = \mathcal{S} - \{S_i, S_j\} \cup \{S_i \cup S_j\}$
  - FindSet(x): return $S_i \in \mathcal{S}$ such that $x \in S_i$
- Before discussing implementation details, we look at example application: MSTs

Kruskal's Algorithm

```
Kruskal()
{
    T = ∅;
    for each v ∈ V
        MakeSet(v);
    sort E by increasing edge weight w
    for each (u,v) ∈ E (in sorted order)
        if FindSet(u) ≠ FindSet(v)
            T = T U {{u,v}};
            Union(FindSet(u), FindSet(v));
}
```

Kruskal's Algorithm

Run the algorithm:

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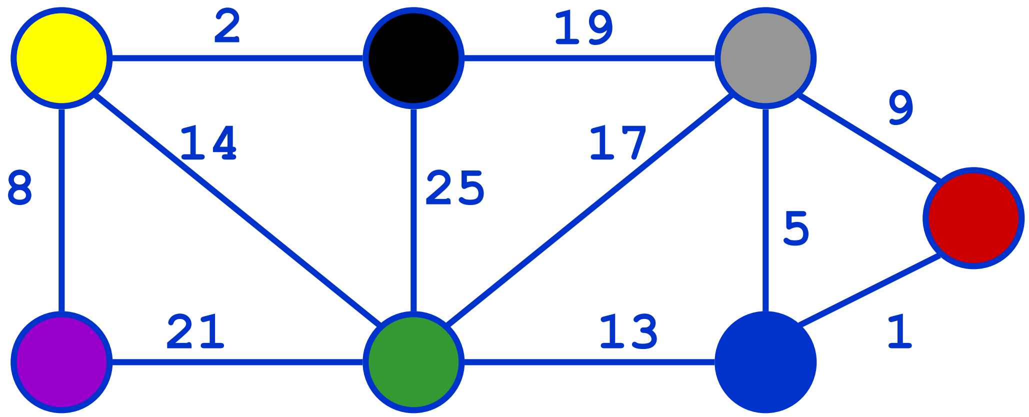


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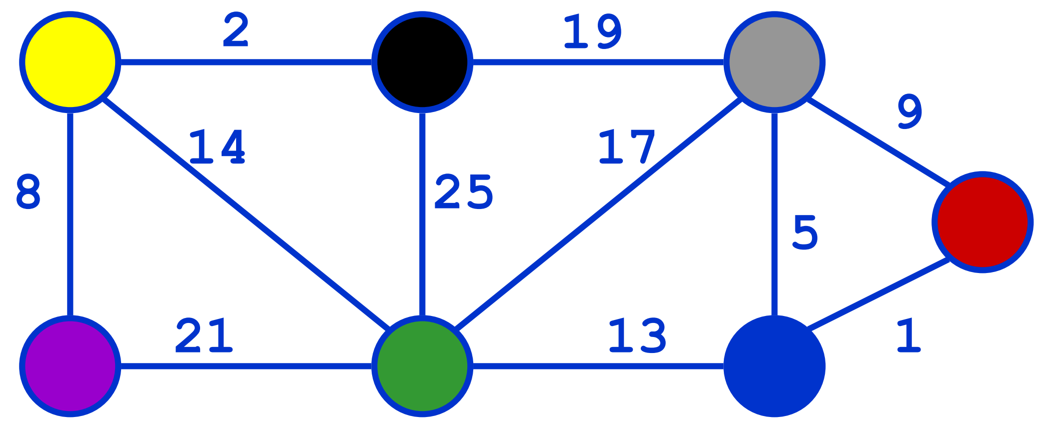


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Kruskal's Algorithm

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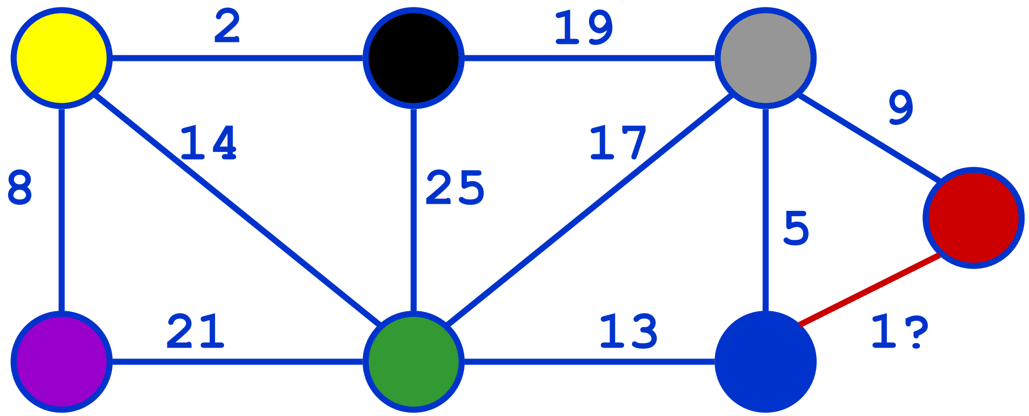


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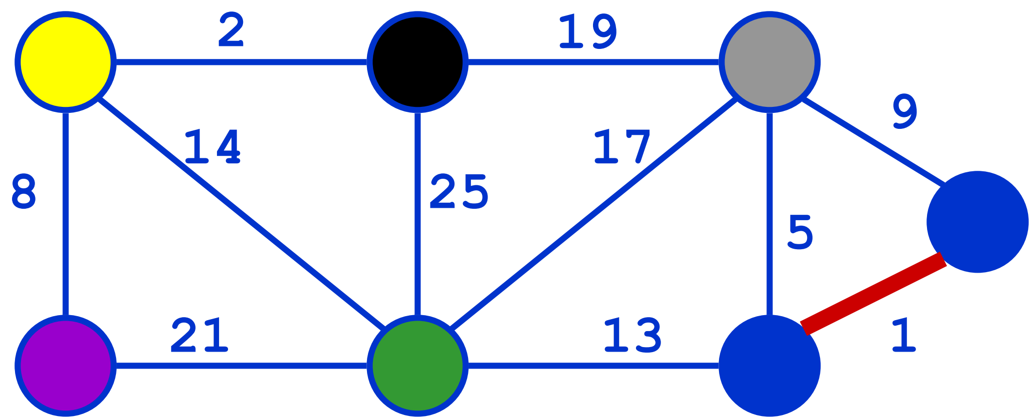


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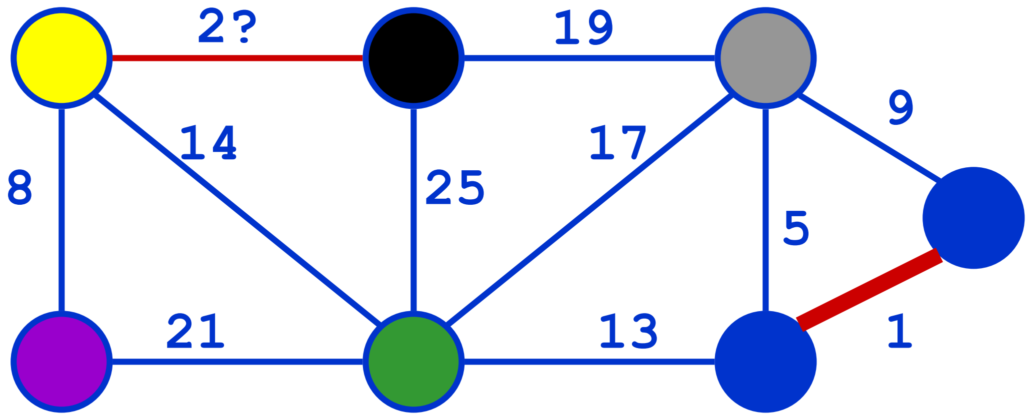


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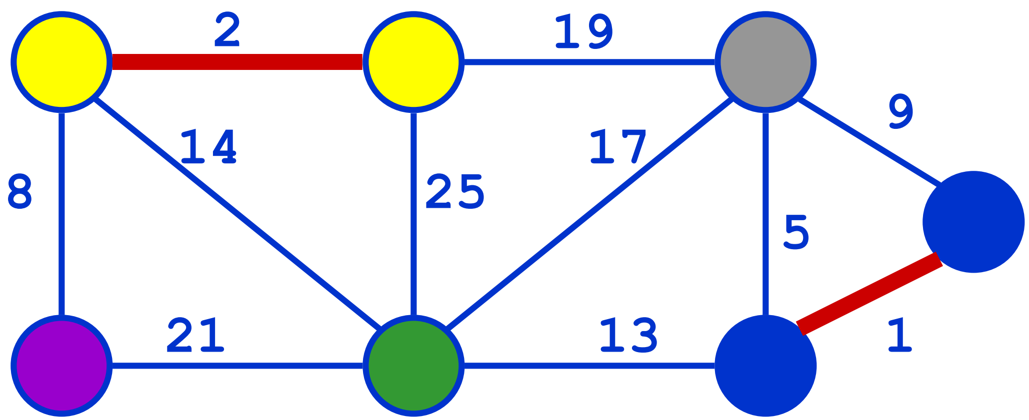


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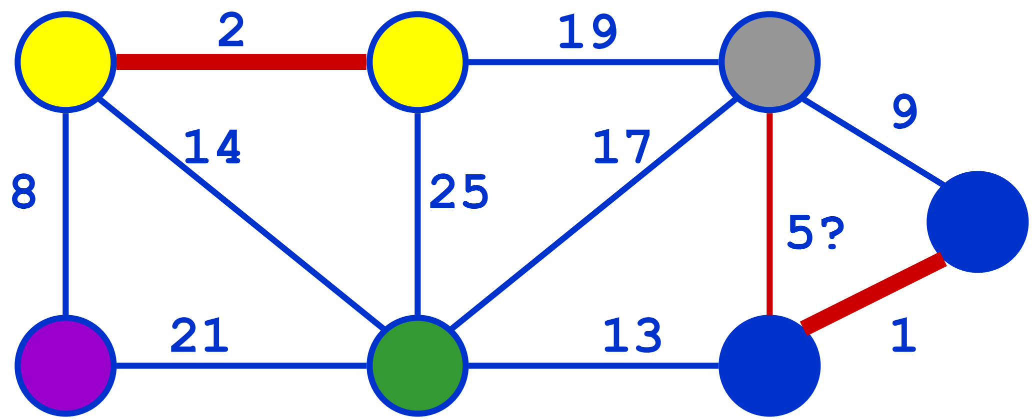

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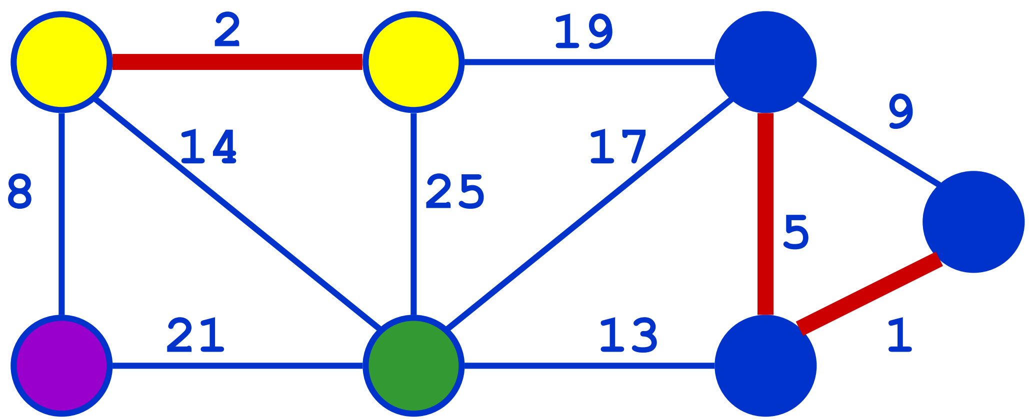


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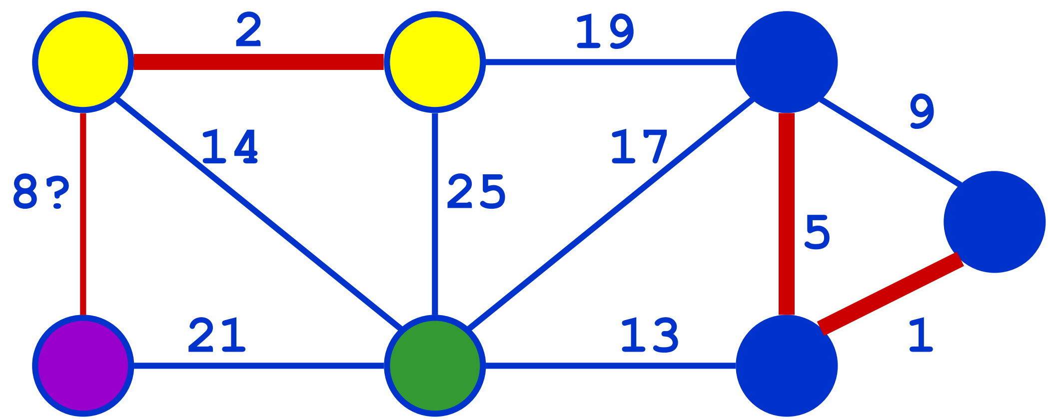

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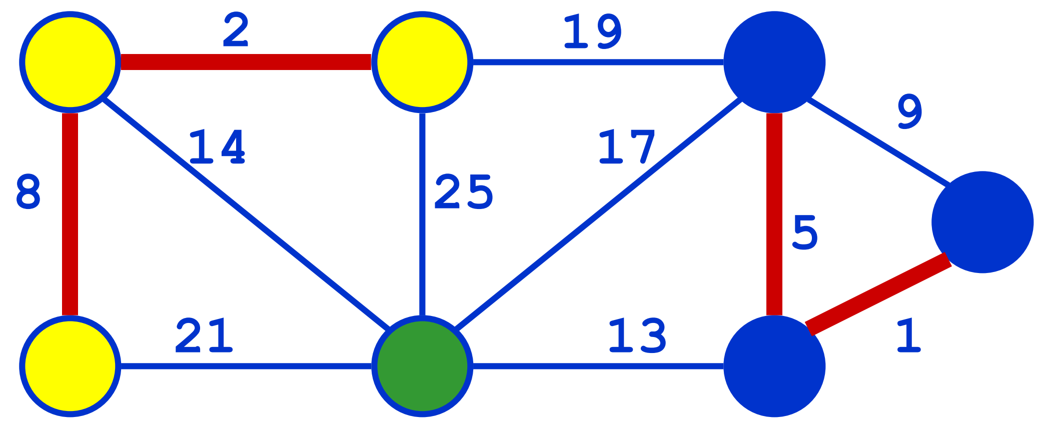

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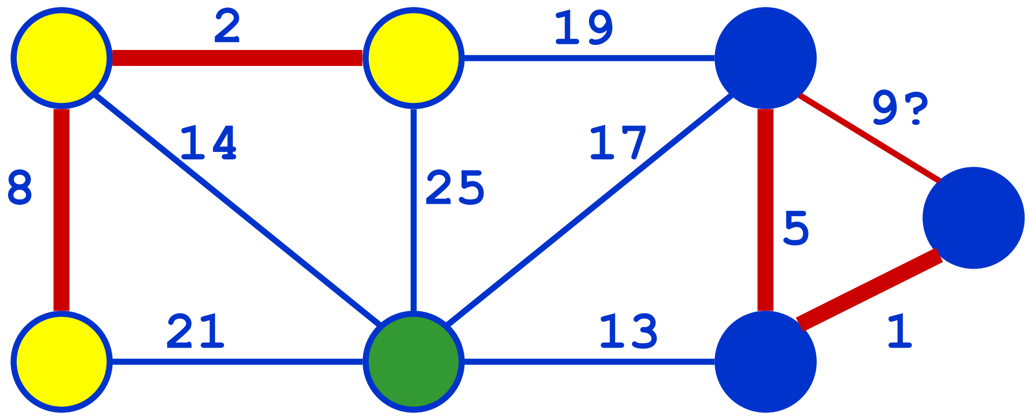

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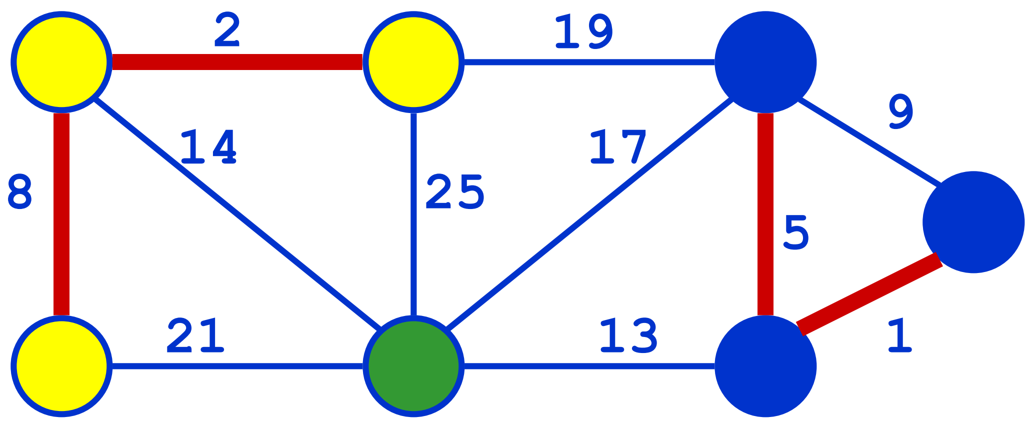


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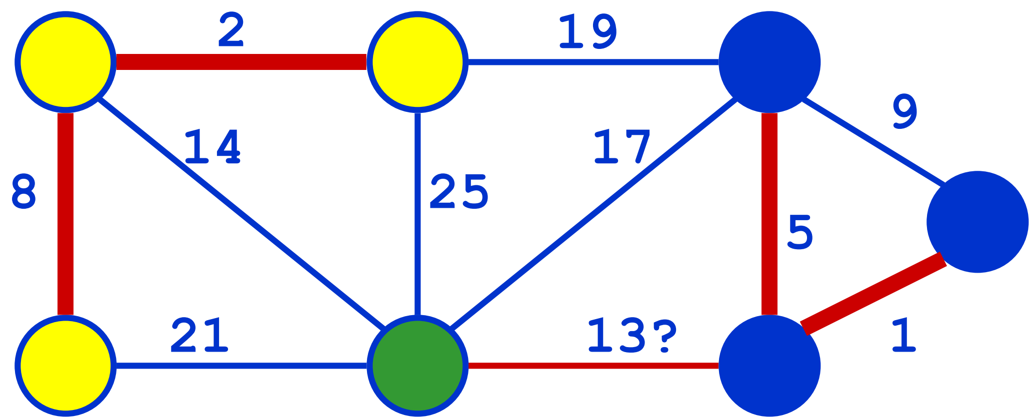


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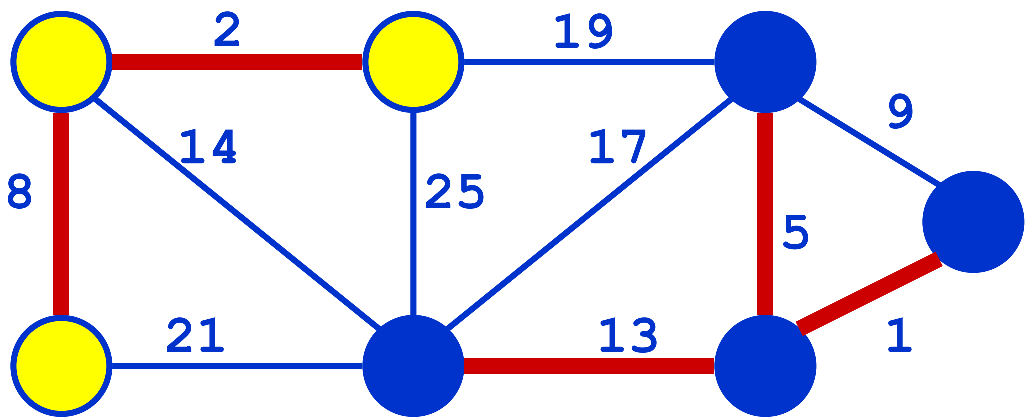

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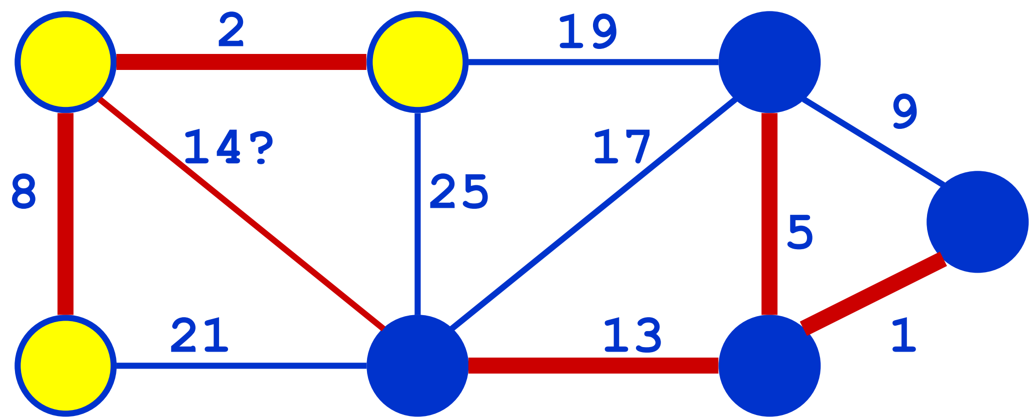

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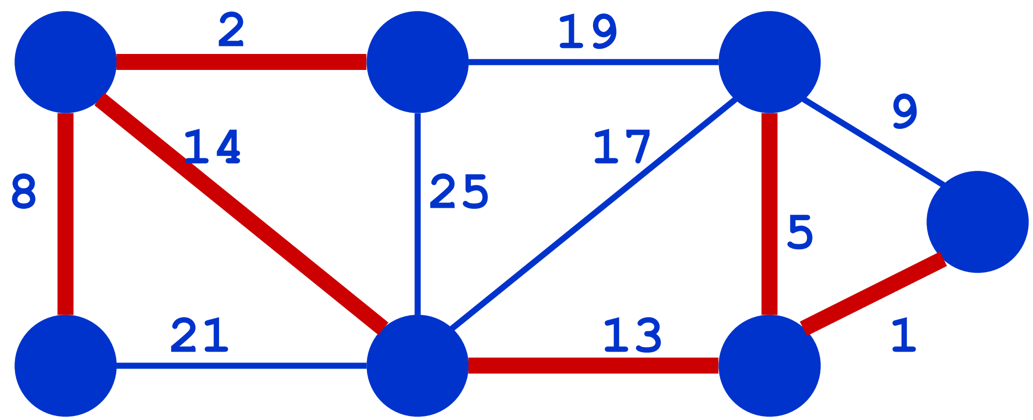


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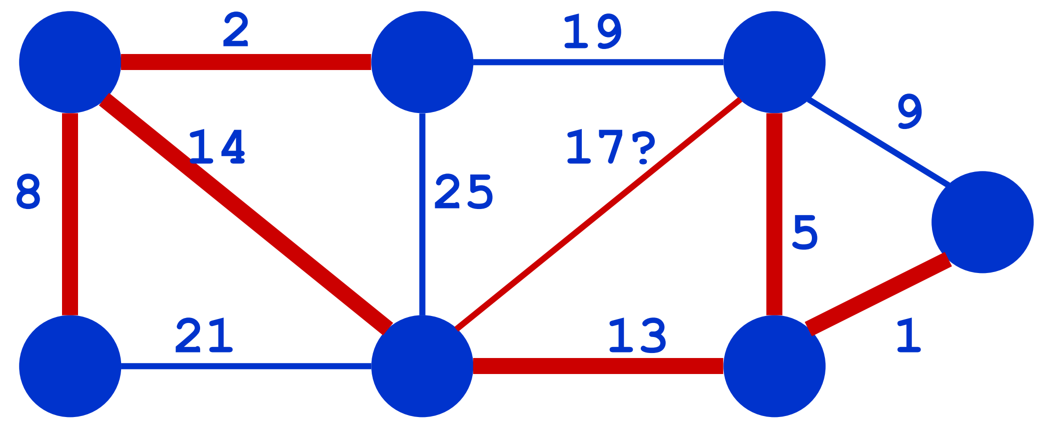


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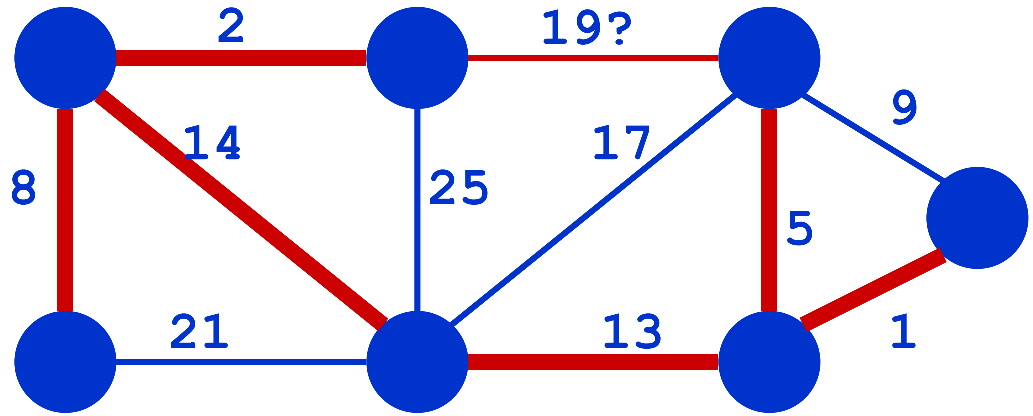

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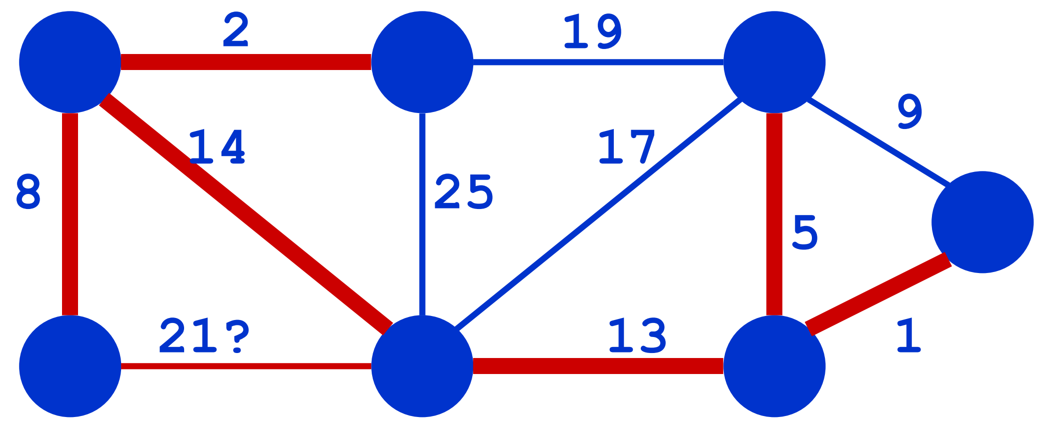

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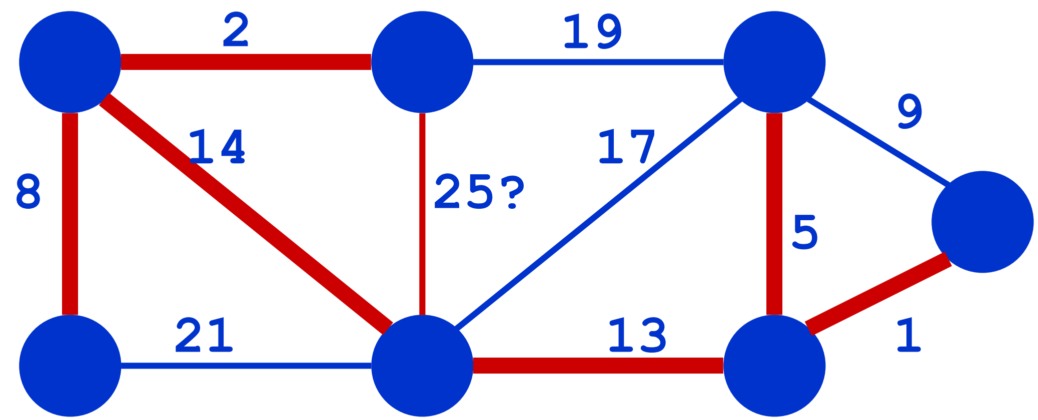


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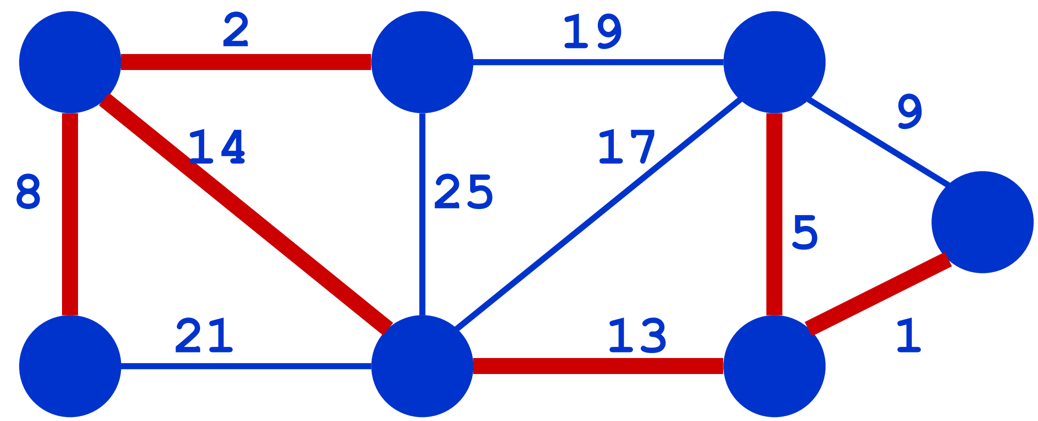

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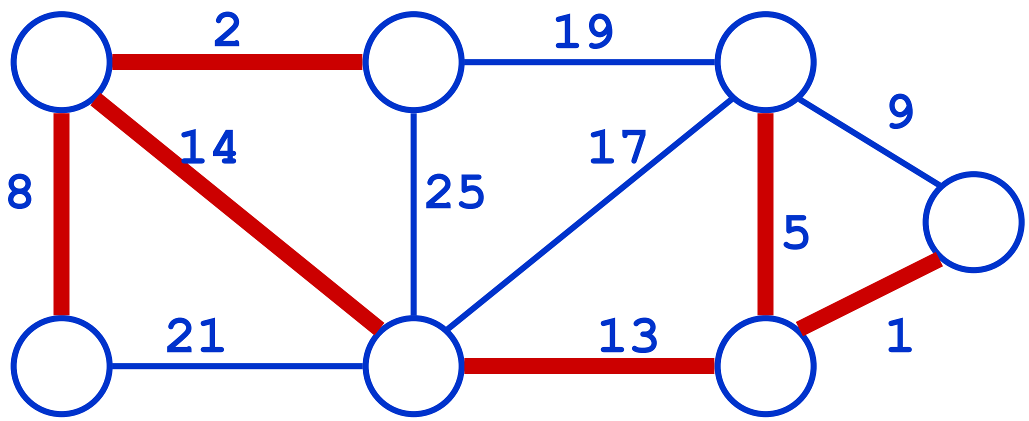

Kruskal's Algorithm

What will affect the running time?

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Kruskal's Algorithm

What will affect the running time?

1 Sort

O(V) MakeSet() calls

O(E) FindSet() calls

O(V) Union() calls

(*Exactly how many Union()s?*)

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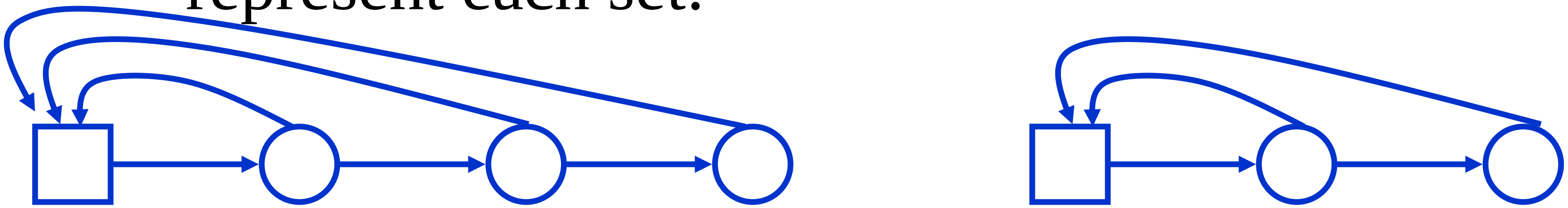
Kruskal’s Algorithm: Running Time

- To summarize:
  - Sort edges: O(E lg E)
  - O(V) MakeSet()’s
  - O(E) FindSet()’s
  - O(V) Union()’s
- Upshot:
  - Best disjoint-set union algorithm makes above 3 operations take $O(E \cdot \alpha(E,V))$, α almost constant
  - Overall thus O(E lg E), almost linear w/o sorting

Disjoint Set Union

- So how do we implement disjoint-set union?
  - Naïve implementation: use a linked list to represent each set:



    - MakeSet(): ??? time
    - FindSet(): ??? time
    - Union(A,B): "copy" elements of A into B: ??? time

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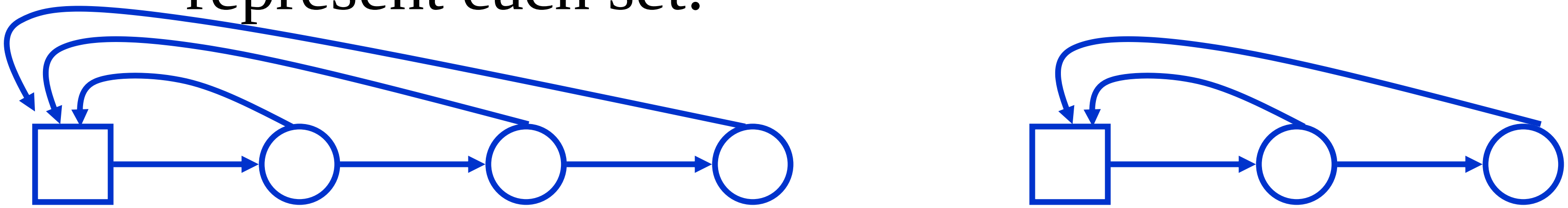

    - MakeSet(): O(1) time
    - FindSet(): O(1) time
    - Union(A,B): "copy" elements of A into B: O(A) time
  - *How long can a single Union() take?*
  - *How long will n Union()'s take?*

Disjoint Set Union: Analysis

- Worst-case analysis: $O(n^2)$ time for n Union's

Union(S_1, S_2)	"copy"	1 element
Union(S_2, S_3)	"copy"	2 elements
…		
Union(S_{n-1}, S_n)	"copy"	n-1 elements
		$O(n^2)$

- Improvement: always copy smaller into larger
  - *Why will this make things better?*
  - *What is the worst-case time of Union()?*
- But now n Union's take only $O(n \lg n)$ time!