Dynamic Programming

Dynamic Programming

- Another strategy for designing algorithms is *dynamic programming*
  - A metatechnique, not an algorithm
    (like divide & conquer)
  - The word "programming" is historical and predates computer programming
- Use when problem breaks down into recurring small subproblems

Properties of a problem that can be solved with dynamic programming

- Simple Subproblems
  - We should be able to break the original problem to smaller subproblems that have the same structure
- Optimal Substructure of the problems
  - The solution to the problem must be a composition of subproblem solutions
- Subproblem Overlap
  - Optimal subproblems to unrelated problems can contain subproblems in common

Dynamic Programming Example: Longest Common Subsequence

- *Longest common subsequence* (*LCS*) problem:
  - Given two sequences x[1..m] and y[1..n], find the longest subsequence which occurs in both
  - Ex: x = {A B C B D A B }, y = {B D C A B A}
  - {B C} and {A A} are both subsequences of both
    - *What is the LCS?*
  - Brute-force algorithm: For every subsequence of x, check if it's a subsequence of y
    - *How many subsequences of x are there?*
    - *What will be the running time of the brute-force alg?*

LCS Algorithm

- Brute-force algorithm: 2^m subsequences of x to check against n elements of y: $O(n\ 2^m)$
- We can do better: for now, let's only worry about the problem of finding the *length* of LCS
  - When finished we will see how to backtrack from this solution back to the actual LCS
- Notice LCS problem has optimal substructure
  - Subproblems: LCS of pairs of *prefixes* of x and y

Finding LCS Length

- Define c[*i*,*j*] to be the length of the LCS of x[1..*i*] and y[1..*j*]
  - *What is the length of LCS of x and y?*
- Theorem:

$$c[i,j]=\begin{cases} c[i-1,j-1]+1 & \text{if } x[i]=y[j], \\ \max(c[i,j-1],c[i-1,j]) & \text{otherwise} \end{cases}$$

- *What is this really saying?*

Longest Common Subsequence (LCS)

- Define X_i, Y_j to be prefixes of X and Y of length i and j; m = |X|, n = |Y|
- We store the length of LCS(X_i, Y_j) in c[i,j]
- Trivial cases: LCS(X_0 , Y_j) and LCS(X_i, Y_0) is empty (so c[0,j] = c[i,0] = 0)
- Recursive formula for c[i,j]:

$$c[i,j] = \begin{cases} c[i-1, j-1]+1 & \text{if } x[i]=y[j], \\ \max(c[i,j-1], c[i-1,j]) & \text{otherwise} \end{cases}$$

c[m,n] is the final solution

Longest Common Subsequence (LCS)

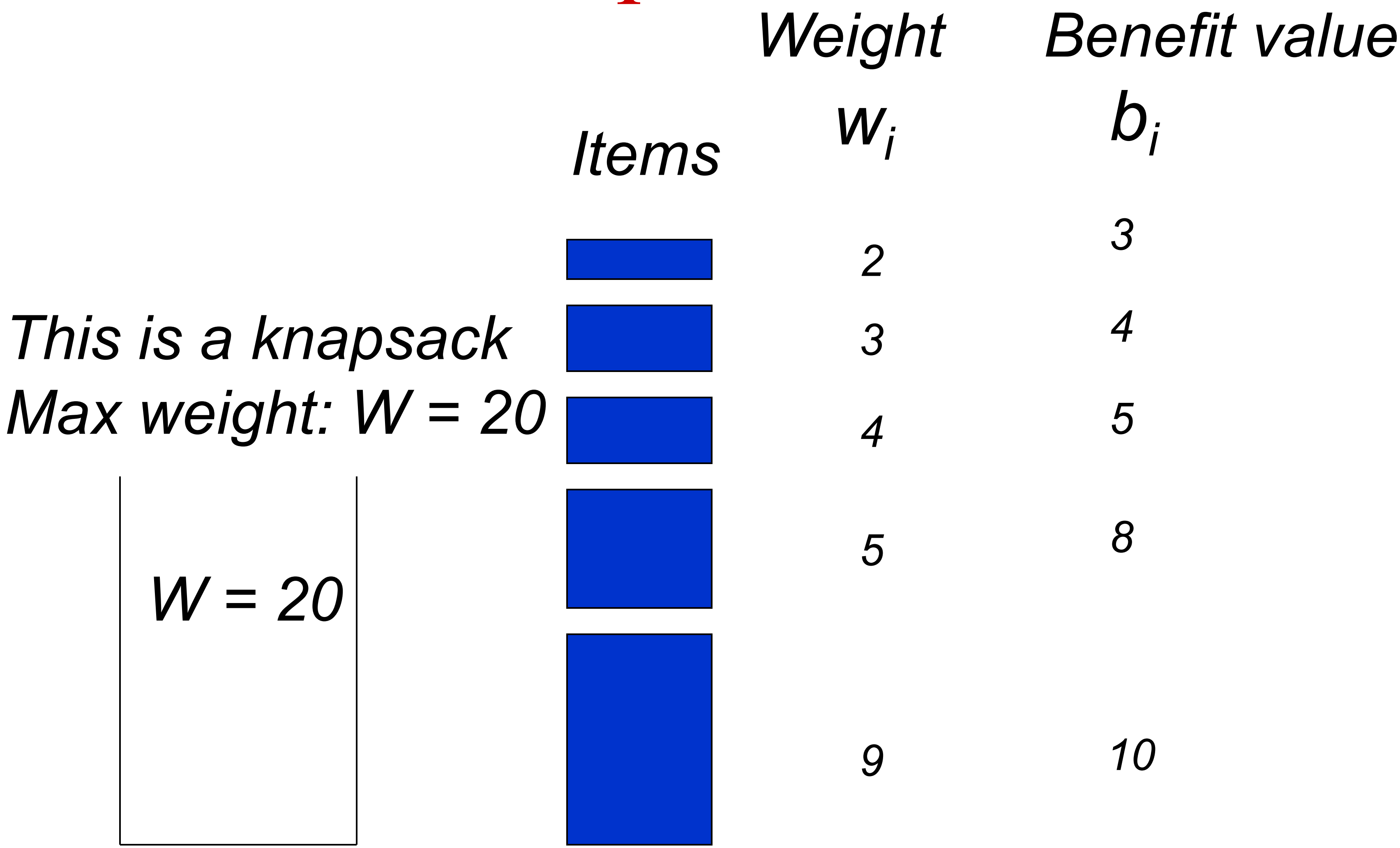
- After we have filled the array *c[ ]*, we can use this data to find the characters that constitute the Longest Common Subsequence

- Algorithm runs in $O(m*n)$, which is *much* better than the brute-force algorithm: $O(n\ 2^m)$

0-1 Knapsack problem

- Given a knapsack with maximum capacity W, and a set S consisting of n items
- Each item i has some weight w_i and benefit value b_i (all w_i , b_i and W are integer values)
- <u>Problem</u>: How to pack the knapsack to achieve maximum total value of packed items?

0-1 Knapsack problem: a picture



0-1 Knapsack problem

- Problem, in other words, is to find

$$\max \sum_{i \in T} b_i \text{ subject to } \sum_{i \in T} w_i \le W$$

- *The problem is called a "0-1" problem, because each item must be entirely accepted or rejected.*
- *Just another version of this problem is the "Fractional Knapsack Problem", where we can take fractions of items.*

0-1 Knapsack problem: brute-force approach

Let’s first solve this problem with a straightforward algorithm

- Since there are n items, there are 2^n possible combinations of items.
- We go through all combinations and find the one with the most total value and with total weight less or equal to W
- Running time will be $O(2^n)$

0-1 Knapsack problem: brute-force approach

- Can we do better?
- Yes, with an algorithm based on dynamic programming
- We need to carefully identify the subproblems

Let's try this:

If items are labeled 1..n, then a subproblem would be to find an optimal solution for S_k = {items labeled 1, 2, .. k}

Defining a Subproblem

If items are labeled *1..n*, then a subproblem would be to find an optimal solution for S_k = *{items labeled 1, 2, .. k}*

- This is a valid subproblem definition.
- The question is: can we describe the final solution (S_n) in terms of subproblems (S_k)?
- Unfortunately, we <u>can't</u> do that. Explanation follows….

Defining a Subproblem

$w_1 = 2$ $b_1 = 3$	$w_2 = 4$ $b_2 = 5$	$w_3 = 5$ $b_3 = 8$	$w_4 = 3$ $b_4 = 4$	

?

Max weight: W = 20

For S_4:

Total weight: 14;

total benefit: 20

$w_1 = 2$ $b_1 = 3$	$w_2 = 4$ $b_2 = 5$	$w_3 = 5$ $b_3 = 8$	$w_4 = 9$ $b_4 = 10$

For S_5:

Total weight: 20

total benefit: 26

Item #	Weight w_i	Benefit b_i
1	2	3
2	3	4
3	4	5
4	5	8
5	9	10

S_4

S_5

Solution for S_4 is not part of the solution for S_5!!!

Defining a Subproblem (continued)

- As we have seen, the solution for S_4 is not part of the solution for S_5
- So our definition of a subproblem is flawed and we need another one!
- Let' s add another parameter: w, which will represent the <u>exact</u> weight for each subset of items
- The subproblem then will be to compute $B[k,w]$

Recursive Formula for subproblems

- *Recursive formula for subproblems:*

$$B[k,w] = \begin{cases} B[k-1,w] & \text{if } w_k > w \\ \max\{B[k-1,w], B[k-1,w-w_k]+b_k\} & \text{else} \end{cases}$$

> It means, that the best subset of S_k that has total weight w is one of the two:

1) the best subset of S_{k-1} that has total weight w, **or**

2) the best subset of S_{k-1} that has total weight $w-w_k$ plus the item k

Recursive Formula

$$B[k,w] = \begin{cases} B[k-1,w] & \text{if } w_k > w \\ \max\{B[k-1,w], B[k-1,w-w_k]+b_k\} & \text{else} \end{cases}$$

- The best subset of S_k that has the total weight w, either contains item k or not.
- First case: $w_k > w$. Item k can't be part of the solution, since if it was, the total weight would be $> w$, which is unacceptable
- Second case: $w_k <= w$. Then the item k <u>can</u> be in the solution, and we choose the case with greater value

0-1 Knapsack Algorithm

for $w = 0$ to W

    $B[0,w] = 0$

for $i = 0$ to n

    $B[i,0] = 0$

    for $w = 0$ to W

        if $w_i <= w$ // item i can be part of the solution

            if $b_i + B[i-1,w-w_i] > B[i-1,w]$

                $B[i,w] = b_i + B[i-1,w- w_i]$

            else

                $B[i,w] = B[i-1,w]$

        else $B[i,w] = B[i-1,w]$ // $w_i > w$

Running time

```
for w = 0 to W
    B[0,w] = 0
for i = 0 to n
    B[i,0] = 0
    for w = 0 to W
        < the rest of the code >
```

O(W)

Repeat n times

O(W)

What is the running time of this algorithm?

*O(n*W)*

Remember that the brute-force algorithm takes $O(2^n)$

Example

Let's run our algorithm on the following data:

n = 4 (# of elements)
W = 5 (max weight)
Elements (weight, benefit):
(2,3), (3,4), (4,5), (5,6)

Example (2)

W \ i	0	1	2	3	4
0	0				
1	0				
2	0				
3	0				
4	0				
5	0				

```
for w = 0 to W
        B[0,w] = 0
```

Example (3)

W \ i	0	1	2	3	4
0	0	0	0	0	0
1	0				
2	0				
3	0				
4	0				
5	0				

for i = 0 to n

B[i,0] = 0

Example (4)

Items:

1: (2,3)
2: (3,4)
3: (4,5)
4: (5,6)

W \ i	0	1	2	3	4
0	0	0	0	0	0
1	0	→ 0			
2	0				
3	0				
4	0				
5	0				

$i=1$

$b_i=3$

$w_i=2$

$w=1$

$w-w_i=-1$

if w_i <= w // item i can be part of the solution
if b_i + B[i-1,w-w_i] > B[i-1,w]
B[i,w] = b_i + B[i-1,w- w_i]
else
B[i,w] = B[i-1,w]
else ***B[i,w] = B[i-1,w]*** *// w_i > w*

Example (5)

Items:

1: (2,3)
2: (3,4)
3: (4,5)
4: (5,6)

W \ i	0	1	2	3	4
0	0	0	0	0	0
1	0	0			
2	0	**3**			
3	0				
4	0				
5	0				

$i=1$

$b_i=3$

$w_i=2$

$w=2$

$w-w_i = 0$

if $w_i <= w$ // item i can be part of the solution
    if $b_i + B[i-1,w-w_i] > B[i-1,w]$
        $B[i,w] = b_i + B[i-1,w-w_i]$
    else
        $B[i,w] = B[i-1,w]$
else $B[i,w] = B[i-1,w]$ // $w_i > w$

Example (6)

Items:

$\boxed{1: (2,3)}$

2: (3,4)

3: (4,5)

4: (5,6)

W \ i	0	1	2	3	4
0	0	0	0	0	0
1	0	0			
2	0	3			
3	0	**3**			
4	0				
5	0				

$i=1$

$b_i=3$

$w_i=2$

$w=3$

$w-w_i=1$

if $w_i \le w$ // item i can be part of the solution

if $b_i + B[i-1,w-w_i] > B[i-1,w]$

$B[i,w] = b_i + B[i-1,w- w_i]$

else

$B[i,w] = B[i-1,w]$

else $B[i,w] = B[i-1,w]$ // $w_i > w$

Example (7)

Items:

1: (2,3)
2: (3,4)
3: (4,5)
4: (5,6)

W \ i	0	1	2	3	4
0	0	0	0	0	0
1	0	0			
2	0	3			
3	0	3			
4	0	**3**			
5	0				

$i=1$
$b_i=3$
$w_i=2$
$w=4$
$w-w_i=2$

if w_i <= w // item i can be part of the solution
if $b_i + B[i-1,w-w_i] > B[i-1,w]$
$B[i,w] = b_i + B[i-1,w- w_i]$
else
$B[i,w] = B[i-1,w]$
else $B[i,w] = B[i-1,w]$ // $w_i > w$

Example (8)

Items:

1: (2,3)
2: (3,4)
3: (4,5)
4: (5,6)

W \ i	0	1	2	3	4
0	0	0	0	0	0
1	0	0			
2	0	3			
3	0	3			
4	0	3			
5	0	**3**			

$i=1$

$b_i=3$

$w_i=2$

$w=5$

$w-w_i=2$

if $w_i <= w$ *// item i can be part of the solution*

    if $b_i + B[i-1,w-w_i] > B[i-1,w]$

        $\mathbf{B[i,w] = b_i + B[i-1,w- w_i]}$

    else

        $B[i,w] = B[i-1,w]$

else $B[i,w] = B[i-1,w]$ *//* $w_i > w$

Example (9)

Items:

1: (2,3)
2: (3,4)
3: (4,5)
4: (5,6)

W \ i	0	1	2	3	4
0	0	0	0	0	0
1	0	0	→ **0**		
2	0	3			
3	0	3			
4	0	3			
5	0	3			

$i=2$

$b_i=4$

$w_i=3$

$w=1$

$w-w_i=-2$

if $w_i <= w$ *// item i can be part of the solution*

    if $b_i + B[i-1,w-w_i] > B[i-1,w]$

        $B[i,w] = b_i + B[i-1,w- w_i]$

    else

        $B[i,w] = B[i-1,w]$

else **$B[i,w] = B[i-1,w]$** *//* $w_i > w$

Example (10)

w \ i	0	1	2	3	4
0	0	0	0	0	0
1	0	0	0		
2	0	3	→ **3**		
3	0	3			
4	0	3			
5	0	3			

Items:

1: (2,3)
2: (3,4)
3: (4,5)
4: (5,6)

$i=2$

$b_i=4$

$w_i=3$

$w=2$

$w-w_i=-1$

if $w_i <= w$ *// item i can be part of the solution*
    if $b_i + B[i-1,w-w_i] > B[i-1,w]$
        $B[i,w] = b_i + B[i-1,w- w_i]$
    else
        $B[i,w] = B[i-1,w]$
else **B[i,w] = B[i-1,w]** *//* $w_i > w$

Example (11)

w \ i	0	1	2	3	4
0	0	0	0	0	0
1	0	0	0		
2	0	3	3		
3	0	3	**4**		
4	0	3			
5	0	3			

Items:

1: (2,3)
2: (3,4)
3: (4,5)
4: (5,6)

$i=2$

$b_i=4$

$w_i=3$

$w=3$

$w-w_i=0$

if $w_i <= w$ *// item i can be part of the solution*

if $b_i + B[i-1,w-w_i] > B[i-1,w]$

$\mathbf{B[i,w] = b_i + B[i-1,w- w_i]}$

else

$B[i,w] = B[i-1,w]$

else $B[i,w] = B[i-1,w]$ *//* $w_i > w$

Example (12)

Items:

1: (2,3)
2: (3,4)
3: (4,5)
4: (5,6)

W \ i	0	1	2	3	4
0	0	0	0	0	0
1	0	0	0		
2	0	3	3		
3	0	3	4		
4	0	3	**4**		
5	0	3			

$i=2$

$b_i=4$

$w_i=3$

$w=4$

$w-w_i=1$

if $w_i <= w$ // item i can be part of the solution

if $b_i + B[i-1,w-w_i] > B[i-1,w]$

$B[i,w] = b_i + B[i-1,w- w_i]$

else

$B[i,w] = B[i-1,w]$

else $B[i,w] = B[i-1,w]$ // $w_i > w$

Example (13)

Items:

1: (2,3)
2: (3,4)
3: (4,5)
4: (5,6)

W \ i	0	1	2	3	4
0	0	0	0	0	0
1	0	0	0		
2	0	3	3		
3	0	3	4		
4	0	3	4		
5	0	3	**7**		

$i=2$

$b_i=4$

$w_i=3$

$w=5$

$w-w_i=2$

if $w_i <= w$ *// item i can be part of the solution*

    if $b_i + B[i-1,w-w_i] > B[i-1,w]$

        $\mathbf{B[i,w] = b_i + B[i-1,w- w_i]}$

    else

        $B[i,w] = B[i-1,w]$

else $B[i,w] = B[i-1,w]$ *//* $w_i > w$

Example (14)

W \ i	0	1	2	3	4
0	0	0	0	0	0
1	0	0	0	**0**	
2	0	3	3	**3**	
3	0	3	4	**4**	
4	0	3	4		
5	0	3	7		

Items:

1: (2,3)
2: (3,4)
3: (4,5)
4: (5,6)

$i=3$

$b_i=5$

$w_i=4$

$w=1..3$

if $w_i <= w$ *// item i can be part of the solution*

    if $b_i + B[i-1,w-w_i] > B[i-1,w]$

        $B[i,w] = b_i + B[i-1,w- w_i]$

    else

        $B[i,w] = B[i-1,w]$

else **B[i,w] = B[i-1,w]** *//* $w_i > w$

Example (15)

W \ i	0	1	2	3	4
0	0	0	0	0	0
1	0	0	0	0	
2	0	3	3	3	
3	0	3	4	4	
4	0	3	4	**5**	
5	0	3	7		

$i=3$
$b_i=5$
$w_i=4$
$w=4$
$w- w_i=0$

Items:

1: (2,3)
2: (3,4)
3: (4,5)
4: (5,6)

if w_i <= w // item i can be part of the solution
if b_i + B[i-1,w-w_i] > B[i-1,w]
B[i,w] = b_i + B[i-1,w- w_i]
else
B[i,w] = B[i-1,w]
else B[i,w] = B[i-1,w] // w_i > w

Example (15)

Items:

1: (2,3)
2: (3,4)
3: (4,5)
4: (5,6)

W \ i	0	1	2	3	4
0	0	0	0	0	0
1	0	0	0	0	
2	0	3	3	3	
3	0	3	4	4	
4	0	3	4	5	
5	0	3	7	→ **7**	

$i=3$

$b_i=5$

$w_i=4$

$w=5$

$w- w_i=1$

if $w_i <= w$ *// item i can be part of the solution*

if $b_i + B[i-1,w-w_i] > B[i-1,w]$

$B[i,w] = b_i + B[i-1,w- w_i]$

else

$\boldsymbol{B[i,w] = B[i-1,w]}$

else $B[i,w] = B[i-1,w]$ *//* $w_i > w$

Example (16)

Items:

1: (2,3)
2: (3,4)
3: (4,5)
4: (5,6)

W \ i	0	1	2	3	4
0	0	0	0	0	0
1	0	0	0	0 →	**0**
2	0	3	3	3 →	**3**
3	0	3	4	4 →	**4**
4	0	3	4	5 →	**5**
5	0	3	7	7	

$i=3$

$b_i=5$

$w_i=4$

$w=1..4$

if $w_i <= w$ *// item i can be part of the solution*
    if $b_i + B[i-1,w-w_i] > B[i-1,w]$
        $B[i,w] = b_i + B[i-1,w- w_i]$
    else
        $B[i,w] = B[i-1,w]$
else $\mathbf{B[i,w] = B[i-1,w]}$ *//* $w_i > w$

Example (17)

Items:

1: (2,3)
2: (3,4)
3: (4,5)
4: (5,6)

W \ i	0	1	2	3	4
0	0	0	0	0	0
1	0	0	0	0	0
2	0	3	3	3	3
3	0	3	4	4	4
4	0	3	4	5	5
5	0	3	7	7 →	**7**

$i=3$

$b_i=5$

$w_i=4$

$w=5$

if $w_i \le w$ // item i can be part of the solution

if $b_i + B[i-1,w-w_i] > B[i-1,w]$

$B[i,w] = b_i + B[i-1,w-w_i]$

else

$B[i,w] = B[i-1,w]$

else $B[i,w] = B[i-1,w]$ // $w_i > w$

Comments

- This algorithm only finds the max possible value that can be carried in the knapsack
- To know the items that make this maximum value, an addition to this algorithm is necessary
- Refer to previous algorithms to figure out a strategy to extract this data from the table we built

Conclusion

- Dynamic programming is a useful technique of solving certain kind of problems
- When the solution can be recursively described in terms of partial solutions, we can store these partial solutions and re-use them as necessary
- Running time (Dynamic Programming algorithm vs. naïve algorithm):
  - LCS: **$O(m*n)$** vs. **$O(n * 2^m)$**
  - 0-1 Knapsack problem: **$O(W*n)$** vs. **$O(2^n)$**